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European Puts, Credit Protection, and Endogenous Default*

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Abstract

In a default corridor $[0, B]$ that the stock price can never enter, a deep out-of-the-money American put option replicates a pure credit contract (Carr and Wu, 2011). Assuming discrete (one-period-ahead predictable) cash flows, we show that an endogenous credit-risk model generates, along with the default event, a default corridor at the cash-outflow dates, where $B > 0$ is given by these outflows (i.e., debt service and negative earnings minus dividends). In this endogenous setting, however, the put replicating the credit contract is not American, but European. Specifically, the crucial assumption that determines an endogenous default corridor at the cash-outflow dates is that equityholders' deep pockets absorb these outflows; that is, no equityholders's fresh money, no endogenous corridor.

Keywords: default corridor; endogenous default; equity puts; credit default swaps; tail risk

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1 Introduction

A key result of the Merton (1974) model is that a corporate bond is the sum of a riskless bond and a short put on the firm's assets. This insight has been extended along many avenues, including by Carr and Wu (2011, hereafter CW), who show that the cash flows associated with credit default swaps (CDS) can be replicated with deep out-of-the-money (DOOM) equity puts. Specifically, CW assume the existence of a default corridor, which is essentially a method for simultaneously describing the firm's default boundary and the loss given default (LGD). Their default corridor has an upper barrier B, which the stock price stays above before default, and a lower barrier A, which the stock price stays below after default. CW show that the price of a CDS contract on a firm with default corridor $[A, B]$ can be linked to the prices of two American put options with strike prices within the corridor. The corridor may exist because stock price dynamics include a jump to default as a result of the underlying firm value jumping to default, as in Zhou (2001), Huang and Huang (2012), and Cremers, Driessen, and Maenhout (2008), or because of strategic default, as in Anderson and Sundaresan (1996). A special case is the default corridor $[0, B]$, where the LGD is 100% and a single DOOM American put becomes a digital put.[1] The $[0, B]$ default corridor can occur if equity prices have the potential to jump to zero, as in Merton (1976), Carr and Wu (2007, 2010), Carr and Linetsky (2006), and Le (2015).

Numerous researchers have shown that a serious flaw of the Merton (1974) model is its tendency to underpredict spreads of investment-grade debt.[2] This has led to a large literature on structural bond pricing where subsequent researchers have proposed new models that incorporate features that are assumed away in the Merton (1974) model. Among these models are a set that incorporate endogenous default (e.g., Geske, 1977;

[1] The digital put is an Arrow-Debreu security that pays off one dollar in the event of default. It is similar to the Credit Event Binary Options that once traded on the Chicago Board Options Exchange (/www.cboe.com/Institutional/DOOM.aspx). See CW for more details.

[2] See Eom, Helwege, and Huang (2004), Jones, Mason and Rosenfeld (1984), Huang and Huang (2012), Schaefer and Strebulaev (2008), Du, Elkamhi, and Ericsson (2019), Bai, Goldstein, and Yang (2020).

Leland and Toft, 1996). In an endogenous default model, equityholders have the ability to absorb the firm's negative cash flows, which implies that default is not a random event outside anyone's control, but a decision by the firm's shareholders not to put more money into the firm. Unlike the Merton (1974) setup, cash flow to shareholders is either positive (an inflow in the form of a cash dividend) or negative (an outflow in the form of a coupon paid to bondholders from the deep pockets of the equityholders).

In this paper, we study endogenous credit-risk models in the presence of a default corridor. In our model, the default corridor only exists on coupon dates and at maturity, as endogenous default does not occur at other times (the same assumption as made in Geske 1977). In this setting the equity put that can replicate a credit default security is not DOOM American, but rather European. It expires on bond coupon dates, holds for any moneyness, and has a strike price lower than the coupon. Therefore, our model can simultaneously generate a default event and a corridor.

Given our assumptions, equity prices can be arbitrarily small without triggering default and a deep in-the-money American put is optimally exercised before expiration (Duffie, 2001). Therefore, only the European put counterpart is able to replicate an Arrow-Debreu credit security. Moreover, contrary to CW, where in-the-money American/European puts struck within the corridor $[0, B]$ cannot exist (because the equity price can never enter in the corridor), in our setting, in-the-money puts exist. That is, the put replicating the credit contract is not necessarily DOOM, but rather a low strike-price (LSP) European put.

This result means that the price of European (not American) puts that expire at the coupon date is linear in the strike price falling within the corridor. This linearity leads to an implied-volatility skew for low strike prices. The volatility soars because, within the corridor, the put payoff is not the capped difference between the strike and equity prices as in a benchmark setting, but rather the strike price. It follows that for riskier firms, such as speculative-grade firms, we can have linear-in- the-strike-price put prices and a very steep skew at certain maturities. By contrast, for investment-grade firms, these linear-in-the-strike put prices should correspond to a thinner corridor.

Finally, we provide several extensions of the model to incorporate short-term rollover of debt, the acquisition of a distressed firm, or the following one. Perhaps in normal times, that is, when defaulting is not an option for equityholders, the firm can easily absorb/finance any outflows. However, perhaps in bad times, that is, when defaulting is an option, debt markets are tapered, and therefore only fresh injections of new capital can support the outflows and daily operations of the firm, keeping the same firm alive.[3] In this scenario, that is, in the bad times, we show a default corridor exists.

Section 2 relates our work to the literature. Section 3 motivates our endogenous-default corridor in a coupon-bond model. Section 4 prices European puts in a default corridor. Section 5 provides two extensions of the corridor and Section 6 concludes. An appendix contains omitted proofs and shows the corridor in a general discrete-time setting.

2 Related literature

This paper is related to two strands of the literature. First, it is related to the literature on the link between tail risk, credit risk, and equity derivatives and on the spanning property of option markets (Cremers et al., 2008; Coval et al., 2009; Carr and Wu, 2009 and 2010; Collin-Dufresne et al., 2012; He et al., 2017). Kelly et al. (2016) study the US financial-sector tail risk during the 2007-2009 crisis from the price of out-of-the-money puts. Culp et al. (2018) empirically extend Merton's put insights. Siriwardane (2019) uses Carr and Wu's default corridor to infer credit-risk spreads. Ibáñez (2020) develops a measure of default risk based on Leland-type models. This measure is linked to the default corridor/event of the same endogenous model.

Second, it is related to the literature on the valuation of derivative securities in

[3] In bad times, retained earnings are exhausted and short-term financing soars. Rolling over the full face value of debt leads to a maturity rat race (Brunnermeier and Oehmke, 2013). Binding covenants may limit further indebtness. Directly selling the firm's assets may be expensive, carrying a discount.

structural models, a problem that until recently (Geske et al., 2016; Bai et al., 2018) has received limited attention (Toft and Prucik, 1997). Bai et al. (2018) indeed show Merton's model explains the price of put options on a risky financial sector better than a benchmark setting (Kelly et al., 2016). Specifically, all these works emphasize that even if the asset volatility is constant as in Merton's model, structural models generate a leverage effect in a natural way, because equity is a call option on leveraged assets. In all extant models, however, a default corridor does not exist, which is in contrast to Carr and Wu (2011) as well as this paper.

3 An endogenous default model

We consider a two-period model $N = 2$, $n \in \{1, 2\}$ and respective times $0 < T_1 < T_2$. We denote by V_t the value of the firm's assets, whose dynamic is left unspecified, $0 \leq t \leq T_2$. The firm issues a two-period coupon bond (Merton, 1974; Geske, 1977), where T_2 is the maturity, $D > 0$ is the face value, and $c \times D > 0$ is the coupon. In this structural setting, we denote by r the riskless rate and assume a Q risk-neutral measure exists.

As in any endogenous credit-risk model, the debt service (coupon or face value) is absorbed by equityholders' deep pockets (e.g., Leland, 1994; Leland and Toft, 1996's rollover model; Manso et al., 2010's performance-sensitive debt model; Carr and Wu, 2011's structural model). Otherwise, if the debt service is subtracted from the firm's assets or is refinanced, default is delayed until these assets are entirely depleted. Then,

$$B_1 = cD > 0 \text{ and } B_2 = (1 + c) D > 0$$

are the two respective cash outflows at T_1 and T_2.

We denote by C_t the equity continuation value. We assume equityholders' limited liability, which implies $C_t \geq 0$, $0 \leq t \leq T_2$. It follows that because $B_n > 0$, defaulting at T_n is optimal if and only if

$$C_n \leq B_n,\ n = 1, 2,$$

with indifference between defaulting and paying the cash outflow if $C_n = B_n$. This default choice maximizes equity value; that is, it is endogenous.

3.1 Endogenous equity pricing

Given the terminal assets value ($V_2 \geq 0$), the continuation value of equity is defined recursively as follows (where, in an abuse of notation, $C_n = C_{T_n}$, $n = 1, 2$):

$$\begin{aligned} C_2 &= V_2 \geq 0, \\ C_t &= E_t^Q \left[e^{-r(T_2 - t)} \max \{0, C_2 - (1 + c) \times D\} \right] \geq 0,\ T_1 \leq t < T_2, \\ C_t &= E_t^Q \left[e^{-r(T_1 - t)} \max \{0, C_1 - c \times D\} \right] \geq 0,\ 0 \leq t < T_1. \end{aligned} \tag{1}$$

In particular, C_1 is the price of a European call, whereas C_0 is the price of a compound option. The definition of C_1 and C_2 recognizes the debt service (i.e., the coupon cD and $(1 + c) D$, respectively) is absorbed by equityholders' deep pockets and is never subtracted from the firm's assets (i.e., from V_1 and V_2).

Importantly, the process C_t is always discontinuous at T_1 (and T_2); that is,

$$\lim_{t \uparrow T_1} C_t \to \max \{0, C_1 - c \times D\} < C_1 \text{ if } c \times D > 0. \tag{2}$$

Although the left-hand-side limit is only well defined if C_t does not jump at $t = T_1$ (where $t \uparrow T_1$ means $t \to T_1$, $t < T_1$), the inequality is always correct (if $C_1 > 0$).

In addition, we assume the equity continuation value is strictly positive; that is,

$$C_t > 0,\ t \in [0, T_2]. \tag{3}$$

As shown next, this assumption implies equity prices are also strictly positive between cash-outflow dates (i.e., conditional on no previous default), and default is never optimal outside the outflow dates.

To define the ex-cash-flow equity price (denoted by E), which is subject to default risk, we introduce an auxiliary binary process, $a \in \{0,1\}$. Namely, $a_0 = 1$ and

$$a_n = a_{n-1} \times 1_{\{C_n > B_n\}},\ n = 1, 2, ..., N, \tag{4}$$

and hence $a_n = 0$ indicates the company has defaulted (i.e., $a_j = 0$, $j = n, n+1, ..., N$). Consequently,

$$\begin{aligned} E_2 &= a_2 \times C_2 = 1_{\{C_1 > B_1\}} \times 1_{\{C_2 > B_2\}} \times C_2, \\ E_t &= a_1 \times C_t = 1_{\{C_1 > B_1\}} \times C_t,\ T_1 \leq t < T_2, \\ E_t &= a_0 \times C_t = C_t,\ 0 \leq t < T_1. \end{aligned} \tag{5}$$

The (ex-cash-flow) equity-price function $E_n(C_n)$ is discontinuous at T_1 and T_2. That is,

$$\begin{aligned} E_1 &= 0 \text{ if } C_1 \leq c \times D, \\ E_1 &= C_1 > c \times D > 0 \text{ otherwise,} \end{aligned} \tag{6}$$

both with positive probability.[4] Likewise, either

$$E_2 = 0 \quad \text{or} \quad E_2 = C_2 > (1+c) \times D > 0.$$

In particular, because equityowners absorb the entire debt service, conditional on non-

[4] The process E_t is also discontinuous at T_1 if the firm survives (otherwise, is zero); that is,

$$\lim_{t \uparrow T_1} E_t \rightarrow \max\{0, C_1 - c \times D\} < 1_{\{C_1 > c \times D\}} \times C_1 \quad \text{if } C_1 > c \times D \text{ and } c \times D > 0,$$

where the limit is only well defined if C_t does not jump at $t = T_1$.

default ($a_2 = 1$), the equity value equals the asset value at T_2; that is, $E_2 = V_2$.

Moreover, the discontinuity at T_1 implies the equity-price function is also close to discontinuous right after T_1 (which may approximate a corridor for puts expiring after this instant). That is, if $a_1 = 1$ and $T_1 < t < T_2$, $E_t = C_t$, where $C_t > 0$ can be arbitrarily close to zero. However, in the limit $t \downarrow T_1$, $E_1 = C_1 > c \times D$, which implies that for $t > T_1$ and t not far away from T_1, the probability that $C_t \in [0, c \times D]$ should be relatively small (i.e., a function of a small $t - T_1$).

Importantly, $C_t > 0$ implies $E_t = C_t > 0$ between outflow dates, conditional on no previous default. Strictly positive equity prices imply default is not optimal between outflow dates. Hence, we next focus on outflow dates.

3.2 The endogenous default event and the default corridor

Implicit in the definition of the equity value (i.e., equation (5)) are two endogenous-default events at periods T_1 and T_2, that is,

$$\{C_1 \leq c \times D\} \text{ and } \{C_2 \leq (1+c) \times D\}, \tag{7}$$

respectively. These two events are endogenous because they maximize equity value, and are default events because they imply equity value becomes zero. These two default events define two respective optimal default thresholds, Y_1 and Y_2. That is,

$$\begin{aligned} V_1 &= Y_1 : \; C_1(Y_1) = c \times D \text{ and } \{C_1(V_1) \leq c \times D\} \equiv \{V_1 \leq Y_1\}, \\ V_2 &= Y_2 : \; C_2(Y_2) = (1+c) \times D \text{ and } \{C_2(V_2) \leq (1+c) \times D\} \equiv \{V_2 \leq Y_2\}. \end{aligned}$$

In general, $Y_1 > 0$ and $Y_2 > 0$ are unique because (i.e., we assume if necessary that) call-type payoffs are increasing functions in V. If V depends on stochastic parameters, Y_1 and Y_2 are threshold functions.

Moreover, because the outflows are absorbed by equityholders' deep pockets, from equation (6), these two endogenous default events lead to two default corridors,

$$[0, c \times D] \quad \text{and} \quad [0, (1+c) \times D], \tag{8}$$

respectively, in which the ex-cash-flow equity price cannot enter. That is, $E_1 \notin (0, c \times D]$ and $E_2 \notin (0, (1+c) \times D]$, at periods T_1 and T_2, respectively. Thus, at T_1, the following four default events are equivalent:

$$\{V_1 \leq Y_1\}, \ \{C_1 \leq c \times D\}, \ \{E_1 \leq c \times D\}, \text{ and } \{E_1 = 0\},$$

which depend, respectively, on the firm's low asset value, low continuation value, low equity value, and exhausted equity value.

Remark 1. As in Merton (1974), $C_1 > 0$ is the premium of a European call that expires at T_2 (with a strike price equal to $(1+c) \times D$). However, if we consider that $C_2 = \{V_2 - D\}^+$, a default corridor exists at T_2, but it is very thin; that is, $[0, c \times D]$. Note that if we define

$$C_t = E_t^Q \left[e^{-r(T_2 - t)} \times \max\{0, C_2 - c \times D\} \right], \ T_1 \leq t < T_2,$$

the continuation value (C) is the same process as in equation (1), because

$$\{V_2 - (1+c) \times D\}^+ = \left\{ \{V_2 - D\}^+ - c \times D \right\}^+,$$

implying the definition of the equity price, default events, and default corridor are robust and carry over for $t < T_2$.

Remark 2. We provide an example in which a default corridor is empty. If $c \times D$ is exclusively paid from the firm's assets, the default event at T_1 is trivially given in terms of the asset value, namely, by $\{V_1 \leq c \times D\}$. This specific default event implies a default corridor does not exist at T_1, because the equity-price function is continuous (i.e., zero or larger than zero). That is, if $\lim_{t \uparrow T_1} V_t = c \times D + \xi$, with $\xi > 0$ and $t < T_1$, $V_1 = \xi$ (after the coupon-payment date), and we assume the value of equity E is arbitrarily close to zero if $\xi \to 0$.

Moreover, if $C_1 \leq V_1$, that is, if equity value is bound by the value of the assets, endogenous default leads to an earlier default than if the firm's managers entirely deplete the firm's assets or holdings prior to default.

3.3 European puts, digital puts, and pure credit contracts

We denote by K the strike price of puts and calls. At T_1, we show a low strike-price European equity put becomes a digital put, which replicates a pure credit contract. That is, for a put with maturity T_1, the payoff reduces to

$$\begin{aligned} \max\{0, K - E_1\} &= (K - E_1) \times 1_{\{E_1 \leq K\}} \\ &= K \times 1_{\{E_1 = 0\}} + (K - E_1) \times 1_{\{c \times D < E_1 \leq K\}} \\ &= K \times 1_{\{E_1 = 0\}} \text{ if } K \leq c \times D, \end{aligned} \tag{9}$$

which is a digital option in the case of low strike-price (LSP) puts, namely, $K \leq c \times D$. The second equality follows from equation (6).

Then, from equation (6), $E_n = E_n \times 1_{\{E_n > B_n\}}$, from which follows $1_{\{E_1 = 0\}} = 1_{\{E_1 \leq c \times D\}}$, and hence

$$\max\{0, K - E_1\} = K \times 1_{\{E_1 \leq c \times D\}} \text{ if } K \leq c \times D, \tag{10}$$

which replicates a pure credit contract, in which $\{E_1 \leq c \times D\}$ is the endogenous-default event. In particular, in this endogenous setting, a DOOM put (that replicates a pure credit contract) is rather an LSP put, for all moneyness.

A similar result follows for T_2, in which $B_2 = (1+c) \times D$; namely,

$$\max\{0, K - E_2\} = K \times 1_{\{E_2 \leq (1+c)\times D\}} \text{ if } K \leq (1+c) \times D. \tag{11}$$

4 The price of European puts in a default corridor

We study the pricing of European puts/calls in a default corridor.

1) Under the Q–measure, from equation (10), the price of an LSP put with maturity T_1 is given by

$$E_0^Q\left[e^{-rT_1} K \times 1_{\{E_1 \leq c\times D\}}\right] = e^{-rT_1} K \times Q\left(V_1 \leq Y_1\right), \, K \leq c \times D. \tag{12}$$

Like DOOM puts in Carr and Wu (2011), in our setting, LSP European-put prices are linear in the strike price falling within the corridor (i.e., $K \leq c \times D$), a straightforward empirical prediction. The forward price of this European put, scaled by the strike price, gives the one-period risk-neutral default probability.

2) In terms of the implied volatility σ, where $P^{BS}(E_0, \sigma)$ denotes the Black-Scholes-Merton put-price formula, we have that

$$e^{-rT_1} K \times Q\left(V_1 \leq Y_1\right) = P^{BS}\left(E_0, \sigma\right). \tag{13}$$

Then, the implied-volatility curve $\sigma(K)$ holds that (see the Appendix)

$$\sigma'(K) \times \sqrt{T_1} = \frac{-N(-d_1)}{K \times N'(d_1)} < 0, \, K \leq c \times D, \tag{14}$$

where $N()$ is the cumulative Gaussian-distribution function.

A negative skew (i.e., $\sigma'(K) < 0$) implies LSP puts are expensive. First, they are more expensive in a default corridor than in a setting of no corridor. That is, compared to a benchmark setting (in which the put payoff is $\max\{0, K - C_1\}$), LSP European puts that expire at T_1 are overpriced by the following amount (which follows from $E_1 = 1_{\{C_1 > B_1\}} \times C_1$):

$$e^{-rT_1} \times E_0^Q \left[e^{-rT_1} C_1 \times 1_{\{C_1 \leq \min\{K, B_1\}\}} \right] + e^{-rT_1} K \times E_0^Q \left[1_{\{K < C_1 \leq B_1\}} \right] > 0.$$

Conditional on $C_1 \in [0, B_1]$, the first (second) term covers the in-the-money (out-of-the-money) part of the put in the benchmark setting. Second, the deeper out of the money the put is, the more expensive this put is in implied-volatility units. For instance, in our numerical exercise, $\sigma(K)$ is unbounded when $K \to 0$.

3) The same linear (in the strike price) result happens in the case of the second-outflow date T_2, in which $B_2 = (1 + c) \times D$. The price of an LSP put with maturity T_2 is given by

$$\begin{aligned} & E_0^Q \left[e^{-rT_2} K \times 1_{\{E_2 \leq (1+c) \times D\}} \right] & (15) \\ = \; & E_0^Q \left[e^{-rT_2} K \times 1_{\left\{ 1_{\{C_1 > c \times D\}} \times 1_{\{C_2 > (1+c) \times D\}} \times V_2 \leq (1+c) \times D \right\}} \right] \\ = \; & e^{-rT_2} K \times \left(Q(V_1 \leq Y_1) + Q(V_1 > Y_1) \times Q(V_2 \leq Y_2) \right), \; K \leq (1 + c) \times D. \end{aligned}$$

Scaled by the discounted strike price $e^{-rT_n} K$, the price difference between two LSP European puts with the same strike but different maturity equals the probability of default at T_2, namely,

$$Q(V_1 > Y_1) \times Q(V_2 \leq Y_2) > 0, \; K \leq c \times D,$$

given that $c \times D < (1 + c) \times D$.

4) For no LSP European puts at T_1 (i.e., $K > c \times D$),

$$\begin{aligned} & E_0^Q \left[e^{-rT_1} \max\{0, K - E_1\} \right] & (16) \\ = \; & e^{-rT_1} K \times Q(V_1 \leq Y_1) + E_0^Q \left[e^{-rT_1} (K - E_1) \times 1_{\{c \times D < E_1 < K\}} \right], \; K > c \times D, \end{aligned}$$

and no LSP puts are also more expensive than in a benchmark setting.

5) Lastly, for European equity puts and calls (with respective prices p_t and c_t), with the same strike price (K) and expiring at the first-outflow date (T_1),

$$K + \max\{0, E_1 - K\} = E_1 + \max\{0, K - E_1\},$$

which follows from put-call parity at T_1. Then, from the law of one price (see the Appendix), put-call parity at T_0 becomes

$$e^{-rT_1}K + c_0 = E_0 + e^{-rT_1}B_1 \times Q(V_1 > Y_1) + p_0, \tag{17}$$

where $B_1 = c \times D$ and $Q(V_1 > Y_1) = Q(E_1 > B_1)$. Similar to the case of a paying-dividend stock, put-call parity is also adjusted, in this case, by $e^{-rT_1}B_1 \times Q(V_1 > Y_1)$.

From the last equation, the call price is given by

$$c_0 = E_0 + e^{-rT_1}B_1 \times Q(V_1 > Y_1) + p_0 - e^{-rT_1}K. \tag{18}$$

Specifically, for DOOM puts (i.e., $K \leq B_1$), from equation (12),

$$c_0 = E_0 - e^{-rT_1}(K - B_1) \times Q(V_1 > Y_1). \tag{19}$$

For example, consider a spread between two co-terminal European calls struck within the corridor, with respective strike prices K_1 and K_2, $K_1 < K_2 \leq B_1$. Then,

$$e^{rT_1} \times \frac{c_0(K_1) - c_0(K_2)}{K_2 - K_1} = Q(V_1 > Y_1), \tag{20}$$

which is the one-period risk-neutral surviving probability.

4.1 Numerical example

For simplicity, we present a one period version of our model that is based on Merton (1974). That is, we consider a setting in which equity is a European call on a firm with value V, and has a strike price of $B > 0$ and a maturity of T_1 (which are the face value and maturity of a zero-coupon bond). We assume a lognormal asset value, $\ln \frac{V_1}{V_0} \sim \mathcal{N}\left((r - \frac{\sigma^2}{2})T_1, \sigma\sqrt{T_1}\right)$, where r is the riskless rate and σ is volatility. From the Black-Scholes-Merton formula, the equity price is equal to

$$\begin{aligned} E_0(V_0) &= V_0 \times N(d_{1B}) - e^{-rT_1} B \times N(d_{2B}), \\ d_{1B} &= \frac{\ln \frac{V_0}{B} + (r + \sigma^2/2) \times T_1}{\sigma\sqrt{T_1}} \text{ and } d_{2B} = d_{1B} - \sigma\sqrt{T_1}, \end{aligned}$$

However, we assume equityholders absorb the outflow $B > 0$, which implies a default corridor $[0, B]$ exists at T_1. That is,

$$E_1 = \max\{0, V_1 - B\} + B \times 1_{\{V_1 > B\}} = V_1 \times 1_{\{V_1 > B\}}.$$

First, we emphasize that although puts with the same maturity as debt are expensive if a default corridor exists, in which case $B_1 = B$, they are more expensive in the original Merton's noncorridor model. That is, if V_1 is the asset value, although the put payoff in a benchmark setting is $\max\{0, K - V_1\}$, this payoff, which equals the strike price K within the corridor $[0, B]$ if $K \leq B$, is increased by $(K + B - V_1) \times 1_{\{B < V_1 \leq K+B\}}$ in the latter noncorridor model. Namely, if $K \leq B$, these three payoffs, where $\{V_1 \leq B\}$ is the unique default event, hold:

$$\underbrace{\overbrace{(K - V_1) \times 1_{\{V_1 \leq K\}}}^{\text{noncorridor exists}}}_{\substack{\textbf{benchmark setting} \\ (V_1 \text{ is the asset value;} \\ K \text{ is the put strike price})}} < \underbrace{\overbrace{\underbrace{K \times 1_{\{V_1 \leq B\}}}_{\text{equityholders absorb debt}}}^{\substack{\text{here, a default corridor} \\ \text{exists } (B_1 = B)}} \leq \underbrace{K \times 1_{\{V_1 \leq B\}} + \overbrace{(K + B - V_1) \times 1_{\{B < V_1 \leq K+B\}}}^{\text{this part implies noncorridor exists } (B_1 = 0)}}_{\text{debt is repaid by selling the firm's assets (original model)}}}_{\textbf{Merton's structural model } (B \text{ is the face value of a zero-coupon bond})}.$$

Moreover, for put maturities that are shorter than the debt maturity, a corridor does not exist—the two put payoffs in Merton's model are the same.

Consider a European equity put, with strike price $K > 0$ and maturity T_1 as well. Given the same maturity of the equity claim (or Merton's call) and this equity-put derivative, and given that $E_1 = V_1 \times 1_{\{V_1 > B\}}$, the price of this equity put simplifies to

$$\begin{aligned} p_0 &= E_0^Q \left[e^{-rT_1} \max\{0, K - E_1\} \right] \qquad (21) \\ &= e^{-rT_1} K \times E_0^Q \left[1_{\{V_1 \le B\}} \right] + E_0^Q \left[e^{-rT_1} (K - V_1) \times 1_{\{B < V_1 \le K\}} \right] \\ &= e^{-rT_1} K \times N(-d_{2B}) \\ &\quad + \left(e^{-rT_1} K \times (N(-d_{2K}) - N(-d_{2B})) - V_0 \times (N(d_{1K}) - N(d_{1B})) \right) \times 1_{\{K > B\}}, \end{aligned}$$

where $d_{1,2K}$ are defined akin to $d_{1,2B}$ with K instead of B. In particular,

$$p_0 = e^{-rT_1} K \times N(-d_{2B}), \; K \le B.$$

Conversely, for a reciprocal call with the same maturity and strike price, the payoff in terms of the asset value is given by

$$\begin{aligned} \max\{0, E_1 - K\} &= \max\left\{0, V_1 \times 1_{\{V_1 > B\}} - K\right\} \\ &= \max\{0, V_1 - K\} \times 1_{\{K \ge B\}} + (V_1 - K) \times 1_{\{K < B < V_1\}}, \end{aligned}$$

where the first (second) term corresponds to strike prices higher (lower) than B. It follows that, in contrast to a benchmark setting, low strike-price (i.e., $K < B$) calls are underpriced, because they pay nothing if $V_1 \in (K, B]$.

Following Carr and Wu (2011), we define B as a low strike price, $B = 3$. We assume $\sigma = 30\%$, $r = 2\%$, a maturity of $T_1 = 6$ months, and four equity prices $E_0 = \{2.03, 3.03, 4.03, 6.03\}$, which are associated with the asset values of $V_0 = \{5, 6, 7, 9\}$, respectively. Each price implies a risky, healthier, sound, and super sound firm. For asset values lower than 5, the implied volatility of low strike-price equity puts quickly soars above 100%. In Figure 1, we show the four implied-volatility curves, for a range of strike prices $K \in [1, 20]$. Hence, the volatility-smile function, $\sigma(K)$, solves

$$p_0(\sigma = 0.3) = e^{-rT_1} K \times N(-d_{2K}, \sigma(K)) - V_0 \times N(-d_{1K}, \sigma(K)), \tag{22}$$

and in particular, for $K \leq B$,

$$e^{-rT_1} K \times N(-d_{2B}, \sigma = 0.3) = e^{-rT_1} K \times N(-d_{2K}, \sigma(K)) - V_0 \times N(-d_{1K}, \sigma(K)).$$

From Figure 1, one can see that for risky firms (i.e., $V_0 \leq 6$), the default corridor generates a clear volatility smile, with large implied-volatility levels away from the money. However, for sound firms (i.e., $V_0 \geq 7$), the default corridor generates more of a volatility smirk, where the implied volatility approximates the asset volatility for strike prices higher than $B = 3$. We also see that the riskier the firm, namely, the lower V_0, the larger the implied volatility, which is an example of a leverage effect. These results are robust to the maturity, $T_1 \in \{3, 6, 12\}$ months.

***** to include Figure 1 *****

5 Extensions of the Endogenous Default Corridor

We now provide two extensions. In the first we amend the model so that outflows occur in one of two regimes: a good state of the world and a bad one. In the second extension, we allow for control of the firm's assets to pass from equityholders to creditors.

Extension 1: That equityholders absorb cash outflows in bad times is sufficient for the existence of an endogenous default corridor. Specifically, because we assume a firm cannot refinance any outflow $B_1 > 0$, the default corridor and the defaulting region are given by the same interval $[0, B_1]$. However, if we split the surviving region (B_1, ∞) into two complementary regions, $(B_1, b]$ and $[b, \infty)$, where $B_1 \leq b$, but we allow refinancing in the good-times region $[b, \infty)$, the default corridor is given by $[0, \min(B_1, b - B_1)]$.

That is, the equity price is now given by

$$E_1(C_1) = C_1 \times 1_{\{C_1 > B_1\}} - B_1 \times 1_{\{C_1 > b\}},$$

which implies $E_1(C_1 \downarrow B_1) = B_1$ and $E_1(C_1 \downarrow b) = b - B_1 \geq 0$.

It follows that if good times start soon (i.e., if $b \leq 2B_1$), we have a thinner corridor $[0, b - B_1]$ at the first-outflow date; otherwise, the corridor is $[0, B_1]$. How equity-holders' deep pockets absorb cash outflows explains the *size* of the default corridor.

Extension 2: As endogenous models assume, substantial evidence shows default is not entirely random, but rather firms default in poor economic conditions or with expired debt (Asquith et al., 1994; Campbell et al., 2008; Giesecke et al., 2011; Davydenko, 2012). However, because a bankruptcy has severe economic consequences from layoffs to large distress costs, creditors may have a say in default (Carey and Gordy, 2007).

Therefore, assume the following scenario in which endogenous default is rather delayed. In hard times, in the interval $C_1 \in [0, B_1]$, the equity price is either 0 or larger than $B_{\min}$ (where $0 < B_{\min} \leq B_1$) until all uncertainty is resolved—in which case, either $E_1 = 0$ or, if the firm survives, $E_1 > B_1$. This setting supports Carr-Wu's corridor $[0, B_{\min}]$ after the outflow date and until uncertainty is solved, as well as an endogenous corridor $[0, B_1]$ if the uncertainty has an expiring time. In this setting, the key assumption is the equity price cannot slip in the corridor, namely, if $B_{\min}$ is meaningful. We illustrate this example with the (tentative) takeover of a distress firm.[5]

The distressed firm needs fresh capital, and a large shareholder (L1) announces a capital-injection plan, but only if (in addition to banks extending a new credit line and bondholders increasing the maturity of expiring loans so all stakeholders contribute) it gets control of the firm (>50% of shares). L1 offers a price per share of 0.67—an offer

[5] Specifically, we offer the example of the Spanish retailer DIA, with thousands of employees and a wide presence in Europe. Like many other retailers facing serious competition, it has suffered a decline in demand (poor sales) which has led to greater financial pressures as its debt matures. See for example, https://elpais.com/economia/2019/04/26/actualidad/1556263740_217086.html

expiring in six months. In six months, two mutually exclusive scenarios are possible: Either L1 does not get control and hence (no capital injection but firm default and) the equity price sinks to zero, or L1 gets control and the price per share is 0.67 or above.

However, because other equityholders want a better deal than 0.67, they not only put at risk the L1-control plans and solvency of the firm, but also push down the price per share to a low 0.34 during this six-month period. Associated with this potential takeover, two default corridors exist for the troubled firm. First is a Carr and Wu's (2011) corridor $[0, 0.34]$ during the six months. Second is a corridor in six months, at the offer expiring, in which the price is either zero or above 0.67. The following (DIA Spanish retailer) stock price in Figure 2 seems consistent with both corridors.

***** to include Figure 2 *****

For this distressed firm, in a perfect world, the quotes of low strike-price American/European puts or credit default swaps expiring in six months can be used to get the risk-neutral probability of L1 not gaining control, and the firm stock price jumping to zero. Actually, a price per share of 0.34 represents a 50% risk-neutral probability of each of the two scenarios (either zero or 0.67).

6 Empirical prediction

The striking insight of Carr and Wu (2011) is that the protection leg of a CDS contract can be replicated by rolling over American DOOM puts (up to a scale factor that depends on the expected recovery value of the bond and the strike price of the put). Moreover, Carr and Wu show empirically that the change in CDS spreads and the change in American DOOM put prices are cointegrated, a result that supports the equivalence between both contracts.

We do not conduct a similar test to price contracts in this paper, but we do lay out a possible approach for analyzing the empirical usefulness of our model. Specifically, we suggest examining a set of CDS-spread quotes and a set of prices for short-term DOOM American puts on the same firms. Assume the recovery value is zero, the strike price is 1, and the interest rate is also zero (otherwise, we can scale every contract). Assuming a strategy of rolling over the puts, and collecting data on the initiation of the next put contract, we can compute four averages: (i) The CDS payment leg, which is given by the sum of quarterly payments if the firm did not default, and fewer in the case of default. (ii) The CDS protection leg, which is either zero or 1 (give zero bond recovery). (iii) The rollover cost of the puts; (iv) The realized payoff (i.e. exercise value) of the puts, which depends on whether default occurred. We expect the DOOM puts will finish (near) out of the money for non-defaulting firms, whereas we also expect their payoff will be (close to) the strike price in the event of default.

Because American-style puts can be early exercised—even suboptimally exercised (Ibáñez and Paraskevopulos, 2009)—we consider European instead of American DOOM puts. Fortunately, the former is easily synthesized from the latter counterpart. That is, detaching the early-exercise-premium from the American put price, we obtain the European put price. For DOOM puts or low riskfree interest rates, this premium is indeed small (and can be approximated by zero). The payoff of the European put is given by assuming exercise only at the option maturity. In this latter case, we do not have to observe when every American put option was exercised.

CDS contracts and DOOM puts are similar if the averages based on (i) and (iii), as well as the averages based on (ii) and (iv), are similar.[6] Further, we can obtain two estimates of the credit-risk premium by comparing the averages in (i) and (ii) (in (iii) and (iv)) for CDS securities (DOOM puts). This credit-risk premium is clearly related to a negative-jump risk premium extracted from equity and option prices alone.

[6]See Cheng et al., 2018, for a credit-equity relative value strategy based on these concepts.

7 Concluding remarks

This paper shows an endogenous credit-risk model generates a default corridor $[0, B]$. This corridor is linked to the endogenous-default event, $\{E < B\}$, in which E is the ex-cash-flow equity price and $B > 0$ is given by the outflows (i.e., debt service and negative earnings minus cash dividends). In this setting, the default corridor only necessarily happens at coupon dates, which implies the low strike-price put (that replicates a pure credit contract) is not DOOM American but rather European style. The corridor $[0, B]$ is especially relevant for speculative-grade firms, which are more leveraged and therefore have larger coupons ($B > 0$). If the early-exercise premium is a small fraction of the American price, such as in a DOOM case, the exchanged-listed American price can provide a good approximation to the European counterpart, namely, a pure credit contract.

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8 Appendix

8.1 The implied volatility of low strike-price European equity puts

For a put option, from the Black-Scholes-Merton formula,

$$\begin{aligned} P^{BS}(E_0) &= e^{-rT_1}K \times N(-d_2) - E_0 \times N(-d_1), \\ d_1 &= \frac{\ln \frac{E_0}{K} + (r + \sigma^2/2) \times T_1}{\sigma\sqrt{T_1}} \text{ and } d_2 = d_1 - \sigma\sqrt{T_1}, \end{aligned}$$

which implies the following two Greeks:

$$\frac{\partial P^{BS}}{\partial K} = e^{-rT_1} \times N(-d_2) \quad \text{and} \quad \frac{\partial P^{BS}}{\partial \sigma} = E_0 N'(d_1)\sqrt{T_1},$$

where $N()$ is the cumulative Gaussian-distribution function.

In terms of the implied volatility σ, the low strike-price (LSP) put price verifies that

$$e^{-rT_1}K \times Q(V_1 \leq Y_1) = P^{BS}(E_0), \; K \leq c \times D.$$

Then, the implied-volatility function, $\sigma(K)$, holds that

$$e^{-rT_1} \times Q(V_1 \leq Y_1) = e^{-rT_1} \times N(-d_2) + \frac{\partial P^{BS}(E_0)}{\partial \sigma} \times \sigma'(K), \; K \leq c \times D, \tag{23}$$

implying

$$\begin{aligned} \sigma'(K) \times \sqrt{T_1} &= e^{-rT_1} \times \frac{Q(V_1 \leq Y_1) - N(-d_2)}{E_0 \times N'(d_1)} \\ &= \frac{-N(-d_1)}{K \times N'(d_1)} < 0, \; K \leq c \times D, \end{aligned} \tag{24}$$

which is equation (14). ■

8.2 European put-call parity and call pricing

We derive the initial value of the ex-cash-flow equity price at T_1, where $B_1 = cD$ and $a_1 = 1_{\{C_1 > B_1\}}$. From equation (5), where $E_1 = a_1 C_1$,

$$\begin{aligned} E_0^Q\left[e^{-rT_1} E_1\right] &= E_0^Q\left[e^{-rT_1} a_1 C_1\right] \\ &= E_0^Q\left[e^{-rT_1} a_1 (C_1 - B_1)\right] + e^{-rT_1} B_1 \times E_0^Q[a_1] \\ &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1), \end{aligned} \tag{25}$$

because B_1 is predictable at T_0.

For European equity puts and calls (with respective prices p_t and c_t), with the same strike price (K) and expiring at the first-outflow date (T_1), from put-call parity at T_1,

$$K + \underbrace{\max\{0, E_1 - K\}}_{=c_1} = E_1 + \underbrace{\max\{0, K - E_1\}}_{=p_1}.$$

Then, from the law of one price, put-call parity at $t = 0$ becomes

$$e^{-rT_1} K + c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0, \tag{26}$$

The call price is given by

$$c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0 - e^{-rT_1} K.$$

As in the case of a paying-dividend stock, put-call parity is also adjusted, in this case, by $e^{-rT_1} B_1 \times Q(V_1 > Y_1)$.

Specifically, for LSP puts (i.e., $K \leq B_1$), from equation (12),

$$\begin{aligned} c_0 &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + e^{-rT_1} K \times Q(V_1 \leq Y_1) - e^{-rT_1} K \qquad (27) \\ &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) - e^{-rT_1} K \times Q(V_1 > Y_1) \\ &= E_0 - e^{-rT_1} (K - B_1) \times Q(V_1 > Y_1). \end{aligned}$$

For example, consider a spread between two co-terminal European calls struck within the corridor, with respective strike prices K_1 and K_2, $K_1 < K_2 \leq B_1$. Then,

$$c_0(K_1) - c_0(K_2) = e^{-rT_1} \times (K_2 - K_1) \times Q(V_1 > Y_1), \qquad (28)$$

which implies the following surviving probability, $\frac{c_0(K_1) - c_0(K_2)}{K_2 - K_1} = e^{-rT_1} \times Q(V_1 > Y_1)$.

For no LSP puts (i.e., $K > B_1$), from equation (16),

$$\begin{aligned} c_0 &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) \qquad (29) \\ &\quad + e^{-rT_1} K \times Q(V_1 \leq Y_1) + E_0^Q \left[e^{-rT_1} (K - E_1) \times 1_{\{B_1 < E_1 < K\}} \right] - e^{-rT_1} K \\ &= E_0 - e^{-rT_1} (K - B_1) \times Q(V_1 > Y_1) + E_0^Q \left[e^{-rT_1} (K - E_1) \times 1_{\{B_1 < E_1 < K\}} \right]. \blacksquare \end{aligned}$$

8.3 The general model: A default corridor in an endogenous setting

We show the link between European puts and credit protection is given by endogenous default.

1. The endogenous credit-risk model We study a general discrete-time setting with stochastic cash flows. Consider a model with N periods, $n = 1, 2, ..., N$, and respective times $0 < T_1 < T_2 < ... < T_N$. We denote by B_n the negative payout rate of the firm. Thus, B_n is an outflow that is paid if $B_n > 0$ (an inflow that is collected if $B_n < 0$) by equityholders' deep pockets, in Leland's tradition.

The equity continuation value is denoted by C_t. We assume equityholders' limited liability, implying $C_t \geq 0$, $0 \leq t$. It follows that if $B_n > 0$, defaulting at T_n is optimal if and only if

$$C_n \leq B_n,\ n = 1, 2, ..., N,$$

with indifference between defaulting and paying the cash outflow if $C_n = B_n$. This default choice maximizes equity value; that is, it is endogenous.

In addition, we also assume strictly positive values for the process C_t; that is,

$$C_t > 0,\ t \in [0, T_N],$$

which implies default is never optimal between outflow dates.

2. Endogenous equity pricing Given a terminal value $C_N \geq 0$, the equity continuation value is defined recursively as follows:

$$C_t = E_t^Q \left[e^{-r(T_n - T_{n-1})} \{C_n - B_n\}^+ \right] \geq 0,\ T_{n-1} \leq t < T_n, \tag{30}$$

for $n = 1, 2, ..., N$. The process C_t is discontinuous at T_n; that is,

$$\lim_{t \uparrow T_n} C_t \rightarrow \{C_n - B_n\}^+ < C_n \text{ if } B_n > 0,\ n = 1, 2, ..., N. \tag{31}$$

Similarly, $\{C_n - B_n\}^+ > C_n > 0$ if $B_n < 0$.

The ex-cash-flow equity price, as in the previous coupon-bond model, is given by

$E_N = a_N \times C_N$ and

$$E_t = a_{n-1} \times C_t,\ T_{n-1} \leq t < T_n, \tag{32}$$

for $n = 1, 2, ..., N$, where $a_0 = 1$ and a_n is in equation (4). In particular, $E_t = C_t$, $0 \leq t < T_1$. The (ex-cash-flow) equity-price function $E_n(C_n)$ is also discontinuous at T_n. That is, either

$$E_n = 0 \text{ or } E_n = C_n > B_n > 0 \text{ if } B_n > 0,\ n = 1, 2, ..., N, \tag{33}$$

both with positive probability. By contrast, in the case of a cash inflow (i.e., $B_n < 0$), and conditional on no previous default (i.e., $a_{n-1} = 1$), $E_n = C_n > 0$ if $B_n < 0$.

Then, $C_t > 0$ implies $E_t = C_t > 0$ between outflow dates, conditional on no previous default. It follows that default is never optimal between outflow dates. Hence, we focus on outflow dates. First, we provide an example.

Example In Figure 3, we provide a typical equity path that ends in default at T_3. The firm's assets mature in four periods. Three deterministic cash flows exist, namely, $B_1 = -5$, $B_2 = 4$, and $B_3 = 4$ (i.e., $B_1 < 0$ is a cash dividend and B_2, $B_3 > 0$ are debt service or outflows). The assets are risky and have an expected value of 7.5 at T_4. The value of the firm is $C_0 = E_0 = 6$, which equals the intrinsic value (i.e., $5 - 4 - 4 + 7.5 = 4.5$) plus some option/upside value (i.e., $6 - 4.5 = 1.5$).

***** to include Figure 3 *****

Equity value increases in the first period from $E_0 = 6$ to 7. Namely, right before T_1, the value of equity is 7, that is, $C_1 = E_1 = 7 - 5 = 2$, which is the (downward-jumping) ex-dividend equity price. After this large dividend, most of the firm value is option value ($C_1 = 2$). From T_1 to T_2, the firm remains stable. Right before T_2, the value of equity is also 2, that is, $C_2 = E_2 = 2 + 4 = 6$, and the equity price jumps upwards. However, after T_2 and a high-volatility period, the firm quickly loses value and defaults at T_3 because $C_3 = 3$ is less than $B_3 = 4$. Hence, we advance that a default corridor $[0, 4]$ exists at T_2 and T_3.

At T_3, it is optimal to equityholders to not absorb the debt service but default, which implies $E_t = 0$, $t \geq T_3$. From $C_3 = 3$, and given no additional cash-flows, V_3 is also close to 3 (which is less than the initial expected value of 7.5). In brief, after this poor path/performance, all stakeholders lose. Equityholders get $5 - 4 = 1$, which is less than the initial equity value of $E_0 = 6$, and debtholders get 4 and 3, instead of the promised cash flows of 4 and 4, which implies a loss of 1/8 for them.

3. The endogenous default event and the default corridor Implicit in the definition of the equity value (i.e., equation (32)) are N endogenous-default events; that is,

$$\{C_n \leq B_n\}, \, n = 1, 2, ..., N. \tag{34}$$

These N endogenous-default events lead to the N default corridors, in which the ex-cash-flow equity price cannot enter. That is, from equation (33),

$$E_n \notin (0, B_n] \, , \, n = 1, 2, ..., N.$$

In particular, the event $\{C_n \leq B_n\}$ is equivalent to $\{E_n \leq B_n\}$, $n = 1, 2, ..., N$. Naturally, the event/corridor is empty if the cash flow is an inflow, that is, if $B_n < 0$.

However, because the payout rate B_n follows a random process, the N default corridors are conditional on B_n. Therefore, assuming B_n is one-period predictable, the only possible default corridor is at time T_1 and is given by $[0, B_1]$ (i.e., $E_1 \notin (0, B_1]$). In a general setting with operational leverage, earnings are stochastic; hence, we assume predictability. In the case of financial leverage, (no floating) coupons and principal are known since issuance time, and only refinancing costs are random.

Naturally, time advances, and after the first period ends (and conditional on non-default), the second period becomes a new first period, and we again have a default corridor. That is, if $C_1 > B_1$ at T_1, we have a new default corridor at T_2 if $B_2 > 0$, because $E_2 \notin (0, B_2]$. So, without loss of generality, we assume $B_1 > 0$.

4. European puts, digital puts, and pure credit contracts At T_1, an LSP European put becomes a digital put, which replicates a pure credit contract. That is, for a put with maturity T_1, the payoff reduces to

$$\begin{aligned} \max\{0, K - E_1\} &= (K - E_1) \times 1_{\{E_1 \leq K\}} \\ &= K \times 1_{\{E_1 = 0\}} + (K - E_n) \times 1_{\{B_1 < E_1 \leq K\}} \\ &= K \times 1_{\{E_1 = 0\}} \text{ if } K \leq B_1, \end{aligned} \tag{35}$$

which is a binary option in the case of LSP puts, namely, $K \leq B_1$. The second equality follows from equation (33). As emphasized above, the latter result only happens for $n = 1$, because B_n is predictable yet stochastic for $n > 1$.

Then, from equation (33), $E_n = E_n \times 1_{\{E_n > B_n\}}$, from which follows $1_{\{E_1=0\}} = 1_{\{E_1 \leq B_1\}}$ and hence

$$\max\{0, K - E_1\} = K \times 1_{\{E_1 \leq B_1\}} \text{ if } K \leq B_1, \tag{36}$$

which replicates a pure credit contract, where $\{E_1 \leq B_1\}$ is the endogenous-default event at T_1. In this setting, the DOOM put (that replicates a pure credit contract) is rather an LSP put.

5. The price of European puts in a default corridor Similar to the coupon-bond model, in which leverage is only financial (and $B_1 = cD$), from equation (36), the price of an LSP European put with maturity T_1 is given by

$$E_0^Q \left[e^{-rT_1} K \times 1_{\{E_1 \leq B_1\}} \right] = e^{-rT_1} K \times E_0^Q \left[1_{\{E_1 \leq B_1\}} \right], K \leq B_1, \tag{37}$$

and the same implications follow as in section 3.

That is, LSP European-put prices are linear in the strike price, and the implied-volatility skew is negative, $\sigma'(K) < 0$, $K \leq B_1$. Put options are more expensive in a default corridor than in a benchmark setting of no corridor. Put-call parity is adjusted by the cash outflow (for $t < T_1$), and from this parity link, we price call options. All these results happen for a maturity T_1 that is equal to the first-outflow date, in which the outflow is assumed to be predictable.

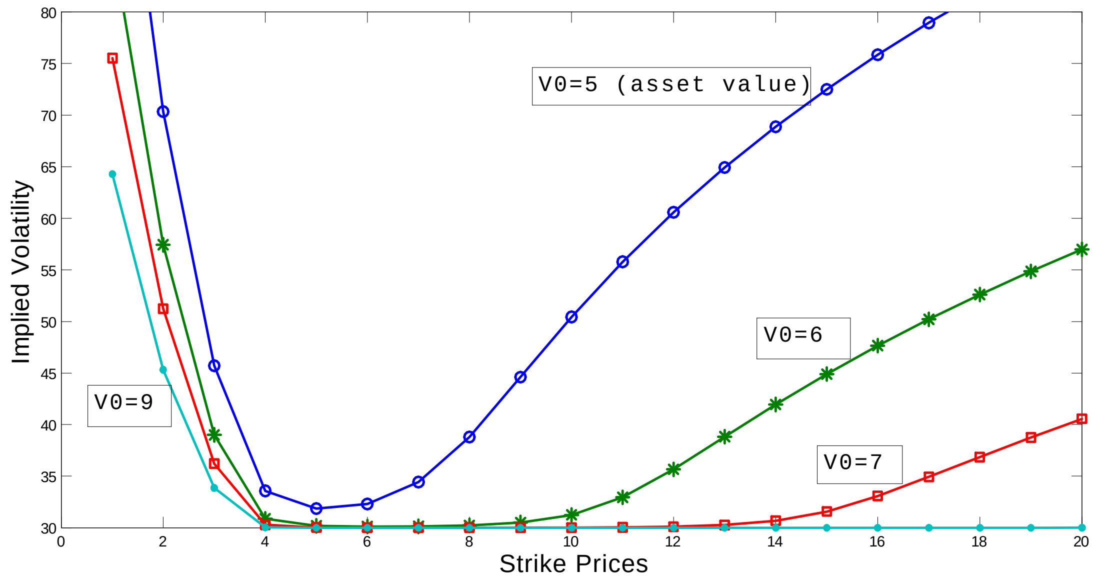

Figure 1: From a default corridor $[0, B]$ at T_1, we show the implied-volatility curves generated by European put options. At T_1, the value of equity equals $V_1 \times 1_{\{V_1 > B\}}$. We define the corridor by assuming $B = 3$. The lognormal assets volatility is $\sigma = 30\%$, the interest rate is $r = 2\%$, and the maturity is $T_1 = 6$ months. Equity and the European equity put have the same maturity. We consider four equity prices, $E_0 = \{2.03, 3.03, 4.03, 6.03\}$, corresponding to the four asset values, $V_0 = \{5, 6, 7, 9\}$, respectively.

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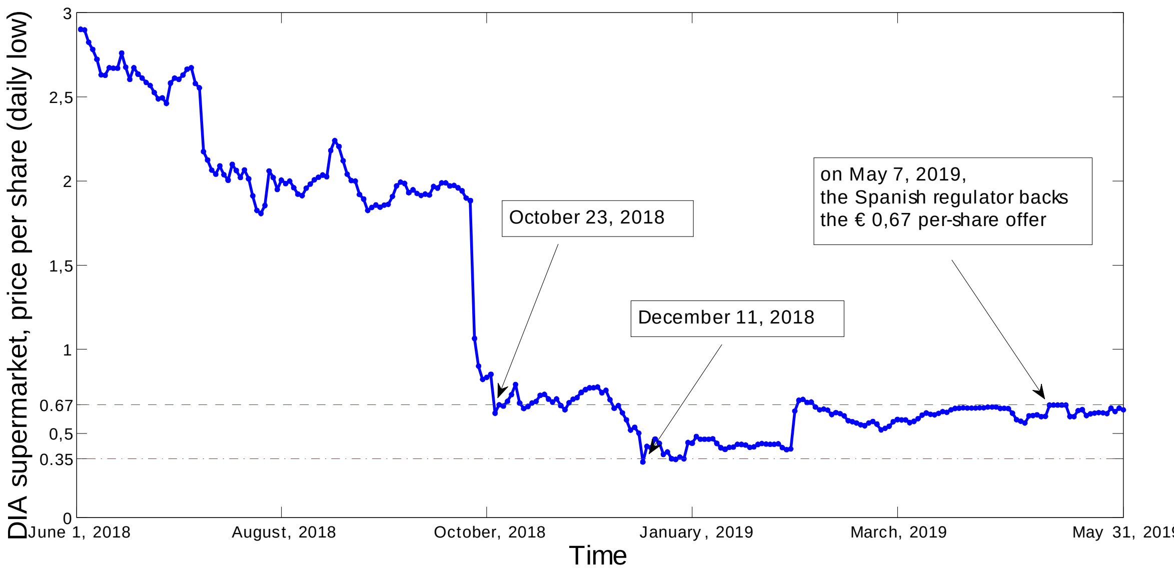

Figure 2: Price per share of Spanish retailer DIA (in Euros) from June 1, 2018, to May 31, 2019. Since October 23, 2018, the company's stock price looks consistent with a default corridor.

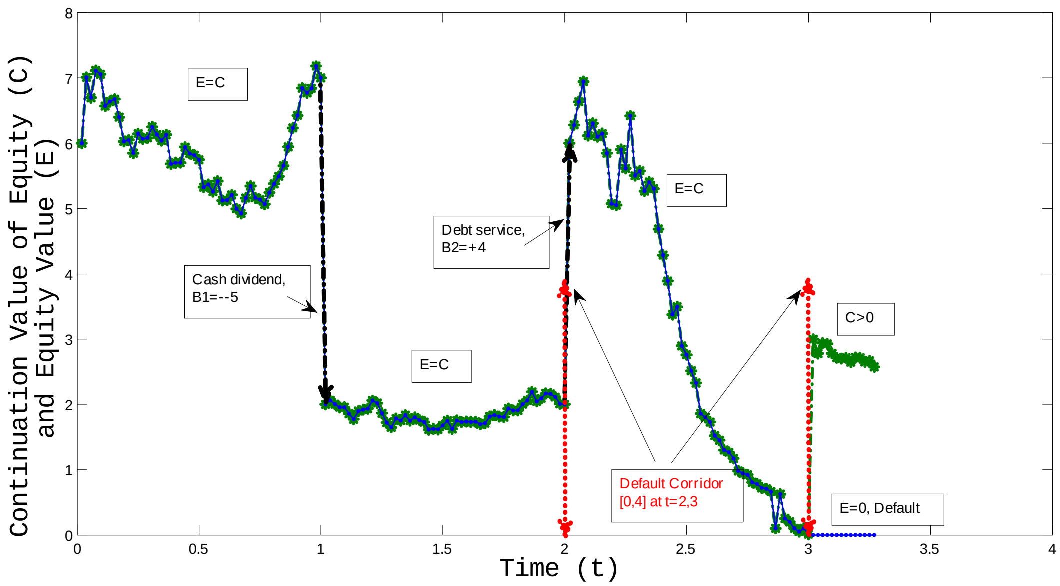

Figure 3: In a default corridor, a typical equity path that ends in default. The firm's assets mature in four periods. Three deterministic cash outflows exist, $B_1 = -5$, $B_2 = 4$, and $B_3 = 4$, where $B_1 < 0$ is a dividend and B_2, $B_3 > 0$ are debt payments. The initial value of the firm is $E_0 = 6$. At T_1, equityholders receive a dividend of 5, and equity falls from 7 to 2. T_1 to T_2 is a calm period, equityholders pay a debt service of 4, and equity jumps from 2 to 6. After T_2 and a high-volatility period, the firm quickly loses value, and equityholders choose defaulting at T_3 because the value of the assets is 3, which is lower than the debt service of 4. That is, $E_t = 0$, $t \geq T_3$.