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STATE-CONSTRAINED SEMILINEAR ELLIPTIC OPTIMIZATION PROBLEMS WITH UNRESTRICTED SPARSE CONTROLS

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Dedicated to Prof. Dr. Fréderic Bonnans on the occasion of his 60th birthday

Abstract. In this paper, we consider optimal control problems associated with semilinear elliptic equation equations, where the states are subject to pointwise constraints but there are no explicit constraints on the controls. A term is included in the cost functional promoting the sparsity of the optimal control. We prove existence of optimal controls and derive first and second order optimality conditions. In addition, we establish some regularity results for the optimal controls and the associated adjoint states and Lagrange multipliers.

1. **Introduction.** In this paper, we analyze the following optimal control problem

$$\text{(P)} \quad \min_{u\in\mathcal{U}_{ad}} J(u)$$

with $J(u) = F(u) + \kappa j(u)$, $\kappa > 0$,

$$F(u) = \frac{1}{2}\int_\Omega (y_u(x) - y_d(x))^2\,dx + \frac{\nu}{2}\int_\Omega u^2(x)\,dx \quad \text{and} \quad j(u) = \int_\Omega |u(x)|\,dx,$$

where $y_d \in L^2(\Omega)$ and $\nu > 0$ are given. We define

$$\mathcal{U}_{ad} = \{u \in L^2(\Omega) : |y_u(x)| \le \gamma\ \forall x \in \bar\Omega\}$$

with some $\gamma > 0$, where y_u is the solution of the semilinear elliptic partial differential equation

$$\begin{cases} Ay + a(x, y(x)) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

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Above, $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a bounded open set with a Lipschitz boundary Γ. For the differential operator A we assume that

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}[a_{ij}(x)\partial_{x_i} y], \quad a_{ij} \in L^\infty(\Omega) \text{ for } 1 \le i, j \le n \tag{1.2}$$

and

$$\exists \Lambda > 0 \ \text{ such that } \ \Lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ \ \forall \xi \in \mathbb{R}^n \text{ and for a.a. } x \in \Omega. \tag{1.3}$$

We also assume that $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable satisfying

$$a(\cdot, 0) \in L^2(\Omega), \quad 0 \le \frac{\partial a}{\partial y}(x, y) \ \ \forall y \in \mathbb{R}, \tag{1.4}$$

$$\forall M > 0 \ \exists C_M > 0 : \Big|\frac{\partial^j a}{\partial y^j}(x, y)\Big| \le C_M \ \ \forall |y| \le M \text{ and } j = 1, 2, \tag{1.5}$$

$$\left\{ \begin{array}{l} \forall \rho > 0 \text{ and } \forall M > 0 \ \exists \varepsilon > 0 \text{ such that} \\ \Big|\dfrac{\partial^2 a}{\partial y^2}(x, y_2) - \dfrac{\partial^2 a}{\partial y^2}(x, y_1)\Big| < \rho \ \ \forall |y_i| \le M \text{ with } |y_2 - y_1| < \varepsilon, \end{array} \right. \tag{1.6}$$

for almost all $x \in \Omega$.

There are many papers devoted to the analysis of control problems with pointwise state constraints. As far as we know, control constraints are also included in all of them, except for a few papers dealing with error estimates for the numerical approximation; see [10], [14], [17]. Of course, the analysis is more involved when both type of constraints are present. However, stronger results can be proved if there is no explicit constraint on the controls. In this paper, we want to show these results that are not available under explicit control constraints.

After a second section, where we present some preliminary results, we prove first order optimality conditions for local solutions in §3. A linearized Slater assumption is usually made to derive the first order optimality conditions in a qualified form. However, for our control problem we prove that this condition is automatically fulfilled, it does not have to be assumed. An additional difficulty to get the optimality conditions is the presence of the non-differentiable term $j(u)$ in the cost functional that promotes the sparsity of the optimal control. This difficulty was overcome in [11] by using an abstract result of [3, Theorem 2.1]. In this paper, we provide a new abstract result under less restrictive assumptions; see Theorem 3.2. The presence of j in the cost functional promotes the sparsity of the optimal control; see Corollary 3.5. Actually, the size of the support of the optimal control can be monitored by the sparse parameter κ.

Assuming that the set of points where the state constraint is active has a zero Lebesgue measure, we prove the uniqueness of the Lagrange multiplier associated to the state constraints. This assumption is typically satisfied. Indeed, the set of points x such that $\bar{y}(x) = \gamma$, where $\bar{y}$ denotes an optimal state, is most of the times reduced to a finite number of points or (most frequently, see Theorem 3.8) it defines a line if $n = 2$ or a surface if $n = 3$. As a conclusion, we obtain also the uniqueness of the adjoint state $\bar{\varphi}$ and the multiplier $\bar{\lambda}$ corresponding to the non-differentiable term $j(u)$. Finally, under a very weak assumption on the nonlinear term of the state equation, we prove that the adjoint state belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$. This regularity is transferred to the optimal control. All these results are presented in Section 3.

In §4 we derive second order conditions for local optimality. We also prove that the different notions of local solution are equivalent for our control problem. Furthermore, we obtain that the usual quadratic growth inequality for local solutions is satisfied in an $L^2(\Omega)$-neighborhood of an optimal control if and only if it is satisfied in an $L^\infty(\Omega)$-neighborhood of the optimal state.

2. **Preliminary results.**

In this section we analyze the state equation and the cost functional J. The results are already known, but we recall them to fix the notation and for convenience of the reader. We start with a well known theorem on existence, uniqueness and regularity of the solution of (1.1).

Theorem 2.1. *Under the assumptions (1.2)–(1.5), for every $u \in L^2(\Omega)$ the state equation (1.1) has a unique solution $y_u \in H_0^1(\Omega) \cap C^{0,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$. Moreover, there exists a constant K_y such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C^{0,\alpha}(\bar\Omega)} \le K_y(\|u\|_{L^2(\Omega)} + \|a(\cdot,0)\|_{L^2(\Omega)}) \quad \forall u \in L^2(\Omega). \tag{2.1}$$

In addition, if $u_k \rightharpoonup u$ in $L^2(\Omega)$, then $y_{u_k} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar\Omega)$.

The proof of existence of a unique solution $y_u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is well known; see, for instance, [9, page 7] or [21, § 4.2]. The Hölder regularity follows from [16, Theorem 8.29]. The last part of the theorem is a consequence of the compactness of the embeddings $L^2(\Omega) \subset H^{-1}(\Omega)$ and $C^{0,\alpha}(\bar\Omega) \subset C(\bar\Omega)$.

Remark 2.2. The theorem is still valid, for different numbers $\alpha \in (0,1)$ and K_y, if the assumption $u, a(\cdot,0) \in L^2(\Omega)$ is replaced by $u, a(\cdot,0) \in L^p(\Omega)$ with $p > \frac{n}{2}$, see [21, § 4.2] and [16, Theorem 8.29].

Now, we prove the differentiability of the control-to-state mapping. Let us denote by $G : L^2(\Omega) \longrightarrow H_0^1(\Omega) \cap C^{0,\alpha}(\bar\Omega)$ the mapping defined by $G(u) = y_u$

Theorem 2.3. *Assume that (1.2)–(1.5) hold. Then G is of class C^2; furthermore, if $u, v \in L^2(\Omega)$ and $z_v = DG(u)v$, then z_v is the solution in $H_0^1(\Omega) \cap C^{0,\alpha}(\bar\Omega)$ of the Dirichlet problem*

$$\begin{cases} Az + \dfrac{\partial a}{\partial y}(x, y_u(x))z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Finally, for every $u, v_1, v_2 \in L^2(\Omega)$, $z_{v_1,v_2} = D^2G(u)(v_1, v_2)$ is the solution of

$$\begin{cases} Az + \dfrac{\partial a}{\partial y}(x, y_u(x))z + \dfrac{\partial^2 a}{\partial y^2}(x, y_u(x))z_{v_1}z_{v_2} = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma, \end{cases} \tag{2.3}$$

where $z_{v_i} = DG(u)v_i$ for $i = 1, 2$.

Proof. This result is an easy consequence of the implicit function theorem. Indeed, we define

$$V = \{y \in H_0^1(\Omega) \cap C^{0,\alpha}(\bar\Omega) : Ay \in L^2(\Omega)\}$$

and endow this space with the graph norm. Then V is a Banach space. Now, we define the mapping $\mathcal{F} : V \times L^2(\Omega) \longrightarrow L^2(\Omega)$ by $\mathcal{F}(y, u) = Ay + a(x, y) - u$. It is obvious that $\mathcal{F}$ is a C^2 mapping and its derivative $\dfrac{\partial \mathcal{F}}{\partial y}(y,u) : V \longrightarrow L^2(\Omega)$,

$$\frac{\partial \mathcal{F}}{\partial y}(y,u)z = Az + \frac{\partial a}{\partial y}(x, y(x))z,$$

is an isomorphism. Indeed, the injectivity and continuity of $\frac{\partial \mathcal{F}}{\partial y}(y,u)$ is obvious. Due to the open mapping theorem it is enough to prove that $\frac{\partial \mathcal{F}}{\partial y}(y,u) : V \longrightarrow L^2(\Omega)$ is surjective. Given an element $v \in L^2(\Omega)$, the existence of an element $z \in H_0^1(\Omega)$ satisfying $\frac{\partial \mathcal{F}}{\partial y}(y,u)z = v$ is a consequence of the Lax-Milgram theorem. The Hölder regularity of z follows from [16, Theorem 8.29]. Moreover, we have $Az = v - \frac{\partial a}{\partial y}(x, y(x))z \in L^2(\Omega)$ and therefore $z \in V$. Finally, since $\mathcal{F}(G(u), u) = 0$, the implicit function theorem implies the desired differentiability of G and simple computations lead to the equations (2.2) and (2.3). □

By a straightforward application of the chain rule we deduce the differentiability of the functional F. Recall that F is the first summand of the cost functional J.

Theorem 2.4. *The functional $F : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 and for every $u, v, v_1, v_2 \in L^2(\Omega)$ we have*

$$F'(u)v = \int_\Omega (\varphi_u + \nu u)v \, dx, \tag{2.4}$$

$$F''(u)(v_1, v_2) = \int_\Omega \left(\left[1 - \varphi_u \frac{\partial^2 a}{\partial y^2}(x, y_u(x))\right] z_{v_1} z_{v_2} + \nu v_1 v_2 \right) dx, \tag{2.5}$$

where $z_{v_i} = DG(u)v_i$, $i = 1, 2$, and $\varphi_u \in H_0^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ is the adjoint state, solution of

$$\begin{cases} A^*\varphi + \dfrac{\partial a}{\partial y}(x, y_u(x))\varphi = y_u - y_d & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{2.6}$$

Above, A^* denotes the adjoint operator of A given by

$$A^*\varphi = -\sum_{i,j=1}^n \partial_{x_j}[a_{ji}(x)\partial_{x_i}\varphi].$$

We finish this section by recalling some known properties of the functional j. Obviously, j is convex and Lipschitz. A simple computation shows that $\lambda \in \partial j(u)$ if and only if

$$\begin{cases} \lambda(x) = +1 & \text{if } u(x) > 0, \\ \lambda(x) = -1 & \text{if } u(x) < 0, \\ \lambda(x) \in [-1, +1] & \text{if } u(x) = 0 \end{cases} \tag{2.7}$$

holds a.e. in Ω. Further, j has directional derivatives given by

$$j'(u; v) = \lim_{\rho \searrow 0} \frac{j(u + \rho v) - j(u)}{\rho} = \int_{\Omega_u^+} v \, dx - \int_{\Omega_u^-} v \, dx + \int_{\Omega_u^0} |v| \, dx, \tag{2.8}$$

for $u, v \in L^1(\Omega)$, where Ω_u^+, Ω_u^- and Ω_u^0 represent the sets of points where u is positive, negative or zero, respectively. Finally, the following relation holds

$$\max_{\lambda \in \partial j(u)} \int_\Omega \lambda \, v \, dx = j'(u; v) \le \frac{j(u + \rho v) - j(u)}{\rho} \quad \forall\, 0 < \rho \le 1. \tag{2.9}$$

We refer to Clarke [12, Chapter 2] and Bonnans and Shapiro [4, § 2.4.3] for more details.

3. **Existence of optimal controls and first order optimality conditions.** The goal of this section is to prove the first order optimality conditions satisfied by any local solution of (P) and to deduce some important conclusions from the optimality system. First we observe that the classical approach of taking a minimizing sequence of (P) and to deduce its boundedness in $L^2(\Omega)$, which follows from the structure of the cost functional, produces weak limits that are solutions of (P). It is enough to use Theorem 2.1 in the last step. Hence, we have the following existence theorem.

Theorem 3.1. *Under the hypotheses* (1.2)–(1.5) *and assuming that $\mathcal{U}_{ad} \neq \emptyset$, the control problem* (P) *has at least one solution $\bar u$.*

In the sequel, $\bar u$ will denote a local solution of (P). Before proving the first order optimality conditions satisfied by $\bar u$, we establish two preliminary results. The first result is concerned with the optimality system for an abstract optimization problem that covers (P) as a particular case.

Let U and Y be two topological vector spaces and $K \subset U$ and $C \subset Y$ two convex sets. Given the mappings $G : U \longrightarrow Y$, $f : U \longrightarrow \mathbb{R}$ and $g : U \longrightarrow (-\infty, +\infty]$, we consider the optimization problem

$$\text{(Q)} \quad \min\{f(u) + g(u) : u \in K \text{ and } G(u) \in C\}.$$

The next theorem provides the optimality conditions satisfied by any local solution of (Q). The reader is referred to [3] for a related result.

Theorem 3.2. *Let $\bar u$ be a local solution of* (Q). *Assume that f and G are Gâteaux differentiable at $\bar u$, g is convex and continuous at some point of K, and $\mathrm{int}\, C \neq \emptyset$. Then there exist a real number $\bar\mu_0 \geq 0$, a multiplier $\bar\mu \in Y^*$, and $\bar\lambda \in \partial g(\bar u)$ such that*

$$(\bar\mu_0, \bar\mu) \neq (0, 0), \tag{3.1}$$

$$\langle \bar\mu, y - G(\bar u)\rangle_{Y^*,Y} \leq 0 \quad \forall y \in C, \tag{3.2}$$

$$\langle \bar\mu_0[f'(\bar u) + \bar\lambda] + [DG(\bar u)]^*\bar\mu, u - \bar u\rangle_{U^*,U} \geq 0 \quad \forall u \in K. \tag{3.3}$$

Moreover, if the linearized Slater condition

$$\exists u_0 \in K : G(\bar u) + DG(\bar u)(u_0 - \bar u) \in \mathit{int}\, C \tag{3.4}$$

is satisfied, then (3.3) *holds with $\bar\mu_0 = 1$.*

Proof. Let us define the sets

$$\begin{aligned} A = \{(y, t) \in Y \times \mathbb{R} : \exists u \in K \text{ such that } y = G(\bar u) + DG(\bar u)(u - \bar u) \\ \text{and } t \geq f'(\bar u)(u - \bar u) + g(u) - g(\bar u)\}, \\ B = \mathit{int}\, C \times (-\infty, 0) \subset Y \times \mathbb{R}. \end{aligned}$$

A and B are convex sets and B is open. Moreover, we have $A \cap B = \emptyset$. Indeed, if $(y_0, t_0) \in A \cap B$, then there exists $u_0 \in K$ such that $y_0 = G(\bar u) + DG(\bar u)(u_0 - \bar u) \in \mathit{int}\, C$ and $f'(\bar u)(u_0 - \bar u) + g(u_0) - g(\bar u) \leq t_0 < 0$. From here we infer the existence of a number $\rho_0 \in (0, 1)$ such that $\forall 0 < \rho < \rho_0$ and $u_\rho = \bar u + \rho(u_0 - \bar u)$

$$G(\bar u) + \frac{G(u_\rho) - G(\bar u)}{\rho} \in \mathrm{int}\, C \quad \text{and} \quad \frac{f(u_\rho) - f(\bar u)}{\rho} + g(u_0) - g(\bar u) < 0.$$

Then we have

$$G(u_\rho) = \rho\Big[G(\bar u) + \frac{G(u_\rho) - G(\bar u)}{\rho}\Big] + (1-\rho)G(\bar u) \in int\, C,$$

and with the convexity of g

$$\frac{f(u_\rho) - f(\bar u)}{\rho} + \frac{g(u_\rho) - g(\bar u)}{\rho} \le \frac{f(u_\rho) - f(\bar u)}{\rho} + g(u_0) - g(\bar u) < 0.$$

The above inequalities imply $f(u_\rho) + g(u_\rho) < f(\bar u) + g(\bar u)$ for every $0 < \rho < \rho_0$. Since we have $u_\rho \in K$ and $G(u_\rho) \in C$ for every $0 < \rho < \rho_0$, this contradicts the local optimality of $\bar u$. Therefore, we can separate the sets A and B (see, for instance, [5, pp. 5–7]) by a continuous linear form $(\bar\mu, \bar\mu_0) \in Y^* \times \mathbb{R}$:

$$\langle \bar\mu, y_1\rangle_{Y^*,Y} + \bar\mu_0 t_1 > \langle \bar\mu, y_2\rangle_{Y^*,Y} + \bar\mu_0 t_2 \quad \forall (y_1,t_1) \in A \text{ and } \forall (y_2,t_2) \in B. \tag{3.5}$$

From the strict inequality, (3.1) follows. Taking $y_1 = G(\bar u)$ and $t_1 > 0$ arbitrarily large, and fixing an element $(y_2,t_2) \in B$, (3.5) yields that $\bar\mu_0 \ge 0$. Now, by density of B in $C \times (-\infty, 0]$ and continuity of $\bar\mu$, we deduce from (3.5)

$$\langle \bar\mu, y_1\rangle_{Y^*,Y} + \bar\mu_0 t_1 \ge \langle \bar\mu, y_2\rangle_{Y^*,Y} + \bar\mu_0 t_2 \; \forall (y_1,t_1) \in A,\; \forall (y_2,t_2) \in C \times (-\infty, 0]. \tag{3.6}$$

Taking in (3.6) $y_1 = G(\bar u)$, $t_1 = t_2 = 0$, and $y_2 = y \in C$ arbitrarily, we obtain (3.2). To prove (3.3), we first set $y_1 = G(\bar u) + DG(\bar u)(u - \bar u)$, $y_2 = G(\bar u)$, $t_2 = 0$ and $t_1 = f'(\bar u)(u - \bar u) + g(u) - g(\bar u)$ with arbitrary $u \in K$. Thus we get

$$\langle \bar\mu_0 f'(\bar u) + [DG(\bar u)]^*\bar\mu, u - \bar u\rangle_{U^*,U} + \bar\mu_0[g(u) - g(\bar u)] \ge 0 \quad \forall u \in K. \tag{3.7}$$

This implies that $\bar u$ is solution of the optimization problem

$$\min_{u\in U} I(u) = \bar\mu_0 f'(\bar u)u + \bar\mu_0 g(u) + \langle [DG(\bar u)]^*\bar\mu, u\rangle_{U^*,U} + I_K(u),$$

where I_K denotes the indicator function of K. Since g is continuous at some point of K, we can apply the subdifferential calculus to obtain

$$0 \in \partial I(\bar u) = \bar\mu_0[f'(\bar u) + \partial g(\bar u)] + [DG(\bar u)]^*\bar\mu + \partial I_K(\bar u).$$

Hence, there exists an element $\bar\lambda \in \partial g(\bar u)$ such that (3.3) holds. Finally, we assume the linearized Slater condition and prove that $\bar\mu_0 > 0$, then we take $\frac{1}{\bar\mu_0}\bar\mu$ as new Lagrange multiplier. Renaming this multiplier by $\bar\mu$, we conclude the proof. To show that $\bar\mu_0 > 0$, we argue by contradiction. Assume that $\bar\mu_0 = 0$. Inserting $u = u_0$ in (3.3) and $y = y_0 = G(\bar u) + DG(\bar u)(u_0 - \bar u)$ in (3.2), we infer that $\langle \bar\mu, DG(\bar u)(u_0 - \bar u)\rangle_{Y^*,Y} = 0$. Let V be a neighborhood of 0 in Y such that $y_0 + V \subset int\, C$. Then, with (3.2) we get

$$\langle \bar\mu, y\rangle_{Y^*,Y} = \langle \bar\mu, y + DG(\bar u)(u_0 - \bar u)\rangle_{Y^*,Y} = \langle \bar\mu, y + y_0 - G(\bar u)\rangle_{Y^*,Y} \le 0 \quad \forall y \in V.$$

This implies that $\bar\mu = 0$, contradicting (3.1). □

Before establishing the optimality system (3.1)–(3.3) for our control problem (P), we prove that the linearized Slater condition is satisfied by any local solution $\bar u$ of (P). We say that $\bar u$ is a local solution or a local minimizer of (P) if there exists $\varepsilon > 0$ such that

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|u - \bar u\|_{L^2(\Omega)} \le \varepsilon.$$

The reader is referred to Section 4, Definition 4.1, Theorem 4.2 and Remark 4.3 for additional comments on this definition.

Theorem 3.3. *Let $\bar{u}$ be a local solution of* (P) *with associated state $\bar{y}$ and assume that* (1.2)–(1.5) *hold. Then the linearized Slater condition*

$$\exists u_0 \in L^2(\Omega) : |\bar{y}(x) + z_{u_0-\bar{u}}(x)| < \gamma \quad \forall x \in \bar{\Omega} \tag{3.8}$$

is fulfilled, where $z_{u_0-\bar{u}} = DG(\bar{u})(u_0 - \bar{u})$ is the solution of (2.2) *for $v = u_0 - \bar{u}$.*

Proof. Let us set $v = A\bar{y} + \dfrac{\partial a}{\partial y}(x, \bar{y}(x))\bar{y}$. Obviously we have that $v \in L^2(\Omega)$. Fix $\rho \in (0, 1)$ and take $u_0 = \bar{u} - \rho v$. Then we have $z_{u_0-\bar{u}} = -\rho\bar{y}$. Indeed, it is enough to check that $z_{u_0-\bar{u}}$ and $-\rho\bar{y}$ satisfy the same equation, namely $Az + \dfrac{\partial a}{\partial y}(x, \bar{y}(x))z = -\rho v$. Therefore, it follows

$$|\bar{y}(x) + z_{u_0-\bar{u}}(x)| = (1 - \rho)|\bar{y}(x)| \le (1 - \rho)\gamma < \gamma \quad \forall x \in \bar{\Omega}.$$

□

Now, the optimality system satisfied by $\bar{u}$ follows.

Theorem 3.4. *If $\bar{u}$ is a local solution of* (P) *with associated state $\bar{y}$ and* (1.2)–(1.5) *hold, then there exist $\bar{\varphi} \in W_0^{1,p}(\Omega)$ for all $1 \le p < \frac{n}{n-1}$, $\bar{\lambda} \in \partial j(\bar{u})$, and $\bar{\mu} \in M(\Omega)$ such that*

$$\begin{cases} A^*\bar{\varphi} + \dfrac{\partial a}{\partial y}(x, \bar{y}(x))\bar{\varphi} = \bar{y} - y_d + \bar{\mu} & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma, \end{cases} \tag{3.9}$$

$$\int_\Omega (y(x) - \bar{y}(x))\, d\bar{\mu}(x) \le 0 \quad \forall y \in \overline{B_\gamma(0)}, \tag{3.10}$$

$$\bar{\varphi} + \nu\bar{u} + \kappa\bar{\lambda} = 0. \tag{3.11}$$

Proof. We apply Theorem 3.2 taking $K = U = L^2(\Omega)$; $Y = C_0(\Omega) = \{y \in C(\bar{\Omega}) : y = 0 \text{ on } \Gamma\}$; $C = \overline{B_\gamma(0)}$ the closed ball of $C_0(\Omega)$ with center 0 and radius γ; $f = F$ and $g = \kappa j$. Moreover, $G : L^2(\Omega) \longrightarrow C_0(\Omega)$ is the control-to-state mapping $u \mapsto y_u$. We recall that the dual of $C_0(\Omega)$ is the space $M(\Omega)$ of real and regular Borel measures in Ω. Notice that under this identification of sets and functions, (3.8) is the Slater assumption (3.4) corresponding to (P). Hence, the existence of $\bar{\mu} \in M(\Omega)$ and $\bar{\lambda} \in \partial j(\bar{u})$ satisfying the conditions (3.2) and (3.3) with $\bar{\mu}_0 = 1$ follows from theorems 3.2 and 3.3. In our setting, the inequality (3.2) is the same as (3.10). Moreover, since $K = L^2(\Omega)$ is the whole space, (3.3) can be written as an identity

$$\int_\Omega [F'(\bar{u}) + \kappa\bar{\lambda} + [DG(\bar{u})]^*\bar{\mu}]v\, dx = 0 \quad \forall v \in L^2(\Omega),$$

or equivalently with (2.4)

$$\int_\Omega (\bar{y} - y_d)z_v\, dx + \int_\Omega z_v\, d\bar{\mu} + \int_\Omega (\nu\bar{u} + \kappa\bar{\lambda})v\, dx = 0 \quad \forall v \in L^2(\Omega),$$

where $z_v = G'(\bar{u})v$ is the solution of (2.2). Finally, let $\bar{\varphi} \in W_0^{1,p}(\Omega)$ for all $1 \le p < \frac{n}{n-1}$ be the solution of (3.9). Then the above identity can be written as

$$\int_\Omega (\bar{\varphi} + \nu\bar{u} + \kappa\bar{\lambda})v\, dx = 0 \quad \forall v \in L^2(\Omega),$$

which is equivalent to (3.11). □

From this theorem, we get the following properties.

Corollary 3.5. *Under the assumptions of Theorem 3.4, the following properties hold*

(1) $\bar{\lambda}$ *and* $\bar{\varphi}$ *are related by the formula*

$$\bar{\lambda}(x) = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\kappa}\bar{\varphi}(x)\Big) \quad \text{for a.a. } x \in \Omega. \tag{3.12}$$

Further, we have that $\bar{\lambda} \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

(2) *The regularity* $\bar{u} \in W_0^{1,p}(\Omega)$ *for every* $1 \le p < \frac{n}{n-1}$ *is fulfilled.*

(3) *The sparsity relation* $\bar{u}(x) = 0 \Leftrightarrow |\bar{\varphi}(x)| \le \kappa$ *for a.a.* $x \in \Omega$ *holds.*

(4) *If* $\bar{\mu} = \bar{\mu}^+ - \bar{\mu}^-$ *is the Jordan decomposition of* $\bar{\mu}$*, then* $\operatorname{supp}\mu^+ \subset \Omega_\gamma^+$ *and* $\operatorname{supp}\mu^- \subset \Omega_\gamma^-$*, where*

$$\Omega_\gamma^+ = \{x \in \Omega : \bar{y}(x) = +\gamma\} \quad \text{and} \quad \Omega_\gamma^- = \{x \in \Omega : \bar{y}(x) = -\gamma\}.$$

(5) *Let* y_0 *be the state associated to the null control. We assume that* y_0 *belongs to the open ball* $B_\gamma(0) \subset C_0(\Omega)$*. Then, there exists* $\kappa_0 > 0$ *such that* $\bar{u} \equiv 0$ *is the unique solution of* (P) *for every* $\kappa \ge \kappa_0$.

(6) *If* $y_0 \in B_\gamma(0)$*, then there exists* $C_\mu > 0$ *such that* $\|\bar{\mu}\|_{M(\Omega)} \le C_\mu \ \forall \kappa \ge 0$.

Proof. The reader is referred to [8] for the proof of (3.12) and (3). The $H_0^1(\Omega)$-regularity of $\bar{\lambda}$ follows from Lemma 3.6 below. Note that $\bar{\lambda} \in L^\infty(\Omega)$ is implied by $\bar{\lambda}(x) \in [-1,1]$ for a.a. x. This regularity and the one of $\bar{\varphi}$ established in Theorem 3.4 along with (3.11) imply (2). A proof of (4) can be found in [6]; see also [10].

Proof of (5). Let us denote by (P_κ) the control problem corresponding to a fixed value κ. With u_κ and y_κ we denote a local solution of (P_κ) and its associated state. Thanks to Theorem 3.4 we know the existence of $(\lambda_\kappa, \mu_\kappa, \varphi_\kappa)$ satisfying (3.9)–(3.11). Since $0 \in \mathcal{U}_{ad}$, we have

$$J(u_\kappa) \le J(0) = \frac{1}{2}\|y_0 - y_d\|^2_{L^2(\Omega)} \quad \forall \kappa \ge 0,$$

where y_0 is the state associated with the control 0. From the above inequality we infer

$$\|y_\kappa - y_d\|_{L^2(\Omega)} \le \|y_0 - y_d\|_{L^2(\Omega)}, \tag{3.13}$$

$$\|u_\kappa\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\nu}}\|y_0 - y_d\|_{L^2(\Omega)}, \tag{3.14}$$

$$\|u_\kappa\|_{L^1(\Omega)} \le \frac{1}{2\kappa}\|y_0 - y_d\|^2_{L^2(\Omega)}. \tag{3.15}$$

From (3.14) and (3.15), and Hölder's inequality with $q = \frac{4}{3}$ and $q' = 4$, we obtain

$$\|u_\kappa\|_{L^{7/4}(\Omega)} \le \|u_\kappa\|^{6/7}_{L^2(\Omega)}\|u_\kappa\|^{1/7}_{L^1(\Omega)} \le \frac{1}{\sqrt[7]{2\nu^3\kappa}}\|y_0 - y_d\|^{8/7}_{L^2(\Omega)}.$$

This implies that $u_\kappa \to 0$ strongly in $L^{7/4}(\Omega)$ as $\kappa \to \infty$. Since $\frac{7}{4} > \frac{n}{2}$ for $n \in \{2,3\}$, we deduce from Theorem 2.1 and Remark 2.2 that $y_\kappa \to y_0$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$. Due to our assumption $y_0 \in B_\gamma(0)$, we conclude the existence of $\kappa_\gamma > 0$ such that $y_\kappa \in B_\gamma(0) \ \forall \kappa \ge \kappa_\gamma$. Hence, the state constraint is not active for $\kappa \ge \kappa_\gamma$ and (4) implies that $\mu_\kappa = 0$. Now, from (3.9), Theorem 2.1, and (3.13), we infer the existence of $C_1 > 0$ such that

$$\|\varphi_\kappa\|_{C_0(\Omega)} \le C_1\|y_\kappa - y_d\|_{L^2(\Omega)} \le C_1\|y_0 - y_d\|_{L^2(\Omega)} \quad \forall \kappa \ge \kappa_\gamma. \tag{3.16}$$

Finally, we take

$$\kappa_0 = \max\left\{\kappa_\gamma, C\|y_0 - y_d\|_{L^2(\Omega)}\right\}.$$

We have proved that $\|\varphi_\kappa\|_{C_0(\Omega)} \le \kappa$ for every $\kappa \ge \kappa_0$. Then, (3) implies that $u_\kappa \equiv 0$ $\forall \kappa \ge \kappa_0$.

Proof of (6). First we prove that

$$\exists C > 0 \;\text{ such that }\; \|\varphi_\kappa\|_{L^2(\Omega)} \le C \quad \forall \kappa \ge 0. \tag{3.17}$$

To this end, first we assume that $\kappa \le \kappa_0$. Then, from (3.11), (3.14) and the fact that $\|\lambda_\kappa\|_{L^\infty(\Omega)} \le 1$, we deduce

$$\|\varphi_\kappa\|_{L^2(\Omega)} \le \nu\|u_\kappa\|_{L^2(\Omega)} + \kappa\|\lambda_\kappa\|_{L^2(\Omega)} \le \sqrt{\nu}\|y_0 - y_d\|_{L^2(\Omega)} + \kappa_0\sqrt{|\Omega|} = C_2.$$

If $\kappa \ge \kappa_0$, then from (3.16) it follows

$$\|\varphi_\kappa\|_{L^2(\Omega)} \le \sqrt{|\Omega|}\|\varphi_\kappa\|_{C_0(\Omega)} \le \sqrt{|\Omega|}C_1\|y_0 - y_d\|_{L^2(\Omega)} = C_3.$$

Hence, if we take $C = \max\{C_1, C_3\}$, the estimate (3.17) is proved. To get the estimate for μ_κ we use (3.10), (3.9) and (1.1) as follows

$$\gamma\|\mu_\kappa\|_{M(\Omega)} = \sup_{\|y\|_{C_0(\Omega)} \le \gamma} \int_\Omega y\, d\mu_\kappa \le \int_\Omega y_\kappa\, d\mu_\kappa = \int_\Omega \Big(A^*\varphi_\kappa + \frac{\partial a}{\partial y}(x, y_\kappa(x))\varphi_\kappa\Big) y_\kappa\, dx$$
$$- \int_\Omega (y_\kappa - y_d) y_\kappa\, dx = \int_\Omega \Big(A y_\kappa + \frac{\partial a}{\partial y}(x, y_\kappa(x)) y_\kappa\Big)\varphi_\kappa\, dx - \int_\Omega (y_\kappa - y_d) y_\kappa\, dx$$
$$= \int_\Omega \Big(u_\kappa - a(x, y_\kappa(x)) + \frac{\partial a}{\partial y}(x, y_\kappa(x)) y_\kappa\Big)\varphi_\kappa\, dx - \int_\Omega (y_\kappa - y_d) y_\kappa\, dx.$$

Finally, using (3.14), (3.17) and the fact that $\|y_\kappa\|_{C_0(\Omega)} \le \gamma$, we conclude that the last two integrals are bounded by a constant C independent of κ. □

Lemma 3.6. *Let $\mu \in \mathcal{M}(\Omega)$ and let $\varphi \in W_0^{1,p}(\Omega)$ for all $p < n/(n-1)$ be the solution of*

$$\begin{cases} A^*\varphi + a_0(x)\varphi = \mu & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma \end{cases}$$

with $a_0 \ge 0$ and $a_0 \in L^\infty(\Omega)$. Then, $\mathrm{Proj}_{[-M,M]}(\varphi)$ belongs to $H_0^1(\Omega)$ for every $M > 0$.

This result can be deduced from [13, Th. 10.1 and Eq. (2.22)] or [15, Eq.(7)]. See also [10] for a detailed proof.

Next we analyze the uniqueness of the Lagrange multiplier $\bar\mu$. If $y_0 \in B_\gamma(0)$, as assumed in Corollary 3.5, (5) and (6), and $\bar u \equiv 0$, then $\bar\mu = 0$ is obviously the only Lagrange multiplier associated with $\bar u$. In the next theorem, we consider the case of a non zero locally optimal control. First, we introduce some notation. We assume that $\bar y$ is the state associated with $\bar u$ and

$$K_\gamma = \{x \in \Omega : |\bar y(x)| = \gamma\}$$

denotes the set of points where the state constraint is active. Because of the continuity of $\bar y$ and the boundedness of Ω, the set K_γ is compact.

Theorem 3.7. *Let the assumptions (1.2)–(1.5) be fulfilled and suppose that $a_{ij} \in C^{0,1}(\bar\Omega)$ for $1 \le i, j \le n$, $\bar u$ is a nonzero local minimizer of (P), and K_γ has a zero Lebesgue measure. Then the Lagrange multiplier $\bar\mu$ satisfying the optimality conditions (3.9)–(3.11) is unique. As a consequence, $\bar\varphi$ and $\bar\lambda$ are unique as well.*

Proof. We introduce the set

$$\Omega_{\bar u} = \{x \in \Omega \setminus K_\gamma : |\bar\varphi(x)| > \kappa\},$$

where $\bar\varphi$ is the adjoint state corresponding to $\bar u$. We observe that equation (3.9) implies that $\bar\varphi$ is continuous in the open set $\Omega \setminus K_\gamma$; see [20, Theorem 9.3]. Hence, $\Omega_{\bar u}$ is also an open set. The identity (3.12) implies that $\bar\lambda \in C(\Omega \setminus K_\gamma)$ as well. Finally, we get from (3.11) that $\bar u$ is also continuous in $\Omega \setminus K_\gamma$. Moreover, according to Corollary 3.5-(3), we have that $\bar u(x) \neq 0$ holds for all $x \in \Omega_{\bar u}$. Now, we define the linear operator

$$T : L^2(\Omega_{\bar u}) \longrightarrow C(K_\gamma) \ \text{ by } \ T(v) = z_v,$$

where the elements $v \in L^2(\Omega_{\bar u})$ are extended by 0 to Ω and $z_v = G'(\bar u)v$ is the solution of (2.2) for $u = \bar u$. The remaining proof is split into two parts.

1.- $\mathcal{R}(T) = \{Tv : v \in L^2(\Omega_{\bar u})\}$ is dense in $C(K_\gamma)$. We argue by contradiction: if $\overline{\mathcal{R}(T)} \neq C(K_\gamma)$, then we deduce the existence of a measure $\mu \in M(K_\gamma) = C(K_\gamma)^*$ with $\mu \neq 0$ such that

$$\int_{K_\gamma} z_v \, d\mu = 0 \quad \forall v \in L^2(\Omega_{\bar u}). \tag{3.18}$$

Let $\psi \in W_0^{1,p}(\Omega)$ for all $p < n/(n-1)$ be the solution of

$$\left\{\begin{array}{l} A^*\psi + \dfrac{\partial a}{\partial y}(x, \bar y(x))\psi = \mu \ \text{ in } \Omega, \\ \psi = 0 \ \text{ on } \Gamma. \end{array}\right. \tag{3.19}$$

Then, for every $v \in L^2(\Omega_{\bar u})$ we have with (3.18)

$$\begin{aligned} \int_{\Omega_{\bar u}} \psi v \, dx &= \int_\Omega \psi v \, dx = \int_\Omega \Big[A z_v + \frac{\partial a}{\partial y}(x, \bar y(x)) z_v\Big]\psi \, dx \\ &= \int_\Omega \Big[A^*\psi + \frac{\partial a}{\partial y}(x, \bar y(x))\psi\Big] z_v \, dx = \int_{K_\gamma} z_v \, d\mu = 0, \end{aligned} \tag{3.20}$$

hence, ψ vanishes in $\Omega_{\bar u}$. Moreover, we get from (3.19)

$$A^*\psi + \frac{\partial a}{\partial y}(x, \bar y(x))\psi = 0 \ \text{ in } \Omega \setminus K_\gamma.$$

Then, we infer from a uniqueness result of [19] that $\psi = 0$ holds in $\Omega \setminus K_\gamma$. Since $|K_\gamma| = 0$, we conclude that $\psi = 0$ in Ω and, hence, $\mu = 0$, which contradicts our assumption.

2.- Uniqueness of the Lagrange multiplier. Let us assume that $\bar\mu_1, \bar\mu_2 \in M(\Omega)$ are Lagrange multipliers satisfying (3.9)–(3.11) with adjoint states $\bar\varphi_1$ and $\bar\varphi_2$, respectively. We will prove that $\bar\mu_1 = \bar\mu_2$. We know from Corollary 3.5-(4) that $\operatorname{supp} \bar\mu_i \subset K_\gamma$ holds for $i = 1, 2$. Take an arbitrary $v \in L^2(\Omega_{\bar u})$ that is extended by zero to Ω. Then, by definition of $\Omega_{\bar u}$, (3.11) and (2.7) we have for $i = 1, 2$

$$\int_\Omega \bar\varphi_i v \, dx = -\nu \int_\Omega \bar u v \, dx - \kappa \int_\Omega \bar\lambda_i v \, dx = -\nu \int_\Omega \bar u v \, dx - \kappa\Big(\int_{\Omega_{\bar u}^+} v \, dx + \int_{\Omega_{\bar u}^-} v \, dx\Big),$$

where $\Omega_{\bar u}^+$ and $\Omega_{\bar u}^-$ denote the sets of points of $\Omega_{\bar u}$ where $\bar u$ takes positive and negative values, respectively. Taking $\psi = \bar\varphi_2 - \bar\varphi_1$, $\mu = \bar\mu_2 - \bar\mu_1$, and subtracting the equations for $\bar\varphi_i$, $i = 1, 2$, we get that ψ satisfies (3.19). Moreover from the above equality and (3.20) we infer

$$0 = \int_\Omega \psi v\, dx = \int_\Omega \Big[Az_v + \frac{\partial a}{\partial y}(x, \bar y(x))z_v\Big]\psi\, dx = \int_{K_\gamma} z_v\, d\mu \quad \forall v \in L^2(\Omega_{\bar u}).$$

Due to the density $\overline{\mathcal{R}(T)} = C(K_\gamma)$ we obtain that $\langle \mu, y\rangle_{M(K_\gamma),C(K_\gamma)} = 0$ for all $y \in C(K_\gamma)$, hence $\mu = 0$, as desired. Finally, $\bar\varphi$ is uniquely determined by the equation (3.9) and $\bar\lambda$ is given by (3.12). □

We finish this section by a surprising regularity result.

Theorem 3.8. *Suppose that the assumptions (1.2)–(1.5) hold. Let $\bar u$ be an element of $\mathcal{U}_{ad}$ satisfying the optimality system (3.9)–(3.11) along with its associated state $\bar y$. Assume in addition that at least one of the following two conditions holds*

$$a(\cdot, 0) \in L^\infty(\Omega), \tag{3.21}$$

$$\exists\varepsilon \in (0, \gamma) : a(x, -s) \le 0 \text{ and } a(x, s) \ge 0 \text{ for a.a. } (x, s) \in \Omega \times [\gamma - \varepsilon, \gamma]. \tag{3.22}$$

Then, we have that $\bar u, \bar\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $\bar\mu \in H^{-1}(\Omega) \cap M(\Omega)$.

The assumption (3.22) means that $a(x, s) \le 0$ is satisfied in a certain neighborhood of $-\gamma$, and $a(x, s) \ge 0$ in a neighborhood of γ. In particular, this holds if $a(x, 0) = 0$ in Ω. Before proving this theorem, we recall a preliminary result whose proof can be found in [11, Lemma 2.9]; see also [18].

Lemma 3.9. *Let $\mu \in M(\Omega)$ be a positive measure with a compact support in Ω and let g_A be the Green's function corresponding to the Dirichlet problem associated with $A^* + \frac{\partial a}{\partial y}(x, \bar y(x))I$. Define φ_μ as the solution to the problem*

$$\begin{cases} A^*\varphi_\mu + \dfrac{\partial a}{\partial y}(x, \bar y(x))\varphi_\mu = \mu & \text{in } \Omega,\\ \varphi_\mu = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\varphi^*_\mu(x) := \int_\Omega g_A(x, \xi)\, d\mu(\xi) \quad \forall x \in \Omega.$$

Then it holds $\varphi^*_\mu(x) = \varphi_\mu(x)$ a.e. in Ω and

$$\varphi_\mu \in L^\infty(\Omega) \Leftrightarrow \sup_{x\in \operatorname{supp}\mu} \varphi^*_\mu(x) < \infty.$$

Proof of Theorem 3.8. It is enough to prove that $\bar\varphi$ belongs to $L^\infty(\Omega)$. Then, the regularity $\bar\varphi \in H^1_0(\Omega)$ is an immediate consequence of Lemma 3.6. Indeed, if we take $M = \|\bar\varphi\|_{L^\infty(\Omega)}$, then we immediately deduce that $\bar\varphi = \mathrm{Proj}_{[-M,M]}(\bar\varphi) \in H^1_0(\Omega)$. The regularity $\bar u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ follows from (3.11) and Corollary 3.5-(1). Finally, the adjoint state equation (3.9) implies that $\bar\mu \in H^{-1}(\Omega)$.

The proof of the boundedness of $\bar\varphi$ is inspired by the proof of [11, Theorem 2.6]. Let us write $\bar\varphi = \bar\varphi^0 + \bar\varphi^+ - \bar\varphi^-$, where $\bar\varphi^0$, $\bar\varphi^+$ and $\bar\varphi^-$ are the solutions of the adjoint equation (3.9) with right hand side $\bar y - y_d$, $\bar\mu^+$, and $\bar\mu^-$, respectively. Since $\bar y - y_d \in L^2(\Omega)$, we know that $\bar\varphi^0 \in H^1_0(\Omega) \cap C(\bar\Omega)$. We will prove that $\bar\varphi^+$ and $\bar\varphi^-$ belong to $L^\infty(\Omega)$. We observe that both functions are nonnegative. Thus, we only need to prove that they are bounded from above. Let us define the open sets

$$\Omega^+ = \{x \in \Omega : \bar y(x) > \gamma - \varepsilon\} \qquad \text{and} \qquad \Omega^- = \{x \in \Omega : \bar y(x) < -\gamma + \varepsilon\},$$

where $\varepsilon = \gamma/2$ if (3.21) is satisfied, otherwise ε is given by (3.22). Recall that $\operatorname{supp}\bar\mu^+ \subset \Omega^+$ and $\operatorname{supp}\bar\mu^- \subset \Omega^-$. Then, following Stampacchia [20, Theorem. 9.3 and proof], we know that $\bar\varphi^- \in C(\overline{\Omega^+})$, $\bar\varphi^+ \in C(\overline{\Omega^-})$, and

$$\exists C_\gamma > 0 \text{ such that } \max(\|\bar\varphi^-\|_{C(\overline{\Omega^+})}, \|\bar\varphi^+\|_{C(\overline{\Omega^-})}) \le C_\gamma \|\bar\mu\|_{M(\Omega)}.$$

If assumption (3.21) holds, then we deduce from (1.5)

$$\begin{aligned}|a(x,t)| &\le |a(x,0)| + \operatorname{ess\,sup}_{(x,s)\in\Omega\times[-\gamma,+\gamma]}\Big|\frac{\partial a}{\partial y}(x,s)t\Big| \\ &\le \|a(\cdot,0)\|_{L^\infty(\Omega)} + C_\gamma\gamma = M_\gamma \quad \text{for a.a. } (x,t)\in\Omega\times[-\gamma,+\gamma]. \end{aligned} \tag{3.23}$$

Let us set

$$M = \nu M_\gamma + \kappa + C_\gamma\|\bar\mu\|_{M(\Omega)} + \|\bar\varphi^0\|_{L^\infty(\Omega)},$$

where we define $M_\gamma = 0$ if (3.22) holds. Let us prove that $\bar\varphi^-$ is bounded. Analogously, we can prove the boundedness of $\bar\varphi^+$. We argue by contradiction. Define

$$\bar\varphi^{*+} = \int_\Omega g_A(x,\xi)d\bar\mu^+(\xi) \text{ and } \bar\varphi^{*-} = \int_\Omega g_A(x,\xi)d\bar\mu^-(\xi),$$

and $\bar\varphi^* = \bar\varphi^0 + \bar\varphi^{*+} - \bar\varphi^{*-}$. If $\bar\varphi^-$ is not bounded, then we deduce from Lemma 3.9 the existence of $x_0 \in \operatorname{supp}\bar\mu^-$ such that $\bar\varphi^{*-}(x^0) > M$. The function $\bar\varphi^{*-}$ is lower semicontinuous. This follows from Fatou's Lemma and the integral representation of $\bar\varphi^{*-}$. In view of this, a $\rho > 0$ exists such that

$$\varphi^{*-}(x) > M \quad \forall x\in\overline{B_\rho(x^0)} \text{ and } \overline{B_\rho(x^0)}\subset\Omega^-.$$

This is possible due to

$$\operatorname{supp}\bar\mu^- \subset \{x\in\Omega : \bar y(x) = -\gamma\}.$$

This implies

$$\begin{aligned}\bar\varphi^*(x) &= \bar\varphi^0(x) + \bar\varphi^{*+}(x) - \bar\varphi^{*-}(x) < \|\bar\varphi^0\|_{L^\infty(\Omega)} + \sup_{x\in\overline{\Omega^-}}\bar\varphi^{*+}(x) - M \\ &\le \|\bar\varphi^0\|_{L^\infty(\Omega)} + C_\gamma\|\bar\mu\|_{M(\Omega)} - M = -\nu M_\gamma - \kappa \quad \forall x\in\overline{B_\rho(x^0)}.\end{aligned}$$

The last identity follows from the definition of M above. Therefore, since $\bar\varphi(x) = \bar\varphi^*(x)$ holds for almost all $x\in\Omega$ and $\|\bar\lambda\|_{L^\infty(\Omega)} \le 1$, we have

$$\bar u(x) = -\frac{1}{\nu}(\bar\varphi(x) + \kappa\bar\lambda(x)) > M_\gamma \quad \text{for a.a. } x\in\overline{B_\rho(x^0)}.$$

Under the assumption (3.22) we have that $a(x,\bar y(x)) \le 0$ for almost all $x \in \overline{B_\rho(x^0)} \subset \Omega^-$ and $M_\gamma = 0$. On the other hand, assumption (3.21) along with (3.23) implies that $M_\gamma - a(x,\bar y(x)) \ge 0$ for almost all $x\in\overline{B_\rho(x^0)}$. In any case we have

$$A\bar y = \bar u - a(x,\bar y) > 0 \quad \text{a.e. in } \overline{B_\rho(x^0)}.$$

From the maximum principle for elliptic equations, we deduce

$$-\gamma = \bar y(x_0) \ge \min_{x\in\overline{B_\rho(x^0)}}\bar y(x) = \min_{x\in\partial B_\rho(x^0)}\bar y(x) \ge -\gamma,$$

thus $\bar y(x) \equiv -\gamma$ in $\overline{B_\rho(x^0)}$. Hence, we have

$$a(x,-\gamma) = A\bar y + a(x,\bar y(x)) = \bar u(x) > M_\gamma \quad \text{in } B_\rho(x^0),$$

which contradicts (3.22) and (3.23). Therefore, we have shown that $\bar\varphi\in L^\infty(\Omega)$. $\square$

4. **Second order optimality conditions.** The goal of this section is to set up sufficient second order optimality conditions for a local solution of (P). First, let us define the notion of local solution depending of the selected topology.

Definition 4.1. We say that $\bar{u}$ is an $L^p(\Omega)$-weak local solution of (P), $p \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|\bar{u} - u\|_{L^p(\Omega)} \leq \varepsilon. \tag{4.1}$$

We say that $\bar{u}$ is a strong local solution if there exists some $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|y_{\bar{u}} - y_u\|_{L^\infty(\Omega)} \leq \varepsilon. \tag{4.2}$$

We say that $\bar{u}$ is a strict (weak or strong) local solution if the above inequalities are strict for $u \neq \bar{u}$.

As far as we know, the concept of strong local solution was introduced for the first time in the framework of control of partial differential equations in [1]; see also [2].

Theorem 4.2. *The following statements hold:*

1. *If $\bar{u}$ is an $L^2(\Omega)$-weak local solution, then $\bar{u}$ is an $L^1(\Omega)$-weak local solution.*
2. *If $\bar{u}$ is an $L^p(\Omega)$-weak local solution, then $\bar{u}$ is an $L^q(\Omega)$-weak local solution for every $p < q \leq \infty$.*
3. *If $\bar{u}$ is a strong local solution, then $\bar{u}$ is an $L^p(\Omega)$-weak local solution for every $p \in [1, \infty]$.*
4. *If $\bar{u}$ is an $L^2(\Omega)$-weak local solution, then it is a strong local solution.*

Proof. Proof of 1.- Since $\bar{u}$ is an $L^2(\Omega)$-weak local solution, there exists $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|\bar{u} - u\|_{L^2((\Omega))} \leq \varepsilon. \tag{4.3}$$

We argue by contradiction. If $\bar{u}$ is not an $L^1(\Omega)$-weak local solution, then there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad}$ such that

$$J(u_k) < J(\bar{u}) \quad \text{and} \quad \|u_k - \bar{u}\|_{L^1(\Omega)} \leq \frac{1}{k} \quad \forall k \geq 1. \tag{4.4}$$

From the first inequality we infer

$$\frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)} \leq J(\bar{u}) \quad \forall k \geq 1.$$

This estimate and the second inequality of (4.4) imply the existence of a subsequence, denoted in the same way, such that $u_k \rightharpoonup \bar{u}$ in $L^2(\Omega)$. We get with Theorem 2.1 and the weak lower semicontinuity of the last two terms of J

$$J(\bar{u}) \leq \liminf_{k\to\infty} J(u_k) \leq \limsup_{k\to\infty} J(u_k) \leq J(\bar{u}).$$

This implies the convergence $J(u_k) \to J(\bar{u})$. Since $y_k = y_{u_k} \to \bar{y}$ in $L^\infty(\Omega)$, we deduce

$$\lim_{k\to\infty}\Big(\frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)} + \kappa\|u_k\|_{L^1(\Omega)}\Big) = \frac{\nu}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \kappa\|\bar{u}\|_{L^1(\Omega)}. \tag{4.5}$$

From the weak convergence $u_k \rightharpoonup \bar{u}$ in $L^2(\Omega)$ we obtain

$$\|\bar{u}\|^2_{L^2(\Omega)} \leq \liminf_{k\to\infty}\|u_k\|^2_{L^2(\Omega)} \quad \text{and} \quad \|\bar{u}\|_{L^1(\Omega)} \leq \liminf_{k\to\infty}\|u_k\|_{L^1(\Omega)}. \tag{4.6}$$

From (4.5) and (4.6) it follows

$$\begin{aligned}
&\frac{\nu}{2}\|\bar u\|^2_{L^2(\Omega)} \le \liminf_{k\to\infty}\frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)} \le \limsup_{k\to\infty}\frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)}\\
&\le \limsup_{k\to\infty}\Big(\frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)} + \kappa\|u_k\|_{L^1(\Omega)}\Big) - \liminf_{k\to\infty}\kappa\|u_k\|_{L^1(\Omega)} \le \frac{\nu}{2}\|\bar u\|^2_{L^2(\Omega)},
\end{aligned}$$

hence we have the convergence of the norms $\|u_k\|_{L^2(\Omega)} \to \|\bar u\|_{L^2(\Omega)}$. This convergence and the weak convergence $u_k \rightharpoonup \bar u$ in $L^2(\Omega)$ are equivalent to the strong convergence $u_k \to \bar u$ in $L^2(\Omega)$. Therefore, there exists k_ε such that that $\|u_k-\bar u\|_{L^2(\Omega)} \le \varepsilon$ for all $k \ge k_\varepsilon$. Hence, (4.3) implies that $J(\bar u) \le J(u_k)$ for all $k \ge k_\varepsilon$, wich contradicts (4.4).

Proof of 2.- Since $\bar u$ is an $L^p(\Omega)$-weak local solution, there exists $\varepsilon > 0$ such that

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|\bar u - u\|_{L^p(\Omega)} \le \varepsilon. \tag{4.7}$$

Taking into account that

$$\|u-\bar u\|_{L^p(\Omega)} \le |\Omega|^{1-\frac{q}{p}}\|u-\bar u\|_{L^q(\Omega)} \quad \forall q \in (p,\infty],$$

(4.7) implies that

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|\bar u - u\|_{L^q(\Omega)} \le \varepsilon',$$

where $\varepsilon' = \varepsilon/|\Omega|^{1-\frac{q}{p}}$.

Proof of 3.- We have that $\bar u$ satisfies (4.2) for some $\varepsilon > 0$. Let us take $u \in \mathcal{U}_{ad}$ arbitrarily. First we estimate $\|y_u - \bar y\|_{L^\infty(\Omega)}$ in terms of $\|u-\bar u\|_{L^2(\Omega)}$. To this end, we set $z = y_u - \bar y$. Subtracting the equations satisfied by y_u and $\bar y$ and applying the mean value theorem we find

$$\left\{\begin{array}{l} Az + \dfrac{\partial a}{\partial y}(x,\hat y(x))z = u - \bar u \ \text{ in } \Omega,\\ z = 0 \ \text{ on } \Gamma,\end{array}\right.$$

where $\hat y(x) = \bar y(x) + \theta(x)(y_u(x) - \bar y(x))$ for some measurable function $0 \le \theta(x) \le 1$. Then, there exists a constant C_2 independent of u such that

$$\|y_u - \bar y\|_{L^\infty(\Omega)} = \|z\|_{L^\infty(\Omega)} \le C_2\|u-\bar u\|_{L^p(\Omega)};$$

see [20, §4]. Next we select $\varepsilon' = \varepsilon/C_2$ and deduce with (4.2)

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|u - \bar u\|_{L^2(\Omega)} \le \varepsilon'.$$

Hence $\bar u$ is a local solution of (P). Then, 1 and 2 imply that $\bar u$ is an $L^p(\Omega)$-weak local solution of (P) for all $p \in [1,\infty]$.

Proof of 4.- Since $\bar u$ is an $L^2(\Omega)$-weak local solution, (4.3) holds for some $\varepsilon > 0$. We argue again by contradiction. If $\bar u$ is not a strong local solution, then there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad}$ such that

$$J(u_k) < J(\bar u) \quad \text{and} \quad \|y_k - \bar y\|_{L^\infty(\Omega)} \le \frac{1}{k} \quad \forall k \ge 1, \tag{4.8}$$

where y_k is the state associated with u_k. Now we can argue as in the proof of (1) and to deduce that $\bar u_k \to \bar u$ strongly in $L^2(\Omega)$. Hence, (4.3) implies that $J(\bar u) \le J(u_k)$ for all sufficiently large k, which contradicts (4.8) □

Remark 4.3. Let us discuss a few consequences of this result, first for the problem (P) posed here, where control constraints are not given and the feasible set is not bounded in general. Observe that we assumed $\nu > 0$ from the very beginning.

Since $J(u) < \infty$ if and only if $u \in L^2(\Omega)$, this is the natural space to study the control problem (P). In $L^2(\Omega)$, $\|\cdot\|_{L^p(\Omega)}$ is a norm if and only if $p \in [1, 2]$. Hence, the natural concepts for a local solution of (P) are the $L^p(\Omega)$-weak local solutions with $p \in [1, 2]$ and strong local solutions. However - for $p \in [1, 2]$ - all these concepts are equivalent as we deduce from Theorem 4.2.

Let us complete this observation by discussing the case of control problems with pointwise control constraints and L^∞-bounds, where the set of admissible controls is bounded in $L^\infty(\Omega)$. Here, all the norms $\|\cdot\|_{L^p(\Omega)}$ with $p \in [1, \infty]$ are reasonable. Hence, it makes sense to consider $L^p(\Omega)$-weak local solutions and strong local solutions for arbitrary $p \in [1, \infty]$. We can distinct between two cases:

Case $\nu > 0$**.** Arguing as in Theorem 4.2, we find that for $p \in [1, \infty)$ any $L^p(\Omega)$-weak local solution is a strong local solution and any strong local solution is an $L^p(\Omega)$-weak local solution for all $p \in [1, \infty]$, i.e. even for $p = \infty$. Therefore, here the notions of all weak local solutions and strong local solutions are equivalent with one exception: The property of being an $L^\infty(\Omega)$-weak local solution does not imply the other types of local solution. Having in mind this exception, all the notions of weak and strong local solutions are equivalent. We only have to distinguish between $L^\infty(\Omega)$-weak local and strong local solutions.

Case $\nu = 0$**.** In this case, to assure the existence of a solution, we have to assume some constraints on the controls. If the set of feasible controls is bounded in $L^\infty(\Omega)$, then – a priori – there are three different type of local solutions: $L^\infty(\Omega)$-weak local solutions, $L^p(\Omega)$-weak local solutions with $1 \le p < \infty$, and strong local solutions. Since all the topologies induced in $\mathcal{U}_{ad}$ for the $L^p(\Omega)$ norms with $1 \le p < \infty$ are equivalent in an L^∞-bounded set, the concepts of local solutions in the $L^p(\Omega)$ sense are equivalent for $p \in [1, \infty)$. Once again arguing as in the proof of Theorem 4.2, we are able to deduce that any strong local solution is an $L^p(\Omega)$-weak local solution for every $p \in [1, \infty]$, and every $L^p(\Omega)$-weak local solution is an $L^\infty(\Omega)$-weak local solution, but the converses are maybe false: We cannot deduce that an $L^p(\Omega)$-weak local solution is a strong local solution or that an $L^\infty(\Omega)$-weak local solution is an $L^p(\Omega)$-weak local solution for $p < \infty$.

In order to establish a sufficient second order condition for local optimality, we introduce the cone of critical directions. Thus, given $\bar{u} \in \mathcal{U}_{ad}$ satisfying the first order optimality conditions (3.9)–(3.11), we define

$$C_{\bar{u}} = \Big\{ v \in L^2(\Omega) :$$

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})\, v + \kappa j'(\bar{u}; v) = 0, \tag{4.9}$$

$$z_v(x) \left\{ \begin{array}{ll} \le 0 & \text{if} \quad \bar{y}(x) = +\gamma \\ \ge 0 & \text{if} \quad \bar{y}(x) = -\gamma, \end{array} \right. \tag{4.10}$$

$$\int_\Omega |z_v|\, d|\bar{\mu}| = 0 \Big\}, \tag{4.11}$$

where $z_v = G'(\bar{u})v$ is the solution of (2.2) for $u = \bar{u}$. $|\bar{\mu}| = \bar{\mu}^+ + \bar{\mu}^-$ is the total variation measure associated with $\bar{\mu}$, and $\mathcal{L} : L^2(\Omega) \times M(\Omega) \longrightarrow \mathbb{R}$ is the Lagrangian function defined by

$$\mathcal{L}(u,\mu) := F(u) + \int_\Omega y_u \, d\mu. \tag{4.12}$$

From Theorems 2.3 and 2.4, we deduce that $\mathcal{L}$ is of class C^2 and

$$\frac{\partial \mathcal{L}}{\partial u}(u,\mu)v = F'(u)v + \int_\Omega z_v \, d\mu = \int_\Omega (\varphi_u + \nu u)v \, dx, \tag{4.13}$$

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v^2 = \int_\Omega \left\{ \left[1 - \frac{\partial^2 a}{\partial y^2}(x, y_u)\, \varphi_u\right] z_v^2 + \nu v^2 \right\} dx, \tag{4.14}$$

where $\varphi_u \in W_0^{1,p}(\Omega)$, for all $p < \frac{n}{n-1}$, is the solution of

$$\left\{ \begin{array}{l} A^*\varphi + \dfrac{\partial a}{\partial y}(x, y_u)\varphi = y_u - y_d + \mu \ \text{ in } \Omega, \\ \varphi = 0 \ \text{ on } \Gamma. \end{array} \right. \tag{4.15}$$

According to (4.13) and the identity (3.11), we have

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu)v + \kappa \int_\Omega \bar\lambda v \, dx = 0 \quad \forall v \in L^2(\Omega). \tag{4.16}$$

Moreover, from (2.9) we also infer

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu)v + \kappa j'(\bar u; v) \, dx \ge 0 \quad \forall v \in L^2(\Omega). \tag{4.17}$$

The next theorem provides sufficient second order conditions for local optimality.

Theorem 4.4. *Suppose that (1.2)–(1.6) hold. Assume that $\bar u$ along with its associated state $\bar y$ satisfies (3.9)–(3.11). We also assume that*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}. \tag{4.18}$$

Then there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in \mathcal{U}_{ad} \cap \overline{B_\varepsilon(\bar u)}, \tag{4.19}$$

where $\overline{B_\varepsilon(\bar u)}$ denotes the $L^2(\Omega)$ closed ball centered at $\bar u$ with radius ε.

Proof. This proof is inspired in [7] and [11]. We argue by contradiction. Suppose that $\bar u$ does not satisfy the quadratic growth condition (4.19). Then there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad}$ such that

$$\|u_k - \bar u\|_{L^2(\Omega)} < \frac{1}{k} \ \text{ and } \ J(\bar u) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)} > J(u_k) \ \ \forall k \ge 1. \tag{4.20}$$

Let us take

$$\rho_k = \|u_k - \bar u\|_{L^2(\Omega)} \quad \text{and} \quad v_k = \frac{1}{\rho_k}(u_k - \bar u).$$

Since $\|v_k\|_{L^2(\Omega)} = 1$, we can extract a subsequence, denoted in the same way, such that $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$. Now we split the proof into several steps.

Step 1. $\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu)v + \kappa j'(\bar u; v) = 0$. In the following, we write $y_k = y_{u_k}$. Since u_k is feasible, it holds that $|y_k(x)| \le \gamma$ for every $x \in \bar\Omega$. By (4.20) and (3.10), we obtain

$$F(\bar u) + \kappa j(\bar u) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)} > F(u_k) + \kappa j(u_k),$$

$$\int_\Omega \bar y(x) \, d\bar\mu(x) \ge \int_\Omega y_k(x) \, d\bar\mu(x).$$

Adding both inequalities we infer

$$\mathcal{L}(\bar u,\bar\mu)+\kappa j(\bar u)+\frac{1}{k}\|u_k-\bar u\|^2_{L^2(\Omega)}>\mathcal{L}(u_k,\bar\mu)+\kappa j(u_k)\quad\forall k\ge 1. \tag{4.21}$$

From the mean value theorem and (2.8)–(2.9), it follows

$$\mathcal{L}(u_k,\bar\mu)=\mathcal{L}(\bar u,\bar\mu)+\rho_k\frac{\partial\mathcal{L}}{\partial u}(\hat u_k,\bar\mu)v_k,$$
$$j(\bar u+\rho_k v_k)-j(\bar u)\ge\rho_k j'(\bar u;v_k),$$

where $\hat u_k=\bar u+\theta_k(u_k-\bar u)$ for some number $0\le\theta_k\le 1$. From these relations and (4.21), we obtain

$$\frac{\partial\mathcal{L}}{\partial u}(\hat u_k,\bar\mu)v_k+\kappa j'(\bar u;v_k)<\frac{1}{\rho_k k}\|u_k-\bar u\|^2_{L^2(\Omega)}=\frac{1}{k}\|u_k-\bar u\|_{L^2(\Omega)}<\frac{1}{k^2}.$$

Then, with (2.8) we get

$$\frac{\partial\mathcal{L}}{\partial u}(\bar u,\bar\mu)v+\kappa j'(\bar u;v)\le\lim_{k\to\infty}\frac{\partial\mathcal{L}}{\partial u}(\hat u_k,\bar\mu)v_k+\kappa\liminf_{k\to\infty}j'(\bar u;v_k)\le 0. \tag{4.22}$$

This inequality and (4.17) prove that $\frac{\partial\mathcal{L}}{\partial u}(\bar u,\bar\mu)v+\kappa j'(\bar u;v)=0$.

Step 2. $v\in C_{\bar u}$. We have to confirm (4.10) and (4.11). Let us prove (4.10). From Theorem 2.3 we have

$$z_v=G'(\bar u)v=\lim_{k\to\infty}\frac{y_{\bar u+\rho_k h_k}-\bar y}{\rho_k}\quad\text{in } C(\bar\Omega)\cap H^1_0(\Omega),$$

which implies for every $x\in\bar\Omega$ such that $\bar y(x)=+\gamma$

$$z_v(x)=\lim_{k\to\infty}\frac{y_{\bar u+\rho_k h_k}(x)-\bar y(x)}{\rho_k}\le 0.$$

The last inequality follows from the fact that $u_k\in\mathcal{U}_{ad}$, $\bar u+\rho_k h_k=u_k$, and consequently $y_{\bar u+\rho_k h_k}(x)=y_{u_k}(x)\le\gamma$ for every $x\in\bar\Omega$. Analogously we prove that $z_v(x)\ge 0$ for every $x\in\Omega$ such that $\bar y(x)=-\gamma$.

Now, we confirm (4.11). Taking $y=y_{u_k}$ in (3.10), we get

$$\begin{aligned}\int_\Omega z_v(x)\,d\bar\mu(x)&=\lim_{k\to\infty}\frac{1}{\rho_k}\int_\Omega(y_{\bar u+\rho_k v_k}(x)-\bar y(x))\,d\bar\mu(x)\\&=\lim_{k\to\infty}\frac{1}{\rho_k}\int_\Omega(y_{u_k}(x)-\bar y(x))\,d\bar\mu(x)\le 0.\end{aligned} \tag{4.23}$$

On the other hand, from (4.20) we find

$$\begin{aligned}F'(\bar u)v+\kappa j'(\bar u;v)&\le\liminf_{k\to\infty}\left\{\frac{F(\bar u+\rho_k v_k)-F(\bar u)}{\rho_k}+\kappa\frac{j(\bar u+\rho_k v_k)-j(\bar u)}{\rho_k}\right\}\\&=\liminf_{k\to\infty}\frac{J(u_k)-J(\bar u)}{\rho_k}\le\liminf_{k\to\infty}\frac{\rho_k}{k}=0.\end{aligned} \tag{4.24}$$

Then, the established identity $\frac{\partial\mathcal{L}}{\partial u}(\bar u,\bar\mu)v+\kappa j'(\bar u;v)=0$, (4.23), (4.24), and (4.13) imply that

$$F'(\bar u)v+\kappa j'(\bar u;v)=\int_\Omega z_v(x)\,d\bar\mu(x)=0.$$

Finally, from (4.10) and Corollary 3.5-(4) it follows

$$\int_\Omega|z_v(x)|\,d|\bar\mu|(x)=\int_\Omega z_v(x)\,d\bar\mu(x)=0.$$

Thus (4.11) holds, and we have that $v \in C_{\bar u}$.

Step 3. $v = 0$. Taking into account (4.18), it is enough to prove that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) v^2 \le 0. \tag{4.25}$$

To this end, we evaluate the Lagrangian. By a second-order Taylor expansion, we derive

$$\mathcal{L}(u_k, \bar\mu) = \mathcal{L}(\bar u, \bar\mu) + \rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu) v_k + \frac{\rho_k^2}{2} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu) v_k^2,$$

where $\hat u_k = \bar u + \vartheta_k(u_k - \bar u)$ with $0 \le \vartheta_k \le 1$. From here we get

$$\begin{aligned}\rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu) v_k + \frac{\rho_k^2}{2} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) v_k^2 - \frac{\rho_k^2}{2}\left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right] v_k^2 \\ = \mathcal{L}(u_k, \bar\mu) - \mathcal{L}(\bar u, \bar\mu).\end{aligned} \tag{4.26}$$

For the functional j we deduce from (2.8)–(2.9)

$$\rho_k j'(\bar u; v_k) \le j(u_k) - j(\bar u). \tag{4.27}$$

Now, we write (4.21) as follows

$$\mathcal{L}(u_k, \bar\mu) - \mathcal{L}(\bar u, \bar\mu) + \kappa[j(u_k) - j(\bar u)] < \frac{\rho_k^2}{k}.$$

Inserting (4.16) and (4.27) we infer

$$\begin{aligned}\rho_k \left\{\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu) v_k + \kappa j'(\bar u; v_k)\right\} + \frac{\rho_k^2}{2} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) v_k^2 \\ - \frac{\rho_k^2}{2}\left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right] v_k^2 < \frac{\rho_k^2}{k}.\end{aligned}$$

Using (4.17) and dividing the above inequality by $\rho_k^2/2$, we get

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) v_k^2 - \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right] v_k^2 < \frac{2}{k}. \tag{4.28}$$

Since $\mathcal{L}(\cdot, \bar\mu) : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 and $\hat u_k \to \bar u$ in $L^2(\Omega)$, we infer

$$\begin{aligned}&\left|\left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right] v_k^2\right| \le \left\|\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right\|_{B(L^2(\Omega))} \|v_k\|^2_{L^2(\Omega)} \\ &= \left\|\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right\|_{B(L^2(\Omega))} \to 0 \quad \text{as } k \to \infty,\end{aligned} \tag{4.29}$$

where $B(L^2(\Omega))$ is the space of quadratic continuous forms in $L^2(\Omega)$. This convergence and (4.28) imply (4.25).

Step 4. *Final contradiction.* We have proved that $v_k \rightharpoonup 0$ in $L^2(\Omega)$, then $z_{v_k} \to 0$ strongly in $C(\bar\Omega)$. From this convergence, along with (4.14), (4.28), (4.29), and the identity $\|v_k\|_{L^2(\Omega)} = 1$, it follows

$$0 < \nu = \lim_{k\to\infty} \left\{\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) v_k^2 - \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\hat u_k, \bar\mu)\right] v_k^2\right\} \le 0,$$

which is a contradiction. $\square$

Finally, let us notice that the quadratic growth property (4.19) holds, eventually with a different radius ε, if we consider different neighborhoods of $\bar u$.

Corollary 4.5. *Suppose that* (1.2)–(1.6) *hold and let* $\bar{u}$ *be an element* $\mathcal{U}_{ad}$. *Let* $p \in [1, 2)$ *be arbitrary. Then for all* $\delta \in (0, \nu)$ *the following statements are equivalent*

1. *There exists* $\varepsilon > 0$ *such that* (4.19) *is fulfilled.*
2. *There exists* $\varepsilon_p > 0$ *such that*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in \mathcal{U}_{ad} : \|u - \bar{u}\|_{L^p(\Omega)} \le \varepsilon_p. \tag{4.30}$$

3. *There exists* $\varepsilon_\infty > 0$ *such that*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in \mathcal{U}_{ad} : \|y_u - \bar{y}\|_{L^\infty(\Omega)} \le \varepsilon_\infty. \tag{4.31}$$

Proof. Let us consider the optimization problem

$$(\mathrm{P}_\delta) \quad \min_{u \in \mathcal{U}_{ad}} J_\delta(u)$$

with

$$J_\delta(u) = \frac{1}{2}\int_\Omega (y_u - y_d)^2\,dx + \frac{\nu - \delta}{2}\int_\Omega u^2\,dx + \kappa\int_\Omega |u|\,dx + \delta\int_\Omega u\bar{u}\,dx.$$

Notice that $J_\delta(u)$ is obtained from J by subtracting $\frac{\delta}{2}(\|u - \bar{u}\|^2_{L^2(\Omega)} - \|\bar{u}\|^2_{L^2(\Omega)})$.

The statements of the corollary are equivalent to the claims that $\bar{u}$ is an $L^2(\Omega)$-weak local solution of (P_δ), an $L^p(\Omega)$-weak local solution of (P_δ), and a strong local solution of (P_δ), respectively. But, from Theorem 4.2, we know that these three notions are equivalent for $p \in [1, 2)$. Indeed, the last term of J_δ is linear and continuous with respect to u and we have that $\nu - \delta > 0$. Consequently, the proof of Theorem 4.2 is exactly the same if we replace the cost functional J by J_δ. ∎

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