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Towards a Field Theory of the Plateau Transitions in the Integer Quantum Hall Effect

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We suggest a procedure for calculating correlation functions of the local densities of states (DOS) at the plateau transitions in the Integer Quantum Hall effect (IQHE). We argue that their correlation functions are appropriately described in terms of the SL$(2,\mathbb{C})$/SU(2) WZNW model (at the usual Kač–Moody point and with the level $6 \leq k \leq 8$). In this model we have identified the operators corresponding to the local DOS, and derived the partial differential equation determining their correlation functions. The OPEs for powers of the local DOS obtained from this equation are in agreement with available results.

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Key words: localization, Quantum Hall effect, multifractality, conformal symmetry.

I. INTRODUCTION

Under the conditions of low temperature and strong perpendicular magnetic field, a two-dimensional electron gas exhibits a striking macroscopic manifestation of a quantum phenomenon, namely the quantum Hall effect [1, 2]: the Hall conductivity exhibits quantized plateaus at well defined multiples of e^2/h (a fundamental constant). In samples where

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the rôle of random impurities (disorder) is more important than electron-electron interactions, the plateaus occur at integer multiples of e^2/h giving rise to the so-called integer quantum Hall effect (IQHE). It is widely believed that in the absence of a magnetic field, all wavefunctions for non-interacting, disordered electrons in two dimensions are localized. In the presence of a magnetic field however, a delocalized state occurs at the centre of the (disorder broadened) Landau level, with energy E_c. As one tunes the electron energy E (by varying the magnetic field), through the centre of a Landau level, the localization length, ξ, diverges as $\xi = |E - E_c|^{-\nu}$, where numerical simulations indicate that $\nu \sim 2.3$.[1] The plateaus with differing σ_{xy} are separated by these critical points. A theoretical description of these points remains one of the most challenging unresolved problems in the theory of disordered systems.

In the field-theoretic approach to the problem of disordered electrons, the presence of a static potential preserves the one-body energy, so the frequency becomes simply a parameter of the action and one may consider each frequency separately. This reduces the problem of disordered electrons in d spatial dimensions to a d-dimensional Euclidean field theory. The natural description of the plateau transitions in the IQHE, should therefore be in terms of the critical point of some two-dimensional Euclidean field theory, to which one may apply the powerful machinery of conformal field theory (CFT) — see for example the books [3–5]. Despite persistent efforts over the last fifteen years, the form of the effective action describing the critical point is still uncertain.

The first field-theoretical description of the disordered Landau level was given by Levine, Libby and Pruisken [6] and Pruisken [7] in the form of a non-linear sigma model with a topological term. On the basis of this model, Khmelnitskii suggested a two-parameter scaling theory [8] of the IQHE. In this theory, all the (renormalization group) flow lines merge into one of a number of fixed points occurring at integer multiples of the Hall conductance (measured in units of e^2/h) corresponding to the existence of plateaus in the IQHE. In addition, the flow-diagram contains unstable fixed points, corresponding to the transition states occurring between plateaus. These flows were justified in a dilute instanton gas approximation by Levine, Libby and Pruisken [9] and independently by Knizhnik and Morozov [10]. Unfortunately, the extrapolation towards the conjectured fixed points lies at strong coupling, and quantitative results are lacking. In the original derivation of the model, Pruisken and collaborators employed the method of replicas, leading to a sigma model defined on the manifold $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$ with $n \to 0$. A more rigorous formulation using Efetov's supersymmetry approach [11,12] was given in [13], and led to a sigma model on the manifold $\mathrm{SU}(1,1|2)/\mathrm{U}(1|1)\times\ \mathrm{U}(1|1)$. The action may be written in the following form,

$$S = \int d^2x \,\mathrm{str}\left[-\frac{1}{8\alpha}\left(\partial_\mu Q\right)^2 + \frac{1}{8}\sigma^0_{xy}\left(Q\left[\partial_x Q, \partial_y Q\right]\right) + \frac{1}{2}\pi\rho_0\eta\Sigma^3 Q\right], \tag{1.1}$$

[1] We note, that in a system of size L, the number of states with wave functions that reach the boundaries is $\sim \rho(E)L^{2-1/\nu}$. Since the density of states, $\rho(E)$, remains a smooth function of energy, this number is always macroscopic. However, the density of delocalized states $\sim L^{-1/\nu}$ and goes to zero in the limit of infinite sample size.

where Q is a 4×4 supermatrix satisfying the conditions,

$$\mathrm{str} Q = 0, \quad Q^2 = 1, \tag{1.2}$$

$\Sigma^3 = \mathrm{diag}(1, -1, 1, -1)$ (in the boson-fermion supermatrix representation), $\sigma^0_{xy}(E)$ is the bare Hall conductance at energy E, ρ_0 is the average density of states at energy E, and η is the imaginary frequency which serves as a symmetry breaking field. At weak disorder, $\alpha \ll 1$, the bare value of the inverse coupling constant, α^{-1}, coincides with the bare longitudinal conductance $\sigma^0_{xx}(E)$ at energy E. To be more precise, one can directly relate the bare parameters of the action to experimentally observable quantities of a mesoscopic system of size $L \sim l$ (the mean free path). At $E = E_c$, one has $\sigma^0_{xy} = \frac{1}{2}$, and the model is expected to be critical at large distances.

The second term appearing in (1.1) is topological, and despite the fact that its presence is crucial for the critical behaviour, its effect cannot be spotted in a perturbative expansion in powers of α; it does not contribute to the equations of motion and hence does not contribute to the loop expansion of the beta function. The effects of the topological term become visible only for samples of size greater than $\xi \sim l \exp[\pi {\sigma^0_{xx}}^2]$, where l is the electron mean free path. In the model (1.1), with $\sigma^0_{xy} = 0$, the length scale ξ corresponds to the localization length; with $\sigma^0_{xy} = 1/2$, this scale is the *transmutation length* (in field-theoretic jargon) and signifies a crossover to the regime of *universal* critical fluctuations.

It is widely believed that the model (1.1) has a non-trivial infrared fixed point at some α^* and $\sigma_{xy} = 1/2$. However, one cannot simply substitute α^* with the experimental value of $(\sigma^*_{xx})^{-1}$ (the inverse longitudinal conductance) in order to obtain the effective action at the critical point. The reason for this is that the fixed point occurs at strong coupling, where the fundamental fields of the model (1.1) are strongly fluctuating. In this regime, the coupling constant α, being dependent on the regularization procedure, can no longer be identified with any measurable quantity. This phenomenon is well known in asymptotically free gauge theories — the gauge coupling constant has meaning only at short distances, and cannot be defined in a universal way in the infrared.[2] In QCD, for example, gluons do not exist as asymptotic states in the infrared — there is a mass transmutation phenomenon and the emerging degrees of freedom are massive. In the quantum Hall effect, however, we expect the emerging infrared theory to be a massless CFT, and so one should not stretch the analogy too far. A better analogy is the Seiberg duality [14] appearing in $N = 1$ supersymmetric gauge theories, with N_c colours and N_f flavours, in the so-called conformal window, $3N_c/2 < N_f < 3N_c$. There is an infrared (IR) fixed point in this theory, in the vicinity of which, the theory can be described as an infrared limit of *another* gauge theory (with the same number of flavours and $N_f - N_c$ colours). If N_f is close to $3N_c/2$ the original theory is strongly coupled near the IR fixed point, whereas the second is only weakly coupled; the appropriate IR behaviour is given by the second (dual) formulation.

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[2]Notable exceptions to the rule are theories with spontaneously broken gauge symmetry (in which case zero-charge U(1) can also emerge) where the coupling constant can be defined in the infrared through either an effective Fermi four-fermion interaction or Thompson scattering.

Another example of this nature occurs in the deformation of the minimal model M_p of two-dimensional CFT, by the operator $\Phi_{1,3}$ [15, 16]. The theory in the infrared can be described either as a strongly fluctuating asymptotically free M_p model, or as a weakly fluctuating infrared free M_{p-1} model, deformed by the irrelevant operator $\Phi_{3,1}$. Again, it is the second (dual) description which is required in the infrared.

These analogies encourage us to believe that the theoretical description of the plateau transitions in the IQHE will ultimately lie in a reformulation of the model (1.1), in terms of degrees of freedom more appropriate for the infrared region.

An alternative approach to the study of the plateau transitions is based on the Chalker–Coddington Network Model [17]. This model has been reformulated in Hamiltonian form as a Replica Spin Chain [18, 19], and as a Superspin Chain [20–23]. The Superspin Chain, is a lattice model with nearest neighbour antiferromagnetic exchange between 'spins', which are generators of the gl(2|2) algebra. The advantage of such reformulations is that one may approach the critical point more accurately by tuning the parameters of the network. This makes the model indispensable for numerical simulations. However, the Chalker–Coddington model (or the spin chain models) are lattice theories, and one still needs to derive their continuum limits in order to describe the critical fluctuations. The derivation of such limits remains an open problem.

Exasperated by long and futile efforts to find a rigorous derivation of the effective action at the critical point, Zirnbauer has drawn upon a great many sources and conjectured its possible form [24][3]. Such a conjecture ought to be justified, amongst other considerations, by its ability to reproduce existing results for physical quantities at the critical point. In [24] this has been successfully done for the two-point conductance. In the present paper we shall concentrate on the correlation functions of the local density of states (DOS).

The structure of our paper is as follows: in section II we discuss a possible form for the correlation functions at the transitions. In particular, we discuss the operator product expansion (OPE) for moments of the local density of states ρ^q, which reflects the multifractality of the critical wavefunctions. The two-point correlation function of the local DOS is given as a particular limit of a four-point function. The latter function includes two operators with anomalous dimension zero (vacuum insertions) which serve to redefine the ground state of the theory. In section III we discuss more fully the important rôle played by these operators. In section IV we discuss a possible connection between the properties of the required *two-dimensional* conformal field theory, and non-perturbative results [27] obtained by studying the *one-dimensional* limit of model (1.1). The analysis of the one-dimensional case brings us to the idea that one can study the behaviour of the local DOS in terms of the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ Wess–Zumino–Novikov–Witten (WZNW) model. In sections V and VI we study one of the candidate theories for describing the plateau transitions, namely the PSL(2|2) WZNW model [24]. The non-compact bosonic sector of this theory is the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model. In section VII we use the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model to study the correlation functions of the local DOS. Finally we present concluding remarks

[3]Here we would like to mention the recent paper [25] containing interesting suggestions with respect to derivation of the critical theory.

(including open problems) and technical appendices.

II. CONSTRAINTS ON CANDIDATE THEORIES

Any conformal field theory purporting to describe the plateau transitions must satisfy a number of constraints, some of which have been discussed by Zirnbauer [24]. For the purpose of this paper, we believe the following are important requirements:

1. The model should be defined over a manifold associated with some supergroup, and must contain a non-trivial (non-unit) operator with zero scaling dimension (see, for example [11]).

2. The model must reproduce all known scaling dimensions.

3. Restricting our two-dimensional CFT to a finite-width strip geometry (in a suitable limit to be discussed below, and in section IV) one should recover certain known quasi-one-dimensional results.

We shall now take each of these constraints in turn, and consider some of their consequences.

A. Supermanifold and Zero-Dimension Operator

Given our belief that a (presently unknown) non-perturbative treatment of model (1.1) will ultimately describe the plateau transitions in the IQHE, it is entirely natural to expect the resulting effective action to be defined over some supermanifold. However, in order to combine conformal symmetry with bosonic and fermionic degrees of freedom, one will ultimately face a zero-mode problem: if the effective action at the critical point contains only terms with derivatives, the integral over the zero modes (constant fields) leaves the partition function ill-defined. Explicitly,

$$\int dQ_0 = \int dB \int dF \tag{2.1}$$

is indeterminate since the integral over the fermionic fields, F, is zero, whereas the integral over the bosonic fields, B, is infinite (assuming that manifold is non-compact as in the original model (1.1)). In the sigma model (1.1) this problem is solved by the presence of the additional term

$$S_\eta = \eta \int d^2x \, \mathrm{str}(\Sigma^3 Q) \tag{2.2}$$

This term is responsible for energy level broadening in the original theory. The corresponding operator has zero scaling dimension[4] and its value does not change in all orders of perturbation theory. The term is highly relevant and generates a *length scale* $L_\eta \sim \eta^{-1/2}$.

[4]We note that the present theory is a purely spatial one, and all dimensions are expressed in terms of units of length, not time. From this point of view time is dimensionless and the frequency, $\eta \sim 1/[\mathrm{Time}]$, corresponds to a dimensionless operator.

The zero modes are also eliminated when one considers an open system. The possibility of escape through the boundaries gives rise to non-uniform (i.e. energy-dependent) broadening of the energy levels. For this reason closed and open systems may require slightly different approaches. In this paper we consider only *closed* systems and shall not discuss any problems related to the conductance. In a closed system one has to introduce boundary conditions as was suggested by Zirnbauer [26] and subsequently used by various authors [27, 28]. In the field theory approach these boundary conditions can be realized by the insertion of certain operators into correlation functions. Such insertions naturally lead to a dependence of the correlation functions on the distance L_B from the boundary. However, if the boundary operator contains a non-trivial zero dimension operator (we shall call this operator Ψ_0), one can obtain a finite answer for the correlation function on sending the boundaries to infinity.

B. Local DOS

In a closed system one cannot study conductance. The only remaining quantities to study are the wave functions. Since wave functions are not observables, the original sigma model does not generate them directly, but rather allows one to calculate correlation functions of the local density of states (DOS). We believe that these correlation functions can be expressed solely in terms of the (non-compact) bosonic sector of the full theory. (For the one-dimensional case this point is discussed in [11], and for two dimensions in [28].)

In a system of finite size where all energy levels are discrete, the local DOS is defined as

$$\rho(x, E) = \sum_a |\psi_a(x)|^2 \delta(E - E_a) \tag{2.3}$$

where a denotes eigenstates of the system. However, this definition is not suitable for calculation of averages of the local DOS because products of delta functions yield infinities. These infinities are removed if one introduces a finite level broadening replacing the delta functions by the Lorentzians:

$$\delta(E - E_a) \to \frac{1}{\pi}\frac{\eta}{(E - E_a)^2 + \eta^2} \tag{2.4}$$

As we have mentioned above, in the sigma model the level broadening comes from term (2.2). This term does not break conformal invariance provided the level broadening is much smaller than the mean level spacing, $\eta L^2 << 1$. Even such small broadening does the trick of removing infinities from correlators of the local DOS. In what follows we consider *normalized* powers of the DOS defined as

$$[\rho]^q = (\eta\rho)^q \tag{2.5}$$

In the context of the sigma model (1.1) the normalized $[\rho]^q$ is identified with $\{\eta \mathrm{str}(\Sigma Q k)\}^q$ where k is a certain constant matrix.

In a closed system the wave function on the boundary does not depend on the fields, that is equal to unity. However, it may occur (and we shall further explain why this is the case for the theories in question) that the unit operator is not a primary field of the problem. Then one needs to decompose the unit operator in terms of the primaries:

$$I(z) = \Psi_0(z) + \int d\mu(p)\Psi_p(z) \tag{2.6}$$

In this decomposition Ψ_0 has zero conformal dimension and the other fields $\Psi_p(z)$ have positive dimensions [5].

Working with the normalized DOS and the boundary operators present one can define the correlation functions in the infinite system (we imagine this system as a long strip of length $2L$ with periodic boundary conditions in the transverse direction)). In the limit

$$\lim_{\substack{L\to\infty \\ \eta L^2 \to 0}} \langle I(-L)(\eta\rho)^q(x)(\eta\rho)^q(y)I(L)\rangle = \lim_{L\to\infty} \langle \Psi_0(-L)[\rho^q](x)[\rho^q](y)\Psi_0(L)\rangle \tag{2.7}$$

only Ψ_0 operators survive as $L \to \infty$. Taking this into account and also that expression (2.7) is valid in the strip geometry, one obtains the following expression for the two-point correlation function in the infinite plane:

$$\overline{[\rho^q](\mathbf{r}_1)[\rho^q](\mathbf{r}_2)} = \lim_{|\mathbf{r_3}|\to 0, |\mathbf{r}_4|\to\infty} \langle [\rho^q](\mathbf{r}_1)[\rho^q](\mathbf{r}_2)\Psi_0(\mathbf{r}_3)\Psi_0(\mathbf{r}_4)\rangle \tag{2.8}$$

Here we use the overbar to denote disorder averaging, and use the angular brackets to denote the correlation functions of our field theory. In the case of strip geometry which can be obtained from the plane by a conformal transformation, the point of origin is mapped onto minus infinity. We shall return to a more detailed discussion of this procedure towards the end of Section IV.

C. Scaling Dimensions

The theory we seek should reproduce all known scaling dimensions associated with the plateau transition in the IQHE. The most famous of these is the localization length exponent, ν, mentioned in the introduction.[6]

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[5] An analogue of this decomposition is equation (3.7) of [27]:

$$W^{(1)}(X,\tau) = 2X^{1/2}\left[K_1(2X^{1/2}) + \frac{2}{\pi}\int_0^\infty dp \frac{p}{1+p^2}\sinh\frac{\pi p}{2} K_{ip}(2X^{1/2})\exp(-\frac{1+p^2}{4}\tau)\right]$$

where K_n is the modified Bessel function of the third kind and $W^{(1)}(X,\tau)$ is a solution of a differential equation

$$\left(X^2\frac{\partial^2}{\partial X^2} - X\right)W^1(X,\tau) = \frac{\partial W^{(1)}(X,\tau)}{\partial \tau}$$

with the boundary condition $W^{(1)}(X,0) = 1$; here τ plays the role of z.

We see that at zero τ this is indeed the decomposition of unity and at infinite τ only the first term $2X^{1/2}K_1(2X^{1/2})$ survives - which is an analogue of $\Psi_0(z)$.

[6] If we assume that the correlation length is generated by a single operator, of scaling dimension d coupled to $(\sigma_{xy} - 1/2)$ and assume further a linear relationship between the Hall conductance and

Other available information comes from the behaviour of the local DOS, $\rho(E, \mathbf{r})$, in a two-dimensional sample of side L. At the critical point, $\rho(\mathbf{r}) = \rho(E = E_c, \mathbf{r})$, they are known to satisfy the following fusion rules[7] [29]:

$$\rho^p(\mathbf{r}_1)\rho^q(\mathbf{r}_2) \sim |\mathbf{r}_1 - \mathbf{r}_2|^{-d(p)-d(q)+2d(p+q)}\rho^{p+q}(\mathbf{r}_2) \tag{2.9}$$

$$\overline{\rho^p(\mathbf{r}_1)} \sim L^{-d(q)} \tag{2.10}$$

where, once again, the bar stands for disorder averaging; we reserve angle brackets for field-theoretic correlation functions. (Recall that the local DOS are normalized in such a way that their correlation functions are finite in the limit of vanishing level broadening.) Numerical simulations (see [30] and references therein) indicate that[8] (see Appendix A 3),

$$d(q) = 2q(1-q)/k, \quad 2/k = 0.28 \pm 0.03. \tag{2.11}$$

This gives $6 < k < 8$.

The quadratic dependence of $d(q)$ gradually becomes linear for $|q| > 2.5$. We note that the scaling dimensions become negative for $q < 0$ and $q > 1$, well inside the interval of validity (2.11). This suggests that the conformal field theory we are looking for is *non-unitary*.

D. Quasi-One-Dimensional Results

Transfer matrix calculations for model (1.1) in a quasi-one-dimensional, closed (no external leads), infinite sample, yield the following form for the two-point correlation function of the local DOS (see equations (3.58), (3.63) and (3.64) of [27]):

$$\overline{[\rho^q](x_1)[\rho^q](x_2)} = \int d\mu(p)\, |\langle 0|Q^q|p\rangle|^2 \exp\Big(-|x_{12}|\tilde{d}(p)/4\xi\Big), \tag{2.12}$$

where

$$\tilde{d}(p) = 1 + p^2 \tag{2.13}$$

$$d\mu(p) = \frac{2p\sinh(\pi p)}{\pi^2}dp \tag{2.14}$$

$$\begin{aligned} \langle 0|Q^q|p\rangle &= \int_0^\infty dX\, X^{q-2}\, W_0(X) W_{(-1+ip)/2}(X) \\ &\sim |\Gamma[q + (1+ip)/2]|^2\, |\Gamma[q - (1+ip)/2]|^2 \end{aligned} \tag{2.15}$$

$|E - E_c|$ in the vicinity of the transition, we conclude that the scaling dimension of the operator is $d = 2 - 1/\nu \sim 1.57$.

[7] We adopt the usual CFT nomenclature in anticipation of the Virasoro-primary nature we shall subsequently motivate for the local DOS.

[8] The proportionality constant in this expression is denoted as $2/k$ with some hindsight: the parameter k will ultimately be identified as the level of the SL(2,$\mathbb{C}$)/SU(2) WZNW theory.

and ξ is the correlation length. The functions W_j satisfy the following eigenvalue equation

$$\left(X^2\frac{\partial^2}{\partial X^2} - X\right) W_j(X) = j(j+1)W_j(X). \tag{2.16}$$

The solutions to this equation may be written in the form

$$W_j(X) = 2\sqrt{X}K_{(1+2j)}(2\sqrt{X}) \tag{2.17}$$

where K_p is the modified Bessel function of the third kind. The relevance of these quasi-one-dimensional calculations for the critical theory in two dimensions, will be discussed in detail in section IV. We simply note here that the correspondence emerges when one places the CFT on a strip of width $2\pi R$. If the infinite plane is parametrized by the coordinates $(z, \bar{z})$, and the strip by $(w, \bar{w})$, one may effect this mapping by means of the conformal transformation $w = R\ln z$. Under certain conditions (in particular, one has to consider $R \sim \xi$) such a strip may be regarded as a quasi-one-dimensional wire.

III. DISCRETE AND CONTINUOUS REPRESENTATIONS

To the reader familiar with conformal field theory, equations (2.9) and (2.12) may appear contradictory. If ρ^q is a primary field, under what representation of the symmetry group of the theory does it transform? Whatever the symmetry group of the 'grand' theory turns out to be, we expect it to be some subgroup (or coset) of Gl(2|2), and as such may support both discrete and continuous representations. Equation (2.9) suggests that ρ^q transforms according to a discrete representation, whereas equation (2.12) suggests that ρ^q is not a single primary field, but a linear combination of fields with different scaling dimensions.[9]

We suggest a resolution of this paradox which is independent of any particular form of the plateau theory. The solution is intimately connected to the zero-mode problem and the existence of a (non-trivial) zero-dimension operator discussed in section II A. We suggest that the two-point function of the *normalized* (see the discussion around Eq.(2.5)) local DOS is given by the limit of the *four-point* function of some CFT,

$$\overline{[\rho^q](\mathbf{r}_1)[\rho^q](\mathbf{r}_2)} = \lim_{|\mathbf{r_3}|\to 0, |\mathbf{r}_4|\to\infty} \langle [\rho^q](\mathbf{r}_1)[\rho^q](\mathbf{r}_2)\Psi_0(\mathbf{r}_3)\Psi_0(\mathbf{r}_4)\rangle. \tag{3.1}$$

Let us discuss this suggestion in detail. Invariance under the projective transformations of the plane[10], restricts the above four-point function (which we denote by $G_q(1,2,3,4)$) to have the form

$$G_q(1,2,3,4) = \frac{1}{|z_{12}|^{4h_q}}\mathcal{F}_q(z,\bar{z}); \qquad z = \frac{z_{32}z_{41}}{z_{31}z_{42}}, \tag{3.2}$$

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[9] A simple resolution of the paradox that the integral (2.12) generates simple powers is denied by the analytical properties of the exponential function which prevent one from deforming the contour of integration onto the poles of the matrix element (2.15).

[10] Namely transformations of the form $w = (az+b)/(cz+d)$, where $ad - bc = 1$.

where we have used the fact that ρ^q has conformal dimension $h_q(=d_q/2)$, and Ψ_0 has zero conformal dimension. By definition, the function (3.2) is invariant under the permutation of z_1 and z_2 which implies

$$\mathcal{F}_q(z,\bar{z}) = \mathcal{F}_q(1/z, 1/\bar{z}). \tag{3.3}$$

In order to reproduce the fusion rules of the local DOS (2.9) one needs to consider the limit, $z_{12} \to 0$. The three-point correlation functions are fixed by conformal invariance, their holomorphic dependence being given by

$$\langle \mathcal{O}_{h_1}(1)\mathcal{O}_{h_2}(2)\mathcal{O}_{h_3}(3)\rangle = C_{123}\, z_{12}^{-h_1-h_2+h_3} z_{13}^{-h_1-h_3+h_2} z_{23}^{-h_2-h_3+h_1} \tag{3.4}$$

where C_{123} are the so-called structure constants of the theory. Using this fact, one obtains the desired limit

$$\begin{aligned}\langle [\rho^q](1)[\rho^q](2)\Psi_0(3)\Psi_0(4)\rangle &\to |z_{12}|^{-4h_q+2h_{2q}} \langle [\rho^{2q}](2)\Psi_0(3)\Psi_0(4)\rangle \\ &= |z_{12}|^{-4h_q+2h_{2q}}\, C_{2q}^{00}\, |z_2|^{-2h_{2q}},\end{aligned} \tag{3.5}$$

(recall that $z_3 = 0, z_4 \to \infty$). This result fixes the asymptotics of $\mathcal{F}_q(z)$ as $1 - z \to 0$ in the following manner:

$$\mathcal{F}_q(z) \sim (1-z)^{h_{2q}}. \tag{3.6}$$

As we shall now demonstrate, the two-point function of the local DOS in the strip geometry explores different asymptotics of $\mathcal{F}_q(z)$. One can map the plane to the strip by the transformation $w = R\ln z$; point 3 goes to $-\infty$ and point 4 goes to $+\infty$. As a result one obtains (the holomorphic part of) the correlation function as follows:

$$G_q(1,2,3,4) = [2R\sinh(w_{12}/2R)]^{-2h_q}\mathcal{F}_q\left[\exp(w_{12}/R)\right]. \tag{3.7}$$

Notice that in the strip geometry the correlation function is translationally invariant. In the limit $\mathrm{Re}\, w_{12} \gg R$ the behaviour of this function is governed by the asymptotics of $\mathcal{F}_q(z)$ for $|z| \gg 1$. In the limit $\mathrm{Re}\, w_{21} \gg R$ it is governed by the asymptotics for $|z| \ll 1$. (These two limits are related by the crossing invariance condition (3.3).) We see that the fusion rule (2.9) and the expansion in continuous representations (2.12) appear in *different channels* of the four-point function.

Thus the two-point function of the local DOS explores different asymptotics of the function $\mathcal{F}$ in different geometries. In this way one may to expect to resolve the paradox.

We conclude this Section with a list of Operator Product Expansions (OPE) which are necessary to reproduce the above results. First we shall specify our normalization conventions. We normalize all two-point correlation functions of primary fields $\mathcal{O}_h$ from *discrete* representations as follows:

$$\langle \mathcal{O}_h(1)\mathcal{O}_{h'}(2)\rangle = \delta_{h,h'}\, z_{12}^{-2h}. \tag{3.8}$$

The operators from continuous representations $\mathcal{V}_p(z)$ are normalized by the invariant measure $d\mu/dp$ on the group[11]:

$$\langle \mathcal{V}_p(1)\mathcal{V}_{-p'}(2)\rangle = \frac{\delta(p-p')}{d\mu/dp} z_{12}^{-2h(p)}. \tag{3.9}$$

Adopting these conventions, one may write the OPEs in the following manner:

$$[\rho^q](1)[\rho^p](2) = C_{qp}^{(q+p)}\, |z_{12}|^{-2h_q-2h_p+2h_{(q+p)}}\, [\rho^{(q+p)}](2) + \cdots \tag{3.10}$$

$$[\rho^q](1)\Psi_0(2) = |z_{12}|^{-4h_q}\, C_{00}^q\, \Psi_0(2) + \int d\mu(p)|z_{12}|^{-4h_q+2h(p)} C_{0,q}^q(p)\mathcal{V}_p(2) + \cdots \tag{3.11}$$

where the ellipsis stand for less singular terms. With these conventions one finds the following asymptotics for $\mathcal{F}(z)$:

$$\mathcal{F}(z\to 0) = C_{qq}^{2q}\, C_{00}^{2q}\, |z|^{2h_{2q}} + \cdots \tag{3.12}$$

$$\mathcal{F}(z\to\infty) = |z|^{4h_q}\left[[C_{00}^q]^2 + \int d\mu(p)\, |C_{0,p}^q|^2\, |z|^{-2h(p)} + \cdots \right]. \tag{3.13}$$

IV. CFT ON THE STRIP

A. General Considerations

In the absence of a rigorous derivation of the critical model describing the plateau transitions in the IQHE, one is forced to make some assumptions about the general form of the action. Given the form of the model (1.1), it is natural to assume that this action is of the sigma model type, and probably includes the Wess–Zumino term. The reason for including the Wess–Zumino term is related to the fact that nearly all critical sigma models that we know of require such a term to ensure criticality. Models of this kind are known as Wess–Zumino–Novikov–Witten (WZNW) models; their actions either have a full group symmetry, G, or are defined on a coset space, G/H, with H being a subgroup of G. The symmetry manifold of the required WZNW model is some supergroup manifold, whose symmetry is almost certainly greater than that of the original model (1.1).

The WZNW action may be written in the form

$$S = \int d^2x \left[\sqrt{g} g^{\mu\nu} G_{ab}[X]\partial_\mu X^a \partial_\nu X^b + \epsilon^{\mu\nu} B_{ab}[X]\partial_\mu X^a\partial_\nu X^b\right] \tag{4.1}$$

where X^a are fields representing the coordinates on some group (or coset) manifold, G_{ab} is the metric tensor on this manifold, and B_{ab} is an antisymmetric tensor. We define the action on a curved (world sheet) surface with a metric $g_{\mu\nu}$ ($\mu,\nu = 1,2$).

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[11]It is highly likely that for the true theory, the measure coincides with the one given by equation (2.14) and $h(p) = c(1+p^2)/8$. However, in this general discussion we do not need to be so specific.

A very important feature of the action (4.1) is that the second term does not contain the world sheet metric. Consequently, the classical stress-energy tensor, $T_{\mu\nu} = \delta S/\delta g^{\mu\nu}$, is determined solely by the first term. In particular, the most important components for the critical model are given by,

$$T_{zz} = G_{ab}[X]\partial_z X^a \partial_z X^b, \quad T_{\bar{z}\bar{z}} = G_{ab}[X]\partial_{\bar{z}} X^a \partial_{\bar{z}} X^b \tag{4.2}$$

where $z = x_0 + ix_1, \bar{z} = x_0 - ix_1$. Here, the reader should not get the false impression that the Wess–Zumino term is not important. Unlike the *topological term* in action (1.1) it does contribute to the equations of motion. Since the model (4.1) is supposed to be *critical*, these equations (to be understood as identities for correlation functions in the quantum theory) are

$$T_{z\bar{z}} = 0, \quad \partial_{\bar{z}} T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0. \tag{4.3}$$

Their fulfillment depends on the Wess–Zumino term through the dynamics of the underlying fields X^a.

The smallness of $1/k$ means that there are many fields in the theory with conformal dimensions much smaller than unity. This gives weight to the idea that the critical point occurs in the region where the coupling constant of the sigma model is relatively small. Thus, one may attempt to describe the critical point using the semiclassical approximation. We shall formulate the semiclassical approximation with the specific aim of establishing contact between our calculations, and the calculations of correlation functions of the local DOS for quasi-one-dimensional systems performed by Mirlin [27] and Fyodorov and Mirlin [32].

In the infinite plane, parametrized by coordinates $(z, \bar{z})$, conformal invariance restricts the two-point function of primary fields of conformal dimension $(h,\bar{h})$ to be of the form

$$\langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\rangle = (z_1 - z_2)^{-2h}(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}}. \tag{4.4}$$

Under a conformal transformation of the plane, $w = w(z)$, this correlation function transforms like a tensor of rank $(h, \bar{h})$:

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle = \prod_{i=1}^{2} \left(\frac{dz}{dw}\right)_{w_i}^{h} \left(\frac{d\bar{z}}{d\bar{w}}\right)_{\bar{w}_i}^{\bar{h}} \langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\rangle. \tag{4.5}$$

One may pass from the infinite plane to a strip of width $2\pi R$ by means of the conformal transformation $w = R \ln z$. Combining this transformation with (4.4) and (4.5), one obtains the two-point function in the strip geometry:

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle = \left[2R \sinh\left(w_{12}/2R\right)\right]^{-2h} \left[2R \sinh\left(\bar{w}_{12}/2R\right)\right]^{-2\bar{h}}. \tag{4.6}$$

Introducing coordinates (τ,σ) along and across the strip respectively ($w = \tau + i\sigma$, $\bar{w} = \tau - i\sigma$; $-\infty < \tau < \infty$, $0 < \sigma < 2\pi R$), one may expand (4.6) in the following manner,

$$\langle \phi(\tau_1, \sigma_1)\phi(\tau_2, \sigma_2)\rangle = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} C_{nm}\, e^{-(h+\bar{h}+n)|\tau_{12}|/R}\, e^{-i(h-\bar{h}+m)|\sigma_{12}|/R} \tag{4.7}$$

One may also obtain the two-point function in the operator formalism, leading to the Lehmann expansion:

$$\langle \phi(\tau_1, \sigma_1)\phi(\tau_2, \sigma_2)\rangle = \sum_\alpha |\langle 0|\hat{\phi}|\alpha\rangle|^2 \, e^{-E_\alpha|\tau_{12}|-iP_\alpha|\sigma_{12}|} \tag{4.8}$$

where E_α and P_α are the eigenvalues of the Hamiltonian and the momentum operator respectively, in the state $|\alpha\rangle$. Comparing (4.7) and (4.8) one obtains a relationship between the eigenvalues of the Hamiltonian and the momentum operator, and the scaling dimensions in the corresponding CFT:

$$E_\alpha = \frac{h+\bar{h}+n}{R}, \quad P_\alpha = \frac{h-\bar{h}+m}{R}. \tag{4.9}$$

Restricting our attention to fields with $h = \bar{h} = d/2$, one may rewrite (4.6) in the form

$$\langle \phi(\tau_1, \sigma_1)\phi(\tau_2, \sigma_2)\rangle = R^{-2d} \left[2\cosh\left(\tau_{12}/R\right) - 2\cos\left(\sigma_{12}/R\right)\right]^{-d}. \tag{4.10}$$

One observes that for $\tau_{12} \gg R$ the asymptotic form of the correlation function is *independent* of σ,

$$\langle \phi(\tau_1, \sigma_1)\phi(\tau_2, \sigma_2)\rangle \sim R^{-2d} \exp\left(-d\tau_{12}/R\right) \quad (\tau_{12} \gg R) \tag{4.11}$$

and in this limit one should set $n = m = 0$ in equations (4.7) and (4.9).

Let us now place the model (4.1) on a thin strip of width $2\pi R$, and neglect any σ-dependence of the fields X_a. From our considerations in the previous paragraph, such a procedure preserves the (large τ) asymptotics of the correlation functions. The action (4.1) becomes

$$S = 2\pi R \int d\tau \, G_{ab}[X]\partial_\tau X^a \partial_\tau X^b, \tag{4.12}$$

which may be recognised as the action for a free, non-relativistic particle, of mass $m = 4\pi R$. The corresponding Hamiltonian is the Laplace–Beltrami operator (multiplied by $-1/2m$),

$$\hat{H} = \frac{-1}{8\pi R\sqrt{G}} \frac{\partial}{\partial X^a}\left(\sqrt{G}G^{ab}\frac{\partial}{\partial X^b}\right) \tag{4.13}$$

As we have already established, the eigenvalues of this Hamiltonian are related to the spectrum of scaling dimensions in our CFT by equation (4.9). Moreover, solution of this Schrödinger equation allows one to obtain explicit expressions for the eigenstates, $|\alpha\rangle$, appearing in (4.8).

B. Emergence of the $\mathbf{SL}(2,\mathbb{C})/\mathbf{SU}(2)$ symmetry

We now wish to establish contact with the one-dimensional calculations based on the original model (1.1). As is well established, the scaling trajectories for different values of σ_{xy}^0 only start to deviate at the scale $\xi \gg l$ (the mean free path). For the sigma model this scale is in the deep infrared, whereas for the (unknown) critical theory this scale serves as an ultraviolet cut-off. The critical theory is conformally invariant, and one may map it to the strip in the manner described above; we assume that the general form of such expressions holds for strips as narrow as $R \sim \xi$. Similarly, we assume that the results [27] hold for strips as wide as ξ.

On the one hand, the functions (2.17) entering into the matrix elements (2.15) should be eigenfunctions of the Laplace–Beltrami operator for the critical model we seek. On the other hand, as we are going to show, these functions are solutions of the eigenvalue problem for the Laplace–Beltrami operator on the manifold $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. This is striking because the $\mathrm{SL}(2,\mathbb{C})$ symmetry was not the classical symmetry of the original sigma model and in fact appears due to the elaborate limit $\eta \to 0$ described in the previous sections.

Let us describe the details. An arbitrary element $h \in \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ admits the following decomposition (see equation (18) of [31] and make the trivial replacements $(\phi, \mu_+, \mu_-,) \to (\theta, \mu, \mu^*)$),

$$h = \begin{pmatrix} 1 & 0 \\ \mu^* & 1 \end{pmatrix} \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \tag{4.14}$$

where $\theta \in \mathbb{R}$, $\mu \in \mathbb{C}$, and μ^* is the complex conjugate of μ. Adopting this parameterization, one may write the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW action in the following form (see equation (19) of [31]),

$$S = \frac{k}{4\pi} \int d^2x \left[4\partial\theta\bar{\partial}\theta + e^{2\theta}\partial\mu\bar{\partial}\mu^* \right]. \tag{4.15}$$

One may now read off the corresponding metric G_{ab} (c.f. (4.1)) and deduce the form of the Laplace–Beltrami operator (4.13) on the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ symmetric space. The resulting eigenvalue equation reads ($X = \exp(-\theta)$):

$$-\left(X^2 \frac{\partial^2}{\partial X^2} + X \frac{\partial^2}{\partial \mu^* \partial \mu} \right) F_\lambda(\mu, \mu^*, X) = \lambda F_\lambda(\mu, \mu^*, X). \tag{4.16}$$

The functions W_j satisfying equation (2.16) are related to the following eigenfunctions of (4.16):

$$F_j = \exp[ik\mu^* + ik^*\mu] W_j(X/|k|) \tag{4.17}$$

with eigenvalues $\lambda = -j(j+1)$. At the same time the μ-independent solutions of Eq.(4.16) represented by the functions X^{-j} have the same eigenvalues. We suggest that these functions with $j = -q$ represent $[\rho]^q$ operators. The matrix element (2.15) is then just a Clebsh-Gordan coefficient.

These facts establish a relationship between the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW theory and the results of the one-dimensional calculations described in section II. The functions W_j appearing in the matrix elements (2.15) are related to the eigenfunctions F_j of the Hamiltonian (4.16). Both the vacuum state $j = 0$ and the excited states (the states with $j = -1/2 + ip/2$) belong to the coherent state representations of the $\mathrm{SL}(2,\mathbb{R})$ group (see Appendix A 4 and [33] for details). The representations of the excited states are distinct in the respect that their angular momentum is a complex number with continuously varying imaginary part. The scaling dimension of the vacuum state is zero whereas those of the excited states are proportional to $\lambda = (1+p^2)/4$. They coincide with the exponents in equation (2.13). The constant wave function, though being a solution of equation (4.16), is not orthogonal to the eigenfunctions and therefore does not belong to the basis of eigenstates. In the subsequent sections we shall study the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model in more detail, and show how it may emerge as an independent subsector of some 'grand' (supergroup manifold) theory.

The appearance of the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ symmetry absent in the original sigma model where it is explicitly broken at finite frequencies is a highly non-trivial fact and deserves comment. The calculations of the local DOS performed in [27, 32] employ the general regularization procedure described in Section II. Taking the limit $\eta \to 0$ and keeping (ηQ) constant involves working in the limit of very large values of fields. In this limit the η-term (2.2) in the action becomes of order of unity (such that η-dependence disappears) and contributes the term linear in X to the Schrödinger equation (2.16). One can say that this term is some sort of "quantum anomaly".

Thus, in the original sigma model the term linear in X is generated as a potential contribution. On the other hand, as we have seen from equations (4.16) and (4.17), equation (2.16) can be interpreted as a subsector of the pure Laplace-Beltrami-type equation (4.16), which contains no potential terms. This subsector specifies representations of the particular type (4.17). It is important that the representations of $[\rho]^q$ (that is the discrete series) are solutions of the same equation (4.16), but in the μ-independent subsector. This fact allows one to consider (4.17) and $[\rho]^q$ as representations of the same group. Thus the one-dimensional limit has a *hidden* dynamical symmetry and this fact gives another argument in favour of our conjecture that the critical point possesses the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ symmetry.

V. GAUGED WZNW MODELS

In this section we consider the WZNW model on the supergroup manifold PSL(2|2)=SL(2|2)/GL(1) (see Appendix IX for more details on supergroups). These two-parameter models are quite remarkable in that they are believed to be conformal along (marginal) lines of parameter space, rather than isolated points [34, 35]. Zirnbauer has proposed a theory describing the plateau transitions in a region of this parameter space not endowed with the Kač–Moody symmetry [24]. For the purposes of this paper we consider the model where the Kač–Moody symmetry is present (the 'Kač–Moody' point). In particular, in section VI, we demonstrate that the bosonic and fermionic sectors *decouple*; the non-compact bosonic sector is described by the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model.

The standard Goddard–Kent–Olive (GKO) procedure [36] for dealing with such coset spaces may be formulated as a gauged WZNW model (for more details see [3–5]). The equations for correlation functions can be obtained by the gauge dressing of the conventional Knizhnik–Zamolodchikov equations [37]. We consider the WZNW model on the SL(2|2) group first and then gauge away the GL(1) subsector. In order to make our arguments transparent, we discuss some general aspects of the theory of WZNW models.

A convenient starting point for discussing the coset G/H, is the (Euclidean) WZNW action on the supergroup manifold G,

$$\hat{S}[g] = \hat{S}_0[g] + k\hat{\Gamma}[g], \tag{5.1}$$

where

$$\hat{S}_0[g] = \frac{1}{4\lambda^2}\int d^2\xi \,\mathrm{str}\left(\partial^\mu g^{-1}\,\partial_\mu g\right) \tag{5.2}$$

$$\hat{\Gamma}[g] = \frac{-i}{24\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{str}\left(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\rho g\right). \tag{5.3}$$

The matrix field $g(\xi)$ is taken to be an element of some supergroup G, the hats indicate the presence of the supertrace[12], $\xi^\mu = (\xi^1, \xi^2)$ are the coordinates of our two-dimensional (Euclidean) space, λ^2 and k are dimensionless parameters ($\hbar = 1$). The Wess–Zumino term, $\hat{\Gamma}[g]$, is defined by the integral over the three-dimensional ball with coordinates x^α; the boundary being identified with our two-dimensional space. Since the Wess–Zumino term contributes to the equations of motion (see below), and hence to the perturbative beta function, it is *not* to be confused with the topological term.

The action (5.1) satisfies the Polyakov–Wiegmann identity[13] [38],

$$\hat{S}[ab] = \hat{S}[a] + \hat{S}[b] + \int d^2\xi\,\omega^{\mu\nu}\,\mathrm{str}(a^{-1}\partial_\mu a\, b\partial_\nu b^{-1}); \quad \omega^{\mu\nu} = \frac{\delta^{\mu\nu}}{2\lambda^2} - \frac{ik\epsilon^{\mu\nu}}{8\pi}. \tag{5.4}$$

The classical equation of motion follows from the requirement that the action be stationary ($0 = \delta S \equiv S[g+\delta g] - S[g]$) under the replacement $g \to g + \delta g \equiv g(1 + g^{-1}\delta g)$. Substituting the latter form of the variation into (5.4), and keeping terms of order δg,

$$0 = \delta S = -\int d^2\xi\,\omega^{\mu\nu}\,\mathrm{str}(g^{-1}\partial_\mu g\,\partial_\nu(g^{-1}\delta g)) \Rightarrow \partial_\nu(\omega^{\mu\nu} g^{-1}\partial_\mu g) = 0. \tag{5.5}$$

Thus for $\lambda^2 = 4\pi/k$ the field equation becomes

$$\partial(g^{-1}\bar{\partial}g) = 0 \Longleftrightarrow \bar{\partial}(\partial g g^{-1}) = 0. \tag{5.6}$$

We shall denote the action $\hat{S}[g]$ at the Kač–Moody point ($\lambda^2 = 4\pi/k$) by the symbol $\hat{W}[g]$:

$$\hat{W}[g] \equiv k\hat{I}[g] = \frac{k}{16\pi}\int d^2\xi\,\mathrm{str}(\partial^\mu g^{-1}\partial_\mu g) + k\hat{\Gamma}[g]. \tag{5.7}$$

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[12]In the subsequent analysis removal of the hats in any term corresponds to the replacement of supertrace by ordinary trace. Our conventions for supertrace (and superdeterminant) are defined in Appendix IX.

[13]In the proof of the Polyakov–Wiegmann identity we exploit the cyclic property of the trace. For supermatrices, it is the supertrace which has this property (see Appendix IX). The form of the Polyakov–Wiegmann identity for supermatrices is the same as that for ordinary matrices, providing we replace trace by supertrace.

The action (5.7) is invariant under the semi-local transformation

$$g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z}) \tag{5.8}$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are arbitrary elements of G. This invariance is made manifest by the Polyakov–Wiegmann identity (5.4) with $\lambda^2 = 4\pi/k$,

$$\hat{W}[ab] = \hat{W}[a] + \hat{W}[b] + \frac{k}{2\pi}\int d^2\xi \,\mathrm{str}(a^{-1}\bar{\partial}a\, b\partial b^{-1}). \tag{5.9}$$

One may consider promoting this semi-local symmetry to a true local symmetry, with the introduction of auxiliary *gauge fields*. In particular, the action

$$\hat{W}[g,h,\tilde{h}] = \hat{W}[h^{-1}g\tilde{h}] - \hat{W}[h^{-1}\tilde{h}] \tag{5.10}$$

is clearly invariant under the combined (gauge) transformations

$$g \to \lambda(z,\bar{z}) g \lambda^{-1}(z,\bar{z}), \quad h \to \lambda(z,\bar{z}) h, \quad \tilde{h} \to \lambda(z,\bar{z})\tilde{h}. \tag{5.11}$$

Applying the Polyakov–Wiegmann identity (5.9) and defining the following *gauge fields*

$$A = \tilde{h}\partial\tilde{h}^{-1}, \quad \bar{A} = h\bar{\partial}h^{-1} \tag{5.12}$$

one may rewrite the action (5.10) in the following form,

$$\hat{W}[g,A,\bar{A}] = \hat{W}[g] + \frac{k}{2\pi}\int d^2\xi \,\mathrm{str}\left(Ag^{-1}\bar{\partial}g - \bar{A}\partial g g^{-1} + Ag^{-1}\bar{A}g - A\bar{A}\right) \tag{5.13}$$

$$= \hat{W}[g] + \frac{k}{2\pi}\int d^2\xi \,\mathrm{str}\left(A\bar{J}_A + \bar{A}J_A\right) \tag{5.14}$$

where we have introduced the gauge invariant generalizations of the conserved currents (5.6):

$$\bar{J}_A = g^{-1}(\bar{\partial} + \bar{A})g, \quad J_A = -(\partial + A)gg^{-1}. \tag{5.15}$$

The gauge fields A and $\bar{A}$ are non-propagating, and play the rôle of Lagrange multipliers forcing certain currents to vanish. Choosing the gauge fields so that the currents associated with the group H are set to zero, we may describe the coset space G/H. The action (5.13) will be our staring point in the next section.

VI. DECOUPLING AT THE KAČ–MOODY POINT

In order to see how the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW may emerge as an independent subsector of the Zirnbauer model (at the Kač–Moody point) or some other 'grand' supermanifold theory, we parametrize the supergroup manifold using the Gauss decomposition. The use of such decompositions is well established in the the free-field approach to WZNW models (see, for example [39] and references therein). In particular, such an approach has been followed in the study of the gl$(n|n)$ current algebras and associated topological field theories [40]. We consider the gauged WZNW model[14],

$$\hat{W}[g,A,\bar{A}] = \hat{W}[g] + \frac{k}{2\pi}\int \mathrm{str}\left(Ag^{-1}\bar{\partial}g - \bar{A}\partial g g^{-1} + Ag^{-1}\bar{A}g - A\bar{A}\right) \tag{6.1}$$

in which the supergroup element, $g \in \mathrm{SL}(2|2)$, admits the following Gauss decomposition (see equation A7)

$$g = e^{\Phi}\gamma; \quad \gamma = \begin{pmatrix} 1\!\!1 & 0 \\ \lambda & 1\!\!1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1\!\!1 & \chi \\ 0 & 1\!\!1 \end{pmatrix} \tag{6.2}$$

where $\Phi \in \mathbb{C}$, λ and χ are arbitrary 2×2 Grassmann-odd matrices, and a and b are arbitrary 2×2 *unimodular* matrices ($\det(a) = \det(b) = 1$) with Grassmann-even entries. One may recover the Zirnbauer's 'base manifold' by taking $a \in \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ and $b \in \mathrm{SU}(2)$. In order to set the gl(1) currents equal to zero, and thus describe the PSL$(2|2)$ coset, we choose (see equation A10)

$$A = \frac{\mu}{2}\Sigma, \quad \bar{A} = \frac{\bar{\mu}}{2}\Sigma; \quad \Sigma = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \tag{6.3}$$

Applying the Polyakov–Wiegmann identity (5.9) to the first term of (6.1),

$$\hat{W}[g] = \hat{W}[e^{\Phi}] + \hat{W}[\gamma] + \frac{k}{2\pi}\int \bar{\partial}\Phi\,\mathrm{str}\left(\gamma\partial\gamma^{-1}\right) = \hat{W}[\gamma] \tag{6.4}$$

where we have used the fact that $\hat{W}[e^{\Phi}] = 0$ and that $\mathrm{str}\left(\gamma\partial\gamma^{-1}\right) = 0$. Repeated application of the Polyakov–Wiegmann identity (5.9) shows that

$$\hat{W}[\gamma] = W[a] - W[b] + \frac{k}{2\pi}\int \mathrm{tr}(b^{-1}\bar{\partial}\lambda a\partial\chi). \tag{6.5}$$

Using the fact that $\mathrm{str}(\Sigma\cdots) = \mathrm{tr}(\cdots)$ for arbitrary arguments, the second term in (6.1) may be written

$$\frac{k}{2\pi}\int \left[\mathrm{tr}\left(\frac{\mu}{2}\gamma^{-1}\bar{\partial}\gamma - \frac{\bar{\mu}}{2}\partial\gamma\gamma^{-1} + \frac{\mu\bar{\mu}}{4}\gamma^{-1}\Sigma\gamma\right) + 2\mu\bar{\partial}\Phi - 2\bar{\mu}\partial\Phi\right]. \tag{6.6}$$

Straightforward matrix manipulation of γ, together with the fact that $\mathrm{tr}(a^{-1}\bar{\partial}a) = 0$, $\mathrm{tr}(\partial a a^{-1}) = 0$, and likewise for b, yields

$$\mathrm{tr}(\gamma^{-1}\bar{\partial}\gamma) = 2\mathrm{tr}\left(b^{-1}\bar{\partial}\lambda a\chi\right), \tag{6.7}$$
$$\mathrm{tr}(\partial\gamma\gamma^{-1}) = 2\mathrm{tr}\left(b^{-1}\lambda a\partial\chi\right), \tag{6.8}$$
$$\mathrm{tr}(\gamma^{-1}\Sigma\gamma) = -4\mathrm{tr}\left(b^{-1}\lambda a\chi\right). \tag{6.9}$$

[14]Throughout this section $\int \equiv \int d^2\xi$.

Combining the above results we may write

$$\hat{W}[g,A,\bar{A}] = W[a] - W[b] + \frac{k}{2\pi}\int \left[\mathrm{tr}\left(b^{-1}(\bar{\partial}-\bar{\mu})\lambda a(\partial+\mu)\chi\right) + 2\mu\bar{\partial}\Phi - 2\bar{\mu}\partial\Phi\right]. \quad (6.10)$$

It is convenient to introduce new fermionic fields λ' and χ' via

$$\lambda = b\lambda', \quad \chi = a^{-1}\chi'. \quad (6.11)$$

Since a and b are unimodular matrices, the corresponding Jacobian is unity. The trace term becomes

$$\mathrm{tr}\left[\left(\bar{\partial}-\bar{\mu}+b^{-1}\bar{\partial}b\right)\lambda'\left(\partial+\mu+a\partial a^{-1}\right)\chi'\right] = \mathrm{tr}\left[(\bar{\partial}+\bar{A}^b)\lambda'(\partial+A^a)\chi'\right] \quad (6.12)$$

in which we have reparametrized, $\mu = \partial\alpha$, and $\bar{\mu} = \bar{\partial}\beta$, and introduced the gauge fields

$$\bar{A}^b = (e^{\beta}b^{-1})\bar{\partial}(e^{-\beta}b), \quad A^a = (e^{-\alpha}a)\partial(e^{\alpha}a^{-1}). \quad (6.13)$$

We make a further change of fermionic variables

$$\lambda'' = \left(\bar{\partial}+\bar{A}^b\right)\lambda', \quad \chi'' = (\partial + A^a)\chi' \quad (6.14)$$

and note that one may express the corresponding Jacobian in terms of the WZNW action on the matrices a and b (not b^{-1}) respectively [38, 41]. Recalling that the fermionic fields are 2×2 matrices, this leads to a level shift of 2 in the WZNW models defined over a and b. Such level shifts were also encountered in [34, 40]. One obtains the following action for the PSL(2|2) WZNW model,

$$k\hat{\mathrm{I}}[\mathrm{PSL}(2|2)] = (2+k)\mathrm{I}[a] + (2-k)\mathrm{I}[b] + \cdots \quad (6.15)$$

in which the ellipsis indicates terms *independent* of a and b; these include the contributions of the bosonic fields α and β, the fermions λ'' and χ'' and ghosts. In the case of the Zirnbauer 'base manifold', $a \in$SL$(2,\mathbb{C})$/SU(2) and $b \in$SU(2). The level shifts are important here, and the resulting contribution to the central charge from these (renormalized) WZNW models is level-independent:

$$C = \frac{3(k+2)}{(k+2)-2} + \frac{3(k-2)}{(k-2)+2} = 6 \quad (6.16)$$

In particular, one observes that the SL$(2,\mathbb{C})$/SU(2) WZNW model emerges as an *independent* subsector of the theory.

VII. CORRELATION FUNCTIONS OF LOCAL DENSITY OF STATES

We have provided evidence to suggest that the correlation functions of the *normalized* local DOS may be obtained from the *critical* SL$(2,\mathbb{C})$/SU(2) WZNW model, in which the level k is fixed by the known scaling dimensions (2.11). This model has been studied before, even in the context of the theory of disorder [31]. From the preceding sections, we have seen how this theory may be embedded, for example, into a larger theory such as the PSL(2|2) model (at the Kač–Moody point). Taking into account the level shifts induced in this embedding one may write the action for the SL(2, $\mathbb{C}$)/SU(2) WZNW model (4.15) in the following form

$$S = \frac{(k+2)}{4\pi} \int d^2x \left[4\partial\theta\bar{\partial}\theta + e^{2\theta}\partial\mu\bar{\partial}\mu^*\right] \tag{7.1}$$

A comparison between the decomposition of SL(2, $\mathbb{C}$)/SU(2) appearing in (4.14) and the Gauss decomposition of SL(2, $\mathbb{R}$)

$$g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \tag{7.2}$$

where $\gamma, \theta, \phi \in \mathbb{R}$ is convenient at this stage. In particular, one may obtain the parametrization of SL(2, $\mathbb{C}$)/SU(2) from that of SL(2, $\mathbb{R}$), by first complexifying the real parameters γ and ψ, and then making the identification $\gamma^* = \psi$. In this way, one may obtain the SL(2, $\mathbb{C}$)/SU(2) correlation functions from the (much studied) SL(2, $\mathbb{R}$) WZNW model[15] one treats γ and ψ as *independent real* fields and performs the appropriate continuation wherever necessary.

The action (7.1), is characterized by a single coupling constant k, which determines the scaling dimensions:

$$d_j = -2j(j+1)/k \tag{7.3}$$

There are no restrictions on the value of k, and the values of j are related to representations of the SL(2,$\mathbb{R}$) group (see Appendix A 4).

As we have discussed in Section III, the available information about the local DOS appears to be self-contradictory. On the one hand, the fusion rules for $[\rho(\mathbf{r})]^q$ suggest that these are operators belonging to the sl(2,$\mathbb{R}$) representation with $j = -q$ and with scaling dimensions given by (7.3). On the other hand, the two-point correlation function in the strip geometry is given by an integral over *continuous* representations of sl(2,$\mathbb{R}$), with $j = -1/2 + ip/2$. As we have demonstrated in Section III, this paradox is resolved if, in fact, the two-point function of the local DOS is given as the limit of a *four-point* function in our field theory:

$$G_q(1,2) = \lim_{|r_3|=0, |r_4|\to\infty} \langle [\rho^q](1)[\rho^q](2)\Psi_0(3)\Psi_0(4)\rangle \tag{7.4}$$

The four-point correlation function appearing in this relation (which we shall subsequently denote by $G_q(1,2,3,4)$) is determined by the Knizhnik–Zamolodchikov equations

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[15] Despite the fact that the SL(2, $\mathbb{R}$) model cannot be defined as an Euclidean path integral, it can be defined algebraically. Following this procedure one can derive the Knizhnik- Zamolodchikov equation for the correlation functions. This equation is not different from the corresponding equation for the SL(2, $\mathbb{C}$)/SU(2) model.

for the SL(2,$\mathbb{R}$) WZNW model; these equations have been derived in [37, 42]. The operators in (7.4) are matrix elements of infinite-dimensional matrices realizing particular representations of the group (information about representation theory of sl(2,$\mathbb{R}$) is given in Appendix 4). Instead of working with these particular matrix elements, it is more convenient to calculate the correlation function between *all possible* tensors belonging to representations with $j = 0, j = -q$. We emphasise that one does not need to worry about exact definition of Ψ_0 because the Knizhnik–Zamolodchikov equations together with the asymptotics of the solution *automatically select* the necessary primary fields. By that token Ψ_0 standing in the correlation function is *the* very primary field with zero scaling dimension we need. It is necessary to mention here that there may be several fields with zero dimensions with different symmetry properties and unusual OPE [43] leading to other solutions of Knizhnik–Zamolodchikov equations for correlation function (7.4). However these solutions are excluded since they do not lead to the desirable quasi-one-dimensional asymptotics.

We introduce the fields $\rho^q(z,y), \Psi_0(z,y)$ as described in Appendix 4. Now each point in (7.4) is characterized by two complex coordinates z and y. Invariance under the projective transformations of the plane[16], together with the sl(2,$\mathbb{R}$) Ward identities, restricts the four-point function to have the form,

$$G_q(1,2,3,4) = \frac{1}{|z_{12}|^{4h_q}|y_{12}|^{4q}}\mathcal{F}_q(z,\bar{z};t,\bar{t}); \qquad z = \frac{z_{32}z_{41}}{z_{31}z_{42}}, \quad t = \frac{y_{32}y_{41}}{y_{31}y_{42}}, \tag{7.5}$$

where we have used the fact that $[\rho^q]$ has conformal dimension $h_q(= d_q/2)$, and sl(2,$\mathbb{R}$) spin $j = -q$, whereas Ψ_0 has zero conformal dimension and $j = 0$. By definition, the function (7.5) is invariant under the permutation of z_1 and z_2 which implies

$$\mathcal{F}_q(z,\bar{z};t,\bar{t}) = \mathcal{F}_q(1/z, 1/\bar{z}, 1/t, 1/\bar{t}). \tag{7.6}$$

In the limit, $z_{12} \to 0$, corresponding to the fusion of the local DOS, one obtains

$$\begin{aligned} \langle[\rho^q](1)[\rho^q](2)\Psi_0(3)\Psi_0(4)\rangle &\to |z_{12}|^{-4h_q+2h_{2q}}\langle[\rho^{2q}](2)\Psi_0(3)\Psi_0(4)\rangle \\ &= |z_{12}|^{-4h_q+2h_{2q}}\, C^{00}_{2q}\, |z_3 - z_2|^{-2h_{2q}}, \end{aligned} \tag{7.7}$$

where C^{00}_{2q} is a structure constant of the conformal field theory. This fixes the asymptotics of $\mathcal{F}_q(z)$ at $1 - z, 1 - y \to 0$:

$$\mathcal{F}_q(z,y) \sim (1-z)^{h_{2q}} y^{2q} \tag{7.8}$$

We may explore different asymptotics of $\mathcal{F}_q(z)$ in the strip geometry. Under the transformation $z = \exp(w/R)$ the point 3 goes to $-\infty$ and the point 4 goes to $+\infty$. Considering only the holomorphic part and omitting the y-dependence, one obtains

$$G_q(1,2,3,4) = [2R\sinh(w_{12}/2R)]^{-2h_q}\mathcal{F}_q\left[\exp(w_{12}/R)\right] \tag{7.9}$$

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[16]Namely transformations of the form $w = (az+b)/(cz+d)$, where $ad - bc = 1$.

Thus in the strip geometry the correlation function is translationally invariant. The limit $\mathrm{Re}\, w_{12} \gg R$ is governed by the asymptotics of $\mathcal{F}_q(z)$ as $z \to \infty$ and the limit $\mathrm{Re}\, w_{21} \gg R$ is governed by the asymptotics as $z \to 0$. Both limits are related by the crossing invariance condition (7.6).

Thus, the fusion rule (2.9) and the expansion in continuous representations (2.12) appear in *different channels* of the four-point function, thus resolving the apparent paradox.

According to [37, 42] the Knizhnik–Zamolodchikov equation for model (7.1) is given by

$$\left\{k\frac{\partial}{\partial z_i} - \sum_{j\neq i} \frac{1}{z_{ij}} \left[(y_{ij})^2 \frac{\partial^2}{\partial y_i \partial y_j} + 2(y_{ij}) \left(j_j \frac{\partial}{\partial y_i} - j_i \frac{\partial}{\partial y_j}\right) - 2 j_i j_j\right]\right\} G(1\cdots 4) = 0 \quad (7.10)$$

as may be seen by combining equations (A30) and (A45) together with a suitable redefinition of k. Substituting (7.5) into this equation, and using the global conformal invariance to map $(z_1, z_2, z_3, z_4) \to (1, z, 0, \infty)$ (and similarly for y using the sl(2,$\mathbb{R}$) Ward identities) one arrives at the following partial differential equation for $\mathcal{F}(z, t)$

$$\partial_t^2 \mathcal{F} \frac{(t-z)t(t-1)}{z(z-1)} + \partial_t \mathcal{F} \left[\frac{(1-t)^2}{z-1} + \frac{t(2q-t)}{z}\right] + k\partial_z \mathcal{F} = 0 \quad (7.11)$$

¿From equation (7.8) we conclude that in the 'discrete' channel $|z-1| \ll 1$, $\mathcal{F} = (z-1)^{h_{2q}} f(t)$. Equation (7.11) may now be written as

$$(1-t)^2 \partial_t [t \partial_t f] + k h_{2q} f = 0. \quad (7.12)$$

The solution of this equation takes the form

$$\begin{aligned} f[t = 1/(1-x)] &= F(2q, 1-2q, 1; x), \\ &= [\partial_x]^{2q-1} \left[x^{4q-1}(1-x)^{-1}\right], \\ &= (2q-1)!(t^{-1}-1)^{2q} + \sum_{k=0}^{2q-1} (1-t^{-1})^{-k} k! C_{4q-k}^{k+2q}. \end{aligned} \quad (7.13)$$

Since equation (7.12) is invariant under the transformation $t \to 1/t$ the second linearly independent solution is

$$\tilde{f}(t) = f(1/t) \quad (7.14)$$

and the entire crossing invariant solution at $|z-1| << 1$ is given by

$$\mathcal{F}(z, \bar{z}; t, \bar{t}) = |1-z|^{2h_{2q}} \{A[f(t)f(\bar{t}) + f(1/t)f(1/\bar{t})] + B[f(t)f(1/\bar{t}) + f(\bar{t})f(1/t)]\} \quad (7.15)$$

Following equation (3.6) at $|z| \ll 1$ (the 'continuous' channel) we look for the solution in the form of a linear combination of power law solutions:

$$\mathcal{F} = \int d\mu(p) |z|^{2D(p)/k} F_p(t, \bar{t}) \quad (7.16)$$

where

$$D(p) = (1+p^2)/4 - kh_q = (q-1/2)^2 + p^2/4. \tag{7.17}$$

The function F_p obeys the equation:

$$t^2(1-t)\partial_t^2 F + t(2q-t)\partial_t F + D(p)F = 0. \tag{7.18}$$

After the change of variables $t = 1/\tau$ one obtains the hypergeometric equation:

$$\tau(1-\tau)\partial_\tau^2 F + [1-(2-2q)\tau]\partial_t F - D(p)F = 0 \tag{7.19}$$

There are two independent solutions of this equation:

$$\begin{aligned} F_p^{(1)}(t) &= F(j, j^*, 1; 1/t), \\ F_p^{(2)}(t) &= \ln t\, F(j, j^*, 1; 1/t) + R(1/t), \end{aligned} \tag{7.20}$$

where $j = 1/2 - q + ip/2$ and $R(x)$ is regular as $x \to 0$. Thus, the general solution (7.16) at small $|z|$ is

$$\mathcal{F} = \int d\mu(p)|z|^{2D(p)/k} C_{ab}(p) F_p^{(a)}(t) F_p^{(b)}(\bar{t}) \tag{7.21}$$

where the matrix C_{ab} $(a, b = 1, 2)$ is determined by the crossing symmetry (3.3). At the moment we do not have a complete solution of equation (7.11) which prevents us from determining all the constants in our equations and OPEs. It appears, however, that the solution is possible. Here we shall just outline the method leaving the detailed analysis for our next publication.

The general solution contains the parameters y_i which replace matrix indices for operators belonging to infinite dimensional representations. Presumably the local DOS corresponds to certain matrix elements. In order to identify ρ^q it is instructive to take a closer look at the solution on the strip. Combining equation (7.9) with equation (7.21) we obtain the following expression for the asymptotics of the two-point correlation function of operators $\rho^q(w, y)$:

$$\overline{\rho^q(w_1, y_1)\rho^q(w_2, y_2)} = |y_{12}|^{-4q} \int d\mu(p) \exp\left[-(1+p^2)|\tau_{12}|/2kR\right] C_{ab}(p) F_a(t) F_b(\bar{t}) \tag{7.22}$$

where $w = \tau + i\sigma$ $(0 < \sigma < 2\pi R)$. In the limit $|\tau_{12}| \gg R$ the prefactor of the integral becomes unity. One recovers the correlation function (2.15) leaving in the expansion of (7.22) in $t, \bar{t}$ only the term $t^{-2q}\bar{t}^{1-2q}$.

We use this fact to determine which matrix element represents ρ^q. Therefore let us assume that this is a matrix element of $\rho^q(y)$ containing y^n. In order to extract it from the y-dependent correlation function one has to integrate it with the weight y^{-1-n} over a contour in the y-plane surrounding zero. Then we have

$$\begin{aligned} &\int \prod_{i=1}^4 dy_i\, y_1^{-1-n} y_2^{-1-n} y_{12}^{-2q} \mathcal{F}(t) = \\ &\qquad \int d\xi_1 d\xi_2 (\xi_1\xi_2)^{-(1+n)} \xi_{12}^{2(1-q)} \int dt\, t^{-2n-2q+2} \mathcal{F}(t) \int dx \frac{x}{(1-x)^2(x-t)^2} \sim \\ &\qquad\qquad \int d\tau F(j.j^*, 1; \tau) \frac{1+\tau}{(1-\tau)^3} \tau^{2(q+n-1)} \end{aligned}$$

where we have performed the transformation

$$x = \frac{y_{32}}{y_{31}}, \qquad y_4 = \frac{y_1 y_{32} - t y_2 y_{31}}{y_{32} - t y_{31}}.$$

Notice that we integrate over y_3, y_4 with unit weight. In order to recover the term τ^{2q-1} in the expansion of the hypergeometric function corresponding to the necessary matrix element, one has to put $n = -2q$. Thus we identify ρ^q with the *highest weight* of the $j = -2q$ representation.

The Knizhnik-Zamolodchikov equation (7.11) can be solved by the method suggested in [42]. Let us make the change of variables:

$$\mathcal{F}(z,t) = t^a (t-1)^b (t-z)^c z^d (z-1)^e Y(t,z) \tag{7.23}$$

where

$$\begin{gathered} a = -(1+k)/2, \quad b = q - (k+1)/2, \quad c = 1/2 - q, \\ d = -(2q-1)/2k, \quad e = (2q-1)^2/2k. \end{gathered} \tag{7.24}$$

In terms of Y, equation (7.11) becomes

$$\begin{aligned} -k^{-1}\partial_t^2 Y = & \frac{z(z-1)}{t(t-1)(x-t)}\partial_z Y + \frac{1-2t}{t(t-1)}\partial_t Y + \\ & + [h_1(t-z)^{-2} + h_2 t^{-2} + h_3(t-1)^{-2} - \kappa[t(t-1)]^{-1}]Y \end{aligned} \tag{7.25}$$

where

$$\kappa = h_1 + h_2 + h_3 - h_4 - (3k+2)/4 \tag{7.26}$$

This equation coincides with the equation for the 5-point function of vertex operators in the Liouville theory

$$S_L = \int d^2x \left[\frac{k}{4\pi}(\partial_\mu \phi)^2 + 2(k+1)\mathcal{R}\phi + \eta \exp(-2\phi)\right] \tag{7.27}$$

($\mathcal{R}$ is the Riemann curvature of the world sheet) with the central charge equal to

$$\hat{C} = 1 + 6(\sqrt{k} + 1/\sqrt{k})^2 \tag{7.28}$$

The operators of the WZNW model are identified as vertex operators of the Liouville theory:

$$V_n = \exp(n\phi) \tag{7.29}$$

with certain ns.

This establishes a link between the Liouville theory and the theory of the local DOS at the plateau transitions in the IQHE.

VIII. CONCLUSIONS

We summarize here the results of this paper:

1. We have described a general procedure for calculating correlation functions of local DOS at the plateau transitions in the IQHE. This procedure requires insertions of additional vacuum operators into any correlation function, so as to modify the ground state.

2. We have provided arguments, that the correlation functions of the local DOS are appropriately described in terms of the SL$(2,\mathbb{C})$/SU(2) WZNW model at the usual Kač–Moody point and with a suitably chosen level k. The comparison with the numerics gives $6 < k < 8$. In this model we have identified the operators corresponding to the properly normalized local DOS, and calculated their correlation functions.

3. We have demonstrated that the SL$(2,\mathbb{C})$/SU(2) WZNW model may emerge as an independent subsector of some supergroup theory, and may indeed describe a disordered system.

There are many unresolved problems which require further study. A number of these are outlined below.

The model contains a free parameter k which we fix by comparison with the known scaling dimensions. It is not clear at the moment what mechanism is responsible for pinning k to this particular value. The example given in Appendix A 6 demonstrates the kind of surprises one can expect in dealing with theories on non-compact manifolds.

We have studied in some detail only the correlation functions and scaling dimensions of the local DOS. We are not in a position to discuss the correlation function exponent ν which is, presumably, determined by an operator which does not belong to the non-compact bosonic sector of the theory.

Even in the non-compact bosonic sector of the theory there are important unsolved problems. For example, we have not yet solved the Knizhnik–Zamolodchikov equation (7.11) which determines the two-point correlation function of the local DOS. In subsequent publications we shall describe a procedure for calculation of the multi-point correlation functions. This will allow us to explain the termination of the multifractal spectrum. As we have mentioned in Section II C the multifractal spectrum of the local DOS terminates at $q \sim 2.5$. If the SL(2,$\mathbb{C}$)/SU(2) WZNW model is the correct model for the local DOS, it must possess some termination mechanism. This may be a mechanism similar to the one described in [31] in the context of the theory of Dirac fermions with gauge potential disorder. The latter mechanism was related to the logarithmic nature of the theory.

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APPENDIX:

1. Supergroups

In this section we provide a brief introduction to supergroups. For the reader interested in further details we refer them to the book [12] and references therein. For the purposes of this paper we consider matrices of the form

$$M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \tag{A1}$$

where a and b are respectively $n \times n$ and $m \times m$ matrices containing even elements of a Grassmann algebra, and σ and ρ are respectively $n \times m$ and $m \times n$ matrices consisting of odd elements of a Grassmann algebra. Such matrices are called *supermatrices*. The set of such complex (respectively real) square supermatrices is denoted by $\mathrm{M}(m|n;\mathbb{C})$ (respectively $\mathrm{M}(m|n;\mathbb{R})$). The supertrace of M is defined as

$$\mathrm{str}(M) = \mathrm{tr}(a) - \mathrm{tr}(b) \tag{A2}$$

where the symbol tr stands for the conventional trace. This definition provides the invariance under cyclic permutations:

$$\mathrm{str}(M_1 M_2 \cdots M_n) = \mathrm{str}(M_n M_1 \cdots M_{n-1}) \tag{A3}$$

for arbitrary supermatrices $M_1, \cdots, M_n$. The superdeterminant (or Berezinian) of M is defined as

$$\mathrm{sdet}(M) = \frac{\det(a - \sigma b^{-1} \rho)}{\det(b)}. \tag{A4}$$

This definition provides the factorization property of the superdeterminant:

$$\mathrm{sdet}(M_1 M_2 \cdots M_n) = \mathrm{sdet}(M_1)\mathrm{sdet}(M_2) \cdots \mathrm{sdet}(M_n). \tag{A5}$$

The *general linear supergroup* $\mathrm{GL}(m|n;\mathbb{C})$ (respectively $\mathrm{GL}(m|n;\mathbb{R})$) is the supergroup of invertible complex (respectively real) supermatrices, the group composition law being the product of supermatrices. In particular, an arbitrary element $g \in GL(n|n)$ admits the *Gauss decomposition* [40]

$$g = \begin{pmatrix} 1\!\!1 & 0 \\ \lambda & 1\!\!1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1\!\!1 & \chi \\ 0 & 1\!\!1 \end{pmatrix} \tag{A6}$$

where A and B (λ and χ) are arbitrary $n \times n$ Grassmann-even (respectively Grassmann-odd) matrices.

The *special linear supergroup* $\mathrm{SL}(m|n;\mathbb{C})$ is the subsupergroup of supermatrices $M \in \mathrm{GL}(m|n;\mathbb{C})$ such that $\mathrm{sdet}(M) = 1$. This requirement forces $\det(A) = \det(B)$ in the Gauss decomposition (A6). Thus, an arbitrary element $g \in \mathrm{SL}(n|n;\mathbb{C})$, may be decomposed as

$$g = \exp(\Phi) \begin{pmatrix} \mathbb{1} & 0 \\ \lambda & \mathbb{1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \mathbb{1} & \chi \\ 0 & \mathbb{1} \end{pmatrix} \tag{A7}$$

where $\Phi \in \mathbb{C}$, and a and b are arbitrary $n \times n$ *unimodular* matrices ($\det(a) = \det(b) = 1$) with Grassmann-even entries.

2. Superalgebras

In addition to supergroups, one may also introduce the notion of a Lie superalgebra, in which the Lie bracket is replaced by a generalised bracket (commutator/anticommutator) that depends on whether the generators considered are 'bosonic' or 'fermionic'. A particularly useful dictionary of Lie superalgebras has been compiled in [44].

The superalgebra $\mathrm{gl}(2|2)$ is generated by sixteen independent matrices, which in a suitable basis may be chosen as the twelve off-diagonal matrices consisting of a single entry of unity, together with the matrices

$$T_{13} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & 0 \end{pmatrix} \quad T_{14} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix} \quad I = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \Sigma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{A8}$$

where $\sigma^3 = \mathrm{diag}(1, -1)$, and $\mathbb{1}$ denotes the 2×2 unit matrix. The Lie algebra $\mathrm{sl}(2|2)$ is generated by supertraceless matrices, and may be obtained from the above representation by removing Σ. An arbitrary $\mathrm{sl}(2|2)$ valued current may be expanded as

$$J(z) = i(z)I + \sigma(z)\Sigma + \sum_{i=1}^{14} t_i(z)T_i. \tag{A9}$$

The Lie algebra $\mathrm{sl}(2|2)$ contains a non-trivial centre; the unit matrix I commutes with all the generators. The algebra $\mathrm{psl}(2|2)$ is obtained by removing this generator. One may isolate the contribution to the $\mathrm{sl}(2|2)$ current arising from the identity component by the following relation

$$i(z) = \frac{1}{4}\mathrm{str}(\Sigma J) = \frac{1}{4}\mathrm{tr}J \tag{A10}$$

3. Scaling Dimensions

We take a d-dimensional sample of material of side L and divide it into $N = (L/l)^d$ boxes of side l. The probability of finding an electron in box i (the so called *box-probability*) is given by

$$P_i = \int d^d\mathbf{r}_i \, |\psi(\mathbf{r})|^2. \tag{A11}$$

The average value of the box probability scales in the following manner

$$\langle P \rangle = \frac{1}{N} \sum_{i=1}^{N} P_i \sim \left(\frac{l}{L} \right)^d \tag{A12}$$

for all normalized wavefunctions ($\sum_i^N P_i = 1$) and is not useful in distinguishing between localized, extended, and critical wavefunctions. We are led to consider the scaling of the *moments* of the box-probabilities

$$\langle P^q \rangle \sim \left(\frac{l}{L} \right)^{d+\tau_q} \tag{A13}$$

which serves to define new exponents τ_q. Consistency with (A12) requires[17]

$$\tau_1 = 0 \quad \tau_0 = -d. \tag{A14}$$

It is often advantageous (especially in numerical simulations) to introduce the Legendre transform, f, of $\tau(q)$,

$$f(\alpha_q) \equiv q\alpha_q - \tau_q, \tag{A15}$$

where

$$\alpha_q \equiv \frac{d\tau_q}{dq}. \tag{A16}$$

It follows that

$$\frac{df(\alpha)}{d\alpha} = q \tag{A17}$$

Thus, the function $f(\alpha)$ has a maximum at $q = 0$. Using (A14) and (A15),

$$f(\alpha)_{\max} = f(\alpha_0) = d. \tag{A18}$$

The slope of $f(\alpha)$ is unity at the value of α corresponding to $q = 1$,

$$\left. \frac{df(\alpha)}{d\alpha} \right|_{q=1} = 1. \tag{A19}$$

Combining (A14) and (A15) we also find that,

$$f(\alpha_1) = \alpha_1 \tag{A20}$$

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[17] For a uniform electron distribution $\langle P^q \rangle_{\text{uniform}} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N}\right)^q \sim \left(\frac{l}{L}\right)^{dq}$ and one finds that $\tau_q = (q-1)d$. It is conventional to define *generalised dimensions*, D_q, such that $\tau_q \equiv (q-1)D_q$. The scaling relation (A13) may equivalently be written $\langle P^q \rangle \sim \left(\frac{l}{L}\right)^{d+(q-1)D_q}$. We note that (A14) implies $D_0 = d$.

The constraints (A18), (A19) and (A20) allow us to write a parabolic *approximation* in terms of one parameter, α_0 (the position of the maximum)[18]

$$f(\alpha) = d - \frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - d)}. \tag{A21}$$

This parabolic approximation together with the definitions (A15) and (A16) gives rise to the relation

$$q\frac{d\tau}{dq} - \tau = d - \frac{(\frac{d\tau}{dq} - \alpha_0)^2}{4(\alpha_0 - d)} \tag{A22}$$

This equation is solved *exactly* by the polynomial

$$\tau_q = -d + \alpha_0 q - (\alpha_0 - d)q^2. \tag{A23}$$

Recalling the scaling relation (A13) one obtains

$$\langle P^q \rangle \sim \left(\frac{l}{L}\right)^{\alpha_0 q - (\alpha_0 - d)q^2}. \tag{A24}$$

The local density of states is given by

$$\rho = \delta^{-1} P \sim L^d P. \tag{A25}$$

Hence

$$\langle \rho^q \rangle \sim \frac{1}{L^{(\alpha_0 - d)q(1-q)}}. \tag{A26}$$

Numerical simulations for $d = 2$ give $\alpha_0 \approx 2.28$.

4. Representations of the SL(2,$\mathbb{R}$) Group

The Lie algebra sl(2) is generated by three independent traceless matrices, which, in the *spin basis* may be taken as

$$J^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{A27}$$

Their commutation relations read

$$[J^+, J^-] = 2J^0, \quad [J^0, J^\pm] = \pm J^\pm, \tag{A28}$$

[18]Assume $f(\alpha) = d - A(\alpha - \alpha_0)^2$ where A is a constant to be determined; this takes into account (A18). Condition (A19) implies $\alpha_1 = \alpha_0 - 1/(2A)$. Substituting this into the equation following from (A20), namely $\alpha_1 = d - A(\alpha_1 - \alpha_0)^2$, determines A.

and should be familiar from the theory of angular momentum. It is readily seen that the differential operators

$$J^- = \frac{\partial}{\partial y}, \quad J^0 = y\frac{\partial}{\partial y} - j, \quad J^+ = 2jy - y^2\frac{\partial}{\partial y} \tag{A29}$$

obey the same commutation relations (A28) when acting on the space of differentiable functions.

In this differential realisation, the monomial y^{j+m} plays the rôle of the state vector $|j,m\rangle$. One observes that $\partial/\partial y$ acts as a kind of lowering operator, and y acts as a kind of raising operator. It is convenient to modify our basis slightly by defining $t^{+1} = -J^+$, $t^0 = J^0$, $t^{-1} = J^-$. This enables us to write the generators in the following compact form

$$t^l = y^{(l+1)}\frac{\partial}{\partial y} - (l+1)jy^l; \quad l = -1, 0, +1 \tag{A30}$$

Their commutation relations read

$$[t^+, t^-] = -2t^0, \quad [t^0, t^\pm] = \pm t^\pm \tag{A31}$$

That is to say we have structure constants,

$$f^{0+}{}_{+} = 1 = -f^{0-}{}_{-}, \quad f^{+-}{}_{0} = -2 \tag{A32}$$

The Killing form reads (with the ordering $+, 0, -$, and $c_{\mathrm{v}} = 2$)

$$\eta^{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad \eta_{ab} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \tag{A33}$$

More generally, one can construct representations defining their action on functions $f(y)$:

$$\hat{A}f(y) = |cy+d|^{2j}\,\mathrm{sign}\,(cy+d)^{2\epsilon} f\left(\frac{ay+b}{cy+d}\right) \tag{A34}$$

where $ad - cb = 1$.

The operator ρ^q discussed in the text belongs to the $j = -q$ representation and is annihilated by t^-:

$$t^+_{j=-q}\rho^q = 0, \quad \rho^q t^- = 0 \tag{A35}$$

In other words one may perform the following expansion

$$\rho^q(y, \bar{y}) = \sum_{k,\bar{k}=0} y^{-2q-k}\bar{y}^{\bar{k}}\Phi^{(-q)}_{k,\bar{k}} \tag{A36}$$

In this representation t^3 is diagonal :

$$t^3\Phi_{k,\bar{k}} = -(q+k)\Phi_{k,\bar{k}} \tag{A37}$$

The operator Φ_0 belongs to the representation where t^3 cannot be diagonalized:

$$t^+\Phi_0 = \mu\Phi_0, \quad \Phi_0 t^- = \mu\Phi_0 \tag{A38}$$

For more information see, for example [33].

5. The Knizhnik-Zamolodchikov Equation

The stress-energy tensor of WZNW models is proportional to the scalar product of currents. Therefore the Virasoro generator with $n = -1$ acting on any primary field gives

$$L_{-1}|\phi_i\rangle = \frac{2}{k+c_{\rm v}}\eta_{ab}J^a_{-1}J^b_0|\phi_i\rangle = \frac{-2}{k+c_{\rm v}}\eta_{ab}J^a_{-1}t^b_i|\phi_i\rangle \tag{A39}$$

We consider the insertion of the zero vector

$$|\chi\rangle = \left[L_{-1} + \frac{2}{k+c_{\rm v}}\eta_{ab}J^a_{-1}t^b_i\right]|\phi_i\rangle = 0 \tag{A40}$$

inside the correlation function of a set of primary fields. We note that the insertion of the operator J^a_1 in the correlator can be expressed as

$$\langle\phi_1(z_1)\cdots(J^a_{-1}\phi_i)(z_i)\cdots\phi_n(z_n)\rangle = \frac{1}{2\pi i}\oint_{z_i}\frac{dz}{z-z_i}\langle J^a(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{A41}$$

$$= -\sum_{j\neq i}\frac{1}{2\pi i}\oint_{z_i}\frac{dz}{(z-z_i)}\frac{t^a_j}{(z-z_j)}\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{A42}$$

$$= -\sum_{j\neq i}\frac{t^a_j}{(z_i-z_j)}\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{A43}$$

Therefore

$$\langle\phi_1(z_1)\cdots\chi(z_i)\cdots\phi_n(z_n)\rangle = \left[\partial_{z_i} - \frac{2}{k+c_{\rm v}}\sum_{j\neq i}\frac{\eta_{ab}\,t^a_i\otimes t^b_j}{z_i-z_j}\right]\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{A44}$$

and by construction this must vanish:

$$\left[\partial_{z_i} - \frac{2}{k+c_{\rm v}}\sum_{j\neq i}\frac{\eta_{ab}\,t^a_i\otimes t^b_j}{z_i-z_j}\right]\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0. \tag{A45}$$

This is the *Knizhnik–Zamolodchikov equation.* The solutions to this equation are the correlation functions of primary fields.

6. BRST Invariance and $\Delta = 0$ Conformally Invariant Deformation

Let us consider a WZNW model at level $-k$[19] deformed by its kinetic term:

$$S(g) = S_{WZNW}(g^{-1}) \;-\; \epsilon\int d^2z\Omega(z,\bar{z}) \tag{A46}$$

[19]The notation is choosen in such a way that for $k > 0$ the action for the SL$(N,\mathbb{C})/$SU(N) model is positive.

where

$$\Omega(z,\bar{z}) = J^a(z)\bar{J}^b(\bar{z})\phi^{ab}(z,\bar{z}), \tag{A47}$$

and

$$J = \frac{1}{2}kg^{-1}\partial g, \qquad \bar{J} = \frac{1}{2}k\bar{\partial}gg^{-1}, \qquad \phi^{ab} = \mathrm{tr}g^{-1}t^a gt^b. \tag{A48}$$

The perturbation operator Ω possesses a number of interesting properties. Let us compute the commutator of the Kač–Moody current $J^a(z)$ with the operator Ω. Denoting its chiral part $O(z)$ we get

$$\begin{aligned}
[J^a(y), O(z)] &= \oint \frac{d\zeta}{2\pi i}\frac{1}{\zeta - z}\{[J^a(y), J^b(\zeta)]\phi^b(z) + J^b(\zeta)[J^a(y),\phi^b(z)]\} \\
&= \oint \frac{d\zeta}{2\pi i}\frac{1}{\zeta - z}\{f_c^{ab}J^c(\zeta)\phi^b(z)\delta(y,\zeta) + \frac{1}{2}k\phi^a(z)\delta'(y,\zeta) + f_c^{ab}J^b(\zeta)\phi^c(z)\delta(y,z)\} \\
&= \oint \frac{d\zeta}{2\pi i}\frac{1}{\zeta - z}\left\{\frac{f_c^{ab}f_d^{cb}}{\zeta - z}\phi^d(z)\delta(y,\zeta) + \frac{1}{2}k\phi^a(z)\delta'(y,\zeta) + \frac{f_c^{ab}f_d^{bc}}{\zeta - z}\phi^d(z)\delta(y,z)\right\} \\
&+ \oint \frac{d\zeta}{2\pi i}\frac{1}{\zeta - z}[f_c^{ab}\Psi^{cd}(z)\delta(y,\zeta) + f_b^{ac}\Psi^{cb}(z)\delta(y,z)].
\end{aligned} \tag{A49}$$

Here

$$\Psi^{cb}(z) =: J^c(z)\phi^b(z) : . \tag{A50}$$

By taking contour integrals, we obtain

$$[J^a(y), O(z)] = (-\frac{1}{2}k + c_V)\phi^a(z)\delta'(y,z). \tag{A51}$$

Thus for the entire operator Ω we get find

$$[J^a(z), \Omega(w,\bar{w})] = \left(-\frac{1}{2}k \ + \ c_V\right)\bar{J}^b(\bar{w})\phi^{ab}(w,\bar{w})\delta'(z,w). \tag{A52}$$

For the $\mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$ model where $c_V = N$ this commutator vanishes when $k = 2N$. (as in the main text the level k is defined such that the central charge is given by $C = k(N^2-1)/(k-N)$).

In other words, when $k = 2N$, the operator Ω is invariant under the Kač–Moody symmetry. Since the Virasoro operators are quadratic combinations of the Kač–Moody currents, the operator Ω is automatically invariant under the conformal transformations. In particular,

$$[L_0, \Omega] = \Delta\Omega = 0. \tag{A53}$$

Indeed,

$$\Delta(k = 2N) = 0. \tag{A54}$$

The operator Ω forms a closed OPE algebra

$$\Omega(z)\Omega(0) = \Omega(0). \tag{A55}$$

The constant term on the right hand side is absent because the operator Ω has zero norm; this follows from equation (A52).

Thus, a the $\mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$ WZNW model at level $k = 2N$ perturbed by the operator Ω must be conformally invariant for arbitrary parameter ϵ. This can be seen in the following way: The property (A55) implies that away from the conformal point the following relation holds

$$\bar{\partial}T = \partial\Theta, \tag{A56}$$

where T and Θ are the components of the stress-energy tensor. Moreover, the trace is given as follows

$$\Theta = \beta_\Omega \, \Omega, \tag{A57}$$

where β_Ω is the renormalization group beta-function of the coupling ϵ. Since Ω is not a marginal operator, $\beta_\Omega \neq 0$. However, the following is true

$$\partial\Theta = [L_{-1}, \Theta] = \beta_\Omega [L_{-1}, \Omega]. \tag{A58}$$

Since $[L_n, \Omega] = 0$ for any n, we arrive at

$$\bar{\partial}T = 0, \tag{A59}$$

i.e. the stress-energy tensor component T is still a holomorphic function even away from the critical point. This means that the perturbed theory remains conformal for an arbitrary perturbation. This also means that the most stable point in this case is not $k = 1$ as for the SU(N) WZNW model, but $k = 2N$.

The described possibility is not realized for the PSL(2|2) model where $c_V = 0$. Though the scaling dimension of the Ω operator in this case is equal to 2 and it may appear being exactly marginal, the fact that it does not commute with J's may be an indication that the line of critical points obtained by studying *perturbative* series [34, 35] is destroyed by non-perturbative effects. This is an open interesting question.

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