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Examining the Role of Assessment in a Transition-to-proof Course: Teaching Practices and Evaluation

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EXAMINING THE ROLE OF ASSESSMENT IN A TRANSITION–TO–PROOF COURSE: TEACHING PRACTICES AND EVALUATION

by

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Under the Direction of Draga Vidakovic, Ph.D.

ABSTRACT

Over the past decade, research about students' proof capabilities has been a prevalent topic in collegiate mathematics education. Also, while not as prevalent, there has been interest in research about the teaching practices of the introduction to proof and other proof–based collegiate mathematics courses. To investigate the link between these two topics, this dissertation examined the assessment and teaching practices of Dr. Wyatt, a research mathematician who participated in mathematics education research alongside mathematics educators from multiple universities, utilized as the instructor of a Transition–to–proof course. An analysis of responses of his former students, observations of his instruction, the examination of a variety of types of assessments used during the course, and an interview at the end of the semester are used to determine the impact his participation in mathematics education research had on his beliefs about teaching and the assessment of students' mathematical understanding/knowledge. This dissertation utilizes an assessment framework developed by Mejia-Ramos et al. (2012) (which focuses on students' proof comprehension) and a framework about teaching practices at the collegiate level developed by Speer et al. (2010). The findings in this dissertation indicate that Dr. Wyatt uses several types of assessment that focus on the foundational aspects of mathematical proof while providing targeted feedback to students' responses. Further, Dr. Wyatt's teaching practices have been enhanced through the use of a new assessment question type modeled on what he learned from the mathematics education research project.

INDEX WORDS: Mathematical Proof, Teaching Practices, Assessment Practices, Transition–to–proof.

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DEDICATION

In memory of **Nana**

(Jessie Louise Simmons Thomas Spell)

Your love of reading sparked my curiosity, your kindness nurtured my compassion, and your love of learning got me to where I am today.

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To think the thinkable, that is the mathematicians aim.– C.J.Keyser

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LIST OF ABBREVIATIONS

- CUPM – Committee on the Undergraduate Program in Mathematics
- DTP – Definition–Theorem–Proof
- ACE – Activity, Classroom Discussion, and Exercise
- RCGP – Reading Comprehension of Geometric Proof
- PCT – Proof Comprehension Test
- UM – University Mathematicians
- ET – Expert Teachers
- PT – Proficient Teachers
- ROG – Resources–Orientations–Goals
- CTW – Critical Thinking Through Writing
- TQ – Teacher Question
- LQ – Learner Question
- GaP – Grade a Proof
- AaP – Analyze a Proof
- PMI – Principle of Mathematical Induction
- PCI – Principle of Complete Induction

PART 1

INTRODUCTION

Mathematicians use proof for a variety of reasons. Reid & Knipping (2010) identify verification, explanation, exploration, and communication as some of the aspects of proof; further, proof can be used for falsification and to illustrate new methods of deduction (Stylianides et al., 2017). The *2015 Committee on the Undergraduate Program in Mathematics (CUPM) Curriculum Guide to Majors in the Mathematical Sciences* further solidifies the importance of proof in mathematics education by suggesting students "should learn to read, understand, analyze, and produce proofs" (Shumacher & Siegle, 2015) that increase in depth throughout their studies. In this chapter we will discuss the importance of examining the teaching practices used in a transition to proof course, state the research questions this study hopes to answer, and discuss the theoretical perspectives used to analyze assessments and teaching practices.

1.1 Statement of the problem

In recent decades proof has been a central topic in collegiate mathematics education research. Indeed, empirical research in this area have focused on "how students at different levels understand, typically *mis*understand, proof" (Stylianides et al., 2017); further, Mejia-Ramos, Lew, de la Torre, & Weber (2017) developed an assessment to measure a student's comprehension of proofs. However, there is "very little empirical research" (Speer, Smith III, & Horvath, 2010) focusing on the general teaching practices used in collegiate mathematics.

For this study, teaching practices are defined, generally as "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer et al., 2010). I began this study with the following in mind. First, instructional activities are the methods an instructor uses to communicate information to students. Typically, these methods include lecture, cooperative group activities, class discussion, and review assignments (Speer et al., 2010; Mason, 2002; Weber, 2004; Hanna & Barbeau, 2010). As noted by Weber (2004); Mejia-Ramos et al. (2017), lecture in the style of definition-theorem-proof (DTP) is the most common instructional activity used in upper level courses in collegiate mathematics courses. Secondly, how instructors evaluate students' work, both written and oral, is an essential task both in and outside of the classroom; this can include answering questions during the instructional activity, assisting students during their office hours, and examining formal assessments from students (Mason, 2002; Weber, 2004; Mejia-Ramos, Fuller, Weber, Rhodes, & Samkoff, 2012). Thirdly, and the final aspect considered in this study, self-reflections about instructional activities and student interactions during lessons are explained because these have a direct impact on future instructional activities (Speer et al., 2010; Le Fevre, 2014; van Kan et al., 2013).

In addition to the research focus on students' learning and teaching practices in collegiate mathematics, there is a small number of research studies that focus on factors that inhibit educational change. For example, Fullan (2007) notes that educational change always has "a number of things at stake – changes in goals, skills, philosophy or beliefs, behavior, and so forth;" in short, educational change is a multidimensional problem. One of the most vital of these factors, teacher beliefs, can be very difficult to change. One reason for this, in general, is that teachers have to "question the effectiveness of their previous and current beliefs" (Le Fevre, 2014). This is particularly difficult with regards to mathematical proof at the collegiate level because proof is "the heart of what most mathematicians do" (Cilli-Turner, 2017) and is used for many purposes including those listed above. For example, Lew, Fukawa-Connelly, Mejia-Ramos, & Weber (2016) suggest the DTP instructional paradigm is so prevalent in advanced mathematics because it establishes the logical patterns used in mathematics, or worse, the lecturer believes most students are incapable of learning the material. For any mathematician to alter their teaching practices, their method either needs to fit into their beliefs or their beliefs have to alter.

Further, instructional activities take time to develop. While preservice K–12 teachers have courses focusing specifically on pedagogy, this type of course is not an essential part, if at all, of a mathematician's training (Lew et al., 2016; Speer et al., 2010). Because it takes time to develop pedagogical skills, the experiences over the course of a mathematician's career shape and develop their beliefs about certain pedagogical methods (Speer et al., 2010; Weber, 2004). For example, the Moore Method was popularized and refined "during Moore's forty-nine year tenure" (Zitarelli, 2004) at the University of Texas.

Finally, the purpose of assessment within collegiate mathematics classes is examined. Steen (2006) identified the state of assessment in collegiate mathematics.

> Prodded by persistent questions, mathematicians have begun to think afresh about content and pedagogy. In assessment however, mathematics still seems firmly anchored in hoary traditions. More than most disciplines, mathematics is defined by its problems and examinations, many with histories that are decades or even centuries old. (p. 13)

He further noted that "most tests have a high proportion of template problems" (p. 14), requiring students to be able to mimic responses. This mimicry allows for the possibility that students can correctly generate the solution without understanding the concept. This early analysis of assessment practices lead to several frameworks being developed to improve assessment methods, specifically with regard to identifying how students comprehend proofs (Mejia-Ramos et al., 2012; Pinto & Karsenty, 2018; Yang & Li, 2018; Herizal et al., 2019; F N et al., 2019). Further, Miller et al. (2018) examined how professors assign points to proofs. They identified that correctness is not the only criteria used when grading proofs and that some generic proofs, though incomplete, are considered correct and appropriate as a lecture format (Miller et al., 2018, p. 31–32). This study used the assessment framework developed by Mejia-Ramos et al. (2012), which is discussed in Section 1.3.1. This framework expanded upon the model developed by Lin & Yang (2007) which focused on reading comprehension of proofs.

1.1.1 Significance of the study

Keith Weber (2004) made the following conclusion in a case study, which will be discussed in detail in the next chapter, examining a mathematics professor's lecture style.

> Leading professors to improve their teaching of advanced mathematics courses requires leading these professors to adjust their goals for the course and their beliefs about mathematics education. In [the subject]'s case, his beliefs were coherent and stable, and hence would likely not be changed easily. It follows from basic constructivist principles that simply telling professors the beliefs that mathematics educators would like them to have would most likely do little good. Rather, perhaps the best way for mathematics educators to meaningfully change the way that mathematics professors teach is for both groups to engage in a mutual negotiation about goals for advanced mathematics courses and appropriate beliefs about mathematics education.

This study was conducted following that belief, that is to say, the collaboration of mathematicians and mathematics educators is an important factor in improving mathematics education at the collegiate level. The subject of this study, Dr. Wyatt, is a mathematics professor at a research university in the southeastern United States. At the time of this study, he was working with collegiate mathematics education specialists and other mathematicians in a NSF supported project to develop and implement research based activities and assessments designed to enhance undergraduate students' understanding of mathematical proof, including the ability to comprehend, use, and write mathematical proof. One of Dr. Wyatt's roles as a participant in this project was to help design assessment items using an assessment framework developed by Mejia-Ramos et al. (2012), which will be described in Section 1.3.1. Further, Dr. Wyatt was expected to use various items from this assessment throughout his transition to proof course. Therefore, this study focuses on examining the teaching practices of a mathematics professor who is working with mathematics educators and analyze how these practices have been affected by his involvement in this project.

1.1.2 Purpose of the study

Schoenfeld (2000) notes "basic knowledge of how something works can, overtime, yield tremendous practical dividends." However, few studies have examined teaching practices at the collegiate level; that is to say, there is little knowledge of how the act of teaching works in university classrooms (Weber, 2004; Wagner et al., 2007; Speer & Wagner, 2009; Speer et al., 2010; Johnson et al., 2017; Pinto & Karsenty, 2018). Dr. Wyatt expressed his desire to incorporate new assessment methods into his transition to proof course that were developed through the NSF supported project. The goal of the project is to improve the student proof capabilities, that is improve students' ability to comprehend, use and write mathematical proofs. The project engages mathematicians in systematic reflection on the nature of student proof capabilities. Specifically, mathematicians and mathematics educators collaborate to devise and implement pedagogical changes in transitional undergraduate proof courses that lead to measurable improvements in students' proof capabilities. This opportunity allows an examination of how Dr. Wyatt implements these changes and how they affect his teaching practices as a whole, providing "basic knowledge" of how this process works.

1.2 Research questions

In this study, I examine the impact of Dr. Wyatt's participation in the project focusing on specific assessment method on individual instructional practices. In particular, this study is designed to answer the following specific research questions:

1. In what ways does Dr. Wyatt use the ideas of a particular assessment method focusing on students' thinking with respect to mathematical proofs in his teaching of transition to proof class?

2. How do Dr. Wyatt's current instructional practices compare to his previous method(s) used?

3. What impact does Dr. Wyatt's participation in the project have on his core beliefs

about teaching and the value of research in undergraduate mathematics education?

To answer these question, I used the assessment framework developed by Mejia-Ramos et al. (2012) to examine the types of assessment methods used by the instructor in his transition to proof class and the framework developed by Speer et al. (2010) to examine the instructional practices used in this course.

1.3 Theoretical perspectives

One key factor about assessment is that it must be cyclical in nature (Shumacher & Siegle, 2015). At the departmental level, this involves examining the learning strategies, examining the assessment methods, and evaluating the assessment process itself; this ensures that the program continually adapts to the needs at the time. While the CUPM discussed this specifically with regard to assessing how the department runs as a whole, the principles can be examined at the course level. In this section, I will discuss the assessment framework and the teaching practices framework used in this study.

1.3.1 Assessment framework

National Council of Teachers of Mathematics (2000) notes that assessments "inform and guide teachers" while they are developing instructional activities. Specifically, assessments help teachers by providing insight into how well the student understands the content of the assessment. Therefore, a well developed assessment will provide good information to use while reflecting on instructional activities. The proof assessment framework developed by Mejia-Ramos et al. (2012) is the framework used in the project in which Dr. Wyatt participates. Therefore, this framework will be used in this study. This framework has identified seven types of cognitive difficulties students face while producing and understanding mathematical proof. These seven types are broken into two groups: *local* and *holistic*.

The local group of proof comprehension. The local group of proof comprehension examines the local properties of a proof; that is to say, the "understanding that can be discerned either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (Mejia-Ramos et al., 2012). The local group of proof comprehension consists of three domains: the meaning of terms and statements, the logical status of statements, and the justification of claims.

If the definition of the key terms in a proof, or any form of writing, are not understood, then students will struggle with the other concepts within the proof. Similarly, while a student may understand the key terms, they may not comprehend a statement within the proof. Mejia-Ramos et al. (2012) describe five actions to assess students comprehension of the key terms (first two actions) and the statements of a proof:

1. State the definition of the term
2. Identify examples that illustrate the term
3. State a statement in an equivalent way
4. Identify the implications of the statement
5. Identify examples that illustrate the statement (p. 8)

It is important to note that while understanding the key terms and statements of a proof is needed to fully understand a proof, it is not (always) essential to understand the proof to be able to perform these types of actions.

Understanding the logical status of statements within a proof refers to the aspects of a proof. The first is the logical status of various assertions made within the proof; for example, identifying an assertion as an axiom, postulate, or another theorem. The second is recognizing the "logical relationship between the statement being proven and the assumptions and conclusions of the proof" (Mejia-Ramos et al., 2012). Seldon & Seldon (1995) refer to this as the *proof framework* or "a representation of the 'top level' logical structure of a proof, which does not depend on the relevant mathematical concepts." For example, a proof by contradiction or a proof by induction have specific, recognizable structures. Two actions are given in Mejia-Ramos et al. (2012) to asses students' understanding of the logical status of statements:

1. Identify the purpose of a sentence in a proof framework

2. Identify the type of proof framework (p. 9)

The final local aspect of proof comprehension is the justification of claims. In a proof, students need to make inferences about what statements and mathematical principles are not explicitly stated in the body of a proof but are required for the given statement to be deduced (Mejia-Ramos et al., 2012). Three actions are listed as assessments of a student's comprehension of justification in proof:

1. Make an implicit statement in a proof explicit

2. Identify the specific data supporting a claim

3. Identify the specific claims that are supported by a given statement. (p. 10)

What follows is a description of the second subgroup of cognitive difficulties students face while producing and understanding mathematical proof.

The holistic group of proof comprehension. The holistic group of proof comprehension deals with proofs being "understood in terms of its main ideas, methods, and application to other contexts" (Mejia-Ramos et al., 2012). In essence, this group can be assessed by looking at four cognitive tasks: summarizing via high-level ideas, identifying the modular structure within a proof, transferring the general ideas or methods to another context, and using examples to illustrate the proof.

It is important to note that logical detail in a proof, while vital to mathematics, can overwhelm students and hinder their ability to recognize the main ideas of the proof (Weber, 2004; Mejia-Ramos et al., 2012). These main ideas can refer to the proof as a whole, or to an essential step that must be taken to complete the proof. However, understanding these ideas does not necessarily equate to understanding all of the logic steps within the proof. Mejia-Ramos et al. (2012) identify two actions as appropriate means of assessing this domain:

1. Identify or provide a good summary of the proof
2. Identify a good summary of a key sub-proof within the proof (p. 11 – 12)

Proofs can often be broken down into individual components or modules. In an interview one mathematician noted "a good proof often has a number of interesting lemmas and corollaries and sub theorems... longer proofs can get pretty complicated" (as cited in Mejia-Ramos et al., 2012). Therefore, when working with a longer proof, being able to break the proof into separate modules is a good method for increasing comprehension. Three actions are listed for assessing students' comprehension of the separate modules within a proof:

1. Partition the proof into modules.
2. Identify the purpose of a module in a proof.
3. Identify the logical relation between separate modules in a proof. (p. 12 – 13)

Mathematicians examine proofs not only for the reasons mentioned above, but also to understand key concepts or techniques that can be applied to other proofs Mejia-Ramos et al. (2012). Weber (2004) uses proof on limits from analysis as an example, summarizing how the instructor taught various techniques that could be applied to any proof about limits. Three actions are listed to measure this domain:

1. Transferring the method.
2. Identifying the method.
3. Appreciate the scope of the method. (p. 13 – 14)

Finally, students' understanding of a proof can be developed and assessed by examining their use of examples. Mejia-Ramos et al. (2012) notes generating examples in this fashion helps mathematicians check the logic within the proof; further, examples and diagrams often enhance students understanding of a proof. Two actions to assess how students use examples to illustrate the proof are given by Mejia-Ramos et al. (2012):

1. Illustrate a sequence of inferences with a specific example.

2. Interpret a statement or its proof with a diagram. (p. 15)

Next, I will discuss the framework developed by Speer et al. (2010) to examine the teaching practices of collegiate mathematics instructors based on examples used in K–12 teaching.

1.3.2 Framework for examining teaching practices.

In their discussion on the lack of empirical studies of teaching practices at the collegiate level, Speer et al. (2010) describe teaching practices that have been “productive foci for research on K–12 teaching.” While acknowledging the seven domains discussed below are not sufficient to cover all aspects of collegiate level teaching, they identified these domains to cover the core aspects of collegiate mathematics teaching.

Although there are seven domains identified by Speer et al. (2010), this section will only discuss in detail five: allocating time within lessons; posing questions, using wait time, and reacting to student responses; motivating specific content; representing mathematical concepts and relationships; and evaluating and preparing for the next lesson (p. 107). We will not examine the sequencing of content within the lessons or designing assessment problems as the former is established by the syllabus and the latter was discussed previously.

Allocating time within lessons. The length of the course sessions are predetermined, in this study each session will last one hour and fifteen minutes. The majority of these decisions are made while planning the lesson, but are altered during the sessions to meet the needs of students. Further,

> time allocation decisions are crucial for teachers who use multiple instructional activities because issues of sequence (e.g., which comes first?) and transition (how do I move between them?) must be addressed. But even where lecture is

> the sole instructional activity, teachers must decide how long to spend on each element of their presentation (e.g., on a definition, method, worked example, and/or theorem) in order to know what is feasible in one class session (Speer et al., 2010, p. 108)

Allocation of time within a lesson, while visible during classroom observations, cannot be fully understood without examining the reasoning behind the allocation. In fact, Speer et al. (2010) suggest interviewing the teacher to obtain data on this.

Motivating specific content. Speer et al. (2010) define motivating content as "providing a rationale for a sequence of topics to increase students' engagement with that content." Motivation can be established from examining the structure, both externally and internally, or discussing the historical development of the content. Regardless, motivating parts of content requires a time commitment and is directly related to how time is allocated during a session.

Asking questions, using wait time, and reacting to students responses. Essentially, questions are commonly used in most classrooms. Asking questions during sessions requires the instructor to decide "what to ask, how long to wait for student responses, and how to react to and evaluate those responses" (Speer et al., 2010). Through the examination of K–12 research on questioning, Speer et al. (2010) emphasized four components (frequency, character and intent, wait time, and reaction/evaluation) of questioning that should be considered by collegiate mathematics instructors.

First, the instructor needs to establish how often he or she ask questions; regardless of the instructor, this decision is often spontaneous. Second, the character and goals of the questions determine the value of the question. Speer et al. (2010) note that instructors ask questions either to keep students engaged or to provide insight regarding overall understanding and make decisions regarding time allocation. Third, wait time between asking a question and reacting to or providing the answer to the question. Finally, how instructors react to the responses is a key factor in developing a classroom culture and ultimately determines how students react to further questioning.

Representing mathematical concepts and relationships. The key aspects to consider about representing mathematical content is that it "includes both *what* is displayed and *how* it is displayed" (Speer et al., 2010). Both of these questions are deeply rooted in the teachers beliefs. For example, the Activity, Classroom Discussion, and Exercise (ACE) teaching cycle is a pedagogical approach used along with APOS theory (Arnon et al., 2014, Chapter 5), a Constructivist theory developed from Piaget's theory of learning. With this approach, students first work on a cooperative assignment designed to help them develop mental constructions without an emphasis on correct answers. Then, classroom discussions "involves small group and instructor-led class discussion, as students work on paper and pencil tasks that build on the lab activities completed in the Activities" portion of the lesson, these discussions allow students to reflect on and solidify there mental constructions from the previous part of the lesson. Finally, the homework exercises are designed to support the construct developed during the session. This process would then repeat in later sessions. The ACE teaching cycles' answer to *what* mathematical content is displayed is based on the genetic decomposition of the topic. A genetic decomposition is "hypothetical model that describes the mental structures and mechanisms that a student might need to construct" (Arnon et al., 2014) in order to truly understand a topic. The genetic decomposition is then examined and the necessary mathematical content needed for a student to develop these structures is displayed. The answer to *how* mathematical content is displayed in the ACE teaching cycle can vary greatly, including computer or paper based activities.

While the ACE teaching cycle is more student driven, traditional DTP lectures are teacher driven. However, the presentation of content can vary greatly. For example, the subject of the study by Weber (2004) used three distinct strategies to present the content of a real analysis course; this study will be discussed in more detail later. Further, Mason (2002) discusses in various approaches to organizing a lecture and suggests methods to use various screens (overhead projectors, whiteboards, etc) to display content coherently. Regardless, the method of presentation will be chosen by the instructor based on his/her beliefs.

Evaluating and preparing for the next lesson. Prior to a session, the instructor makes decisions about various aspects of his/her lesson. It is equally important to reflect on the previous lesson "evaluating their plans, particular actions and choices, and their students' contributions" Speer et al. (2010) to make necessary adjustments to his/her next lesson and course. This aspect of teaching practices is best examined through interviews.

1.3.3 Relating the assessment and teaching practices frameworks

As noted earlier, this study is focusing on the affect of Dr. Wyatt's participation in the NSF supported project has on his teaching practices. Therefore, a clear relationship between the two frameworks must be identified. The assessment framework provides a means to determine how students comprehend proofs; this information is then used to inform instructional methods. However, how does the information obtained from the assessment affect the teaching practices?

Figure 1.1 provides an overview of how these frameworks are related. First, the local group of proof comprehension forms the basis for students' proof capabilities, therefore it determines the initial time allocation needed to learn various topics in, and methods of, mathematical proofs. Second, the holistic group of proof comprehension identifies the central goals for students' proof capabilities. So, the holistic group of proof comprehension provides techniques that can be used to motivate the content of an upper level mathematics course. Thirdly, all aspects of the assessment framework provide a general guideline for questioning in the classroom; the assessment framework potentially helps the instructor determine what to ask and how to ask specific questions to identify students' comprehension. Finally, when reflecting on past instructional methods and preparing for future course sessions, the instructor must determine how they will represent the topics being covered. The assessment framework provides a list of what students need to develop in order to have strong proof

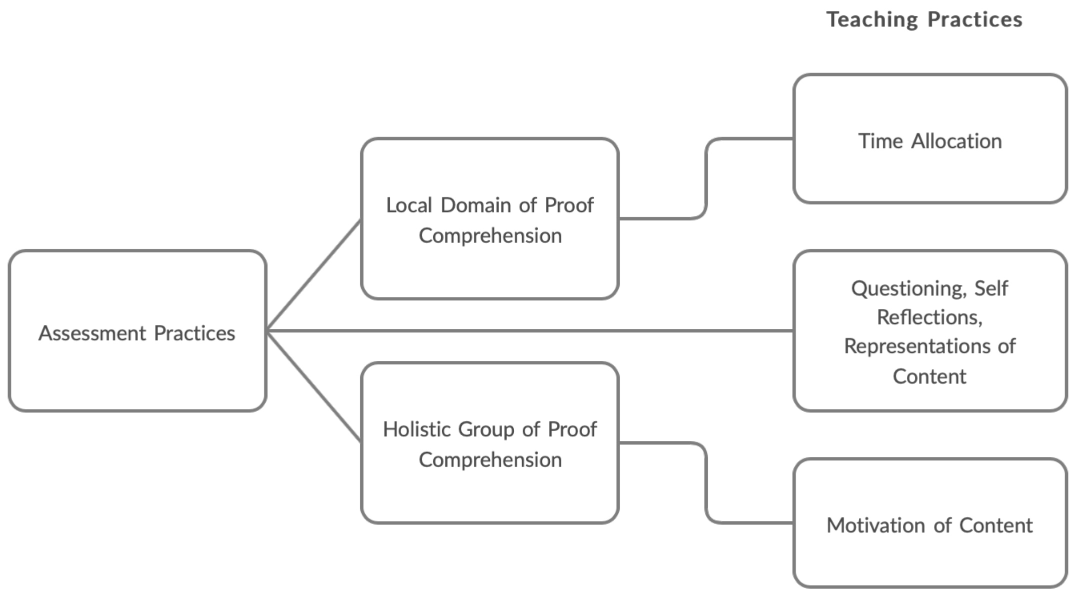


Figure (1.1) General overview of the connections between the teaching practices and assessment frameworks.

capabilities. Thus, to determine the best ways to represent mathematical content to a class, an instructor needs to focus on what methods will promote the various aspects of proof comprehension discussed in the assessment framework.

1.4 Chapter summary

In this chapter we examined the knowledge base on teaching practices at the collegiate level. Three key aspects of teacher practices, instructional activities, method of assessment, and self-reflections, were identified and defined. Further, teacher beliefs and the time it takes to generate instructional activities were noted as factors inhibiting change in collegiate mathematics education. The importance of mathematicians and mathematics educators working together was identified as well as the opportunity currently present with Dr. Wyatt's transition to proof course. Research questions relating assessment of proof comprehension and teacher practices were established. Finally, the frameworks, created by Mejia-Ramos et al. (2012) and Speer et al. (2010),used to generate questions and assess students' comprehension of proofs and evaluate teaching practices were briefly introduced.

PART 2

LITERATURE REVIEW

In chapter 1, I discussed some of the key issues collegiate mathematics educators face when teaching proofs and provided a framework for assessing students reading comprehension of proofs and for analyzing teaching practices. In this chapter, I will discuss the literature related to this study. First, the focus will be on the nature of proof, including how it is used in mathematics education and how students comprehend proofs. Then literature about assessment in mathematics education, including some factors limiting its adoption, will be examined. Also, an example of how the assessment framework is used to create assessment items will be described. Finally, literature discussing current teaching practices will be examined.

2.1 Literature on the nature of proof

Smith et al. (2011) describe the main role of proof is to "demonstrate that our conclusions [about a theory] are true" (p. 1), that is to say for verification purposes. While this is not incorrect, it is not as in depth as is needed. In addition to the role of verification described above, mathematical proofs can be used to explain a theory, provide the mathematical tools needed to allow for the discovery of mathematical knowledge, promote the communication of mathematical knowledge, and systemize or organize mathematical concepts (Lew et al., 2016). In this section the role proof plays in mathematics education and what is meant by "understanding a proof" will be examined.

2.1.1 Proof and mathematics education

There are several roles mathematical proof can have to mathematicians and mathematics students; however, according to Rocha (2019), there is a distinct difference between the two. She notes that the role of proof in *school mathematics*[1] is not as diverse as their role in mathematics in general, specifically that "in school mathematics the explanation function [of mathematical proofs] is the most relevant" (Rocha, 2019, p. 10) aspect for students to learn.

Rocha (2019) makes several points about factors on learning mathematical proofs. A key factor on how students learn mathematical proof is based on the experience they have with proofs, that is to say, how their teachers approach the instruction of the proofs. For example, while *the sum of of the first n odd numbers is n^2* can be proved using the principle of mathematical induction, it can also be shown using the diagram in Figure 2.1[2]. Further, she states the following:

> [P]roofs do not have to be restricted to formal proofs, and this means that the simplicity of the language used has to be present. Also, the modes of argumentation need to be appropriate to the level of the students, and once again this means some simplicity is required (Rocha, 2019, p. 9).

In other words, the language and methods used by teachers must meet the students at their level of understanding.

A large portion of research on the role of proof in mathematics education has focused "primarily with the logical aspects of proof and with the problems encountered in having students follow deductive arguments" (Hanna & Barbeau, 2010) Further, Hanna & Barbeau (2010) note that proof can be used to demonstrate new methods and introduce topics. They examined two proofs found in secondary mathematics; the first is used in the development of the quadratic formula and the second when discussing inscribed angles in a semi-circle.

Though not technically a theorem, the quadratic formula is a statement of a result. Nevertheless, this formula does permit itself to being introduced not by statement of the formula but through the question "how can we solve a quadratic equation?" (Hanna &

[1] This phrase is used to indicate mathematics at the primary and secondary levels of education.

[2] A larger image with this same format can be found in Rocha (2019), p. 3.

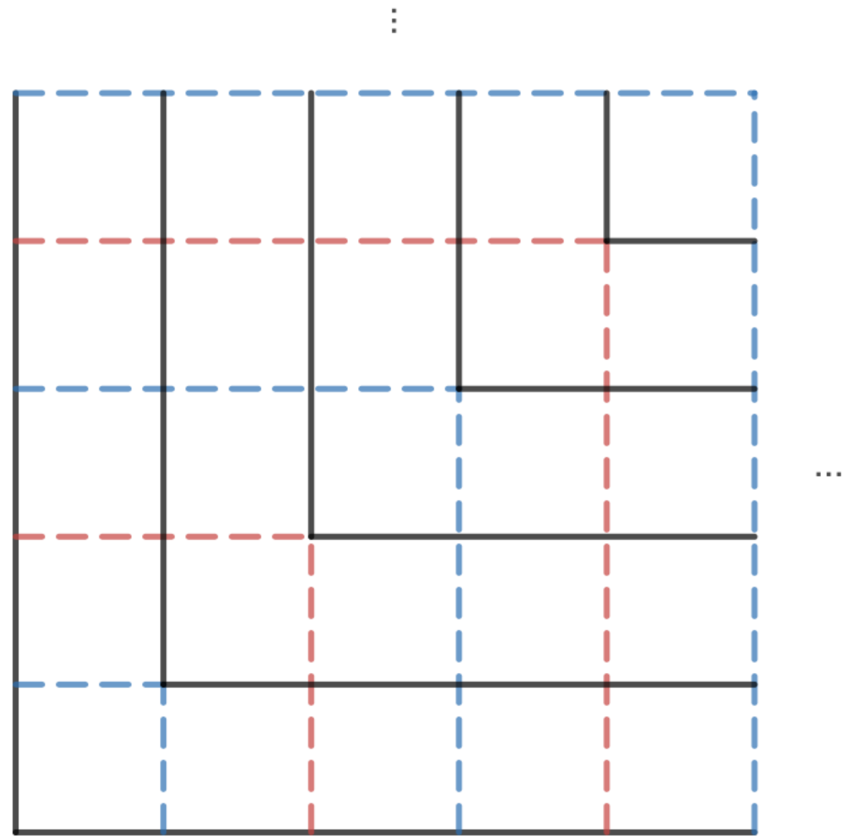


Figure (2.1) A visual proof of the sum of the First n odd numbers is n^2.

Barbeau, 2010, p. 91). Hanna & Barbeau (2010) describe a key benefit gained from this method.

> This may be the first time that secondary school students see this general technique of adding and then subtracting a term in an expression, a useful technique that they will see frequently as they advance their study of mathematics. We note here that completing the square does not stem logically from a previous statement or axiom. Rather it is a topic specific move and an additional mathematical tool for the students to use in other similar situations (p. 92).

Further, this method can generate a discussion about a specific aspect of algebra, finding information from an expression, and how to perform the necessary algebraic manipulations to get an expression into a form where the information can be easily read.

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Method:

1. Examine cases where $b = 0$
2. Obtain $ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$
3. Complete the square on $x^2 + \frac{b}{a}x$

Figure (2.2) A possible use of proof as an introduction of the quadratic formula (Hanna & Barbeau, 2010).

Similarly, several proofs of the following proposition lead to interesting discussions about geometry. A common approach makes use of concepts about isosceles triangles and properties of a circle; however, this result can be extended from a semicircle to a circle more easily using isosceles triangles (Hanna & Barbeau, 2010). This demonstrates how proofs can be used to unify other arguments within mathematics.

Proposition. *"Let A and B be opposite ends of the diameter of a circle and let C be a point on its circumference. Then angle ACB is right" (Hanna & Barbeau, 2010, p. 93).*

2.1.2 Proof comprehension

During the discussion on the local group of proof comprehension, the concept of a proof framework (Seldon & Seldon, 1995) was defined as the relationship between statement and the structure of a proof. Let's examine that statement a little more, using the example (provided by Seldon & Seldon) of how to prove the $\lim_{x \to 3} x^2 = 9$. After going through the entirety of the proof, Seldon & Seldon provide the proof framework with blanks representing material that changes depending on what limit you are proving. However, the list in Figure 2.3 is comprised of the aspects of the proof that remain constant. They describe proof framework as containing the exact same information as the formal statement; however, these two formats are linguistically different. Further, a single theorem can have many proof frameworks associated with it depending on the specific proof technique being used, but regardless of the proof framework, the logic can still be translated.

1. Let $\epsilon > 0$.
2. Provide a positive δ in terms of ϵ.
3. Suppose the distance between some number x as it approaches the given value is less than δ.
4. Perform the necessary algebraic manipulations.
5. Therefore, the conclusion is true

Figure (2.3) Constants that form the proof framework for limit proofs (Seldon & Seldon, 1995).

The ability to translate the logic between statements and the proof framework requires that the information is read and comprehended. Lin & Yang (2007) described reading as a process that requires students to identify and understand definitions and statements and integrate them into the main theme of the section (p. 730). To be able to express this more clearly, they developed the four levels of understanding that described reading comprehension of geometric proof (RCGP). The first three levels compose a local understanding of the proof, that is to say, comprehension of the surface, elements, and logic behind how elements are connected. The fourth level, comprehension of encapsulization, is "characterized as interiorizing a proposition and its proof as a whole, which implies that one can apply it, as well as distinguish different premises related to other similar propositions" (Lin & Yang, 2007, p. 730) and is the foundation of a holistic understanding of the proof. Figure 2.4 relates the RCGP with the assessment framework used in this study.

The RCGP has two paths describing how understanding is developed at each level. Lin & Yang (2007) describe one form as *relational comprehension.* This type of comprehension arises as a lower level directly leads to the next level, that is to say, lower levels are related to the next level directly, and students need to pass through each level accordingly (Lin

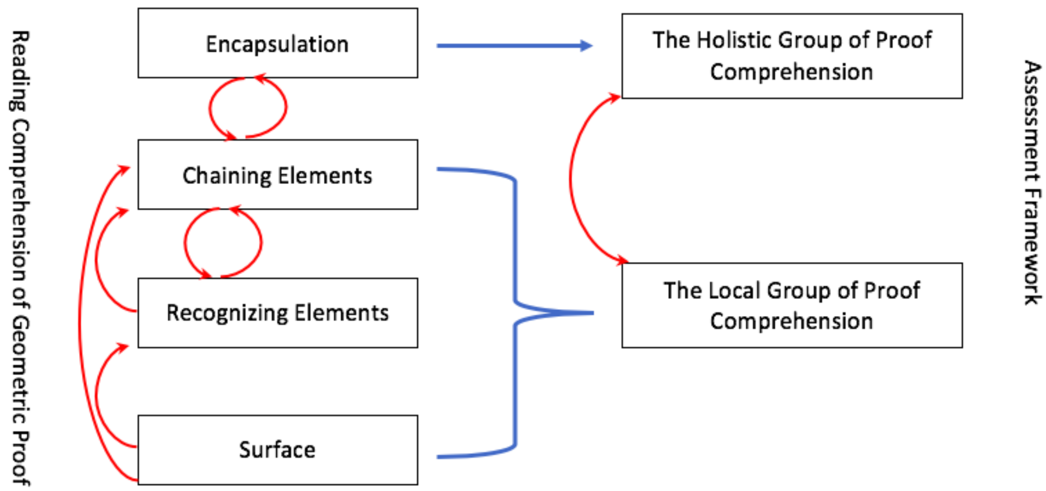


Figure (2.4) The RCGP and assessment framework (Lin & Yang, 2007; Mejia-Ramos et al., 2012).

& Yang, 2007). The second path describes *instrumental comprehension* which is signified by students "skipping" the step of either recognizing or chaining of elements level(s) and proceeding directly to the next level. In the case where students skip both the recognizing or chaining of elements levels, Lin & Yang (2007) describe students following instrumental comprehension as having an understanding of "the mathematical terms or concepts in these proofs and what these proofs validated, and can apply these statements properly in other similar situations" (p. 740) but can lack the ability to identify some key local aspects of the proof.

How students' understand mathematical proof is a topic of study at the secondary level as well as the collegiate level. One such study, described in Herizal et al. (2019), tested ten students' mathematical proof comprehension ability from from a private secondary school in Indonesia. To accomplish this task, they constructed four essay questions based on trigonometric topics designed to assess how students justify claims and can transfer ideas from one proof to another[3].

Herizal et al. (2019) identified four key results, as shown in Figure 2.5. They concluded

[3]Aspects of the local domain and holistic domain of proof comprehension, respectively.

"teachers have to teach about proving (mathematical proof comprehension and mathematics proof construction)" (Herizal et al., 2019, p. 6) at the secondary level, especially when discussing topics where proofs can be explored. This echos the recommendations of Hanna & Barbeau (2010), and both provide examples (the quadratic formula and the sine/cosine rules) of how mathematical proofs can be studied at the secondary level.

1. Students have no previous experience with questions of this level.
2. Students have difficulties identifying data that supported specific claims.
3. Students were able to apply the steps from one proof to a similar situation.
4. Students have difficulties beginning a proof.

Figure (2.5) A summary of students' understanding of proof (Herizal et al., 2019)

Harel (1999) identifies three categories of students understanding of proof in relation to geometry and linear algebra.

Category 1 Understanding parallels that of the Greek conception of math

Category 2 Understanding parallels 16-17th century conception of math

Category 3 Understanding the result of faulty instruction in primary and secondary schools

At the Category 1 level of understanding, students understand the roles axioms play in mathematics. Category 1 can be further broken down into three subcategories: intuitive axiomatic, structural, and axiomatizing. With an intuitive axiomatic understanding, axioms must align with the student's intuition for the student to accept the axiom as true; a structural understanding implies the student understands how various structures are consistent across proofs. If the student has a structural understanding, then the student can progress to axiomatizing, that is the student can "investigate the implications of varying a set of axioms" (p. 603) or understand the axioms of a specific field of study.

Category 2 is identified by a focus on why the object being proved is true. Harel (1999) provides the reasoning behind some students behaviors:

> We tend to associate misconception and missing conceptions only with mathematically weak students. But in fact, all students, the weak and the able, in their desire to understand and make sense of the mathematical concepts we intend to teach them, encounter difficulty, and demonstrate as a result behaviors that in many cases are difficult to explain (p. 606).

One of the behaviors encountered is the distrust associated with any form of proof by contradiction and the reliance on proof being causal, that is a proof should adequately show how the antecedent is the cause of the conclusion. Harel (1999) provides an explanation of issues produced by the need for a proof to be causal by considering the case "A if and only if B;" if a requirement of a proof is that must show some form of causality, then "A if and only if B" inadvertently states that A is the cause of A, which is illogical.

Category 3 misconceptions are not the direct fault of the student's conceptual limitations, but the result of how they learned mathematics in their primary and secondary education. Harel & Sowder (1998) describes Category 3 as being an "external conviction proof scheme" and further subdivides this into three categories: ritual, authoritarian, and symbolic. Harel & Sowder (1998) believe a key cause of Category 3 misconceptions is premature formalization of mathematics, that is, the emphasis on simply using "formulas to solve problems" and the teacher and textbook being the only source of mathematical knowledge breeds an environment where students learn that memorization and lack of creativity are the keys to success in mathematics.

The ritual proof scheme is formed when there is an "over-emphasis in schools on proof writing prior to and even in place of proof understanding, production, and appreciation" (Harel & Sowder, 1998). This over-emphasis leads to students' conception of proof being focused around how proofs are presented; two indicators of this are false-proof verification and uncertainty whether a justification can be considered a proof (Harel & Sowder, 1998). False-proof verification means an incorrect proof is analyzed and determined to be correct; in other words, the logic of a proof is not comprehended correctly because the form of the proof is correct (Harel & Sowder, 1998). Similarly, if correct justification is presented in a way students do not consider to be a proof, Harel & Sowder (1998) note students may have difficulties accepting the justification because it does not match their expectations of what constitutes a proof.

According to Harel & Sowder (1998), an authoritarian proof scheme is the result of "the fact that current mathematics curricula emphasize truth rather than the reason for truth" (p. 247). An authoritarian proof scheme leads to students expecting proofs to be provided because they "view mathematics [as] a collection of truths" (Harel & Sowder, 1998) and do not recognize need for proof because an authority figure, a teacher or a textbook, explicitly state conjectures as true. In short, students require affirmation from an external source when working with mathematical topics.

Harel & Sowder (1998) describe symbolic reasoning as "thinking of symbols as though they possess a life of their own without reference to their possible function or quantitative reference" (p. 250). Symbolic reasoning is developed throughout primary and secondary education; however, the symbolic proof scheme is identified by "approaching the solution to a problem without first comprehending its meaning" (Harel & Sowder, 1998, p. 251). However, Harel & Sowder (1998) note this definition of symbolic reasoning can be productive and lead to mathematical discoveries.

2.1.3 What constitutes a proof?

Proof is essential to the practice of mathematics. Weber & Czocher (2019) noted some disagreements about exactly what is acceptable as a proof. In this study, an internet survey tasked mathematicians to evaluate five proofs for validity. The five proofs were arranged into three categories: *prototypical proofs, empirical proofs*, and *non-prototypical proofs*. Prototypical proofs are "arguments using standard mathematical notation" (p. 258); there were two prototypical proofs. One empirical proof, that is a proof by observation, and two non–prototypical (visual and computer generated proofs) were also provided in the survey. The goal was to identify any disagreements (or consistencies) with the types of proofs mathematicians deemed valid.

One key disagreement occurred: there was not a consensus view on the validity of the non–prototypical proofs[4]. Weber & Czocher (2019) note that this presents a problem for people who conduct mathematics education research and believe that non–prototypical proofs are "not controversial" (p. 262). They describe this disagreement as demonstrating a *pluralistic view* of mathematical proofs. In short, there are many ways to define a correct proof.

2.2 Literature on assessment in mathematics education

The *CUMP Guidelines for Assessment of Student Learning* (2006) defines assessment as "the process of gathering and interpreting information about student learning" (p. 230). Though this definition is simplistic in nature, Steen (2006) describes assessment in collegiate mathematics education as "a minority culture beset by ignorance, prejudice, and the power of a dominant discipline backed by centuries of tradition" (p. 18); Madison (2006) notes that assessment was viewed skeptically, with "lack of enthusiasm and inevitably" (p. 3) as key indicators. This section will begin by discussing some of the factors inhibiting growth in assessment, then examining the purpose of assessment, and finally examining one assessment created using the assessment framework discussed developed by Mejia-Ramos et al. (2012).

2.2.1 Tensions and tethers

According to Madison (2006), the arguments against assessment fall into two broad categories: tensions and tethers. Tensions are actions or beliefs that favor easier models of assessment at the cost of effectiveness. Tethers are practices that are bound in tradition that prevents the creation of meaningful assessment (p. 3 – 4). Generally, to ease tensions, the risk involved with developing assessment practices must be reduced or alleviated while the values and beliefs of mathematicians have to be considered when addressing tethers.

The primary tension toward assessment involves practicality and effectiveness (Madison, 2006). Enrollment in Universities increased 14% between 2005 and 2015 to 20.0 million

[4]They agreed that the prototypical proofs were valid and the empirical proof was invalid.

students (U.S. Department of Education, National Center for Educational Statistics, 2016); since the majority of these students have to be assessed in at least one mathematics course, it is much more practical to use multiple-choice style assessments that may not be the most effective means of gathering understanding. I will discuss the effort to alleviate this tension that Mejia-Ramos et al. (2017) have developed.

Another tension arises because of philosophical differences about validity. Madison (2006) notes the following:

> Mathematicians are confident of their disciplinary knowledge and generally agree on the validity of research results. However, their research paradigm of reasoning logically from a set of axioms and prior research results is not the empirical methodology of educational practice where assessment resides. This tension *between ways of knowing* in very different disciplines often generates disagreements that prompt further evidence gathering and caution in drawing inferences from assessment evidence (p. 6).

This tension is not one that can be easily overcome. Educational research often "fall far short of mathematically rigorous standards" (Steen, 2006, p. 15) because it often is derived from observational studies.

All tethers are founded in traditional practices in collegiate mathematics. One such tradition is the DTP lecture style. Lectures often constrict the ability to perform meaningful assessments, especially formative assessments Madison (2006). Lew et al. (2016) describe one such constriction as pertaining to how students interpret the material being covered in the lecture. When examining the main idea in a lecture proof, students will interpret some of the arguments being used as support, for example algebraic arguments used when proving the limit of a sequence, as the main idea behind the proof if this is the focus of the "board proof" being constructed in class (p. 184). This disconnect between the instructor's intentions and the students' interpretation of a lecture creates a situation where assessment is naturally difficult because the goal of the lecture was not made explicit.

2.2.2 The purpose of assessment

Good classroom assessment will simultaneously enhance instruction by monitoring its effectiveness and audit learning by reliably identifying what students understand Ohlsen (2007). Further, assessment can act as an extrinsic motivational tool for students, as a record keeping device via grades (p. 8), and to troubleshoot and improve prerequisite developmental mathematics courses Cavanaugh et al. (2006). Assessments are naturally subjective because they measure a mental construct; however, a well designed and developed assessment is both reliable and valid, that is, consistent in the results it provides and it measures what it is designed to measure (Romagnano, 2001).

Assessment in proof–oriented courses are “comprised of proving task” (Miller et al., 2018). In fact, Miller et al. (2018) reports that approximately 80% of assessments in real analysis textbooks are proving tasks. This has an affect on what students deem important; assessments “provide students with a clear indication” (p. 25) of what is essential to learn in a mathematics courses. In other words, if assessment items are primarily focussed on understanding and comprehending mathematical proof, then students will identify this as the goal of the course. Section 2.2.3 describes how the framework developed by Mejia-Ramos et al. (2012) is used to create an assessment that is focussed on students comprehension of proofs.

In addition to recognizing how students identify the essential information for a course, how instructors evaluate these assessments is important. Miller et al. (2018) note how some mathematics professors will assign full credit to incomplete proofs, that is proofs without a full justification. Due to this inconsistency in evaluating the correctness of a proof, students may struggle with identifying what types of justifications constitute a proof. One method of eliminating this confusion is to require students to complete revisions of proofs as discussed in Pinto & Karsenty (2018). The subject of this study would grade students’ “term paper” weekly; the students were required to revise their previous proofs while completing the next submission.

2.2.3 The development of a proof comprehension test

Table (2.1) Actions taken and purposes of each phase.

Phase	Action	Purpose
Generating open-ended items	For each facet of our proof comprehension assessment model, we generated up to four open-ended questions.	To generate the questions for the items of our long multiple-choice tests.
Conducting pre-test interviews	We interviewed 12 students, asking them to answer each item generated in stage 1.	To generate the choices (correct answers and foils) for our long multiple-choice tests.
Reviewing items	We asked other math educators and mathematicians to review the long multiple-choice tests generated after stage 2.	To improve items, especially items that were mathematically inaccurate or ambiguous.
Conducting pilot interviews	We interviewed 12 students, asking them to think aloud as they answered the long multiple- choice tests refined in stage 3.	To identify and improve items in which students' responses were not indicative of their understanding of the proof.
Administering the test to a large population	We gave the long multiple-choice tests to approximately 200 students.	To verify that these tests had high internal reliability, to identify problematic items with poor discriminatory power, and to identify items that can be removed to generate shorter multiple-choice tests.
Conducting validating interviews	We interviewed 12 students, asking them to think aloud as they answered the shorter multiple-choice tests generated after stage 5.	To verify that the final, shorter multiple-choice tests accurately measured students' understanding.

Mejia-Ramos et al. (2017) used the assessment framework discussed in Chapter 1 to develop a Proof Comprehension Test (PCT) modeled on concept inventories such as the Force Concept Inventory used in physics education and the Precalculus Concept Assessment

(Mejia-Ramos et al., 2017, p. 9). Unlike the concept inventories mentioned above, the PCT did not focus on concepts from a specific course; however, the PCT uses a multiple choice format similar to both concept inventories. They describe the six phases used in the development of the PCT, as reproduced in Table 2.1[5]. These questions were illustrated for three theorems; however, I will examine only one here. The first three phases of development focused solely on generating and editing the initial multiple choice questions and responses. Consider the following theorem and proof reproduced from Mejia-Ramos et al. (2017).

Theorem. *There are infinitely many prime numbers.*

Proof. "Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < ... < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \cdots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q. Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, q is one of the p_i in the list. Thus, q divides the product $p_1 p_2 p_3 \cdots p_k$. Since q divides n,q divides the difference $n - (p_1 p_2 p_3 \cdots p_k)$. But this difference is 1,so $q = 1$. From the contradiction $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite." (Mejia-Ramos et al., 2017, p. 11). □

One open-ended question generated for this proof is "why is it valid to conclude n is a natural number?" (Mejia-Ramos et al., 2017, p. 14). This question was posed during the pretest interview and the answers provided were analyzed to develop the individual answer choices shown in Figure 2.6. The best explanation is item (a); then, this question was reviewed by both mathematicians and mathematics educators to insure accuracy, that is to say that the question was valid (p. 16). During this review, it was acknowledged that items (a) and (d) were very similar; therefore, option (d) was changed to "The set of integers greater than 4.5 and less than 9999"(Mejia-Ramos et al., 2017, p. 17) at the suggestion of a mathematician.

[5]Reproduced from Mejia-Ramos et al. (2017).

MC1. In the proof, why is it valid to conclude that n is a natural number? Please select **<u>the best</u>** option.

(a) Because the product and sum of natural numbers is a natural number.

(b) Because n is greater than 0.

(c) Because $1, p_1, p_2, \ldots, p_k$ are all integers.

(d) Because it is a given in the proof that n is a natural number.

Figure (2.6) Multiple choice question developed from an open ended question (Mejia-Ramos et al., 2017, p. 15).

The last three phases examined the state of the PCT after the initial revisions and then administer and examine how well the PCT performed. An interview was conducted with 12 undergraduate mathematics students; focusing on situations where

1. The correct answer was selected with incorrect reasoning

2. The wrong answer was selected with correct reasoning (Mejia-Ramos et al., 2017, p. 18)

These contradictory statements provided information to, once more, ensure the question was measuring the correct item, that is to say the question was valid. The test was then administered to 200 undergraduates in a transition to proof course (p. 19). They then proceeded to interview several students who completed the assessment to validate their responses.

In the end, Mejia-Ramos et al. (2017) identified several important reasons for the PCT as a resource. First, these assessments can enhance the students learning by leading them to the important aspects of a proof; second, they provide a simple resource for instructors and they believe questions can be generated to examine a more general understanding of proof (p. 22 – 24). While the question in Figure 2.6 is focused on identifying a local understanding of the proof, the information provided can direct future assessment towards a more holistic understanding.

2.3 Literature on general teaching practices

This section will examine teaching practices in general, that is to say the studies are not necessarily restricted to the university level. In total, three articles will be discussed here including the connections between these studies and the framework discussed by Speer et al. (2010) in chapter 1. The first study by Locke & Schattke (2018) examines various types of motivation. Next, the study by Maulana et al. (2015) examining the affects of time on lesson structure in Indonesian secondary mathematics classrooms. Third, a study examining participation in mathematics classrooms, including instructional and questioning methodology.

Speer et al. (2010) describe the main component of motivation being a rationale for a topic. While this definition is succinct, it can be expanded upon in several ways. Locke & Schattke (2018) study the concepts of both *intrinsic* and *extrinsic* motivation; however, they specify different aspects of each that has an impact on teaching practices.

While intrinsic motivation has been traditionally linked specifically with personal enjoyment (Locke & Schattke, 2018). However, enjoyment of a task does not necessarily equate to improving at the task. So Locke & Schattke (2018) suggest "to get a high level of skill at anything requires another form of motivation, *achievement motivation*" (p. 4); this type of motivation is not focused on the enjoyment of the task. Instead, this type of motivation prioritizes improvement toward a goal.

Locke & Schattke (2018) describe extrinsic motivation as being a "means–end" (p. 6) relationship. This means that factors outside of the task being performed are key motivators; however, these outside factors do not necessarily affect the individual directly. For example, exercising is a means, but the goal is not to complete the exercise but instead attempting to remain healthy. So extrinsic motivation requires a specific goal to be set.

Speer et al. (2010) do not directly discuss the structure of lessons, instead they discuss the preparation and sequencing of lessons. Maulana et al. (2015) examine the specific structure of the lesson, which they divide into six parts. *Introduction* only pertains to "activities occurring during the start of a lesson that are not related to the content of the lesson" (p. 847); for example, greeting students and returning assignments. Maulana et al. (2015) define *review* as discussing topics from the previous lesson[6]; that is to say assessing how students responded to the previous lesson. *Introduction of new content* is defined traditionally, but the lesson can be either teacher or student lead. Similarly, *student work time* focusses on when students are actively working with the topic. *Closing* the lesson focuses on how the instructor transitions from the main topic of the lesson to the end of the lesson. The final category Maulana et al. (2015) define includes everything that has not already been covered by the previous categories; this category is referred to as *other*.

The conclusions from their study are given here: time allocated to each part of the lesson structure and how teaching experience affects this structure. The majority of time was allocated to student work time and this part defines the format of the lesson; introduction of new topics was allocated the next highest amount of time. Further, more experienced teachers achieved better results[7] especially with regard to how they format student work time.

While developing an instrument focusing on questions and prompts in the mathematics classroom, Watson (2007) goes into detail about several taxonomies (e.g., Bloom's and Biggs and Collis' SOLO[8]) as well as frameworks describing the structure of mathematical knowledge. There are several key points she considered that are relevant to this study. She notes

> [The] disagreement between the teacher's intentions and the learners' perceptions confounds any attempt to use 'learning outcome' taxonomies to categorize teaching, and yet without complex articulation of learning, teachers cannot sensibly create or select tasks (Watson, 2007, p. 115).

[6]This means that material from several weeks prior or review for test is not covered in this part.
[7]That is to say, the lesson structure was more consistent
[8]Structure of Observed Learning Outcomes.

This indicates questioning needs to be carefully focused and worded to ensure the teacher's intentions are not perceived incorrectly by students. She suggests "the minimal assumption is that if [specific goals and practices] are explicitly encouraged, named, and valued" (Watson, 2007, p. 118) then students will actively participate to accomplish these goals.

2.4 Literature on university teaching practices

This section will begin with a description of an article by Schoenfeld et al. (2016) examining the current understanding of university mathematics education and thoughts on how to improve these methods. Then, I will examine eight empirical studies pertain teaching university mathematics. The first study focuses on the relationship between the amount of material to cover in a calculus course and the instructional methods used (Johnson et al., 2016). The next two studies, one on how students unpack logical aspects of mathematical statements by Seldon & Seldon (1995) and another on students note-taking in proof based mathematics courses by Fukawa-Connelly, Weber, & Mejia-Ramos (2017), are examined because these studies provide insight into how students identify and understand informal content with regard to proof. The fourth study by Syamsuri et al. (2018) discusses strategies used to teach mathematical proof at the undergraduate level. The fifth study, by Johnson et al. (2017), pertains to lecturing in an abstract algebra course. Then a study that examines two approaches to an introduction to proof course, a comparative study between a traditional DTP and an inquiry based approach by Cilli-Turner (2017) is described. The last two studies both examine the instruction of a Real Analysis course at the undergraduate level; one examining the uses of lecture presentation of proofs by Weber (2004) because of its focus on one of the most common instructional activities in collegiate mathematics education and a study examining the use of a 'term paper' to enhance students understanding of proof by Pinto & Karsenty (2018).

2.4.1 Understanding and improving the teaching of university mathematics

Schoenfeld et al. (2016) categorized collegiate mathematics instructors into three groups based on their primary consideration when making instructional decisions: *university mathematicians (UM), expert teachers (ET)*, and *proficient teachers (PT)*. The UM's primary consideration is with the mathematical content itself, they state:

> Part of what makes [the authors] mathematicians is the understanding that claims must be justified and justifiable, if [UMs] make a claim in an instructional context, [UMs] must be able to back it up (Schoenfeld et al., 2016, p. 4).

ETs differ from UMs in two ways; the depth of ETs' content knowledge will be less than that of UMs and ETs' pedagogical knowledge will be greater than UMs' pedagogical understanding. Lastly, PTs are described as a blending of the previous categories; that is to say, where UMs' primary consideration is the content and ETs primary consideration is the pedagogy, PTs' make teaching decisions based on both factors equally.

Schoenfeld et al. (2016) used the *Resources–Orientations–Goals (ROGs)* decision making theory. An instructor's resources include knowledge of the content as well as physical resources (e.g., access to technology). Meanwhile, his/her orientations are based on their beliefs on what is important for the students to understand. Finally, his/her goals are described as "what he or she is trying to achieve" (p. 3). In practice, instructors will form their goals based on their orientations while keeping in mind the resources available to them. These goals are used to make instructional decisions.

In short, the classifications are based on mathematical content knowledge (UM) and pedagogical content knowledge (ET) in decision making. A strong relationship between these two sets of knowledge results in the PT classification. This information was used to develop a professional development framework, which I will only briefly mention. They found that forming a small group of colleagues to view and discuss small segments of a lecture (chosen and given by one member of the group) allows for a focused discussion on how to improve the pedagogical considerations being demonstrated.

2.4.2 Coverage and instructional practices in college calculus.

Johnson et al. (2016) is primarily concerned with the affect, if any, of the amount of material being covered in a calculus course on the instructional methods used. That is to say they examined if the "adage that high coverage demands encourage (or even necessitate) more teacher–centric" (p. 501) instructional practices. While the short answer is no, I will elaborate using their findings.

In regard to course pacing being the determining factor on instructional methods, Johnson et al. (2016) found this not to be a factor. In fact, the university with the slowest and fastest course pacing both primarily used teacher–centric methods of instruction. Further, pressure to cover material quickly did not differentiate instructional practices; regardless of the pressure felt to cover material quickly, the majority of the respondents reported using teacher–centric instructional methods. Therefore, while they acknowledge time constraints are factors that affect instruction, it is not the primary factor to consider. That is to say examining time constraints is not an indicator of instructional practices.

2.4.3 Unpacking the logic of mathematical statements

Seldon & Seldon (1995) focused on how university students interpret informal mathematical statements[9] and the role of informal mathematical statements in validating proofs. They defined validation as "the process an individual carries out to determine whether a proof is correct and actually proves the particular theorem it claims to prove" (p. 127). Further, they claimed informal statements, statements of mathematical content excluding formal mathematical vocabulary (127), to be an indicator of a student's ability to validate proof.

Unpacking an informal statement is accomplished by translating it into an equivalent

[9]For example, the Pigeonhole Principle, stated informally, is if there are more objects than places to put them, then more than one object will share a spot. Formally, let $n, r \in \mathbb{N}$ and $f : \mathbb{N}_n \to \mathbb{N}_r$, if $n > r$ than f is not injective.

formal mathematical statement. This, along with proof frameworks (defined on page 7), Seldon & Seldon consider to be key skills used by mathematicians while validating proofs. To examine students' abilities to unpack statements, they used simplified informal statements on timed assessments and photocopied statements as an untimed (homework) assessment. They found informal statements on timed assessments to be unpacked successfully only 8.5% (p. 138) of the time; these statements were unpacked successfully only 5% (p. 139) on untimed assessments.

Seldon & Seldon used these results to recommend three instructional activities. First, informal statements should be presented alongside their formal statement; second, receive advice on methods for validating proofs; and third, distinguish between proofs and supplementary comments. I point these out not because they will be specifically identified in this study, but because they related to the local group of proof comprehension from the assessment framework developed by Mejia-Ramos et al. (2012). For example, stating a statement in equivalent ways is directly related to their first recommendation and identifying the proof framework is an aspect of the second recommendation.

2.4.4 Note-taking in advanced mathematics classes

Fukawa-Connelly et al. (2017) examined how informal statements are used by instructors in a transition to proof class and the importance given to these statements by students. They observed, transcribed items written on the board, and audio recorded lectures, then compared these to copies of students' notes. Though informal statements are most often (73%) presented orally, students only recorded 3% of the oral statements.

Two components examined in the assessment framework is to successfully justify claims and transfer general ideas or methods to other situations. Generally, instructors present formal mathematical content on the board but discuss informal content, including methodology, verbally (Fukawa-Connelly et al., 2017, p. 15). Since these oral statements are not noted by students often, they may not be practicing the skills necessary to understand written proofs; essentially making two of the components more difficult.

Weber et al. (2016) state that the main goal of advanced mathematics instruction is twofold: demonstrate how to construct proofs and characterize what makes a good proof. They proceed to identify steps mathematics instructors should take to ensure that lecture notes are useful for students. Since "students focus on what is written on the blackboard; this is a traditional way by which teachers emphasize importance" (p. 1190), instructors should make sure the key ideas they want to communicate are written down. Mason (2002) reinforces this idea by explicitly identifying that students have "to make sense" (p. 41) of these items, identifying the key ideas is essential. Similarly, since students and mathematicians think of understanding mathematics differently, expectations should be clearly communicated from the beginning of both the course and the lecture itself (p. 1191). Further, instructors need data on how students understand their lectures; Weber et al. (2016) note "[mathematics professors] rely on indirect measures such as their students' performance on exams and their comments on student evaluations, where it is difficult to posit causal links between specific actions of the instructor and the responses of students" (p. 1192).

2.4.5 Strategies for teaching mathematical proof.

Syamsuri et al. (2018) provides recommendations for instructional methods used to teach mathematical proof. Their recommendations are based on students' understanding of mathematical proof; students' understanding was assessed using the framework developed by Mejia-Ramos et al. (2012) and then categorized into a quadrant system. What follows is a description of the instrument used to assess students' understanding, a description of the quadrant system, and the recommendations for instruction they suggest.

The assessment used in this study was conducted primarily via interview; however, first the students needed to construct a proof for the claim given in Figure 2.7. After the

Prove: For any positive integer m and n, if m^2 and n^2 are divisible by 3, then $m + n$ is divisible by 3.

Figure (2.7) Proving task used in Syamsuri et al. (2018)

students completed their proofs, interview questions were constructed using the assessment framework by Mejia-Ramos et al. (2012). Each interview instrument was comprised of seven questions, one focusing on each of the seven aspects[10] of the framework.

After they had the response, they categorized them appropriately. If the proof was correct and the interview was consistent, then the student was considered to be in quadrant I; Syamsuri et al. (2018) note that any instructional method will benefit students in this quadrant because they possess a complete understanding of the proof structure and concepts. If a student knows how to begin the proof, then he/she was placed in Quadrant II; one recommendation for instructional method that will benefit students in Quadrant II is an adapted two column method with emphasis placed on the justifications. Students in Quadrant III did not know how to begin the proof but did attempt to use inductive reasoning; recommendations for instruction include having students constructed concrete examples of a concept as well as analyzing a complete proof. All remaining students are categorized in Quadrant IV, Syamsuri et al. (2018) recommend using *structured proofs*. Structured proofs are organized by levels, that is to say the main ideas and concepts are in one level while the justifications are in separate levels.

2.4.6 Lecturing in an abstract algebra course.

Both individual (primarily based on beliefs) and situational factors help determine how an instructor makes decisions on the methodology they use. Johnson et al. (2017) identified three categories of instruction with regards to lectures in abstract algebra. *Traditional instruction* is lecture–oriented, that is to say, very teacher centric. *Alternative instruction* uses lecture, but "class time is split (fairly evenly)" between teacher–centric (lecture, demonstrations) and student–centric (cooperative learning, student presentations) methods. *Mixed instruction* blends the two previous methods, with a moderate amount of class being devoted to lecture.

[10]Three questions pertained to the local domain of proof comprehension and four questions pertained to the holistic domain of proof comprehension

Further, Johnson et al. (2017) found key factors in determining which type of instruction used by a specific mathematics educator is based on "teaching experience, beliefs about teaching and learning, and interest in various types of scholarly activity." Traditional instruction was primarily used by the most experienced group who typically had a "stronger interest in mathematics research that educational research;" additionally, this group had a strong belief in the benefits of lecture, including the necessity of it with learning mathematics. Alternative instruction was primarily used by the least experienced teachers; coincidently, these teachers place a higher value on educational research. Instructors who used the a mixed instruction method valued the mathematical and educational research almost equally.

2.4.7 Inquiry pedagogy and students' conceptions of the function of proof

Cilli-Turner (2017) conducted a teaching experiment with a control group to examine the affect of inquiry pedagogy on a transition-to-proof course. The instructional methods for the inquiry course were extremely student focused with students constructing proofs individually and then collaborating with their peers to gather information and suggestions before completing a revision (p. 17). To build this type of classroom, the instructor first had to create the appropriate culture for the classroom and provide guidance on how to collaborate. Then, starting in the third week, students began to use the methods described above (p. 17 – 18). In essence, the students' proofs were constantly being assessed. The traditional group used the DTP format; with students being encouraged to ask questions and the vast majority of the time was spent completely on lecture (p. 18).

The main goal of this study was to determine which instructional method lead to a better understanding of the purpose of proof. By the very format of the course, students in the inquiry course showed a better appreciation for the communicative aspects of proof (Cilli-Turner, 2017). This relates directly to holistic aspects of the comprehension of proof; the students were constantly providing good summaries of the proofs.

2.4.8 Lecture presentation in real analysis

Keith Weber (2004) conducted a case study on one teacher's approach to teaching real analysis using a traditional DTP approach. The goals of his study were to describe this instructor's lecture style in detail as well as examining the reason why this instructional method was used and the effectiveness of this style (p. 116). This study was conducted at a university in the southern United States.

Data was collected through classroom observations and interviews. For the classroom observations, Weber primarily transcribed and described notes from the course; however, in order to obtain a detailed analysis on the instructional methods, some sessions were video recorded (p. 118). The interviews occurred on a weekly basis and focused on what goals were being set for the following week and explaining why certain instructional methods were used during the previous week (p. 117). Aspects of these methods will be adopted by and expanded on in this study.

Weber identified three primary categories of lecture styles used in this course: logico-structural, procedural, and semantic. Logico-structural proofs focused on the importance of definitions and organization in writing proofs; characteristics of this style included partitioning the board into proof and scratch work areas, seldom had diagrams accompanying the proofs, and were often constructed by unpacking and applying the definitions to both the given assumptions and desired conclusions with the goal of the two ends meeting in the middle (p. 121). Procedural proofs focused on the proof framework and various "techniques and heuristics" students could use to construct and identify key elements of the proof (p. 124). Primarily used when discussing proofs about limits, procedural proofs used in this course were characterized by first examining the proof structure, then finding the appropriate values to complete the proof (p. 125). Semantic proofs focus less on definitions and techniques and more on an overall conceptual understanding, often by employing diagrams (p. 127).

Weber notes that the instructional methods, though all derived from the DTP format, varied depending on the topic (p. 131). Further, four core beliefs were identified as reasons for the use of these instructional methods.

1. If students become frustrated, they will give up.

2. Students must have a good foundation in logic prior to being able to learn advanced mathematics.

3. Students must master basic symbolic techniques before progressing to harder concepts.

4. Students must gain experience working at the symbolic level in order to gain an intuitive understanding of advanced mathematics. (p. 128)

As discussed when considering the significance of this study, these beliefs are well established by this professor and will not easily change, so the instructional methods would not be easy to change if needed.

During the final interview with the subject of his study, Weber (2004) learned the reasoning behind the structure of the course, that is, why logico-structural proofs were presented early in the course, then proceedural proofs were introduced, and semantic proofs were introduced at the end. Logico-structural proofs were used early on to ensure students had a strong foundation to construct proofs and "sets and functions were topics that were particularly amenable to [the subject's] logico-structural lectures" (Weber, 2004). The focus on procedural proofs was two fold: it provided "step-by-step descriptions and hints" (Weber, 2004) and the ability to use the techniques acquired from working with the inequalities present in procedural proofs "becomes second nature" (p. 128), greatly improving the chance for success in analysis. Semantic proofs we built off of the foundation of logico-structural and procedural proofs, and signify the point in analysis when "rote strategies" (Weber, 2004) are no longer sufficient to proceed in the analysis.

Essentially, even if using DTP instructional format, Weber (2004) documents that this method does not prohibit a wide variety of methods. The logico-structural, proceedural, and semantic methods are very different in structure, methodology, and purpose (Weber, 2004). Also, Weber (2004) highlights a key factor as it pertains to the significance of this study, “mathematics professors must choose to employ [pedagogical] methods” (131) and new pedagogical methods are successful if they correspond with the professors’ belief structure.

2.4.9 Attending to student independent proof reading

Pinto & Karsenty (2018) describe one professor’s methods in an introduction to analysis course. Specifically, this professor altered what the authors refer to as *course image*; a course image is “an instructor’s full set of intensions and expectations concerning what will take place in a course” (p. 133). This includes the types of instructional activities and assessment methods used during the course. Specifically, this instructor was altering his course from a traditional weekly homework assignment to using a course long *term paper*; that is to say, students created detailed course notes (including full formal proofs) that they turned in weekly and were revised as needed throughout the course.

The goal of this term paper was for students to create their ”own personal version of the course book” (p. 137). This is meant to be more detailed than a simple summary; the instructors feedback on one students’ first draft is given below:

> [The summary] is correct, but the goal is not to summarize the lectures, but [create an] exposition of mathematical theory. So I would remove (or change) this introduction accordingly (Syamsuri et al., 2018, p. 138)

Further, the instructor emphasized the need for students to produce rigorous proofs in their paper. During the course, he presented incomplete proofs during his lecture; it was the students responsibility to provide a rigorous proof. He believed having students write and revise proofs weekly via the term paper would provide a structure that will improve their ability to construct mathematical proofs.

PART 3

METHODOLOGY

The purpose of this qualitative study is to understand how Dr. Wyatt's participation in a mathematics education research project with mathematics educators has affected his teaching practices. More specifically, this study specifically seeks to answer three questions

1. In what ways does Dr. Wyatt use the ideas of a particular assessment method focusing on students' thinking with respect to mathematical proofs with his teaching of a transition to proof course?

2. How do these instructional practices compare to his previous method(s) used?

3. What impact does the instructor's participation in the project have on his core beliefs about and the value of research in undergraduate mathematics education?

In this chapter, I will describe in detail the research setting, the overall design of the study, the methods of data collection, and the method of data analysis. Through this analysis, I hope to gain insight into Dr. Wyatt's beliefs about and methodology of teaching and assessment in a transition to proof course.

3.1 Research setting

3.1.1 Subject of the study

Dr. Wyatt is an Associate Professor in the Mathematics and Statistics Department at an urban research university in the southeastern United States with sixteen years of collegiate teaching experience. He has been an active faculty member in the NSF supported project from the beginning and has participated in the development of assessment instruments that align to the assessment framework developed by Mejia-Ramos et al. (2012), the implementation of some of the assessment items, and the evaluation of students' responses to those instruments. At the time of this study, he had participated in the research project for three years.

Dr. Wyatt's primary research interest lie outside of mathematics education. This is a key factor in his recruitment for this study. Since his specialization is not in mathematics education, his participation in a collegiate mathematics research project potentially exposed Dr. Wyatt to completely different methods of research and thinking about mathematics. Further, he was listed as the primary instructor for a section of the transition–to–proof course at his university. For all of these reasons, Dr. Wyatt was selected for this study.

The proof capabilities of students are affected by the pedagogical practices professors use in upper level mathematics courses. The subject of this study, Dr. Wyatt, is a participant in a project that emphasizes the importance of assessment and instructional practices when teaching proofs; to that end, the goal of this study is to examine the impact of his participation in this project. Dr. Wyatt is teaching a transition-to-proof course that focuses on several proof techniques common in advanced mathematics courses during the fall semester of 2018. In this section, I will provide a brief description of the transition–to–proof course and the learning outcomes of this course.

3.1.2 Description of the course

The transition–to–proof course at the university, from a topical standpoint, includes concepts from set theory, real numbers, analysis, and algebra. This subset of mathematics was chosen because it "illustrates a formal approach to the presentation and development" (*Course Description*, 2018) of advanced mathematics. This course has two prerequisites, *Linear Algebra I* and *Discrete Mathematics*, and is the required Critical Thinking Through Writing (CTW) course in the mathematics major. The following are the departmental learning outcomes for the transition–to–proof course:

1. Develop a truth table for a logical expression

2. Express the negation of a logic statement

3. Correctly decide if two statements are logically equivalent
4. Express the converse, inverse and contrapositive of a logic statement
5. Express universally and existentially quantified statements, and their negations
6. Understand the definition of a set
7. Correctly express the union, intersection and complement of sets
8. Do a direct proof
9. Correctly decide if a given proof is valid
10. Do a proof by contrapositive, contradiction or exhaustion
11. Understand indexed families of sets, their unions, intersections and complements
12. Do a proof using mathematical induction: the statement to be proved may be an equality or an inequality
13. Correctly decide if a given relation is an equivalence relation
14. Correctly determine the equivalence classes of an equivalence relation
15. Understand the division algorithm and its implications in divisibility problems
16. Correctly express the power set of a given set, and its cardinality
17. Correctly decide if a function is one-to-one, onto, or has an inverse
18. Correctly formulate the composition of two functions
19. Correctly decide if a set is finite, countable or uncountable
20. Correctly use the epsilon definition of greatest lower bound and least upper bound in proofs
21. Correctly apply the concepts of open and closed sets to proofs

22. Correctly apply the concepts of limit points, deleted neighborhoods and closure to proofs

23. Correctly decide if a sequence is monotone and/or bounded

24. Prove that a sequence converges to a limit, using the definition of convergence

25. Correctly decide if a function is bounded or monotone (*Learning Outcomes*, 2014)

It is clear from these learning outcomes that a key focus of this course is for students to develop the ability not only to construct proofs but to be able to carefully read, analyze, and comprehend proofs.

3.2 Data collection

This study focuses on Dr. Wyatt's teaching practices, with special interest on his use of assessment throughout the course. His teaching practices, specifically his motivational techniques, preparation and self reflection, representation of concepts and relations, and questioning techniques will be analyzed using the framework developed by Speer et al. (2010). His assessment techniques, including assessment construction and feedback methods, will be analyzed using the framework developed by Mejia-Ramos et al. (2012). Further, since the primary goal of this study is to compare the affect Dr. Wyatt's participation in the NSF mathematics education research study had on his teaching method, data pertaining to previous teaching and assessment practices will need to be collected. What follows is a description of the methods used to collect the data on Dr. Wyatt's teaching and assessment practices, including both his previous and current practices.

3.2.1 Dr. Wyatt's previous teaching and assessment practices

In order to collect data on Dr. Wyatt's previous teaching and assessment practices, an electronic questionnaire was created and distributed to students enrolled in at least one of Dr. Wyatt's classes during the previous three semesters (Fall and Spring 2017, Spring 2018)[1].

[1]Dr. Wyatt was participating in the NSF project during these semesters.

The full questionnaire is given in Appendix A and was distributed using two steps. First, the questionnaire was created using technological resources available through the university; second, a link to the questionnaire was distributed by Dr. Wyatt via email to his former students.

The questionnaire was eleven questions split into two parts: four fixed–response questions and seven narrative–responses questions. The fixed–response questions provided either four or five options for the respondents to choose from based on the question; this method was selected because it would provide a clear overview of students' recollections about Dr. Wyatt's teaching. The narrative–response questions were designed to allow respondents an opportunity to elaborate on their selections to the fixed–response questions and describe their personal experiences with Dr. Wyatt's teaching and assessment practices. McGuirk & O'Neil (2016) notes narrative–questions invite "respondents to recount understandings, experiences, and opinions in their own style;" thus, providing good details about their setting.

3.2.2 Classroom observations

I observed the final eight[2] class sessions of Dr. Wyatt's transition–to–proof course in the fall semester of 2018. During these observations, I maintained field notes that included the notes he used during his course, how he communicated with students during the lesson, and general observations.

To supplement these notes, audio recordings were created during each observation. Portions of these audio files were transcribed[3] as needed based on the data analysis. As noted in Speer et al. (2010), classroom observations provide insight into several teaching practices; for example, time allocation, methods of representation of content, and questioning practices.

[2]There were nine class sessions during the time I was observing Dr. Wyatt's class; however, a test was administered during one of these sessions and was not observed.

[3]The portions transcribed are provided in the data analysis.

3.2.3 An interview with Dr. Wyatt

I conducted an interview with Dr. Wyatt during the spring semester following my observations of his transition–to–proof course. The interview was semi–structured; Cohen & Crabtree (2006) gives the following characteristics of a semi-structured interview.

1. The interviewer and respondents engage in a formal interview.
2. The interviewer develops and uses an 'interview guide.' This is a list of questions and topics that need to be covered during the conversation, usually in a particular order.
3. The interviewer follows the guide, but is able to follow topical trajectories in the conversation that may stray from the guide when he or she feels this is appropriate.

The goals of the interview were to gain insight into Dr. Wyatt's beliefs about being an educator, the reasoning behind his pedagogical decisions[4], and the goals of various assessment items. The semi–structured interview format was used because it allows for open ended questions that are essential to examining Dr. Wyatt's beliefs. Further, this format allowed for topics to be included that were not considered during the development of the interview guide.

The interview was audio recorded. A transcript of the interview was created from the audio file after the interview; then the audio file was used to proofread the transcript. After the transcript was created and proofread, the audio file was deleted. The instrument used during the interview is included in the appendices.

3.2.4 Formal assessments

Dr. Wyatt used three types of formal assessments in this course: homework, test, and a final exam. As mentioned previously, the transition–to–proof course is considered a CTW course by the university. A CTW course places emphasis on how written content is presented

[4] This includes both anticipatory and immediate decisions.

in addition to the correctness of the content. Dr. Wyatt identified this as a vital part of what he expects from student's solutions and proofs throughout the course.

One of the goals of this study was to identify items from the assessments used in the course that resemble the type of questions developed for the project, I collected a copy of all assessments *after* they have been graded. This provided not only insight into the style of questions used, but also how he provides feedback on those assessment items. I identified how each question aligns with the principles of proof capabilities project, and then closely examine his comments about students work. Please note, in order to maintain anonymity, all identifying information was removed from from the assessments.

3.3 Data analysis

To attempt an answer for the first research question, provided below, the data collected from the formal assessments as well as part of the data collected from the interview were used.

1. In what ways does the instructor use the ideas of a particular assessment method focusing on students' thinking with respect to mathematical proofs in his teaching of transition to proof class?

Data collected from the questionnaire, the observations, and the interview were used to answer the second research question, provided below.

2. How do these instructional practices compare to his previous method(s) used?

The interview data and formal assessments were used to answer the third research question, given below.

3. What impact does the instructor's participation in the project have on his core beliefs about teaching and the value of research in undergraduate mathematics education?

The data presented in this report include observational field notes, sections of an interview with Dr. Wyatt, graded assessments with feedback, and comments from former students via the questionnaire. All of this data was analyzed, with regards to the research questions, with several passes through the data. This was to ensure both consistency and accuracy of the interpretation of the data.

After transcription was completed and proof read for accuracy, the data was initially categorized according to the framework of Speer et al. (2010) or labels as 'assessment practices'; that is to say the information was distributed into six categories. On the second pass of the interview, the data was examined and sub–codes were created based on the language used by Dr. Wyatt. The second pass gave me an opportunity to translate the teaching practices framework from Dr. Wyatt's perspective. Also, during the second pass the assessment practices category was further organized by question type and individual aspects of proofs. On the third pass of the data, a final organization method was established linking Dr. Wyatt's language to the frameworks used in this study.

On the first pass of analyzing the formal assessments, short phrases describing the assigned tasks were formulated without regard to the students' solutions. On the second pass through the data, mathematical concepts were assigned to each question and the tasks were redefined by a single phrase. This allowed for the questions to be categorized by the type of tasks being asked of the students. With the questions now categorized, my analysis turned to the students' solutions. On the first pass of the solutions I categorized them according to points earned on each question. The second time examining the solutions I focused my attention on Dr. Wyatt's feedback; specifically how he provided feedback on certain types of errors. Next, I analyzed the questions and the solutions simultaneously, identifying Dr. Wyatt's feedback as it related to the assigned tasks. Finally, I categorized the goal of each question using the assessment framework.

The field notes and audio files of the observation were analyzed simultaneously. First, I synced the two types of data by comparing time stamps in the audio files with the notes he provided to the class. On the second pass, I categorized the parts of the observations as they correlated to each part of the teaching practices framework. During the third pass, I identified moments that were exemplary of Dr. Wyatt's teaching practices for each part of the framework by Speer et al. (2010). Lastly, I reexamined all of the examples from the third pass to verify they were properly categorized.

I first analyzed the questionnaire's fixed–response questions and collected data into a tabular format; then I organized this data by how they correlated with Dr. Wyatt's teaching practices. The narrative–response questions where analyzed in a similar fashion. The reasoning for this method of analysis is because the questionnaire's results are only truly related to the second research question. Further, this research question requires a strict comparison; thus, categorizing this data directly against what is it being compared to is essential.

3.4 Chapter summary

In this chapter, a description of Dr. Wyatt and the research setting was provided. This included a description and the goals of the course as defined by the university. Further, I detailed how the data was collected for this study; in particular, some of the benefits of using each method of data collection was provided.

I detailed the method by which the data was analyzed. Specifically, I explained what types of data pertained to each of the research questions; then, the process of analyzing each data type was described. I chose to describe the data analysis process in terms of item type (instead of by research question) because most of the data types are used to answer multiple research questions. In chapter 4, the results are provided.

PART 4

RESULTS

Chapter 3 discussed the methods that will be used for the data analysis. This chapter will perform the data analysis in two parts. First, the results from the questionnaire examining Dr. Wyatt's previous teaching practices will be analyzed. Then, the teaching practices used by Dr. Wyatt in this transition–to–proof course will be analyzed using the framework proposed by Speer et al. (2010). Third, the assessments used in this course will be analyzed using the framework developed by Mejia-Ramos et al. (2012). As needed, additional literature will be used to emphasize specific parts of this analysis.

4.1 Dr. Wyatt's Previous Teaching and Assessment Practices.

The questionnaire consisted of a total of eleven questions, the first four of which were fixed–response (students selected from a list of four to five categories depending on the question) while the remainder where narrative–response. Figure 4.1 states the fixed–response questions; the narrative–response questions are given in Figure 4.2. The complete questionnaire is included in Appendix A.

1. How would you rate the instructor's excitement in teaching the course?
2. How often did the instructor make you explain your responses to questions posed in class?
3. How often did the instructor make you explain your responses to questions posed on assessments (Homework/Test)?
4. Overall, how high were the instructors expectations of you?

Figure (4.1) Fixed–response questions from the questionnaire.

In this section I will analyze the data from the questionnaire sent to Dr. Wyatt's former

5. Describe how the instructor paced the course. Did his pacing change during the course?
6. Describe how the instructor covered essential material in the course, that is, the material that counted as the foundation for the rest of the course.
7. Describe the types of questions used by the instructor during the course, that is, the types of questions he would ask not appearing on an assessment.
8. Describe how your instructors assessments (homework/test) impacted your learning in the course.
9. Describe how your instructors assessments (homework/test) provided insight into mathematical proofs.
10. What is the one thing the instructor did that you wish all teachers did? Please explain.
11. What is the one thing the instructor could have done to improve your learning? Please explain.

Figure (4.2) Narrative–response questions from the questionnaire.

students. Of his former students, a total of eleven responded to the fixed–response questions. Of these eleven respondents, six completed the narrative–response questions as well.

4.1.1 Fixed–response questions

The first question was designed to provide insight into Dr. Wyatt's personality with regards to the content of the course. The second and third questions were designed to examine his questioning and assessment techniques, respectively. The final question gives insight into how students perceive Dr. Wyatt's goals for the course. Of his former students, eleven responded to the questionnaire. Tables 4.1, 4.3, and 4.2 provide a distribution of students responses to question 1, questions 2 and 3, and question 4 from the questionnaire, respectively.

Locke & Schattke (2018) state "motivation orients, energizes, and selects behaviors" (p. 2) that ensure the individual pursues specific goals regardless of if they are internal or external. One role of instructors is to motivate their students to study and learn the content of the course. Speer et al. (2010) refers to this as "providing [a] rationale" for the study of

Table (4.1) Distribution of Prior Students Response to Question 1

Category	Count
Excellent	9
Very Good	1
Good	1
Poor	0

Table (4.2) Distribution of Prior Students Response to Question 4

Category	Count
Extremely High	0
Very High	4
High	5
Average	2
Low	0

a topic. Further, Khalilzadeh & Khodi (2018) note that an instructors personality has an affect on how students perceive the goals being presented by teachers. They state

> [T]eachers who were dutiful, disciplined, considered, competent, and achievement striving would strongly influence those students who desired to carry out an activity (Khalilzadeh & Khodi, 2018).

Therefore, Dr. Wyatt's personality (excitement about the material and his expectations of students) play a role in their motivation. From the responses to questions 1 and 4, students view Dr. Wyatt as being very passionate about his teaching and feel he places high expectations on his students.

Questions 2 and 3 focus on Dr. Wyatt's questioning and assessment methods. Specifically, students disagree on how much Dr. Wyatt expected them to justify their responses. This is an important aspect of his teaching and assessment practices as noted by both the

Table (4.3) Distribution of Prior Students Response to Questions 2 and 3

Question	Always	Often	Sometimes	Rarely	Never
2	3	2	3	2	1
3	2	5	2	1	1

teaching practices and assessment practices framework. Based on the collected data, Dr. Wyatt required justifications regularly, but not excessively. Speer et al. (2010) note teachers could ask the student to clarify responses or even ask for a second opinion, both options are eliciting a justification. Therefore, it is possible that instead of requiring justification initially, Dr. Wyatt may have opted to seek justification only when the answer was not fully correct.

4.1.2 Narrative–response questions

The narrative-response questions can be broken into four broad categories: assessment design and purpose (questions 8 and 9), questioning practices (question 7), time allocation methods (question 5), and methods of representing content (questions 6, 10, and 11). In the following paragraphs I will analyze each of these four categories. These questions were given in Figure 4.2; while focussing on each category, students responses will be provided.

Figures 4.3 and 4.4 provide students' comments to the assessment based questions from the survey. There are a few things that emerge from these responses. First, Dr. Wyatt's

- The format of the assignments was extremely helpful in order to tackle the questions part by part, although it made it difficult [for] me to keep the big picture.
- Homework and test were basically landmarks on how to think about the subject. It forced [the student] to mull over complex but apparently very deep and simple ideas for a long time. I LOVED IT![1]
- The assessments helped me to better understand the materials and were mainly examples of the concepts with some proofs.
- Assessments made the material easy to grasp.
- Homeworks and tests were straight–forward and easy to understand if a student had a strong understanding of what was covered in class.
- He was a great professor, I prefer his teaching style over any other professor I've had so far. He teaches what we need to know and tests what he taught. His assessments are not too bad if you pay attention in class and understand what he has done.

Figure (4.3) Responses to question 8.

- it made it very easy to tackle proofs systematically part by part.
- It forced me to memorize the basic definitions and foundational propositions and theorems so that it was like my name.
- the insight was provided by giving concrete examples of abstract materials. Also, some of the examples helped differentiate ideas.
- I am not sure there's much to say. Most homework problems were proofs. Usually the extra credit problems were proof questions.
- His assessments do provide mathematical proofs. Questions like "if statement is true, provide a proof; otherwise, give a counter example." [The questions make] you think because it is one or the other, cannot be both.

Figure (4.4) Responses to question 9.

assessments were organized in a way that allowed students to approach questions part by part; that is to say, they provided scaffolding for the students responses. Second, the students recognized the need to have a strong grasp of the fundamentals, which were heavily focused on during class. Third, the types of examples given on the assessments helped students learn the material. Finally, students hold Dr. Wyatt's assessments in high regard, noting they are essential to understanding the mathematical content. The local domain of proof comprehension from the framework in Mejia-Ramos et al. (2012) focuses on the fundamental aspects of proofs; Dr. Wyatt's question design from previous classes appears to follow this mindset.

Students' responses concerning Dr. Wyatt's questioning practices are provided in Figure 4.5. While the intent of this question was to examine Dr. Wyatt's previous questioning practices, the responses provide better insight into how Dr. Wyatt presents information. Specifically, the examples used his in class are very similar to those that appear on his homework. However, it is also clear that the questions he does ask during class help students learn how to think actively.

Figure 4.6 provides students' responses to how Dr. Wyatt allocated time during lessons. Students found the pacing of the course beneficial to their learning. However, there appears

- I don't remember clearly
- He basically taught us how to teach ourselves.
- He would ask similar questions to [those on the] assessment that would serve as examples.
- homework questions
- There were no real surprises. The main graded part is covered during the lectures. The occasional extra credits were over taught and learned concepts but required much more thinking.

Figure (4.5) Responses to question 7.

to be a disagreement on if his pacing changed during the course. Most of the responses comment that his pace was relatively consistent, if not a little slower than needed. One student commented that Dr. Wyatt's pacing was "inversely proportional" with the difficulty level of the content; that is to say, his pace was faster while covering "basic" material than it was covering "complex" material. While this alteration of pace makes sense, it is difficult to identify if this is what was happening in Dr. Wyatt's class because it contradicts the other responses.

Responses pertaining to how Dr. Wyatt represents content during the course is given in Figures 4.7, 4.8, and 4.9   Students responses here paint an interesting picture about Dr. Wyatt's teaching. First, they acknowledge that he is not just instructing the materials, but also helped them develop the skills needed to study mathematics in general. He provides clear examples, ensures students have a strong foundation with which to study the content, and provide guidance necessary for thinking mathematically. Dr. Wyatt teaches the students how to teach themselves; that is to say, his focus is on providing students a foundation that allows them to pursue mathematics on their own within the context of the course. Further, students acknowledge that Dr. Wyatt's style of instruction is, while not necessarily how every student learns best, is the best method he can apply because of his personality.

- His pace was fairly regular throughout the semester.
- Yes, rate of pacing was inversely proportional to the complexity of the material.
- The professor kept with a consistent pace that allowed us to finish the material that we were being tested on and learn even more just for our information.
- Steady and moderate.
- The pace of the course was a little slower than usual, but all content was covered which meant that rather than quickly covering all necessary content plus other content, the instructor thoroughly covered solely the required content.
- Very well paced. Always something new every day. [He] didn't linger on too much unless students were not grasping the concept well enough.

Figure (4.6) Responses to question 5.

- He explained the essential material very clearly, and I feel he spent the right amount of time on it.
- Very heavy emphasis was laid on the foundational material.
- The professor dedicated around two classes to cover essential materials prior to starting the course material.
- Clearly and always with examples.
- Thoroughly. The instructor frequently repeated the essentials.
- He does a great job on the background building up to the harder material.

Figure (4.7) Responses to question 6.

- Know the material in depth.
- Teach us how to teach ourselves so that we are on the way to becoming an expert ourselves.
- He works really hard. He hand–writes the notes as he discusses ideas in his lectures. I know this can seem a bit old–fashion. However, I believe it really helped to pace the lectures and helped me to keep up with the materials. Also, he communicated very well and ensured that we understood by reinforcing a communication often.
- Providing examples of each theorem taught.
- His homework followed a consistent format. All assignments were linked online, so it was easy to study from the homework when preparing for test. The test were in the same format as homework, but with different problems.
- Weekly assignments where material was covered thoroughly in class.

Figure (4.8) Responses to question 10.

- I like in–class discussions, but I have no way of knowing if it would have improved my learning given the personality of the teacher, the subject, and the curriculum.
- More homework and test. High frequency retrieval in multiple domains with a mixture of cues help foster deeper learning.
- After formally introducing material, I really learn when ideas are communicated in layman terms. Perhaps, clarifying ideas without the formal confusing jargon.
- Offering extra credit for correcting errors or mistakes from homework.
- I think sometimes class went too slowly.
- He gets sidetracked a bit, but he knows it, and tries to get back on topic.

Figure (4.9) Responses to question 11.

4.2 Teaching Practices

Speer et al. (2010) define teaching practices as "what teachers do and think daily, in class and out, as they perform their teaching work." While many factors affect what teaching practices an instructor uses, one key factor are his/her personal beliefs about education. In this section, I will analyze Dr. Wyatt's teaching practices using information from classroom observations, Dr. Wyatt's statements from a personal interview, and comments from former students collected via an electronic questionnaire. Before I begin with the analysis of Dr. Wyatt's teaching practices, I will define each of the domains of teaching practices I will examine.

The seven domains of teaching practices identified by Speer et al. (2010) are (a) the motivation of content, (b) preparation and self reflection, (c) time allocation within lessons, (d) representation of content, (e) questioning methods, (f) sequencing of content within lessons, and (g) designing assessments. Of these seven domains, the sequencing of content within lessons and designing assessments are not covered in this analysis because the sequence of the content is determined by the syllabus and assessment design will be covered in a separate section using the assessment framework developed by Mejia-Ramos et al. (2012). *The motivation of content* includes all methods used to "provide rationale" (Speer et al., 2010) for information studied during the course. *Preparation and self reflection* pertain to the work Dr. Wyatt does outside of the classroom to construct lessons and examine his previous teaching experiences. The *time allocation* domain focuses on the structure of individual lessons, including transitioning between topics and teaching methods. *Representation of content* includes "both what [content] is displayed and how [content] is displayed" (Speer et al., 2010), that is to say, methods used for the presentation of content. *Questioning methods* include an examination of the types of questions being asked, the amount of wait time provided after the question, and how Dr. Wyatt reacts to student responses. All of these domains of teaching practices have significant impact on how students learn and understand mathematical content; therefore, understanding Dr. Wyatt's teaching practices with regards to each of the domains will provide an outline of his beliefs on teaching. I will begin by analyzing how Dr. Wyatt motivates the content within his lessons.

4.2.1 Motivation of content

Speer et al. (2010) define motivation of content as "providing a rationale for a sequence of topics to increase students' engagement with that topic." Specifically, motivation of content is focused on how the instructor chooses to introduce the material to students. This is due to the fact that mathematicians have a deep understanding of the "logical structure, internal connections, and historical development" (Speer et al., 2010) of the course content that students will not possess. Weber et al. (2016) describe a goal of collegiate mathematics instructors, especially in upper level courses, as "[engaging] students with high-level or intuitive ways to understand the course content;" in other words, collegiate mathematics instructors want to motivate students to understand the content deeply.

Motivating content can be difficult in collegiate mathematics classrooms. Speer et al. (2010) note the amount of content that must be covered can limit the amount of time an instructor devotes to motivation. Weber et al. (2016) comment that, despite instructors' desire for students to have a deep understanding of mathematical content, often the assessments used in the course focuses on students' "ability to produce formal mathematics" which leads students to be motivated to learn the formal aspects, not the rationale, of collegiate mathematics.

In the following pages, I will analyze Dr. Wyatt's beliefs and methods of motivation in two ways. First, I will describe Dr. Wyatt's view of the importance of motivation followed by illustrative excerpts. Second, a description of the topics, both the logical structure and content, covered over multiple days will be discussed.

Dr. Wyatt's beliefs about motivation. When describing what prerequisite knowledge students need to succeed in a Transition-to-Proof course, Dr. Wyatt stated the following.

> [I]f you ask me what is truly helpful, I think just, you know, a mathematical way of thinking. So all of the officially listed prerequisites, sure they will help, but I always have the belief [that] some people can just learn [the course content] without those prerequisite... I always have the feeling that some high school students can just learn [the content of a Transition-to-Proof course] pretty well (Personal Interview, February 12, 2019)

In short, Dr. Wyatt believes that some people have a natural drive and desire to learn mathematical content, that is, are motivated intrinsically to learn mathematics. Further this drive can act as a substitute for a strong mathematical background. From a historical point of view, several mathematicians (as well as others from various academic fields) have had little to no formal mathematics training yet were motivated to learn mathematical content to an extent allowing them to formulate remarkable findings. In essence, Dr. Wyatt is acknowledging the same type of drive that allowed these self-taught mathematicians to succeed.

However, Dr. Wyatt is not sure how to foster this type of motivation, acknowledging how cultural differences often play an important role in student motivation, citing his beliefs about Chinese mathematics students.

> [Many] Chinese students [have this drive], some of the drive might come from their parents or their peers because everyone is trying to be good at mathematics, so they feel pressure. So there is some kind of pressure, or whatever, that pushes them to learn... I don't know how to generate, how to create that pressure. (Personal Interview, February 12, 2019)

This is a form of *extrinsic motivation*; Locke & Schattke (2018) define extrinsic motivation as "means–ends relationship; it is doing something to get a future value (avoid a future disvalue)." In the situation posed by Dr. Wyatt, the value is praise from parents or respect from peers; that is the culture, an external factor to the learning of mathematics, places a high value on learning mathematics. Thus, students want to learn mathematics to gain this value. Dr. Wyatt's belief on the cultural impact of motivation is again mentioned at the end of the interview.

> [Students who do well in academics] are the most popular person. Just imagine if sometimes the math guy or the physics guy or whatever, if [academics] got that type of attention; then the students [will] all try to study mathematics and physics... all the students, as a whole, will be better at mathematics. (Personal Interview, February 12, 2019)

This demonstrates Dr. Wyatt's strong belief that culture is a key aspect of determining students' motivation to learn mathematical concepts.

Even though Dr. Wyatt admits he is uncertain as to how to develop the drive to learn mathematics in students, he does provide several examples about what can be done in the classroom to possibly develop this drive. The primary method he mentioned was "practice and [completing] problems." This coincides with Dr. Wyatt's method of preparation, that is, his focus on using examples, both purely mathematical and in real world examples, which will be discussed in more detail in Section 4.2.2 (Personal Interview, February 12, 2019). Since he views the use of examples as the "best way" (for him) to instruct mathematics, he naturally feels that practicing and completing problems, i.e. completing examples, will motivate students to learn mathematical content. However, he notes that every instructor teaches in their own way and comments the methods used by other instructors are valid, so it can be said that he believes there are many ways of fostering the drive for students to learn mathematics.

Evidence of motivation of content from the lectures. As stated previously, Speer et al. (2010) define motivation of content specifically with regards to topics or methods that take place over several class meetings. During the eight lectures two topics, relations/functions and cardinality, took multiple class sessions to complete. Of these two topics, I only observed the introduction to cardinality, as my first observation occurred during the discussion of relations and functions. Therefore, from a topical standpoint, I only observed the rationale for the study of cardinality.

Table 4.4 provides a topical outline[2] by date of the eight lectures I observed in Dr. Wyatt's Transition-to-proof course. *Primary topic* identifies the overarching category the lesson belongs, that is to say, a topic that may span multiple class sessions. *Sub-topic* identifies the specific information covered in the lesson. *Prerequisite* notes information from previous lessons required to study the subtopic based on Dr. Wyatt's emphasis during the lecture. This table shows that the subtopic *injective and surjective functions* is a prerequisite for the majority of the observed lessons. Therefore, the rationale provided for this subtopic will be examined.

Table (4.4) Topical Breakdown of Dr. Wyatts Lectures

Date	Primary Topic	Subtopic	Prerequisite
Oct. 29	Functions and relations	Injections and surjections	Definition of functions
Nov. 5	Functions and relations	Injections and surjections	Function compositions
Nov. 7	Cardinality	Equivalent sets	Bijective functions
Nov. 12	Cardinality	Countability	Bijective functions
Nov. 14	Cardinality	Comparing sets	Bijective functions
Nov. 26	Division algorithm	Greatest common divisor	None specified
Nov. 28	Pigeonhole Theorem	Importance of assumptions	Bijective functions
Dec. 3	Exam review	various	various

Therefore, what follows are three examples of how Dr. Wyatt uses motivation. First, the rationale for cardinality of a set[3] will be identified. Then, the rationale for studying injective and surjective[4] functions will be presented. Thirdly, the rationale behind the use of the division algorithm will be examined because it is a self contained lesson, that is none of the material covered in the observed lessons were noted as prerequisites for this topic.

One example used frequently in Dr. Wyatt's lectures on cardinality is the example of the two infinite armies, the natural numbers and the integers. Specifically, he used this topic to introduce how to compare the cardinality of two infinite sets while emphasizing the

[2]This list only pertains to the observed lessons. The topical breakdown of previous lessons were not collected.

[3]The "size" of a set

[4]Will be defined when discussed

role students' intuition can play in creating misunderstandings. Figure 4.10 reproduces the diagram Dr. Wyatt used when posing the original question and the modifications he made to the original diagram while explaining the solution to the problem.

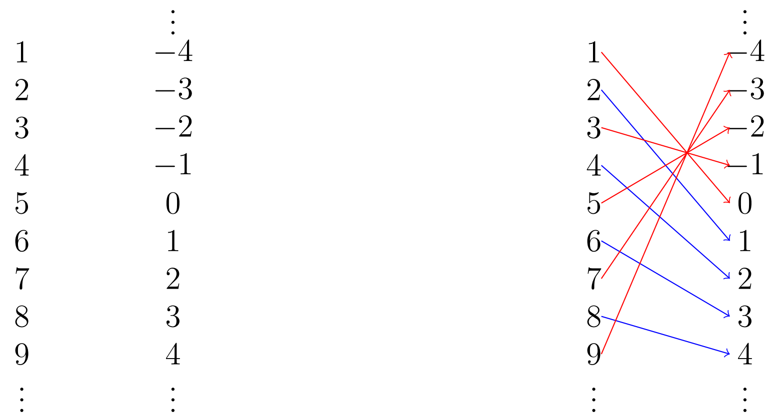


Figure (4.10) Diagrams for the "Infinite Armies" example.

Dr. Wyatt continuously returned to the example of the infinite armies as he covered cardinality, therefore I conclude that he used this example as motivation for the topic of cardinality. He introduced the above example by first discussing what he believed students' prior experiences had taught them, specifically

> I think we all know how to compare finite sets. You know which one has more. One set has a finite number of people, say 10, and the other set, another collection has 100 horses, you all know 100 is more, and I believe you all know how to compare finite sets, you know, which one has more elements.

Further, he states "we will define what is the definition of being bigger." These statements imply the rationale Dr. Wyatt provides for learning cardinality is to compare the size of sets and identify which one is larger. In other words, Dr. Wyatt provides a specific goal for learning the material, Locke & Schattke (2018) would describe this as *achievement motivation*. Achievement motivation is reliant on a goal being set, in this case comparing the sizes of sets, and then the tools needed to achieve the goal and the expected standards completing the task are known.

Thus, Dr. Wyatt's lectures on cardinality should revolve around this goal. This is evident through the examples he used during the three lectures on cardinality. Figure 4.11 provides a list of the examples he used during these lectures, in order of presentation. Notice

1. Let $A = \{a\}$, $B = \{a, b\}$, and $C = \{a, x, \lambda\}$. Show these are equivalent to N_1, N_2, and N_3[5] respectively.
2. Show $\mathbb{N} \approx \mathbb{Z}$ (the natural numbers and integers are the same size)
3. Show the interval $(0, 1)$ is equivalent to the real numbers $\mathbb{R}$.
4. Show $\mathbb{N}$ is not equivalent to the interval $(0, 1)$.
5. Show $\mathbb{Z}$ is equivalent to the Rational number $\mathbb{Q}$.
6. Is $\mathbb{Z} \times \mathbb{Z}$ countable[6] or uncountable?
7. Show the cardinality of any set is less than the cardinality of the associated power set.

Figure (4.11) List of examples used when studying cardinality.

that every example Dr. Wyatt used while covering cardinality pertained to comparing the sizes of sets. Hence, he motivated the topic by examining the infinite armies. Next, I will identify the motivation behind injective and surjective functions, a topic required to study cardinalities of sets.

Unlike the discussion of cardinality, Dr. Wyatt does not provide an example as motivation, and instead comments that from the homework he can tell "some [students] already know about injective functions" and then provides the definition of both injective and surjective functions, given below.

Definition. Let $f : A \to B$ be a function from A to B. We say

a) A function is surjective, or onto, if the range of f is B, that is $\text{Rng}(f) = B$.

b) A function is injective, or 1–1, if, for all elements in A, $f(a_1) = f(a_2)$ if and only if $a_1 = a_2$.

Initially, it appears that no rationale is provided for this topic, and if one only examines the beginning of the topic, they would be correct. However, the goal of discussing these types of functions is to emphasize the necessity of focusing on the required criteria for a function to be injective or surjective, that is to say, he focused on the logical structure of the topic instead of the topic itself.

Again, the examples used identify the goal is to focus on the logical structure of the definitions; however, the emphasis is demonstrated by how Dr. Wyatt explains the solutions to the examples. The following is illustrative of the majority of examples he used in discussing injective and surjective functions, that is most of the examples involved finite sets.

Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{a, b, c, d\}$. Define $f = \{(1, b), (2, a), (3, c)\}$ and $g = \{(1, a), (2, b), (3, a)\}$.

1. Is f onto B? Is f onto C?
2. Is f injective?
3. Is g onto A? Is g onto C?
4. Is g 1–1?
5. Is f^{-1} a function?
6. Is g^{-1} a function?

As Dr. Wyatt answered these questions, he continually was asking the students why the given answer was correct. His emphasis was on analyzing the definition of injective and surjective functions to provide the reasoning. For example, the justification for the question required the students to include information about the range of f in their explanation. Further, the following exchange occurred while justifying f^{-1} was a function.

Dr. Wyatt(W) Therefore, from this example... what do you think? What is the so called criteria for an inverse to be a function?

Student(s) It has to be onto and 1–1

W Ok, so onto and 1–1. So uh, so if I want you to drop a condition which one would you drop? Between 1–1 and onto, which one would you drop?

S Onto?

W Yes, onto. So we only need 1–1. If f is a function from A to B, then it is bijective and we have both conditions. But if f is a function from A to C, f is not onto C. It does not matter if it is onto.

In other words, Dr. Wyatt's rationale for teaching injective and surjective functions is to focus on the logical requirements for the situation to occur. Further, he expands this logical reasoning of injective and surjective functions to how they are used, that is to say, what are the minimum requirements for this to occur. Bijective[7] functions form the foundation for justifying that two sets are equivalent, so it is used throughout his lectures on cardinality; however, he still reiterates the minimum criteria for bijective functions often.

The final example of rationale for a topic is demonstrated in the lecture about the division algorithm. Before I continue, this does not directly correlate with the definition provided by Speer et al. (2010) because this lecture is self contained, that is occurs entirely in one class session. However, this is also the only observed topic that Dr. Wyatt did not explicitly state a prerequisite among other topics from the Transition-to-Proof course itself.

Dr. Wyatt begins this lecture by giving the basic definition of a *common divisor* and the *greatest common divisor (gcd)*[8]. Then, he finds the gcd of 24 and 42 using the definition. First, he emphasizes that they must list "all, again I will state all, we have to get all the divisors." Then, he proceeds to list all of the divisors for both 24 and 42. Third, he asks the students to identify the common divisors between the two numbers. Finally, he identifies the largest of the common divisors.

During this example, Dr. Wyatt continually referred to the need to list everything. After the completion of the example, he commented

[7]Both injective and surjective

[8]The gcd of two numbers is the maximum of their common divisors

> So this is one way, this is by definition, but there are other ways. Other ways are actually more efficient, there is nothing wrong with [completing the solution] by definition, but it is good to be efficient.

Therefore, the rationale for the division algorithm is that it is a more efficient tool for finding the gcd. He then goes on to define the division algorithm and supply examples. The first example he completes using the division algorithm is the same question he used to open the lecture, ensuring students see the algorithm generates the same solution in a more efficient way.

Summary of motivation of content. Dr. Wyatt has various methods of motivating topics. The rationale for the topics he uses vary depending on the topic; specifically, there is evidence that he motivates topics by focussing on the goal of the content, the structure of the content, and the efficiency of the content. This focus allows him to structure examples and lectures around reiterating the desired goal; this builds a structure for the students to follow in the lecture. Further, Dr. Wyatt notes the impact culture has on students motivation to learn mathematics specifically.

The motivation behind Dr. Wyatt's approach to these topics is based in his beliefs about common misconceptions students possess. To identify potential misconceptions, he relies on his prior experiences as an instructor as well as other tools, such as textbooks and internet resources, to inform his decisions on specific examples to use in class. Next, I will analyze the methods Dr. Wyatt uses to prepare for his courses and discuss how he reflects on his prior teaching methods as well as examine hypothetical methods of instruction.

4.2.2 Preparation and self reflection

Speer et al. (2010) state that collegiate mathematics instructors must "evaluate their plans, particular actions and choices, and their students' contributions and questions before they teach their next lesson" to ensure that other aspects of teaching practices are formulated to best benefit the students. For this section, preparation includes choosing what information to present, how to display that information to the class, how to pose questions, and how to allocate time; preparation with regards to the construction of assessments will be discussed in a later section. While the exact information to be covered in a course is prescribed prior to an instructor's individual involvement, how the instructor chooses to cover the material is formed by that individual's beliefs about teaching and learning.

Below I will examine Dr. Wyatt's methods of preparation and his opinions about his instructional methods. First, I will look at what Dr. Wyatt views to be the best way for him to act as an instructor, including how he chooses what types of examples to use and when he uses both real world and purely mathematical examples. Second, I will look at how Dr. Wyatt reflects on his experiences as an instructor, including his ability to examine how he would use a reading centric method to teach proof techniques, how students' responses to assessment questions help guide his instruction, and how he views his methods of helping students identify appropriate proof frameworks to use for certain proofs.

The use of examples. Dr. Wyatt described his role as an instructor in terms of finding his "best way" to structure the material so that the students have the tools to fully understand the topics being presented. Dr. Wyatt states

> I just teach the material and find my best way to teach that. So in terms of how to explain material and new idea and also how to, yeah, explain a new idea so that the students can understand it better. To assign my homework problem, how to make my homework assignment problems in a way so that it, uh, it helps the students digest the material, the lesson... all of the components of my teaching are just, ah, they are geared up to make sure the student can learn the materials (Dr. Wyatt, Personal Interview, February 12, 2019)

Two questions arise from his statement. First, what processes does Dr. Wyatt use while preparing for his courses? Second, why are these the methods Dr. Wyatt trusts will help the students understand the material fully? To answer these questions, I will examine evidence from an interview held with Dr. Wyatt on February 12, 2019, observations of several lessons taught by Dr. Wyatt, and feedback from his former students about his teaching methods.

When asked how he starts his preparation, Dr. Wyatt responded the "examples I use in class to explain the idea... both the truly mathematical problem and the so called real world problem" (Personal Interview, February 12, 2019). Essentially, his examples were used as a guide, but he did not restrict himself to a specific type of example. The extent to which Dr. Wyatt focuses on the examples was noted by a former student, who states that Dr. Wyatt covered the essential material of the course "clearly and always using examples." First, I will examine how he determines which type of example to use for a given topic; then, specific instances of "real world examples" from the interview and his lectures will be identified. Thirdly, some strictly mathematical examples from his lecture will be examined.

Dr. Wyatt understands that time constraints prohibit him from "attach[ing] a real world example" to every topic that arises in his classes (Personal Interview, February 12, 2019). He uses the internet to decide about the appropriate example by identifying the topics that are not only confusing to students but also to the "general public."

> They don't understand very well, just like if you say 'only if,' at least some people, they are asking 'what does only if mean?' They are asking online which means, yeah, not just the students in my class but also many people in the general public still struggle with that thing. So therefore, yeah, I will try because of that; I try to fully explain more using, uh, real world examples. (Personal Interview, February 12, 2019)

So topics that he feels are not well understood generally are the topics he will use a real world example to discuss. He further stresses that he will always discuss the purely mathematical information, but he feels that real world examples are essential for students understanding. Topics that are "very fundamental in mathematics and also in [the students'] real life" will include more real world examples to support the formal mathematics (Personal Interview, February 12, 2019).

Real world examples. During the interview, Dr. Wyatt provided several examples of the types of real world questions he uses in class. When dealing with subsets, he explained "I write $A \subset B$ you know, mathematically, but I also use a real world example; I like to use the toy collection to explain the idea." A toy collection is something that most students will be familiar with; for example, linking all of the dinosaur toys as belonging to the entire collection while still being their own set is a foundation for students to begin to understand subsets. Also, Dr. Wyatt mentioned comparing the process of induction to that of dominos, where you "make one falling and knock down the next and the next," stressing the importance of having a true base case in order to construct an inductive proof.

The following are two instances from Dr. Wyatt's lectures using real world examples in a similar manner to what he discussed in the interview above. The first instance discusses the cardinality of sets, specifically how Dr. Wyatt links the students intuition about the cardinality of finite sets to that of infinite sets. The second discusses the Pigeonhole Principle and how he uses diagrams and students' access to technology to emphasis the basics of the Pigeonhole Principle.

Prior to formally discussing cardinality, Dr. Wyatt discussed comparing cardinalities of sets in a less formal way. He provides the students with three separate comparisons, the first two with focus on how students' intuition is a useful tool but can sometimes be misleading. In the first, he examines how to compare finite sets to other finite sets; then, he compares finite sets to infinite sets. Third, he compares the infinite sets of the natural numbers and the integers.

He began by discussing the intuition that naturally lends itself to finite sets.

> I think we all know how to compare finite sets. You know which one has more. One set has a finite number of people, say 10, and the other set, another collection has 100 horses, you all know 100 is more, and I believe you all know how to compare finite sets, you know, which one has more elements.

Also, he describes how he is confident that the students all know an infinite set is larger than a finite set. He is acknowledging and emphasizing that the students already have the tools needed to understand how to compare anything having to deal with a finite set because it is “one of the basic things for human beings to see the world.” In short, all humans have an understanding of the size of finite collections, so students’ intuition can be used as a guide for these types of situations.

Dr. Wyatt then presents the situation where a comparison between two infinite sets is required. Specifically, he presents a situation where the natural numbers and the integers are compared to each other.

> Of course, we will define what is the definition of being bigger, right now we will just call it a tricky question. But, uh, so let us say it this way. Say I am a general, all that means is I control an army. And I have [the natural numbers as my] soldiers, so 1,2,3,4,... And you are a general controlling [the integer] army. They are all true soldiers, they all have the equal fighting power, fighting ability, its just that their name is -1,-2 or 1, 2 and so on.

So the question becomes, if these two armies were to do battle, who would win? He describes how their intuition might cause them to believe the integer army would win because “from a certain point of view it makes sense;” the integer army “extends in both directions,” are both infinitely positive and infinitely negative, so the integers appear to be twice as large as the natural numbers. However, using the diagram in Figure 4.10, Dr. Wyatt presents a pattern that will allow him to position his army to create “one on one combat,” touching on the need for an injective relationship between the infinite sets. Figure 4.10 is duplicated below.

In short, he used the above example to demonstrate two aspects about cardinality of infinite sets. First, students’ intuition may be misleading when dealing with infinite sets, that is to say, it “looks like” the integers are larger than the natural numbers. Second, for two infinite sets to have equal cardinalities requires a “one on one” situation to be identified. Dr. Wyatt continues to reference the above example as he continues his discussion about cardinality with infinite sets. He continues to use diagrams when discussing the cardinality of the rational numbers, real number, and several cross products, often reverting back to

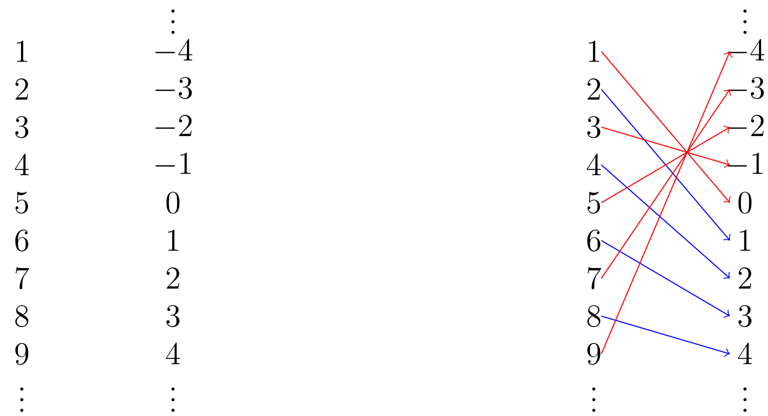


Figure (4.12) Diagrams for the "Infinite Armies" example.

this example to remind students they need to use their intuition wisely.

The next instance of Dr Wyatt's use of real life examples occurs during the discussion on the Pigeonhole Principle[9] in his November 28th lecture, Dr. Wyatt used three examples based off of real world situations. The first example pertained to a small group of people, specifically any group with at least eight people. In such a group, he claimed that "there exist at least two people with the same birth weekday," that is at least two people born on the same day of the week. To direct this example toward the mathematical definition, that is there is no injective function that exist between a set of at least eight people and the set of days of the week, which has exactly seven members, Dr. Wyatt presented the diagram in Figure 4.13, noting the cardinalities of both sets. That is to say, the cardinality of set A is 8 and the cardinality of set B is 7. Since $\bar{\bar{A}} > \bar{\bar{B}}$, two of the points in A would have to be sent to the same day of the week in B.

For the second example focussing on the Pigeonhole Principle, Dr. Wyatt gave the assumption that "humans can live to be at most 150 years old" and asked the students if the statement "every group of more that 151 people contains two people with the same age" was true. A student responded that this statement is true; then Dr. Wyatt emphasized that this would not necessarily be true without the assumption about the maximum human age.

[9] Let A and B be non-empty sets such that $\bar{\bar{A}} > \bar{\bar{B}}$. Then there exist no injective function from A to B.

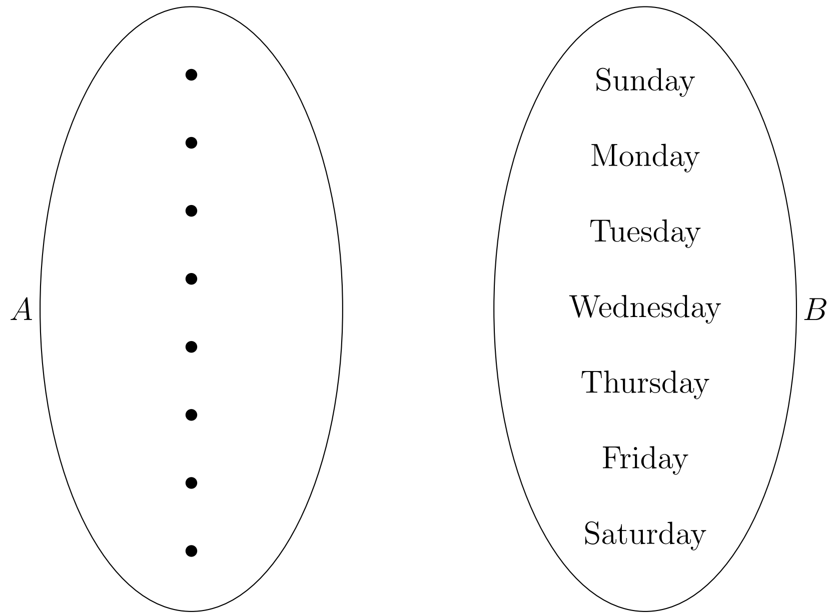


Figure (4.13) Diagram showing that in a set of eight people, two people must share a birthday.

Thirdly, Dr. Wyatt proposed a situation where the students had to look up information prior to answering.

> There exist at least two people in the greater-Atlanta area, of course, so far this is correct as I haven't finished the example, who have the same number of hair, the word the same [number of] strands of hair... so how to do this, again, using the Pigeonhole Principle. How many people we have in greater-Atlanta?... How many strands of hair does a human have?

After searching the internet, one student provided the population of greater-Atlanta (486,290) and another student provided the number of hair follicles on the human head (100,000). The last three examples highlight various aspects of the Pigeonhole Principle, the need to identify the cardinality of the sets in question, the importance of specific assumptions before proceeding with a solution, and how questions that seem relatively complex (the third example) can be reduced down to simply an examination of the cardinality of sets.

Purely mathematical examples. As mentioned previously, Dr. Wyatt knows "I don't have all the time to teach... therefore I don't have all the time to just do the real world examples" (Personal Interview, February 12, 2019) for everything he teaches. Therefore, the topics that tend to cause less confusion, he presents using purely mathematical examples. Here I will focus on two instances that show Dr. Wyatt is very conscious of the students needs when it comes to looking at purely mathematical examples. The first instance occurred during a discussion about the restrictions of domain of functions; the second occurred when examining the division algorithm.

The first instance demonstrates how Dr. Wyatt will alter his initial example to benefit the students needs. His initial example, given in Figure 4.14, pertains to functions with restrictions to their domains.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x + 2$.

Under $h_1 = f_{|(-\infty,2]} \cup g_{|(2,\infty)}$, find $h_1(1)$.

Figure (4.14) An example of infinite functions with restricted domains.

Dr. Wyatt did restate that a restriction of a function is still a function and pointed out that the two intervals used in h_1 did not overlap; thus, h_1 itself is a union of two functions with disjoint domains, that is to say, a function. However, a student mentioned that $h_1(1)$ did not exist because both of the original functions produced an answer, so the question was ambiguous. This prompted Dr. Wyatt to alter his initial plan because he recognized that not all of the students completely understood how restrictions on infinite domains (discussed in a previous class) operated. Therefore, he began by looking at the same concept using finite instead of infinite domains, as shown in Figure 4.15.

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Define $f = \{(1, b), (2, d), (3, a), (4, a)\}$

and $g = \{(1, d), (2, a), (3, c), (4, d)\}$.

Then $h = f_{|\{1,2\}} \cup g_{|\{3,4\}} = \{(1, b), (2, d), (3, c), (4, d)\}$ (Provided by students)

Figure (4.15) Finite example of functions with restricted domains.

Then he proceeded to ask for the values of $h(1), h(3)$, and $h(4)$. This example still accomplished the original goal Dr. Wyatt had planned while returning to a prior topic and reinforcing students knowledge. After this example was discussed in detail, he returned to the original example; Figure 4.16 provides both numeric and graphical solutions to the questions posed in Figure 4.14.

$h_1(1) = f(1) = (1)^2 = 1$
$h_1(2) = f(2) = (2)^2 = 4$
$h_1(3) = g(3) = (3) + 2 = 5$

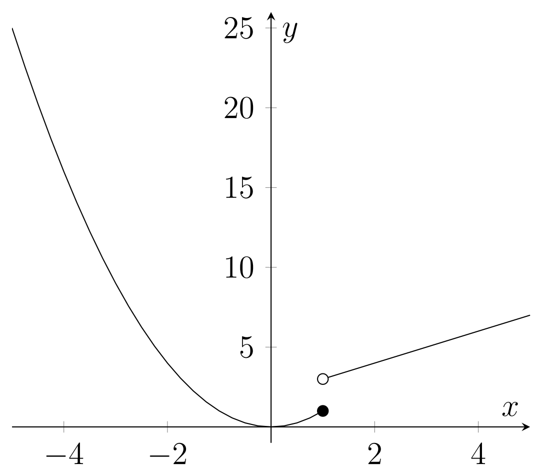


Figure (4.16) Solutions to the example in Figure 4.14.

The next instance of purely mathematical examples used by Dr. Wyatt, he focuses on the division algorithm [10] and how it can be used to find the greatest common divisor of two numbers. Dr. Wyatt used the definition of the greatest common divisor, a topic the students are familiar with, to build up to how the division algorithm can be used. Using the definition, he examined

$$\gcd(24, 42)$$

that is all of the divisors of 24 and 42 were listed, then the maximum of the values was selected. So

$$\text{The divisors of 24 are } \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24\}$$

$$\text{The divisors of 42 are } \{\pm1, \pm2, \pm3, \pm6, \pm7, \pm14, \pm21, \pm42\}$$

[10]For all $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist $q, r \in \mathbb{Z}$ $(0 \leq r < b)$ such that $a = bq + r$

Thus gcd$(24, 42) = 6$. While working through this example, he discusses how this method is not the most concise method and that it provides you with little additional information. He then presents the division algorithm and demonstrates how it can be used to calculate the gcd$(24, 42)$. Specifically,

$$\begin{array}{r|c|c} 42 & 24 & 42 = 24(1) + 18 \\ 24 & 18 & 24 = 18(1) + 6 \\ 18 & 6 & 18 = 6(3) + 0 \end{array}$$

Since 6 is the last non-zero remainder, it is the greatest common factor. Then he proceeds to discuss how, using substitution, this can be used to write 6 as a linear combination of 42 and 24, that is

$$6 = 24(2) + 42(-1)$$

While Dr. Wyatt discusses this, he does not provide the reasons why being able to write the GCD as a linear combination of the first two terms is important. I believe this is due to the fact that this was a stand alone lesson so time constraints required Dr. Wyatt to focus on the primary topic, the division algorithm.

Self reflection. When responding to a question regarding what Dr. Wyatt could have done differently to improve their learning, two things former students mentioned were the use of more in class discussion and that Dr. Wyatt has a tendency to become sidetracked. With regards to the first comment, the student is not positive "if [more in class discussion] would have improved my learning given the personality" of Dr. Wyatt. The second student noted that Dr. Wyatt is aware that he gets sidetracked, and works to get back on topic quickly. Both of these comments note that teaching is a very personal activity, and that Dr. Wyatt ultimately has to make the final decision as to whether or not certain practices are used in his classroom.

Dr. Wyatt is very reflective about his teaching. He admits that he uses his methods because they are his best way of teaching, but he is able to examine other methods and look at how he may implement such methods in his class. Also, he freely admits when he is “lacking” a particular aspect of his teaching. Dr. Wyatt’s reflection practices will be analyzed by seeing how Dr. Wyatt views the use of reading as an instructional method; then, I will examine the importance Dr. Wyatt places on the details of proof writing. Finally, Dr. Wyatt discusses how he demonstrates when to use various proof techniques.

Reading as an instructional method. When discussing the students’ role in learning proof techniques, Dr. Wyatt states

> I almost don’t assign any like reading, you know like “for the next class you should read,” you know I almost never do that. But since you are asking this question, say if I want to do that, say if I plan to do this kind of assignment, yeah, I don’t have any novel idea you know you can just read the thing, read, uh, yeah more than once that would definitely help. Read the material, the next material, once and, uh, yeah. make sure you understand the critical steps. (Personal Interview, February 12, 2019)

When asked why he does not assign reading generally, he responded simply that he does not “know if it would help” (Personal Interview, February 12, 2019). Further he notes that the students have “lots of classes, [so] they don’t have the time” to read the material (Personal Interview, February 12, 2019). In his mind, time constraints could prevent students from gaining further understanding of proof techniques from reading assignments; therefore, he prefers to work on the assumption that he will be teaching the material from scratch.

Dr. Wyatt proceeds to discuss how his teaching methods may change if he were to assign reading assignments. He comments that if he assigned reading, then he would not “do as much lecturing”, providing fewer examples than he currently uses. Instead, he would assign more exercises because the students should have obtained the basics from the reading. He acknowledges that the students would still need guidance with regards to the exercises, but believes this could be “a way of teaching the material.”

That being said, Dr. Wyatt then expressed the fact this may be a cycle of his own creation.

> The way I teach does not require reading, so therefore they do not read [the material]. So it might be, you know, you can call it a cycle, um, yeah. I think that is the thing, I don't know, the other instructors, they might [require reading] but I can't. I thought on it but I do not have the nerve to do it. (Personal Interview, February 12, 2019)

This connects to the comment made by his former student concerning Dr. Wyatt's personality, specifically, it does not tend itself toward assigning reading because he prefers to teach the material from scratch, where he knows all of the students are on equal ground. The risk associated with fundamentally changing educational methods described by Le Fevre (2014). She states "altering the way [schools] have always done carries cost of not only risk and failure but sadness and loss" (p. 57). That is to say, Dr. Wyatt cannot find the "nerve" to alter his teaching methods because it increases the chance of failing to properly instruct students. Dr. Wyatt's methods are very detail oriented and designed to work with his beliefs and personality, which I will analyze next.

Teaching the details. While discussing students' responses to questions on assessments, Dr. Wyatt commented on the fact that grading these questions "helps with my teaching" (Personal Interview, February 12, 2019). One reason is he makes a note on what he needs to emphasize either the next time he teaches the topic or in the next class depending on the severity of the error. However, he notes that looking at the students' responses provides a guideline for how much detail is needed when looking at proofs.

Dr. Wyatt comments

> I can be honest, some textbooks give, uh, some textbooks give very sketchy proofs, you know. If I don't [examine students responses to identify how they can connect material] then I might think, oh this kind of sketchy proofs are

> perfect, I don't need to do anything. In class I just also do sketchy proofs, you know this is the proof, just learn it (Personal Interview, February 12, 2019).

By examining students solutions to various assignments, he notes the level of detail the students require to learn a proof. Thus, he can ensure that the examples he uses provide enough detail to both guarantee students understand the material and to help the student develop reasoning skills so they can fill in the gaps of "sketchy proofs" they encounter in future courses. This includes the examination of proof techniques and logical thinking in general.

Demonstrating when to use various proof techniques. Dr. Wyatt describes how he scaffolds instructing students to use specific proof techniques on assignments as stages. In the first stage, a specific proof technique is prescribed, that is a question will state specifically to use, for example, the principle of mathematical induction or a proof by contradiction. The second stage of these assignments still provides details on methods to be used; however, it is now requested more generally, such as "use [an] indirect method... [the students] at least know it could be by contradiction or by contraposition" (Personal Interview, February 12, 2019). The third stage, which Dr. Wyatt admits does not appear much on his homework assignment for this course, involves no actual guidance on technique and instead just asks the student to prove the statement.

When asked how he provides information to the students later in the course on why he chose to use a specific proof method in class, Dr. Wyatt states

> I think that the truth is maybe I, if I recall, maybe I don't do that, so called I don't explain as much as [is ideal]. So later on I might just say it this way, yeah this is a theorem and ask how to prove it. I might just say, we prove it by contradiction, so uh no, I know you are here you are asking how do I explain. Why do it this way, the truth is maybe [I do not explain my why]. (Personal Interview, February 12, 2019).

He goes on to state that this part is “hard to explain,” but he notes that sometimes he will follow up a proof with the statement that this method was chosen because “it works,” it is elegant. Further he states that maybe a method would be to ask the students to try and prove the statement using a different framework, but acknowledges this would be a difficult thing to do in many cases. Dr. Wyatt concludes the discussion on this topic with a request to be informed about a good method for doing this if one existed or is discovered.

Dr. Wyatt’s preparation focuses on which examples will best help students understand the material presented in the course. These choices occur because Dr. Wyatt takes the time to identify common misconceptions about specific mathematical topics. Further, he has strong beliefs how he should present the material while acknowledging his methods are not the only way of instruction. He is able to reflect on other teaching styles and even formulate a possible structure of the way he would implement such a style. The next topic I will analyze is how Dr. Wyatt allocates time during a lesson.

4.2.3 Time allocation

Content coverage requirements do affect the methods used by mathematics instructors, specifically when considering how an instructor allocates time to various aspects of their teaching. Speer et al. (2010) note, while the content of a course is determined by the institute, the amount of time devoted to a specific topic and instructional practices used is ultimately left to the instructor. Further, they note that the amount of content in a course “forces teachers to make hard choices about what to include and exclude” with regards to the amount of time spent on any given task. Further, Johnson et al. (2016) found that not only instructors, but also students, in Calculus I courses feel “there was not enough time to understand difficult ideas.” Also, Johnson et al. (2017) identify time constraints as having an affect on instructional practices. In short, the amount of time an instructor can devote to a given task has a strong effect on how an instructor allocates their time. Dr. Wyatt describes his teaching method through the use of examples. In fact, Dr. Wyatt’s methods of preparation have been adapted around this purpose; he states:

> When I first got into teaching, when I first teach in the [United States], I write my notes in very detail, almost sentence by sentence, but now I don't do that. Now I just think about what examples I use in class to explain the idea, what example includes both the truly mathematical problem and the so called real world examples. (Personal Interview, February 12, 2019)

Further, he acknowledges "I have [a] time limit," because he has to teach the required content for the course, so the constraint on time forces Dr. Wyatt to choose when to use specific examples (Personal Interview, February 12, 2019). The following subsections discuss how Dr. Wyatt distributes his time throughout the Transition-to-Proof course and Dr. Wyatt's description of an ideal situation and how it would restructure his teaching practices.

Dr. Wyatt's time allocation. Through classroom observations, I noticed several patterns in Dr. Wyatt's instructional methods, specifically how he presented material and the emphasis on different categories of the presentation of material. The majority of this discussion will focus on the latter, while the former will be analyzed in detail in Section 4.2.4: Representations of Concepts and Relations. Table 4.5 provides a breakdown of Dr. Wyatt's instructional methods by time, in minutes. I identified four categories Dr. Wyatt prioritized, specifically Review, Mathematical Content, Examples, and Other, each of which will be discussed below. The following paragraphs define each of the categories individually, including reasoning for the categorical structure.

Table (4.5) Dr. Wyatt's Allocation of Time (in minutes) by Category.

Date	Oct 29	Nov 5	Nov 7	Nov 12	Nov 14	Nov 26	Nov 28	Dec 3	Time per Category	Precent
Review	5	3	7	10	10	0	10	45	90	15.0
Content	15	21	11	30	7	17	4	0	105	17.5
Examples	39	45	50	34	43	45	31	20	307	51.2
Other	16	6	7	1	15	13	30	10	98	16.3
Course Time	75	75	75	75	75	75	75	75	600	100

One of Dr. Wyatt's former students stated "[Dr. Wyatt] frequently repeated essentials," that is material that builds the foundation of a course. A second student reiterated this stance, describing Dr. Wyatt as placing "very heavy emphasis on the foundational material." Essentially, Dr. Wyatt reviews material throughout the courses he teaches, especially if he believes that material as essential to the topic being discussed currently. Maulana et al. (2015) define *review* as involving "all activities during which the class discuss material/topic of the previous lesson." In Dr. Wyatt's classroom, this definition works well when considering a multi–day lesson, such as the discussion on cardinality of sets, which took place over three days. However, the lesson pertaining to the division algorithm was covered entirely in one day, and as a result, did not have a review portion opening the lesson. Methods of presentation of review materials will be discussed when analyzing representation of materials.

New content was discussed every class session, as observed over the course of all lectures accept the exam review; further, this was reiterated by one former student who commented the course was "very well paced... [Dr. Wyatt] didn't linger on [one topic] too much unless students were not grasping the concept well." Time allocated to covering new material, that is the "lesson components where new material is introduced" (Maulana et al., 2015) is shown in the table above as *content*. Here Dr. Wyatt would either introduce a new primary topic, such as cardinality of sets or the division algorithm or would introduce a subtopic, such as equivalent sets or injective and surjective functions. Types of content covered in this category include formal definitions and theorems. The actual methods used to represent and discuss this content will be described Section 4.2.4.

Maulana et al. (2015) group the use of examples as being part of the introduction of new content. I have chosen to consider the examples, including those relating to proof techniques, used by Dr. Wyatt as being separate from the content of the course. There are two reasons for this decision: (a) Dr. Wyatt, and several of his former students, describe his instructional method as relying heavily on the use of examples, both real world and purely mathematical and (b) the primary goal of this course is to introduce students to and provide examples of proper proof techniques.

The category *other* describes "all components that are outside of the lesson content but occur during lesson" (Maulana et al., 2015). In Dr. Wyatt's case, this refers to transitioning between review of previous topics, presenting new content, and demonstrating the content through the use of examples and proofs. One exception to this rule occurred on November 28; on this day, an assessment was administered to Dr. Wyatt's class as part of the NSF funded research study in which Dr. Wyatt participated.

Table 4.5 shows that Dr. Wyatt allocates over 50% of the course time to providing examples. Given that Dr. Wyatt has described examples as his "best way" of covering material, this distribution makes sense (Personal Interview, February 12, 2019). Further Dr. Wyatt spends 17.5% of the course time to covering new content; again this aligns well with comments about Dr. Wyatt's teaching from former students and the constraints that arise from the amount of content to be covered in the course. Time allocated for other activities (16.3%) and reviewing previous content (15.0%) are possibly skewed by three factors. First, on November 28, an assessment administered as part of the NSF research study occurred during the last thirty minutes of class time. Second, the fact that one of the class sessions observed featured the review for the final exam does not exemplify the amount of time Dr. Wyatt would spend reviewing previous material. Finally, there was one class session that featured no review of previous content, which by analyzing the other observed lessons does not appear to be the norm for Dr. Wyatt's class. However, with those three instances aside, it does appear that Dr. Wyatt tends to allocate less time to the review of previous material than other activities that may occur during the class. Next, we will analyze how Dr. Wyatt's ideal amount of time would alter the structure of his class.

Time constraints shaping Dr. Wyatt's instruction. Dr. Wyatt described himself as being "old fashioned" in his approach to teaching mathematics, specifically he states

> I think I am still doing, you can call me old fashioned, I still think the so called just doing the problems. [The students] just do the problem, think and digest, then do more problems. I am still this kind, one of those kinds of people (Personal Interview, February 12, 2019).

From this statement, it seems Dr. Wyatt feels that students learn best when they are working through and struggling with problems. One of his former students also referred to Dr. Wyatt's instructional methods as "old fashioned," commenting

> [Dr. Wyatt] works really hard. He handwrites the notes as he discusses the ideas in lecture. I know this can seem a bit old-fashioned; however, I believe it really helped to pace the lecture and helped me to keep up with the material.

These two statements, along with evidence from the observations, identify the primary method of instruction Dr. Wyatt used in class was through lecture; specifically, Dr. Wyatt allocates time to present content using a traditional lecture method incorporating problems and examples to work through with the students as they work with the content.

Johnson et al. (2017) categorized the structure of lessons in an Abstract Algebra course as either *traditional, mixed*, or *alternative* based on the amount of time devoted to lecture. Their description of traditional instruction is given below

> Traditional instruction is characterized by heavy use of lecture. During lectures... [instructors] are are showing students how to write specific proofs and pausing to ask students questions.

Traditional instruction accurately describes Dr. Wyatt's instruction as almost all of the time in each of his lessons is devoted to some form of lecture, where he asks questions of students while presenting and completing examples and providing proofs of theorems to the class. The primary reason Dr. Wyatt chose to use a traditional instructional method is due to the time constraints of the course.

Dr. Wyatt commented that he has "to teach the materials that are required in the course, so therefore I don't have all of the time" to attach a real world example to each topic in the course. He also notes, when explaining why he does not assign reading outside of class, that students have time constraints of their own, commenting

> [Students] have lots of classes, they don't have time [to complete reading assignments]. My feeling is it just, [the students] are counting on the fact that I will be

> teaching the material in the class; therefore, they don't need to just prepare [by previewing the content for the lectures] (Personal Interview, February 12, 2019).

It is evident that Dr. Wyatt feels he and the students have certain time constraints that determine, in part, how Dr. Wyatt structures his instructional time. However, Dr. Wyatt described two situations that would possibly alter his instructional methods. Both of these situations, specifically a master-apprentice and the students having the Transition-to-Proof course as their sole responsibility, are dependent on the elimination of students' time constraints.

The hypothetical method of a master-apprentice relationship was discussed by Dr. Wyatt while responding about how to foster a desire to learn in students. He began by describing a "blacksmith, say they just follow their master like their whole life... the early part of the study they just watch what the master does" before practicing the material under the "blacksmith's" supervision (Personal Interview, February 12, 2019). Then, he stated

> If this is the way, I know this is not practical, you know I fully understand that, but then I wouldn't teach all the time. I would let them struggle with a problem. Understand, sure I would teach occasionally, once in a while. You know the key steps I would give some hint or whatever, but most of the time I would just let them, uh, do the thing (Personal Interview, February 12, 2019).

Though having a true master-apprentice situation is not practical, Johnson et al. (2017) describe the *alternative* category of instruction being evident by "having students work in small groups... have students work individually... having students explain their thinking." Further, Johnson et al. (2017) note some lecture and teacher-oriented are included; however, they are limited when compared to student-oriented activities. This could be used as a substitute for what Dr. Wyatt describes as the master-apprentice relationship. The students still struggle with the material, but within the confines of the time restraints.

As mentioned previously, one reason Dr. Wyatt does not assign reading because he does not believe the students have the time to truly focus on the reading. However, if he felt the students had the time outside of class to read the material fully, he described how his teaching methods would alter.

> Say I gave a reading assignment, at the end of the class I give a reading assignment for the next class. You know, 'oh read this,' so in the next class I don't do as much lecturing. I let [the students], you know, just say you have read those, then let's do some exercises (Personal Interview, February 12, 2019).

This method coincides well with the *mixed* instructional method described by Johnson et al. (2017). As the name suggests, this method is a combination of the *traditional* and *alternative* instructional methods, specifically

> Mixed instruction is characterized by moderate use of lecture... Additionally, there is some class time devoted to students working alone and in small groups, giving presentations, and explaining their thinking.

Assuming that students read that material, the class could begin with groups of students working together to ensure they understand the key points of the reading; then the instructor could have a short lecture on some of the applications of the material. Finally, students can work in groups or individually on exercises. This method could approximate the hypothetical method presented by Dr. Wyatt.

These two hypothetical situations demonstrate that Dr. Wyatt is open to considering different methods of instruction, given the time, and that he believes a good way for students to learn material is by struggling with problems. When we discuss the format of the assessments (homework and exams) Dr. Wyatt uses, I will analyze this concept of struggling with problems but having hints to guide the students' thought process. Next, I will describe Dr. Wyatt's methods of representing and presenting content.

4.2.4 Representations of concepts and relations

Speer et al. (2010) discuss representing mathematical content as consisting of "both *what* is displayed and *how* it is displayed." They note several factors that affect the representations used in mathematics classrooms, including the textbook used for the course, representation methods that have worked well in the past for the students, and the prevalence of technology in the classroom. Dr. Wyatt makes the selection of his representations based partly on his personal beliefs about teaching and learning, the textbook used in the course, the needs of the students in the course, and information he obtains from other resources, including the internet.

I will begin analysis of how Dr. Wyatt represents mathematical content with discussing the primary instructional method used in the course, that is to say, how Dr. Wyatt uses lecture. Then, I will describe the way notation, algebraic expressions, diagrams, and verbal discussions are used in his course. Next, the role the textbook plays in informing how Dr. Wyatt represents mathematical content will be examined. Finally, how the students needs and technology inform Dr. Wyatt's choice of representations will be analyzed.

The structure of Dr. Wyatt's lecture. The primary method by which Dr. Wyatt presents information in this course is through traditional means, that is to say in a teacher–centric way. As noted when discussing how Dr. Wyatt prepares for his class and how he allocates time within his lesson, he considers his "best way" of communicating mathematical content is through the use of examples and explanations of those examples. Several times throughout the interview, Dr. Wyatt describes how he just teaches the material, "in my class I, most of the time, I teach [the material]" (Personal Interview, February 12, 2019). In short, Dr. Wyatt is using the word *teach* to mean using lecture, or teacher centric explanations. Johnson et al. (2016) discuss in their examination of content coverage and instructional methods in Calculus I that "lecture is still overwhelmingly predominant method of instruction." Further, Weber (2004) and Lew et al. (2016) both identify lecture as the predominant method of instruction in collegiate mathematics.

Weber (2004) notes, while lecture is the primary method of instruction used by Dr. T in his analysis course, the methods he uses vary based on the goal he is trying to accomplish and the topic being discussed. For example, Dr. T uses a *logico–structural* method early in the analysis course, which is characterized by the "[careful use of] the definitions to understand how to begin and conclude proofs" (Weber, 2004) and seldom uses diagrams; one of Dr. T's goals was to improve the logical thinking skills of his students and allow students to become comfortable with proofs. In contrast, Dr. T used a *semantic* teaching style, characterized by an emphasis on conceptual understanding and the use of diagrams; Dr. T's primary goal for the students when using the semantic teaching style was "for students to have rich imagery they could associate with the concepts being taught" (Weber, 2004). In short, well structured lecture based on specific goals using task to enhance those goals is a strong instructional method.

Dr. Wyatt has well formed beliefs that drive his lectures, and the methods he uses to represent the material varies based on his beliefs and the students' needs. As evident from the way Dr. Wyatt allocates time during lessons, 51.2% of his instructional time is devoted to the use of examples and proof techniques. In five of the eight observed lectures, Dr. Wyatt used "real world" examples. In all eight lectures, He used formal mathematical examples. The methods Dr. Wyatt used to present the examples varied based on topic, and the methods included the construction of sets using set notation, the use of graphs on the Cartesian plane, diagrams, and verbal discussion. In the following paragraphs, we will discuss each of these methods with examples from the lectures themselves.

The role of set notation. The first topic I observed Dr. Wyatt teach was *injective and surjective functions*. For this topic, which were covered over the course of two class sessions, finite sets were the primary objects used as examples. One example used is provided below

Let $A = \{1, 2\}$, $B_1 = \{a, b, c\}$, and $B_2 = \{a, b\}$. Consider the functions $f_1 : A \to B_1 = \{(1, a), (2, b)\}$, $f_2 : A \to B_2 = \{(1, a), (2, b)\}$, and $f_3 : A \to B_2 = \{(1, a), (2, a)\}$.

Figure (4.17) An example from a lecture on injective and surjective functions.

Dr. Wyatt then described each of the three functions in Figure 4.17 in regards to their status as injective and surjective functions. Specifically, he identified f_1 as injective but not surjective on B_1, f_2 is bijective[11], and f_3 possess neither of these qualities. While he wrote the status of each function on the board, he verbally provided the reasoning for each status. Weber et al. (2016) discuss various methods for improving lectures in upper level undergraduate mathematics courses; one of their suggestions is to "write down the key points [mathematics instructors] want the students to learn." Their reasoning for this suggestion is given below:

> It is natural for students to focus on what is written on the blackboard; this is a traditional way by which teachers emphasize importance, and written comments have a permanence that oral comments lack (Weber et al., 2016).

If one assumes Dr. Wyatt is using this knowledge as he is teaching this topic, then it appears that Dr. Wyatt's goals for this example are to provide guidance on how to list out functions of finite sets and that these specific functions fit their status in regards to injective and surjective functions. However, it is clear that Dr. Wyatt is emphasizing the reason why these functions have their specific statuses. Therefore, Dr. Wyatt is not maximizing the use of this example, that is he is not representing the importance of the reasons why functions are injective or surjective because he is not writing down the reasoning explicitly.

Graphs of algebraic functions. When discussing the definitions of image and preimage, Dr. Wyatt asked the students to study the function $f(x) = x^2 - 2$ and consider the following

1. $f((-2, 1)) = [-2, 2)$
2. $f^{-1}([2, 7)) = (-3, -2] \cup [2, 3)$

To explain the former, he used the diagram given in Figure 4.18 Notice that on the graph he notes the interval that is being examined and highlights image of the function by thickening the y–axis. He takes a similar approach for the preimage by using the diagram in Figure 4.19 Notice that Dr. Wyatt again highlights the interval being considered, and then thickens the

[11] Both injective and Surjective on B_2

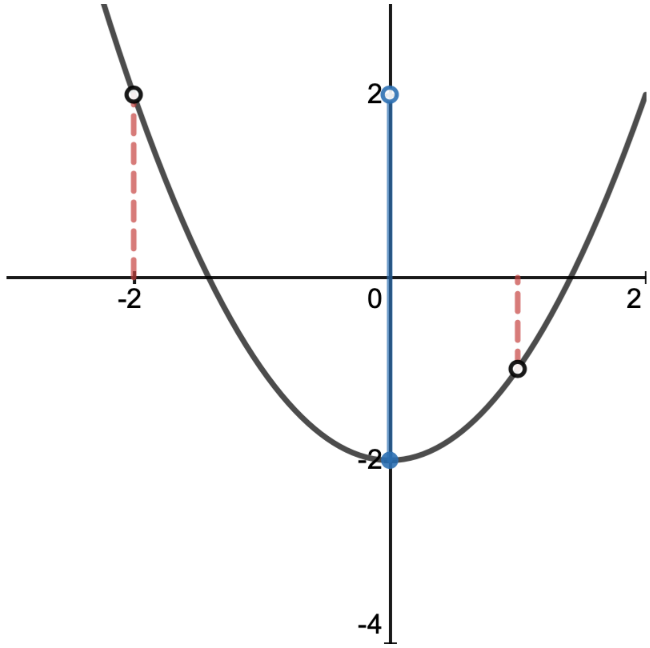


Figure (4.18) Graph used to describe $f((-2, 1))$.

line on the x–axis to highlight the preimage.

In both of the graphs above, Dr. Wyatt highlights two aspects: the interval being examined by the image or preimage and the interval that is the given image or preimage. The goal of this example is to get the students to think about infinite sets (the intervals) and a method for identifying these sets. Again, he verbally provides the reasoning behind each interval, that is where the endpoints come from, but what he wants the students to understand (how to visualize the image and preimage) is displayed clearly on the board.

How Dr. Wyatt uses diagrams. In our discussion about Dr. Wyatt's teaching practices to this point, we have analyzed several diagrams including the diagram of "infinite armies" used to motivate the discussion on how the natural numbers and integers are equivalent and the diagram used while discussing the Pigeonhole principle. In these two cases, the diagrams are not needed to find the solution to the given example. The former could be described using the bijective function, which was used in a proof of the equivalence later in the lecture.

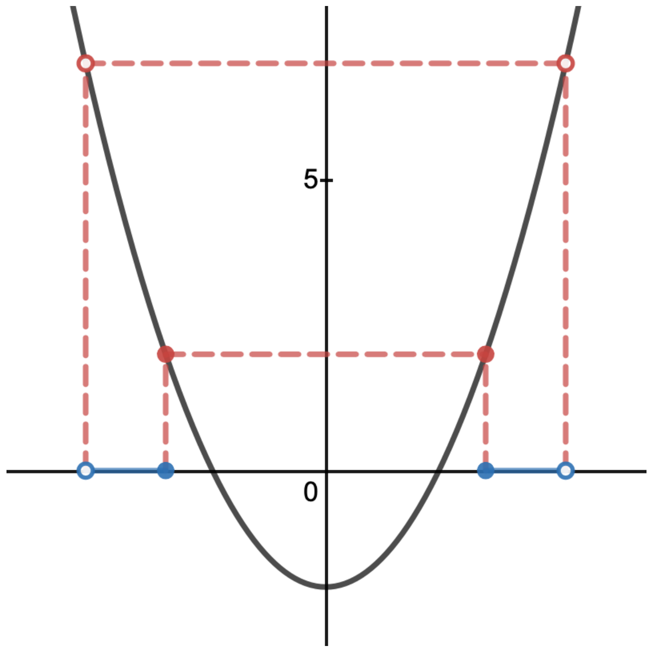


Figure (4.19) Graph used to describe $f^{-1}([2,7])$.

$$f : \mathbb{N} \to \mathbb{Z} = \begin{cases} -\frac{n-1}{2} \text{ if } n \text{ is odd} \\ \frac{n}{2} \text{ if } n \text{ is even} \end{cases}$$

However, this algebraic function, while accurate, does not promote understanding the concept of why the two sets are equivalent. Weber et al. (2016) made the following statement about how students think about proofs

> When hearing a proof in lecture, students often focus on the calculation and logical detail, [mathematics instructors] should shift what students attend to by assessing their understanding of other aspects of proof.

In short, students view proofs as procedural; since Dr. Wyatt wants the students to understand why this is true conceptually, he avoided using a function that the students would focus on calculating more than understanding. The latter could have been studied using set notation and list. Again, Dr. Wyatt used a diagram because it can represent multiple sets at once, not limiting the understanding to a specific case.

In short, Dr. Wyatt uses diagrams to emphasize the underlying concepts of mathematics. This method helps students develop a deep conceptual understanding of the topics because students are shown how to visualize certain topics. This is very similar to Dr. T's semantic teaching style as described in Weber (2004). Dr. T wants the students to visualize the material because he wants "students to have a sense that something occurs" (Weber, 2004) when they hear certain topics.

The use of the textbook and technology. Dr. Wyatt, as discussed previously, does not assign the students any reading during the course. Further, Dr. Wyatt does not reference the textbook often in class, and when he does he generally provides only a chapter number from the textbook or references the notation used in the textbook. For example, the textbook for this course denotes the cardinality of set A using $\overline{\overline{A}}$. As a result, Dr. Wyatt uses this notation throughout the course. Therefore, the primary affect the textbook has on Dr. Wyatt's representations is by setting the notation that he uses.

Technology provides a format for mathematics instructors to "explore more dynamic" (Speer et al., 2010) representation methods than the traditional static representations. Another use of technology is to allow students to quickly look up information for real world examples. This is how Dr. Wyatt uses technology during the observed lectures.

For example, while discussing the Pigeonhole principle, Dr. Wyatt presented examples that required more information than he had, specifically the number of hair follicles on the human head and the population of a metropolitan city in the United States. While this does engage students in the content, it is not using technology to represent mathematical content. There are many computer programs that Dr. Wyatt could use to enhance the diagrams used in this course; however, I observed no evidence of any technology used for representing content.

Summary of representations of content. Mathematics instructors have to decide what they display and how they display mathematical content. Dr. Wyatt chooses to use a more traditional lecture format for his lessons. He relies heavily on the use of examples, but those examples take many different forms, including formal mathematical notation, Cartesian graphs, and various diagrams. The notation he uses remains consistent and follows the standards set by the required text of the course. He does not use technology to display information; however, technology is used during some classroom discussions. Dr. Wyatt will highlight important information verbally and written; however, according to the literature he does not always use written information effectively.

Dr. Wyatt uses several modes of representation to enhance students understanding of the topics being discussed. However, he must identify what students need in order to succeed so he can structure his examples efficiently. The next section discusses the questioning techniques Dr. Wyatt uses in his course.

4.2.5 Questioning

Questions are an integral part to mathematics instruction. Questions can be used to motivate topics, identify misconceptions, and guide instruction. Speer et al. (2010) notes that identify four components of questioning that have shown impact in K–12 mathematics education research: Frequency, character and intent, wait time, and reactions. Dr. Wyatt used questioning often during his instruction; however, not all questions are created equal. Watson (2007) notes that several frameworks exist for developing questions. Likely the most well known, Bloom's Taxonomy, focuses on "learning as a whole-school issue" (Watson, 2007); while still useful for developing questions in the mathematics classroom, it does have key weaknesses, primarily in what constitutes a successful outcome. I will begin by categorizing Dr. Wyatt's questions based on on the framework posed by Speer et al. (2010), then specific examples of Dr. Wyatt's questioning will be examined.

A question can "differ widely in *character* and in the teachers' *goals or intentions*" (Speer et al., 2010), that is to say that questions posed in a class reflect the instructors' beliefs about teaching. However, they did provide two broad categories of questions that will be used here. The first is what Speer et al. (2010) refer to as *Teacher Questions (TQ)* which are designed to promote the structure of a lecture and engage students more than to inform about students' understanding of a concept. An essential trait of TQ is that the teacher knows the answer to the question; for example, when an instructor ask students if a quadratic function is concave up or concave down.[12]

The second type of question Speer et al. (2010) describe is concerned with identifying the students current level of understanding. They do not provide a label for this category of question, so I will refer to them as *Learner Questions (LQ)* because how the learner responds will guide the actions of the instructor. Speer et al. (2010) describe a key aspect of LQ is that the teacher does not know how the student will respond, so the teacher uses this information to assess understanding and adjust the prepared lesson accordingly.

I will examine Dr. Wyatt's questioning using these two categories. To begin, I will examine the frequency and wait time Dr. Wyatt used during the observed lessons. Table 4.6 shows the frequency for both question time broken into three categories of wait time in seconds. It is clear Dr. Wyatt asks primarily TQs, with approximately 91.3% of his

Table (4.6) Frequency of Question by Type and Wait–time.

Category	Wait Time	Oct. 29	Nov. 5	Nov. 7	Nov. 12	Nov. 14	Nov. 26	Nov. 28
	$t < 3$ s	13	9	8	12	15	10	3
TQ	$3 < t < 10$ s	2	0	0	3	2	4	1
	$t > 10$ s	0	0	1	1	0	0	0
	$t < 3$ s	0	0	0	2	0	0	0
LQ	$3 < t < 10$ s	1	3	0	0	0	1	0
	$t > 10$ s	0	0	0	0	1	0	0

questions being of this variety. Further, most of these questions are accompanied by a wait time of less that three seconds. This suggests the primary goal of Dr. Wyatt's questioning is to keep the lecture moving at a good pace as opposed to assessing students' understanding with these questions.

What follows are examples of Dr. Wyatt's questioning practices. This includes examining the types of questions (TQ and LQ) being asked as well as examples demonstrating Dr. Wyatt's beliefs about questions. I will analyze two major aspects of his method: using questioning as a tool to build a students' confidence and how he uses questioning to build

[12]The concavity of a quadratic function determines the range of the function.

students' understanding of mathematical proofs.

Questions as confidence building tools. Dr. Wyatt describes the questions he asks as "immediately relevant to the material [being covered currently] but not very hard" (Personal Interview, February 12, 2019). This statement indicates that he is assessing the students to determine their immediate understanding of the connections between previous and current content while attempting to build their confidence. Watson (2007) comments that questions that enhanced the "belief, persistence and courage" of students in mathematics was essential when counteracting underachievement. Also, Dr. Wyatt's beliefs about questioning appear to coincide with that of the subject of Weber's (2004) case study, who used questioning as transitioning points in proofs (immediately relevant) but were processes that students should understand from previous instruction. I will begin by categorizing questions aimed at improving students' confidence, including his expectation of students. Then, I will analyze how Dr. Wyatt adjusts his expectations based on students' responses.

When asked about his expectations of students with regards to questioning, Dr. Wyatt discussed both the reason he asks questions and what he expects from the students. He states

> The students feel, you know, mathematics is too hard. I know [students are] not likely to answer any of the questions, so they don't even try. But by [asking immediately relevant questions], if some student can answer it, then [the other students] can see "oh yeah, he or she just answered it so it is not that hard." So next time [one of the other students will] answer similar type of question, or at least I hope they will actively thinking for themselves (Personal Interview, February 12, 2019).

In the above statement, Dr. Wyatt is describing TQs. They are designed to progress the lecture forward as they are immediately relevant to the material being discussed. In short, they are questions he knows the answer already and is expecting someone to be able to answer the question quickly and accurately.

Figure 4.20 provides the definitions for the image of a relation and the preimage of a relation Dr. Wyatt presented in lecture. To facilitate students understanding of these

Let $f : A \to B$ be a function. Let $C \subset A$ and $D \subset B$.

Then, the *image* of C under f, written $f(C)$, is defined as

$$f(C) = \{f(x)|x \in C\}.$$

Similarly, the *pre-image* of D under f, denoted $f^{-1}(D)$ is defined as

$$f^{-1}(D) = \{a \in A|f(a) \in D\}.$$

Figure (4.20) Definitions for the image and preimage of a relation

definitions, Dr. Wyatt uses two examples. The first example is "a small one," that is to say, pertains to finite sets; the second example is a "calculus example," that is, pertains to an infinite set. These are good choices of examples to use for the purpose of explaining the above definitions for several reasons. The first example is discrete, so the items can be listed out; this ensures the students can see exactly why each item is included. Further, the first example reinforces some of the basic concepts of set theory, a topic not typically discussed in more computationally driven courses. The second example revisits, indirectly, a topic that gave the student problems in the previous class meeting about restrictions on the domain of a function discussed in Section 4.2.4.

The first example is shown in Figure 4.21 tasks students to find the image and preimage of a function f over specific sets. Dr. Wyatt asked the class as a whole for the answers [13] to each of the six questions posed in the above example. These questions are immediately relevant, as they are discussing the definitions just presented. Also, these questions are "not very hard" in the sense they require the student to use skills they have developed throughout the course (Personal Interview, February 12, 2019). Therefore, they help build the confidence of the students in the class. Additionally, the students answered the questions quickly, with

[13] $f(C_1) = \{c\},\ f(C_2) = \{d, a\},\ f(C_3) = \{c, d\},\ f^{-1}(D_1) = \{1, 2, 3\}, f^{-1}(D_2) = \{1, 2, 4\}$, & $f^{-1}(D_3) = \emptyset$. All answer provided by students.

Let $A = \{1, 2, 3, 4\},\ B = \{a, b, c, d, e\}$ and define $f : A \to B = \{(1, c), (2, d), (3, a), (4, c)\}$ and let

$$C_1 = \{1, 4\}, C_2 = \{2, 3\}, C_3 = \{2, 4\}, D_1 = \{a, b, c\}, D_2 = \{b, c, d\}, \text{ and } D_3 = \{b\}$$

Find

$$f(C_1),\ f(C_2),\ f(C_3),\ f^{-1}(D_1), f^{-1}(D_2), \text{ and } f^{-1}(D_3)$$

Figure (4.21) Finite example discussing the image and preimage of a function.

a wait time between 3 and 10 seconds. However, it is clear that these questions help to keep the flow of the lecture consistent; that is to say, these questions fall squarely into the category of teacher questions as described by Speer et al. (2010).

The second example, given in Figure 4.22, has the same task as the previous example, though is taken over infinite sets. Like the previous example, Dr. Wyatt asked the class as

$$\text{Let } f(x) = x^2 - 2. \text{ Find } f((-2, 1)),\ f^{-1}([2, 7)), \text{ and } f^{-1}((-10, 14)).$$

Figure (4.22) Infinite example discussing the image and preimage of a function.

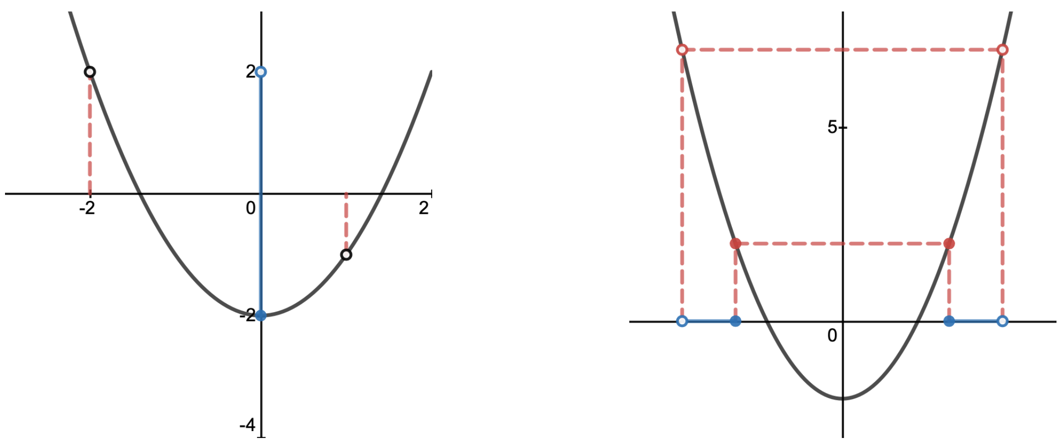
a whole for the answers, [14] but this example he had to do a little extra scaffolding to obtain answers from the students. Initially, the students were not sure how to begin with any of these statements; the questions in this example, though intended to be TQs, became LQs because the answer was not what Dr. Wyatt had anticipated.

> [Asking questions] is like knowledge you know. First, someone can answer it; if no one can answer it I will answer it. In that case it is just learning, but I think most of the time I just want to get them involved, to let them think actively (Personal Interview, February 12, 2019).

This example informed Dr. Wyatt about the students understanding, or misunderstandings in this case. The difficulty for the students came from the fact they were evaluating the

[14] $f((-2, 1)) = [-2, 2),\ f^{-1}([2, 7)) = (-3, -2] \cup [2, 3),\ \&\ f^{-1}((-10, 14)) = f^{-1}([-2, 14)) = (-4, 4)$

function over an interval, not at a specific value, that is to say, they did not initially think of the question graphically. Dr. Wyatt used the graphs in Figure 4.23 to help the students understand how to approach this example; the graph on the left illustrates $f((-2,1))$ while the graph on the right illustrates $f^{-1}([2,7))$. The students, once seeing the graph, began to show a deeper understanding; however, a similar question posed the following class session once again gave students difficulties. Fukawa-Connelly et al. (2017) describe the process of using the diagrams in Figure 4.23 as an informal representation, or a "presentation that gave meaning to the content beyond what was stated in the formal definition" (p. 573). Dr. Wyatt altered his goal for the question (keeping the students engaged) to give deeper meaning to the formal definitions.



(a) The image of $(-2,1)$). (b) The preimage of $[2,7)$

Figure (4.23) The graphs of $f(x) = x^2 - 2$ used to answer the example.

Both of these examples demonstrate three aspects of Dr. Wyatt's questioning methods. First, these questions both were designed to help build students' confidence because they have worked, at this time in the course, extensively with finite sets and with relations both finite and infinite. Second, he expected these questions to keep the pace of the lecture going, that is these are teacher questions. Finally, he adapted quickly when he discerned his students' had a misconception about evaluating functions over intervals, changing his approach to this example. Essentially, the second example acted as a LQ, as it evaluated students' understanding of a topic.

The concept of proof. Weber et al. (2016) describe two key benefits of using questioning with regards to proof writing. Questions can "give [students] a better sense of how the material can be understood" (Weber et al., 2016) as well as inform the teacher of misconceptions. In other words, questioning can help students construct the ideas behind the proof. Dr. Wyatt states

> [My questions] almost always present some kind of twist. Either it is like some small trap, yeah or if there is some special techniques involved. You know in the middle of a proof I might say 'here, what to do next.' In other words, when I ask questions I almost always [have] some twist there, or there involves some trap or there is some new technique (Personal Interview, February 12, 2019)

Dr. Wyatt uses the words "twist" and "trap" when describing the types of questions he poses. These words are likely referencing how, while many of his examples are straight forward, he does ask students questions where their intuition may hinder their comprehension. For example, two questions posed in his November 12 and November 14 lectures on cardinality exemplify this type of twist or trap: when identifying the cardinality of the rational numbers and determining whether or not $\mathbb{Z} \times \mathbb{Z}$ [15] is countable.

On November 12, Dr. Wyatt posed the question

What is the cardinality of $\mathbb{Q}$?

and instantly noted

$$\mathbb{Q} \approx \mathbb{N}$$

so the cardinality of the rational numbers is the same as the cardinality of the natural numbers. He then asked the class what needed to be true in order for the rationals and natural numbers to be equivalent. The students noted an injective relationship had to exist. The next question was how can this correspondence be found? Dr. Wyatt used the diagram in Figure 4.24 to demonstrate the way a correspondence can be created. In this diagram,

[15] The set of ordered pairs with entries belonging to the integers.

the x–axis consist of the Integers while the y—axis is the Natural numbers. Each point represents the value $\frac{x}{y}$; for example, the point $(-1, 2)$ represents $-\frac{1}{2}$ and the point $(2, 3)$ represents $\frac{2}{3}$.

Dr Wyatt described the process as starting at the point $(0, 1)$ to obtain zero. Then, move right one to the point $(1, 1)$; here he created a square around the starting point, circling all points that produced a new rational number. This process would continue on, but he notes one can count the number of circled points as they go, omitting those that are unnecessary. For example, $(0, 2)$ is not circled because it is equivalent to zero and the point $(-3, 3)$ is not circled because it is equivalent to -1.

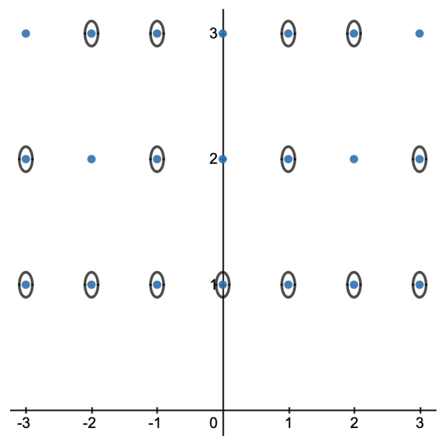


Figure (4.24) Diagram illustrating the cardinality of the rational numbers.

The next class began with the following question: Is $\mathbb{Z} \times \mathbb{Z}$ countable or uncountable? He gave the students a lot of wait time with this question[16], letting them analyze the question for almost a minute prior to asking them for their response. Dr. Wyatt used the diagram in Figure 4.25 to demonstrate that $\mathbb{Z} \times \mathbb{Z} \approx \mathbb{N}$, that is to say $\mathbb{Z} \times \mathbb{Z}$ behaves in the same way as $\mathbb{N}$, so it is countable. However, this did lead to the necessity of having to clarify the meaning of denumerable. One student stated that though the natural numbers were denumerable, not countable. This student had a misconception about the definitions of countable and denumerable, that is, they did not connect that a set must be countable to

[16]In fact, it is the only LQ to have a wait time longer than 10 seconds

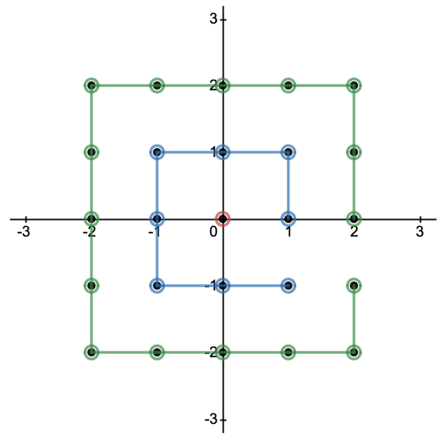


Figure (4.25) Diagram illustrating the cardinality of $Z \times Z$.

be denumerable. This question was designed to provide Dr. Wyatt with insight on how the students thought about the logic behind the concept, but the primary benefit came from identifying a misconception about what denumerable meant. This was clarified reiterating the definition and stressing the requirement that the set be infinite.

Dr. Wyatt uses questioning to achieve two primary goals: increase students' confidence and promote active thinking. Though a limited number of examples are provided here, he uses questioning several times in every class; however, the questions are often questions that the answer is known, such as investigating a key step in the process of writing a proof. His wait time fluctuates from non-existent to possibly too much; that being said, often times this aspect of his questioning reaches these extremes for a specific reason, such as demonstrated in the first two examples about cardinality above. Dr. Wyatt understands that questioning is a vital practice that promotes students' learning.

4.3 Assessment Practices

Currently, assessment design is a key topic being explored at the collegiate level, especially with regard to assessing students' understanding of proofs. Mejia-Ramos et al. (2017) have taken their framework, as developed in Mejia-Ramos et al. (2012), to generate and validate sample proof comprehension test. Further, research groups have formed over the past decade to study the construction of assessments and students' comprehension of proof. Dr. Wyatt works with one of those groups, and this experience has had an impact on his assessment practices.

In this section I will analyze Dr. Wyatt's assessment practices, including how he constructs assessments and provides feedback to students. I will begin by examining his homework assignments; including identifying length of the assignment, the tasks being assigned to students, and his feedback on their solutions. Then, I will discuss two specific types of questions that appear on his homework assignments, specifically questions asking students to grade a proof and those asking them to analyze a proof. Finally, I will analyze the second test from his course including the overall format and students' solutions with feedback.

4.3.1 General format of assessments

During the time I conducted my observations of Dr. Wyatt's Transition-to-Proof course, a total of four homework assignments(Homework 8, 9, 10, and 11), one test (Test II), and the final exam were administered and graded. Of these assignments, I collected data, specifically the students' work after Dr. Wyatt graded and provided feedback on the assignments, for the homework assignments and test. In this section, I will describe the overall format of the homework assignments and the test, relating each question to the framework developed by Mejia-Ramos et al. (2012) and analyzing how Dr. Wyatt's beliefs about assessment affected the format of these assessments.

Dr. Wyatt assigned a total of eleven homework assignments throughout the course of the semester. Homework was always assigned on the last day of class for the week, and then collected one week later. After the assignments are collected; Dr. Wyatt makes the solutions available to his students. The only exceptions to this rule were the weeks before Test 1 and Test 2, where no new homework was assigned; further, homework was assigned the day of a test, given to the students as they turned in their test.

Every homework assignment was constructed in a similar way. There were a total of four questions per homework; the first three questions focused on calculations, definitions, and justifications of various claims, that is to say, they focused on the Local Domain of Proof Comprehension described by Mejia-Ramos et al. (2012). The fourth questions was always formulated in one of two ways, either as a “Grade a Proof” or an “Analyze a Proof” question, both of which will be discussed in detail in later sections. The remainder of this section will be devoted to the examination of the first part of the homework assignments. To accomplish this, I will analyze the various types of questions used, look at example responses by students, and examine some comments in response to students’ solutions by Dr. Wyatt.

Categorization of questions from the homework. Excluding the “Grade a Proof” and “Analyze a Proof” questions, there were a total of twelve questions from homework assignments that were collected by me after Dr. Wyatt graded them. Though these twelve questions all focused on the Local Domain of Proof Comprehension, the questions fell into one of three broad types of task being assigned to the students: *construction of an example, evaluation of a statements validity*, and *calculation of a result*. What follows is a description of the category, followed by an example question; after all categories are defined, I will analyze the student assessments themselves.

Construction of an example questions all took some variation of the following form: the students were given a list of properties that needed to be satisfied and asked to produce a an item the exemplified properties. For example, question 8.1, shown in Figure 4.26, asked students to construct non–empty relations[17] that satisfy specific properties. It is clear that the objective of this question is to test students understanding about the definitions of symmetric and antisymmetric relations[18]. In order to create an accurate example, a student needs to be very comfortable with these definitions. While Mejia-Ramos et al. (2012) believe that students should be able to identify examples that illustrate a specific statement, they excluded creating an example with regards to definitions.

There were two questions that asked students to evaluate a statements validity and both

[17]Let A and B be sets. A relation from set A to B is a subset of $A \times B$, the cross–product of A and B. That is to say, the relation is the collection of relevant ordered pairs (a, b).

[18]A relation R is symmetric if, for all x and y in set A, $xRy = yRx$. A relation R is antisymmetric if, for all x and y in set A, $xRy = yRx$ implies $x = y$.

Construct concrete non–empty relations R, S, and T on $X = \{1, 2\}$, as well as a non–empty relation U on $Y = \{1, 2, 3\}$, such that the following properties are satisfied.

1. A relation R on X such that R is symmetric and is antisymmetric.
2. A relation S on X such that S is symmetric and is not antisymmetric.
3. A relation T on X such that T is not symmetric and is antisymmetric.
4. A relation U on Y such that U is not symmetric and is not antisymmetric.

Figure (4.26) Homework 8 question 1.

questions appeared on homework 10. Further, both of these questions pertained to finite and infinite sets. In these questions, Dr. Wyatt tasked students with categorizing various sets as either finite, infinite, countable, denumerable, and uncountable or a combination of these concepts. For example, homework 10 question 3 asked students to evaluate statements as either true or false, as shown in Figure 4.27. The purpose of this question is to evaluate

Determine whether each of the statements are true or false. No Justification is necessary.

1. If a set A is countable, then A is infinite.
2. If a set B is denumerable, then B is infinite.
3. If a set C is uncountable, then C is infinite.
4. If a set D is denumerable, then D is countable.
5. If a set E is not denumerable, then E is uncountable.

Figure (4.27) Homework 10 question 3.

students comprehension of the various types of sets. Specifically, this question asks students to interpret the "trivial implications of a given statement" (Mejia-Ramos et al., 2012), that is to say, what is required, by definition, to be a specific kind of set.

Most of the questions on Dr. Wyatt's homework assignments require students to perform some sort of calculation. Here calculation means students are given specific information and asked to find a specific value or range of values. However, all of these questions still focused on students' understanding of various definitions. Figure 4.28 is a calculation question from Homework 9. In order to perform these calculations, students must understand the concept

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$.

1. Find $f([-1, 0) \cup (2, 4])$.
2. Find $f^{-1}([-2, 3))$.
3. Find $f^{-1}((-1, 5] \cup (17, 26])$.

Figure (4.28) Homework 9 question 3

of evaluating a function over an interval and the definitions of the pre–image of a function. Prior to entering this course, students should have an understanding of how to evaluate functions; however, the notation f^{-1} is commonly used when discussing the inverse of a function. Since invertable functions are injective and $f(x) = x^2 + 1$ is not injective over all real numbers, students may have difficulty evaluating f^{-1} across the various intervals.

In short, most of the questions Dr. Wyatt asks on homework assignments fall into one of three categories. However, all of these questions focus on the basic definitions of terms, that is to say, they assess the terms introduced in the course. It is my belief this demonstrates Dr. Wyatt's desire to make sure students have a good foundation of the basic concepts of course content. To ensure this, he focuses on the meaning of terms and statements, one of the categories in the Local domain of proof comprehension described by Mejia-Ramos et al. (2012).

Next I will analyze Dr. Wyatt's feedback on students' homework assignments based on category. First, I will examine students' responses to five construction problems, two from each of homework 8 and homework 9 as well as one from homework 10. Then I will do the same with the two evaluation questions from homework 10. Third, students' responses to the calculation based question from homework 8 and homework 9 will be considered.

Samples of students' responses to construction problems. There were a total of five construction questions on the homework assignments; Table 4.7 shows how the points were distributed by question. With the exception of homework 10 question 2, the majority

Table (4.7) Score Distribution of Construction Problems

Range	Question 8.1	Question 8.2	Question 9.1	Question 9.2	Question 10.2
$(4,5]$	7	8	6	6	3
$(3,4]$	4	3	2	4	3
$(2,3]$	0	0	2	0	4
$(1,2]$	0	0	0	0	0
$(0,1]$	0	0	0	0	0

of students scored in the highest range possible; while some of the solutions in this category earned full credit, others earned a 4.5 with one case were a person scored a 4.9. This demonstrates that Dr. Wyatt is very precise with his grading, noting when students are extremely close to perfect in their reasoning. Further, the majority of scores in the range $(3,4]$ were actually scores of 4, with the only other score in this range being being a 3.5. This suggest, once students solutions leave the highest range, Dr. Wyatt does not differentiate in as much detail between the quality of solutions. Students seldom scored in the (2,3] range; further, of those that did, the lowest score was a 2.5.

Figure 4.29 states both construction questions from homework 8[19]. The definitions being assessed in Figure 4.29a are *symmetric* and *antisymmetric*[20]. Specifically, students must understand the relationship between these two types of relations. The definitions being assessed in Figure 4.29b are a *relation* and a *function*[21]. Again, students must understand how these two concepts are related and how they are different.

Figure 4.30 states both of the construction questions from homework 9. Both of these questions have two tasks, that is to construct a set and a function. Further, both questions are focused on the definitions of a function, an injective function, and a surjective function.

[19]Homework 8 Question 1 is also given in Figure 4.26.

[20]Let A be a set and R a relation on A. A function is symmetric if and only if for all $x \in A$ and $y \in A$, if xRy, then yRx. A function is antisymmetric if and only if for all $x \in A$ and $y \in A$, if xRy and yRx, then $x = y$.

[21]Let A and B be sets. R is a relation from A to B if and only if R is a subset of $A \times B$. A function f from A to B is a relation from A to B such that the domain of f is A and if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$.

Construct concrete non–empty relations $R,\ S,$ and T on $X = \{1,2\}$, as well as a non–empty relation U on $Y = \{1,2,3\}$, such that the following properties are satisfied.

1. A relation R on X such that R is symmetric and is antisymmetric.
2. A relation S on X such that S is symmetric and is not antisymmetric.
3. A relation T on X such that T is not symmetric and is antisymmetric.
4. A relation U on Y such that U is not symmetric and is not antisymmetric.

(a) Question 1

Construct concrete relations $r,\ s,\ t$, and u from $A = \{3,4\}$ to $B = \{a,b\}$ with the following properties.

1. Relation r is not a function.
2. Relation s is a function, but not a function from A to B.
3. Relation t is a function from A to B with $\text{Rng}(t) = B$.
4. Relation u is a function from A to B with $\text{Rng}(u) \neq B$.

(b) Question 2

Figure (4.29) Construction questions from homework 8.

The second question (Figure 4.30b) has the added condition of understanding compositions of functions[22]. Students with a below average understanding of these concepts will struggle to answer these questions, but their responses will supply Dr. Wyatt with valuable information.

Figure 4.31 states question 2 from homework 10, the only construction problem on this assignment. This question assesses students' understanding of finite and infinite sets, the union of two sets, the intersection of two sets, and cardinality. Again, if the students have an understanding of the definitions, they will be able to answer the question. However, like the questions in Figure 4.30, there are several tasks the students need to complete. First, they must consider the validity of the statement. Second, the must create an example that supports the statement. Third, they will need to verify their examples.

[22]Let R be a relation from A to B and S be a relation from B to C. Then $S \circ R = \{(a,c) :$ there exist a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S\}$.

Let $A = \{1, 2\}$. Construct sets B_i and functions $f_i : A \to B_i$, for $i \in \{1, 2, 3, 4\}$, with the following properties.

1. f_1 is both one–to–one and onto B_1.
2. f_2 is one–to–one but not onto B_2.
3. f_3 is onto B_3 but not one–to–one.
4. f_4 is neither one–to–one nor onto B_4.

(a) Question 1

Let $A = \{1, 2\}$ and $C = \{x, y\}$. For each $i = 1, 2$, construct sets B_i as well as functions $f_i : A \to B$ and $g_i : B \to C$ satisfying the following properties.

1. $g_1 \circ f_1$ is onto C but f_1 is not onto B_1.
2. $g_2 \circ f_2$ is one–to–one but g_2 is not one–to–one.

(b) Question 2

Figure (4.30) Construction questions from homework 9.

Provide a concrete example of each of the following cases, if ever possible. If a case is never possible, then state so:

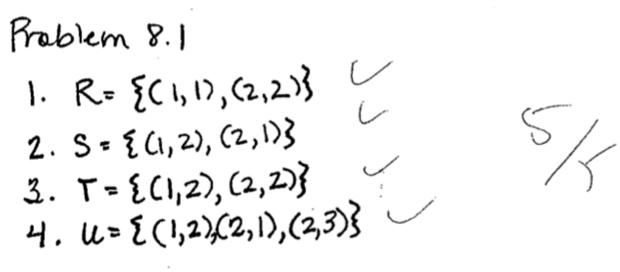
1. An infinite subset X of a finite set Y.
2. A collection $\{A_i :, i \in \mathbb{N}\}$, with each A_i non–empty, such that $\bigcup_{i \in \mathbb{N}} A_i$ is finite.
3. Finite non–empty sets A and B such that $\overline{\overline{A \cup B}} \neq \overline{\overline{A}} + \overline{\overline{B}}$
4. Finite non–empty sets C and D such that $\overline{\overline{C \cup D}} = \overline{\overline{C \cap D}}$.

Figure (4.31) Homework 10 question 2.

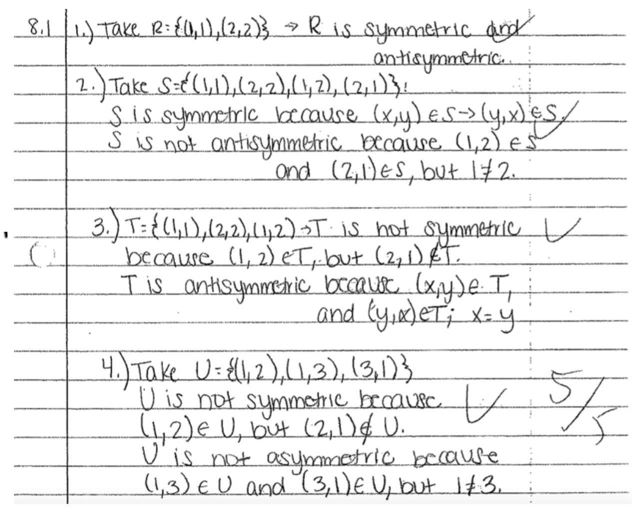
In all of these questions, Dr. Wyatt is assessing students' understanding of definitions. Mejia-Ramos et al. (2012) identified students understanding of the meanings of terms in a statement as part of the local domain of proof comprehension. The way they suggest assessing students' understanding of definitions is to have students identify examples or restate a statement in a different but equivalent way (Mejia-Ramos et al., 2012, pg. 9). Dr. Wyatt tasked the students to create an example of a given statement. As part of the holistic domain of proof comprehension, Mejia-Ramos et al. (2012) discussed the importance of being able to generate examples. Thus, Dr. Wyatt's construction questions are assessing two aspects of students' knowledge: their understanding of definitions and their ability to provide examples. Both of these aspects are skills vital to being able to interpret a proof. Therefore, while not asking students to construct a proof, Dr. Wyatt is assessing skills students will use when constructing proofs.

Students' solutions to construction questions with feedback. Dr. Wyatt returned graded assignments the first class session after they were submitted. He provided feedback on most of the assessments, that is he would comment or otherwise point out students' mistakes on their work. Further, the types of feedback he left was concise. What follows are examples of students' solutions for the construction questions from the homework. This will be discussed by grade range, with examples of perfect solutions, solutions in the highest range that did not receive full credit, solutions scoring in the range $(3, 4]$, and solutions scoring in the range of $(2, 3]$.

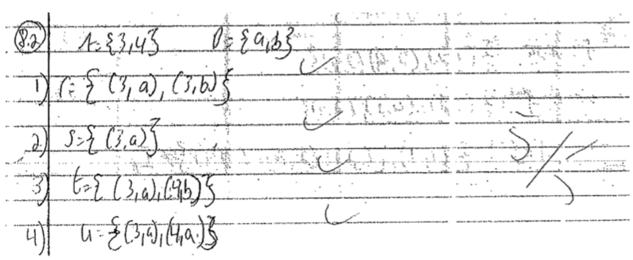
A sample of solutions that earned a perfect score are given in Figure 4.32. There are a few things to note about these solutions. First, with the exception of Kimberly's solution to homework 8 question 1, all of these solutions are very concise and without justification. In fact, almost all perfect solutions follow this format. This shows, that at least on homework questions, Dr. Wyatt does not require his students to justify their solution. This seems to contradict what former student's said about Dr. Wyatt's assessments. Out of the eleven respondents to the questionnaire, seven responded that he either always or often required



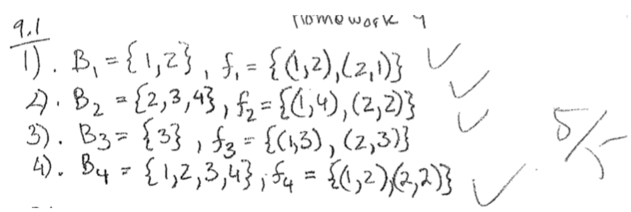
(a) Payton: Homework 8 question 1.



(b) Kimberly: Homework 8 question 1.



(c) Damian: Homework 8 question 2.

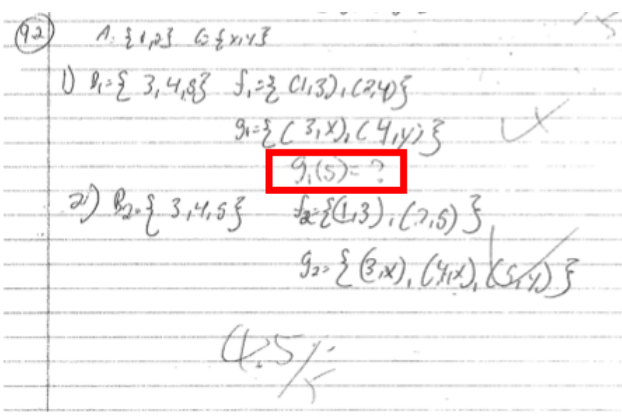


(d) Vaughn: Homework 9 question 1.

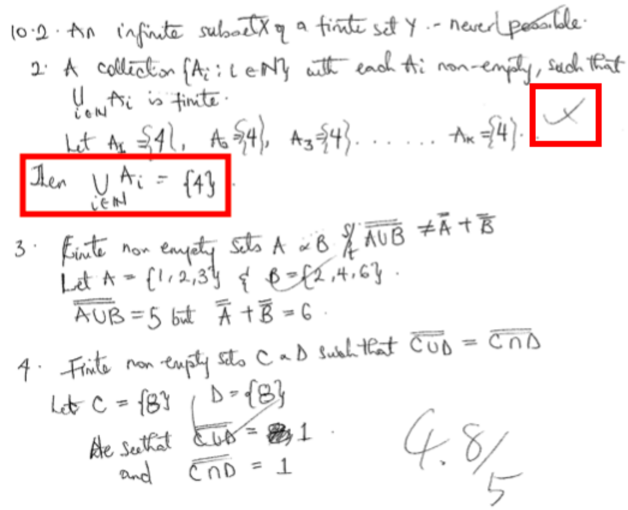
Figure (4.32) Solutions to construction questions earning a perfect score

justifications for responses on assessment. In contrast, only one student responded that he never required justifications for responses on assessments. This contrast could be due to the fact his former students surveyed may have had Dr. Wyatt for a different course, where he may require additional justifications.

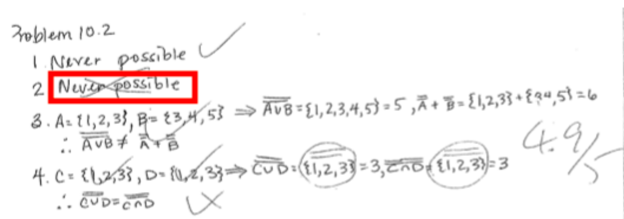
A large number of students scored in the range $(4, 5]$. Most of these solutions where perfect scores, but there were some exceptions, as shown in Figure 4.33. On Payton's solution to homework 10 question 2 received nearly full marks; Dr. Wyatt's only feedback was he crossed out the incorrect portion. It is important to notice that Payton's error was that she identified a situation that can exist as not existing. Therefore, he did not need to provide any further feedback. On Michaela's solution to homework 10 question 2, her only error was that she did not write the final solution. That is to say, she created



(a) Damian: Homework 9 question 2.



(b) Michaela: Homework 10 question 2.

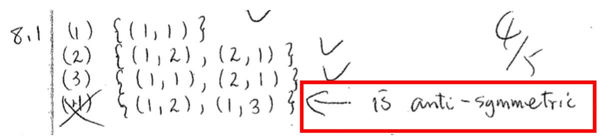


(c) Payton: Homework 10 question 2.

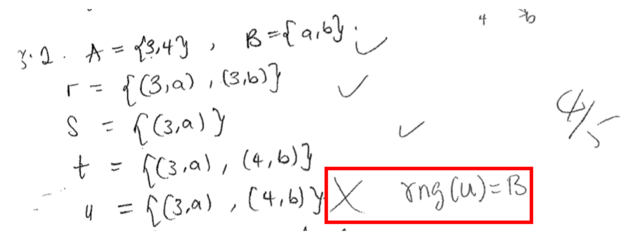
Figure (4.33) Solutions to construction questions scoring in the range $(4, 5]$

sets and provided reasoning for her solution, but forgot to write "then $\bigcup_{i\in\mathbb{N}} A_i$". Here, Dr. Wyatt provided the final statement, highlighted in Figure 4.33b, as the only feedback to her solution. On Damian's solution to homework 9 question 2, Dr. Wyatt provided feedback in the form of a question, specifically "$g_1(5) = ?$", which is highlighted in Figure 4.33a. So Dr. Wyatt's feedback for students who almost received perfect scores is to ask for the specific missing item. However, the feedback is generally a small statement or question regarding the situation.

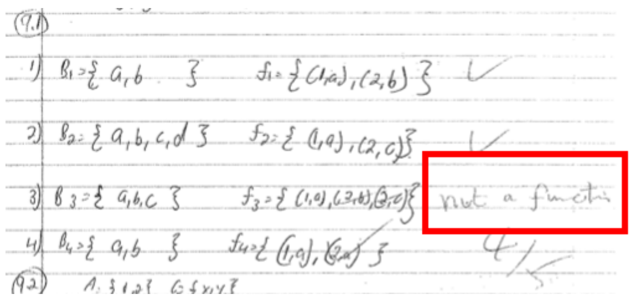
Some students scored in the $(3, 4]$ range; a few of these examples are given in Figure 4.34. With the exception of Jordyn's solution in Figure 4.34d, all of these solutions received a score of 4. The feedback provided by Dr. Wyatt is targeted and concise. On three of the solutions (Figure 4.34a, 4.34b, and 4.34c) Dr. Wyatt simply states the error in their logic, most of which are due to misconceptions about the definitions. The feedback to solution



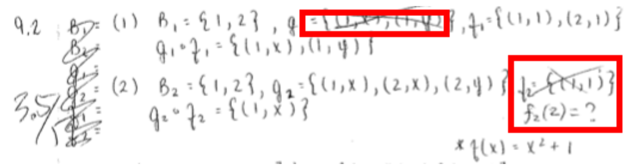
(a) Jordyn: Homework 8 question 1.



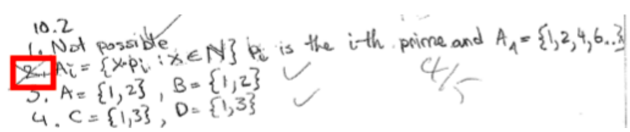
(b) Michaela: Homework 8 question 2.



(c) Damian: Homework 9 question 1.



(d) Jordyn: Homework 9 question 2.

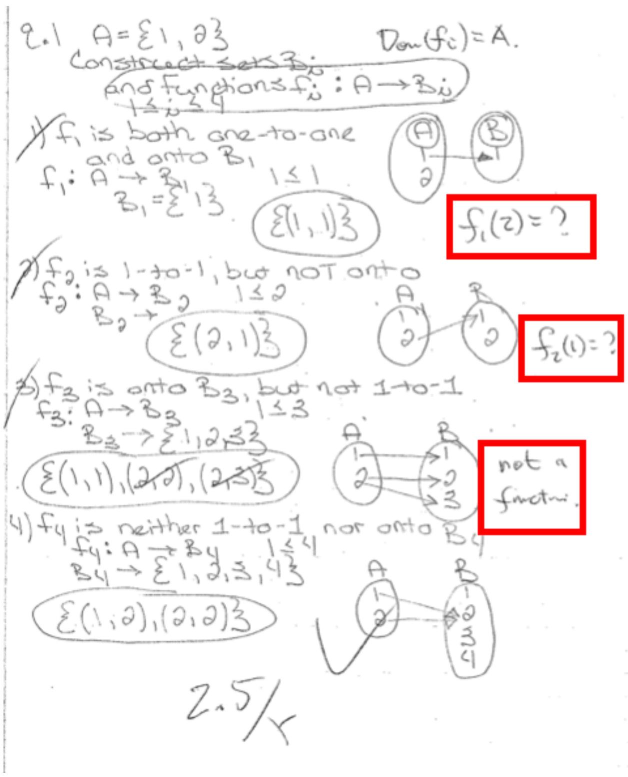


(e) Vaughn: Homework 10 question 2.

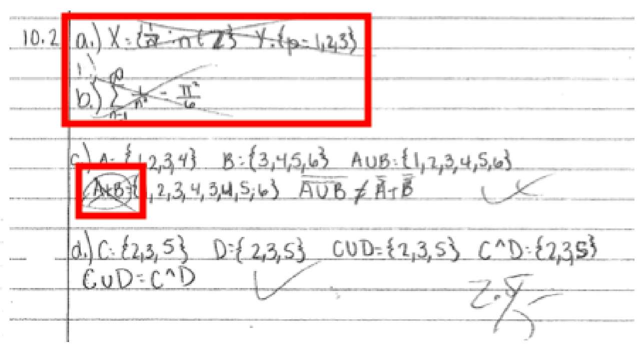
Figure (4.34) Solutions to construction questions scoring in the range $(3, 4]$.

in Figure 4.34d is a question, specifically "$f_2(2) =?$". Vaughn's solution to homework 10 question 2 is only marked incorrect; this is due to Vaughn having a binary error, specifically he produced an infinite set instead of a finite one. At this point, Dr. Wyatt appears to have a consistent method of feedback. His feedback is generally a statement or a question. He provides statements when students have used a definition incorrectly or made a false statement. He provides a question when the solution is missing a key fact.

On homework 9 questions 1 and 2 and homework 10 question 2, there are examples of solutions in the range $(2, 3]$. Two of those examples are given in Figure 4.35. Again it is clear that Dr. Wyatt emphasizes short questions or statements pertaining to definitions. Nadia's solution (Figure 4.35a) has two questions ("$f_1(2)$" and "$f_2(2)$") and a statement ("not a function") from Dr. Wyatt. Again, the questions are when something is missing and the statement is when a definition was not followed. For Kimberly's solution (Figure 4.35b)



(a) Nadia: Homework 9 question 1.



(b) Kimberly: Homework 9 question 1.

Figure (4.35) Solutions to construction questions scoring in the range $(2, 3]$.

Dr. Wyatt provided no feedback, just marked the solutions as incorrect. This is consistent for the first part of the solution because the answer was binary; however, Dr. Wyatt does not provide an example for the other. This is because Dr. Wyatt provides the students solutions to the homework assignments after they are collected.

Dr. Wyatt uses construction questions to assess students' knowledge of the definitions being presented in lecture. Any time there is a fundamental problem with the definition, that is the student did not meet all of the requirements, Dr. Wyatt wrote a statement focused on the misconception. Speer et al. (2010) pointed out that teachers have some basic choices when reacting to questions in class, two of which are asking for more detail and evaluate students' responses in class. While they make these statements based on in class questioning, it appears Dr. Wyatt is following these guidelines when assessing homework. Next, I will analyze the students' performance on the evaluation questions.

Students' responses to evaluation questions. There were a total of two evaluation questions on the homework assignments. Table 4.8 provides a breakdown of students scores by range. The first thing of note is all students scored in the highest range on

Table (4.8) Score Distribution for Evaluation Questions

Range	Question 10.1	Question 10.3
$(4,5]$	10	4
$(3,4]$	0	1
$(2,3]$	0	5
$(1,2]$	0	0
$(0,1]$	0	0

question 1; in fact, all of the students received a perfect score. This indicates that students have a strong understanding of those concepts. On the other hand, question 3 had a majority of the students scoring in the $(2,3]$ range. Further, those that scored in the highest range on question 3 all received perfect scores. Clearly, there is a significant gap in student understanding of the concepts in question 3.

Figure 4.36 shows question 1. To answer this question successfully, you needed to

Determine whether the sets are finite or infinite. No justification is needed.

1. $\mathbb{Q}$, the set of all rational numbers.
2. $\{x \in \mathbb{R} : x^2 + 1 = 0\}$
3. The set of all turkeys eaten in the year 1620.
4. $\{x \in \mathbb{N} : x \text{ is composite}\}$
5. $\{x \in \mathbb{R} : 4x^8 - 5x^6 + 12x^4 - 18x^3 + 26x^2 - 13x + 100 = 0\}$

Figure (4.36) Homework 10 question 1.

understand the concept of finite and infinite sets as well as the ability to identify the number of solutions to an equation. Mejia-Ramos et al. (2012) stated that another aspect of the local domain of proof comprehension is the ability to provide justification for claims. More specifically, students need the ability to "identify the specific claims that are supported by a given statement" (p. 10). Even though students are not required to justify their reasoning, they are identifying a claim (the set is finite, the set is infinite) that identifies the status of the statement (the various sets given). Hence, this is why I am referring to these questions as evaluation questions.

Question 3 is given in figure 4.37. Students had significant difficulty with this ques-

Determine whether the following statements are true or false. No justification is necessary.

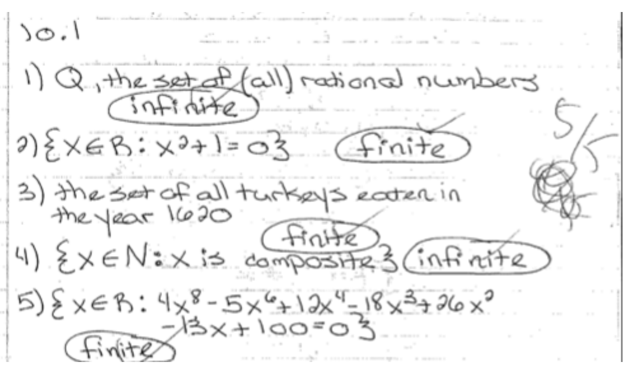
1. If a set A is countable, then A is infinite.
2. If a set B is denumerable, then B is infinite.
3. If a set C is uncountable, then C is infinite.
4. If a set D is denumerable, the D is countable.
5. If a set E is not denumerable, then E is uncountable.

Figure (4.37) Homework 10 question 3.

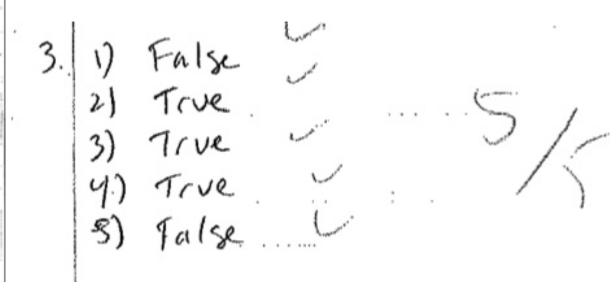
tion. The question itself is assessing students knowledge of the terms infinite, denumerable, countable, and uncountable; further, judging from the scores on question 10.1, students have a good understanding of infinite sets. So the misconception the students have should be about the other terms, or at least, how they relate to each other.

Students' solutions to evaluation questions with feedback. Student's solutions to evaluation questions are very dichotomous. In fact, looking at the scores, it is clear that these two questions were viewed as completely different by the students, even though they do share some common features. In the following paragraphs, I will provide examples of students' responses to these questions. The examples of perfect solutions will be examined first to identify what Dr. Wyatt was expecting. Then I will examine the one solution from question 10.3 that received a score in the $(3, 4]$ range. Then, some of the solutions from the $(2, 3]$ range will be given and analyzed.

Figure 4.38 shows two responses that earned perfect scores. As expected, the solutions



(a) Nadia: Homework 10 question 1.



(b) Jeremy: Homework 10 question 3.

Figure (4.38) Solutions to evaluation questions with perfect scores

are essentially one word responses. Still, to determine which response to use, students must reason through the statement and make a determination.

The one response on question 3 that was in the $(3, 4]$ range is shown in Figure 4.39. Like with the Construction questions, Dr. Wyatt's feedback is minimal, specifically the incorrect response is just crossed out. Vaughn's misconception has to do with the relationship of countable and denumerable sets. Specifically, not all countable sets are denumerable. Dr. Wyatt can identify this misconception strictly based off of the wording of the question.

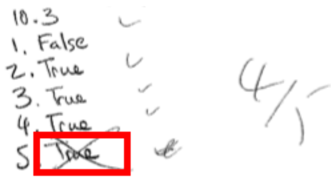
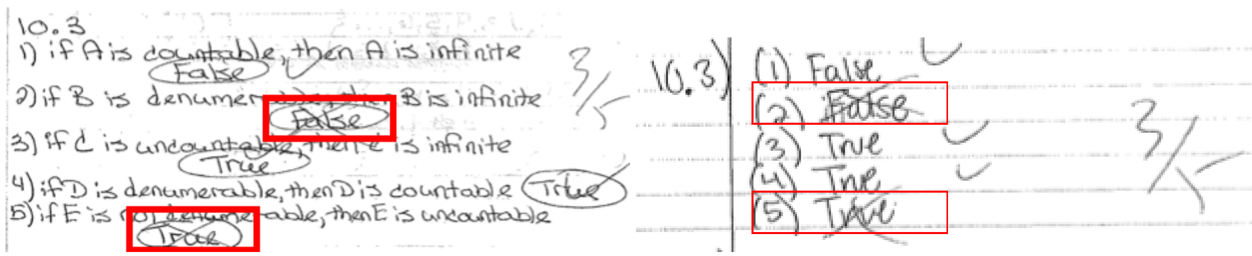


Figure (4.39) Vaughn's solution to homework 10 question 3.

Figure 4.40 shows two solutions from the range $(2, 3]$. Again, Dr. Wyatt provides



(a) Nadia: Homework 10 question 3.

(b) Olivia: Homework 10 question 3.

Figure (4.40) Solutions to evaluation questions scoring in the range $(2, 3]$.

minimal feedback for this question. This is consistent with his grading style in general, that is to simply mark questions incorrect if the answers are binary. Further, given that he provides solutions to the students, Dr. Wyatt does not need to provide explanation with binary responses. Further, in his solutions he provides students with justifications and examples, which will help students identify their misconceptions[23].

Evaluation questions, judging from these samples, provide students with a statement and a binary response. The tasks students must complete is to categorize the statements, generally based off of definitions. Therefore, Dr. Wyatt continues to focus his assessments on the pertinent definitions for the course.

Students responses to calculation based questions. There were a total of two calculation questions on the observed homework assignments[24]. Table 4.9 provides a break-

[23]Dr. Wyatt does not revisit these misconceptions or verify if students have identified their error using his solutions.

[24]This is not counting homework 11, which had three calculation questions, because I do not have student samples from that assignment.

down of the grade distribution of these questions. The scores on question 8.3 are mostly

Table (4.9) Score Distribution for Calculation Questions

Range	Question 8.3	Question 9.3
$(4, 5]$	10	6
$(3, 4]$	1	4
$(2, 3]$	0	0
$(1, 2]$	0	0
$(0, 1]$	0	0

in the $(4, 5]$ range; actually, they all received full credit. The one student who scored in the $(3, 4]$ range earned a 4. For question 9.3, some of the scores in the $(4, 5]$ range were not perfect. However, all scores in the range $(3, 4]$ for question 9.3 were 4. This distribution suggest that most students are comfortable with performing calculations.

Figure 4.41 states question 8.3, which is concerned with images and pre-images. Even

Consider the real function $f(x) = x^2 - 1$. Calculate each of the following.

1. The image/value of 5 under f
2. All the pre-images of 15 under f, if they exist.
3. All the arguments associated with the value of 20, if they exist.
4. All the pre-images of -10 under f, if they exist.

Figure (4.41) Homework 8 question 3.

though this is a question involving calculations, it is still a test of a students' knowledge of definitions, specifically image and pre–image. As far as the actual calculations are concerned, they are actually simplistic in nature.

Figure 4.42 states question 9.3; like question 8.3, question 9.3 is testing students' knowledge of the definitions for image and pre–image. The difference between these two questions is the values students are calculating, that is to say, an interval as opposed to a value. Therefore, the way the calculations are made are vastly different, and while students should be comfortable with these kinds of calculations, it is not guaranteed they are.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$

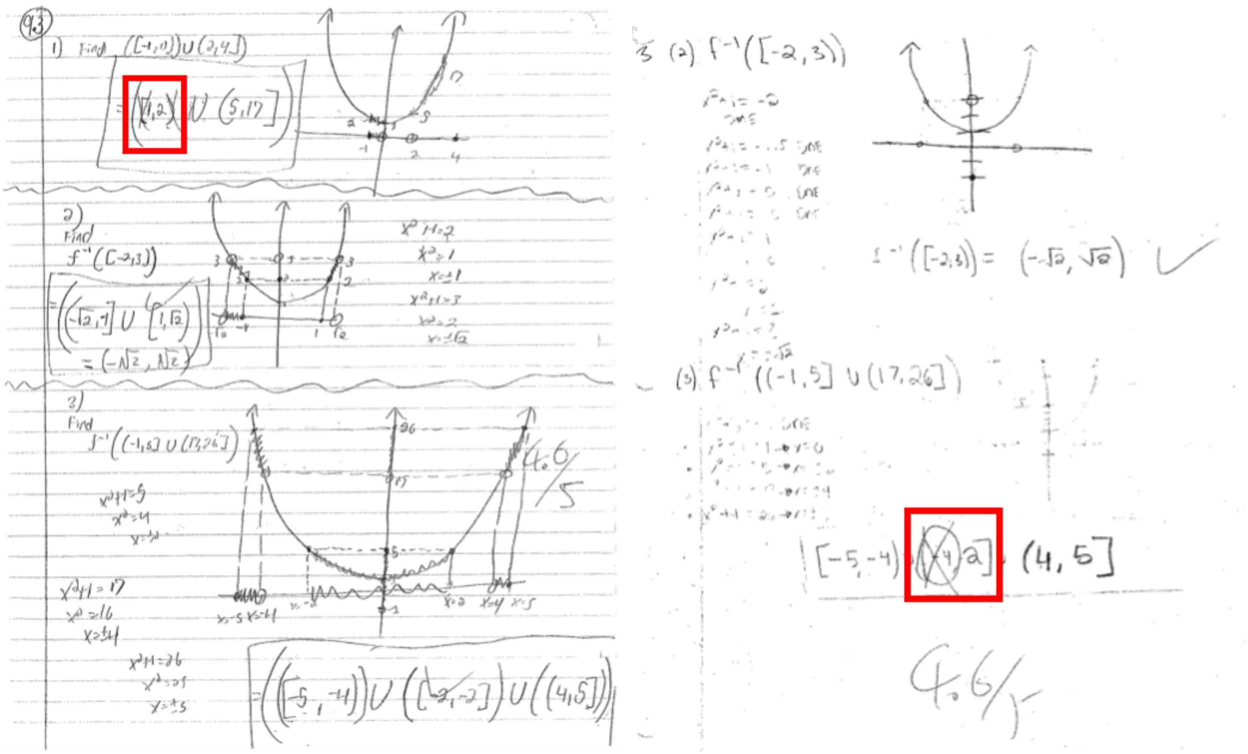
1. Find $f\left([-1, 0) \cup (2, 4]\right)$.
2. Find $f^{-1}\left([-2, 3)\right)$.
3. Find $f^{-1}\left((-1, 5] \cup (17, 26]\right)$.

Figure (4.42) Homework 9 question 3.

Students' solutions to calculation questions with feedback. The solutions to the calculation questions provide insight into how the students performed on certain tasks. Specifically, it demonstrates how students viewed the calculations they were familiar with (question 8.3) to the calculations they may not be (question 9.3). Ten students on question 8.3 and six students on question 9.3 scored in the highest range. All of the students in the range $(4, 5]$ had perfect scores on 8.3, but there were several who were in this range and did not earn a perfect score.

Figure 4.43 provides two solutions that did not receive full credit for question 9.3. Both students made the same error, just in different places. Specifically, they did not label one of their intervals correctly. Damian (Figure 4.43a had an incorrect boundary on an interval; he label the interval inclusive when it should have been exclusive. Olivia (Figure 4.43b) also had an incorrect boundary; however, her mistake was an incorrect sign on a number. Dr. Wyatt's feedback on these questions was to simply cross out the incorrect part of their solutions because their errors were binary.

Figure 4.44 provides examples of two students with scores in the range $(3, 4]$. Dr. Wyatt's feedback again comes in the form of just marking an answer incorrect. While this makes sense in Payton's case (Figure 4.44a) as her error is essentially binary, it does not make sense in Jordyn's case (Figure 4.44b). Jordyn's errors are non–binary, as in the calculation is incorrect. The error in the calculation could be at any point, and in these instances, up to this point, Dr. Wyatt has asked questions or provided statements. That being said, since she did not justify her answers with the necessary work, Dr. Wyatt may have simply not known where the error occurred.

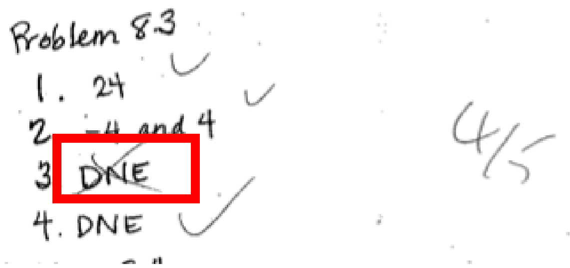


(a) Damian's solution. (b) Olivia's solution.

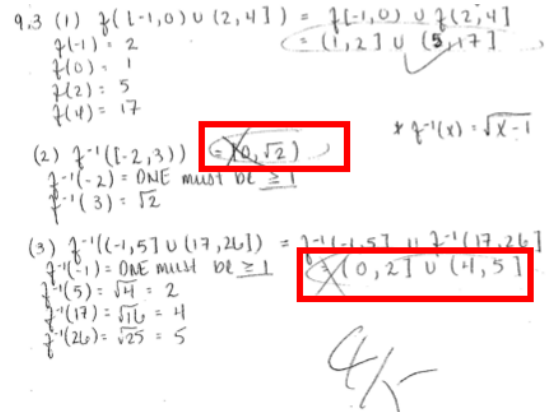
Figure (4.43) Solutions to homework 9 question 3 with scores in the range $(4, 5]$.

Summary of basic homework questions. Dr. Wyatt's primary goal with his standard homework questions is to ensure students have a solid understanding of the fundamentals of the course. This is especially true with his emphasis on definitions. Further, he wants students to be able to categorize information as well as generate examples that support given statements, both of which are emphasized by Mejia-Ramos et al. (2012) in their discussion on the local domain of proof comprehension.

Also, Dr. Wyatt has very specific methods of providing feedback to students. If the question has a binary response, then Dr. Wyatt does not provide a comment, but instead just marks the binary part incorrect. For non binary questions, Dr. Wyatt exhibits two types of responses. The first method is to ask a simple question about an item that is missing from the student's solution; generally these questions take the form of asking about a specific function value that is missing. I believe Dr. Wyatt prefers this method because it contributes to one of his teaching goals, that improve students ability to think actively, as discussed along side



(a) Payton: Homework 8 question 3.



(b) Jordyn: Homework 9 question 3.

Figure (4.44) Solutions to calculation questions in the range $(3, 4]$.

his teaching methodologies. The second method is just to state what specifically is wrong with the solution; this statement generally referenced a specific definition, such as identifying that a students' example is antisymmetric when it was suppose to not be antisymmetric. While his feedback on these questions are simple, they direct students toward a better understanding of the fundamentals. In the next part, I will analyze the "Grading a Proof" question, one of the two types of questions that appeared as the fourth homework problem on an assignment.

4.3.2 "Grading a Proof" question type

Dr. Wyatt stated "it seems that some, at least sometimes, some students don't know what's right and what's wrong" with regards to how they provide proof for various claims. Further, he commented

> I as a teacher grade [the students'] solutions, you know I point out that it is wrong, sure sure, but I don't know if that is the best... maybe there are other ways for them to realize they are, they the students, making mistakes. So therefore I just do it this way, I make them, make the students [the] grader (Personal Interview, February 12, 2019)

Essentially, Dr. Wyatt believes that while it is important for teachers to carefully grade students' work and point out the mistakes that are evident, it is equally (if not more) important for students to realize when they are making these mistakes. In order to foster the thought processes needed for students to identify when they are making mistakes, Dr. Wyatt uses the **Grading a Proof** (GaP) question type. In the following pages I will identify Dr. Wyatt's reasoning for choosing GaP questions, the way these questions are structured, provide examples of students' solutions to these questions, and analyze the effectiveness of this type of question.

The purpose of GaP questions. GaP questions are designed specifically to "make the student the grader, or teacher". Dr. Wyatt elaborates on the previous statement

> After you teach something, you will be a better student. So I am hoping after the students have graded other proofs, then when the next time they are doing their own homework and trying to write their own proofs they will be more careful, especially when a similar situation comes up. They will know "oh, the last time we tried that grading the proof problem, [the GaP proof] did that and it was wrong, so I better not repeat that kind of mistake." (Personal Interview, February 12, 2019)

There are multiple goals Dr. Wyatt wants to achieve with the GaP question type. For example, he wants students to carefully go step by step through a given proof and grade it. Even if the student would grade it incorrectly, Dr. Wyatt hopes he/she would remember it and use it in writing a proof on his/her own.

When examining the given proof in a GaP question, the student must identify several components within the proof to determine the proof's accuracy. The first of these is to understand the meaning of specific statements and terms in both the statement being proven and the proof itself. Mejia-Ramos et al. (2012) describe this as the first component of the *local domain of proof comprehension*. That is to say that before students can begin to examine the proof as a whole, the students need to comprehend the individual aspects of the proof. For example, in the GaP question from Homework 8, students were asked to determine the validity of a given proof of the claim “for non-empty sets A and B, if $f : A \to B$ is a function then $f \circ f^{-1} \subseteq I_B$.”. The proof is given in Figure 4.45. In order to determine if this proof

Let $(b, b) \in f \circ f^{-1}$. Then there exists $a \in A$ such that $(b, a) \in f^{-1}$ and $(a, b) \in f$. Thus $(b, b) \in f \circ f^{-1}$ and $(b, b) \in I_B$. Therefore, $f \circ f^{-1} \subseteq I_B$.

Figure (4.45) The proof used in homework 8’s GaP.

is rigorous, students need to know the definitions and notations for functions, compositions, preimage, and identity functions. If a student is lacking any of these, then they will not be able to examine the proof as a whole.

If students understand all of the necessary terms, then they can begin to study the logical relationships and logical structure of the given proof. This structure is what Seldon & Seldon (1995) describe as a *proof framework*, that is “the ‘top level’ logical structure of a proof, which does not depend on detailed knowledge of the relevant concepts” (p. 129). Being able to identify the proof framework used in a proof is important because “different proofs are, of course, different, but in this difference, one of the proofs might have some special quality” (Rocha, 2019). These special qualities help determine the flow of a proof. In both proof by contradiction and proof by contraposition, the consequence is negated; however, a proof by contradiction will create an impossible statement where as a proof by contraposition uses the negated consequence to show the negation of the antecedent[25]. The proof given in Figure 4.45 attempts to prove the claim directly; however, it violates the proof framework for a direct proof because it, in essence, assumes the consequence is true.

Therefore, Dr. Wyatt uses GaP questions because they require the students to think carefully about the basic aspects of a proof. Students need to take the time to read the proof; in order for the students to read the proof successfully, they need to understand the meaning behind the terms used in both the claim and proof as well as the proof framework being used in the given proof. Next, the task Dr. Wyatt assigns the students for a GaP question is analyzed.

[25]A statement with one truth value that logically precedes the consequence.

The format of GaP questions. Of the four homework assignments collected from Dr. Wyatt's course, three of them contain a GaP question. The given information is presented in the same format every time: a claim followed by a proof of the claim. Then, Dr. Wyatt assigns three tasks presented as shown in Figure 4.46. In the following paragraphs

1. Determine whether the "proof" is rigorous. **Identify the issues in the "proof",** if any.
2. Determine whether the statement is **true** or **false**.
3. If the claim is true and the "proof" is not rigorous, then provide a **correct and rigorous proof**. If the claim is false, give a **concrete counterexample**.

Figure (4.46) The tasks assigned to GaP questions in Dr. Wyatt's assessments.

I analyze each task individually and analyze Dr. Wyatt' s reasoning for the order presented in Figure 4.46.

The first task for the students requires them to examine the proof for correctness. During the interview, Dr. Wyatt stated "for grading a proof, at least for last semester, the ["proof"] is always wrong, there is something wrong with it. " Figure 4.47 provides the given information for all three GaP questions I collected as part of this study. While it is known that all of the "proofs" are incorrect, it is important to identify what makes each of them incorrect. The "proof" shown in Figure 4.47a has an error within the proof framework itself, specifically, in order to construct this type of proof one chooses an arbitrary element from $f \circ f^{-1}$ and shows the element exists in I_B. The "proof" provided has chosen a very specific element (one that is, by definition, in I_B), that is to say, their argument is circular. So, the issue with the proof is that they did not choose an arbitrary element.

While similar to the error described in the previous paragraph, the information provided in Figure 4.47c, the "proof" is not rigorous due to the fact it does not consider arbitrary elements. However, unlike the previous example, the error occurs not because the argument is circular, but because the argument is not general. The "proof" *illustrates the claim with an example*, which Mejia-Ramos et al. (2012) describe as a form of holistic understanding

Claim: For non–empty sets A and B, if $f : A \to B$ is a function, then $f \circ f^{-1} \subset I_B$.

"Proof": Let $(b, b) \in f \circ f^{-1}$. Then there exists $a \in A$ such that $(b, a) \in f^{-1}$ and $(a, b) \in f$. Thus, $(b, b) \in f \circ f^{-1}$ and $(b, b) \in I_B$. Therefore, $f \circ f^{-1} \subset I_B$.

(a) GaP from homework 8.

Claim: If A and B are infinite sets, then $A \approx B$..

"Proof": Let A and B be infinite sets. Then we can describe

$$A = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \text{ and } B = \{b_1, 2_2, \ldots, b_n, b_{n+1}, \ldots\}$$

Define a function $f : A \to B$ by $f(a_n) = b_n$ for all $a_n \in A$. Clearly, f is one-to-one and onto B. Therefore, $A \approx B$, finishing the proof.

(b) GaP from homework 10.

Claim: For all sets A and B, if $A \subset B$ and $A \neq B$, then $A \not\approx B$.

"Proof": Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ The $A \subsetneqq B$; and in this case it is clear that $A \not\approx B$.

In general, if $A \subset B$ and $A \neq B$, then B clearly has more elements than A has, hence $A \not\approx B$. Therefore, for all sets A and B, if $A \subsetneqq B$ then $A \not\approx B$.

(c) GaP from homework 11.

Figure (4.47) GaP questions used in Dr. Wyatt's transition–to–proof course.

of a general proof. They note the importance of working with specific examples stating "comprehending a proof often involves understanding how the proof could be illustrated by a specific example" (p. 14), but a specific example does not formulate a proof of a general claim. Therefore, while it is important to be able to use examples to demonstrate understanding of a proof, it cannot act as a proof itself.

For the final "proof" being used as an example, the error is a result of not completely understanding the difference between infinite and denumerable sets[26]. As shown in Figure 4.47c, the proof labels the infinite sets as

$$A = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \text{ and } B = \{b_1, b_2, \ldots, b_n, b_{n+1}, \ldots\}$$

[26]Denumerable sets are equivalent to the Natural Numbers, that is they are infinite and countable

which associates each element of A and B with a Natural number. Therefore, A and B are both denumerable. Mejia-Ramos et al. (2012) note the failure to understand key terms hinder students' "ability to comprehend other aspects of a proof" (p. 8); in this case, the proof demonstrates the claim for denumerable sets, but not all infinite sets. So for students to correctly identify this error, they must have a good understanding of the necessary definitions.

The second task asked students to identify if the claim is correct or not. Dr. Wyatt stated:

> For GaP, [the purpose of] the question is to determine if the statement is true or false, one of the explicit questions is asking [the students] to provide a proof or a counterexample. So they should know that for GaP, they should literally know that the claim could be false, could be true or could be false. (Personal Interview, February 12, 2019)

That is to say, even without the directed task of identifying if the claim is correct, the third task from Figure 4.46 tells the students to complete this task. Dr. Wyatt's reasoning for having this task emphasized is because "having the idea part" is important before writing a proof or a counterexample; therefore, explicitly assigning this task to students ensures they think carefully before they begin writing their proof or counterexample.

Of the three claims provided in Figure 4.47, the two claims from 4.47b and 4.47c are incorrect. The claim in 4.47b is incorrect because it supposes A and B are **infinite** sets. This claim does not apply to all infinite sets, but it does apply to **denumerable** sets. Therefore, the error in the claim is similar to the error in the "proof", that is to say the error deals specifically with the definitions of terms, part of the *local domain of proof comprehension* provided by Mejia-Ramos et al. (2012). So the claim "If A and B are denumerable, then $A \approx B$" is true and could be proven using similar methods provided in the "proof."

The claim in Figure 4.47c is incorrect because it applies to all sets. There exists sets that make the conclusion in this claim true (in fact, the sets $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ used in the "proof" from 4.47c are an example); however, it does not exist for all sets. Again, this error can be described by Mejia-Ramos et al. (2012) as "identifying trivial implications of a given statement" (p. 8) which illustrates understanding the meanings of terms and statements. Again, this is part of the local domain of proof comprehension.

The last task assigned in GaP questions is to write a rigorous proof or provide a counterexample. Since the claim in Figure 4.47a is correct, a proof can be constructed. Dr. Wyatt provided the proof in Figure 4.48. The key to constructing this proof requires students to understand three aspects: the proof framework requirement of choosing an arbitrary element in $f \circ f^{-1}$, the relationship between f and f^{-1}, and the definition of a function. The proof framework for a direct proof showing $f \circ f^{-1} \subset I_B$, or generally any proof showing one set is a subset of another, requires the choice of an arbitrary element. This stems from the definition of a subset, A is a subset of B if and only if every element in A is in B. So any element from $f \circ f^{-1}$ will be in I_B; therefore, one should complete the proof using an element representative of all other elements.

Let A and B be non-empty sets and assume that $f : A \to B$ is a function. In order to prove $f \circ f^{-1} \subset I_B$, let $(x, y) \in f \circ f^{-1}$. Then there exist an $a \in A$ such that $(x, a) \in f^{-1}$ and $(a, y) \in f$. Consequently

$$(a, x) \in f \text{ and } (a, y) \in f$$

which implies $x = y \in B$ because f is a function from A to B. Therefore $(x, y) = (x, x) \in I_B$.

Figure (4.48) Dr. Wyatt's solution for the GaP question on homework 8.

The relationship between f and f^{-1} is related to the definition of a function and preimage. The definition of a function from A to B requires each element in A to have exactly one element it maps to in B. Further, the preimage of f, f^{-1}, examines an element in B and maps it to the appropriate element in A. Without these definitions, students cannot construct a logical proof. In short, the key information for writing the proof comes from the local domain of proof comprehension.

The claims in Figures 4.47b and 4.47c are, as discussed previously, incorrect. Therefore, a concrete counterexample is required. It is important to note that both counterexamples provided by Dr. Wyatt's solutions were examples used in class *before* the assignment was collected. Therefore, students had the solution for the third task on both of these questions available to them. In other words, the counterexamples needed could have been *transferred* from one context to another. Mejia-Ramos et al. (2012) identify the ability to transfer concepts from one proof to another situation as being part of the *holistic domain of proof comprehension.* The holistic domain deals with the understanding of a proof as a whole; this includes understanding how applications relate to other contexts.

That being said, to find a counterexample, students need to know what is required in the structure of the counterexample. First, students need to understand what information is given to them, including definitions of terms. Then, they choose a specific example that meets the criteria of the given information. Thirdly, they demonstrate how the chosen example does not produce the desired consequence. While much of the information students need to form counterexamples can be identified in terms of the local domain of proof comprehension, illustrating a proof or, in this case, the falseness of a claim is part of the holistic domain of proof comprehension. Therefore, students need to think actively about the information given in the statement. Dr. Wyatt stated "[students] sometimes provide a counterexample that they think is a counterexample, but it is not" (Personal Interview, February 12, 2019). He said it is often a case of students choosing an example that does not meet the necessary criteria. This can occur due to a misunderstanding of a component in either the local domain or holistic domain. Next, I will discuss students responses to GaP questions discussed above.

Students responses to GaP questions. Dr. Wyatt graded each GaP question out of a total of five points. Table 4.10 shows the grading distribution for the GaP question from two assignments. It is evident that the students performed much better on homework 10 than they did on homework 8. I will analyze why this is the case, providing examples of students work with feedback from Dr Wyatt for each question and how Dr. Wyatt used class time with regards to this type of question. For the GaP from Homework 8, I collected eleven responses from Dr. Wyatt after his grading was complete; ten responses were collected from Homework 10. I will begin by providing examples from each grade range in Table 4.10 for both assessments.

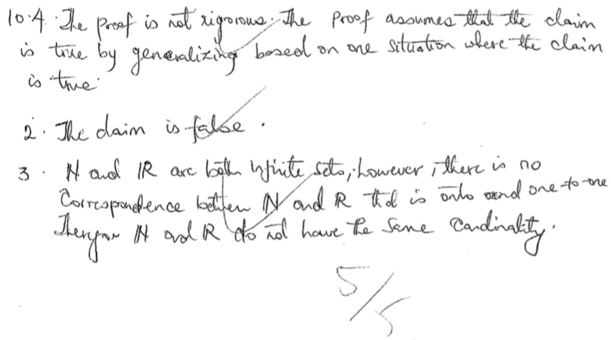
Table (4.10) Point Distribution of Students Grade for GaP Questions

Point Range	Homework 8	Homework 10
(4,5]	2	9
(3,4]	2	0
(2,3]	7	1
(1,2]	0	0
(0,1]	0	0

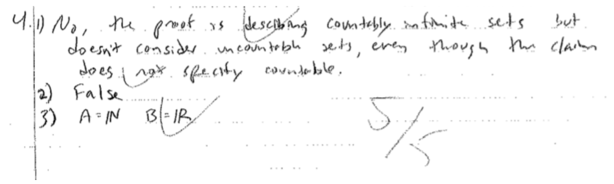
For grades in the $(4, 5]$ range, I first want to note that zero students on homework 8 and seven students on homework 10 received full credit, as such, there is no feedback provided by Dr. Wyatt; however, it is still important to analyze these solutions because it provides a guideline for what constitutes a correct proof. Figure 4.49 shows the solutions of Michaela, Jeremy, and Damian; the remainder of the perfect scores fall into one of these three solutions.

. Notice that Michaela's and Damian's solutions both provide the justification for their counter examples, whereas Jeremy's does not. This appears to be true of Dr. Wyatt's past classes as well. When asked if Dr. Wyatt required students to justify their response to questions on assessments, while the majority of students responded with often having to explain their responses, some students did respond with rarely or never. These solutions suggest that Dr. Wyatt does not require full explanations as long as a correct counterexample is provided. In a study focusing on how mathematicians assign points when grading proofs, Miller et al. (2018) found that half the participants "would not deduct points for omissions from a proof if it was clear to them that the student understood the proof" (p. 30). Therefore, it can be interpreted that because Jeremy provided two sets that are indeed a counterexample, Dr. Wyatt believed the proof was completely understood.

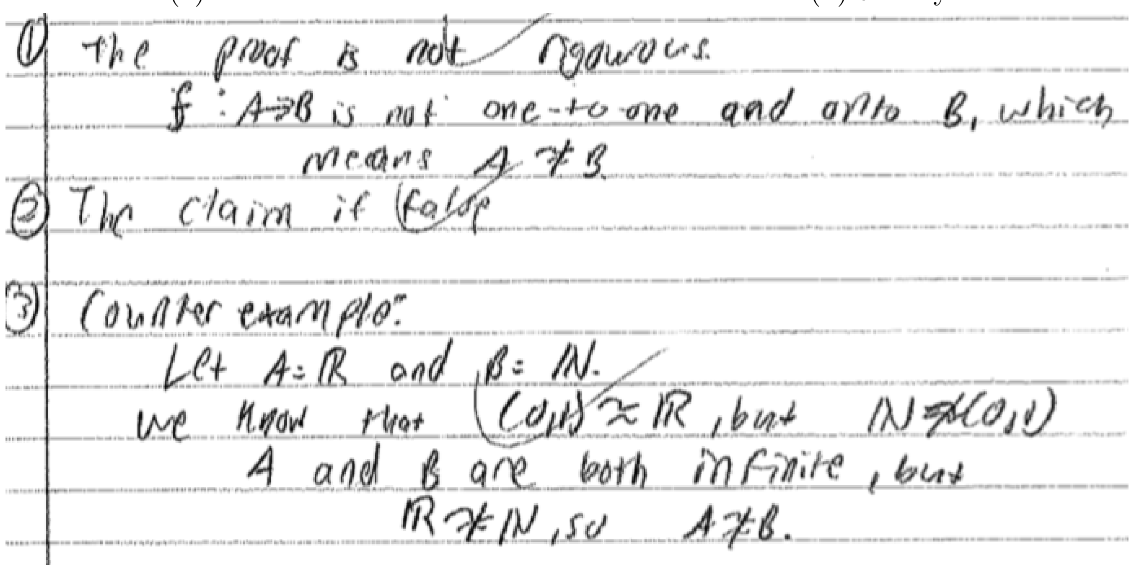
Further, each example states the error in the "proof" in a different way. Michaela described the error as assuming the claim, that is, not choosing an arbitrary element. Jeremy describes the error as the proof only referring to countably infinite sets. Damian states "$f : A \to B$ is not one-to-one and onto B, which means $A \not\approx B$."; however, the function


10.4 The proof is not rigorous. The proof assumes that the claim is true by generalizing based on one situation where the claim is true.
2. The claim is false.
3. N and R are both infinite sets, however, there is no correspondence between N and R that is onto and one-to-one. Therefore N and R do not have the same cardinality.
5/5

(a) Michaela's solution.


4. 1) $\aleph_0$, the proof is describing countably infinite sets but doesn't consider uncountable sets, even though the claim does not specify countable.
2) False
3) A = ℕ B = ℝ
5/5

(b) Jeremy's solution.


1) the proof is not rigorous.
f : A→B is not one-to-one and onto B, which means A ≉ B.
2) The claim is false
3) Counter example:
Let A = ℝ and B = ℕ.
We know that (0,1) ≈ ℝ, but ℕ ≉ (0,1)
A and B are both infinite, but ℝ ≉ ℕ, so A ≉ B.

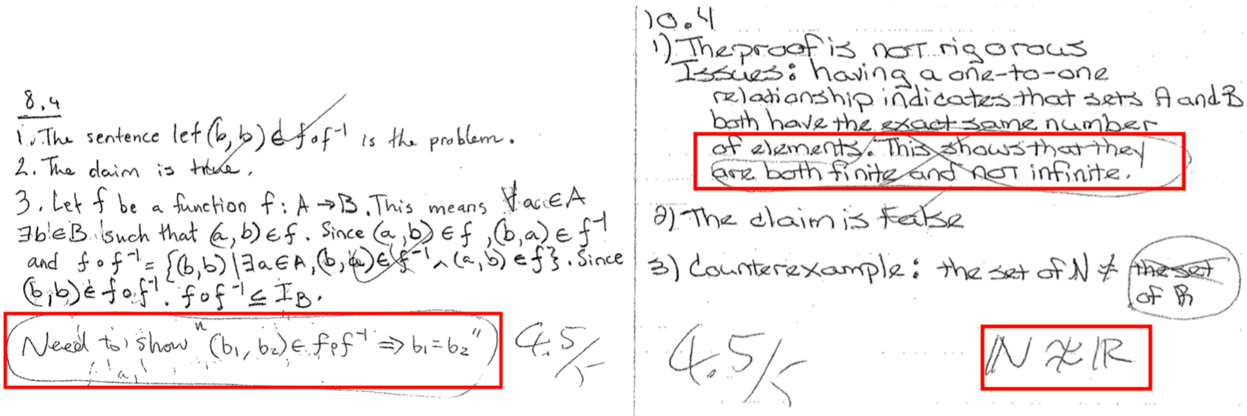
(c) Damian's solution.

Figure (4.49) GaP perfect scores from homework 10.

provided in the "proof" is bijective[27]. Therefore, it is difficult to claim why Dr. Wyatt marked this with full credit.

For solutions that almost earned 5 points, Dr. Wyatt does provide some feedback and general guidance. Figure 4.50 shows Vaughn's and Nadia's solutions from homework 8 and 10 respectively. For both of these instances, Dr. Wyatt provides very short and specific feedback addressing the issues with the solutions. For Vaughn's solution, Dr. Wyatt states "need to show '$(b_1, b_2) \in f \circ f^{-1} \implies b_1 = b_2$'". This tells Vaughn that his work is correct, but he did not finish the proof. Dr. Wyatt commented "$\mathbb{N} \not\approx \mathbb{R}$" after Nadia's counter

[27] $f : A \to B$ is bijective if f is one-to-one and onto B, which means $A \not\approx B$.



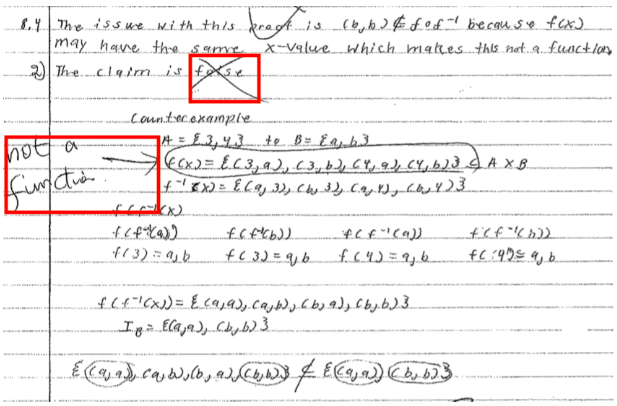
(a) Vaughn's solution form homework 8. (b) Nadia's solution form homework 10.

Figure (4.50) Strong solutions to GaP questions.

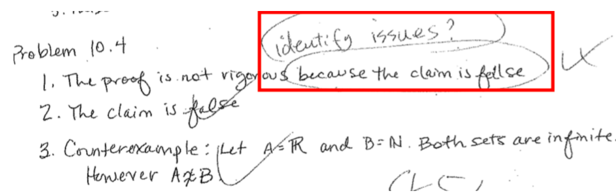
example. This tells her to be aware of her notation. Further, he crossed out part of her reasoning as to why the proof is incorrect, that is, he let her know the additional statement was incorrect. Therefore, Dr. Wyatt's feedback clearly addresses the needs of the students.

The responses from Dalton and Payton received more than half credit on the GaP from homework 8. That being said, their errors were very different, as shown in Figure 4.51. Dalton incorrectly identified the claim as false; Dr. Wyatt's feedback was to identify Dalton's error in his counterexample, specifically that Dalton did not provide a function in his counter example. On the other hand, Payton did not justify why the "proof" was incorrect and did not produce an accurate proof after identifying the claim as correct; in fact, she repeated the mistake from the "proof." Both instances, however, stem from not having a complete understanding of the local domain of proof comprehension. Specifically, Dalton did not correctly apply the definition of a function and Payton did not show evidence of understanding the appropriate proof framework.

Seven people on homework 8 and one on homework 10 provided weak responses to the the GaP question. The common aspect to all of these solutions is that no one received full credit on any part. For the first task, most students provided an incorrect reason for why the "proof" was invalid; however, there were two that provided no reasoning. The third task, while an attempt was made to provide a solution, there were small errors in the work.



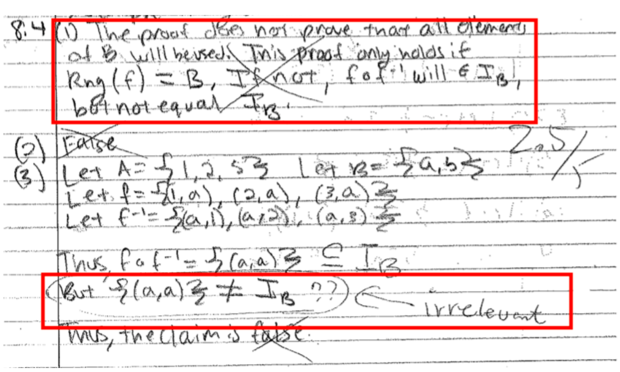
(a) Dalton's solution form homework 8.



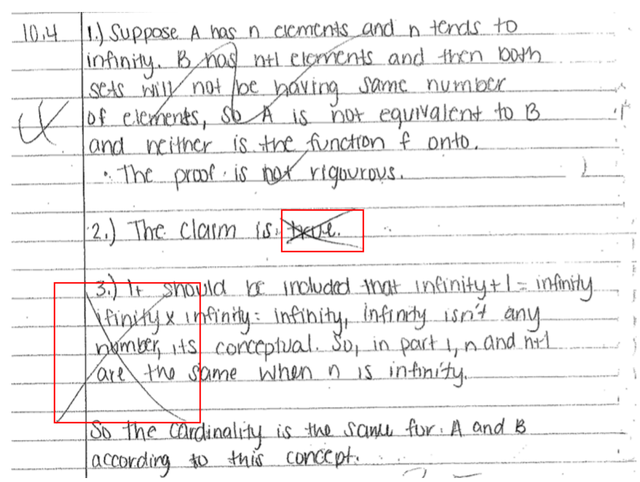
(b) Payton's solution form homework 10.

Figure (4.51) Average solutions to GaP questions.

Olivia's and Kimberly's responses, shown in Figure 4.52, exemplify these types of responses. It is important to note that Dr. Wyatt provided little feedback on these solutions, often doing no more that crossing out irrelevant information.



(a) Olivia's solution form homework 8.



(b) Kimberly's solution form homework 10.

Figure (4.52) Weak solutions to GaP questions.

Overall, Dr. Wyatt seems to be very clear with his feedback, offering just enough information to highlight the students' error. For the students' solutions themselves, he requires just enough information to be able to infer the students understand the topic. As noted by Miller et al. (2018), this appears to be a common mindset for mathematicians as they grade proofs. In the next part, I will analyze a different style of question that still relies heavily on students being able to read and comprehend proofs and statements.

4.3.3 "Analyzing a Proof" question type

Dr. Wyatt was intentionally selected as a subject for this study because of his involvement in the NSF research project alongside mathematics education specialists from multiple universities. As mentioned previously, the projects' purpose is, in part, to develop assessments that accurately measure students ability to comprehend various proofs. Dr. Wyatt modeled the questions he refers to as **Analyze a Proof (AaP)** after the types of questions used the NSF project.

In fact, Dr Wyatt noted "I included this example because in the so called [NSF project] they have that type of question, type of problem. So therefore, I think "yeah, I will include it." But nevertheless, this is a very good type of problem to be included in my homework assignments." Therefore, his involvement in [NSF project] is the sole reason he thought to include this type of question. However, it is also evident that Dr. Wyatt would not use AaP questions if he did not feel they had a purpose that other questions he already used did not cover. In the following pages I will analyze the purpose of AaP questions, the format of AaP questions, and students solutions to the AaP question from homework 9.

The purpose of AaP questions. Dr. Wyatt stated the purpose of AaP questions is " to help students, uh learn the proof structure of different kind[s of proofs,] to help them recognize different proof structures and also be able to help them, uh fill in the so called gaps within a proof. " Further, Dr. Wyatt acknowledges AaP questions inform him about the amount of detail he needs to use when presenting proofs. He feels this is vital because " some textbooks give very sketchy proofs. If I don't do [AaP] questions I might just think 'oh, these sketchy proofs are perfect, I don't need to do anything in class I can just do the sketchy proof' you know, this is the proof, just learn it " (Personal Interview, February 12, 2019). That is to say, Dr. Wyatt believes AaP questions help students learn how to read and comprehend "sketchy" proofs given in textbooks and, as noted Lew et al. (2016) and Weber et al. (2016), sometimes in lecture. Therefore, Dr. Wyatt acknowledged the existence of these "sketchy" proofs and, due to his participation with the NSF project group, recognized that posing the AaP questions is a useful way to counteract "sketchy" proofs.

The format of AaP questions. Of the collected assessments, only homework 9 had an AaP question included. This question appeared as the last of four questions, that is, in the same location as the GaP questions. According to Mejia-Ramos et al. (2017), questions similar to Dr. Wyatt's AaP questions can be generated by using both the local and holistic domains of proof comprehension; in fact, the framework that provides the outline of AaP questions was developed by Mejia-Ramos et al. (2012) and used in the NSF study. Therefore, Dr. Wyatt needed to decide on the claim to prove, the proof to provide students, and what tasks to assign to measure students comprehension.

Mejia-Ramos et al. (2017) acknowledge AaP questions can be "time consuming to generate and grade," so they opted to formulate questions around claims that are well known[28]. Dr. Wyatt takes a similar approach with his questions. He states " the claims in the AaP, they are all so called well known claims... in other words it is very likely [students] have seen these claims ". Figure 4.53 provides the claim and proof that Dr Wyatt used on homework 9.

Further, the proof is a "popular" proof for the given claim. The reason for this is similar to that of the claim, the type of proof the students analyze is likely something they have seen. Dr. Wyatt prefers the well known claims and proofs because "if [the students] have the right exposure, they know, they should know that [AaP questions use] correct proofs and correct claims". The simple fact that students can look up these claims and proofs, if they desire, separates AaP and GaP questions.

The tasks assigned in AaP questions are based on the local and holistic domains of proof comprehension. For this question, Dr. Wyatt formulated the tasks, shown in Figure 4.54

[28]For example, there exist an infinite amount of prime numbers.

Claim: $\sqrt{2}$ is not a rational number.

Proof: We prove the statement using the following steps

(a) Suppose that $\sqrt{2}$ is a rational number.

(b) So there exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z}\backslash\{0\}$ such that $\sqrt{2} = \frac{a}{b}$

(c) We can further assume that a and b do not have a common divisor with absolute value greater that 1.

(d) Note that a^2 is even.

(e) Consequently, we conclude that a is even. So write $a = 2k$, with $k \in \mathbb{Z}$.

(f) Then $b^2 = 2k^2$, which shows that b^2 is even.

(g) Thus b is even.

(h) But this is impossible.

(i) So $\sqrt{2}$ is not a rational number.

Figure (4.53) The claim and proof provided in the AaP question.

1. Explain why (d) holds. Provide your justification.
2. Explain why (d) implies (e). Provide your justification
3. Explain what is impossible, as claimed in (h). Provide your justification.

Figure (4.54) The tasks assigned in the AaP question.

to focus on the justification of claims, part of the local domain of proof comprehension. I believe justification of claims was the focus of this question because of his view on "sketchy" proofs; by evaluating students in this regard ensures Dr. Wyatt that the students can read textbooks with these kinds of proofs.

Student solutions to the AaP question. Dr. Wyatt graded the AaP question out of a total of five points. Table 4.11 shows the grading distribution for the AaP question from homework 9. It is important to note that two students received perfect scores while one student did not attempt the question. The remainder of the students score either a 4 or a 3.5. The following paragraphs will analyze examples of a perfect score, a score of a 4, and a score of a 3.5.

Table (4.11) Point Distribution of Students Grade for GaP Questions

Point Range	Homework 8
(4,5]	2
(3,4]	7
(2,3]	0
(1,2]	0
(0,1]	1

Both of the perfect scores provided very thorough solutions, one of which is provided in Figure 4.55. For the first task, both showed the complete calculation to proceed from (b) to (d), that is they began with the definition of a rational number and solved for a^2 to show that $a^2 = 2b^2$. For the second task, both students examined what happens to a^2 when a is odd and a is even; that is, they produced a mini proof examining the parity of a. With the third task, both used the fact that if both a and b are even, then $\frac{a}{b}$ is reducible, which contradicts the assumption in (c). In other words, both took the provided "sketchy" proof and added all of the detail required to create a full proof. Thus, they created the necessary modules (an aspect of the holistic domain) by providing justifications for the claims (an aspect of the local domain) to complete the proof.

All of the students who scored a 4 out of 5 did not correctly complete the second task, that is explaining why a^2 being even implies that a is even. Further, all of the errors occurred because the students could not determine how to correctly justify this claim. As exemplified in Figure 4.56, most students recognized the connection as true, but could not formulate an argument demonstrating this fact. Further, Dr. Wyatt provided minimal feedback to this difficulty, in essence stating the information provided by Vaughn with the phrase "prove this."

Similarly, students that scored 3.5 out of 5 tended to have parallel difficulties. All three students with this score did not provide a full justification for the second task; with the exception of Nadia's solution, Dr. Wyatt did not provide any feedback to guide the students toward the correct justification. Further, the students did not correctly justify the first task,

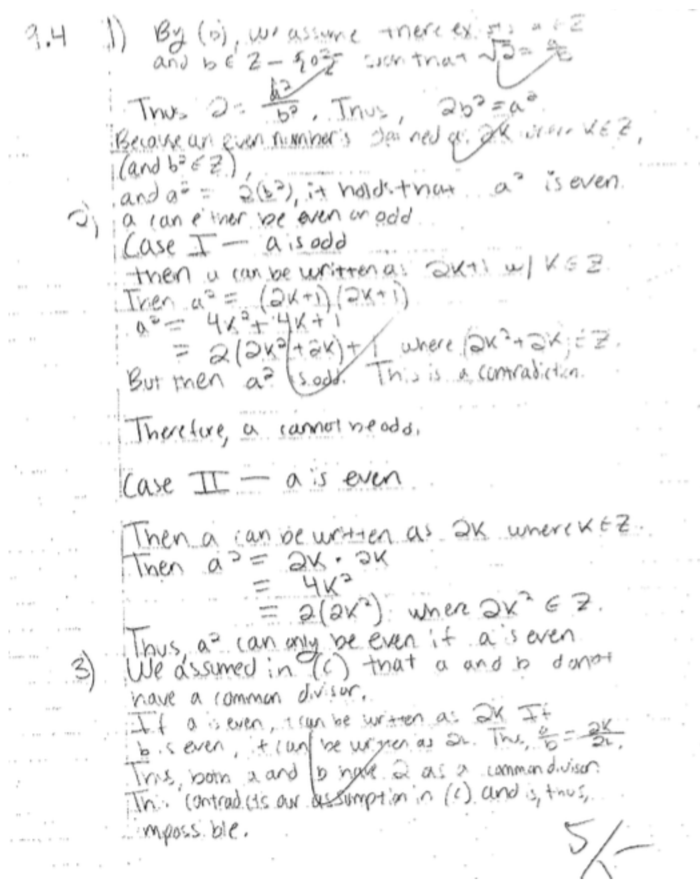


Figure (4.55) Olivia's solution to the AaP question.

as shown in Figure 4.57. In the case of Payton, Dr. Wyatt supplied an example to help fix the misconception, but did not provide further feedback.

I believe the primary reason Dr. Wyatt did not provide much feedback on the assessment itself is because he provided a complete solution guide to the students. That being said, the students have to look up that document, which some may not do. Therefore, some students may be relying solely on the feedback from Dr. Wyatt.

Overall, the AaP provided Dr. Wyatt with a glimpse into the students misunderstanding of this proof. Most of the students had difficulty filling in the missing parts of the proof, meaning that they are not as prepared for dealing with "sketchy" proofs. However, with this knowledge, he can evaluate how he discusses proofs in class to assist the students. Next, I will analyze the questions from Test II, looking at their format, score distribution, and compare them to the homework questions.

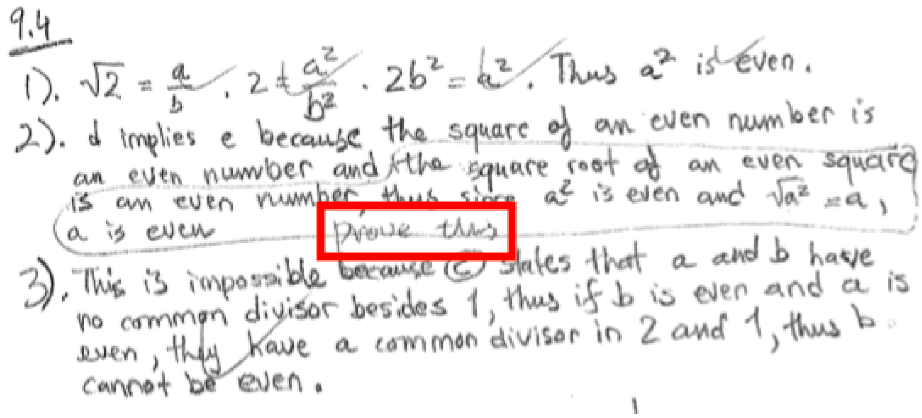
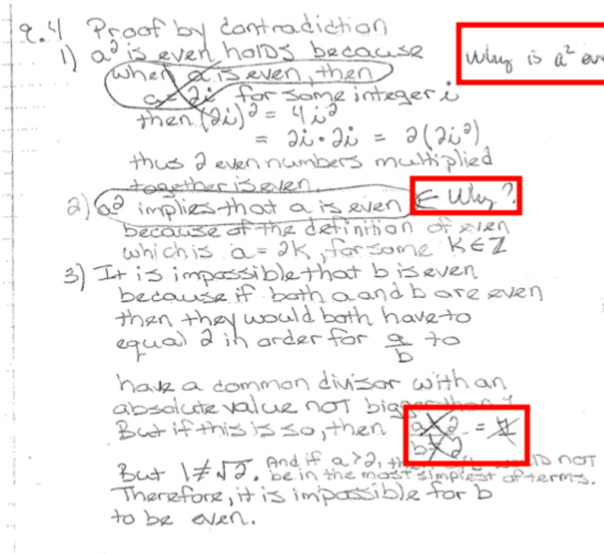


Figure (4.56) Vaughn's solution to the AaP question.

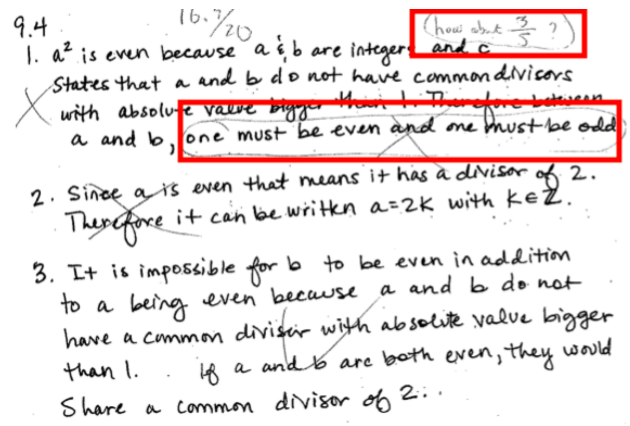
4.3.4 Test II

Test II was administered in late October, one week after my observations began. When he assigned Homework 8, Dr. Wyatt also gave his class a review sheet for the exam. The way this review sheet was formatted was to inform the students not only what topics were on the test, but also which questions from their homework assignments corresponded to each topic. For example, one of the topics listed was "Reflexive, (anti)symmetric, and transitive relations" followed by the list of problems associated with the topic, specifically Questions 6.1, 6.2, 6.3, 7.1, 7.4, 8.1, 8.4.

The test itself was similar to the homework assignments, though it consisted of five questions (each worth five points) and an extra credit question (worth one bonus point). The primary focus of this test was to assess students ability to use the Principle of Mathematical Induction (PMI) and the Principle of Complete Induction (PCI). Further, there was also a question focusing on calculations of composite relations, similar to the calculation questions from the homework. The fifth question was an evaluation question, where five statements were given and the students needed to determine if they were true or false, without the need to justify. The extra credit question asked students to either prove or disprove a claim. In the following sections, I will analyze the questions on the test and student responses, with feedback. To start, I will look at the format of each question (or type of question). Then I will provide a point distribution for the students responses as well as sample solutions for



(a) Nadia's solution.



(b) Payton's solution.

Figure (4.57) Solutions to the AaP question earning 3.5 out of 5 points.

each type of question.

Format of the questions from Test II. Each question, worth a total a five points, can be described as either a *proof writing question, a calculation question*, or an *evaluation* question. Proof writing questions require students to write a complete proof of a statement from scratch; on this test the first three questions all required students to write an inductive proof, either using the PMI or the PCI. The fourth question was a calculation question where the students were given several sets and asked to calculate composite relations. The evaluation question tasked the students with examining the statements pertaining to transitivity and (anti)symmetric relations. The extra credit problem was a combination problem in the sense that students needed to first evaluate the validity of a statement and then either write a proof or provide a concrete counter example.

Figure 4.58 states each of the first three questions from Test II. It is important to note that each of these questions provide the proof framework for the students to use, that is to say the various types of inductive proofs. Further, the ordering of the questions is the same order in which the students would have learned the different types of inductive proofs. In

Use the Principle of Mathematical Induction (PMI) to prove $3|(n^3 - 19n)$ for all $n \in \mathbb{N}$.

(a) Question 1.

Use the Principle of generalized Mathematical Induction (PMI) to prove $\prod_{i=2}^{n} \frac{i^2-1}{i^2+2i} = \frac{3}{n^2+2n}$ for all $n \geq 2$.

(b) Question 2.

Define $a_1 = 5$, $a_2 = 25$, and $a_{n+1} = 9a_n - 20a_{n-1}$ for all $n \geq 2$. Use the Principle of Complete Induction (PCI) to prove $a_n = 5^n$ for all $n \in \mathbb{N}$.

(c) Question 3.

Figure (4.58) Proof writing questions from Test II.

order to complete these questions, students need to completely understand how to construct an inductive proof as well as the any pertinent definitions.

The homework assignments I have access to did not discuss PMI and PCI specifically; those topics were covered prior to the start of my observations. However, judging from the review Dr. Wyatt provided his students, the entirety of homework 5 was designed around inductive reasoning and proofs. This suggests, that unlike the homework assignments I collected, it focused less on the definitions as the basis of the content and had a more proof focused approach. Also, there would have been one GaP or AaP question constructed around inductive reasoning.

Figure 4.59 states the calculation question from Test II. The purpose of this question is

Let $X = \{1, 2, 3, 6, 7, 8\}$, $R = \{(1, 3), (2, 1), (6, 8), (8, 2)\}$, and $S = \{(1, 1), (3, 7), (6, 2), (7, 8)\}$. Determine each of the following explicitly.

1. $S \circ S$
2. $R \circ S$
3. $S \circ R$
4. $R^{-1} \circ S^{-1}$
5. $S^{-1} \circ R^{-1}$

Figure (4.59) The calculation question from Test II.

to assess students' understanding of composite functions and inverse functions. As with the homework questions discussed earlier, in order to be successful in answering these questions students need a complete understanding of the definitions involved. Also, the calculations are similar to those of the homework questions in the sense that students should be familiar with the processes involved from previous courses.

The final required question and the evaluation question from Test II is given in Figure 4.60. This question is designed to assess students' knowledge of the definitions of transitive,

Determine if the following statements are true or false, in which X and Y are non–empty sets.

1. $D = \{(a, b) \in \mathbb{N} \times \mathbb{N} : \ a|b\}$ is an antisymmetric relation on $\mathbb{N}$.
2. $R = \{(1, 2), (2, 3), (3, 1)\}$ is a transitive relation on $\mathbb{R}$.
3. $S = \{(1, 4), (2, 5), (3, 6)\}$ is an antisymmetric relation on $\mathbb{Q}$.
4. If a relation T on X is not antisymmetric, then T is symmetric.
5. A relation U on Y is transitive if and only if $U \circ U \subseteq U$.

Figure (4.60) The evaluation question from Test II.

symmetric, and antisymmetric relations and how they are related. Similar to the homework questions, Dr. Wyatt left a hint for the students on this question instructing them they do not need to justify their responses.

The extra credit question is given in Figure 4.61. There are a few things to note

Prove or Disprove: For *all* sets $A,\ B,$ and C, if $A \times B = A \times C$ then $B = C$.

Figure (4.61) Extra credit question from Test II.

about the expectations Dr. Wyatt has of students for this question. First, he clearly states that this question is worth one point and that no partial credit will be given. Second, he does provide a hint instructing students to "provide either a rigorous proof or a concrete counterexample;" this was the same tasks as assigned to the students on part three of GaP questions. The main difference between this question and the GaP questions is that the students do not have a sample proof to evaluate first.

As far as the overall format of the test, the order of the questions is an important factor to consider. Kuhn & Kiefer (2013) comment that one consideration of the Australian Educational Standards with regards to mathematical assessments is the placement of questions on assessments. Specifically, “easier items [appear first] in an attempt to raise test motivation in students” (p. 196); this concept is also mentioned by Speer et al. (2010) who state “harder problems placed early in [the assessment] may well lower overall performance” (p. 111) because students either mismanage their time or lose confidence because of the perceived difficulty of the questions. Therefore, it could be argued that beginning the assessment with three proofs covering PMI and PCI may cause students to become unmotivated. This is due to the fact students will have had limited experience constructing proofs, where as performing the calculations in question 4 (Figure 4.60) may be easier, and thus increase confidence, because students have extensive experiences performing calculations, even if the basic format of the calculation is new.

The questions themselves are consistent with a transition to proof course, or proof heavy courses in general. Miller et al. (2018) note that, through a personal conversation with Annie Seldon, approximately 80% of questions in an average real analysis book are proof oriented, that is require students to construct proofs. Therefore, transitioning to more questions involving proof construction on assessments is beneficial. Further, the questions are similar to those found on Dr. Wyatt’s homework assignments. Overall, the students should have expected the assessment to be structured in this manner. Next, I will analyze students solutions to these questions, with an emphasis on the feedback provided by Dr. Wyatt.

Samples of students’ responses to the questions from Test II. Table 4.12 shows the scoring distribution for each of the questions on Test II. Students performed well on questions 1, 2, and 4, with perfect scores being earned by 4 students total between questions

Table (4.12) Score Distributions for Test II

Range	Question 1	Question 2	Question 3	Question 4	Question 5
$(4,5]$	9	7	3	7	2
$(3,4]$	1	3	2	3	7
$(2,3]$	1	1	4	1	2
$(1,2]$	0	0	2	0	0
$(0,1]$	0	0	0	0	0

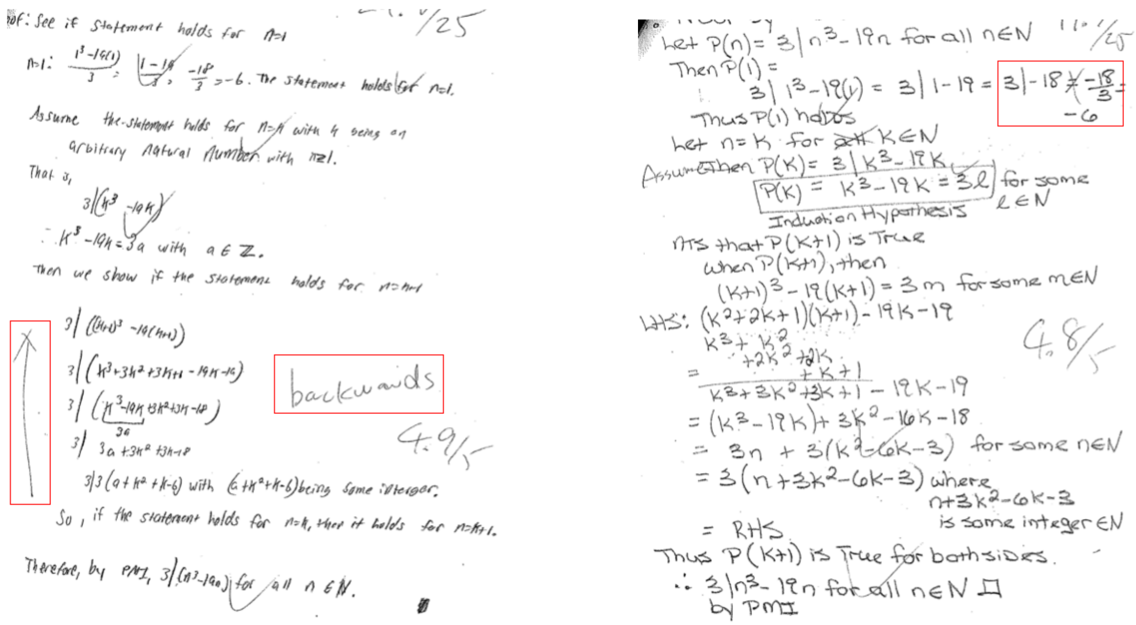
1 and 2 and by 3 students on question 4. Students struggled on questions 3 and 5; one student earned a perfect score on question 3. Both students who scored in the highest range on question 5 had perfect scores. The other scores in the range $(4,5]$ on question 3 were a 4.5 and a 4.9. Further, question 3 had the only scores in the range $(1,2]$[29].

In the following paragraphs I will examine students solutions from Test II. I am ordering the discussion based off of performance instead of topic. Therefore, I will begin by examining solutions to questions 1 and 2. Next, I will analyze question 4. Questions 3 and 5 will be discussed last because they will have the most opportunities to discuss Dr. Wyatt's feedback.

Test II questions 1 and 2. The skill being assessed in the first two questions is how to construct a proof using PMI. That being said, question 1 is examining just students' ability to construct a proof using PMI while question 2 is assessing the same skill using the generalize version of PMI. Figures 4.62 and 4.63 provide students responses to questions 1 and 2 in the $(4,5]$ range. The errors that occurred in these two examples both derive from the local domain of proof comprehension. Figure 4.62a shows the only error made by Damian was he work the algebraic part of his proof backwards; in other words, the error was in the structure of the proof itself, not with the content of the proof. So Damian's error is with the proof framework itself. On the other hand, Nadia's error in Figure 4.62b is with the definition of the of divisibility itself; to correct this, Dr. Wyatt simply eliminates the equality statement made by Nadia. Otherwise, her proof is correct.

Figure 4.63 shows Jordyn's solution to the second test question. Her error is a nota-

[29] Both earned a score of two.



(a) Damian's solution.

(b) Nadia's solution.

Figure (4.62) Solutions in the range $(4, 5]$ for question 1.

tional one. Specifically, she states that she is writing $P(k+1)$ in general terms using product notation, but then writes out a correct, but misplaced statement. Dr. Wyatt corrects this by identifying the problem and crossing it out. However, this error does not affect the structure of the proof, just how the structure was being demonstrated. Thus, Dr. Wyatt scored this as a good solution.

Figure 4.64 shows Dalton's solution to question 2; this response scored in the interval $(3, 4]$ and is typical of a solution within this range. Dalton has some key errors in his solution. First, he does not verify that the result is true in the base case, that is when $n = 2$. Instead, he shows the case when $n = 3$. Further, Dalton does not provide an induction hypothesis. Both of these instances shows a misunderstanding of the structure of a generalized PMI proof. That is to say, he does not fully understand the proof framework, an essential part of the local domain of proof comprehension. Mejia-Ramos et al. (2012) state "a reader needs to not only identify the logical status of statements in proofs but also recognize the logical relationship between the statement being proven and the assumptions and the conclusions of a proof" (p. 9). It appears Dalton does not understand this relationship. Dr. Wyatt

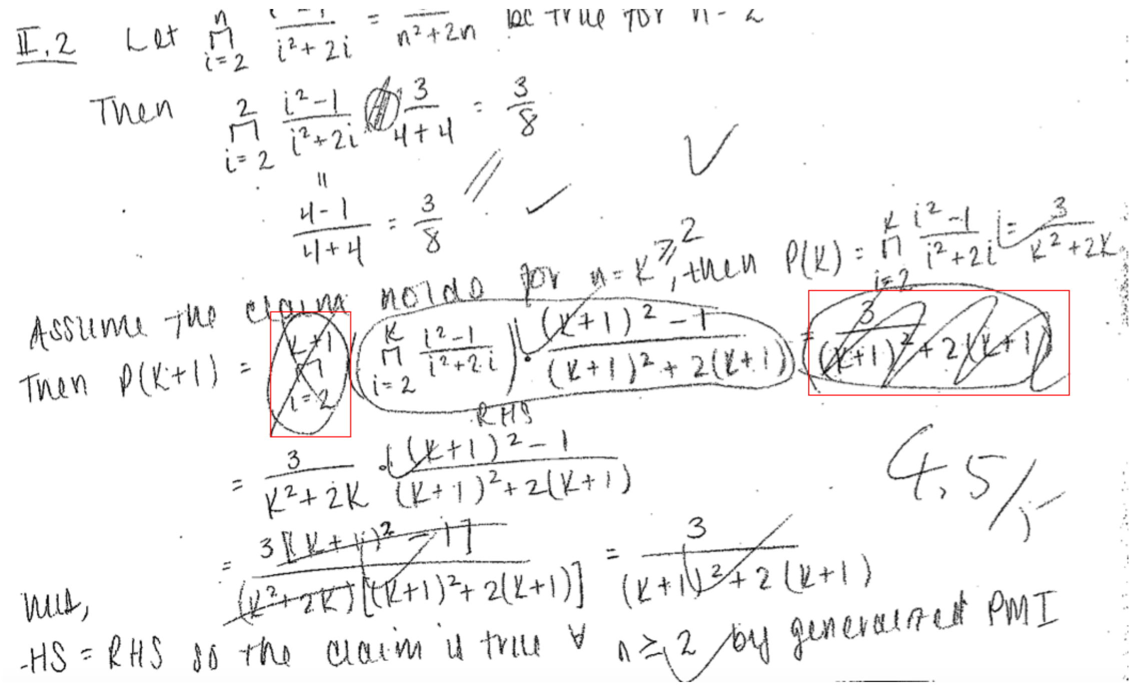


Figure (4.63) Jordyn's solution to question 2.

provides feedback to guide Dalton with how to complete the proof correctly; specifically, that Dalton needs to "verify $P(2)$" (the base case) and to provide an induction hypothesis. Dr. Wyatt does not go into detail on how to accomplish these tasks.

Figure 4.65 provide Jordyn's solution to question 1 and Allyssa's solution to question 2 from Test II. The primary error in both solutions is concerned with the induction hypothesis, but they are very different errors. Jordyn's solution in 4.65a stated in her induction hypothesis that "$3|(n^3 - 19n)$, when $n = k$ **[for] some** $\mathbb{N}$;" this statement is incorrect for an induction proof because it implies that the result is true for specific values of k. The structure of the PMI proofs is that the result holds for all natural numbers; Dr. Wyatt corrects this by writing "$k \in \mathbb{N}$". All other errors in this problem are computational. In contrast, Dr. Wyatt comments on Allyssa's solution to question 2 (Figure 4.65b) that she provided the "wrong induction hypothesis," but he does not provide a correct response. All other errors on these two questions appear to be computational.

Overall, the primary issue students had with PMI and the generalized PMI is the induction hypothesis. Therefore, it appears students are struggling with the overall proof

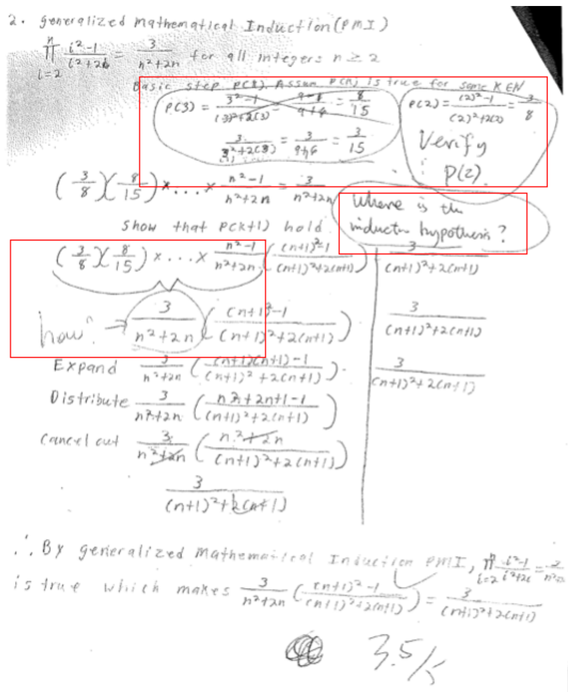


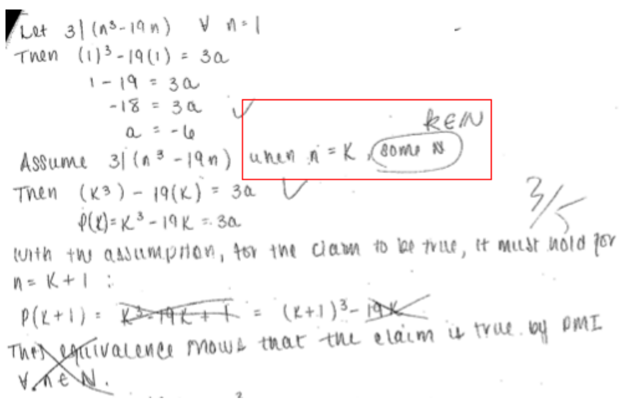
Figure (4.64) Dalton's solution to question 2.

framework associated with induction questions. I will now move to the analysis of Question 4.

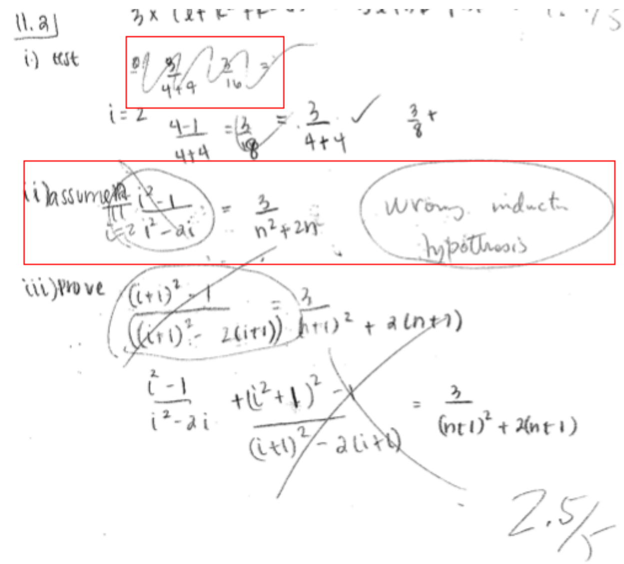
Test II question 4. For students to be successful in answering this question, they must have an understanding of inverse relations and compositions of functions. Specifically, they must know how to find the inverse relation and how to calculate composition of two relations. Most of the errors committed by students was either because of an omission or an incomplete understanding of key definitions. In the following paragraphs I will provide examples of students' solutions that scored in the $(4, 5]$ and $(3, 4]$ range.

Figure 4.66 shows the solutions provided by Kimberly and Michaela. Kimberly only provided a partial solution to $S^{-1} \circ R^{-1}$. The full solution is $\{(3,1),(1,6),(2,7)\}$ and she only listed $(3,1)$ and $(2,7)$. Similarly, Kimberly omitted the ordered pair $(1,1)$ from her solution to $S \circ S$. Dr. Wyatt does not provide feedback to these answers other than to mark them partially incorrect.

Jordyn's and Jeremy's solution to question 4 both earned four points and are provided

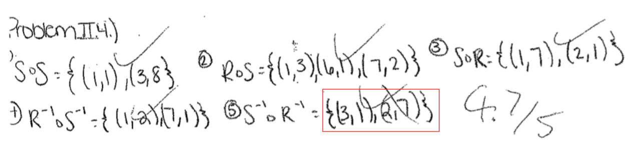


(a) Jordyn's solution to question 1.

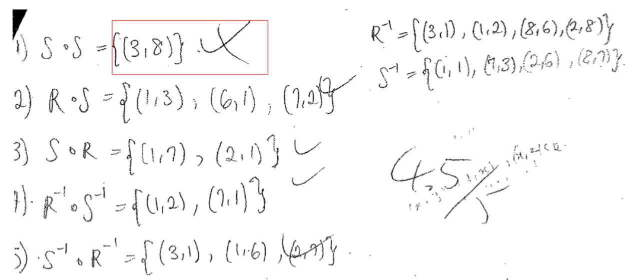


(b) Allyssa's solution to question 2.

Figure (4.65) Solutions to questions 1& 2 scoring in the range $(2, 3]$.



(a) Kimberly's solution.

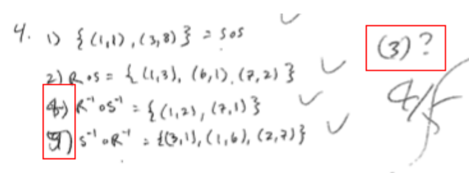


(b) Michaela's solution.

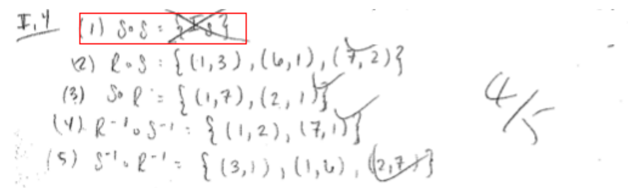
Figure (4.66) Solutions to question 4 scoring in the $(4, 5]$ range.

in Figure 4.67. Jordyn's error stems from a misunderstanding of a definition, which is consistent with most errors on this question. She lists "$S \circ S = \{I_S\}$," that is she claims the solution is the identity of set S. While I believe she was referring to the ordered pair $(1, 1)$; however, this is a misunderstanding of the definition of an identity. In fact, S does not have an identity element.

Jeremy's error was included because it is unique in terms of every solution on every assignment. He made no mathematical errors; however, he omitted a part of the question. While omissions have been examined, this one occurred because he numbered the parts



(a) Jeremy's solution.



(b) Jordyn's solution.

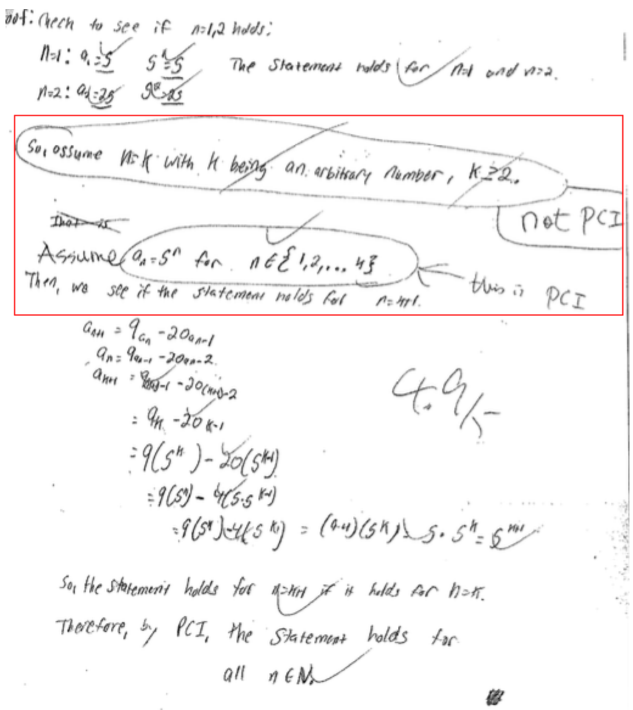
Figure (4.67) Solutions to question 4 scoring in the range $(3, 4]$.

incorrectly; specifically, he numbered part 4 and 5 as parts 3 and 4 respectfully. Since his "part 4" was the last calculation being asked, he believed he had provided all the necessary solutions. Dr. Wyatt provided feedback by changing Jeremy's numbering and then asking where is part 3. This demonstrates how Dr. Wyatt reacts to a student not reading the question correctly.

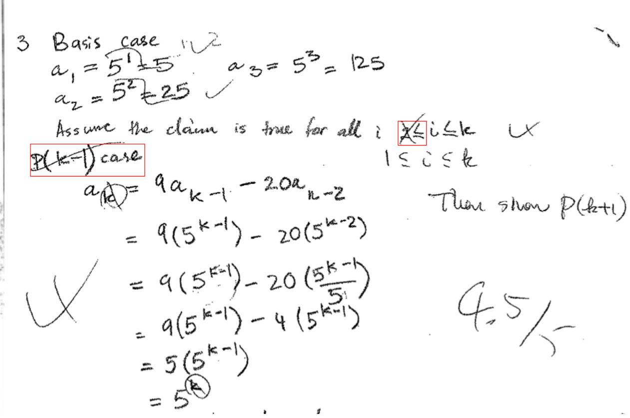
With the exception of Jeremy, all of the errors on question 4 arose from a misunderstanding of the definitions being assessed. With both Kimberly's and Michaela's solution, the definition in question is that of a composition of functions itself. Clearly, they know how to calculate compositions, but each omitted an element from one of their sets. Jordyn does not properly understand the definition of an identity. Like his homework assignments, Dr. Wyatt placed an emphasis on the definitions being used. Next, questions 3 and 5 will be analyzed.

Test II questions 3 and 5. Questions 3 and 5 had the fewest scores in the highest range of any questions. Also, question 3 is the only question with scores in the range $(1, 2]$. The following paragraphs discuss some of the solutions from each range of scores. The range $(4, 5]$ will be analyzed first. Next, the middle ranges of $(3, 4]$ and $(2, 3]$ will be analyzed together because the errors are similar. Lastly, the range of $(1, 2]$ will be examined.

The solutions provided by Damian and Vaughn, shown in Figure 4.68, both contain minor notational errors. Damian stated two induction hypotheses that he believed said the same thing, but where in fact different. He first list an induction hypothesis that would be used in PMI questions; Dr. Wyatt points out that this is not the induction hypothesis



(a) Damian's solution.



(b) Vaughn's solution.

Figure (4.68) Solutions to question 3 with minor errors.

needed for PCI. Damian then list the proper induction hypothesis and Dr. Wyatt points out that this is what is needed for PCI.

Similarly, Vaughn's errors are purely notational. He lists the set from the induction hypothesis incorrectly and denotes the third step as $P(k-1)$ instead of $P(k+1)$. In fact, he performs all of the calculations correctly for $P(k)$, which he assumed to be true. Dr. Wyatt simply marked most of the incorrect notation wrong except for the definition of the set, which he wrote the correct version.

Olivia's solution, provided in Figure 4.69, has errors that are minor but significant. Dr. Wyatt comments on the significance of these errors in a complete and concise way. He comments that her choice of k eliminates the possibility of $P(4)$, that is to say, the way she defined the set that PCI is being completed on makes it impossible to calculate the fourth case. Further, she showed the solution for "$P((k+1)+1)$" which omits the case $P(k+1)$ based on her definition of the set. These errors suggest Olivia was struggling with the basic

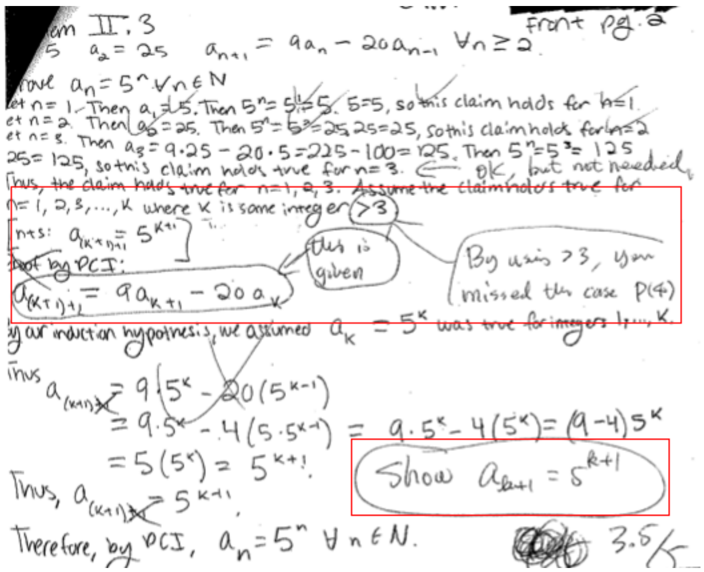


Figure (4.69) Olivia's solution to question 3.

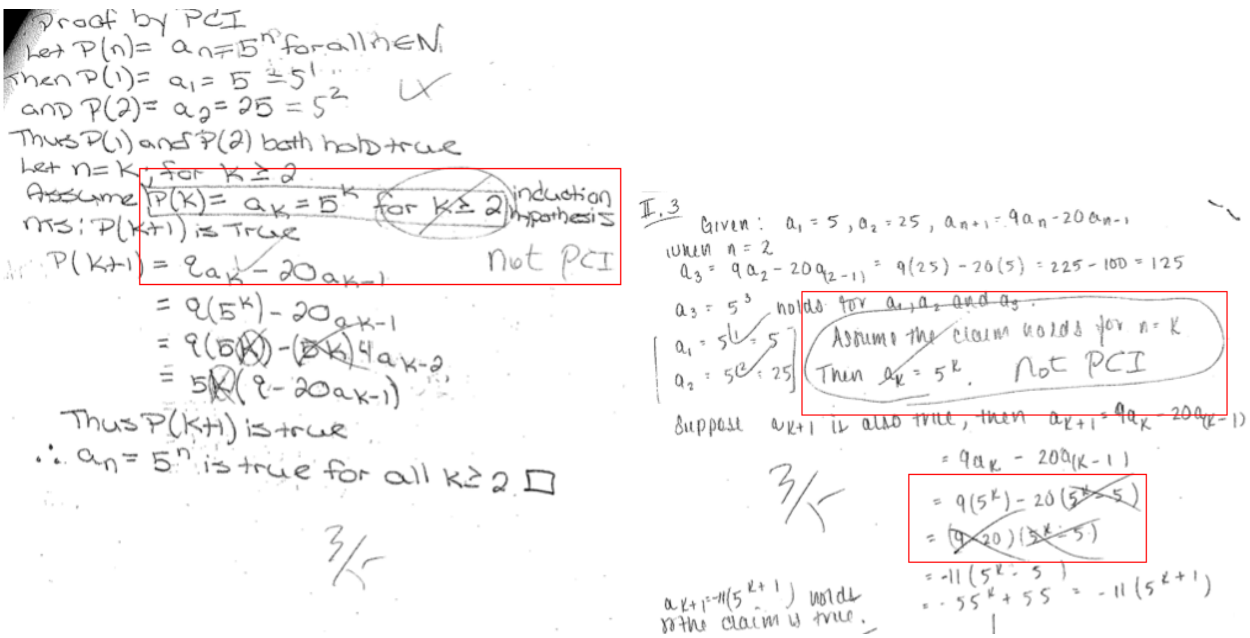
structure of the PCI proof method.

Figure 4.70 shows solutions from Nadia and Jordyn[30]. Both students scored in the $(2, 3]$ range and had similar errors. Specifically, neither used the correct induction hypothesis for PCI. Also, in both cases Dr. Wyatt remarked that their solutions were "not PCI." Again, this shows a misconception about the structure of PCI in general.

Both Dalton and Allyssa earned a score of 2, as shown in Figure 4.71. Dalton had three major errors. First, he did not verify both $P(1)$ and $P(2)$, the base case of this proof. Second, he did not provide a visible induction hypothesis. Finally, he proved a statement that he was not meant to prove. For the first two of these errors, it again appears Dalton did not understand the basic structure of a PCI question; Dr. Wyatt comments on the former by writing "verify $P(1)$ and $P(2)$" and the latter by asking where is the induction hypothesis. Both of these comments are consistent with Dr. Wyatt's methods of feedback overall, that is they are short statements.

Allyssa's solution has errors in two parts. First, she does not state an induction hypothesis consistent with PCI. Second, she does not provide a proof. Dr. Wyatt comments that

[30]Jordyn's solution to this question spanned more than one page. I included the page that had feedback.



(a) Nadia's solution.

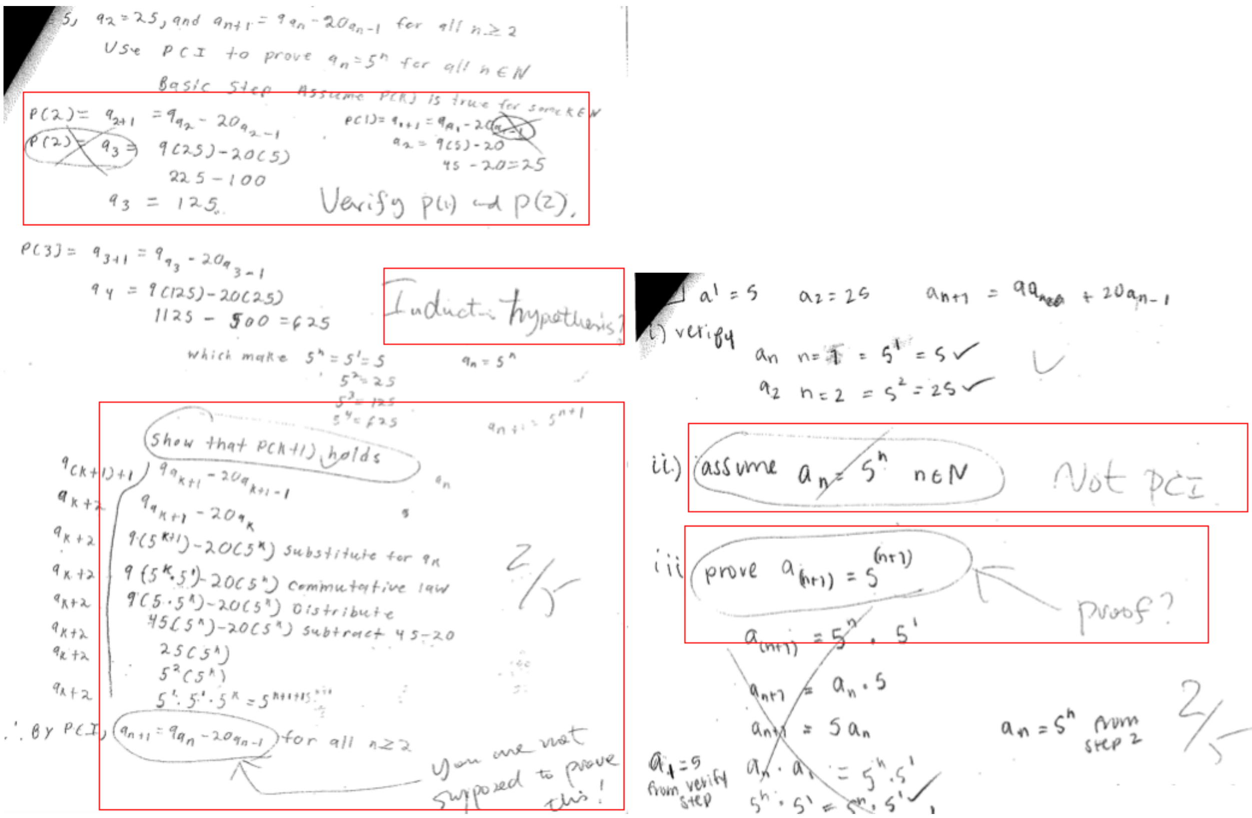
(b) Jordyn's solution.

Figure (4.70) Solutions to question 3 in the range $(3, 5]$.

the induction hypothesis was incorrect, but only asks where is the proof. She did provide a correct base case, demonstrating she has an idea about the proper structure of PCI; however, with an incorrect induction hypothesis, this shows that there are still misconceptions about this structure.

Question 5 tasks students with identifying statements as either true or false. Due to the binary nature of these responses, Dr. Wyatt did not provide feedback on these solutions past marking them incorrect. For this reason, none of the solutions are provided; however, I wanted to note that Dr. Wyatt's methods of providing feedback on these types of questions was consistent with those from his homework.

In short, the errors students had with their solutions to questions 3 and 5 are a result of a misconception in the local domain of proof comprehension. For question 3, the misconception manifested itself in the form of an incorrect proof structure, that is to say, students did not understand the difference between PMI and PCI. On question 5, the errors were the result of not fully understanding the definitions of antisymmetric, symmetric, and transitive. In both instances, Dr. Wyatt was consistent with his feedback to students, providing very short



(a) Dalton's solution. (b) Allyssa's solution.

Figure (4.71) Solutions to question 3 scoring in the range of $(1, 2]$.

statements when needed or marking a question as incorrect in the case of binary responses.

Summary of Assessment Practices It is clear that Dr. Wyatt has clear goals for every question he uses on assessments. Most often, these goals are focused on the local domain of proof comprehension, that is they are designed to provide him insight into students understanding of definitions, the structure of proofs, and how students understand claims as a whole. For GaP and AaP questions, these goals are present in addition to examining how students understand proofs from a more holistic perspective.

Dr. Wyatt is very consistent in both his construction of assessments and the way he provides feedback to students. All of his homework assignments have the same structure, that is to say they all have four questions with either a GaP or an AaP as the last question. Further, his test is formatted almost exactly like his homework assignments except with regard to the total number of questions. In terms of feedback, Dr. Wyatt uses very short and focused statements and questions to help the students learn from any errors. However, if a statement is binary, then he simply marks the answer as incorrect.

PART 5

DISCUSSIONS AND CONCLUSION

The data analysis presented in the previous chapter provides enough information to answer most of the questions posed by this dissertation. What follows is a discussion of the three research questions that guided this study. Then some implications for instruction as well as limitations of this study are presented. Finally, some recommendations for future studies builds on this study will be discussed.

5.1 Discussions of Results

5.1.1 Research Question 1

The first research question is:

1. In what ways does Dr. Wyatt use the ideas of a particular assessment focusing on students' thinking with respect to mathematical proofs in his teaching of a transition–to–proof course?

To answer this question, the data analysis on how Dr. Wyatt constructs assessments, responds to students work, and the personal interview will be referenced.

Dr. Wyatt's assessments follow a clear format, all homework assignments consist of four questions each, with the last questions being either a GaP or an AaP question. It is clear in the types of questions he uses, Dr. Wyatt's primary focus for the majority of his assessments is on local domain of proof comprehension, often through the use of definitions. As stated during the analysis, this is a key aspect of the framework created by Mejia-Ramos et al. (2012). His focus on the local domain of proof comprehension is to build a solid foundation of the material being covered. The reasoning for this focus is based in Dr. Wyatt's beliefs about teaching. As it pertains to proof by contradiction, for example, Dr. Wyatt stated

> [Students must know] not just the formally negated, you have to know what it means. So it's the negation of A or B [$\sim (A \cup B)$] you have to know it is, how you call it, the negation of A and the negation of B [$\sim A \cap \sim B$].

In other words, Dr. Wyatt understands that students need a strong understanding of the definition of a negation and how it relates to sets. This demonstrates that he believes that definitions are the foundation of constructing proofs. This coincides with the results from Syamsuri et al. (2018)[1]; in their study, they found indicators and recommendations for three of the four quadrants describing students' understanding being based in definitions and logical connections. Additionally, F N et al. (2019) found that "almost 90% of the difficulties experience indicate a lack of knowledge about concepts, definitions, and relevant notation" (p. 6). Therefore, this emphasis on definitions and logical connections on Dr. Wyatt's assessments is targeting a major issue students have with mathematical proofs.

That being said, Dr. Wyatt does not limit himself on the style of question being asked. His assessments include three common types of questions: construction, evaluation, and calculation. The tasks assigned to each of these questions, while still assessing the local domain of proof comprehension, have students think about different aspects of proofs. Construction questions focus on students constructing an object that exemplifies the statement being made. Evaluation questions have students identify the truth value of a statement; this means they are analyzing the statements and forming a conclusion based on the definitions being studied. Calculation questions require students to use the definitions or procedures discussed in class to make some form of calculation. These three question types require students to use different techniques that are helpful in proving statements.

As noted in the analysis, Dr. Wyatt accomplishes having students practice working with proofs without having them actually write proofs. This can be difficult for some instructors, for example the instructor discussed in Pinto & Karsenty (2018). This instructor altered his assessment methods to improve students ability to write proofs in analysis; however,

[1]The questions used in this study were constructed using the framework developed by Mejia-Ramos et al. (2012)

the method he used, a term paper where students create a detailed collection of notes in a DTP format, revolved around students writing (and revising) their proofs throughout the course. Rocha (2019) notes that “proofs do not need to be restricted to formal proofs” (p. 9) and recommends focusing on simplicity; therefore, having students write formal proofs is only one way to improve their proof writing. Further, Dr. Wyatt’s uses of questions that do not require students to construct proofs while focusing on the local domain of proof comprehension eliminates students needing to decide how to begin a proof, something that is difficult for students to complete (F N et al., 2019).

GaP and AaP questions. The two special types of questions, GaP and AaP, further emphasize these basic qualities, though in different ways. GaP questions ask the student to first identify an error in a proof, then determine the validity of claim, and third write a correct proof or provide a counter example. The AaP question, designed using the framework developed by Mejia-Ramos et al. (2012), primarily asked students to justify why steps were correct. While different in format, it is most likely that Dr. Wyatt uses these types of questions because it examines how the students differentiate between “what is right and what is wrong” (Personal Interview, February 12, 2019). This is beneficial because there is evidence that students find it difficult to differentiate correct reasoning and incorrect statements. For example, Herizal et al. (2019)[2] notes “Students still [have] difficulties in giving the reason for the right step and made errors when determining the right from of [the] sine rules[3]” (p. 4) to use when completing proofs. Similarly, F N et al. (2019) noted undergraduate students must “be able to show contradictions and to provide counterexamples” (p. 6), but were unable to do so in many cases. Again, Dr. Wyatt is assessing an aspect of mathematical proof that students struggle in understanding.

In particular, GaP questions were constructed to have the students examine a proof to find errors. Dr. Wyatt stated

[2]A study conducted focusing on geometric proofs in a secondary school in Indonesia
[3]the Law of Sines

> I as a teacher grade [the students'] solutions, you know I point out that it is wrong, sure sure, but I don't know if that is the best... maybe there are other ways for them to realize they are, they the students, making mistakes. So there I just do it this way, I make them, make the students [the] grader (Personal Interview, February 12, 2019)[4]

As mentioned during the analysis, his goal is to help students recognize the difference between correct and incorrect statements. The construction of GaP questions reinforcers this stance, that is to say, since the given proof is always incorrect, students must identify where the logical error occurs. This assesses students knowledge of the local domain of proof comprehension. GaP questions forces students to think about the logical connections and their ability to create proofs or counter examples.

AaP questions were designed using the framework developed by Mejia-Ramos et al. (2012). Specifically, these questions assess students' reading comprehension of proofs. Dr. Wyatt adopted this AaP question, in part, because of his involvement in the mathematics education research project. Again, AaP questions, like all of his questions, require students to have a very strong understanding of the foundational definitions and theorems. That is, AaP questions focus on the local group of proof comprehension. The analysis showed that the primary reason Dr. Wyatt adopted these questions is to help students be able to read "sketchy" proofs, which are prevalent in mathematics text books as well as some lectures (Lew et al., 2016; Weber et al., 2016). The data analysis only focused on one AaP question; therefore, it is difficult to make further conclusions about this question type in general.

Feedback on students' responses. The feedback Dr. Wyatt provides students is both simple and direct. As stated previously, he generally informs students he needs more detail using a simple question or to explicitly state what in the solution was incorrect. This method of providing feedback is suggested by Speer et al. (2010). However, there are examples of Dr. Wyatt responding to a students solution with a counterexample. The subject

[4]Quote appears in Section 4.3.2.

of a study by Pinto & Karsenty (2018) identifies one of the goals for providing feedback in the course is to use counterexamples; the reasoning being “counterexamples may illustrate for students that if they make a mistake in their proof, then their proof is rejected not due to a norm or to some personal opinion, but because it failed to function as a proof” (p. 139). Dr. Wyatt feedback has much the same purpose. He believes that a simple question is an affective way of showing students their mistake objectively. Further, when he does state directly what is wrong, it is when students have omitted an essential part of their solution, for example, the inductive hypothesis when using PMI.

Further, the type of feedback Dr. Wyatt uses is dependent on the question type. For construction questions, the analysis found that Dr. Wyatt uses either a short question requesting the missing information or states why the students’ solution is wrong. For evaluations questions, Dr. Wyatt just marks responses incorrect statements because the responses are binary (true/false); thus, simply marking the response wrong is not ambiguous feedback. Calculation questions include either of the above methods based on the type of response required. For both GaP and AaP questions, Dr. Wyatt uses methods similar to those used in his construction questions.

Test II. Dr. Wyatt’s test questions fall into three categories: proof writing, calculation, and evaluation. The analysis showed students should have expected these types of questions, especially the calculation and evaluation questions as they are very similar to those on the homework assignments. The proof writing question, while not specifically observed on the homework assignments, uses the same type of reasoning as the constructions, GaP, and AaP questions. Therefore the analysis of the homework and test questions have similar goals, that is examine the local domain of proof comprehension.

In terms of feedback, Dr. Wyatt’s methods do not deviate from those found on his homework. The evaluation question, being a situation with a binary response, has incorrect solutions simply marked incorrect. Further, his feedback on calculation questions is almost identical to the feedback he gave on similar questions in the homework. For the proof writing questions, he commented when students did not use the correct proof structure in addition to comments similar to the construction, GaP, and AaP questions.

Concluding thoughts on the first research question. In short, Dr. Wyatt's assessments provide students examples focusing on the local domain of proof comprehension. The emphasis on the foundational aspects about mathematical proofs (definitions and justifications) is evident, as is how each question construction ensures students must have a strong understanding of the definitions to be successful. This same emphasis is also found in the construction of GaP, AaP, and test questions.

All of Dr. Wyatt's feedback is stated simplistically. This simplistic feedback does require students to think about why their solutions are incorrect. Specifically, Dr. Wyatt does not answer the question if students get a solution incorrect; instead, he asks students questions or provides statements that clearly identify what is incorrect. Students must think actively to correct their errors.

Dr. Wyatt focuses almost exclusively on the local domain of proof comprehension. Mejia-Ramos et al. (2012) and Mejia-Ramos et al. (2017) identify the local domain of proof comprehension as the foundational aspects of a proof, including definitions, the proof framework, and how statements are connected. Dr. Wyatt believes a focus on the local domain of proof comprehension will improve students proof writing. Further, by eliminating the need for students to have to start a proof, he is simplifying the content to enhance the focus on the fundamentals.

5.1.2 Research Question 2

The second research question is:

2. How do Dr. Wyatt's current instructional practices compare to his previous method(s) used?

The data collected to answer this question were a questionnaire, observations, and the personal interview with Dr. Wyatt. The questionnaire was sent to students who had a course with Dr. Wyatt in the previous three (Fall and Spring 2017, Spring 2018) semesters prior to my observations. Then, I compared the results from the questionnaire to Dr. Wyatt's observed teaching practices.

What follows is a comparison of Dr. Wyatt's past instructional practices with the methods used in the observed transition–to–proof course. This comparison of this data indicates there were minor differences in Dr. Wyatt's teaching practices for this course. This discussion will be organized by specific teaching practices (e.g., motivation of content, time allocation) and assessment practices before my final conclusion is presented.

Motivation of content. Dr. Wyatt believes that cultural influences on students' motivation is a major factor to their personal desire to learn. The analysis notes this phenomena as a form of extrinsic motivation; specifically students are "doing something to get a future value (avoid a future disvalue)" (Locke & Schattke, 2018). In his words, Dr. Wyatt believes "if [academics] got that type of attention; then students [will] all try to study mathematics and physics;" that is to say if academics were valued as highly as other aspects of society (e.g., athletics) more students would be attracted to academics because there is a value placed by society. He admits that he does not know how to foster this kind of drive in students, but acknowledges cultural motivation is most easily seen as "pressure" from peers.

Motivation can be viewed in terms of the "rationale for a sequence of topics" (Speer et al., 2010); Locke & Schattke (2018) describe this form of motivation as achievement motivation. Dr. Wyatt uses achievement motivation during his courses, providing students with rationale (a goal) for studying the material. The data analysis notes that Dr. Wyatt does not always state the rationale outright; instead, he uses examples or even just emphasizes requirements to suggest what the goal of the lesson is. Therefore, since he does not know how to foster cultural motivation, he instead sets clearly defined goals for the students to follow.

While it was impossible to analyze specific instances of how Dr. Wyatt motivates content in his previous courses, his personality was shown to play a factor. Khalilzadeh &

Khodi (2018) identified teachers personality as playing a role in motivation; it was found that his former students were very aware of Dr. Wyatt's personality in the form of excitement about teaching the course. Nine of the eleven respondents rated Dr. Wyatt's excitement about teaching the course as "excellent;" this was further noted because students recognized that Dr. Wyatt "knows the material [he teaches] in depth" and identified his personality as forming the basis of his instructional methods. Further, his former students recognized Dr. Wyatt has high expectations for them. All of these instances show, while the specific methods could not be analyzed, Dr. Wyatt's previous methods of motivation left an impact on his former students, leading me to conclude that the material was motivated with clear cut goals (suggesting achievement motivation) that his former students easily recognized and recalled.

Preparation and Self Reflection. Dr. Wyatt prepares his lessons keeping in mind what he considers to be his "best way" for instructing mathematics: his use of examples. The analysis determined Dr. Wyatt uses two broad categories of examples. He uses real world examples (examples not focusing on purely mathematical situations) when covering concepts that "not just the students in my class by the general public" have difficulty comprehending. Conversely, he will omit real world examples when covering topics that are less confusing to students or due to time constraints. Regardless of the type, it is very clear Dr. Wyatt believes he teaches best when working with examples.

Further, he acknowledged how he begins preparing his courses. Most of his time is focused on which example he is going to use to explain a concept. This includes examining time constraints and topics that cause general confusion to students. Whichever type of example he settles on presenting, he is mindful of his students needs; he recognizes when a specific example is not working and has prepared well enough to be able to switch to a different example (or type of example).

It is clear that Dr. Wyatt has taught primarily using examples in his past teaching as well. His former students acknowledged Dr. Wyatt's methods, commenting he "always uses examples" and uses a variety of examples to ensure students understand the foundational material well enough to work with the more complex problems. Further, a former student stated a desire for more teachers to use examples as Dr. Wyatt does to clarify the theorems discussed in the course. For example, Pinto & Karsenty (2018) described an analysis course where formal proofs were not presented in the lecture; instead, proofs by example were used in class and students constructed the general proofs outside of class. This professor, like Dr. Wyatt, believes that examples are essential to the instruction of mathematics, even if the courses are structured differently.

Time Allocation. The analysis showed that the majority of Dr. Wyatt's instructional time was spent working with examples. During the eight lectures, a total of 51.2% of the instructional time was spent working with examples. As noted in the analysis, this is consistent with what Dr. Wyatt believes is his best way of teaching mathematics. All other components of the analysis of Dr. Wyatt's time allocation (review, content, and other)[5] were approximately the same (15%, 17.5%, and 16.3% respectively). Therefore, the analysis showed Dr. Wyatt prioritizes examples heavily.

Dr. Wyatt's teaching methodology was described in the analysis as *traditional*, which Johnson et al. (2017) defined by "heavy use of lecture" where the instructor may pause and ask questions but primarily works through examples. He is partially reliant on this method of instruction because of time constraints. During the analysis of the interview Dr. Wyatt provided several examples of different ways he would teach if he (and the students) had the necessary time. He described a master–apprentice method of teaching which correlated heavily with *alternative* instruction as defined by Johnson et al. (2017). He also described a potential use of a *mixed*[6] instructional method that revolved around students reading material before it is presented in class. One of the reasons Dr. Wyatt does not use a form of either of those methods is because of time constraints.

That being said, his former students acknowledged the work Dr. Wyatt puts into his

[5]Described in Maulana et al. (2015).
[6]Categorized by Johnson et al. (2017).

teaching, especially with regard to pacing. His pace was described as "steady and moderate" and that this pace allowed him to "thoroughly [cover] solely the required content." As mentioned previously, the former students recognized Dr. Wyatt's use of examples as being extensive, and while they admit his pace could be too slow at times, the benefits were evident.

Representations of concepts and relations. While it is very evident at this point, Dr. Wyatt used a large number of examples; however, those examples took several different forms. In this course, he used set theoretical notation, graphs of algebraic functions, and diagrams depending on the goal he was emphasizing. While the use of these various representation methods were evident in Dr. Wyatt's teaching, they were analyzed as being weak in some areas in accordance with Weber et al. (2016). Specifically, Dr. Wyatt did not list out key points on the board, which is an indicator to students that the content is important. In other areas, his representation methods ensured students focused on the relationship between content.

As for how he represented content in previous classes, it is very difficult to identify. The analysis shows that students remember the use of many examples throughout Dr. Wyatt's course. However, they do not describe how these examples were presented, only that it was present. Therefore, no determination can be made on how Dr. Wyatt presented content in his previous courses.

Questioning. Analysis of Dr. Wyatt's questioning methods show that he used a higher percentage of TQs than LQs in his course. Recall that TQs are questions that will keep a lecture moving at a steady pace where as LQs assess students current understanding. That being said, Dr. Wyatt described his questioning as focusing primarily on building students confidence.

I determined that Dr. Wyatt's confidence building questions are characteristic of TQs, that is they help progress the lesson efficiently. Dr. Wyatt has a clear expectation of how his questions should be answered and believes this will allow students to see mathematics is "not that hard" because their peers can quickly respond. That being said, sometimes his questions reveal that none of the students fully understand a topic. He described this simply as learning because he will answer the question.

Former students view Dr. Wyatt's questioning as being mostly straight forward. They noted Dr. Wyatt only sometimes requires students to justify their responses to questions in class. The specific structure cannot be determined, as noted in the analysis, because students described the types of examples he used instead the types of questions he asked in class. That being said, given that justification was not required on all of Dr. Wyatt's in class questions, it is likely that they are TQs.

Assessment practices. Dr. Wyatt's assessments were constructed in a consistent manner and had five types of questions: construction, justification, calculation, GaP, and AaP. Of these questions, Dr. Wyatt noted that AaPs were added because it is the same type of assessment used in the project in which he participated. An analysis of how these questions were developed identified one key element. Specifically, these assessments focused primarily on the local domain of proof comprehension; that is to say, they were designed to assess primarily definitions and the relationship between statements. This correlates with Dr. Wyatt's beliefs about instruction; he works to help students think actively.

His former students identified several key factors about his assessment. First, they followed a consistent format; students knew what the assignments would look like prior to receiving it. Second, they demonstrated a step by step method for approaching the tasks. Third, they helped the students learn the material as they focussed on the essential material covered in class. Finally, the assessments helped differentiate concepts by providing examples. Most of his students remember the assessments positively, that is to say they identified the positive impact the questions had on their learning.

While specific questions were impossible to obtain, one student commented that the assessments forced them to "memorize the basic definitions and foundational" theorems. So, a large portion of the assessment in these past courses focused on the local domain of proof comprehension. In essence, the assessments helped students master the foundational material of the course.

Concluding thoughts on the second research question. Recall the purpose of the second research question was to compare Dr. Wyatt's past and present teaching practices. This is important because it will show if there was a visible impact from Dr. Wyatt's participation in the mathematics education research project. Comparing the observations from the transition–to–proof class with the former students' responses to the questionnaire, the answer becomes clear: As a result of participation in the mathematics education project, Dr. Wyatt's teaching and assessment practices have been enhanced with inclusion of the AaP questions on assessment.

His teaching practices have been formed over the entirety of his career and Dr. Wyatt refers to his use of examples as his best way of teaching the material. His allocation of time within lessons supports this belief, and his questioning methods keep the course progressing at a constant pace. The data from his former students supports the observed behaviors, with most of them commenting about his use of examples.

His assessments, both past and present, focus on the local domain of proof comprehension. Students recognize they must understand the fundamentals to be successful. In fact, the only evidence showing that there was a change to Dr. Wyatt's assessment practices is by his own admission that he added the AaP questions because of his involvement.

5.1.3 Research Question 3

The third research question is:

3. What impact does Dr. Wyatt's participation in the research project have on his core beliefs about teaching and the value of research in mathematics education?

Examining an instructors' personal beliefs is difficult because it can be hard for instructors to accurately verbalize their beliefs (Le Fevre, 2014; Barney & Maughan, 2015, e.g.,). However, we can make some conclusions about Dr. Wyatt's beliefs by examining the data gathered during the interview with him.

He made several statements pertaining to his beliefs about teaching. Dr. Wyatt believes that "his best way" of teaching is through the use of examples. He emphasizes real life examples to communicate to the students that "[they] have been using [mathematical logic] in [their] daily life" (Personal Interview, February 12, 2019). Further, he believes that mathematical prerequisites for this course are not as important as the students' desire to learn mathematics, even stating that he believes some secondary students could handle the content. He commented that the best way for students to learn content is by struggling with the material and actively thinking about the implications. However, with the data at hand, it is difficult to see a significant change in his beliefs about teaching.

The data indicates that Dr. Wyatt's value of research in undergraduate mathematics education is high. In his reflections about his teaching strategies (Section 4.2.2), Dr. Wyatt was able to formulate various teaching methods outside of his "best way;" specifically, he describe what Johnson et al. (2017) refer to as an *alternative* method that resembled a master–apprentice relationship. He also describe a teaching strategy that emphasized reading as the primary form of instruction, or learning by reading as described in Yang & Li (2018). This suggests that Dr. Wyatt is very cognizant of possible instructional strategies, that is to say, he is interested in various methodologies on teaching. In addition, the adoption of the AaP question type into his assessment practices shows Dr. Wyatt values the information gathered from the educational research project. These details suggest Dr. Wyatt valuing undergraduate mathematics education research; however, more data is required to truly answer this question.

5.2 Implications for Instruction

In this study, two major aspects of teaching mathematics at the collegiate level were examined in detail, the practices used by Dr. Wyatt for the instruction of material and his assessment practices. Five aspects of the former were analyzed, specifically motivation of topics, preparation and self reflection, time allocation within lessons, representations of concepts and relations, and questioning techniques. Meanwhile, the latter was analyzed with regards to assessment construction and methods of feedback. There are several implications for instruction that can be identified.

First, Dr. Wyatt's methods of questioning can act as a model questioning in general. The goals of the questions he asks are less to identify misconceptions, and more to build students' confidence and probe them to "think actively." This mindset helps students' motivation, promoting a deeper desire to work toward truly comprehending the material.

Second, the format of his assessments provides an example of how to assess students' understanding of the local domain of proof comprehension. Each question examines a specific set of definitions or if statements are valid to be successful at writing mathematical proofs without requiring the students to construct a proof for each question. This coincides with research (Mejia-Ramos et al., 2012; Yang & Li, 2018; Syamsuri et al., 2018; Herizal et al., 2019) that suggest examining how students understand a proof (or parts of a proof) can lead to students constructing better proofs. The GaP and AaP questions demonstrate how to apply the assessment framework developed by Mejia-Ramos et al. (2012).

Lastly, Dr. Wyatt's feedback principle could be applied on a wider basis. Providing feedback through short comments and questions creates an environment where students must think carefully about their errors. Therefore, one is not directly stating the error in logic, but instead is directing the student (or providing an example) that illustrates the logical mistake. That is to say, students are learning by example how to identify their errors thus promoting a deeper understanding of the content.

5.3 Limitations of the Study

Several limitations of this study can be identified. First, the limited amount of time I spent observing the course will, by its nature, create an incomplete image of Dr. Wyatt's beliefs and methodology for teaching and assessing mathematical proofs. For example, I can only infer how Dr. Wyatt emphasized the instruction of PMI based off of the review sheet for the second exam. Further, this time constraint limited the number of assessments I was able to analyze. I believe having examples of the GaP and AaP questions focusing on more topics, especially those acting as the foundation of this transition–to–proof course, would allow for a better analysis of those later in the course.

Second, this study followed only Dr. Wyatt and his experience teaching this course after working with mathematics educators. While he was selected for this very reason, his beliefs on mathematics and teaching are unique to his background. Therefore, this does not provide a clear examination of mathematicians in general.

Lastly, the analysis of the data using the frameworks developed by Speer et al. (2010) and Mejia-Ramos et al. (2012) create their own limitations. There are several other frameworks for identifying students understanding of mathematical proof such as those suggested by Syamsuri et al. (2018) and Herizal et al. (2019). Further, the R/O/G theory developed by Schoenfeld et al. (2016) provides an alternative method for analyzing Dr. Wyatt's teaching practices. That is to say, using these models to analyze the data could lead to different conclusions.

5.4 Future Research

Speer et al. (2010) state that there is little research on examining the teaching practices of instructors at the collegiate level. Since then several studies have examined these practices (Lew et al., 2016; Weber et al., 2016; Johnson et al., 2017, e.g.,); however, the library of research is still small. Similarly, studies concerning how students understand mathematical proof specifically are also increasing (Yang & Li, 2018; Herizal et al., 2019, e.g.,), but this number is also small. Further research could expand on this study by comparing multiple professors, those that have and have not participated in mathematics education research, methods of instruction and assessment of mathematical proof. This would allow researchers to formulate a clearer picture on how participation in mathematics education research alters mathematicians teaching methods.

This report provides examples of assessment types in a transition–to–proof course. This can be expanded on in several ways. For instance, how do questions constructed using a framework other than that described by Mejia-Ramos et al. (2012) compare to those from this study, especially the GaP and AaP questions? Further, a more student based study on their perception of the type of feedback demonstrated by Dr. Wyatt; this could answer the question how instructor feedback on students work affect their understanding of mathematical proof. I recommend further research into these issues.

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Appendix A

QUESTIONNAIRE OF FORMER TEACHING PRACTICES

Instructions: Answer the questions honestly, to the best of your ability. If provided options, please circle the best choice.

1. How would you rate the instructor's excitement in teaching the course?

   Excellent   Very Good   Good   Poor

2. How often did the instructor make you explain your responses to questions posed in class?

   Always   Often   Sometimes   Rarely   Never

3. How often did the instructor make you explain your responses to questions posed on assessments (Homework/Test)?

   Always   Often   Sometimes   Rarely   Never

4. Overall, how high were the instructor's expectations of you?

   Extremely High   Very High   High   Average   Low

5. Describe how the instructor paced the course. Did his pacing change during the course.

6. Describe how the instructor covered the essential material in the course, that is, the material that acted as a foundation for the rest of the course.

7. Describe the types of questions the instructor's used while teaching the course, that is, the types of questions he would ask not appearing on an assessment.

8. Describe how the instructor's assessments (homework/test) impacted your learning in the course.

9. Describe how the instructor's's assessments (homework/test) provided insight into mathematical proofs.

10. What is the one thing that the instructor did that you wish all teachers did? Please explain.

11. What is the one thing that the instructor could have done differently to improve your learning? Please explain.

Appendix B

INTERVIEW PROTOCOL

Good morning/afternoon/evening,

Thank you for agreeing to take part in this interview. Before we begin, I would like to verify that you consent to having this interview audio recorded (Yes: Thank you. Start audio recording. No: Put audio recorder away. Ok, thank you. This interview will be recorded through notes only).

Please note, when answering these questions, be mindful to not share names or any information that will identify another person during this interview.

1. Describe your role as an instructor (for example, in terms of preparation; how you determine the content that would be taught each class period; delivery mode – lecture without or with students' participation, etc.).

2. What prerequisite mathematical knowledge do students need to have when learning proof techniques?

3. What do you believe is the students' role in learning proof techniques (for example, listen the instructor's lecture and study at home; prepare and participate in class in some way, etc.)?

4. What are your expectations from questioning students during class? What do you feel are the best types of questions to ask during class?

5. You use diagrams, logic tables, and both formal and informal examples. How do you determine which representation is the best to use for a specific proof?

6. What information do students' responses to "grade the proof" questions provide? How do you use this information in your instruction?

7. What information do students' responses to "analyze a proof" questions provide? How do you use this information in your instruction?

8. What is the primary difference between "grade a proof" and "analyze a proof" questions? Do you think students differentiate between these two kinds of tasks?

9. What is the importance of having students read and analyze various proofs for correctness? How is this used when evaluating students understanding of various proof techniques?

10. What is the importance of having students determine the truth of a statement before either proving the statement or providing a counter example? What type of information does this provide?

11. Choosing an appropriate proof technique is an essential aspect of mathematical proof. What methods are used to help students identify the proof technique that would be best for a given proof?

12. Is there anything else that you would like to share with me regarding your teaching experience and/or your belief about teaching the introduction to proof class and/or students' difficulties in learning the proof techniques and with mathematical proof in general?

Appendix C

HOMEWORK ASSIGNMENTS

Problem 8.1. Construct concrete non-empty relations R, S and T on $X = \{1, 2\}$, as well as a non-empty relation U on $Y = \{1, 2, 3\}$, such that the following properties are satisfied.

(1) A relation R on X such that R is symmetric and is antisymmetric.
(2) A relation S on X such that S is symmetric and is not antisymmetric.
(3) A relation T on X such that T is not symmetric and is antisymmetric.
(4) A relation U on Y such that U is not symmetric and is not antisymmetric.

Problem 8.2. Construct concrete relations r, s, t and u from $A = \{3, 4\}$ to $B = \{a, b\}$ with the following properties.

(1) relation r is not a function.
(2) relation s is a function, but not a function from A to B.
(3) relation t is a function from A to B with $\text{Rng}(t) = B$.
(4) relation u is a function from A to B with $\text{Rng}(u) \neq B$.

Problem 8.3. Consider the real function $f(x) = x^2 - 1$. Calculate each of the following.

(1) The image/value of 5 under f.
(2) All the pre-images of 15 under f, if they exist.
(3) All the arguments associated with the value 20, if they exist.
(4) All the pre-images of -10 under f, if they exist.

Problem 8.4 (Grade a "Proof")**.** Study the following claim as well as the "proof":

Claim. For non-empty sets A and B, if $f\colon A \to B$ is a function then $f \circ f^{-1} \subseteq I_B$.

"Proof". Let $(b, b) \in f \circ f^{-1}$. Then there exists $a \in A$ such that $(b, a) \in f^{-1}$ and $(a, b) \in f$. Thus $(b, b) \in f \circ f^{-1}$ and $(b, b) \in I_B$. Therefore $f \circ f^{-1} \subseteq I_B$. □

Complete the following questions concerning the above **claim** and ***"proof"***:

(1) Determine whether the *"proof"* is rigorous. **Identify the issues in the *"proof"***, if any.
(2) Determine whether the claim is **true** or **false**. Justify the answer in part (3).
(3) If the the claim is true and the *"proof"* is not rigorous, then provide a **correct and rigorous proof**. If the claim is false, give a **concrete counterexample**.

PROBLEMS HINTS SOLUTIONS

Problem 9.1. Let $A = \{1, 2\}$. Construct sets B_i and functions $f_i\colon A \to B_i$, $1 \leqslant i \leqslant 4$, with the following properties:

(1) f_1 is both one-to-one and onto B_1.
(2) f_2 is one-to-one but not onto B_2.
(3) f_3 is onto B_3 but not one-to-one.
(4) f_4 is neither one-to-one nor onto B_4.

Problem 9.2. Let $A = \{1, 2\}$ and $C = \{x, y\}$. For each $i = 1, 2$, construct sets B_i as well as functions $f_i\colon A \to B_i$ and $g_i\colon B_i \to C$ satisfying the following properties:

(1) $g_1 \circ f_1$ is onto C but f_1 is not onto B_1.
(2) $g_2 \circ f_2$ is one-to-one but g_2 is not one-to-one.

Problem 9.3. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$.

(1) Find $f([-1, 0) \cup (2, 4])$.
(2) Find $f^{-1}([-2, 3))$.
(3) Find $f^{-1}((-1, 5] \cup (17, 26])$.

Problem 9.4 (Analyze a Proof)**.** Study the following statement as well as the proof:

Statement. $\sqrt{2}$ is not a rational number.

Proof. We prove the statement in following steps.

(a) Suppose that $\sqrt{2}$ is a rational number.
(b) So there exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ such that $\sqrt{2} = \frac{a}{b}$.
(c) We can further assume that a and b do not have a common divisor with absolute value bigger than 1.
(d) Note that a^2 is even.
(e) Consequently, we conclude that a is even. So write $a = 2k$, with $k \in \mathbb{Z}$.
(f) Then $b^2 = 2k^2$, which shows that b^2 is even.
(g) Thus b is even.
(h) But this is impossible.
(i) So $\sqrt{2}$ is not a rational number. □

Complete the following questions concerning the above **proof**:

(1) Explain why (d) holds (i.e., why a^2 is even). Provide your justification.
(2) Explain why (d) implies (e). Provide your justification.
(3) Explain what is impossible, as claimed in (h)? Provide your justification.

PROBLEMS HINTS SOLUTIONS

Problem 10.1. Determine whether the sets are finite or infinite. No justification is needed.

(1) $\mathbb{Q}$, the set of (all) rational numbers.
(2) $\{x \in \mathbb{R} : x^2 + 1 = 0\}$.
(3) The set of all turkeys eaten in the year 1620.
(4) $\{x \in \mathbb{N} : x \text{ is composite}\}$.
(5) $\{x \in \mathbb{R} : 4x^8 - 5x^6 + 12x^4 - 18x^3 + 26x^2 - 13x + 100 = 0\}$

Problem 10.2. Provide a concrete example of each of the following cases, if ever possible. If a case is never possible, then state so.

(1) An infinite subset X of a finite set Y.
(2) A collection $\{A_i : i \in \mathbb{N}\}$, with each A_i non-empty, such that $\bigcup_{i \in \mathbb{N}} A_i$ is finite.
(3) Finite non-empty sets A and B such that $\overline{\overline{A \cup B}} \neq \overline{\overline{A}} + \overline{\overline{B}}$.
(4) Finite non-empty sets C and D such that $\overline{\overline{C \cup D}} = \overline{\overline{C \cap D}}$.

Problem 10.3. Determine whether each of the statements is true or false. No justification is necessary.

(1) If a set A is countable, then A is infinite
(2) If a set B is denumerable, then B is infinite
(3) If a set C is uncountable, then C is infinite
(4) If a set D is denumerable, then D is countable
(5) If a set E is not denumerable, then E is uncountable

Problem 10.4 (Grade a "Proof")**.** Study the following claim as well as the "proof":

Claim. If A and B are infinite sets, then $A \approx B$.

"Proof". Let A and B be infinite sets. Then we can describe A and B as follows

$$A = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \quad \text{and} \quad B = \{b_1, b_2, \ldots, b_n, b_{n+1}, \ldots\}.$$

Define a function $f\colon A \to B$ by $f(a_n) = b_n$ for all $a_n \in A$. Clearly, f is one-to-one and onto B. Therefore $A \approx B$, finishing the proof. □

Complete the following questions concerning the above **claim** and ***"proof"***:

(1) Determine whether the *"proof"* is rigorous. **Identify the issues in the *"proof"***, if any.
(2) Determine whether the claim is **true** or **false**. Justify the answer in part (3).
(3) If the the claim is true and the *"proof"* is not rigorous, then provide a **correct and rigorous proof**. If the claim is false, give a **concrete counterexample**.

PROBLEMS HINTS SOLUTIONS

Appendix D

TEST II

Problem II.1 (5 points)**.** Use the Principle of Mathematical Induction (PMI) to prove $3 \mid (n^3 - 19n)$ for all $n \in \mathbb{N}$.

Problem II.2 (5 points)**.** Use the Principle of generalized Mathematical Induction (PMI) to prove $\prod_{i=2}^{n} \frac{i^2-1}{i^2+2i} = \frac{3}{n^2+2n}$ for all integers $n \geqslant 2$.

Problem II.3 (5 points)**.** Define $a_1 = 5$, $a_2 = 25$, and $a_{n+1} = 9a_n - 20a_{n-1}$ for all $n \geqslant 2$. Use the Principle of Complete Induction (PCI) to prove $a_n = 5^n$ for all $n \in \mathbb{N}$.

Problem II.4 (5 points)**.** Let $X = \{1, 2, 3, 6, 7, 8\}$, $R = \{(1, 3), (2, 1), (6, 8), (8, 2)\}$ and $S = \{(1, 1), (3, 7), (6, 2), (7, 8)\}$. Determine each of the following explicitly:

(1) $S \circ S$; (2) $R \circ S$; (3) $S \circ R$; (4) $R^{-1} \circ S^{-1}$; (5) $S^{-1} \circ R^{-1}$.

Problem II.5 (5 points)**.** Determine true or false, in which X and Y are non-empty sets.

(1) $D = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \mid b\}$ is an antisymmetric relation on $\mathbb{N}$
(2) $R = \{(1, 2), (2, 3), (3, 1)\}$ is an transitive relation on $\mathbb{R}$
(3) $S = \{(1, 4), (2, 5), (3, 6)\}$ is an antisymmetric relation on $\mathbb{Q}$
(4) If a relation T on X is not antisymmetric then T is symmetric
(5) A relation U on Y is transitive if and only if $U \circ U \subseteq U$

Extra Credit Problem II.6 (1 point, no partial credit)**. Prove or disprove**: For **all** sets A, B and C, if $A \times B = A \times C$ then $B = C$.

PROBLEMS HINTS SOLUTIONS