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Prime Characteristic Aspects in the Study of Stanley-Reisner Rings and Monomial Ideals

Irina Georgeana Ilioaea

Georgia State University

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by

IRINA ILIOAEA

Under the Direction of Florian Enescu, PhD

ABSTRACT

This dissertation investigates Stanley-Reisner rings and monomial ideals in connection to some important concepts in characteristic p commutative algebra, such as Frobenius complexity, and complexity sequence, and strong test ideals in tight closure theory. The Frobenius complexity of a local ring R measures asymptotically the abundance of Frobenius operators of order e on the injective hull of the residue field of R. It is known that, for Stanley-Reisner rings, the Frobenius complexity is either $-\infty$ or 0. This invariant is determined by the complexity sequence $\{c_e\}_e$ of the ring of Frobenius operators on the injective hull of the residue field. One of our main results shows that $\{c_e\}_e$ is constant for $e \geqslant 2$, generalizing work of Àlvarez Montaner, Boix and Zarzuela. This result settles an open question mentioned by Àlvarez Montaner in [26]. Moreover, we use Cartier algebras to describe a large class of strong test ideals. One of our main results gives a full description of test ideals associated to Cartier algebras in Stanley-Reisner rings. An important consequence of our result states that a bound for the degree of integral dependence that an arbitrary element in the tight closure of an ideal satisfies over the respective ideal is given by a combinatorial invariant, which is the number of facets of the Stanley-Reisner ring considered.

INDEX WORDS: Commutative algebra, Simplicial complexes, Stanley-Reisner rings, Frobenius operators, Frobenius algebras, Complexity sequence, Frobenius complexity, Cartier algebras, Strong test ideals, Tight closure, Skew algebras, Monomial ideals

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IRINA ILIOAEA

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IRINA ILIOAEA

Committee Chair:	Florian Enescu
Committee:	Zhongshan Li
	Mariana Montiel
	Yongwei Yao

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Georgia State University

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DEDICATION

This dissertation is dedicated to my parents.

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CHAPTER 1

INTRODUCTION

The development of characteristic p techniques has been relevant in commutative algebra and birational geometry. Over the past years, positive characteristic commutative algebra became an important area of research. In this dissertation, we will study topics such as Frobenius operators, Cartier algebras and strong test ideals.

The Frobenius map, which is the map that associates to each element of a ring its pth power, plays a crucial role in characteristic p commutative algebra. This map and the concepts associated to it led to important results in commutative algebra. In particular, Frobenius operators on the injective hull have been studied by many commutative algebraists, such as Hochster, Huneke, Smith, Lyubeznik, Singh, Schwede, Enescu, Yao, Sharp, Katzman, to name a few.

This dissertation studies monomial ideals, and in particular Stanley-Reisner rings, in relation to some concepts from positive characteristic commutative algebra. Namely, we study Frobenius complexity, and in particular the complexity sequence, and the notion of strong test ideal.

Lyubeznik and Smith started investigating the ring of Frobenius operators in connection to one of the most intriguing conjectures in tight closure theory, the localization problem. They raised the question about the finite generation of the ring of Frobenius operators on the injective hull of the residue field of a local ring in [25]:

Question 1.0.1 (Lyubeznik, Smith)**.** Is $\mathcal{F}(E_R)$ always finitely generated as a ring over R?

In [21], Katzman found a ring with infinitely generated Frobenius algebra. Enescu and Yao were motivated by this question to introduce a new invariant in [9], called the Frobenius complexity, which gives an asymptotical way of measuring the abundance of Frobenius operators on the injective hull of the residue field of a local ring. Àlvarez Montaner, Boix and Zarzuela considered the finite generation question in the case of Stanley-Reisner rings in [27]. They showed that:

Theorem 1.0.2 (Àlvarez Montaner, Boix and Zarzuela)**.** *The Frobenius algebra $\mathcal{F}(E_R)$ associated to a Stanley-Reisner ring R is either principally generated or infinitely generated.*

They also found the description of the complexity sequence for a class of Stanley-Reisner rings. In [19], I fully described the complexity sequence for any Stanley-Reisner ring. My results settled an open question in the field, mentioned by Àlvarez Montaner in [26].

Theorem 1.0.3. *Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S and $R = S/I$ its Stanley-Reisner ring. Then,*

$$\{c_e\}_{e \geqslant 0} = \{0, \mu + 1, \mu, \mu, \mu, \ldots\},$$

where the ideal J_p is the unique minimal monomial ideal defined in Definition 3.3.2 and $\mu := \mu_S(J_p)$ is the minimal number of minimal monomial generators of J_p.

Tight closure theory was introduced by Craig Huneke and Mel Hochster in 1986. Using tight closure theory, algebraists were able to simplify many proofs by using characteristic p techniques, come up with stronger formulations of well-known existing results and produce new theorems.

Tight closure theory provides a closure operation on ideals and submodules. To every ideal we associate a larger ideal containing it related to the Frobenius map. This new ideal turns out to be helpful in studying the original ideal. It is well-known the fact that computing the tight closure of an ideal in a particular ring can be a very difficult problem. There are not that many examples of such computations in the literature. Huneke introduced the notion of strong test ideal in [18] which helps in providing interesting concrete information about the integral elements that belong to the tight closure of an ideal.

Using Cartier algebras, I found in joint work with Enescu, a large class of strong test ideals in [8]. Having a larger class of strong test ideals is very important because it gives us a better bound for the minimal degree of the equation of integral dependence that an arbitrary element in the tight closure of an ideal satisfies over the respective ideal.

Our results have a combinatorial flavor; in this dissertation, we investigate rings that come from combinatorial commutative algebra in relation to tight closure theory. Stanley-Reisner rings constitute an important class of such rings. These rings are obtained by assigning to combinatorial objects, called simplicial complexes, algebraic objects, called Stanley-Reisner rings.

One theorem we proved in [8] tells us that the number of facets of the simplicial complex associated to our ring represents the minimal number of generators of our test ideal. Hence, we obtained that a bound for the degree of integral dependence that an arbitrary element in the tight closure of an ideal satisfies over the respective ideal is given by a combinatorial invariant, the number of facets of the ring.

Theorem 1.0.4 ([8]). *Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$. Let $I \leqslant S$ be a square-free monomial ideal in S and $R = S/I$ its Stanley-Reisner ring. Let Δ be the simplicial complex associated to the Stanley-Reisner ring R. The ideal given by $(x_F : F \in \mathcal{F}(\Delta))$ is a strong test ideal. Therefore, in the ring R, for every ideal $J \leqslant R$ and every element x belonging to J^*, x satisfies a degree $f_{max}(\Delta)$ equation of integral dependence over J, where $f_{max}(\Delta)$ is the number of facets of the simplicial complex Δ.*

CHAPTER 2

PRELIMINARIES

In this chapter, we set up the notations and introduce the main concepts. Moreover, we will state known results which will be used in the later chapters of the dissertation.

2.1 Rings of Positive Prime Characteristic

Let R be a Noetherian ring of positive prime characteristic p. Let $F : R \to R$ be the Frobenius map, that is, $F(r) = r^p$. We have that

$$(a+b)^p = a^p + b^p,\ (a \cdot b)^p = a^p \cdot b^p,$$

for all $a, b \in R$. Therefore, the Frobenius map is a ring homomorphism. Let $F^e : R \to R$ be the e-th iteration of the Frobenius map, that is, $F^e(r) = r^q$, where $q = p^e, e \in \mathbb{N}$. The ring R is a reduced ring (i.e. it does not have any nilpotent elements) if and only if the Frobenius map $F : R \to R$ is injective, by [7].

If R is a reduced ring and $\mathbb{Q}(R)$ is the total ring of fractions of R, we define the **collection of qth roots** of R as follows:

$$R^{1/q} := \{s \in \overline{\mathbb{Q}(R)} : s^q \in R\}.$$

It is easy to note that $R^{1/q}$ is closed under addition and multiplication and that the map $R \to R^{1/q}$, which sends r to $r^{1/q}$ is an isomorphism of rings. Therefore, $R^{1/q}$ is a ring abstractly isomorphic to R. Moreover, the inclusion $R \subset R^{1/q}$ can be naturally identified with the e-th iteration of the Frobenius endomorphism F^e of R. This gives $R^{1/q}$ the module structure over R as follows: $r * s = r \cdot s$, for any $r \in R$ and $s \in R^{1/q}$.

Definition 2.1.1. A ring R is called F-finite if $R^{1/q}$ is finitely generated as a module over R, for some (or equivalently, any) q.

Example 2.1.2. (i) Any perfect field k of characteristic p (any field which satisfies $k^p = k$) is F-finite.

(ii) Any complete local ring with F-finite residue field is F-finite. For instance, let k be a perfect field of characteristic p and let $S = k[[x_1, \ldots, x_n]]$ be the formal power series ring in n variables over k. Then,

$$S^{1/q} = k^{1/q}[[x_1^{1/q}, \ldots, x_n^{1/q}]] = k[[x_1^{1/q}, \ldots, x_n^{1/q}]] = \bigoplus_{1 \leqslant \lambda_1, \ldots, \lambda_n \leqslant q-1} S(x_1^{\lambda_1} \cdots x_n^{\lambda_n})^{1/q}.$$

Therefore, $S^{1/q}$ is a free S-module with basis $\{x_1^{\lambda_1/q} \ldots x_n^{\lambda_n/q}\}_{0 \leqslant \lambda_i \leqslant q-1}$.

The following properties of F-finite rings are well-known and they are presented in [17].

Proposition 2.1.3. *Let R be an F-finite ring. Then the following hold:*

(i) If S is a multiplicative closed set of R, then $S^{-1}R$ is F-finite.

(ii) If $I \subseteq R$ is an ideal of R, then the quotient ring R/I is F-finite.

(iii) If x is an indeterminate, then the polynomial ring in $R[x]$ and the formal power series ring $R[[x]]$ are F-finite.

The class of F-pure rings has been introduced by Hochster and Roberts in [16]. They play a very important role in tight closure theory and have been studied by Fedder, Goto and Watanabe before tight closure theory came about.

Definition 2.1.4. A monomorphism $f : R \to S$ is **pure** if $f \otimes 1_M : R \otimes M \to S \otimes M$ is injective, for all $R-$modules M. If the Frobenius map $F : R \to R$ is pure, then we say that R is F**-pure**.

It follows from the definition that if R is F-pure then $x^p \in I^{[p]}$ implies that $x \in I$, for any ideal I.

For an F-pure ring R, the Frobenius map is injective, therefore R is reduced (i.e. has no nilpotent elements).

Definition 2.1.5. A ring R is called F-split if the inclusion map $i : R \to R^{1/q}$ splits as a map of R-modules, i.e. if there exists a map $\phi \in \mathrm{Hom}_R(R^{1/q}, R)$ such that $\phi \circ i = id_R$.

Remark 2.1.6. The inclusion map $R \to R^{1/q}$ splits for some q if and only if it splits for all q.

It is easy to note that any F-split ring is F-pure. If R is F-finite or complete local, then R is F-pure if and only if R is F-split.

Proposition 2.1.7. *(i) A ring R is F-pure if and only if R_P is F-pure, for every $P \in$* $\mathrm{Spec}(R)$.

(ii) An F-finite local ring of prime positive characteristic is F-pure if and only if its completion $\widehat{R}$ is F-pure.

Kunz found a way of describing that a ring of prime positive characteristic $p > 0$ is regular in terms of the Frobenius endomorphism on R. His result states the following:

Theorem 2.1.8 ([24], Kunz)**.** *A ring R is regular if and only if the Frobenius endomorphism $F^e : R \to R$ is flat for some e(or equivalently, for any e).*

Kunz Theorem shows that any regular ring is F-pure. In the next chapter, we will present a criterion for F-purity for quotients of regular local rings due to Fedder. By applying Fedder's criteria for F-purity one can show that:

Example 2.1.9. (i) Using Proposition 3.1.7, one can show that Stanley-Reisner rings are F-pure as follows: if $S = k[[x_1, \ldots, x_n]]$, $I \subseteq S$ is a square-free monomial ideal and $R = S/I$ is the Stanley-Reisner ring associated to I, the element $(x_1 \cdots x_n)^{p-1}$ is contained in the colon ideal $(I^{[p]} : I)$ and it does not belong to $m^{[p]}$.

(ii) Let k is a field of prime characteristic p and $R = k[x, y, z]/(x^3 + y^3 + z^3)$. One can show that if $p \equiv 1 (\mathrm{mod}\ 3)$, R is F-pure and if $p \equiv 2 (\mathrm{mod}\ 3)$, R is not F-pure.

2.2 Tight Closure

Let R be a Noetherian commutative ring of prime positive characteristic p. Let $e \geqslant 0$ and $q = p^e$. For any ideal I of R, we denote by $I^{[q]}$ the ideal generated by $\{i^q : i \in I\}$.

For any ideals I and J, we have that $(I+J)^{[q]} = I^{[q]} + J^{[q]}$, $(IJ)^{[q]} = I^{[q]}J^{[q]}$ and $(I^n)^{[q]} = (I^{[q]})^n$, for any positive integer n.

For any ring R, we denote by R^o the set of elements of R not contained in any minimal prime of R. If R is an integral domain, $R^o = R \setminus \{0\}$.

Definition 2.2.1 ([13])**.** Let $I \leqslant R$. Then the **tight closure** of I is the ideal

$$I^* = \{x \in R : \text{there exists } c \in R^o \text{ such that } cx^q \in I^{[q]}, \text{for all } q = p^e \gg 0\}.$$

We recall here the definition of the integral closure of an ideal I:

Definition 2.2.2. Let $I \leqslant R$. Then the **integral closure** of I in the ring R is the ideal denoted by $\overline{I}$ and consisting of elements x in R which satisfy an integral dependence relation

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

such that the coefficients satisfy $a_i \in I^i$, for any $1 \leqslant i \leqslant n$.

A different way of defining the integral closure of an ideal in a ring is the following:

Definition 2.2.3. Let $I \leqslant R$. Then the **integral closure** of I in the ring R is the ideal denoted by $\overline{I}$ and consisting of elements x in R for which there exists $c \in R^o$ such that $cx^n \in I^n$, for infinetely many n.

Since $I^{[q]} \subseteq I^q$ and using the second definition of the integral closure of an ideal in a ring it is easy to note that $I^* \subseteq \overline{I}$. One of the most powerful results connecting tight and integral closure of an ideal in a ring of prime characteristic is the following:

Theorem 2.2.4 (Briançon-Skoda Theorem)**.** *Let R be a Noetherian ring of positive prime characteristic and let I be an ideal of R generated by at most n elements. Then for all $m \geqslant 0$,*

$$\overline{I^{m+n}} \subseteq (I^{m+1})^*.$$

As a consequence of this result:

Proposition 2.2.5. *For any $x \in R$, $(x)^* = \overline{(x)}$.*

Proposition 2.2.6. *For any ideals I and J of R, the following assertions hold:*

(i) $I \subseteq I^* \subseteq \overline{I}$

(ii) *If $I \subseteq J$, then $I^* \subseteq J^*$.*

(iii) $I \subseteq I^* = (I^*)^*$

(iv) $(I + J)^* = (I^* + J^*)^*$

(v) $(I \cdot J)^* = (I^* \cdot J^*)^*$

(vi) $(I \cap J)^* \subseteq I^* \cap J^*$

(vii) $0^* = \sqrt{0}$

(viii) *$x \in I^*$ if and only if the image of x in R/P, $\overline{x}$ lies in $(IR/P)^*$, for any minimal prime ideal P of R.*

Part (*viii*) of the Proposition 2.2.6 above shows that it is enough to study tight closure in integral domains.

Computing the tight closure of an ideal in a given ring is a very difficult problem. The following results give us a way of finding elements in the tight closure of an ideal in a given ring.

Proposition 2.2.7. *If $R \subseteq S$ is a finite R-algebra extension and S is an integral domain, then $IS \cap R \subseteq I^*$.*

Proposition 2.2.8 (Colon capturing)**.** *Let R be a regular domain, S a finite R-module and $x_1, \ldots, x_n \in R$ elements in R which generate a height n ideal in R. Then the following holds*

$$(x_1, \ldots, x_{n-1}) :_S x_n \subseteq ((x_1, \ldots, x_{n-1})S)^*.$$

We will present a class of ideals defined using the tight closure operation on ideals.

Definition 2.2.9. A ring R is called **weakly F-regular** if every ideal in R is tightly closed, i.e., if $I^* = I$, for any $I \subseteq R$ ideal in R. A ring R is called **F-regular** if every localization R_P is weakly F-regular, for every $P \in \mathrm{Spec}(R)$.

Proposition 2.2.10. *Any regular ring is weakly F-regular.*

Proposition 2.2.11. *Any direct summand of a (weakly) F-regular domain is (weakly) F-regular.*

Proposition 2.2.12. *Given a Noetherian ring R of prime characteristic p, the following assertions are equivalent:*

(i) R is weakly F-regular

(ii) R_m is weakly F-regular, for any maximal ideal m of R

(iii) Every m-primary ideal of R is tightly closed.

Therefore, in order to show that a ring is weakly F-regular it is enough to prove that any localization at any maximal ideal of the ring is weakly F-regular. However, an open question in the field, referred to as the localization problem in tight closure theory asks the following question: are weakly F-regular and F-regular rings the same?

For some classes of rings such as Gorenstein rings, $\mathbb{Q}$-Gorenstein rings, images of Gorenstein rings of dimension at most 3, uncountable affine algebras, the two notions coincide. One can actually show that for Gorenstein rings it is enough to check that only some particular classes of ideals of the ring are tightly closed in order to prove that the ring is weakly F-regular:

Proposition 2.2.13. *Let (R, m) be a Gorenstein local ring. Then R is a weakly F-regular ring if and only if every parameter ideal in R is tightly closed. Moreover, R is a weakly F-regular ring if and only if one parameter ideal in R is tightly closed.*

Definition 2.2.14. Let R be a reduced F-finite ring. We will call R a **strongly F-regular ring** if for every $c \in R \setminus R^o$, there exists at least one q such that the map $R \to R^{1/q}$ sending $1 \to c^{1/q}$ splits as a map of R-modules.

It is easy to observe that any strongly F-regular ring is F-split.

Proposition 2.2.15. *(i) A reduced F-finite ring R is strongly F-regular if and only if R_P is strongly F-regular, for every $P \in \operatorname{Spec}(R)$.*

(ii) A reduced F-finite local ring R is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular.

Proposition 2.2.16. *Every strongly F-regular local ring is a domain.*

Theorem 2.2.17. *An F-finite regular ring is strongly F-regular.*

Theorem 2.2.18. *If R is a direct summand of a strongly F-regular ring, then R is strongly F-regular ring .*

These results give us many ways of testing if a ring is strongly F-regular or not.

Example 2.2.19. (i) Stanley-Reisner rings are not domains, hence they are not strongly F-regular.

(ii) Let $R = k[x, y, z]/(xy - z^2)$. One can show that $R \cong k[s^2, st, t^2]$ represents a direct summand of the polynomial ring $k[s, t]$. Since $k[s, t]$ is a strongly F-regular ring (because it is an F-finite regular ring), R is strongly F-regular as well.

Any strongly F-regular ring is weakly F-regular. They are conjectured to be the same for F-finite rings. This conjecture is relevant in the field because it would prove that weakly F-regularity commutes with localization.

2.3 The Algebra of Frobenius Operators

Now we will define a new R-algebra structure on R: for any $e \geqslant 0$, as a ring $R^{(e)}$ equals R while the R-algebra structure is defined by $rs = r^q s$, for all $r \in R$, $s \in R^{(e)}$, where $q = p^e$.

We have that $R^{(e)}$ is the R-bimodule: let $R^{(e)}$ be equal to R as an abelian group, whose left R-module structure is given by the usual multiplication and whose right R-module structure is defined by the Frobenius map as follows $r * s = rs$ and $s * r = r^q s$ for any $r \in R$ and $s \in R^{(e)}$. Let M be a left R-module. Similarly, we will use the Frobenius endomorphism to define a new left R-module structure on M. We will denote the new R-module by $M^{(e)}$. For any $e \geqslant 0$, we let $M^{(e)}$ be equal to M as a set and as an abelian group and the R-module on the left is defined as follows: $r * m = r^q m$, for all $r \in R$, $m \in M^{(e)}$. We can note that $M^{(e)} = R^{(e)} \otimes_R M$ When R is reduced, $R^{(e)}$ is isomorphic to $R^{1/q}$ as left modules over R.

We have the natural R-module isomorphism:

$$R^{(e)} \otimes_R R/I \cong R/I^{[q]}.$$

Definition 2.3.1. An **eth Frobenius operator (or eth Frobenius action)** of M is an additive map $\phi : M \to M$ such that $\phi(rm) = r^q \phi(m)$, for all $r \in R$ and $m \in M$. The collection of eth Frobenius actions on M is an R-module, denoted by $\mathcal{F}^e(E_R)$.

Let $\phi : M \to M$ be an eth Frobenius operator. It is easy to see that this map can be identified with an R-module homomorphism $\phi : M \to M^{(e)}$. Moreover, this Frobenius action naturally defines an R-module homomorphism $f_\phi : R^{(e)} \otimes_R M \to M$, where $f_\phi(r \otimes m) = r\phi(m)$, for all $r \in R$ and all $m \in M$. Note that here we regard $R^{(e)}$ as an $R - R^{(e)}$-bimodule as follows: $R^{(e)}$ has the usual structure as an R-module given by: $R^{(e)} = R$ on the left, while on the right we have the twisted Frobenius multiplication: $r * s = rs^q$, for any $r \in R$ and $s \in R^{(e)}$.

Now we have a natural R-module isomorphism:

$$\mathcal{F}^e(M) = \mathrm{Hom}_R(R^{(e)} \otimes_R M, M) \cong \mathrm{Hom}_R(M, M^{(e)}),$$

defined by $P(\phi) = f_\phi$. The R-module structure on $\mathcal{F}^e(M)$ is given by the natural multiplication by a scalar $(r\phi)(m) = r\phi(m)$, for any $r \in R$, $\phi \in \mathcal{F}^e(M)$ and $m \in M$. It is easy to see that P is additive and $P(s\phi)(r \otimes m) = r((s\phi)(m)) = r(s\phi(m)) = rs\phi(m) =$

$s(r\phi(m)) = sP(\phi)(r \otimes m)$. Therefore, we obtain that $P(s\phi) = sP(\phi)$, for all $s \in R$ and all $\phi \in \mathrm{Hom}_R(M, M^{(e)})$.

Definition 2.3.2. The **algebra of Frobenius operators** on M is defined by

$$\mathcal{F}(M) = \oplus_{e \geqslant 0} \mathcal{F}^e(M).$$

The ring operation on $\mathcal{F}(M)$ is given by the usual composition of maps as multiplication. If $\phi \in \mathcal{F}^e(M)$, $\psi \in \mathcal{F}^{e'}(M)$ then $\phi\psi := \phi \circ \psi \in \mathcal{F}^{e+e'}(M)$. Note that under this multiplication, the ring $\mathcal{F}(M)$ is noncommutative since in general $\phi\psi \neq \psi\phi$.

The ring operation on $\mathcal{F}(M)$ defines a module structure $\mathcal{F}^e(M)$ over $\mathcal{F}^0(M) = End_R(M)$. This makes $\mathcal{F}^e(M)$ an R-module, by restricting the scalars of the canonical map $R \to End_R(M)$. We have that $(\phi \circ r)(m) = \phi(rm) = (r^q\phi)(m)$, for all $r \in R$, $m \in M$ and $\phi \in \mathcal{F}^e(M)$. Hence, $\phi r = r^q \phi$, for all $r \in R$ and $\phi \in \mathcal{F}^e(M)$.

Let $R\{F^e\}$ be the noncommutative associative ring extension of R generated by one variable x which satisfies $xr = r^q x$, for every $r \in R$, where $q = p^e$. There exists a ring homomorphism $R\{F^e\} \to End_{\mathbb{Z}}(R)$ which sends $R \to End_{\mathbb{Z}}(R)$ and $x \to F^e$. Since $R \subset R\{F^e\}$ and R is a subring of $R\{F^e\}$, every $R\{F^e\}-$module is an $R-$module. Conversely, any $R\{F^e\}-$module is an $R-$module with an action F^e. Therefore, in order to define an $R\{F^e\}-$module structure on an $R-$module M, one has to define an additive map $\phi_e : M \to M$ which satisfies $\phi_e(rm) = r^q\phi_e(m)$, for any $r \in R$ and any $m \in M$, which is equivalent to defining a Frobenius operator on the $R-$module M. Hence, we obtain that $\mathcal{F}^e(M)$ represents the sets on $R\{F^e\}-$module structures on M. In [25], Lyubeznik and Smith showed that

(i) $\mathcal{F}(R) \cong R\{F\}$

(ii) $\mathcal{F}(H^d_m(R)) \cong R\{F\}$, where $H^d_m(R)$ denotes the top local cohomology module of a complete local ring (R, m) of positive dimension d which satisfies the Serre's condition S_2.

One can note that E_R has an $\mathcal{F}(E_R)$-module structure defined as follows: $\phi * x = \phi(x)$, for any $\phi \in \mathcal{F}(E_R)$ and $x \in E_R$.

Lyubeznik and Smith characterized strong F-regularity in terms of Frobenius structures in [25]:

Theorem 2.3.3 ([25, Theorem 4.1])**.** *A reduced F-finite local ring is strongly F-regular if and only if E_R is a simple $\mathcal{F}(E_R)$-module.*

Another criteria of F-purity is due to Sharp in [31]:

Theorem 2.3.4 ([31, Theorem 3.2])**.** *Let (R, m, k) be a local ring. Then R is F-pure if and only if there exists an injective Frobenius action on E_R.*

2.4 Frobenius Complexity

2.4.1 Complexity of Skew-Algebras

Let A be a $\mathbb{N}$-graded, noncommutative ring, $A = \oplus_{e \geqslant 0} A_e$, such that $A_0 = R$ is a Noetherian commutative ring. Let A satisfy the following condition: $aR \subseteq Ra$, for all $r \in R = A_0$, $a \in A$ homogeneous. Such a ring is called an $R-$skew-algebra.

Let $G_e := G_e(A)$ be the subring of A generated by elements of degree less than or equal to e. Note that $G_e \subseteq G_{e+1}$, for all e. Moreover, $(G_e)_i = A_i$, for all $0 \leqslant i \leqslant e$ and $(G_e)_{e+1} \subseteq A_{e+1}$. We will denote the minimal number of homogeneous generators of G_e as a subring of A over $A_0 = R$ by k_e.

Proposition 2.4.1. *The minimal number of generators of the R-module $\dfrac{A_e}{(G_{e-1})_e}$ equals $k_e - k_{e-1}$, for all e.*

Definition 2.4.2. The sequence $\{k_e\}_e$ is called the **growth sequence** for A. The **complexity sequence** is given by $\{c_e = k_e - k_{e-1}\}_e$. The **complexity** of A is

$$cx(A) = inf\{n > 0 : c_e = O(n^e)\}.$$

If there is no $n > 0$ with $c_e(A) = O(n^e)$, by convention we have that $cx(A) = \infty$.

It is obvious that $cx(A) = 0$ when A is finitely generated as a ring over R.

Remark 2.4.3. (i) It is easy to note that $cx(A) = 0$ if and only if A is finitely generated as a ring over R, if and only if $\{c_e(A)\}_{e\geqslant 0}$ is eventually zero.

(ii) One can show that $cx(A) > 0$ implies $cx(A) \geqslant 1$.(The sequence $\{n^e\}_e$ converges to 0, for $0 < n < 1$ as $e \to \infty$. Hence, if we assume $cx(A) = n > 0$, then $c_e(A) = O(n^e)$ with $0 < n < 1$. Since $\{n^e\}_e$ converges to 0, as $e \to \infty$, the sequence $\{c_e(A)\}_e$ is eventually 0. But this implies $cx(A) = 0$, which contradicts our assumption. Therefore, we must have $cx(A) \geqslant 1$.)

(iii) We have that $cx(A) = 1$ if the sequence $\{c_e(A)\}_{e\geqslant 0}$ is bounded above, but not eventually zero.

Let $E_R := E_R(k)$ denote the injective hull of the residue field k.

Definition 2.4.4. The **Frobenius complexity** of the ring R is defined by

$$cx_F(R) = log_p(cx(\mathcal{F}(E_R))).$$

Also, let $\{k_e := k_e(\mathcal{F}(E_R))\}_e$ be the **Frobenius growth** sequence and $\{c_e := c_e(\mathcal{F}(E_R))\}_e$ the **complexity sequence**.

Remark 2.4.5. (i) If (R, m, k) is a local, d-dimensional and Gorenstein ring, $E_R = H^d_m(R)$. In [25], Lyubeznik and Smith proved that $\mathcal{F}(H^d_m(R))$ is generated by the canonical Frobenius action F on $H^d_m(R)$. Therefore, $\mathcal{F}(E_R)$ is principally generated as a ring over R and $cx_F(R) = -\infty$.

(ii) Let R a normal, $\mathbb{Q}-$Gorenstein ring of positive dimension d and with the canonical module relatively prime to p. In [23], Katzman, Schwede, Singh and Zhang proved that $\mathcal{F}(E_R)$ is principally generated as a ring over R and $cx_F(R) = -\infty$

(iii) In [21], Katzman gave an example of a ring R such that $\mathcal{F}(E_R)$ is not finitely generated as a ring over R. The ring is $R = k[x, y, z]/(xy, xz)$. One can note that this ring is a Stanley-Reisner ring. Later on, we will see that we can prove that $cx_F(R) = 0$.

(iv) The Frobenius complexity of a ring is 0, if the complexity sequence $\{c_e\}_e$ is bounded but not eventually 0. Àlvarez Montaner, Boix and Zarzuela showed that the Frobenius complexity of the completion of any Stanley-Reisner ring is either 0 or $-\infty$.

(v) Based on Remark 2.4.3 (ii), we have that the Frobenius complexity cannot take negative values.

(vi) In [9], Enescu and Yao computed the Frobenius complexity for the determinantal rings obtained by moding out the 2×2 minors of a 2×3 matrix of indeterminates. They showed that the Frobenius complexity can be positive, irrational and depends on the characteristic.

2.4.2 The T-Construction

In [23], Katzman, Schwede, Singh and Zhang introduced an important example of an R-skew algebra.

Let $\mathcal{R}$ be an $\mathbb{N}$-graded commutative ring of characteristic p with $\mathcal{R}_0 = R$.

Definition 2.4.6. Let $T_e = \mathcal{R}_{p^e-1}$ and $T(\mathcal{R}) = \oplus_e T_e = \oplus_{e \geqslant 0} \mathcal{R}_{p^e-1}$. A ring structure on $T(\mathcal{R})$ is defined by

$$a * b = ab^{p^e},$$

for all $a \in T_e$ and $b \in T_{e'}$.

This operation together with the natural addition inherited from $\mathcal{R}$ defines a noncommutative $\mathbb{N}$-graded ring. Note that $T_0 = R$ and if $a \in T_e$, $r \in R$, then

$$a * r = ar^{p^e} = r^{p^e} a = r^{p^e} * a,$$

for all $e \geqslant 0$. Hence $T(\mathcal{R})$ is a skew R-algebra.

Let (R, m, k) a local normal complete ring. For any divisorial ideal I (i.e. an ideal of height one), we denote by $I^{(n)}$ its nth symbolic power. Let ω denote the canonical ideal of

R. The anticanonical cover of the ring R is defined as

$$\mathcal{R} = \mathcal{R}(\omega) = \bigoplus_{n \geqslant 0} \omega^{(-n)}.$$

In [23], Katzman, Schwede, Singh and Zhang found a new description for the ring of Frobenius operators on the injective hull of the residue field of R using the $T-$construction of the anticanonical cover of the ring R.

Theorem 2.4.7 ([23, Theorem 3.3], Katzman, Schwede, Singh, Zhang)**.** *Let (R, m, k) be a local normal complete ring and ω its canonical ideal. Then there exists an isomorphism of graded rings:*

$$\mathcal{F}(E_R) \cong T(\mathcal{R}(\omega)).$$

2.5 Strong Test Ideals

Let R be a Noetherian ring of characteristic p. In an attempt of understanding which elements in the integral closure of an ideal belong to its tight closure, Huneke introduced the notion of strong test ideal in [18]. He asked whether there exists a uniform bound on the degree of the integral equations satisfied by the elements in the tight closure of an ideal in the given ring.

Test elements are an important tool in tight closure theory since they annihilate all the tight closure operations.

Definition 2.5.1. An element $c \in R$ is called **a test element** if $cI^* \subseteq I$, for all the ideals I of R.

In [15], Hochster and Huneke showed that test elements exist in a large class of rings:

Theorem 2.5.2. *Let R be a reduced excellent local ring. Then for every $c \in R^o$ with R_c regular, there exists n such that c^n is a test element.*

Kunz showed that $F-$finite rings are excellent. Hence, test elements exist in any reduced $F-$finite local ring.

Definition 2.5.3. Let R be a ring in which test elements exist. The ideal generated by all the test elements is called the **test ideal** of the ring R and it is denoted by τ_R.

From the definition, we have that $\tau I^* \subseteq I$.

Definition 2.5.4. Let T be an ideal of R such that $T \cap R^o \neq \emptyset$. Then T is a **strong test ideal** for R if and only if $TI^* = TI$, for all ideals $I \leqslant R$.

Remark 2.5.5. If a strong test ideal exists, $T \subseteq \tau_R$.

Huneke has observed that the minimal number of generators of a strong test ideal T is a bound for the minimal degree of an integral dependence equation that an element $x \in I^*$ satisfies over I, see Theorem 2.1 in [18].

Now, we will give the definition of the tight closure of a module.

For an R-module M, let $F^e(M) = R^{(e)} \otimes_R M$. For a submodule $N \subseteq M$ we denote $N_M^{[q]} = Im(F^e(N) \to F^e(M))$ and for $x \in M$, we let x^q denote the image of $1 \otimes x$ in $F_R^e(M)$.

Definition 2.5.6. Let M be a finitely generated R-module and N a submodule in M, we define the tight closure of N in M as follows:

$$N^* = \{x \in M : \text{there exists } c \in R^o \text{ such that } cx^q \in N_M^{[q]}, \text{for all } q = p^e \gg 0\}.$$

If M is not a finitely generated $R-$module, we will use the notion of finitistic tight closure of N in M, as follows:

Definition 2.5.7. Let M be an R-module and $N \subseteq M$ an R-submodule of M. We call the **finitistic tight closure** of N in M, denoted by N_M^{*fg} the set of elements $u \in M$ for which there exists a finitely generated submodule $L \subseteq M$ such that $u \in (N \cap L)_L^*$. We call N **tightly closed in the finitistic sense** in M if $N_M^{*fg} = N$.

Definition 2.5.8. The **big test ideal** is defined as $\tau_b(R) = \cap_M \text{Ann}_R(0_M^*)$.

The **finitistic test ideal** of R is $\cap_{I \leqslant R} I : I^*$ and is denoted by $\tau_{fg}(R)$.

The big test ideal and the finitistic test ideal are conjectured to be the same.

Remark 2.5.9. Let (R, m, k) be a local ring. In [14], Hochster and Huneke showed that the big test ideal is the annihilator of the finitistic tight closure of 0 in the injective hull of the residue field of R, $\tau_b(R) = \mathrm{Ann}_R(0^{*fg}_{E_R})$. Moreover, we have that the finitistic test ideal can be expressed as $\tau_{fg}(R) = \mathrm{Ann}_R(0^*_{E_R})$. We can note that $\tau_{fg}(R) \subseteq \tau_b(R)$. Moreover, if (R, m, k) is complete by Matlis duality we have that $0^{*fg}_{E_R} = \mathrm{Ann}_{E_R}(\tau_b(R))$ and $0^*_{E_R} = \mathrm{Ann}_{E_R}(\tau_{fg}(R))$.

In [33], Vraciu showed the following:

Theorem 2.5.10. *If (R, m) is a Noetherian local reduced ring of prime characteristic p such that the big test ideal commutes with completion, i.e., $\tau_b(R)\widehat{R} = \tau_b(\widehat{R})$, then the big test ideal $\tau_b(R)$ is a strong test ideal. Moreover, if (R, m) is complete, then the finitistic test ideal $\tau_{fg}(R)$ is a strong test ideal.*

2.6 Cartier Algebras

Definition 2.6.1. A p^{-e}**−linear map** $\psi_e : M \to M$ is an additive map that satisfies $\psi_e(r^{p^e} m) = r\psi_e(m)$, for all $r \in R$, $m \in M$. We denote the set of p^{-e}−linear maps by $\mathscr{C}_e(M)$.

We have that $\mathscr{C}_e(M) = \mathrm{Hom}_R(M^{(e)}, M)$. It is easy to see that if we compose a p^{-e}−linear map and a $p^{-e'}$−linear map we get a $p^{-e+e'}$−linear map. Moreover, each $\mathscr{C}_e(M)$ is a right module over $\mathscr{C}_0(M) = End_R(M)$.

Definition 2.6.2. The **Cartier algebra** on M is

$$\mathscr{C}(M) = \oplus_{e \geqslant 0} \mathscr{C}_e(M) = \oplus_{e \geqslant 0} \mathrm{Hom}_R(M^{(e)}, M).$$

Note that this is a noncommutative ring and $\mathscr{C}_0(M) = End_R(M)$ is not central in R, so this object is not an R-algebra in the classical sense.

Definition 2.6.3. Let $\mathscr{C}$ be the Cartier algebra on R and let $\mathscr{D}$ be a graded-subring of $\mathscr{C}$ such that $\mathscr{D}_0 = \mathscr{C}_0 \simeq R$ and $\mathscr{D}_e \neq 0$ for some $e > 0$. An **Cartier algebra pair** on R is a pair of the form $(R, \mathscr{D})$.

Definition 2.6.4. Let $q = p^e$ and $\phi : R^{1/q} \to R$ be an R-linear map. An ideal $J \leqslant R$ is called **ϕ-compatible** if $\phi(J^{1/q}) \subseteq J$. An ideal J is called **$\mathscr{D}$-compatible** if $\phi(J^{1/q}) \subseteq J$, for all $\phi \in \mathscr{D}_e$ and all $e > 0$.

Schwede has showed how to associate a test ideal to a Cartier subalgebra on R in [29, 30].

Definition 2.6.5. The **test ideal associated to the pair** (R, ϕ), denoted $\tau(R, \phi)$ is the unique smallest ϕ-compatible ideal that intersects nontrivially R^o.

The **test ideal associated to the pair** $(R, \mathscr{D})$, denoted $\tau(R, \mathscr{D})$ is the unique smallest $\mathscr{D}$-compatible ideal that intersects R^o nontrivially.

The existence of test ideals associated to pairs was proved by Schwede based upon a technical result of Hochster and Huneke on test elements.

Lemma 2.6.6 ([30, Lemma 3.6],[15, Theorem 5.10])**.** *Let R and ϕ be as above. Then there exists an element c in R^o such that for all $d \neq 0$ there exists $n \in \mathbb{Z}_{>0}$ with*

$$c \in \phi^n((dR)^{1/p^{ne}}).$$

This allows us to state the existence result for test ideals for R and $\phi : R^{1/q} \to R$, respectively a subalgebra $\mathscr{D}$ of $\mathscr{C}$, mentioned above.

Theorem 2.6.7 ([29, Theorem 3.18],[30, Lemma 3.8 and Theorem 7.13])**.** *Let R, ϕ and c be as in the above Lemma. Then $\tau(R, \phi)$ exists and equals*

$$\sum_{n \geqslant 0} \phi^n((cR)^{1/p^{ne}}).$$

The test ideal of an algebra pair $(R, \mathscr{D})$ is

$$\tau(R, \mathscr{D}) = \sum_{e>0} \sum_{\phi \in \mathscr{D}_e} \tau(R, \phi).$$

In [8], we proved the following:

Theorem 2.6.8 ([8, Theorem 2.4])**.** *Let $\phi : R^{1/q} \to R$ be an R-linear map. Then $\tau(R, \phi)$ is a strong test ideal in R.*

Moreover, if $(R, \mathscr{D})$ is an algebra pair, then the test ideal $\tau(R, \mathscr{D})$ is a strong test ideal.

Remark 2.6.9. This result recovers earlier results of Vraciu and respectively Takagi on strong test ideals. Vraciu showed that the test ideal $\tau_b(R)$ is a strong test ideal in R in [33]. A consequence of a result by Hara and Takagi, Lemma 2.1 in [12], shows that $\tau(R, \mathscr{C}) = \tau_b(R)$.

2.7 Simplicial Complexes and Stanley-Reisner Rings

In this section, we will introduce an important class of rings in combinatorial commutative algebra defined using square-free monomial ideals.

Let $V = \{x_1, \ldots, x_n\}$ be a finite set. We will call a **(finite) simplicial complex** Δ on V a collection of subsets of V such that $F \in \Delta$, whenever $F \subseteq F'$ for some $F' \in \Delta$, and such that $\{x_i\} \in \Delta$, for any $i = 1, \ldots, n$.

The elements of Δ are called **faces** and the maximal faces under inclusion are called the **facets** of the simplicial complex. Let $\mathcal{F}(\Delta)$ denote the set of facets of the simplicial complex Δ. Since any simplicial complex is uniquely generated by its facets, whenever $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we will write $\Delta =< F_1, \ldots, F_m >$.

The **dimension**, $\dim(F)$, of a face F is the number $|F| - 1$. The **dimension** of the simplicial complex Δ is $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$.

By convention, the empty set $\emptyset$ is a face of dimension -1 of any non-empty simplicial complex. Any face of dimension 0 is called **a vertex** and any face of dimension 1 is called **an edge**.

We denote by f_i, the number of faces of Δ of dimension i. We have $f_0 = n$ and $f_{-1} = 1$. The d-tuple

$$f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$$

is called the f**-vector of** Δ.

A **nonface** of Δ is a subset of the vertex set of Δ which is not an element of Δ.

A simplicial complex Δ is called **pure** if all the facets of Δ have the same dimension, namely $\dim(\Delta)$.

A simplicial complex Δ is a Cohen-Macaulay complex over k if $k[\Delta]$ is a Cohen-Macaulay ring.

Proposition 2.7.1. *Any Cohen-Macaulay simplicial complex is pure.*

Definition 2.7.2. Given a finite simplicial complex Δ on the vertex set $\{x_1, \ldots, x_n\}$ and a field k, the corresponding **Stanley-Reisner ring**, denoted $k[\Delta]$ is obtained by taking the quotient ring formed by the polynomial ring $k[x_1, \ldots, x_n]$ and its ideal I_Δ generated by the square-free monomials corresponding to the non-faces of Δ:

$$I_\Delta = (x_{i_1} \cdots x_{i_r} : \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta),$$

$$k[\Delta] = \frac{k[x_1, \ldots, x_n]}{I_\Delta}.$$

.

On the other hand, if I is a square-free monomial ideal, then $k[x_1, \ldots, x_n]/I \cong k[\Delta]$, for some simplicial complex Δ.

For each $F \subset V$ subset of V we denote by $x_F = \prod_{x_i \in F} x_i$ and $P_F = (x_i : x_i \in F)$.

Example 2.7.3. Let Δ be the simplicial complex generated by the facets $F_1 = \{x_1, x_2\}$ and $F_2 = \{x_2, x_3, x_4\}$. Then the nonfaces of Δ are $\{x_1, x_3\}$ and $\{x_1, x_4\}$. The Stanley-Reisner ring associated to Δ is $k[\Delta] = k[x_1, x_2, x_3, x_4]/(x_1x_3, x_1x_4)$.

Proposition 2.7.4. *Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. Then the minimal primary decomposition of the Stanley Reisner ideal associated to Δ is given by*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{F^c},$$

where $\mathcal{F}(\Delta)$ is the set of the facets of Δ and $F^c = V \setminus F$ the complement of F. Moreover,

$$\dim(k[\Delta]) = \dim(\Delta) + 1.$$

Example 2.7.5. In the Example 2.7.3, we have that the minimal primary decomposition of the ideal I_Δ is given by $I_\Delta = (x_1) \cap (x_3, x_4)$.

Definition 2.7.6. Let Δ a simplicial complex. We define **the Alexander dual** of Δ:

$$\Delta^V = \{F^c : F \notin \Delta\}.$$

Remark 2.7.7. One can show that the Alexander dual of the simplicial complex Δ, denoted by Δ^V, is a simplicial complex.

Proposition 2.7.8. *Let $I_\Delta = P_{F_1^c} \cap \ldots \cap P_{F_m^c}$ be the minimal primary decomposition of I_Δ, where $F_1, \ldots, F_m$ are the facets of the simplicial complex Δ. Then the square-free ideal associated to the Alexander dual I_{Δ^V} is generated by the monomials $x_{F_1^c}, \ldots, x_{F_m^c}$.*

Example 2.7.9. The Alexander dual of the simplicial complex Δ in the Example 2.7.3 is generated by the facets $\{x_2, x_3\}$ and $\{x_2, x_4\}$. The square-free ideal associated to the Alexander dual is generated by $I_{\Delta^V} = (x_1, x_3x_4)$ and the Stanley-Reisner ring associated to the Alexander dual is $k[\Delta^V] = k[x_1, x_2, x_3, x_4]/(x_1, x_3x_4)$.

CHAPTER 3

PREPARATORY RESULTS

In this chapter, we will present the preparatory results which will be needed later on in this dissertation.

3.1 Fedder's Lemma

In this section, we will present a result due to Fedder in [10]. Proposition 3.1.7 gives us a nice criteria for F-purity for a quotient of a regular local F-finite ring. We will follow Fedder's work in [10] to provide complete proofs for the results presented.

Throughout this section, let S be an F-finite regular local ring, $I \subseteq S$ an ideal in S and $R = S/I$. We will denote by ω_R the canonical module of the ring R. We will need the following theorem in [7]:

Theorem 3.1.1 ([7, Theorem 3.3.7]). *Let (R, m) be a Cohen-Macaulay ring.*

(a) The following assertions are equivalent:

(i) R is Gorenstein

(ii) ω_R exists and $\omega_R \cong R$.

(b) Let $(R, m) \to (S, n)$ be a local homomorphism of Cohen-Macaulay local rings such that S is a finite $R-$module. If ω_R exists, then ω_S exists and $\omega_S \cong \mathrm{Ext}^t_R(S, \omega_R)$, where $t = \dim(R) - \dim(S)$.

Proposition 3.1.2. *If (S, m) is an F-finite regular local ring of characteristic p, then $S^{(1)}$ is a regular local ring which is free as a left S-module.*

Proposition 3.1.3. *Let (S, m) be a F-finite regular local ring of characteristic p. Then we have that*

$$\mathrm{Hom}_S(S^{(1)}, S) \cong S^{(1)}$$

as left $S^{(1)}$-modules.

Proof. By Proposition 3.1.2, $S^{(1)}$ is a regular local ring which is free as an S-module. Hence, we are under the hypothesis of Theorem 3.1.1 and we obtain that

$$\mathrm{Hom}_S(S^{(1)}, S) \cong \mathrm{Hom}_S(S^{(1)}, \omega_S) \cong \omega_{S^{(1)}} \cong S^{(1)},$$

as left modules over $S^{(1)}$. $\square$

Proposition 3.1.4 ([10, Lemma 1.6])**.** *Let (S, m) be a F-finite regular local ring of characteristic p. Let $I \subseteq S$ be an ideal in S and $R = S/I$. Let f be a generator of $\mathrm{Hom}_S(S^{(1)}, S)$ as a left $S^{(1)}$-module, $J \subseteq S^{(1)}$ an ideal in $S^{(1)}$ and $s \in S^{(1)}$. Then $sf(J) \subseteq I$ if and only if $s \in (IS^{(1)} : J)$.*

Proof. Let $\{s_i\}_i$ be a basis for $S^{(1)}$ as a left module over S. Then we have that $\tilde{s}_i f \in \mathrm{Hom}_S(S^{(1)}, S)$ defined by $\tilde{s}_i f(t) = f(\tilde{s}_i t)$, for any $\tilde{s}_i, t \in S^{(1)}$. Note that for any $s \in S^{(1)}$, the map $sf \in \mathrm{Hom}_S(S^{(1)}, S)$ is defined as $sf(t) = f(st)$, for any $s, t \in S^{(1)}$.

Then $sf(J) \subseteq I$ if and only if $sf(rS^{(1)}) \subseteq I$, for all $r \in J$ if and only if $srf(S^{(1)}) \subseteq I$, for all $r \in J$. Thus, $srf : S^{(1)} \to I$ if and only if $srf(s_i) = r_i \in I$, for all i if and only if $srf = (\sum_i r_i \tilde{s}_i)f$ if and only if $sr = \sum_i r_i \tilde{s}_i \in IS^{(1)}$, for all $r \in J$. Therefore, we showed that $sf(J) \subseteq I$ if and only if $s \in (IS^{(1)} : J)$.

$\square$

Corollary 3.1.5. *Under the assumptions of Proposition 3.1.4, there exists an isomorphism $\psi : (IS^{(1)} : J)/IS^{(1)} \cong \mathrm{Hom}_S(S^{(1)}/J, S/I)$, given by $\psi(\overline{s}) = (\overline{sf})$, where $\overline{sf}$ is the homomorphism defined by $\overline{sf}(\overline{t}) = \overline{sf(t)}$, for $\overline{t} \in S^{(1)}/J$.*

Proof. We will first show that the map ψ is well-defined. Let $\overline{s_1} = \overline{s_2} \in (IS^{(1)} : J)/IS^{(1)}$. This implies that $s_1 - s_2 \in IS^{(1)}$. We have $f((s_1 - s_2)t) \in f(IS^{(1)}) \subseteq If(S^{(1)}) \subseteq IS = I$, which implies $(s_1 - s_2)f(t) \in I$. Hence we obtain $\psi(\overline{s_1}) = \psi(\overline{s_2})$. It is easy to see that the map ψ is a homomorphism. Now we will prove that ψ is injective and surjective. Let $\psi(\overline{s}) = 0$. Then $\overline{sf}(\overline{t}) = \overline{0}$, for all $\overline{t} \in S^{(1)}/J$. This implies $sf(t) \in I$, for all $t \in S^{(1)}$. By

Proposition 3.1.4, we have that $sf(S^{(1)}) \subseteq I$ if and only if $s \in IS^{(1)} : S^{(1)}$. Therefore, $s \in IS^{(1)}$ which proves that $\overline{s} = \overline{0}$. Hence, ψ is injective. Since $S^{(1)}$ is a free S-module, every homomorphism $\phi \in \mathrm{Hom}_S(S^{(1)}/J, S/I)$ induces a commutative diagram

$$\begin{tikzcd}
S^{(1)} \arrow[r, "\pi"] \arrow[d, dotted, "\phi_0"'] & S^{(1)}/J \arrow[r] \arrow[d, "\phi"'] & 0 \\
S \arrow[r, "\overline{\pi}"] & S/I \arrow[r] & 0
\end{tikzcd}$$

There exists $\phi_0 \in \mathrm{Hom}_S(S^{(1)}, S)$ such that $\phi \circ \pi = \overline{\pi} \circ \phi_0$. Moreover, $\phi_0 = sf$ for some $s \in S^{(1)}$ and $\phi(\overline{t}) = \phi(\pi)(t)) = \overline{\pi}(\phi_0(t)) = \overline{\pi}(sf(t)) = \overline{sf(t)}$. Therefore, $\phi = \overline{sf}$. Hence,ψ is surjective. $\square$

Corollary 3.1.6. *Let S be a F-finite regular local ring, $I \subseteq S$ an ideal in S and $R = S/I$. There exists an isomorphism $\psi : (I^{[p]} : I)/I^{[p]} \cong \mathrm{Hom}_R(R^{(1)}, R)$, given by $\psi(\overline{s}) = (\overline{sf})$, where f is any $S^{(1)}$-module generator for $\mathrm{Hom}_S(S^{(1)}, S)$.*

Proof. Since $S^{(1)}$ as a ring is just the ring S, the ideal $IS^{(1)}$ in $S^{(1)}$ becomes identified with $I^{[p]}$ in S. Now the conclusion follows directly from Corollary 3.1.5. $\square$

Proposition 3.1.7 ([10, Proposition 1.7])**.** *Let (S, m) be a F-finite regular local ring, $I \subseteq S$ an ideal in S and $R = S/I$. Then, R is F-pure if and only if $(I^{[p]} : I) \not\subset m^{[p]}$.*

Proof. Let f be the $S^{(1)}$-module generator for $\mathrm{Hom}_S(S^{(1)}, S)$. We know that R is F-pure if and only if the map $R \to R^{1/p}$ splits. Hence it is enough to show that $R \to R^{1/p}$ splits if and only if there exists a map $\phi = \overline{sf} \in \mathrm{Hom}_R(R^{(1)}, R)$ with $Im(\phi) \not\subset m_R$, where m_R denotes the maximal ideal of the ring R. For the first implication, since $F : R \to R^{1/p}$ splits, there exists an R-linear map $\phi : R^{1/p} \to R$ with $\phi \circ F = id$. If $Im(\phi) \subset m_R$ then $Im(\phi) \neq R$. Therefore, $1 \notin Im(\phi)$ which contradicts the fact that $\phi(1) = 1$. For the other implication, since there exists a map $\phi = \overline{sf} \in \mathrm{Hom}_R(R^{(1)}, R)$ with $Im(\phi) \not\subset m_R$, using Proposition 3.1.4 we have that $sf(\phi) \subseteq m_R$ if and only if $s \in (m_R S^{(1)} : S^{(1)}) = m_R S^{(1)} = m^{[p]}$. Therefore, we have that $s \notin m^{[p]}$ which proves that $Im(sf)$ contains a unit. Hence the map $\phi : R^{1/p} \to R$ is surjective which shows that the map $F : R \to R^{1/p}$ splits. $\square$

3.2 The Ring of Frobenius Operators on the Injective Hull

Let (R, m, k) be a complete local ring in positive prime characteristic p. The ring operation on $\mathcal{F}(E_R)$ is given by the usual composition of functions as multiplication. Given $\phi_e \in \mathcal{F}^e(E_R)$ and $\phi_{e'} \in \mathcal{F}^{e'}(E_R)$ we have $\phi_e\phi_{e'}(x) := (\phi_e \circ \phi_{e'})(x)$, for any $x \in E_R$ and $e, e' \geqslant 0$.

Definition 3.2.1. Let (R, m, k) a local ring. Given an R-module M, we call the **Matlis dual** of $M : M^V := \mathrm{Hom}_R(M, E_R(k))$.

One can note that this is a contravariant exact functor from the category of $R-$modules to itself.

Theorem 3.2.2 ([7, Theorem 3.2.13])**.** *(Matlis duality) Let (R, m, k) be a complete Noetherian local ring and $E_R = E_R(k)$ the injective hull of the residue field of the ring R. Then*

(i) $R^V \cong E_R$ and $E_R^V \cong R$

(ii) For every R-module M, there exists a natural map $M \to (M^V)^V$. Under this map, $R \to (R^V)^V$ and $E_R \to (E_R^V)^V$ are isomorphisms.

(iii) If we denote by $\mathcal{A}(R)$ the category of Artinian $R-$modules and by $\mathcal{F}(R)$ the category of finite $R-$modules and if we let $M \in \mathcal{A}(R)$ and $N \in \mathcal{F}(R)$, then $M^V \in \mathcal{F}(R)$ and $N^V \in \mathcal{A}(R)$. Furthermore, $(M^V)^V \cong M$ and $(N^V)^V \cong N$.

Using Theorem 3.2.2(i), we have that $\mathcal{F}^0(E_R) = \mathrm{Hom}_R(E_R, E_R) = E_R^V \cong R$.

The ring of Frobenius operators $\mathcal{F}(E_R)$ is a graded skew R-algebra since it is an $\mathbb{N}$-graded noncommutative ring and $\phi R \subseteq R\phi$, for any $\phi \in \mathcal{F}(E_R)$ homogeneous. One can check that by choosing $\phi \in \mathcal{F}^e(E_R)$, $r \in R$ and $x \in E_R$ as follows:

$$\phi r(x) = \phi(rx) = r^{p^e}\phi(x).$$

Hence we obtain $\phi r = r^{p^e}\phi$, for any $\phi \in \mathcal{F}^e(E_R)$ and any $r \in R$.

3.2.1 The Ring of Frobenius Operators on the Injective Hull of Regular Local Rings

Let (S, m) regular complete local ring, $I \subseteq S$ an ideal of S and $R = S/I$. Let E_S denote the injective hull of the residue field of S and E_R denote the injective hull of the residue field of R. An important result in the literature states that:

Lemma 3.2.3 ([2, Lemma 3.34])**.** *If S is a regular local ring, $I \subseteq S$ an ideal of S and $R = S/I$, then $E_R = \operatorname{Ann}_{E_S}(I) \subseteq E_S$.*

Proof. Using the properties of the Hom functor, we have the following isomorphism of functors:

$$\operatorname{Hom}_S(-, E_S) \cong \operatorname{Hom}_R(-, \operatorname{Hom}_S(S/I, E_S)) = \operatorname{Hom}_R(-, \operatorname{Ann}_{E_S}(I)).$$

Since E_S is an injective S-module, the functor $\operatorname{Hom}_S(-, E_S)$ is exact and by the isomorphism above we obtain that the S-module $\operatorname{Ann}_{E_S}(I)$ is injective as well. Both k and $\operatorname{Ann}_{E_S}(I)$ are S-modules killed by I, so they are R-modules. We have the following extension of S-modules

$$k \subseteq \operatorname{Ann}_{E_S}(I) \subseteq E_S$$

and we know that $k \subseteq E_S$ is an essential extension. This shows that $k \subseteq \operatorname{Ann}_{E_S}(I)$ is an essential extension of R-modules. hence using the fact that $\operatorname{Ann}_{E_S}(I)$ is an injective R-module, we proved that $\operatorname{Ann}_{E_S}(I)$ equals E_R. $\square$

It is a well-known fact presented in Proposition 3.5.4 in [7] that for a regular local ring S, E_S is isomorphic to the top local cohomology module of S, i.e. $E_S \cong H^n_m(S)$, where $n = \dim(S)$. The Frobenius map on S induces a natural canonical Frobenius map on $H^n_m(S)$, denoted by F^e_S. The next result gives a nice interpretation of the Frobenius ring of operators on the injective hull of the residue field of the quotient ring of a local regular ring. Blickle showed the isomorphism below in [2]. Sharp reformulated this result using a different terminology in [31]. We are going to present a proof of this result using our notations. Note that $(I^{[p^e]} :_S I)$ is the ideal generated by the elements $u \in S$ with $uI \subseteq I^{[p^e]}$, for any $e \geqslant 0$.

Corollary 3.2.4 ([2, Proposition 3.36],[31, Lemma 2.5]). *Let (S, m, k) regular complete local ring of positive prime characteristic p, $I \subseteq S$ an ideal of S and $R = S/I$. Let $e \geqslant 0$ and $q = p^e$.*

There exists an isomorphism of R-modules:

$$\mathcal{F}^e(E_R) \cong \frac{I^{[p^e]} :_S I}{I^{[p^e]}}.$$

Therefore,

$$\mathcal{F}(E_R) \cong \bigoplus_{e \geqslant 0} \frac{I^{[p^e]} :_S I}{I^{[p^e]}}.$$

Proof. Since (S, m, k) is a regular local ring, we have that $E_S \cong H_m^{\dim(S)}(S)$, using Proposition 3.5.4(c) in [7]. Moreover, Lyubeznik and Smith showed that $\mathcal{F}^e(H_m^{\dim(S)}(S)) \cong S\{F^e\}$ in Example 3.7 in [25]. Therefore, the Frobenius ring of operators on E_S, $\mathcal{F}^e(E_S)$ is generated by the canonical Frobenius action on E_S, namely $F_S^e : E_S \to E_S$, $F_S^e(x) = x^{p^e}$, for any $x \in E_S$. Hence, each eth Frobenius action ϕ on E_S is of the form $\phi(x) = uF_S^e(x)$, for every $x \in E_S$, for some $u \in S$.

<u>**Claim** 1</u> Let ϕ be a Frobenius action on E_S given by $\phi(x) := uF_S^e(x)$, for any $x \in E_S$. Then ϕ induces a Frobenius action on E_R if and only if $u \in (I^{[p^e]} :_S I)$.

<u>**Proof of Claim** 1</u> By Lemma 3.2.3, we know that $E_R = \mathrm{Ann}_{E_S}(I) \subseteq E_S$. For the implication " $\Longleftarrow$ ", let $x \in E_R$. We want to show that $\phi(x) \in E_R$. Let $a \in I$. Since $u \in (I^{[p^e]} :_S I)$, $au \in I^{[p^e]}$. Hence, there exists $a_1, \ldots, a_t \in I$ and $s_1, \ldots, s_t \in S$ with $au = \sum_{i=1}^{t} s_i a_i^{p^e}$. So, $a\phi(x) = auF_S^e(x) = \sum_{i=1}^{t} s_i a_i^{p^e} F_S^e(x) = \sum_{i=1}^{t} s_i F_S^e(a_i x) = 0$, for any $x \in E_R = \mathrm{Ann}_{E_S}(I)$, because $a_i x = 0$, for any i. Since a was arbitrarily chosen, we obtain that $\phi(x) \in E_R$, for any $x \in E_R$. For the other implication " $\Longrightarrow$ ", we assume that ϕ induces a Frobenius action on E_R, i.e. $\phi : E_R \to E_R$. Let $a \in I$. Since $\phi(x) \in E_R = \mathrm{Ann}_{E_S}(I)$, for any $x \in E_R$, we have that $a\phi(x) = auF_S^e(x) = 0$, for any $x \in E_R$. Since $a \in I$, we have the inclusion $uI \subseteq \mathrm{Ann}_S(F_S^e(E_R))$.

<u>**Claim** 2</u> We will now prove that $\mathrm{Ann}_S(F_S^e(E_R)) = I^{[p^e]}$.

<u>**Proof of Claim** 2</u> We have that $F_S^e(E_R) = F_S^e(\mathrm{Ann}_{E_S}(I)) \cong F_S^e(\mathrm{Hom}_S(S/I, E_S)) \cong S^{(e)} \otimes_S F_S^e(\mathrm{Hom}_S(S/I, E_S)) \cong \mathrm{Hom}_S(S^{(e)} \otimes_S S/I, S^{(e)} \otimes_S E_S)$, because S/I is a finitely

generated $S-$module and S is a regular ring which implies that $S^{(e)}$ is a flat $S-$module. One can note that $S^{(e)} \otimes_S S/I \cong S/I^{[p^e]}$ and using Example 3.7 in [25] we have that $S^{(e)} \otimes_S E_S \cong E_S$. Therefore, we obtain $F_S^e(E_R) \cong \operatorname{Hom}_S(S/I^{[p^e]}, E_S) \cong \operatorname{Ann}_{E_S}(I^{[p^e]})$. Now using Theorem 3.2.2, $\operatorname{Ann}_S(F_S^e(E_R)) = \operatorname{Ann}_S(\operatorname{Ann}_{E_S}(I^{[p^e]})) = I^{[p^e]}$.

Claim 3 If there exists an eth Frobenius action $\phi \in \mathcal{F}^e(E_R)$, then there exists $u \in (I^{[p^e]} :_S I)$ such that $\phi(x) = uF_S^e(x)$, for any $x \in E_R$.

Proof of Claim 3 We recall that $\mathcal{F}^e(E_R) = \operatorname{Hom}_R(R^{(e)} \otimes_R E_R, E_R)$. Since $\phi \in \mathcal{F}^e(E_R)$, the eth Frobenius action ϕ is an R-homomorphism $\phi : R^{(e)} \otimes_R E_R \to E_R$. After tensoring the canonical surjection $\pi : S^{(e)} \to R^{(e)}$ by E_R and composing it with ϕ :

$$S^{(e)} \otimes E_R \to R^{(e)} \otimes E_R \to E_R$$

we obtain the $S-$homomorphism $\alpha := \phi \circ \pi \circ id : S^{(e)} \otimes E_R \to E_R$ defined by $\alpha(s \otimes x) = sx$, for any $s \in S$ and $x \in E_R$. Since S is a regular ring, $S^{(e)}$ is a flat S-module and by tensoring the exact sequence

$$0 \to E_R \xrightarrow{i} E_S| \otimes_S S^{(e)}$$

we obtain the exact sequence

$$0 \to S^{(e)} \otimes_S E_R \xrightarrow{id \otimes_S i} S^{(e)} \otimes_S E_S.$$

Since E_S is an injective S-module there exists an S-homomorphism $\alpha' : S^{(e)} \otimes_S E_S \to E_S$ which makes the diagram below commute

$$\begin{tikzcd} S^{(e)} \otimes E_R \arrow[r, "\alpha"] \arrow[d, "id\otimes i"'] & E_R \arrow[d, "i"'] \\ S^{(e)} \otimes E_S \arrow[r, dotted, "\alpha'"'] & E_S \end{tikzcd}$$

i.e. $\alpha' \circ (id \otimes i) = i \circ \alpha$. Hence, we obtain $\alpha' \in \operatorname{Hom}_S(S^{(e)} \otimes_S E_S, E_S) = \mathcal{F}^e(E_S)$ with $\alpha'(s \otimes x) = sx$, for any $s \in S$ and $x \in E_S$. Using Lyubeznik and Smith result in [25], there exists $u \in S$ with $\phi(x) = uF_S^e(x)$, for any $x \in E_S$. Since $E_R \subseteq E_S$, we have that $\phi(x) = uF_S^e(x)$, for any $x \in E_R$ and using Claim 1 we obtain $u \in (I^{[p^e]} :_S I)$. This proves the desired isomorphism.

$\square$

Given $u \in (I^{[p^e]} :_S I)$ the corresponding Frobenius action on E_R is given by $r \to u \cdot F_S^e(r)$, for any $r \in E_R$, where $F_S^e : E_S \to E_S$ denotes the standard Frobenius operator defined by $F_S^e(r) = r^{p^e}$, for any $r \in E_S$. The $R-$module structure on

$$\mathcal{F}^e(E_R) \cong \frac{I^{[p^e]} :_S I}{I^{[p^e]}}.$$

is given by the usual multiplication, as follows: $r * uF_S^e = r \cdot uF_S^e$.

Let $u \in \dfrac{I^{[p^e]} :_S I}{I^{[p^e]}}$ and $u' \in \dfrac{I^{[p^{e'}]} :_S I}{I^{[p^{e'}]}}$. The algebra multiplication on $\mathcal{F}(E_R)$ via the isomorphism

$$\mathcal{F}(E_R) \cong \bigoplus_{e \geqslant 0} \frac{I^{[p^e]} :_S I}{I^{[p^e]}}.$$

is given by $u * u' = u \cdot (u')^{p^e}$, for any $e, e' \geqslant 0$. One can note that

$$u * u' \in \frac{I^{[p^{e+e'}]} :_S I}{I^{[p^{e+e'}]}},$$

which shows the $R-$algebra structure of $\mathcal{F}(E_R)$.

Remark 3.2.5. Given $0 \neq \phi \in \mathcal{F}^1(E_R)$, there exists $u \in (I^{[p]} :_S I)$ with $\phi = uF$, where $F : E_R \to E_R$ denotes the canonical Frobenius operator on E_R. In Proposition 4.5 in [20], Katzman showed that the $e-$th iteration of ϕ, denoted by $\phi^e = \phi \circ \ldots \circ \phi$ equals $u^{\nu_e} F^e \in \mathcal{F}^e(E_R)$, where $\nu_e = 1 + p + \ldots + p^{e-1}$ and $F^e : E_R \to E_R$ is the canonical eth Frobenius operator on E_R.

The injective hull of the residue field of the formal power series ring $S = k[[x_1, \ldots, x_n]]$ can be described as $E_S(k) = k[x_1^{-1}, \ldots, x_n^{-1}]$. For example, Brodmann and Sharp presented this result in Example 12.4.1 in [6]. We will give a sketch of the proof here. The Čech complex of S with respect to the maximal ideal $m = (x_1, \ldots, x_n)$:

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \to 0$$

can be described as

$$0 \to S \xrightarrow{d^0} \bigoplus_{i=1}^{n} S_{x_i} \xrightarrow{d^1} \ldots \to \bigoplus_{i=1}^{n} S_{y(i)} \xrightarrow{d^{n-1}} S_{x_1 \cdots x_n} \to 0,$$

where $y(i) = x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n$, for any $1 \leqslant i \leqslant n$. The top local cohomology module of S equals

$$H^n_m(S) = \mathrm{Coker}(S_{y(i)} \xrightarrow{d^{n-1}} S_{x_1 \cdots x_n}) \cong \frac{S_{x_1 \cdots x_n}}{Im(\bigoplus_{i=1}^n S_{y(i)} \to S_{x_1 \cdots x_n})}.$$

The k-vector space $S_{x_1 \cdots x_n}$ has $\{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{Z}^n\}$ as a base.

For each $i = 1, \ldots, n$, $Im(S_{y(i)} \to S_{x_1 \cdots x_n})$ is a $k-$vector subspace with base $\{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{Z}^n, a_i \geqslant 0\}$. We denote by $\mathbb{N}^- := \{n \in \mathbb{Z} : n < 0\}$ the set of negative integers. Hence via the isomorphism above the top local cohomology module $H^n_m(S)$ is a k-vector space with base $\{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \in (\mathbb{N}^-)^n\}$. Therefore, we obtain that $H^n_m(S)$ is the polynomial ring $k[x_1^{-1}, \ldots, x_n^{-1}]$. Now since S is a regular local ring $E_S \cong H^n_m(S) = k[x_1^{-1}, \ldots, x_n^{-1}]$. One can note that the S-module structure on E_S can be described as follows:

$$x_i(x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}, & \text{if } a_i < -1 \\ 0, & \text{if } a_i = -1. \end{cases}$$

for any $(a_1, \ldots, a_n) \in (\mathbb{N}^-)^n$ and $1 \leqslant i \leqslant n$.

3.3 The Frobenius Complexity of Stanley-Reisner Rings

Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be an ideal in S and $R = S/I$. In [21], Katzman described the eth Frobenius actions that come from Frobenius actions of lower degree e', with $e' < e$. For any $e \geqslant 0$ denote $K_e := (I^{[p^e]} :_S I)$ and

$$L_e := \sum_{1\leqslant\beta_1,\dots,\beta_s<e,\beta_1+\dots+\beta_s=e} K_{\beta_1}K_{\beta_2}^{[p^{\beta_1}]}\cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}.$$

Proposition 3.3.1 ([21, Proposition 2.1])**.** *For any $e \geqslant 1$, let $\mathcal{F}_{<e}$ be the R-subalgebra of $\mathcal{F}(E_R)$ generated by $\mathcal{F}^0(E_R), \dots, \mathcal{F}^{e-1}(E_R)$. Then*

$$\mathcal{F}_{<e} \cap \mathcal{F}^e(E_R) = L_e.$$

Therefore, $(G_{e-1})_e \cong \dfrac{L_e + I^{[q]}}{I^{[q]}}$ and $c_e = \mu_S\left(\dfrac{I^{[q]} :_S I}{L_e + I^{[q]}}\right)$.

Let $\mathbf{x}^1$ denote the product of all the variables, i.e. $\mathbf{x}^1 = x_1 \cdots x_n$.

Definition 3.3.2. We define J_q to be the unique minimal monomial ideal satisfying the equality

$$(I^{[q]} : I) = I^{[q]} + J_q + (\mathbf{x}^1)^{q-1}.$$

We will consider the case of Stanley-Reisner rings such that the simplicial complex associated to it has no isolated vertices for the remaining part of this section.

Let k be a field of characteristic p, $S = k[[x_1, \dots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S and $R = S/I$ the Stanley-Reisner ring associated to I.

Let $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kn}) \in \{0,1\}^n$, $1 \leqslant k \leqslant r$, be distinct vectors. The support of the vector α_k is defined as $supp(\alpha_k) = \{i : \alpha_{ki} = 1\}$.

Let $I_{\alpha_k} = (x_i : i \in supp(\alpha_k))$, for every $1 \leqslant k \leqslant r$ and $x^{\alpha_k} = x_1^{\alpha_{k1}} \cdots x_n^{\alpha_{kn}}$ such that $I_{\alpha_1} + I_{\alpha_2} + \cdots + I_{\alpha_r} = (x_1, \dots, x_n)$.

In [27], Àlvarez Montaner, Boix and Zarzuela found a formula for the colon ideals $(I^{[q]} : I)$ based on the minimal primary decomposition of the ideal I.

Proposition 3.3.3 ([27, Proposition 3.2])**.** *If $I = I_{\alpha_1} \cap I_{\alpha_2} \cap \dots \cap I_{\alpha_r}$ is the minimal primary decomposition of the ideal I, then*

$$(I^{[q]} :_S I) = (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \dots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}) = (I_{\alpha_1}^{[q]} + (x^{\alpha_1})^{q-1}) \cap \cdots \cap (I_{\alpha_r}^{[q]} + (x^{\alpha_r})^{q-1}).$$

We will present a different proof of this proposition based on a result of Sharp in [31]:

Proposition 3.3.4 ([31, Proposition 2.8]). *Let $I, Q_1, \ldots, Q_s \subsetneq S$ be ideals of S such that $I = Q_1 \cap \ldots \cap Q_s$ is the minimal primary decomposition of I. Then the following assertions hold:*

(i) $I^{[q]} = (Q_1 \cap \ldots \cap Q_s)^{[q]} = Q_1^{[q]} \cap \ldots \cap Q_s^{[q]}$ *is the minimal primary decomposition of* $I^{[q]}$.

(ii) If $P \in \operatorname{Ass}(I)$, *then* $(I^{[q]} :_S I) \subseteq (P^{[q]} :_S P)$.

(iii) Since $0 \neq I \neq S$, *we have* $(I^{[p]} :_S I) \neq S$. *If* $P_1 := \sqrt{Q_1}$ *is a minimal prime ideal of* I, *then* P_1 *is a minimal prime ideal of* $(I^{[p]} :_S I)$ *and the unique* P_1-*primary component of* $(I^{[p]} :_S I)$ *is* $({Q_1}^{[p]} :_S Q_1)$.

Now we will present our alternative proof of Proposition 3.3.3.

Proof. We will use the following assertions which hold in general for any ideals I, J, I_i, J_i :

$$(\bigcap_i J_i) : I = \bigcap_i (J_i : I)$$

$$J : (\bigcap_i I_i) \supseteq \sum_i (J : I_i).$$

Using Proposition 3.3.4 (i), we have that $I^{[q]} = I_{\alpha_1}^{[q]} \cap I_{\alpha_2}^{[q]} \cap \ldots \cap I_{\alpha_r}^{[q]}$ is the minimal primary decomposition of the ideal $I^{[q]}$. Moreover, using the assertions about colon ideals above we obtain that

$$I^{[q]} : I = (I_{\alpha_1}^{[q]} : I) \cap \ldots \cap (I_{\alpha_r}^{[q]} : I) \supseteq \bigcap_{i=1}^{r} \sum_{j=1}^{r} (I_{\alpha_i}^{[q]} : I_{\alpha_j}).$$

Hence,

$$I^{[q]} : I \supseteq (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}).$$

For the other inclusion, one can note that since I is a square-free monomial ideal, the ideals I_{α_i} are generated by variables. Therefore, I_{α_i} are the minimal prime ideals of I and based on Proposition 3.3.4 (ii), we obtain that $(I^{[q]} :_S I) \subseteq (I_{\alpha_i}^{[q]} :_S I_{\alpha_i})$, for any i. Hence we obtain the second inclusion

$$I^{[q]} : I \subseteq (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}).$$

This proves that the equality holds

$$(I^{[q]} :_S I) = (I^{[q]}_{\alpha_1} :_S I_{\alpha_1}) \cap \ldots \cap (I^{[q]}_{\alpha_r} :_S I_{\alpha_r}).$$

It is easy to note that since the ideals I_{α_i} are generated by variables, namely $I_{\alpha_i} = (x_j : j \in supp(\alpha_i))$ we have that the colon ideal equals

$$(I^{[q]}_{\alpha_i} :_S I_{\alpha_i}) = \bigcap_{j \in supp(\alpha_i)} (I^{[q]}_{\alpha_i} : (x_j)) = \bigcap_{j \in supp(\alpha_i)} (I^{[q]}_{\alpha_i} + (x_j^{q-1})) = (I^{[q]}_{\alpha_i} + (x^{\alpha_i})^{q-1}),$$

for any i. This completes the proof of the proposition. □

Remark 3.3.5. (i) Since the ideals in the intersection are monomial ideals, one can compute the minimal monomial generators of the ideal $(I^{[q]} : I)$ by taking the least common multiples of the minimal monomial generators of the ideals $(I^{[q]}_{\alpha_i} + (x^{\alpha_i})^{q-1})$.

In this way, we can see that the minimal generators $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ of $(I^{[q]} : I)$ satisfy $\gamma_i \in \{0, q-1, q\}$.

(ii) One can notice that the formula obtained for $(I^{[q]} : I)$ depends only on q and on the vectors α_i's. Since the vectors α_i are invariants of the ideal I, we can obtain the minimal monomial generators of $(I^{[q]} : I)$ from the minimal monomial generators of $(I^{[p]} : I)$ by changing p into q.

Example 3.3.6. Let $I = (x_1x_5, x_2x_5, x_2x_3, x_2x_4)$. Then

$$(I^{[q]} : I) = (x_1^q x_5^q, x_2^q x_5^q, x_2^q x_3^q, x_2^q x_4^q, x_1^{q-1} x_2^{q-1} x_5^q, x_2^q x_3^{q-1} x_4^{q-1} x_5^{q-1},$$

$$x_1^{q-1} x_2^{q-1} x_4^q x_5^{q-1}, x_1^{q-1} x_2^{q-1} x_3^q x_5^{q-1}, x_1^{q-1} x_2^{q-1} x_3^{q-1} x_4^{q-1} x_5^{q-1})$$

and therefore

$$J_q = (x_1^{q-1} x_2^{q-1} x_5^q, x_2^q x_3^{q-1} x_4^{q-1} x_5^{q-1}, x_1^{q-1} x_2^{q-1} x_4^q x_5^{q-1}, x_1^{q-1} x_2^{q-1} x_3^q x_5^{q-1}).$$

Lemma 3.3.7. *We have that $J_q \neq 0$ if and only if there exists a generator $x^\gamma \in (I^{[q]} : I)$ having $\gamma_i = q$, $\gamma_j = q-1$ and $\gamma_k = 0$ for some $1 \leqslant i, j, k \leqslant n$.*

Proof. It is trivial to see that if there exists $x^\gamma \in (I^{[q]} : I)$ with $\gamma_i = q$, $\gamma_j = q-1$ and $\gamma_k = 0$ for some $1 \leqslant i, j, k \leqslant n$, then $J_q \neq 0$.

Let us assume that $J_q \neq 0$. By Remark 3.3.5, if $x^\gamma \in J_q$, then $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ must have $\gamma_i \in \{0, q-1, q\}$, for all $i \in \{1, \ldots, n\}$.

Moreover, $x^\gamma = lcm(x^{\theta_1}, \ldots, x^{\theta_r})$, where $x^{\theta_i} \in (I_{\alpha_i}^{[q]} + (x^{\alpha_i})^{q-1})$, for $i \in \{1, \ldots, r\}$.

If $\gamma_i \neq 0$, for all $i \in \{1, \ldots, n\}$, then $(\mathbf{x}^1)^{q-1}$ divides x^γ, hence $x^\gamma \in (\mathbf{x}^1)^{q-1}$.

If $\gamma_i \neq q$, for all $i \in \{1, \ldots, n\}$, then we must have $x^{\theta_i} \in ((x^{\alpha_i})^{q-1})$, for all $i \in \{1, \ldots, r\}$. But that happens only if $x^\gamma \in (\mathbf{x}^1)^{q-1}$.

If $\gamma_i \neq q-1$, for all $i \in \{1, \ldots, n\}$, then there exists at least one $x^{\theta_i} \in (I_{\alpha_i}^{[q]})$, hence we have that $x^\gamma \in I^{[q]}$.

Therefore, if $J_q \neq 0$, then there exists at least one generator $x^\gamma \in J_q$ with $\gamma_i = q$, $\gamma_j = q-1$ and $\gamma_k = 0$ for some $1 \leqslant i, j, k \leqslant n$. □

In [27], Àlvarez Montaner, Boix and Zarzuela found that there are only four cases that may occur, considering the minimal primary decomposition of the ideal I and the heights of the ideals I_{α_i} :

Proposition 3.3.8. *There are only four posibilities for the minimal generators of $(I^{[q]} : I)$:*

(i) *Assume $ht(I_{\alpha_i}) > 1$, for all $i = 1, \ldots, r$.*

  (a) $(I^{[q]} : I) = I^{[q]} + (\boldsymbol{x}^1)^{q-1}$.

  (b) $(I^{[q]} : I) = I^{[q]} + J_q + (\boldsymbol{x}^1)^{q-1}$, $J_q \subsetneq I^{[q]} + (\boldsymbol{x}^1)^{q-1}$.

(ii) *Assume $ht(I) = 1$ and and there exists an $i \in \{1, \ldots, r\}$ such that $ht(I_{\alpha_i}) > 1$.*

In this case, $(I^{[q]} : I) = J_q + (\boldsymbol{x}^1)^{q-1}$, with $J_q \subsetneq (\boldsymbol{x}^1)^{q-1}$.

(iii) *Assume $ht(I_{\alpha_i}) = 1$ for all $i \in \{1, \ldots, r\}$.*

Then $(I^{[q]} : I) = (\boldsymbol{x}^1)^{q-1}$.

The Frobenius algebra $\mathcal{F}(E_R)$ is principally generated in cases $(i.a)$ and (iii) and is infinitely generated in cases $(i.b)$ and (ii).

As a direct consequence of this result, Àlvarez Montaner, Boix and Zarzuela showed that:

Theorem 3.3.9 ([27, Proposition 3.4])**.** *The Frobenius algebra $\mathcal{F}(E_R)$ associated to a Stanley-Reisner ring R is either principally generated or infinitely generated.*

Corollary 3.3.10. *If R is a Stanley-Reisner ring, the Frobenius complexity of R is either $-\infty$ or 0.*

Proof. By Theorem 3.3.9, we know that the Frobenius algebra $\mathcal{F}(E_R)$ associated to a Stanley-Reisner ring R is either principally generated or infinitely generated. If $\mathcal{F}(E_R)$ is principally generated, we have that $cx(\mathcal{F}(E_R)) = 0$, which implies $cx_F(R) = -\infty$. In the case when $\mathcal{F}(E_R)$ is infinitely generated, Remark 3.3.12(ii) shows that the complexity sequence $\{c_e\}_{e\geqslant 2}$ is bounded by the minimal number of generators of the ideal J_p. Remark 2.4.3 (iii) implies $cx(\mathcal{F}(E_R)) = 1$, which proves that $cx_F(R) = 0$, in this case. □

As a consequence of Proposition 3.3.8 and Proposition 5.1.9, we obtain the following result:

Corollary 3.3.11. *Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free ideal in S and $R = S/I$ its Stanley-Reisner ring. The ring of Frobenius operators $\mathcal{F}(E_R)$ is principally generated as an R-skew algebra if and only if the R-module $\dfrac{I^{[p^e]} :_S I}{I^{[p^e]}}$ is cyclic. Moreover, if $\mathcal{F}(E_R)$ is principally generated as an R-skew algebra, then it is generated by $(\boldsymbol{x}^1)^{p-1}F$, where $F : E_R \to E_R$ is the canonical Frobenius operator on E_R.*

Proof. Using Remark 5.1.5, we know that in order to prove that $\mathcal{F}(E_R)$ is principally generated as an R-skew algebra it is enough to show that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra. Now the conclusion follows directly from Proposition 5.1.9. Proposition 5.2.3 gives us the second statement of our claim. □

Remark 3.3.12. (i) In the case when $\mathcal{F}(E_R)$ is infinitely generated, $\mathcal{F}^1(E_R)$ has $\mu+1$ minimal generators, μ of them being the minimal generators of J_p and $(\mathbf{x}^1)^{p-1}$. Each graded piece $\mathcal{F}^e(E_R)$ adds up μ new generators coming from J_q.

(ii) The complexity sequence $\{c_e\}_{e\geqslant 2}$ is bounded by the minimal number of generators of the ideal J_p, i.e. $c_e \leqslant \mu_S(J_p)$, for any $e \geqslant 2$. Note that $c_1 = \mu+1$ and $c_0 = 0$.

Definition 3.3.13. Let $Supp(J_q)$ be the set of all the supports of the minimal monomial generators of J_q. We define $\Gamma := Supp(J_q)$ to be the *support set* of the ring R. Then $(\Gamma, \subseteq)$ is a partially ordered set.

Definition 3.3.14. Let Γ be the support set of a Stanley-Reisner ring $R = S/I$. Let $Min(\Gamma)$ be the set of elements in Γ which are minimal with respect to inclusion. We call Γ *minimal* if $\Gamma \neq \emptyset$ and $Min(\Gamma) = \Gamma$.

Definition 3.3.15. Let Γ be the support set of a Stanley-Reisner ring $R = S/I$.

We call Γ *nearly minimal* if $\Gamma \neq \emptyset$ and for every $\gamma \in \Gamma$ which is not minimal in Γ with respect to $\subseteq$ there exists at most one element $\gamma' \in \Gamma$ with $\gamma' \subsetneq \gamma$.

Example 3.3.16. Let $I = (x_1x_2, x_1x_3, x_2x_4)$. Then

$$J_q = (x_1^q x_2^{q-1} x_3^{q-1}, x_1^{q-1} x_2^q x_4^{q-1}).$$

The support set is

$$\Gamma = \{(1,2,3), (1,2,4)\}.$$

In this case, Γ is minimal.

Example 3.3.17. The support set of the ideal in Example 3.3.6 is

$$\Gamma = \{(1,2,5), (2,3,4,5), (1,2,4,5), (1,2,3,5)\}.$$

In this case, Γ is not minimal, but it is nearly minimal.

Example 3.3.18. Let $I = (x_1x_2, x_1x_3x_4, x_1x_3x_5)$. Then

$$J_q = (x_1^{q-1}x_3^q x_5^q, x_1^{q-1}x_3^q x_4^q, x_1^{q-1}x_2^{q-1}x_3^{q-1}x_5^q, x_1^{q-1}x_2^{q-1}x_3^{q-1}x_4^q, x_1^{q-1}x_2^q).$$

The support set is $\Gamma = \{(1,2),(1,3,4),(1,3,5),(1,2,3,4),(1,2,3,5)\}$. Since $(1,2,3,4)$ contains $(1,2)$ and $(1,3,4)$, Γ is not nearly minimal in this case.

Question 3.3.19. Is the Frobenius complexity of a Stanley-Reisner ring preserved by taking the Alexander dual?

We will give an example of a Stanley-Reisner ring R having $cx_F(R) = 0$ such that the Stanley-Reisner ring associated to its Alexander dual has $cx_F(R^V) = -\infty$. This example is presented in [5]. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) = (x_1, x_2) \cap (x_3, x_4)$. Then, the Alexander dual has

$$I^V = (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_2, x_4) = (x_1x_2, x_3x_4).$$

Since $J_q(I^V) = 0$ we obtain $cx_F(R^V) = -\infty$. We obtain that $J_q(I) \neq 0$, hence $cx_F(R) = 0$. Therefore, we have an example of a Stanley-Reisner ring having an infinetely generated Frobenius algebra of operators on E_R, whose Alexander dual has a principally generated Frobenius algebra of operators.

CHAPTER 4

MAIN RESULTS

In this chapter, we will present our main results following our work in [19] and [8].

4.1 On the Frobenius Complexity sequence of Stanley-Reisner Rings

The work in this section will be presented based on our results in [19]. In this section, we will prove that the complexity sequence $\{c_e\}_{e\geqslant 0}$ of the Frobenius algebra of operators of the injective hull of the residue field of any Stanley-Reisner ring with non-empty support set stabilizes starting with $e = 2$.

Let k be a field of characteristic p, $S = k[[x_1, \dots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S and $R = S/I$ the Stanley-Reisner ring associated to I. We will assume that the simplicial complex associated to the ring R has no isolated vertices and use the notations introduced in the previous section.

Lemma 4.1.1. *Let $e \geqslant 0$ an integer and suppose that $J_q \neq 0$. Let x^{γ_e} be a minimal monomial generator of J_q. If there exists a minimal monomial generator $x^{\gamma'_e}$ of J_q with $supp(\gamma'_e) \subsetneq supp(\gamma_e)$, then there exists at least one variable x_k such that*

$$deg_{x_k}(x^{\gamma_e}) = q - 1 \text{ and } deg_{x_k}(x^{\gamma'_e}) = q.$$

Proof. We will prove the lemma by contradiction. Assume not. Then, for all the variables x_k with $deg_{x_k}(x^{\gamma'_e}) = q$, we have that $deg_{x_k}(x^{\gamma_e}) \neq q - 1$, therefore $deg_{x_k}(x^{\gamma_e}) \in \{0, q\}$, by Remark 3.3.5.

But since $supp(\gamma'_e) \subsetneq supp(\gamma_e)$, we have that $deg_{x_k}(x^{\gamma_e}) > 0$. Hence, $deg_{x_k}(x^{\gamma_e}) = q$.

Then, we have that $x^{\gamma'_e}$ divides x^{γ_e}, which is a contradiction. □

Definition 4.1.2. Let $e \geqslant 0$ an integer and suppose that $J_p \neq 0$. Let $x^{\gamma} \in J_p$ a minimal monomial generator. Using Remark 3.3.5, we have a bijective correspondence between the minimal monomial generators of J_p and the minimal monomial generators of J_q. Under this map, there exists $x^{\gamma_e} \in J_q$ which corresponds to $x^{\gamma} \in J_p$. We define $M_e(\gamma) \subseteq K_e$ to be the ideal generated by the minimal monomial generators $x^{\delta} \in J_q$ with $supp(x^{\delta}) \subseteq supp(x^{\gamma_e}) = supp(x^{\gamma})$.

Lemma 4.1.3. *Let $e \geqslant 0$ an integer and suppose that $J_p \neq 0$. Let $x^{\gamma} \in J_p$ a minimal monomial generator. Let $1 \leqslant \beta_1, \ldots, \beta_s < e$ with $\beta_1 + \ldots + \beta_s = e$.*

Then,

$$x^{\gamma_e} \in K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]} \textit{ if and only if}$$
$$x^{\gamma_e} \in M_{\beta_1}(\gamma)(M_{\beta_2}(\gamma))^{[p^{\beta_1}]} \cdots (M_{\beta_s}(\gamma))^{[p^{\beta_1+\cdots+\beta_{s-1}}]}.$$

Proof. Let $x^{\gamma_e} \in M_{\beta_1}(\gamma)(M_{\beta_2}(\gamma))^{[p^{\beta_1}]} \cdots (M_{\beta_s}(\gamma))^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$.

Since $M_{\beta_i}(\gamma) \subseteq K_{\beta_i}$, we obtain that $x^{\gamma_e} \in K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$.

Now, let $x^{\gamma_e} \in K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$. Then for every $i \in \{1, \ldots, s\}$, there exists $m_{\beta_i} \in K_{\beta_i}$ such that $x^{\gamma_e} = m_{\beta_1} m_{\beta_2}^{p^{\beta_1}} \cdots m_{\beta_s}^{p^{\beta_1+\cdots+\beta_{s-1}}} \cdot m$, for some $m \in S$.

If there exists at least an i with $supp(m_{\beta_i}) \nsubseteq supp(x^{\gamma_e})$, there exists at least one $x_k \in supp(m_{\beta_i}) \setminus supp(x^{\gamma_e})$. But this contradicts the equality $x^{\gamma_e} = m_{\beta_1} m_{\beta_2}^{p^{\beta_1}} \cdots m_{\beta_s}^{p^{\beta_1+\cdots+\beta_{s-1}}} \cdot m$. Therefore, we must have that $supp(m_{\beta_i}) \subseteq supp(x^{\gamma_e})$, for all $i \in \{1, \ldots, s\}$.

Hence, $x^{\gamma_e} \in M_{\beta_1}(\gamma)(M_{\beta_2}(\gamma))^{[p^{\beta_1}]} \cdots (M_{\beta_s}(\gamma))^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$. $\square$

Proposition 4.1.4. *Let $e \geqslant 2$ an integer and suppose that $J_q \neq 0$. If all the minimal monomial generators of J_q are not contained in L_e, then $c_e = c_{e+1}$, for all $e \geqslant 2$.*

Proof. Let $e \geqslant 2$ and let x^{γ_e} a minimal monomial generator of J_q. We know that x^{γ_e} is not contained in L_e. So, we obtain that $\bar{0} \neq \overline{x^{\gamma_e}} \in \left(\dfrac{I^{[q]} :_S I}{L_e + I^{[q]}}\right)$. Since x^{γ_e} was arbitrarly chosen in J_q and $c_e = \mu_S\left(\dfrac{I^{[q]} :_S I}{L_e + I^{[q]}}\right)$, we have that $c_e = \mu_S(J_q)$, for all $e \geqslant 2$. Therefore, $c_e = c_{e+1}$, for all $e \geqslant 2$. $\square$

Remark 4.1.5. In order to show that the complexity sequence $\{c_e\}_{e \geqslant 0}$ stabilizes starting with $e = 2$, it is enough to show that all the minimal monomial generators of J_q are not contained in L_e.

We will first show that the complexity sequence $\{c_e\}_{e\geqslant 0}$ stabilizes starting with $e = 2$ for Stanley-Reisner rings with nearly-minimal support set.

Theorem 4.1.6. *Let $e \geqslant 0$ an integer and suppose that $J_q \neq 0$. Let x^{γ_e} be a minimal monomial generator of J_q. If Γ is nearly minimal, then x^{γ_e} is not contained in L_e.*

Proof. Since Γ is nearly minimal, there exists at most one $x^{\gamma'_e} \in J_q$, with $supp(\gamma'_e) \subsetneq supp(\gamma_e)$ and $supp(\gamma'_e)$ minimal with respect to $\subseteq$ in Γ.

By Lemma 3.3.7, we have that

$$x^{\gamma_e} = x_{i_1}^q x_{i_2}^{q-1} x_{i_3}^0 x_{i_4}^{\gamma_4} \cdots x_{i_n}^{\gamma_n}$$

and

$$x^{\gamma'_e} = x_{j_1}^q x_{j_2}^{q-1} x_{j_3}^0 x_{j_4}^{\gamma'_4} \cdots x_{j_n}^{\gamma'_n}.$$

We will show that $x^{\gamma_e} \notin L_e$.

If $x^{\gamma_e} \in L_e$, there exists $1 \leqslant \beta_1, \ldots, \beta_s < e$ with $\beta_1 + \ldots + \beta_s = e$ with $x^{\gamma_e} \in K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$.

Using Lemma 4.1.3, we have that $x^{\gamma_e} \in M_{\beta_1}(\gamma) M_{\beta_2}(\gamma)^{[p^{\beta_1}]} \cdots M_{\beta_s}(\gamma)^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$, where $M_{\beta_i}(\gamma) := (x^{\gamma_{\beta_i}}, x^{\gamma'_{\beta_i}})$, for all $i \in \{1, \ldots, s\}$. In particular, $M_e(\gamma) := (x^{\gamma_e}, x^{\gamma'_e})$.

Then there exists $m_{\beta_i} \in G_{\beta_i}$ such that $m_{\beta_1} m_{\beta_2}^{p^{\beta_1}} \cdots m_{\beta_s}^{p^{\beta_1+\ldots+\beta_{s-1}}}$ divides x^{γ_e}.

By Lemma 4.1.1, there exists at least one variable x_k such that

$$deg_{x_k}(x^{\gamma_e}) = q - 1 \text{ and } deg_{x_k}(x^{\gamma'_e}) = q,$$

for all $e \geqslant 0$.

Since $deg_{x_k}(m_{\beta_i}) \geqslant p^{\beta_i} - 1$,

$$deg_{x_k}(x^{\gamma_e}) = q - 1 \geqslant (p^{\beta_1} - 1) + \ldots + (p^{\beta_s} - 1)p^{\beta_1+\ldots+\beta_{s-1}} = q - 1.$$

Therefore, we should have equality.

Hence, we have that m_{β_i} is divisible by $x^{\gamma_{\beta_i}}$, for all i.

Now, if we look at the degree of x_{i_1}, we have that

$$deg_{x_{i_1}}(x^{\gamma_e}) = q \geqslant p^{\beta_1} + p^{\beta_1}p^{\beta_2} + \ldots + p^{\beta_1+\ldots+\beta_{s-1}}p^{\beta_s},$$

which gives a contradiction.

Therefore, $x^{\gamma_e} \notin L_e$.

$\square$

Now, we will drop the condition on the support set and we will show that the complexity sequence stabilizes for any Stanley-Reisner ring as presented in [19].

Theorem 4.1.7. *Let $e \geqslant 0$ an integer and suppose that $J_q \neq 0$. Let x^{γ_e} be a minimal monomial generator of J_q. Then, x^{γ_e} is not contained in L_e.*

Proof. Let $x^{\delta_e^{(i)}} \in J_q$ be a minimal monomial generator with

$$supp(\delta_e^{(1)}), \ldots, supp(\delta_e^{(k)}) \subsetneq supp(\gamma_e),$$

where $k \geqslant 0$. Note that if $k = 0$, all the minimal monomial generators of x^{γ_e} have minimal support. Note that

$$M_e(\gamma) := (x^{\gamma_e}, x^{\delta_e^{(j)}} : j = 1, \ldots, k).$$

We want to show that $x^{\gamma_e} \notin L_e$.

If $x^{\gamma_e} \in L_e$, there exists $1 \leqslant \beta_1, \ldots, \beta_s < e$ with $\beta_1 + \ldots + \beta_s = e$ with $x^{\gamma_e} \in K_{\beta_1}K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$.

Using Lemma 4.1.3, we have that $x^{\gamma_e} \in M_{\beta_1}(\gamma)(M_{\beta_2}(\gamma))^{[p^{\beta_1}]} \cdots (M_{\beta_s}(\gamma))^{[p^{\beta_1+\cdots+\beta_{s-1}}]}$, where

$M_{\beta_i}(\gamma) := (x^{\gamma_{\beta_i}}, x^{\delta_{\beta_i}^{(j)}} : j = 1, \ldots, k)$, for all $i \in \{1, \ldots, s\}$.

Then there exists $m_{\beta_i} \in M_{\beta_i}(\gamma)$ such that $m_{\beta_1}m_{\beta_2}^{p^{\beta_1}} \cdots m_{\beta_s}^{p^{\beta_1+\ldots+\beta_{s-1}}}$ divides x^{γ_e}.

By Lemma 4.1.1, we have the following:

For any $j \in \{1, \ldots, k\}$, there exists at least one variable $x_{o(j)} \in supp(\delta_e^{(j)})$ with

$$deg_{x_{o(j)}}(x^{\gamma_e}) = q - 1 \text{ and } deg_{x_{o(j)}}(x^{\delta_e^{(j)}}) = q,$$

for all $e \geqslant 0$. If m_{β_s} is a multiple of $x^{\delta_{\beta_s}^{(j)}}$, for some $j \in \{1, \ldots, k\}$, using the Lemma 4.1.1 there exists $x_{o(j)} \in supp(\delta_e^{(j)})$ with

$$deg_{x_{o(j)}}(x^{\gamma_{\beta_s}}) = p^{\beta_s} - 1 \text{ and } deg_{x_{o(j)}}(x^{\delta^{(j)}_{\beta_s}}) = p^{\beta_s}.$$

Then, we obtain that

$$deg_{x_{o(j)}}(x^{\gamma_e}) = q - 1 \geqslant deg_{x_{o(j)}}(m_{\beta_s}) \cdot p^{\beta_1+\ldots+\beta_{s-1}} \geqslant q,$$

which is a contradiction.

Therefore, we must have that m_{β_s} is a multiple of $x^{\gamma_{\beta_s}}$.

Now for those variables x_r with $deg_{x_r}(x^{\gamma_e}) = q$, we have that

$$deg_{x_r}(x^{\gamma_e}) = q \geqslant deg_{x_r}(m_{\beta_s}) \cdot p^{\beta_1+\ldots+\beta_{s-1}} \geqslant q,$$

so we must have equality.

That means that $deg_{x_r}(m_{\beta_1} m_{\beta_2}^{p^{\beta_1}} \cdots m_{\beta_{s-1}}^{p^{\beta_1+\ldots+\beta_{s-2}}}) = 0$, which implies that m_{β_j} is not a multiple of $x^{\gamma_{\beta_j}}$, for all $j \in \{1, \ldots, s-1\}$.

In particular, we have that $m_{\beta_{s-1}}$ is a multiple of $x^{\delta^{(j)}_{\beta_{s-1}}}$, for some $j \in \{1, \ldots, k\}$.

Using the Lemma 4.1.1 again, we know that there exists a variable $x_t \in supp(\delta_e^{(j)})$ with

$$deg_{x_t}(x^{\gamma_{\beta_{s-1}}}) = p^{\beta_{s-1}} - 1 \text{ and } deg_{x_t}(x^{\delta^{(j)}_{\beta_{s-1}}}) = p^{\beta_{s-1}}.$$

Hence

$$deg_{x_t}(x^{\gamma_e}) = q - 1 \geqslant deg_{x_t}(m_{\beta_{s-1}}) \cdot p^{\beta_1+\ldots+\beta_{s-2}} + deg_{x_t}(m_{\beta_s}) \cdot p^{\beta_1+\ldots+\beta_{s-1}}$$

$$\geqslant p^{\beta_{s-1}} \cdot p^{\beta_1+\ldots+\beta_{s-2}} + (p^{\beta_s} - 1) \cdot p^{\beta_1+\ldots+\beta_{s-1}} = q,$$

which gives us a contradiction.

We proved that $x^{\gamma_e} \notin L_e$. □

Corollary 4.1.8. *Let R be a Stanley-Reisner ring such that the simplicial complex associated to it has no isolated vertices. Then the complexity sequence of the Frobenius algebra of operators on the injective hull of the residue field of the ring R is given by*

$$\{c_e\}_{e \geqslant 0} = \{0, \mu+1, \mu, \mu, \mu, \ldots\},$$

where $\mu := \mu_S(J_p)$.

Remark 4.1.9. Corollary 4.1.8 implies Theorem 4.9 in [5].

So far, we worked with Stanley-Reisner rings satisfying $I_{\alpha_1}+I_{\alpha_2}+\cdots+I_{\alpha_r} = (x_1, \ldots, x_n)$, and we showed that for these rings, the complexity sequence stabilizes starting with $e = 2$.

Now our main goal will be to extend this result to all the Stanley-Reisner rings, by dropping the condition on the supports of the minimal prime ideals in the minimal primary decomposition of the ideal I. Let $c_{e,R} := c_e(\mathcal{F}(E_R))$.

Theorem 4.1.10. *Let $(S, m) \to (T, n)$ be a flat, local extension of regular local rings and let $I \leqslant S$ be an ideal in S.*

Let $R := \dfrac{S}{I}$ and $R' := \dfrac{T}{IT}$. Then, $c_{e,R} = c_{e,R'}$, for all $e \geqslant 0$.

Proof. We know that $c_{e,R} = \mu_S\left(\dfrac{I^{[q]} :_S I}{L_{e,R} + I^{[q]}}\right)$ and $c_{e,R'} = \mu_T\left(\dfrac{(IT)^{[q]} :_S (IT)}{L_{e,R'} + (IT)^{[q]}}\right)$, for all $e \geqslant 0$.

Since $S \to T$ is a flat extension of rings, we have that $(IT)^{[q]} :_S (IT) = (I^{[q]} :_S I)T$, for all $e \geqslant 0$. Hence, we obtain that $L_{e,R'} = L_eT$. Moreover, $(IT)^{[q]} = I^{[q]}T$, for all $e \geqslant 0$. Therefore,$c_{e,R'} = \mu_T\left(\dfrac{(I^{[q]} :_S I)T}{(L_{e,R} + I^{[q]})T}\right)$, for all $e \geqslant 0$.

Let $A := (I^{[q]} :_S I)$, $B := (I^{[q]}+L_e)$ and $M := \dfrac{A}{B}$. Now in order to show that $c_{e,R} = c_{e,R'}$, for all $e \geqslant 0$, it sufices to prove that $\mu_S(M) = \mu_T(T \otimes_S M)$.

It is enought to show that $\mu_K\left(\dfrac{M}{mM}\right) = \mu_K\left(\dfrac{T \otimes_S M}{n(T \otimes_S M)}\right)$, where K is the residue field S/m.

Let $K := \dfrac{S}{m}$ and $L := \dfrac{T}{n}$. By tensoring the exact sequence

$$0 \longrightarrow n \longrightarrow T \longrightarrow L \longrightarrow 0| \otimes_S M$$

we obtain the following exact sequence

$$\ldots \longrightarrow Tor_1(L, M) \longrightarrow n \otimes_S M \longrightarrow T \otimes_S M \longrightarrow L \otimes_S M \longrightarrow 0.$$

Hence, we have that $Im(n \otimes_S M \longrightarrow T \otimes_S M) = Ker(T \otimes_S M \longrightarrow L \otimes_S M)$.

By the Fundamental Theorem of Isomorphism,

$$\frac{T \otimes_S M}{Ker(T \otimes_S M \longrightarrow L \otimes_S M)} \cong Im(T \otimes_S M \longrightarrow L \otimes_S M).$$

But the map $T \otimes_S M \longrightarrow L \otimes_S M$ is surjective, therefore

$$Im(T \otimes_S M \longrightarrow L \otimes_S M) = L \otimes_S M.$$

Moreover, it is easy to see that $Im(n \otimes_S M \longrightarrow T \otimes_S M) = n(T \otimes_S M)$.

Hence, we showed that

$$\frac{T \otimes_S M}{n(T \otimes_S M)} \cong L \otimes_S M.$$

In order to complete the proof, we will show that $\mu_L(L \otimes_S M) = \mu_K\left(\frac{M}{mM}\right)$.

We have that

$$L \otimes_S M \cong L \otimes_K \frac{S}{m} \otimes_S M \cong L \otimes_K \frac{M}{mM}.$$

Hence, $\mu_L(L \otimes_S M) = \mu_L(L \otimes_K \frac{M}{mM})$. It is easy to see that

$$\mu_L(L \otimes_K \frac{M}{mM}) = \mu_K\left(\frac{M}{mM}\right),$$

which completes the proof. $\square$

Corollary 4.1.8 and Theorem 4.1.10 give us the following result

Theorem 4.1.11. *Let k be a field of characteristic p, $S = k[[x_1, \dots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S*

and $R = S/I$ its Stanley-Reisner ring. Then,

$$\{c_e\}_{e \geqslant 0} = \{0, \mu + 1, \mu, \mu, \mu, \dots\},$$

where $\mu := \mu_S(J_p)$.

Proof. Let $I_{\alpha_1} + I_{\alpha_2} + \dots + I_{\alpha_r} = (x_1, \dots, x_m)$, where $1 \leqslant m < n$. Since $k[[x_1, \dots, x_m]] \subseteq k[[x_1, \dots, x_n]]$ is a flat local extension of regular local rings, we are under the hypothesis of Theorem 4.1.10. Corollary 4.1.8 combined with Theorem 4.1.10 give us the desired conclusion. $\square$

Remark 4.1.12. Using the notations in the proof of 4.1.11, one can notice that

$$\mu_{k[[x_1,\dots,x_m]]}(J_p) = \mu_S(J_p).$$

Remark 4.1.13. Hence, we showed that the Frobenius complexity sequence, which is a positive characteristic invariant of our ring is in fact a combinatorial invariant introduced by Àlvarez Montaner, Boix and Zarzuela in [5], the number of maximal free pairs of the simplicial complex associated to our ring. Moreover, our result shows that the complexity sequence is independent on the characteristic of the ring in this case.

In [26], Àlvarez Montaner defined the generating function of a skew R-algebra using the complexity sequence.

Definition 4.1.14. (see Definition 2.1 in [26]) The generating function of $\mathcal{F}(E_R)$ is defined as

$$\mathcal{G}_{\mathcal{F}(E_R)}(T) = \sum_{e \geqslant 0} c_e T^e.$$

Note that in [26] the author takes $c_0 = 1$. As a consequence of Theorem 4.1.11, we obtain the generating function of the Frobenius algebra of operators on the injective hull of the residue field of any Stanley-Reisner ring.

Corollary 4.1.15. *Let k be a field of characteristic p, $S = k[[x_1, \dots, x_n]]$ and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S, $R = S/I$ its Stanley-Reisner ring.*

Then the generating function of the Frobenius algebra of operators is

$$\mathcal{G}_{\mathcal{F}(E_R)}(T) = (\mu+1)T + \sum_{e \geqslant 2} \mu T^e = \frac{(\mu+1)T - T^2}{1-T}.$$

Proof. Note that by Definition 2.4.3, $c_0 = 0$. Using the Theorem 4.1.11, we have that $c_1(R) = \mu + 1$ and $c_e = \mu$, for every $e \geqslant 2$. □

We will end this section presenting the formula we obtained for the complexity sequence of the T-construction of Stanley-Reisner rings. Let Δ be a simplicial complex generated by the set of facets $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$. Let k be a field of characteristic p and $\mathcal{R} = k[\Delta] = k[x_1, \dots, x_d]/I_\Delta$ the Stanley-Reisner ring associated to the simplicial complex Δ.

Theorem 4.1.16. *The complexity sequence of the T-construction of the ring $\mathcal{R}$ is given by*

$$c_e = \sum_{i=1}^{m} c_{|F_i|,e} - \sum_{i \neq j} c_{|F_i \cap F_j|,e} + \sum_{1 \leqslant i<j<k \leqslant m} c_{|F_i \cap F_j \cap F_k|,e} - \dots + (-1)^{m-1} c_{|F_1 \cap \dots \cap F_m|,e}.$$

Proof. We recall that the $T-$construction of the Stanley-Reisner ring $\mathcal{R} = k[\Delta]$ is

$$T(\mathcal{R}) = \oplus_e T_e = \oplus_{e \geqslant 0} \mathcal{R}_{p^e - 1}$$

whose ring structure is defined by $a * b = ab^{p^e}$, for all $a \in T_e$ and $b \in T_{e'}$. The zero degree component of $T(\mathcal{R})$ is $\mathcal{R}_0 = k$. It is easy to note that $T_e = \mathcal{R}_{p^e-1}$ is an $k-$vector space with basis given by the monomials in $\mathcal{R} = k[\Delta]$ of total degree $p^e - 1$. Keeping the same notations that we used to define the complexity sequence in the first chapter, we let $G_{e-1} := G_{e-1}(T(\mathcal{R}))$ be the $k-$vector space generated by the monomials that can be written as products of monomials of degree $p^i - 1$, where $i \leqslant e-1$. Moreover, $(G_{e-1})_e$ consists of these monomials having total degree $p^e - 1$. The general term of the complexity sequence of $T(\mathcal{R})$, $c_e := c_e(T(\mathcal{R}))$ is the number of minimal monomial generators of the $k-$vector space $\dfrac{T_e}{(G_{e-1})_e}$ of degree $p^e - 1$ which cannot be written as products of monomials of degree $p^i - 1$ with $i \leqslant e-1$. Since $I_\Delta = (x_F : F \notin \Delta)$, the minimal monomial generators of the $k-$vector space $\dfrac{T_e}{(G_{e-1})_e}$ come from the following sets of monomials

$$\mathcal{M}_j^{(e)} := \{x_{F_j}^{a_j} := x_{i_1}^{a_{i_1}} \cdots x_{i_d}^{a_{i_d}} : a_{i_1} + \dots + a_{i_d} = p^e - 1, a_{i_1}, \dots, a_{i_d} \geqslant 0\},$$

with $1 \leqslant j \leqslant m$ and $1 \leqslant i_1, \dots, i_d \leqslant d$. We will denote by $c_{|F_j|,e}$ the number of minimal monomial generators of $\mathcal{M}_j^{(e)}$, which cannot be written as products of monomials of $\mathcal{M}_j^{(k)}$ with $k \leqslant e-1$. In order to compute $c_e(T(\mathcal{R}))$, we have to consider all these sets of monomials $\mathcal{M}_j^{(e)}$ and exclude the monomials that appear with repetitions. The principle of inclusion and exclusion applied on the sets $\mathcal{M}_j^{(e)}$ states that

$$|\bigcup_{j=1}^{m}\mathcal{M}_j^{(e)}| = \sum_{j=1}^{m}|\mathcal{M}_j^{(e)}| - \sum_{1\leqslant j_1<j_2\leqslant m}|\mathcal{M}_{j_1}^{(e)}\cap\mathcal{M}_{j_2}^{(e)}| + \ldots + (-1)^{m+1}|\mathcal{M}_1^{(e)}\cap\ldots\cap\mathcal{M}_m^{(e)}|.$$

This allows us to compute the complexity sequence of the $T-$construction of $k(\Delta)$ as follows

$$c_e = \sum_{i=1}^{m} c_{|F_i|,e} - \sum_{i\neq j} c_{|F_i\cap F_j|,e} + \sum_{1\leqslant i<j<k\leqslant m} c_{|F_i\cap F_j\cap F_k|,e} - \ldots + (-1)^{m-1}c_{|F_1\cap\ldots\cap F_m|,e}.$$

□

4.2 Strong Test Ideals associated to Cartier Algebras

In this section, I will present results on strong test ideals for Stanley-Reisner rings following joint work with Enescu that has appeared in [8].

Let $(R,\mathfrak{m},k)$ denote a local F-finite reduced ring of prime positive characteristic p. We have stated in the first chapter Huneke's remark which states that the number of minimal generators of a strong test ideal represents an uniform bound for the minimal degree of the equation of integral dependence of an arbitrary element $x\in I^*$ over I, where I is an ideal of R. Therefore, having a larger class of strong test ideals can give a better bound. In [22], Katzman and Schwede have produced an algorithm, which was implemented in Macaulay2, that computes all ϕ-compatible ideals of a surjective R-linear map $\phi : R^{1/q}\to R$.

Let $\phi : R^{1/q}\to R$ a surjective R-linear map. In order to compute the test ideal $\tau(R,\phi)$, which is the smallest ϕ-compatible ideal with respect to inclusion, we have to intersect all the ϕ-compatible prime ideals .

By Fedder's Lemma 4.2.4, we know that there exists an S-linear map $\Phi : S^{1/q}\to S$ which is compatible with I such that $\phi = \Phi/I$. Now, if we want to determine the ϕ-compatible prime ideals, Lemma 2.4 in [22] tells us that it is enough to determine the Φ-compatible prime ideals that contain I, since there is a bijective correspondence between the ϕ-compatible ideals and the Φ-compatible ideals containing I.

Next, we have to eliminate from this list the set of minimal primes of the ideal I, otherwise by intersecting them and moding out the result by the ideal I we obtain the zero ideal. Then the class of the ideal obtained after intersecting the remaining ideals modulo I is the test ideal $\tau(R,\phi)$. Therefore, we have a concrete way of computing strong test ideals for F-pure rings. The following is an example due to Katzman and, further studied by Katzman and Schwede in [22], which illustrates this idea. In the following examples, we will generally use the same letter to denote an element of S and its image in S/I, when it is harmless to do so, to avoid complicating the notation.

Example 4.2.1. Let $k=\mathbb{F}_2$ and let $S=k[[x_1,\ldots,x_5]]$. Let $\mathcal{I}$ be the ideal generated by the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & x_2 & x_5 \\ x_4 & x_4 & x_3 & x_1 \end{pmatrix}.$$

Consider $R=S/\mathcal{I}$. The ring R is Cohen-Macaulay reduced and two dimensional.

Let $\phi: R^{1/2}\to R$ an R-linear map constructed as follows:

Let $S^{1/2}=k[[x_1^{1/2},\ldots,x_5^{1/2}]]$ which is a free S-module with basis $\{x_1^{\lambda_1/2}x_2^{\lambda_2/2}\cdots x_5^{\lambda_5/2}\}_{0\leqslant\lambda_i\leqslant 1}$. Construct $\Phi_S: S^{1/2}\to S$, an S-linear map, by sending $x_1^{1/2}x_2^{1/2}\ldots x_5^{1/2}$ to 1 and the other basis elements to zero.

Now fix $z\in(I^{[2]}:_S I)\setminus\mathfrak{m}^{[2]}$. For an element $\overline{s}\in S/I$ we let $\phi(\overline{s}^{1/2})=\Phi_S(z^{1/2}s^{1/2})$ modulo I.

This defines an R-linear map $\phi: R^{1/2}\to R$.

For the choice $z=x_1^3x_2x_3+x_1^3x_2x_4+x_1^2x_3x_4x_5+x_1x_2x_3x_4x_5+x_1x_2x_4^2x_5+x_2^2x_4^2x_5+x_3x_4^2x_5^2+x_4^3x_5^2$, Katzman and Schwede have applied their algorithm [22] and obtained the list of all ϕ-compatible prime ideals of R. The list of ϕ-compatible prime ideals is as follows

$$R,(x_1,x_4),(x_1,x_4,x_5),$$

$$(x_1+x_2,x_1^2+x_4x_5),(x_1+x_2,x_2^2+x_4x_5),(x_3+x_4,x_1+x_2,x_2^2+x_4x_5),$$

$$(x_1, x_2, x_5, x_3 + x_4), (x_1, x_2, x_4), (x_1, x_2, x_5), (x_1, x_3, x_4),$$

$$(x_1, x_2, x_3, x_4), (x_1, x_2, x_4, x_5), (x_1, x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5).$$

From this list we can easily identify the unique smallest ϕ-compatible ideal. Lemma 2.4 in [22] tells us that we have to keep the Φ-compatible prime ideals that contain the ideal I and eliminate the minimal primes of I from this list.

We have that the set of minimal primes of I is given by

$$Min(I) = \{(x_1 + x_2, x_1^2 + x_4x_5), (x_1, x_2, x_5), (x_1, x_4, x_3)\}.$$

The list of Φ-compatible prime ideals that contain I and are not in the list of the minimal primes of I is given by

$$(x_1, x_2, x_4, x_5), (x_1, x_2, x_3, x_4, x_5), (x_1, x_2, x_5, x_3 + x_4), (x_1, x_2, x_3, x_4), (x_1, x_3, x_4, x_5).$$

Next, by intersecting them and taking the class modulo I we obtain the test ideal of the pair (R, ϕ)

$$\tau(R, \phi) = (x_1, x_2x_5, x_3x_4 + x_4^2).$$

Therefore, in this ring, every element x belonging to I^* satisfies a degree 3 equation of integral dependence over I.

Example 4.2.2. Let $k = \mathbb{F}_2$ and $S = k[[x_1, x_2, x_3, x_4]]$. Let $\mathcal{I} = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ and $R = S/\mathcal{I}$. Let $\phi : R^{1/2} \to R$ an R-linear map constructed as follows:

Let $S^{1/2} = k[[x_1^{1/2}, \ldots, x_4^{1/2}]]$ which is a free S-module with basis $\{x_1^{\lambda_1/2}x_2^{\lambda_2/2}x_3^{\lambda_3/2}x_4^{\lambda_4/2}\}_{0 \leqslant \lambda_i \leqslant 1}$. Construct $\Phi_S : S^{1/2} \to S$, an S-linear map, by sending $x_1^{1/2}x_2^{1/2}x_3^{1/2}x_4^{1/2}$ to 1 and the other basis elements to zero.

Let $z = x_1x_2x_3x_4$ an element contained in $(I^{[2]} : I) \setminus \mathfrak{m}^{[2]}$. The choice of the element z guarantees that the map ϕ is surjective from Fedder's Lemma. By applying the algorithm of Katzman and Schwede [22], we will get the list of ϕ-compatible primes

$$R, (x_4), (x_4, x_3), (x_4, x_3, x_2), (x_4, x_3, x_1), (x_4, x_3, x_2, x_1),$$

$$(x_4, x_2), (x_4, x_2, x_1), (x_4, x_1), (x_3), (x_3, x_2), (x_3, x_2, x_1),$$

$$(x_3, x_1), (x_2), (x_2, x_1), (x_1).$$

Using this list, one can obtain the unique smallest ϕ-compatible ideal. The set of minimal primes of I is $Min(I) = \{(x_1, x_2), (x_3, x_4)\}$.

The Φ-compatible prime ideals that contain the ideal I and are not minimal primes of I are the following

$$(x_4, x_3, x_2), (x_4, x_3, x_2, x_1), (x_4, x_3, x_1), (x_4, x_2, x_1), (x_3, x_2, x_1).$$

After intersecting them in $R = S/I$ we obtain the test ideal of the pair (R, ϕ) is

$$\tau(R, \phi) = (x_3x_4, x_1x_2).$$

Therefore, in this ring, every element x belonging to I^* satisfies a degree 2 equation of integral dependence over I.

We notice that the number of generators of $\tau(R, \phi)$ is actually the number of facets of the simplicial complex Δ associated to the square-free monomial ideal I. In the next section, Corollary 4.2.11 will show that this happens for all Stanley-Reisner rings.

4.2.1 The Case of Stanley-Reisner Rings

Let k be a perfect field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ be the formal power series ring in n variables over k and $q = p^e$, for $e \geqslant 0$. Let $I \leqslant S$ be a square-free monomial ideal in S and $R = S/I$. We denote by Δ the simplicial complex associated to the Stanley-Reisner R. Let $f_{max}(\Delta)$ be the number of facets of the simplicial complex Δ.

The map $\Phi_S : S^{1/q} \to S$ that sends the element $x_1^{q-1/q} \dots x_n^{q-1/q}$ to 1 and all the other basis elements to zero is called ***the trace map***.

Remark 4.2.3. Under the assumptions above, $\mathrm{Hom}_S(S^{1/q}, S)$ is a free $S^{1/q}$-module with generator Φ_S. Therefore, for every S-linear map $\Phi : S^{1/q} \to S$, there is $z \in S$ such that $\Phi(s) = \Phi_S(z^{1/q}s)$, for every $s \in S^{1/q}$.

Now we can apply Corollary 3.1.6 and Proposition 3.1.7 to obtain the following essential result as a consequence of Fedder's work in [10]:

Theorem 4.2.4 (Fedder's Lemma, [10]). *Let $S = k[[x_1, \dots, x_n]]$, where k is a perfect field and $R = S/I$ for some ideal $I \leqslant S$. Then if $\phi : R^{1/q} \to R$ is any R-linear map, then there exists an S-linear map $\Phi : S^{1/q} \to S$ which is compatible with I such that $\phi = \Phi/I$.*

Moreover, ϕ is surjective if and only if $z \notin \mathfrak{m}^{[q]}$, where $\Phi(s) = \Phi_S(z^{1/q}s)$ and Φ_S is the trace map on S. Furthermore, there exists an isomorphism

$$\mathrm{Hom}_R(R^{1/q}, R) \cong \frac{I^{[q]} : I}{I^{[q]}}.$$

Corollary 4.2.5. *Let $I \subseteq S$ be a square-free monomial ideal and $R = S/I$. Then, $z = (\prod_{i=1}^n x_i)^{q-1} \in (I^{[q]} :_S I) \setminus \mathfrak{m}^{[q]}$. Therefore, $z = (\prod_{i=1}^n x_i)^{q-1}$ defines an R-linear surjective map $\phi : R^{1/q} \to R$, $\phi = \Phi/I$ with $\Phi(s) = \Phi_S((\prod_{i=1}^n x_i)^{q-1/q}s)$, for all $s \in S^{1/q}$.*

Proof. Since I is a square-free monomial ideal and the minimal primary decomposition of I can be written as $I = I_{\alpha_1} \cap I_{\alpha_2} \cap \dots \cap I_{\alpha_r}$, where $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kn}) \in \{0,1\}^n$, $1 \leqslant k \leqslant r$, are distinct vectors, and $I_{\alpha_k} = (x_i : i \in supp(\alpha_k))$, for every $1 \leqslant k \leqslant r$.

By using Proposition 3.3.3, we obtain that $lcm((x^{\alpha_1})^{q-1}, (x^{\alpha_2})^{q-1}, \dots, (x^{\alpha_r})^{q-1})$ is an element contained in $(I^{[q]} :_S I)$ that is not in $\mathfrak{m}^{[q]}$.

But $lcm((x^{\alpha_1})^{q-1}, (x^{\alpha_2})^{q-1}, \dots, (x^{\alpha_r})^{q-1})$ divides $(\prod_{i=1}^n x_i)^{q-1}$ because $x^{\alpha_1}, \dots, x^{\alpha_r}$ are square-free monomials. Hence, $(\prod_{i=1}^n x_i)^{q-1} \in (I^{[q]} : I) \setminus \mathfrak{m}^q$.

Therefore, by Theorem 4.2.4 the R-linear map $\phi : R^{1/q} \to R$, given by $z = (\prod_{i=1}^n x_i)^{q-1}$ is a surjective map. $\square$

Proposition 4.2.6 ([4, Corollary 1.5]). *Let $\Phi : S^{1/q} \to S$ an S-linear map and $z \in S$ such that $\Phi(s) = \Phi_S(z^{1/q}s)$, for every $s \in S^{1/q}$. Let $J \leqslant S$ an ideal in S. Then J is Φ-compatible if and only if $J \subseteq (J^{[q]} :_S z)$.*

Definition 4.2.7. Let $K \subset S$ be an ideal in S and $q = p^e$, for $e \geqslant 0$. Then $I_e(K)$ denotes the smallest ideal I such that $I^{[q]} \supseteq K$. The ideal $I_e(K)$ is called the *e-th root ideal* of K.

We have that the following elementary properties of the e-th root ideals hold.

Proposition 4.2.8 ([4, Proposition 1.3]). *Let $K_1, \ldots, K_s \subset S$ ideals in S. Then the following statements hold:*

(a) $I_e(\sum_{i=1}^{s} K_i) = \sum_{i=1}^{s} I_e(K_i)$;

(b) Let $h \in S$ and write

$$h = \sum_{0 \leqslant a_1, \ldots, a_n \leqslant q-1,\ a=(a_1,\ldots,a_n)} h_a^q x_1^{a_1} \cdots x_n^{a_n}.$$

Then $I_e(h)$ is the ideal generated in S by all h_a appearing in the expression above.

Proposition 4.2.9. *Let $S = k[[x_1, \ldots, x_n]]$, where k is a perfect field of characteristic p. Let $\Phi : S^{1/q} \to S$ given by $\Phi(s) = \Phi_S(z^{1/q}s)$, for every $s \in S^{1/q}$ and $z = (\prod_{i=1}^n x_i)^{q-1}$. The set of Φ-compatible prime ideals consists of the set of ideals generated by variables, i.e. $(x_{i_1}, \ldots, x_{i_k})$, where $1 \leqslant i_1, \ldots, i_k \leqslant n$.*

Proof. In order to see that the ideals generated by variables are Φ-compatible we will use Proposition 4.2.6. For example, if we consider the ideal $(x_{i_1}, \ldots, x_{i_k})$, it is easy to see that $(x_1 \ldots x_n)^{q-1}(x_{i_1}, \ldots, x_{i_k}) \subseteq (x_{i_1}^q, \ldots, x_{i_k}^q)$. By using Proposition 4.2.6, we obtained that $(x_{i_1}, \ldots, x_{i_k})$ is Φ-compatible.

On the other hand, we have to show that the ideals generated by variables are the only Φ-compatible prime ideals. In order to prove this, it is enough to show that if an ideal, say J is a prime Φ-compatible ideal then J is monomial since every prime monomial ideal is an ideal generated by variables.

Let J a Φ-compatible prime ideal and let $f \in J$ a polynomial in J. Then $f = \sum_{i=1}^{r} f_i$ is the decomposition of f as a sum of monomials. We have to show that each monomial component f_i of f is contained in J.

Since $zf \in J^{[q]}$, then $I_e(zf) \subseteq J$, where $z = (x_1 \ldots x_n)^{q-1}$. But by Proposition 4.2.8 (a), $I_e(zf) = \sum_{i=1}^{r} I_e(zf_i)$. Moreover, Proposition 4.2.8 (b) gives that $f_i \in I_e(zf_i)$, for $1 \leqslant i \leqslant r$. Hence, each f_i is contained in J. Therefore, J is a monomial prime ideal.

To sum up, all the Φ-compatible ideals are the ideals generated by variables. $\square$

Proposition 4.2.10. *Let $I \subseteq S$ be a square-free monomial ideal and $R = S/I$. Let $\phi : R^{1/q} \to R$ be the R-linear map given by $z = (\prod_{i=1}^{n} x_i)^{q-1}$, i.e. $\phi = \Phi/I$ with $\Phi(s) = \Phi_S((\prod_{i=1}^{n} x_i)^{q-1/q} s)$, for all $s \in S^{1/q}$. Then the test ideal associated to the pair (R, ϕ) is given by*

$$\tau(R, \phi) = (x_F : F \in \mathcal{F}(\Delta)),$$

where Δ is the simplicial complex associated to the ideal I.

Proof. Given $\phi : R^{1/q} \to R$ an R-linear map, there exists an S-linear map $\Phi : S^{1/q} \to S$ which is compatible with I such that $\phi = \Phi/I$ by Theorem 4.2.4, where $\Phi(s) = \Phi_S(z^{1/q} s)$ and Φ_S is the trace map on S. Moreover, ϕ is surjective if and only if $z \notin \mathfrak{m}^{[q]}$,

But according to the Corollary 4.2.5, $z = (\prod_{i=1}^{n} x_i)^{q-1}$ defines an R-linear surjective map $\phi : R^{1/q} \to R$, i.e. $\phi = \Phi/I$ with $\Phi(s) = \Phi_S((\prod_{i=1}^{n} x_i)^{q-1/q} s)$ for all $s \in S^{1/q}$. Using Lemma 2.4 in [22], we have that there is a bijective correspondence between the ϕ- compatible ideals and the Φ-compatible ideals containing I.

Proposition 4.2.9 gives the list of Φ-compatible prime ideals. We want to compute $\tau(R, \phi)$, which is the smallest ϕ-compatible ideal with respect to inclusion. Since, in an F-pure ring, the ϕ-compatible ideals are closed under primary decomposition, we need to intersect all the ϕ-compatible prime ideals. By Lemma 2.4 in [22], to determine the list of all ϕ-compatible prime ideals, we first find the Φ-compatible prime ideals that contain the ideal I. Then we remove the minimal primes of I from the list given by Proposition 4.2.9. After this, $\tau(R, \phi)$ is the image of the ideal obtained after intersecting all these remaining ideals modulo I.

Consider now the simplicial complex Δ associated to the ideal I. Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$ the set of facets of Δ and

$$I = I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{F^c}$$

the primary decomposition of the ideal I.

So we have that the set of minimal primes of I is $Min(I) = \{P_{F^c}\}$. Proposition 4.2.9 tells us that the set of Φ-compatible prime ideals consists of all the ideals generated by variables. Hence, the set of Φ-compatible prime ideals that contain I and are not in the set of minimal primes of I are the following ideals

$$(P_{F_j^c}, x_i : i \in F_j),$$

for every $1 \leqslant j \leqslant m$. Therefore, by intersecting them, we obtain

$$\bigcap_{j=1}^{m} (P_{F_j^c}, \prod_{i \in F_j} x_i)$$

$$= \bigcap_{j=1}^{m} (P_{F_j^c}, x_{F_j}).$$

Now, we obtain the test ideal $\tau(R, \phi)$ by taking the intersection

$$\bigcap_{j=1}^{m} (P_{F_j^c}, x_{F_j})$$

modulo the ideal I. Since $I = I_\Delta = (x_F : F \notin \Delta)$, all the monomials in the intersection

$$\bigcap_{j=1}^{m} (P_{F_j^c}, x_{F_j})$$

are killed by moding out by the ideal I, except $x_{F_1}, \ldots, x_{F_m}$. Hence,

$$\tau(R, \phi) = (x_{F_1}, \ldots, x_{F_m}).$$

$\square$

Corollary 4.2.11. *Let $I \subseteq S$ be a square-free monomial ideal and $R = S/I$. Let $\phi : R^{1/q} \to R$ be the R-linear map given by $z = (\prod_{i=1}^n x_i)^{q-1}$, i.e. $\phi = \Phi/I$ with $\Phi(s) = \Phi_S((\prod_{i=1}^n x_i)^{q-1/q} s)$ for all $s \in S^{1/q}$.*

Then the test ideal associated to the pair (R, ϕ) is $f_{max}(\Delta)$-generated, where Δ is the simplicial complex associated to the ideal I.

Therefore, in this ring, every element x belonging to I^* satisfies a degree $f_{max}(\Delta)$ equation of integral dependence over I.

CHAPTER 5

FURTHER REMARKS AND CONCLUSIONS

In this chapter, we will present directions in which our work could be continued and further results to motivate them. The Frobenius complexity of a local ring in positive prime characteristic p is an important invariant but there are just a few classes of rings in the literature for which we know the answer. When the algebra of Frobenius operators on the injective hull is principally generated, the Frobenius complexity of the ring is known to be equal to $-\infty$. In the future, we would like to find other classes of rings such that their ring of Frobenius operators is principally generated as a skew algebra over the ring considered. We will start by investigating when is an R-skew algebra principally generated and then we will move to the case of monomial ideals.

5.1 Principally Generated Skew R-Algebras

Definition 5.1.1. Let A be an R-skew algebra. We call A **principally generated as an R-skew algebra** if there exists a generator $a_0 \in A$ such that any element $a \in A$ can be expressed as a polynomial in a_0 with coefficients in R. Moreover, the R-skew algebra A is called **homogeneously principally generated as an R-skew algebra** if A is generated by a homogeneous generator a_0 as an R-skew algebra.

Proposition 5.1.2. *Let $A = \oplus_{e \geqslant 0} A_e$ be an R-skew algebra and assume there exists $0 \neq b$ an R-torsion-free element in A_1. If A is principally generated as an R-skew algebra, then A is homogeneously principally generated as an R-skew algebra.*

Proof. Let a_0 be the generator of A as an R-skew algebra. Then we can write $a_0 = a_1 + \ldots + a_k$, with $a_i \in A_i$, for any i and $k \geqslant 1$. We have that there exists a polynomial P with coefficients in R such that $b = P(a_0) = \sum_{i=1}^{n} r_i a_0^i$, $r_i \in R$, for any i. The degree one terms in the left hand side must be equal to the right hand side ones in $b = \sum_{i=1}^{n} r_i (a_1 + \ldots + a_k)^i$.

Hence, we get $b = r_1 a_1$. One can note that since $b = r_1 a_1$ is an R-torsion-free element, a_1 is an R-torsion-free element as well. There exists a polynomial Q with coefficients in R such that $a_1 = Q(a_0) = \sum_{i=1}^{n} s_i a_0^i = \sum_{i=1}^{m} s_i (a_1 + \ldots + a_k)^i$ with $s_i \in R$, for any i. Looking in degree one in the last equality, we obtain $a_1 = s_1 a_1$, which implies $(s_1 - 1)a_1 = 0$. Since a_1 is R-torsion-free, $s_1 = 1$. Now if we look in degree 2 of the equality $a_1 = s_0 + (a_1 + \ldots + a_k) + \ldots + s_m(a_1 + \ldots + a_k)^m$, we get that $a_2 + s_2 a_1^2 = 0$. Hence, a_2 is a polynomial in a_1 with coefficients in R. In the same way looking in degree 3, we obtain that a_3 is a polynomial in a_1 with coefficients in R and so on. In conclusion, each a_k is a polynomial in a_1 with coefficients in R which proves that $a_0 = a_1 + \ldots + a_k$ is a polynomial in a_1 with coefficients in R. This shows that one can assume A principally generated by $a_1 \in A_1$ as an R-skew algebra, i.e. homogeneously principally generated. □

Proposition 5.1.3. *Let R be an F-pure local ring of positive prime characteristic p. If $\mathcal{F}(E_R)$ is a principally generated R-skew algebra, then $\mathcal{F}(E_R)$ is a homogeneously principally generated R-skew algebra.*

Proof. Let ϕ_0 be the generator of $\mathcal{F}(E_R)$ as an R-skew algebra. Then we can write $\phi_0 = f_1 + \ldots + f_k$ with $f_i \in \mathcal{F}^i(E_R)$, for any i. Since R is F-pure local ring, there exists $\phi \in \mathcal{F}^1(E_R)$ an injective Frobenius operator by Theorem 2.3.4. There exists a polynomial Q with coefficients in R such that $\phi = Q(\phi_0) = \sum_{i=1}^{m} r_i (f_1 + \ldots + f_k)^i$ with $r_i \in R$ for any i. The degree 1 terms in each sides should be equal so we get $\phi = r_1 f_1$ with $r_1 \in R$. Since ϕ is an injective Frobenius action, f_1 is injective as well. Moreover, if $r f_1 = 0$ for some $0 \neq r \in R$, then $r^p f_1 = 0$. Hence $f_1(rx) = 0$, for any $x \in E_R$. But f_1 is an injective Frobenius operator on E_R, so $rx = 0$, for any $x \in E_R$. Now since E_R is a faithful R-module, we obtain $r = 0$. There exists a polynomial P with coefficients in R such that $f_1 = P(\phi_0) = P(f_1 + \ldots + f_k) = \sum_{i=1}^{n} a_i (f_1 + \ldots + f_k)^i$ with $a_i \in R$. By setting the degree one Frobenius operators in both sides equal, we obtain $f_1 = a_1 f_1$ and so $(a_1 - 1) f_1 = 0$ which implies $a_1 = 1$. Hence, $f_1 = a_0 + (f_1 + \ldots + f_k) + \ldots + a_n (f_1 + \ldots + f_k)^n$. Now looking in degree 2, we obtain $f_2 + a_2 f_1^2 = 0$ which shows that f_2 is a polynomial in f_1 with coefficients in R. By induction, it follows that every f_i is a polynomial in f_1 with coefficients in R. Therefore,

$\phi_0 = f_1 + \ldots + f_k$ is a polynomial in f_1 with coefficients in R. So one can assume f_1 to be the generator of $\mathcal{F}(E_R)$ as an R-skew algebra, which shows that $\mathcal{F}(E_R)$ is a homogeneously principally generated R-skew algebra. □

Question 5.1.4. Which condition on R implies that $\mathcal{F}(E_R)$ is principally generated as an R-skew algebra if and only if $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra?

Remark 5.1.5. Since Stanley-Reisner rings are F-pure, showing that $\mathcal{F}(E_R)$ is principally generated as an R-skew algebra is equivalent to proving that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by Proposition 5.1.3.

In general, we are interested in finding the answer to the question:

Question 5.1.6. Which condition on R implies that an R-skew algebra A is principally generated if and only if A is homogeneously principally generated as an R-skew algebra?

In Proposition 5.1.2, we showed that the existence of a nonzero torsion free element in degree one guarantees the equivalence between homogeneously principally generated R-skew algebras and principally generated R-skew algebras.

Proposition 5.1.7. *Let R be a local ring of positive prime characteristic p. The ring of Frobenius operators $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if there exists $\phi_0 \in \mathcal{F}^1(E_R)$ such that $\mathcal{F}^e(E_R)$ is an R-cyclic module generated by ϕ_0^e, for any $e \geqslant 1$.*

Proof. First, we assume that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by $\phi_0 \in \mathcal{F}^{e_0}(E_R)$. We will show that $e_0 = 1$. Let $\psi \in \mathcal{F}^1(E_R)$. There exists a polynomial P with coefficients in R such that $\psi = P(\phi_0) = \sum_{i=1}^{m} a_i\phi_0^i$, with $a_i \in R$, for any i. Since $\phi_0^m \in \mathcal{F}^{e_0 m}(E_R)$, the degree of the right hand side of the equality $\psi = a_0 + a_1\phi_0 + \ldots + a_n\phi_0^m$ equals $e_0 m$ and it must be equal to the degree of the left hand side, which is 1. Hence, we obtain $e_0 m = 1$, which implies $e_0 = m = 1$, so $\phi_0 \in \mathcal{F}^1(E_R)$. Moreover, $\psi = a_0 + a_1\phi_0$ which shows that $\mathcal{F}^1(E_R)$ is an R-cyclic module generated by $\phi_0 \in \mathcal{F}^1(E_R)$.

Let $e \geqslant 2$ and $\phi \in \mathcal{F}^e(E_R)$. There exists a polynomial Q with coefficients in R such that $\phi = Q(\phi_0) = b_0 + b_1\phi_0 + \ldots + b_n\phi_0^n$ with $b_i \in R$, for any i. Since, the degrees of the right hand side and left hand side must be equal we obtain $n = e$. Since $\mathcal{F}(E_R)$ is an internal direct sum of $\mathcal{F}^e(E_R)$ over $e \geqslant 0$, we can assume $\mathcal{F}^e(E_R)$ generated by ϕ_0^e. We proved the desired conclusion, i.e. $\mathcal{F}^e(E_R)$ is an R-cyclic module generated by ϕ_0^e, for any $e \geqslant 1$. For the other implication, if $\mathcal{F}^e(E_R)$ is an R-cyclic module with generator ϕ_0^e, for any $e \geqslant 1$ it is easy to see that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by ϕ_0. $\square$

Remark 5.1.8. Using Corollary 3.2.4, one can note that checking the cyclicity of the R-module of eth Frobenius operators on E_R, $\mathcal{F}^e(E_R)$ is equivalent to showing that the R-module $\dfrac{I^{[p^e]} :_S I}{I^{[p^e]}}$ is cyclic. Hence, we can reformulate Proposition 5.1.7, as follows:

Proposition 5.1.9. *Let S be a complete regular local ring of positive prime characteristic p, $I \subseteq S$ an ideal in S and $R = S/I$. The ring of Frobenius operators $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by $\phi_0 = uF \in \mathcal{F}^1(E_R)$, with $u \in (I^{[p]} :_S I)$ and $F : E_R \to E_R$ the canonical Frobenius operator on E_R if and only if the R-module $\dfrac{I^{[p^e]} :_S I}{I^{[p^e]}}$ is cyclic, generated by u^{ν_e}, for any $e \geqslant 1$, with $\nu_e = 1 + p + \ldots + p^{e-1}$.*

Proof. This result follows directly from Proposition 5.1.7 and Corollary 3.2.4. $\square$

This result motivates the following question:

Question 5.1.10. Let S be a complete regular local ring of positive prime characteristic p, $I \subseteq S$ an ideal in S and $R = S/I$. Can one show that the following assertions are equivalent:

(i) The R-module $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic, for any q

(ii) The R-algebra $\mathcal{F}(E_R)$ is principally generated?

In [27], Àlvarez Montaner, Boix and Zarzuela showed that for Stanley-Reisner rings R, the R-algebra $\mathcal{F}(E_R)$ is principally generated if and only if the R-module $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic generated by $(\mathbf{x}^1)^{q-1}$, for any q. An unanswered question in [27] was whether one can read the principally generation of the Frobenius algebra of operators $\mathcal{F}(E_R)$ from the simplicial complex associated to the Stanley-Reisner ring R. This question was addressed by Àlvarez Montaner and Yanagawa in [28]. They found a combinatorial characterization of Stanley-Reisner rings having $\mathcal{F}(E_R)$ principally generated as an R-algebra. Theorem 4 in [28] states that $\mathcal{F}(E_R)$ principally generated as an R-algebra if and only if the simplicial complex Δ associated to the Stanley-Reisner ring R does not have free faces. We recall here the definition of a free face of a simplicial complex presented in [28]:

Definition 5.1.11. Let Δ be a simplicial complex on the vertex set $\{1, \ldots, n\}$. We call a face $F \in \Delta$ a **free face** if $F \cup \{i\}$ is a facet of Δ for some $i \notin F$ and $F \cup \{i\}$ is the unique facet of Δ containing F.

Example 5.1.12. Let Δ be the simplicial complex on the vertex set $\{x_1, x_2, x_3, x_4\}$ generated by the facets $F_1 = \{x_1, x_2\}$ and $F_2 = \{x_3, x_4\}$. One can easily note that the free faces of Δ are the faces: $\{x_1\}$, $\{x_2\}$, $\{x_3\}$ and $\{x_4\}$.

With this question in mind, Boix and Zarzuela asked whether there is any sort of connection between the number of minimal monomial generators of J_q as defined in Definition 3.3.2 and the number of free faces of the simplicial complex associated to the Stanley-Reisner ring given. They proved in Theorem 3.16 in [5] that the number of minimal monomial generators of J_q equals the number of non-empty maximal free pairs of the simplicial complex Δ associated to the Stanley-Reisner ring considered. Boix and Zarzuela defined the notion of a free pair in [5], which extends the notion of free pairs as follows:

Definition 5.1.13 ([5, Definition 3.8])**.** Let Δ be a simplicial complex on the vertex set $\{1, \ldots, n\}$. Let F, G non-empty subsets of $\{1, \ldots, n\}$.

(i) We call (F, G) **a pair** of Δ if $F \cap G = \emptyset$ and if $F \cup G$ is a face of Δ.

(ii) We call (F, G) **a free pair** of Δ if (F, G) is a pair of Δ and if $F \cup G$ is the unique facet of Δ containing F.

Remark 5.1.14. One can note that if F is a free face of Δ and $i \notin F$ is the vertex such that $F \cup \{i\}$ is the unique facet of Δ containing F, then $(F, \{i\})$ is a free pair of Δ.

Boix and Zarzuela introduced a partial order on the set of free pairs of a simplicial complex, as follows:

Definition 5.1.15 ([5, Definition 3.10])**.** Let Δ be a simplicial complex and $FP(\Delta)$ the set of all the free pairs of Δ. Given two free pairs (F, G) and $(F', G') \in FP(\Delta)$, we say that $(F, G) \leqslant (F', G')$ iff $F \cup G = F' \cup G'$, $F \supseteq F'$ and $G \subseteq G'$.

Definition 5.1.16 ([5, Definition 3.11])**.** A free pair $(F, G) \in FP(\Delta)$ is called a **maximal free pair** if it is a maximal element in the poset $FP(\Delta)$.

Theorem 3.16 in [5] describes the bijective correspondence between the number of minimal monomial generators of J_q defined as in Definition 3.3.2 and the number of maximal free pairs of the simplicial complex associated to the Stanley-Reisner ring given, as follows: any maximal free pair (F, G) of Δ, corresponds to the minimal monomial generator of J_q, defined as

$$A(F, G) = \left(\prod_{i \in F} x_i^q\right)\left(\prod_{i \notin F \cup G} x_i^{q-1}\right).$$

Next, we will illustrate using an example this bijective correspondence:

Example 5.1.17. Let k be a field of characteristic p and $q = p^e$, for any $e \geqslant 0$. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subseteq k[[x_1, x_2, x_3, x_4]]$ a square-free monomial ideal and $R = S/I$ its Stanley-Reisner ring. One can compute the colon ideal in Macaulay 2 and obtain

$$(I^{[q]} : I) = I^{[q]} + (x_1^q x_3^{q-1} x_4^{q-1}, x_1^{q-1} x_2^{q-1} x_3^q, x_1^{q-1} x_2^{q-1} x_4^q, x_2^q x_3^{q-1} x_4^{q-1}) + (x_1x_2x_3x_4)^{q-1},$$

for any q, which shows that $J_q = (x_1^q x_3^{q-1} x_4^{q-1}, x_1^{q-1} x_2^{q-1} x_3^q, x_1^{q-1} x_2^{q-1} x_4^q, x_2^q x_3^{q-1} x_4^{q-1})$. One can easily note that the simplicial complex associated to the Stanley-Reisner ring R is generated by the facets $F_1 = \{1, 2\}$ and $F_2 = \{3, 4\}$. The maximal free pairs of Δ are: $(\{1\}, \{2\})$, $(\{3\}, \{4\})$, $(\{4\}, \{3\})$ and $(\{2\}, \{1\})$. Using the bijective correspondence above, we can obtain the minimal monomial generators of J_q without explicitly computing the colon ideal $(I^{[q]} : I)$.

5.2 Monomial Ideals and The Frobenius Algebra of Operators

Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$ the formal power series ring in n variables over k. Let $I \leqslant S$ be a monomial ideal in S and $R = S/I$ the quotient ring. Let

$$M_e := \frac{I^{[p^e]} :_S I}{I^{[p^e]}},$$

for any $e \geqslant 1$.

Lemma 5.2.1. *There exists a bijective correspondence between the minimal monomial generators of $I^{[p]} :_S I$ and the minimal monomial generators of $I^{[q]} :_S I$, for any $e \geqslant 1$. Moreover, this will induce a bijective correspondence between the minimal monomial generators of M_1 and the minimal monomial generators of M_e, for any $e \geqslant 1$.*

Proof. If $I = I_{\alpha_1} \cap I_{\alpha_2} \cap \ldots \cap I_{\alpha_r}$ is the minimal primary decomposition of the ideal I, then since the Frobenius map is flat we have that

$$I^{[q]} = I_{\alpha_1}^{[q]} \cap \ldots \cap I_{\alpha_r}^{[q]}.$$

Therefore,

$$(I^{[q]} :_S I) = (I_{\alpha_1}^{[q]} :_S I) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I).$$

Let $I := (x_{\delta_j}^{a_j} : j \geqslant 1)$, where $x_{\delta_j}^{a_j} = x_{\delta_{j,1}}^{a_{j,1}} \cdots x_{\delta_{j,n}}^{a_{j,n}}$, where $a_{j,i} \geqslant 0$, $1 \leqslant \delta_{j,i} \leqslant n$ for any $1 \leqslant i \leqslant n$.

Let $I_{\alpha_i} := (x_{\beta_{i,k}}^{b_{i,k}} : 1 \leqslant k \leqslant n)$, where $1 \leqslant \beta_{i,k} \leqslant n$ and $b_{i,k} \geqslant 0$, for any $1 \leqslant i \leqslant r$. Then

$$(I^{[q]} :_S I) = \bigcap_{i=1}^{r} (I_{\alpha_i}^{[q]} :_S I) = \bigcap_{i=1}^{r} \bigcap_{j \geqslant 1} (I_{\alpha_i}^{[q]} :_S x_{\delta_j}^{a_j}) =$$

$$= \bigcap_{i=1}^{r} \bigcap_{j \geqslant 1} ((x_{\beta_{i,k}}^{qb_{i,k}} : 1 \leqslant k \leqslant n) : (x_{\delta_j}^{a_j})) =$$

$$= \bigcap_{i=1}^{r} \bigcap_{j \geqslant 1} \left(\frac{lcm(x_{\beta_{i,k}}^{qb_{i,k}}, x_{\delta_j}^{a_j})}{x_{\delta_j}^{a_j}} : 1 \leqslant k \leqslant n \right).$$

We have that

$$\frac{lcm(x_{\beta_{i,k}}^{qb_{i,k}}, x_{\delta_j}^{a_j})}{x_{\delta_j}^{a_j}} = \begin{cases} x_{\beta_{i,k}}^{max(qb_{i,k},a_{j,l})-a_{j,l}}, & \text{if } \beta_{i,k} = \delta_{j,l} \\ x_{\beta_{i,k}}^{qb_{i,k}}, & \text{if } \beta_{i,k} \neq \delta_{j,l} \end{cases}$$

$$= \begin{cases} x_{\beta_{i,k}}^{qb_{i,k}-a_{j,l}}, & \text{if } \beta_{i,k} = \delta_{j,l} \\ x_{\beta_{i,k}}^{qb_{i,k}}, & \text{if } \beta_{i,k} \neq \delta_{j,l}. \end{cases}$$

or

$$= \begin{cases} x_{\beta_{i,k}}^{0}, & \text{if } \beta_{i,k} = \delta_{j,l} \\ x_{\beta_{i,k}}^{qb_{i,k}}, & \text{if } \beta_{i,k} \neq \delta_{j,l}. \end{cases}$$

Hence, any minimal monomial generator of $I^{[q]} :_S I$ is of the form $\prod_{i=1}^{n} x_i^{c_i q - d_i}$, where $c_i, d_i \geqslant 0$, for any $1 \leqslant i \leqslant n$. In this way, we obtain a bijective correspondence between the minimal monomial generators of $I^{[p]} :_S I$ and the minimal monomial generators of $I^{[q]} :_S I$, for any $e \geqslant 1$, which induces a bijective correspondence between the minimal monomial generators of M_1 and the minimal monomial generators of M_e, for any $e \geqslant 1$. $\square$

Proposition 5.2.2. *Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$ the formal power series ring in n variables over k. Let $I \leqslant S$ be a monomial ideal in S and $R = S/I$ the quotient ring. Then the following assertions are equivalent:*

(i) $\mathcal{F}^1(E_R)$ is a principally generated R-module

(ii) There exists e_0 such that $\mathcal{F}^{e_0}(E_R)$ is a principally generated R-module

(iii) $\mathcal{F}^e(E_R)$ is a principally generated R-module, for any $e \geqslant 1$

Proof. Using Proposition 3.1.4, we have that the R-module generated by the eth Frobenius operators on E_R, $\mathcal{F}^e(E_R)$ is principally generated if and only if the R-module $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic.

The implication $(i) \Rightarrow (ii)$ is trivial.

For the implication $(ii) \Rightarrow (iii)$, we will use the bijective correspondence between the minimal monomial generators of M_{e_0} and the minimal monomial generators of M_e, for any

$e_0, e \geqslant 1$ in Lemma 5.2.1.

Let $e \geqslant 1$ an arbitrary integer. Since $\mathcal{F}^{e_0}(E_R)$ is a principally generated R-module, we have that M_{e_0} is a cyclic R-module. Let $x^{\gamma_{e_0}}$ be the minimal monomial generator of M_{e_0} over R. By the bijective correspondence described in the proof of Lemma 5.2.1, we have that the exists a minimal monomial generator x^{γ_e} in M_e which corresponds to $x^{\gamma_{e_0}}$ via this map. Moreover, the bijective correspondence in Lemma 5.2.1 shows that M_e is a cyclic R-module as well.

$(iii) \Rightarrow (i)$ Trivial. $\square$

Proposition 5.2.3. *Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$ the formal power series ring in n variables over k. Let $I \subseteq S$ be a monomial ideal in S and $R = S/I$ the quotient ring. Then the ring of Frobenius operators $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by $x^\gamma F$, with $x^\gamma = \prod_{i=1}^n x_i^{c_i p - d_i}$, where $c_i, d_i \geqslant 0$, for any $1 \leqslant i \leqslant n$ and $F : E_R \to E_R$ the canonical Frobenius operator on E_R if and only if $c_i = d_i$, for any $1 \leqslant i \leqslant n$.*

Proof. We first assume that the ring of Frobenius operators $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra. By Proposition 5.1.7, we can assume that the homogeneous generator ϕ_0 of $\mathcal{F}(E_R)$ as an R-skew algebra is of degree 1, i.e. $\phi_0 \in \mathcal{F}^1(E_R)$. Using Corrolary 3.2.4 and Lemma 5.2.1, we have that $\phi_0 = x^\gamma F = \prod_{i=1}^n x_i^{c_i p - d_i} F$, where $c_i, d_i \geqslant 0$, for any $1 \leqslant i \leqslant n$ and $F : E_R \to E_R$ is the canonical Frobenius operator on E_R given by $F(x) = x^p$, for any $x \in E_R$. Using Lemma 5.2.1, we have a bijective correspondence between the minimal generators of the colon ideals $(I^{[p^e]} :_S I)$, for any $e \geqslant 0$. We will denote by $x^{\gamma_e} = \prod_{i=1}^n x_i^{c_i p^e - d_i}$, for any $e \geqslant 0$. Since $\mathcal{F}(E_R)$ is an R-skew algebra, we have that $\mathcal{F}^e(E_R) \circ \mathcal{F}^{e'}(E_R) \subseteq \mathcal{F}^{e+e'}(E_R)$ must hold, so composing an eth Frobenius action with an e'th Frobenius action will be an $(e+e')$th Frobenius action. For simplicity, we will translate this condition in terms of the generators of the colon ideals using the bijection in

Corollary 3.2.4. Hence, we obtain

$$x^{\gamma_e} * x^{\gamma_{e'}} = x^{\gamma_e} \cdot (x^{\gamma_{e'}})^{p^e} = \prod_{i=1}^{n} x_i^{(c_i p^e - d_i) + (c_i p^{e'} - d_i) p^e} = \prod_{i=1}^{n} x_i^{(c_i p^{e+e'} - d_i)} x_i^{(c_i - d_i) p^e}.$$

Therefore, in order to have $x^{\gamma_e} * x^{\gamma_{e'}} \in \mathcal{F}^{e+e'}(E_R)$, we must have $c_i \geqslant d_i$, for any $1 \leqslant i \leqslant n$. By Proposition 5.1.7, we know that every $\mathcal{F}^e(E_R)$ is cyclic as an R-module. Since x^γ generates $\mathcal{F}^1(E_R)$, its eth iteration $x^\gamma * \cdots * x^\gamma = (x^\gamma)^{\nu_e}$ will generate $\mathcal{F}^e(E_R)$, for any $e \geqslant 1$, where $\nu_e = 1 + p + \ldots + p^{e-1}$. On the other hand, the bijective correspondence described in Lemma 5.2.1 produces an eth Frobenius action $x^{\gamma_e} F^e \in \mathcal{F}^e(E_R)$. Since $x^{\gamma_e} F^e = \prod_{i=1}^n x_i^{(c_i p^e - d_i)} F^e \in \mathcal{F}^e(E_R) = \prod_{i=1}^n x_i^{(c_i p - d_i)\nu_e} F^e$, the following inequalities $c_i p^e - d_i \geqslant (c_i p - d_i)\nu_e$ must hold, for any i. One can note that this is equivalent to $(d_i - c_i)(p + \ldots + p^{e-1}) \geqslant 0$, for any i. This implies that $d_i \geqslant c_i$, for any i. Hence we obtain that $c_i = d_i$, for any i. The other implication follows since if $c_i = d_i$, for any i then the generator of the Frobenius algebra of operators is given by $x^\gamma = \prod_{i=1}^n x_i^{c_i(p-1)}$. Each graded piece $\mathcal{F}^e(E_R)$ of $\mathcal{F}(E_R)$ is generated by $(x^\gamma)^{\nu_e} = \prod_{i=1}^n x_i^{c_i(p^e-1)}$. In conclusion, $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by $x^\gamma F$. $\square$

Using Proposition 3.3.4(iii) due to Sharp in [31], we recover the colon formula 3.3.3 for a large class of monomial ideals, as follows:

Proposition 5.2.4. *Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$ the formal power series ring in n variables over k. Let $I \leqslant S$ be a monomial ideal in S with no embedded associated primes. Let $I = \bigcap_{i=1}^r I_{\alpha_i}$ be the minimal primary decomposition of I. Then*

$$(I^{[q]} :_S I) = (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}) = (I_{\alpha_1}^{[q]} + (x_{\delta_1}^{a_1})^{q-1}) \cap \cdots \cap (I_{\alpha_r}^{[q]} + (x_{\delta_r}^{a_r})^{q-1}),$$

where $I_{\alpha_j} := (x_{\delta_{j,1}}^{a_{j,1}}, \ldots, x_{\delta_{j,n}}^{a_{j,n}})$ and $x_{\delta_j}^{a_j} = x_{\delta_{j,1}}^{a_{j,1}} \cdots x_{\delta_{j,n}}^{a_{j,n}}$, with $a_{j,i} \geqslant 0$, $1 \leqslant \delta_{j,i} \leqslant n$ for any $1 \leqslant i \leqslant n$ and for any $1 \leqslant j \leqslant r$.

Proof. Using Proposition 3.3.4 (i), we have that $I^{[q]} = I_{\alpha_1}^{[q]} \cap I_{\alpha_2}^{[q]} \cap \ldots \cap I_{\alpha_r}^{[q]}$ is the minimal primary decomposition of the ideal $I^{[q]}$. Moreover, using the assertions about colon ideals which we already considered in the proof of Proposition 3.3.3, we obtain the inclusion

$$I^{[q]} : I = (I_{\alpha_1}^{[q]} : I) \cap \ldots \cap (I_{\alpha_r}^{[q]} : I) \supseteq \bigcap_{i=1}^{r} \sum_{j=1}^{r} (I_{\alpha_i}^{[q]} : I_{\alpha_j}).$$

This shows that the following inclusion holds:

$$I^{[q]} : I \supseteq (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}).$$

In order to obtain the other inclusion, we will use the fact that I has no embedded associated primes, i.e. $\mathrm{Min}(I) = \mathrm{Ass}(I)$. Since every $P_i := \sqrt{I_{\alpha_i}} \in \mathrm{Min}(I)$, we can now apply Proposition 3.3.4(iii) to get:

$$I^{[q]} : I \subseteq (I_{\alpha_i}^{[q]} :_S I_{\alpha_i}),$$

for any i. Therefore, we obtain

$$I^{[q]} : I \subseteq (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}).$$

This proves the first desired formula:

$$(I^{[q]} :_S I) = (I_{\alpha_1}^{[q]} :_S I_{\alpha_1}) \cap \ldots \cap (I_{\alpha_r}^{[q]} :_S I_{\alpha_r}).$$

In order to show the second equality, we have to note that since I is a monomial ideal, its primary components are ideals generated by powers of variables, i.e. $I_{\alpha_j} := (x_{\delta_{j,1}}^{a_{j,1}}, \ldots, x_{\delta_{j,n}}^{a_{j,n}})$ and $x_{\delta_j}^{a_j} = x_{\delta_{j,1}}^{a_{j,1}} \cdots x_{\delta_{j,n}}^{a_{j,n}}$, with $a_{j,i} \geqslant 0$, $1 \leqslant \delta_{j,i} \leqslant n$ for any $1 \leqslant i \leqslant n$ and for any $1 \leqslant j \leqslant r$. It is easy to check that

$$I_{\alpha_j}^{[q]} :_S I_{\alpha_j} = (I_{\alpha_j}^{[q]}, (x_{\delta_j}^{a_j})^{q-1}),$$

for any j. Together with the first equality this proves that

$$I^{[q]} : I = \bigcap_{i=1}^{r} (I_{\alpha_i}^{[q]} :_S I_{\alpha_i}) = \bigcap_{i=1}^{r} (I_{\alpha_j}^{[q]}, (x_{\delta_j}^{a_j})^{q-1}),$$

which completes the proof. $\square$

Example 5.2.5. Let $p = 5$ and $S = \mathbb{Z}_5[[x_1, x_2, x_3, x_4]]$.

Let $I = (x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_2^2x_4, x_2x_3x_4)$. The minimal primary decomposition of I is given by $I = (x_1^2, x_2) \cap (x_2^2, x_3) \cap (x_1, x_4)$. Since $\mathrm{Ass}(I) = \mathrm{Min}(I) = \{(x_1, x_2), (x_2, x_3), (x_1, x_4)\}$, I has no embedded associated primes. One can check in Macaulay 2 that $I^{[5]} : I = (I_{\alpha_1}^{[5]} : I_{\alpha_1}) \cap (I_{\alpha_2}^{[5]} : I_{\alpha_2}) \cap (I_{\alpha_3}^{[5]} : I_{\alpha_3})$. Moreover, Proposition 5.2.4 shows that $I^{[q]} : I = (I_{\alpha_1}^{[q]} : I_{\alpha_1}) \cap (I_{\alpha_2}^{[q]} : I_{\alpha_2}) \cap (I_{\alpha_3}^{[q]} : I_{\alpha_3})$ holds for any $q = 5^e$ and $e \geqslant 0$.

Example 5.2.6. Let $p = 5$ and $S = \mathbb{Z}_5[[x_1, x_2, x_3, x_4]]$.

Let $I = (x_1^2x_3^2, x_1^2x_4, x_1x_2x_4, x_2x_3x_4, x_2x_3^2, x_2^2x_4)$. The minimal primary decomposition of I is given by $I = (x_1^2, x_2) \cap (x_1, x_2^2, x_3) \cap (x_3^2, x_4)$. One can easily note that $\mathrm{Min}(I) \neq \mathrm{Ass}(I)$, i.e. I has embedded associated primes. Using Macaulay 2, we found $x_1^3x_2^9x_4^5 \in (I^{[5]} : I) \setminus (I_{\alpha_1}^{[5]} : I_{\alpha_1}) \cap (I_{\alpha_2}^{[5]} : I_{\alpha_2}) \cap (I_{\alpha_3}^{[5]} : I_{\alpha_3})$, which shows that colon formula in Proposition 5.2.4 does not hold for the ideal I.

Now, using the formula for the colon ideal in Proposition 5.2.4 we can describe the colon ideal as follows:

Definition 5.2.7. Let $I \leqslant S$ be a monomial ideal. We define J_q to be the unique minimal monomial ideal satisfying the equality

$$(I^{[q]} : I) = I^{[q]} + J_q.$$

Example 5.2.8. Let k a field of positive prime characteristic p and $S = k[[x_1, x_2, x_3]]$. Let $I = (x_1^2x_2^3, x_2^3x_3, x_1^2x_3)$. We obtain $(I^{[q]} : I) = (x_1^{2q}x_2^{3q}, x_2^{3q}x_3^q, x_1^{2q}x_3^q, (x_1^2x_2^3x_3)^{q-1})$, for any $q = p^e$ and any $e \geqslant 1$. Therefore, we have $J_q = ((x_1^2x_2^3x_3)^{q-1})$, for any q.

Example 5.2.9. Let $p = 3$ and $S = k[[x_1, x_2, x_3, x_4]]$. Let $I = (x_1x_2, x_1^2x_3, x_2x_3, x_2x_4)$. One can compute the colon ideal $(I^{[3]} : I) = (x_1^3x_2^3, x_1^6x_3^3, x_2^3x_3^3, x_2^3x_4^3, x_1^4x_2^2x_3^2, x_1^2x_2^3x_3^2x_4^2)$, hence $J_3 = (x_1^4x_2^2x_3^2, x_1^2x_2^3x_3^2x_4^2)$.

Remark 5.2.10. The complexity sequence $\{c_e\}_{e \geqslant 0}$ is bounded by the minimal number of generators of the ideal J_p, i.e. $c_e \leqslant \mu_S(J_p)$, for any $e \geqslant 0$. Note that $c_1 = \mu$ and $c_0 = 0$.

In the case of Stanley-Reisner rings, Corollary 3.3.11 states that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module. Moreover, Proposition 3.3.8 shows that this R-module is cyclic if and only if it is generated by $(\mathbf{x}^1)^{q-1}$, for any q. We proved a similar result for monomial ideals in Proposition 5.2.3. However, in the case of monomial ideals assuming that $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module does not imply that the generator of this R-module is of the form presented in Proposition 5.2.3 and hence it does not guarantee that the algebra of Frobenius operators $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra.

Question 5.2.11. Let k be a field of characteristic p and $S = k[[x_1, \ldots, x_n]]$ the formal power series ring in n variables over k. Let $I \subseteq S$ be a monomial ideal and $R = S/I$ the quotient ring. Is it enough to assume that $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module, for some q (or equivalently, for any q) in order to show that the Frobenius algebra of operators $\mathcal{F}(E_R)$ is (homogeneously) principally generated as an R-skew algebra?

In Proposition 5.1.9, we have the extra assumption on the generator of $R-$module $\dfrac{I^{[q]} :_S I}{I^{[q]}}$, for any q, which guarantees the principally generation of the Frobenius algebra of operators $\mathcal{F}(E_R)$ by a homogeneous generator in degree one. We do not know if dropping this assumption on the generators would still imply that the Frobenius algebra of operators $\mathcal{F}(E_R)$ is homogeneously principally generated. If this is not true, one should be able to find a counterexample and hence answer the following question:

Question 5.2.12. Can we find examples of ideals I in a complete regular local ring S with $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ cyclic as an R-module, for any q and such that the Frobenius algebra of operators $\mathcal{F}(E_R)$ is not homogeneously principally generated?

If I is a square-free monomial ideal and R is the Stanley-Reisner ring associated to it, we know that this is not possible since if $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module implies that $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is generated by $(\mathbf{x}^1)^{q-1}$ using Definition 3.3.2.

Remark 2.4.5(i) tells us that if R is a Gorenstein ring, the Frobenius algebra of operators $\mathcal{F}(E_R)$ is principally generated as an R-algebra. The converse is not true. There exist examples of rings which are not Gorenstein with principally generated Frobenius algebra of operators on the injective hull of the residue field. Àlvarez Montaner, Boix and Zarzuela gave examples of such rings in [27], Example 4.2. We do not have a good understanding of when is the Frobenius algebra of operators $\mathcal{F}(E_R)$ principally generated as an R-algebra. In [3], Blickle stated that if R is an F-finite normal ring, then the principally generation of the Frobenius algebra of operators $\mathcal{F}(E_R)$ is equivalent to R being Gorenstein. This statement was quoted by Àlvarez Montaner and Yanagawa in [28]. We have a counterexample for this statement. There exists an F-finite normal ring not Gorestein having the Frobenius algebra of operators $\mathcal{F}(E_R)$ principally generated over R. This ring is a quasi-Gorenstein ring, presented in Example 5.1 in [32]. We will first give the definition of a quasi-Gorenstein ring.

Definition 5.2.13 ([1, Definition 2.1])**.** A local ring (R, m, k) is called **quasi-Gorenstein** if $H^d_m(R) \cong E_R(k)$.

Example 5.2.14. Let k be a field of characteristic $char(k) \neq 3$. Let R be the Segre product of the cubic Fermat hypersurfaces: $R := k[x, y, z]/(x^3 + y^3 + z^3)\#k[a, b, c]/(a^3 + b^3 + c^3)$. By [11], R is a quasi-Gorenstein, normal domain of dimension 3 and depth 2. Since R is not a Cohen-Macaulay ring, R is not Gorenstein. Using Definition 5.2.13, we have that the Frobenius algebra of operators $\mathcal{F}(E_R) \cong \mathcal{F}(H^d_m(R))$. In Example 3.7 in [25], Lyubeznik and Smith showed that for any $d-$dimensional local ring satisfying Serre's S_2 condition, $\mathcal{F}(H^d_m(R))$ is principally generated by the canonical Frobenius action as an algebra over R. Our ring R is normal, so it satisfies Serre's S_2-condition. Hence, the Frobenius algebra of operators $\mathcal{F}(E_R)$ is principally generated as an R-algebra. To sum up, the ring R is an example of a normal F-finite ring not Gorenstein having the Frobenius algebra of operators principally generated as an R-algebra.

In fact, there exists another example of a ring which is not Gorestein having the Frobenius algebra of operators $\mathcal{F}(E_R)$ principally generated over R. This example was presented in Example 4.5(1) in [23].

Another question that we would like to answer is the following:

Question 5.2.15. Let k be a field of characteristic p and $S = k[[x_1, \dots, x_n]]$ the formal power series ring in n variables over k. Let $I \subseteq S$ be a monomial ideal with no embedded associated primes in S and $R = S/I$ the quotient ring. Is the ring of Frobenius operators $\mathcal{F}(E_R)$ either (homogeneously) principally generated or infinitely generated?

More specifically, this question asks whether finitely generation implies principally generation in the case of monomial ideals. We do not have a concrete description of the minimal generators of the colon ideals similar to the one in Lemma 3.3.7 for the Stanley-Reisner case. Hypothetically, the ideal $\dfrac{I^{[p]} :_S I}{I^{[p]}}$ could have at least two minimal generators of the form $(x_{\delta_j}^{a_j})^{p-1} = (x_{\delta_{j,1}}^{a_{j,1}} \cdots x_{\delta_{j,n}}^{a_{j,n}})^{p-1}$, with $a_{j,i} \geqslant 0$, $1 \leqslant \delta_{j,i} \leqslant n$ for any $1 \leqslant i \leqslant n$. In this case, the algebra of Frobenius operators $\mathcal{F}(E_R)$ could be finitely generated without being necessarily homogeneously principally generated as an R-skew algebra. However, we do not have an example of such a monomial ideal I, nor do we have a proof showing that this cannot happen. One way of investigating this question is using the formula we found in Proposition 5.2.4 in order to describe the minimal generators of the colon ideal $(I^{[p]} :_S I)$. Moreover, we would like to answer this question in general:

Question 5.2.16. Let k be a field of characteristic p and $S = k[[x_1, \dots, x_n]]$ the formal power series ring in n variables over k. Let $I \subseteq S$ be a monomial ideal and $R = S/I$ the quotient ring. Is the ring of Frobenius operators $\mathcal{F}(E_R)$ either (homogeneously) principally generated or infinitely generated as an R-skew algebra?

To answer this question, one would need to understand better the structure of the colon ideal $(I^{[p]} :_S I)$ for any monomial ideal I. The number of associated primes of the ideal I plays an important role as well. We will illustrate this in the next examples. We start by fixing a Stanley-Reisner ring having two, respectively three associated primes. For each of these rings, we will find conditions on the associated primes which would guarantee the principally generation of the Frobenius algebra of operators.

Example 5.2.17. Let $I = P_1 \cap P_2$, be a square-free monomial ideal with $\mathrm{Ass}(I) = \{P_1, P_2\}$. Since I is a square-free monomial ideal, P_1 and P_2 are ideals generated by variables. Let $P_1 := (x_{i_1}, \ldots, x_{i_s})$ and $P_2 := (x_{j_1}, \ldots, x_{j_t})$, for some $s, t \geqslant 1$. One can note that $(P_1^{[q]} :_S P_1) = (P_1^{[q]}, (x_{i_1} \cdots x_{i_s})^{q-1})$ and $(P_2^{[q]} :_S P_2) = (P_2^{[q]}, (x_{j_1} \cdots x_{j_t})^{q-1})$, for any q. By Proposition 3.3.3, we obtain

$$(I^{[q]} :_S I) = (P_1^{[q]}, (x_{i_1} \cdots x_{i_s})^{q-1}) \cap (P_2^{[q]}, (x_{j_1} \cdots x_{j_t})^{q-1})$$

$$= (I^{[q]}, (x_{i_1} \cdots x_{i_s})^{q-1} x_{j_l}^q, (x_{j_1} \cdots x_{j_t})^{q-1} x_{i_k}^q, (x_{i_1} \cdots x_{i_s} \cdot x_{j_1} \cdots x_{j_t})^{q-1} : 1 \leqslant k \leqslant s, 1 \leqslant l \leqslant t),$$

for any q. One can note that $(x_{i_1} \cdots x_{i_s})^{q-1} x_{j_l}^q \in I^{[q]}$ if and only if $j_l \in \{i_1, \ldots, i_s\}$ and $(x_{j_1} \cdots x_{j_t})^{q-1} x_{i_k}^q \in I^{[q]}$ if and only if $i_k \in \{j_1, \ldots, j_t\}$. Proposition 5.1.9 states that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module by x^{γ_e} and $x^{\gamma_e} = (x^{\gamma_1})^{\nu_e}$, where $\nu_e = 1 + p + \ldots + p^{e-1}$, for any $e \geqslant 1$. Hence, $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $i_k \in \{j_1, \ldots, j_t\}$, for any $k \in \{1, \ldots, s\}$ and $j_l \in \{i_1, \ldots, i_s\}$, for any $l \in \{1, \ldots, t\}$. That happens if and only if $\{i_1, \ldots, i_s\} = \{j_1, \ldots, j_t\}$, i.e. $I = P_1 = P_2 = (x_{i_1}, \ldots, x_{i_s})$. In this case, $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra by $(x_{i_1} \cdots x_{i_s})^{p-1} F$, where $F : E_R \to E_R$ denotes the canonical Frobenius action on E_R. Hence, if I is a square-free monomial ideal with two nonembedded associated primes, the Frobenius algebra of operators cannot be principally generated as an R-skew algebra.

Example 5.2.18. Let $I = P_1 \cap P_2 \cap P_3$, be a square-free monomial ideal with $\mathrm{Ass}(I) = \{P_1, P_2, P_3\}$. Since I is a square-free monomial ideal, P_1, P_2 and P_3 are ideals generated by variables. Let $P_1 := (x_{i_1}, \ldots, x_{i_s})$, $P_2 := (x_{j_1}, \ldots, x_{j_t})$ and $P_3 := (x_{k_1}, \ldots, x_{k_r})$ for some $s, t, r \geqslant 1$. One can note that $(P_1^{[q]} :_S P_1) = (P_1^{[q]}, (x_{i_1} \cdots x_{i_s})^{q-1})$, $(P_2^{[q]} :_S P_2) = (P_2^{[q]}, (x_{j_1} \cdots x_{j_t})^{q-1})$ and $(P_3^{[q]} :_S P_3) = (P_3^{[q]}, (x_{k_1} \cdots x_{k_r})^{q-1})$, for any q. By Proposition 3.3.3, we obtain

$$(I^{[q]} :_S I) = (P_1^{[q]}, (x_{i_1} \cdots x_{i_s})^{q-1}) \cap (P_2^{[q]}, (x_{j_1} \cdots x_{j_t})^{q-1}) \cap (P_3^{[q]}, (x_{k_1} \cdots x_{k_r})^{q-1})$$

$$= (I^{[q]}, (x_{i_1}\cdots x_{i_s})^{q-1}x_{j_l}^q x_{k_m}^q, (x_{j_1}\cdots x_{j_t})^{q-1}x_{i_u}^q x_{k_m}^q, (x_{k_1}\cdots x_{k_r})^{q-1}x_{i_u}^q x_{j_l}^q,$$

$$(x_{i_1}\cdots x_{i_s}\cdot x_{j_1}\cdots x_{j_t})^{q-1}x_{k_m}^q, (x_{j_1}\cdots x_{j_t}\cdot x_{k_1}\cdots x_{k_r})^{q-1}x_{i_u}^q, (x_{i_1}\cdots x_{i_s}\cdot x_{k_1}\cdots x_{k_r})^{q-1}x_{j_l}^q,$$

$$(x_{i_1}\cdots x_{i_s}\cdot x_{j_1}\cdots x_{j_t}\cdot x_{k_1}\cdots x_{k_r})^{q-1} : 1 \leqslant u \leqslant s, 1 \leqslant l \leqslant t, 1 \leqslant m \leqslant r),$$

for any q. By Proposition 5.1.9, $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module by x^{γ_e} and $x^{\gamma_e} = (x^{\gamma_1})^{\nu_e}$, where $\nu_e = 1 + p + \ldots + p^{e-1}$, for any $e \geqslant 1$. Hence, $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is cyclic as an R-module by $(x_{i_1}\cdots x_{i_s}\cdot x_{j_1}\cdots x_{j_t}\cdot x_{k_1}\cdots x_{k_r})^{q-1}$, for any q. This happens if and only if the following conditions hold: $\{i_1,\ldots,i_s,j_1,\ldots,j_t\} = \{1,\ldots,n\}$, $\{j_1,\ldots,j_t,k_1,\ldots,k_r\} = \{1,\ldots,n\}$, $\{i_1,\ldots,i_s,k_1,\ldots,k_r\} = \{1,\ldots,n\}$, $\{i_1,\ldots,i_s,j_l,k_m\} = \{1,\ldots,n\}$ or $\{j_l,k_m\}$ is the support set of one of I's generators, for any $1 \leqslant l \leqslant t, 1 \leqslant m \leqslant r$, $\{j_1,\ldots,j_t,i_u,k_m\} = \{1,\ldots,n\}$ or $\{i_u,k_m\}$ is the support set of one of I's generators, for any $1 \leqslant u \leqslant s, 1 \leqslant m \leqslant r$, and $\{k_1,\ldots,k_r,i_u,j_l\} = \{1,\ldots,n\}$ or $\{i_u,j_l\}$ is the support set of one of I's generators, for any $1 \leqslant u \leqslant s, 1 \leqslant l \leqslant t$. For simplicity, we assume $I \subseteq K[[x_1,x_2,x_3]]$, with K a field of positive prime characteristic p. One can note that the conditions above imply that $\mathcal{F}(E_R)$ is homogeneously principally generated as an R-skew algebra if and only if $I = (x_1,x_2)\cap(x_2,x_3)\cap(x_3,x_1)$, i.e. $I = (x_1x_2, x_2x_3, x_3x_1)$.

In the next example we show that for monomial ideals with two nonembedded associated primes, the Frobenius algebra of operators cannot be principally generated as an R-skew algebra.

Example 5.2.19. Let $I = P_1 \cap P_2$, be a monomial ideal with $\mathrm{Ass}(I) = \mathrm{Min}(I) = \{\sqrt{P_1}, \sqrt{P_2}\}$. Since I is a monomial ideal, P_1 and P_2 are ideals generated by powers of variables. Let $P_1 := (x_{i_1}^{a_1},\ldots,x_{i_s}^{a_s})$ and $P_2 := (x_{j_1}^{b_1},\ldots,x_{j_t}^{b_t})$, for some $s,t \geqslant 1$ and $a_1,\ldots,a_s,b_1,\ldots,b_t \geqslant 0$. One can note that $(P_1^{[q]} :_S P_1) = (P_1^{[q]}, (x_{i_1}^{a_1}\cdots x_{i_s}^{a_s})^{q-1})$ and $(P_2^{[q]} :_S P_2) = (P_2^{[q]}, (x_{j_1}^{b_1}\cdots x_{j_t}^{b_t})^{q-1})$, for any q. By Proposition 3.3.3, we obtain

$$(I^{[q]} :_S I) = (P_1^{[q]}, (x_{i_1}^{a_1}\cdots x_{i_s}^{a_s})^{q-1}) \cap (P_2^{[q]}, (x_{j_1}^{b_1}\cdots x_{j_t}^{b_t})^{q-1})$$

$$= (I^{[q]}, (x_{i_1}^{a_1} \cdots x_{i_s}^{a_s})^{q-1} x_{j_l}^{qb_l}, (x_{j_1}^{b_1} \cdots x_{j_t}^{b_t})^{q-1} x_{i_k}^{qa_k}, (x_{i_1}^{a_1} \cdots x_{i_s}^{a_s} \cdot x_{j_1}^{b_1} \cdots x_{j_t}^{b_t})^{q-1} : 1 \leqslant k \leqslant s, 1 \leqslant l \leqslant t),$$

for any q. Since $\mathrm{Min}(I) = \{\sqrt{P_1}, \sqrt{P_2}\}$, there exists $i_k \notin \{j_1, \ldots, j_t\}$ and $j_l \notin \{i_1, \ldots, i_s\}$ for some $1 \leqslant k \leqslant s$ and $1 \leqslant l \leqslant t$. One can then note that $x_{i_k}^{qa_k}(x_{j_1}^{b_1} \cdots x_{j_t}^{b_t})^{q-1} \notin I^{[q]}$ and $x_{j_l}^{qb_l}(x_{i_1}^{a_1} \cdots x_{i_s}^{a_s})^{q-1} \notin I^{[q]}$. Hence, the R-module $\dfrac{I^{[q]} :_S I}{I^{[q]}}$ is not cyclic and this implies that $\mathcal{F}(E_R)$ is not homogeneously principally generated as an R-skew algebra.

REFERENCES

[1] Y. Aoyama. Some basic results on canonical modules. *J. Math. Kyoto Univ.*, 23:85–94, 1983.

[2] M. Blickle. The intersection homology D-module in finite characteristic. *Mathematische Annalen*, 328:425–450, 2004.

[3] M. Blickle. Test ideals via algebras of $p^{-e}-$linear maps. *J. Algebraic Geom.*, 22:49–83, 2013.

[4] A. Boix and M. Katzman. An algorithm for producing F-pure ideals. *Archiv. der Mathematik*, 103:421–433, 2014.

[5] A. Boix and S. Zarzuela. Frobenius and Cartier Algebras of Stanley-Reisner Rings II. *Acta Mathematica Vietnamica*, 324:1–16, 2019.

[6] M. P. Brodmann and R. Y. Sharp. Local cohomology: an algebraic introduction with geometric applications. *Cambridge University Press*, 60, 1998.

[7] W. Bruns and J. Herzog. Cohen-Macaulay Rings. *Cambridge University Press*, 1993.

[8] F. Enescu and I. Ilioaea. Strong Test Ideals associated to Cartier Algebras. *Journal of Algebra and its Applications*, 19:2050044, 2020.

[9] F. Enescu and Y. Yao. The Frobenius Complexity of a Local Ring of Prime Characteristic. *Journal of Algebra*, 459:133–156, 2015.

[10] R. Fedder. F-purity and rational singularity. *Trans. Amer. Math. Soc.*, 278:461–480.

[11] S. Goto and K. Watanabe. On graded rings I. *Journal of the Mathematical Society of Japan*, 30:179–213, 1978.

[12] N. Hara and S. Takagi. On a generalization of test ideals. *Nagoya Math. Journal*, 175:59–74, 2004.

[13] M. Hochster and C. Huneke. Tight closed ideals. *Bull. Amer. Math. Soc.*, 18:45–48, 1988.

[14] M. Hochster and C. Huneke. Tight closure, invariant theory and the Briançon-Skoda Theorem. *J. Amer. Math. Soc.*, 3:31–116, 1990.

[15] M. Hochster and C. Huneke. F-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346:1–62, 1994.

[16] M. Hochster and J. L. Roberts. The purity of the Frobenius and local cohomology. *Adv. in Math.*, 21:117–172, 1976.

[17] C. Huneke. Tight closure and its applications. *CBMS Regional Conference in Mathematics, American Mathematical Society*, 88, 1996.

[18] C. Huneke. Tight closure and strong test ideals. *J. Pure and Appl. Algebra*, 122:243–250, 1997.

[19] I. Ilioaea. On the Frobenius Complexity of Stanley-Reisner Rings. *Communications in Algebra*, 1:1–10, 2020.

[20] M. Katzman. Parameter test ideals of Cohen-Macaulay rings. *Compositio Mathematica*, 144:933–948, 2008.

[21] M. Katzman. A non-finitely generated algebra of Frobenius maps. *Proc. Amer. Math. Soc.*, 138:2381–2383, 2010.

[22] M. Katzman and K. Schwede. An algorithm for computing compatibly Frobenius split subvarieties. *J. Symbolic Comp.*, 47:996–1008, 2012.

[23] M. Katzman, K. Schwede, A.K. Singh, and W. Zhang. Rings of Frobenius operators. *Math. Proc. Cambridge Phil. Soc.*, 157:151–167, 2014.

[24] E. Kunz. Characterizations of regular local rings of characteristic p. *Amer. J. Math.*, 91:772–784, 1969.

[25] G. Lyubeznik and K. E. Smith. On the commutation of the test ideal with localization and completion. *Trans. Amer. Math. Soc.*, 353:3148–3180, 2001.

[26] J.A. Montaner. Generating functions associated to Frobenius algebras. *Linear Algebra and its Applications*, 528:310–326, 2019.

[27] J.A. Montaner, A. Boix, and S. Zarzuela. Frobenius and Cartier Algebras of Stanley-Reisner Rings. *Journal of Algebra*, 358:162–177, 2012.

[28] J.A. Montaner and K. Yanagawa. Addendum to "Frobenius and Cartier Algebras of Stanley-Reisner Rings". *Journal of Algebra*, 414:300–304, 2014.

[29] K. Schwede. Test ideals in non-$\mathbb{Q}$-Gorenstein rings. *Trans. Amer. Math. Soc.*, 363:5925–5941, 2011.

[30] K. Schwede and K. Tucker. A survey of test ideals. *Progress in commutative algebra 2*, 363:39–99, 2012.

[31] R. Sharp. An excellent $F-$pure ring of prime characteristic has a big tight closure test element. *Trans. Amer. Math. Soc.*, 362:5455–5481, 2010.

[32] K. Shimomoto, N. Taniguchi, and E. Tavanfar. A study of quasi-Gorenstein rings II: deformation of quasi-Gorenstein property. *arXiv preprint arXiv:1810.03181*, 2018.

[33] A. Vraciu. Strong test ideals. *J. Pure and Appl. Algebra*, 167:361–373, 2002.