DISPERSION OF LAMB WAVES PROPAGATING IN A PLATE MADE OF VISCOELASTIC MATERIAL

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ABSTRACT

Dispersion of the Lamb waves propagating in a plate made of viscoelastic material is investigated. The investigations are made by utilizing the exact equations of the theory of linear viscoelasticity. The dispersion equation is obtained for an arbitrary type of hereditary operator of the materials of the constituents and a solution algorithm is developed for obtaining numerical results. Dispersion curves is presented for certain attenuation cases and the influence of the viscosity of the material is studied through the rheological parameters which characterize the characteristic creep time, long-term values and the mechanical behavior of the viscoelastic material around the initial state of the deformation. Numerical results are presented and discussed for the case where the viscoelasticity of the materials is described through fractional-exponential operators by Rabotnov. In particular, how the rheological parameters influence on the dispersion of the Lamb waves are established.

ÖZET

Lineer viskoelastik malzemelerden oluşan plakalarda Lamb dalgalarının dispersyonu incelenmiştir. Araştırmalar, lineer viskoelastik teorisinin kesin denklemlerini uygulayarak yapılmıştır. Dispersyon denklemi, malzemenin herediter operatörünün keyfi türü için elde ettikten sonra incelenen dalgaların dispersyonu üzerine sayısal sonuçlar elde etmek için bir çözüm algoritması geliştirilmiştir. Farklı atenüasyon durumları için dispersyon eğrileri elde edilmiş ve malzemenin viskozitesinin etkisi, elastik sabitlerle karakterize eden reolojik parametreler, yani, Q viskoelastik malzemenin karakteristik akma süresi, d viskoelastik malzemenin mekanik sabitlerinin uzun vadeli değeri ve α viskoelastik malzemenin deformasyonun başlangıç durumu etrafındaki mekanik davranışı aracılığıyla incelenemiştir. Malzemenin viskoelastisitesini Rabotnov'un kısmi eksponansiyel operatörü'nü uygulayarak sayısal sonuçlar sunularak tartışılmıştır. Özellikle, reolojik parametrelerin, incelenen plakada yayılan Lamb dalgalarının dispersyonuna nasıl etkilediği tespit edilmiştir.

INTRODUCTION

The research of Lamb wave propagation problems related to viscoelastic materials have been widely studied in literature [1-6]. Several mathematical models have been used by many authors to study the dispersion behavior of such waves in viscoelastic media. However, in most cases either they have described the viscoelasticity of the materials through some simple models such as the classical Kelvin-Voigt spring-dashpot models or they have used complex elasticity modulus instead of the real one in the stress–strain relations of the viscoelastic materials. Consequently, in general, such a simple viscoelastic models and the numerical results obtained within these models cannot illustrate the real character of the influence of the rheological parameters of the viscoelastic materials on the corresponding wave dispersion and attenuation.

These considerations led the authors to study the Lamb waves dispersion for a system consisting of a viscoelastic material utilizing more realistic mathematical viscoelastic model using Rabotnov [7] fractional exponential operator which are used for the first time in the papers [8-10], where dimensionless rheological parameters characterizing the characteristic creep time (denoted by *Q*), the long-term values of the elastic constants (denoted by *d*) and the form of the creep (or relaxation) function of the materials in the beginning region of deformations (denoted by *α*), are introduced, and through these parameters the viscoelasticity of the materials of the plate on the dispersion curves is studied. An increase in the values of these parameters means a decrease in the viscosity properties of the related material. The numerical results are presented and discussed mainly for the attenuation dispersion case.

FORMULATION OF THE PROBLEM

Figure 1 shows the geometry of the problem and the Lamb waves propagate in the positive direction of Ox_1 axis.

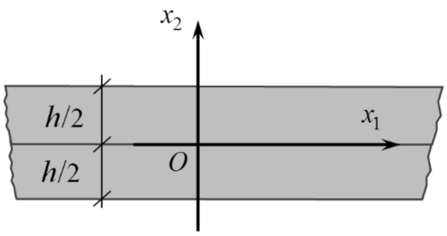


Figure 1. Geometry of the considered problem.

The mechanical relations of the model are as follows:

$$\sigma_{11} = \lambda^* \theta + 2\mu^* \varepsilon_{11}, \quad \sigma_{22} = \lambda^* \theta + 2\mu^* \varepsilon_{22}, \quad \sigma_{12} = 2\mu^* \varepsilon_{12},$$

$$\theta = \varepsilon_{11} + \varepsilon_{22}, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i = 1, 2, \tag{1}$$

and λ^*, μ^* are the following viscoelastic operators:

$$\lambda^* \varphi(t) = \lambda_0 \varphi(t) + \int_0^t \lambda_1 (t-\tau)\varphi(\tau)d\tau,$$

$$\mu^* \varphi(t) = \mu_0 \varphi(t) + \int_0^t \mu_1 (t-\tau)\varphi(\tau)d\tau. \tag{2}$$

We use the fractional exponential operators by Rabotnov [7] for describing the viscoelasticity of the constituents:

$$\mu^{*}\varphi(t)=\mu_{0}\left[\varphi(t)-\frac{3\beta_{0}}{2(1+\nu_{0})}R_{\alpha}^{*}\left(-\frac{3\beta_{0}}{2(1+\nu_{0})}-\beta_{\infty}\right)\varphi(t)\right],$$

$$\lambda^{*}\varphi(t)=\lambda_{0}\left[\varphi(t)-\frac{\beta_{0}}{(1+\nu_{0})}R_{\alpha}^{*}\left(-\frac{3\beta_{0}}{2(1+\nu_{0})}-\beta_{\infty}\right)\varphi(t)\right], \qquad (3)$$

where,

$$R_{\alpha}^{*}\left(x\right)\varphi(t)=\int_{0}^{t}R_{\alpha}\left(x,t-\tau\right)\varphi(\tau)d\tau,$$

$$R_{\alpha}\left(x,t\right)\varphi(t)=t^{-\alpha}\sum_{n=0}^{\infty}\frac{(x)^{n}t^{n(1-\alpha)}}{\Gamma((1+n)(1-\alpha))}, \qquad (4)$$

$\Gamma(x)$ is the gamma function and α, β_0 and β_∞ are the rheological parameters of the viscoelastic material [8].

Representing the displacements and strains as $u_i = v_i(x_2)e^{i(kx_1-\omega t)}$ and $\varepsilon_{ij}=\gamma_{ij}(x_2)e^{i(kx_1-\omega t)}$, and doing mathematical manipulations as described in the papers [9, 10] and introducing notations

$$\lambda_{1c}=\int_{0}^{\infty}\lambda_{1}(s)\cos\left(\omega s\right)ds, \qquad \lambda_{1s}=\int_{0}^{\infty}\lambda_{1}(s)\sin\left(\omega s\right)ds.$$
$$\mu_{1c}=\int_{0}^{\infty}\mu_{1}(s)\cos\left(\omega s\right)ds, \qquad \mu_{1s}=\int_{0}^{\infty}\mu_{1}(s)\sin\left(\omega s\right)ds, \qquad (5)$$

we obtain the following expressions for the stresses

$$\sigma_{11}=\left[\Lambda\left(\omega\right)\vartheta\left(x_{2}\right)+2M\left(\omega\right)\gamma_{11}\left(x_{2}\right)\right]e^{i(kx_{1}-\omega t)},$$
$$\sigma_{22}=\left[\Lambda\left(\omega\right)\vartheta\left(x_{2}\right)+2M\left(\omega\right)\gamma_{22}\left(x_{2}\right)\right]e^{i(kx_{1}-\omega t)},$$
$$\sigma_{12}=2M\left(\omega\right)\gamma_{12}\left(x_{2}\right)e^{i(kx_{1}-\omega t)}, \qquad (6)$$

where

$$\Lambda\left(\omega\right)=\lambda_{0}+\lambda_{1c}\left(\omega\right)+i\lambda_{1s}\left(\omega\right),$$
$$M\left(\omega\right)=\mu_{0}+\mu_{1c}\left(\omega\right)+i\mu_{1s}\left(\omega\right). \qquad (7)$$

Thus, we obtain the complex moduli $\Lambda\left(\omega\right)$ and $M\left(\omega\right)$, instead of Lame constants in relation (1).

Furthermore, according to [8, 9], we introduce the following dimensionless rheological parameters:

$$d=\frac{\beta_{\infty}}{\beta_{0}}, \qquad Q=\frac{c_{2}}{R\left(\beta_{0}+\beta_{\infty}\right)^{\frac{1}{1-\alpha}}}, \qquad (8)$$

The dimensionless rheological parameter d characterizes the long-term value of the viscoelastic material and the rheological parameter Q characterizes the creep time of the viscoelastic material, and finally the rheological parameter α characterizes the form of the creep (or relaxation) function for the viscoelastic material and the case where $\alpha = 0$ corresponds to the "standard solid body" model. Consequently, according to the foregoing expressions, the influence of the viscoelasticity of the material on the dispersion curves is estimated through these three dimensionless rheological parameters.

Finally, considering the boundary conditions, we obtain the system of the linear algebraic equations where from the existence of the non-trivial solution we get the dispersion equation:

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i, j = 1,2,3,4. \tag{9}$$

As the values of the determinant obtained in (9) are complex, the dispersion equation can be reduced to $\left|\det\left\|\alpha_{ij}\right\|\right| = 0$, where $|.|$ means the modulus of the complex number determinant. To find the roots of this dispersion equation here we use the algorithm which is based on direct calculation of the values of the modulus of the dispersion from the criterion $\left|\det\left\|\alpha_{ij}\right\|\right| \le 10^{-9}$, and the values of the wave dispersion velocity are determined under fixed values of the problem parameters.

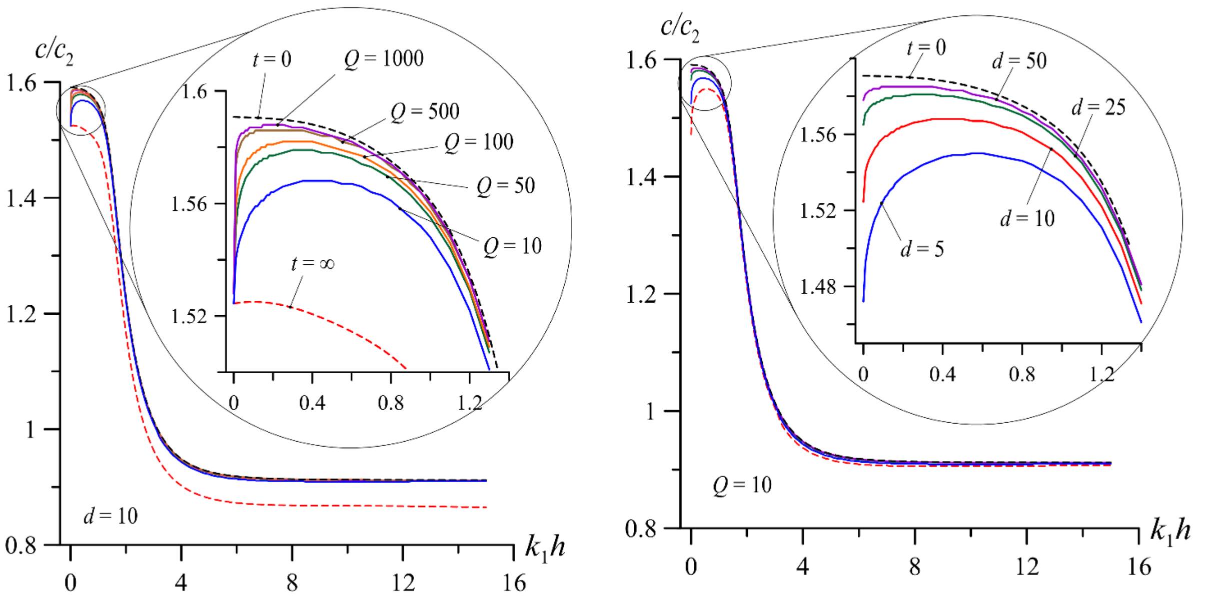
For solution to the dispersion equation (9) we employ the algorithm developed in [9] and under this solution procedure we use the relation $k = k_1 + ik_2 = k_1(1 + i\beta)$ for the complex wavenumber. We analyze the case where the attenuation parameter β, according to [11], is presented as follows:

$$\beta = \frac{1}{2}\frac{\mu_{1s}(\omega)}{\mu_0(\omega) + \mu_{1c}(\omega)}. \tag{10}$$

It is known that using the symmetry and asymmetry of the displacements of the plate, Lamb waves are divided into symmetric (extensional) and asymmetric (flexural) modes. In other words, the conditions $u_1(x_2) = u_1(-x_2)$ and $u_2(x_2) = -u_2(-x_2)$ are used for determination of the symmetric Lamb waves, and the conditions $u_1(x_2) = -u_1(-x_2)$ and $u_2(x_2) = u_2(-x_2)$ are used for determination of the asymmetric Lamb waves.

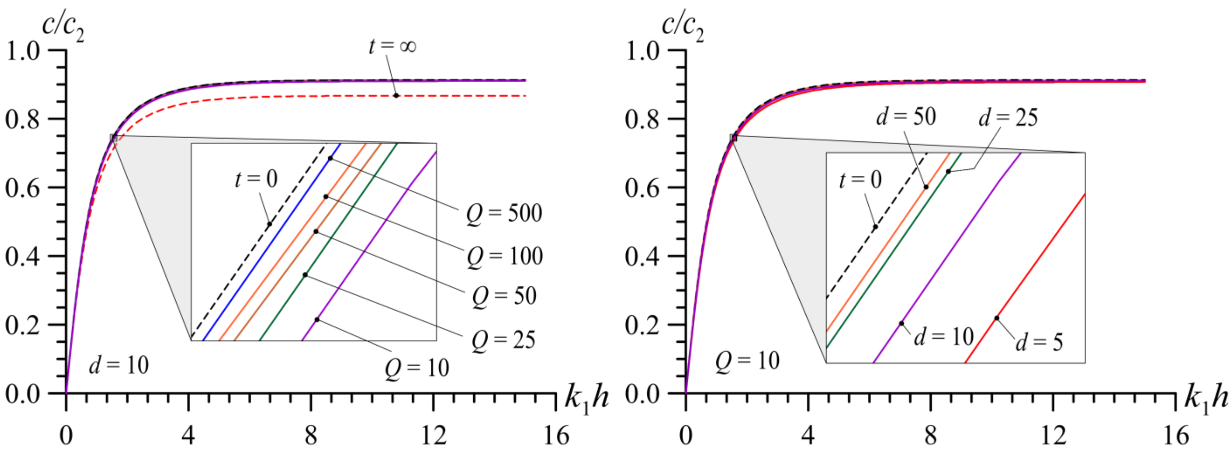
NUMERICAL RESULTS AND DISCUSSIONS

For numerical results we assume that $\upsilon = 0.3$, $\rho = 7190\ \text{kg/m}^3$, $\lambda_0 = 74.2\times10^3\text{MPa}$ and $\mu_0 = 102.5\times10^3\text{MPa}$. Figure 2a (Figure 3a) shows the dispersion curves constructed for various values of the parameter Q under d = 10 for symmetric (asymmetric) Lamb waves and Figure 2b (Figure 3b) gives the results that illustrated the influence of the parameter d on the behavior of the dispersion curves in the case where Q = 10 for symmetric (asymmetric) Lamb waves. Note that these graphs are constructed in the case where $\alpha = 0.5$.



(a) (b)

Figure 2. Symmetric mode: (a) Dispersion curves for different values of parameter Q under d =10. (b) Dispersion curves for different values of parameter d under Q =10.



(a) (b)

Figure 3. Asymmetric mode: (a) Dispersion curves for different values of parameter Q under d =10. (b) Dispersion curves for different values of parameter d under Q =10.

As discussed in the paper by Akbarov et al. [10], we can predict that for the all selected values of the parameter Q and under a fixed value of the parameter d the wave propagation velocity must have the same limit values as $k_1h \to 0$ and this limit velocity coincides with that obtained for the corresponding purely elastic case with long-term values of the elastic constants. As a result, we could say that these limit values of the wave propagation velocity depend on the rheological parameter d, not on the rheological parameters Q and α. Therefore dispersion curves obtained under fixed values of the parameter d are limited with the corresponding dispersion curves obtained for the purely elastic cases under instantaneous values of the elastic constants (upper limits), i.e. under $t=0$, and under long-term values of the elastic constants (lower limits), i.e. under $t=\infty$. It follows from Figure 2 and 3 that, first of all, the viscoelasticity of the plate material causes a decrease in the wave propagation velocity. Moreover, these results show that the dispersion curves obtained for the viscoelastic case approach to the corresponding one obtained for the purely elastic case with instantaneous (long-term) values of the elastic constants at $t = 0$ (at $t = \infty$) with increasing (decreasing) of the rheological parameters d and Q. It should be noted that the mentioned increase (decrease) has monotonic character and considerable effect in this increasing (decreasing) are observed in the cases where $k_1 h \leq 2.0$.

Moreover, according to the character of the dispersion curves obtained for symmetric mode of Lamb wave given in Figure 2 we can conclude that for each value of the rheological parameter Q and for each value of the rheological parameter d there exist the case where

$$\frac{d(c/c_2)}{d(k_1 h)} = 0. \qquad (11)$$

The wave propagation velocity and dimensionless wavenumber related to this case we denote by c_{cr} and $(k_1 h)_{cr}$, respectively. Note that for the dispersion curves related to the purely elastic waves there is not the case where the relation (11) takes place. Consequently, the appearing of the cases where the relation (11) takes place is caused namely with the viscoelasticity of the material of the plate under consideration. Thus, it follows from the foregoing discussions that, the viscoelasticity of the plate material influences on the dispersion curves of the Lamb waves not only in the quantitative sense but also in the qualitative sense.

Now let us consider the results related to the effect of rheological parameters α on the wave dispersion curves. Thus, we consider graphs given in Figures 4 and 5 which illustrate the mentioned influence for symmetric and asymmetric modes of Lamb wave, respectively. Note that the graphs given in Figure 4 (Figure 5) show the effect of the rheological parameter α on the dispersion curves under for a fixed value of the rheological parameter Q (=10) and d (=10). Also note that the case where $\alpha = 0$ coincides with the case where viscoelasticity of the plate material is described through the "standard solid body" model.

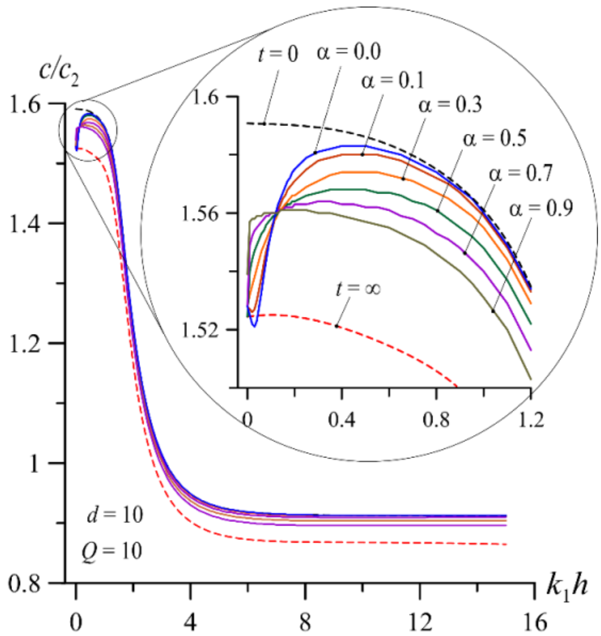


Figure 4. Symmetric mode: Dispersion curves for different values of parameter α under Q =10 and d =10.

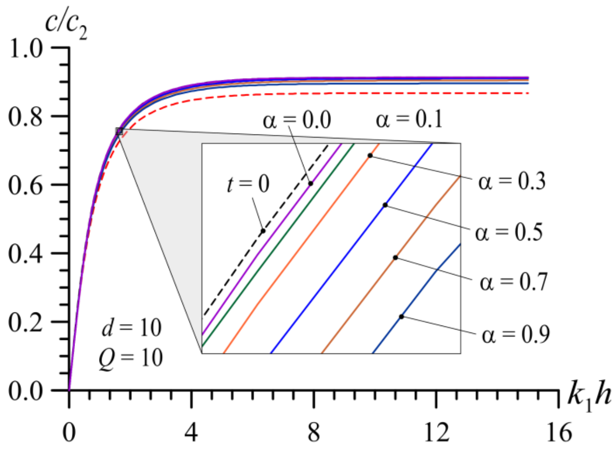


Figure 5. Asymmetric mode: Dispersion curves for different values of parameter α under Q =10 and d =10.

It follows from the results given in Figure 4 that for symmetric mode of Lamb wave there exists such value of the dimensionless wavenumber k_1h (denote it by $(k_1h)^*$) at which the change in the values of the rheological parameter α does not influence the values of the wave propagation velocity. However, in the cases where $k_1h > (k_1h)^*$ ($k_1h < (k_1h)^*$) an increase in the values of the parameter α causes a decrease (an increase) in the wave propagation velocities. According to the aforementioned numerical results, it can be concluded that the $(k_1h)^*$ depends on the values of the rheological parameters Q and d and an increase in the values of these parameters decrease the $(k_1h)^*$. However, for asymmetric mode of Lamb wave such value of the dimensionless wavenumber k_1h does not exist as shown in Figure 5 where dispersion curves have monotonic character.

We can also conclude that the change in the values of the rheological parameter α does not influence the limit values of the wave propagation velocity as $k_1h \to 0$. However, in the near vicinity of this limit case, if to say more precisely in the region $0 < k_1h < (k_1h)^*$, the influence of the rheological parameter α on the dispersion curves is significant not only in the quantitative sense but also in the qualitative sense. So that under small values of α, for instance under $\alpha = 0.1$, the dispersion curves have well-defined minimum in the region $0 < k_1h < (k_1h)^*$ and at this minimum the relation (11) takes place. Moreover, in the near vicinity of this minimum the wave propagation velocity obtained for the viscoelastic cases become less than that obtained for the purely elastic case with long-term values of elastic constants at $t = \infty$. Consequently, in the region $0 < k_1h < (k_1h)^*$ a decrease in the values of α causes to change the character of the dispersion curves. However, with increasing of α the aforementioned minimum disappears in the dispersion curves and wave propagation velocities are limited with the wave propagation velocities obtained for the purely elastic cases with instantaneous values of elastic constants at $t = 0$ (upper limit) and with long-term values of elastic constants at $t = \infty$ (lower limit).

CONCLUSIONS

According to these numerical results, the following main conclusions can be drawn:

- Viscoelasticity of the material of the plate causes both symmetric and asymmetric modes of the Lamb wave propagation velocity to decrease and the magnitude of this decrease increases with a decrease in the aforementioned dimensionless rheological parameters d and Q;
- For symmetric mode of propagation of the Lamb wave the character of the influence of the parameter α on the wave velocities and on the dispersion corves depends on the values of the dimensionless wavenumber $k_1 h$ and on the values of the rheological parameters d and Q.
- In general, the dispersion curves are limited by the purely elastic case with instantaneous values of the elastic constants and by those with long-term values of the elastic constants.
- The high wavenumber limit values of the wave propagation velocity do not depend on the rheological parameters Q and α of the plate material.

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