2020

Pascal's Mystic Hexagon in Tropical Geometry

Hanna Hoffman

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Hanna Hoffman

Dagan Karp, Advisor

Hannah Markwig, Reader

HARVEY MUDD COLLEGE

Department of Mathematics

May, 2020

Abstract

Pascal's mystic hexagon is a theorem from projective geometry. Given six points in the projective plane, we can construct three points by extending opposite sides of the hexagon. These three points are collinear if and only if the six original points lie on a nondegenerate conic. We attempt to prove this theorem in the tropical plane.

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Acknowledgments

I would like to thank my advisor and second reader for their guidance and invaluable feedback. I would also like to thank Cameron, Cienn, and Matty for being the best suitemates I could ask for, and my family and friends for their support in this difficult time.

Chapter 1

Introduction

This thesis considers a classical theorem in a new setting, namely, Pascal's theorem in tropical geometry. Pascal's mystic hexagon is a theorem from algebraic geometry. It states that six points lie on a conic if and only if three points generated by the original six lie on a line. This is remarkable, because we don't expect three general points to be collinear, and because, similarly, we don't expect six general points to lie on the same conic. Recall that conics (also known as conic sections) include such curves as circles, ellipses, hyperbolae, and parabolae in the real plane ($\mathbb{R}^2$). In the projective plane, where Pascal's theorem holds, these curves are equivalent.

This theorem has long been proven, and several different proofs are known. I will attempt to prove an analogue of Pascal's theorem in tropical geometry, which is a relatively young field. Its core ideas have been around for about thirty or forty years, and only since the early 2000s has concerted attention been paid to the subject. For comparison, Pascal's Theorem was first proposed by Blaise Pascal in the 17th century.

Before we get to tropical geometry, however, I'll introduce the tropical semiring. Recall that the real numbers are a ring under addition and multiplication. The tropical semiring has two significant differences from the ring of real numbers: it includes infinity and the operations of addition and multiplication are replaced with tropical addition and tropical multiplication.

Tropical addition is defined to be the minimum: that is, when we add 3 and 5, we find the minimum, or $3 \oplus 5 = \min(3, 5) = 3$. Also, tropical multiplication is addition. So, if we tropically multiply 4 and 7, instead of 28, we get $4 \odot 7 = 4 + 7 = 11$.

Why do we care about this? It turns out that this semiring behaves quite nicely in general, and has several computationally and otherwise

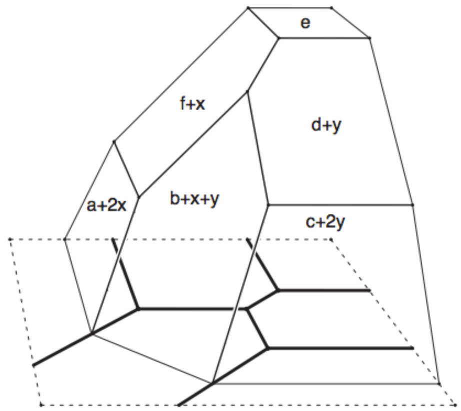

Figure 1.1 The graph of a tropical quadratic, from Maclagan and Sturmfels (2015)

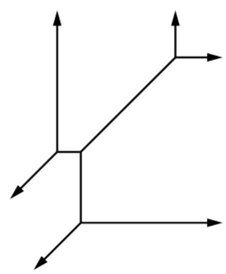

Figure 1.2 A tropical quadratic curve, from Maclagan and Sturmfels (2015)

significant applications beyond the scope of this thesis. This structure has some interesting consequences for algebraic geometry. Algebraic geometry over the tropical semiring is also called tropical geometry, so that it is easy to distinguish the two. While projective geometry, where Pascal's theorem originates, and tropical geometry have some similarities, there are several mathematically significant distinctions.

The easiest of these to illustrate is that tropical curves are piecewise linear. For example, when we have a quadratic equation in two variables, ($h(x, y) = ax^2 + bxy + cy^2 + dy + e + fx$), the graph looks like Figure 1.1. When we take the corner locus, we end up with the curve in Figure 1.2.

These "curves" are much nicer to manipulate than standard curves, in fact, since linear objects are more predictable than nonlinear ones.

Finally, and most relevant to this thesis, we expect this theorem from above to hold, since similar theorems from algebraic geometry have translated over to tropical geometry rather well. Furthermore, similar problems actually look *nicer* to solve in tropical geometry.

Chapter 2

Background, Algebraic Geometry

Algebraic geometry is the study of geometric objects that arise from algebra, and we will be primarily concerned with zero sets of polynomials. We'll start by defining projective space, then move on to homogeneous forms, look at a couple of specific examples (the line and conic) in more depth, and close with several propositions and lemmas leading to the proof of Pascal's theorem, following Reid (1988).

2.1 The Projective Plane

We will work in projective plane for the remainder of this chapter, since it possesses a few particularly nice qualities that $\mathbb{R}^2$ doesn't.

There are a few equivalent definitions of the projective plane, denoted $\mathbb{P}^2$[1], but we'll start with a less technically demanding one:

Definition 2.1.1. $\mathbb{P}^2_{\mathbb{R}}$, the projective plane, is the set of all lines through the origin in $\mathbb{R}^3$.

How can we use this definition to imagine the projective plane? Starting with $\mathbb{R}^3$, we can start with a plane parallel to the xy-plane, say $z = 1$. For each point in this plane, we can draw a distinct line from the origin through that point. So, there exists a 1-to-1 correspondence between points in this plane and lines through the origin in $\mathbb{R}^3$, and we may say that there is a

[1]There are several ways to denote the projective plane over a field: here, since we will be working over $\mathbb{R}$, I will use $\mathbb{P}^2$ for ease, but it is also commonly expressed as $\mathbb{P}^2_{\mathbb{R}}$ and $\mathbb{R}\mathbb{P}^2$.

copy of this plane, or $\mathbb{R}^2$, within $\mathbb{P}^2$. Additionally, we have a collection of lines through the origin that we have not yet considered: those lines in the xy-plane. Each of these lines corresponds to a direction, so we can consider each one as a kind of point "at infinity" in that direction. Hence, we can think of $\mathbb{P}^2$ as $\mathbb{R}^2$ plus a collection of points at infinity.

Before we get to the most difficult definition, let's take a detour into equivalence relations.

Definition 2.1.2. An *equivalence relation* is a relation $\sim$ on a set A that satisfies the following three properties (respectively, the reflexive, symmetric, and transitive properties) for all $a, b, c \in A$:

(i) $a \sim a$ for all a,

(ii) $a \sim b$ implies $b \sim a$,

(iii) $a \sim b$ and $b \sim c$ together imply $a \sim c$.

Equivalence relations let us divide sets into equivalence classes, so that we can more easily consider them.

Definition 2.1.3. Given a set A, an element $a \in A$, and an equivalence relation $\sim$ on A, the *equivalence class* containing a is the set S of all elements of A that are equivalent to a, i.e. $S = \{x \in A : x \sim a\}$.

For example, consider the rational numbers, $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$. We can define an equivalence relation by $\frac{a}{b} \sim \frac{c}{d}$ if and only if $ad = bc$. That is, $\frac{1}{2} \sim \frac{3}{6} \sim \frac{52}{104}$ and so on. Notice that this places fractions in equivalence classes in which each member of a given equivalence class is the same fraction when reduced to lowest terms.

The equivalence relation given in the following theorem is somewhat similar to the prior equivalence relation on $\mathbb{Q}$, except defined over all nonzero points in $\mathbb{R}_3$.

Theorem 2.1.1. *The projective plane is isomophic to* $\mathbb{R}^3 \setminus \{(0,0,0)\}$ *under the equivalence relation* $\sim$*, where* $(x, y, z) \in \mathbb{R}$ *is equivalent to* $(\lambda x, \lambda y, \lambda z)$ *for* $\lambda \in \mathbb{R}, \lambda \neq 0$*.*

Proof. Each equivalence class under $\sim$ corresponds to a unique line through the origin in $\mathbb{R}^3$. Recall that we can define a line L in $\mathbb{R}_3$ by a point p and a vector v through $L = \{p + tv | t \in \mathbb{R}\}$. We can simply let p be the origin, and v be (the vector from the origin to) any point on the line in question. Then for any $\lambda = t$, λv belongs to both the line and the equivalence class containing v. ∎

2.2 Forms and Curves

Now that we have the projective plane, we'll consider curves - for the purposes of this thesis, that will include lines, conics, and cubics. But before we get there, we'll have to talk about polynomials.

Definition 2.2.1. A *homogeneous polynomial* is a polynomial that has the same total degree in each of its terms. The total degree of each term is the *degree* of the polynomial. We will also use the word *form* synonymously.

The general expression for a form in two variables of degree d is

$$F(x, y) = a_0 x^d + a_1 x^{d-1} y + \ldots + a_{d-1} x y^{d-1} + a_d y^d.$$

Note that each term has a total degree of d, as we might expect.

Given a three-variable form F, we can find the set of points (x, y, z) where $F(x, y, z) = 0$, although this set can be empty. We call this set the zero set of F, denoted $V(F)$. We state this more formally below:

Definition 2.2.2. Suppose $F : \mathbb{R}^3 \to \mathbb{R}$ is a form. Then the *zero set*, $V(F)$, of F is the set $V(F) = \{(x, y, z) \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$.

Note that the zero set is well-defined under projective scaling, since for a degree d form F, $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$ always holds. Additionally, we note that *only* homogeneous polynomials have well-defined zero sets over $\mathbb{P}^2$. Otherwise, some representative point $p = (x, y, z)$ might satisfy $F(p) = 0$, but we are not guaranteed that equivalent points $\lambda p = (\lambda x, \lambda y, \lambda z)$ satisfy $F(\lambda p) = 0$.

We have special names for zero sets corresponding to some low-degree forms - lines are the zero sets of degree 1 forms, conics are the zero sets of degree 2 forms, and cubics are the zero sets of degree 3 forms. Let's take a look at an example of each of these.

Example 2.2.3. Consider the form $F(x, y, z) = x + 2y - 14z$. Its zero set is a line, as in Figure 2.1, since each term has total degree 1.

The general form of a degree 2 form over three variables is

$$F(x, y, z) = ax^2 + bxy + cy^2 + dyz + exz + fz^2.$$

Example 2.2.4. Consider the degree 2 form $F(x, y, z) = 2x^2 - xy + 12yz + 3z^2$. This form's zero set is the conic curve in Figure 2.1.

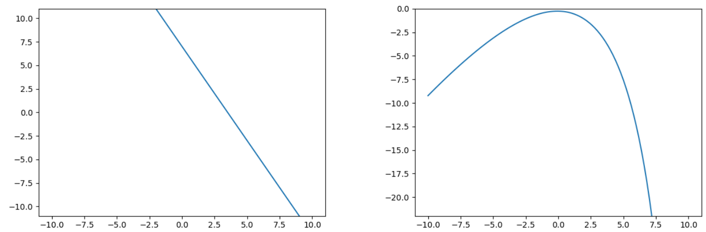

Figure 2.1 A line and a conic, as in Examples 2.2.3 and 2.2.4

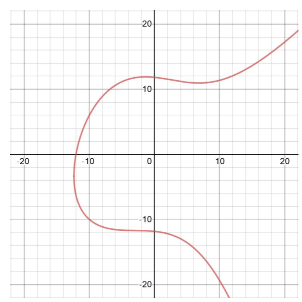

Figure 2.2 A cubic, as in Example 2.2.5

A cubic in $\mathbb{P}^2$ is the zero set of a degree three form. A general cubic form is given by

$$F(x,y,z) = a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xz^2 + a_5xyz + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3.$$

Example 2.2.5. Consider the degree 3 form $G(x,y,z) = 4x^3 - 3x^2y + 5y^3 + xyz - 7z^3$. This corresponds to the cubic curve in Figure 2.2[2].

We can also construct curves by taking the union of two curves of smaller degree.

Proposition 2.2.6. Suppose that we have a line L and a conic Q. Then $L \cup Q$ is a cubic.

Proof. The line L is the zero set of a degree 1 form F, and the conic Q is the zero set of a degree 2 form G. We can then multiply F and G to get a degree 3 form FG, and we will show that the zero set of FG is exactly $L \cup Q$. Note

[2]This figure was made with Desmos

that FG is always zero when at least one of F and G is zero. Also note that since $\mathbb{R}$ has no zero divisors, one of F and G must be zero when FG is zero. Hence, the zero set of FG contains both L and Q, and does not contain any other points, as desired. □

Note also that we can generalize this proposition to any set of curves with total degree d, so that two lines form a conic, three lines form a cubic, two conics form a quartic, and so on.

2.3 Some Helpful Lemmas

Proposition 2.3.1 (Corollary 1.10 in Reid (1988))**.** There exists a unique conic through 5 points, of which no four are collinear, in $\mathbb{RP}^2$.

Additionally, when at least six points lie on the same nondegenerate conic we call them *conconic*. This parallels the vocabulary we use to describe at least three points that lie on the same line. Since through any two distinct points there exists a unique line, when we have three points that lie on a single line we call them *collinear*.

Theorem 2.3.1 (Bézout's theorem)**.** *Suppose C and D are curves in the projective plane with respective degrees m and n. Then C and D intersect in at most mn points.*

While we won't prove Bézout's theorem here (for a proof, see Reid (1988)), we will note that we don't get equality (that is, C and D intersect in exactly mn points) unless the projective plane is over an algebraically closed field and we count intersections with multiplicity.

The next lemma considers the space of cubics through a given set of points and the dimension thereof. For simplicity, we will denote the space of cubics S_3 and the space of cubics through a set of points P as $S_3(P)$. We may restate the first proposition of this section, Proposition 2.3.1, as the following: given general points $P_1, \ldots, P_5$, the dimension of the space of conics through them is 1, or $\dim S_2(P_1, \ldots, P_5) = 1$, where S_2 denotes the space of conics.

Proposition 2.3.2 (Proposition 2.6 in Reid (1988), also known as the Cayley-Bacharach Theorem)**.** Let $P_1, \ldots, P_8 \in \mathbb{P}^2$ be distinct points. Suppose that no 4 of $P_1, \ldots, P_8$ are collinear, and no 7 of them lie on a nondegenerate conic. Then the space of cubics through these 8 points has dimension 2, or $\dim S_3(P_1, \ldots, P_8) = 2$.

Proof. We split this proof up into several cases: the "general position case", where no 3 of our points are collinear, and no 6 of them are conconic; the case where there are 3 collinear points; and the case where there are 6 conconic points.

General Case: We assume to obtain a contradiction that the dimension of the space of cubics is at least three. Suppose there are two more points, P_9 and P_{10} on the line L through P_1 and P_2. Then, the dimension of the space of cubics through $P_1, \dots, P_8, P_9, P_{10}$ is two less than that of the space of cubics through the first 8 points. Since there are at least 3 of the latter, there is at least one nonzero cubic through the ten points. Since the line L goes through P_1, P_2, P_9, P_{10}, this cubic is the union of L and some conic C through the remaining 6 points. If C is nondegenerate, $P_3, \dots, P_8$ are conconic, which contradicts our assumption. Alternately, if C is degenerate it must either be a line pair or a double line, and by the pigeonhole principle, at least three of $P_3, \dots, P_8$ must be collinear, which is also a contraction.

3 collinear points: Suppose without loss of generality that P_1, P_2, P_3 lie on a line L. Let P_9 also be a point on L. Since P_1, P_2, P_3, P_9 are all on L, all cubics through $P_1, \dots, P_9$ must contain L. Then the remaining 5 (noncollinear) points, $P_4, \dots, P_8$ must lie on a conic. By Proposition 2.3.1, we know that the space of conics through 5 general points has only 1 dimension. Thus, the dimension of the space of cubics through $P_1, \dots, P_9$ is 1. Then when we remove P_9, we remove one linear condition, so the space of cubics through $P_1, \dots, P_8$ is at most 2, as desired.

6 conconic points: Suppose without loss of generality that $P_1, \dots, P_6$ lie on a conic Q. Let P_9 also be a (distinct) point on Q. Since $P_1, \dots, P_6, P_9$ are all on Q, all cubics through $P_1, \dots, P_9$ must contain Q. Then the remaining 2 points, P_7 and P_8 must lie on a line. Through any two points there is a unique line, so the dimension of the space of cubics through $P_1, \dots, P_9$ is 1. Then when we remove P_9, we remove one linear condition, so the space of cubics through $P_1, \dots, P_8$ is at most 2, as desired. □

Proposition 2.3.3 (Corollary 2.7 in Reid (1988))**.** Let C_1,C_2 be two cubic curves whose intersection consists of nine distinct points, $C_1 \cap C_2 = \{P_1, \dots, P_9\}$. Then a cubic D that goes through $P_1, \dots, P_8$ also goes through P_9.

Proof. If four of $P_1, \dots, P_8$ were on some line L, then C_1, C_2 would meet L in at least 4 points. By Bézout's theorem, note that a line and a cubic meet in at most 3 points. Thus, if a conic contains 4 collinear points, it must contain the whole line. Hence, if four of $P_1, \dots, P_8$ were collinear on some line L, both cubics would contain L, which contradicts our assumption on $C_1 \cap C_2$.

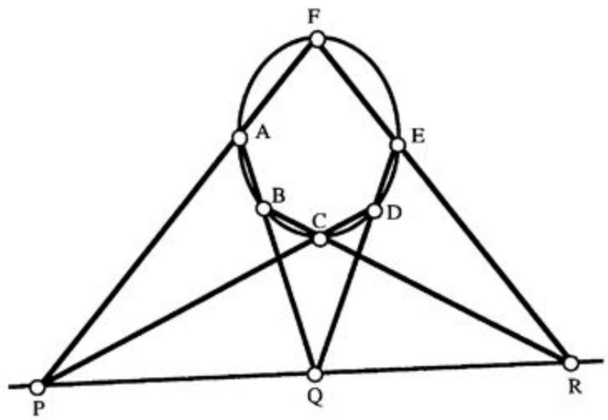

Figure 2.3 Pascal's mystic hexagon; image from Reid (1988)

Similarly, no 7 of the points can be conconic, since a conic and a cubic meet in at most 6 points.

Then the assumptions of Proposition 2.3.2 hold, so the space of cubics through $P_1, \ldots, P_9$ has dimension 2. Suppose that C_1 is the zero set of F_1 and similarly C_2 and F_2. Then F_1 and F_2 are a basis of $S_3(P_1, \ldots, P_8)$. So, $D : (G = 0)$ for $G = \lambda F_1 + \mu F_2$. Since F_1 and F_2 both vanish at P_9, G must also, and hence D goes through P_9. □

2.4 Pascal's Mystic Hexagon

In this section, we will state and then prove Pascal's theorem.

Theorem 2.4.1 (Pascal's Mystic Hexagon, Theorem 2.11 in Reid (1988)). *Given a hexagon $ABCDEF$ in $\mathbb{P}^2$, extend pairs of opposite sides to their intersections to make the points P, Q, and R. Assume that the six lines and nine points are distinct. Then the hexagon $ABCDEF$ is conconic if and only if PQR are collinear.*

See Figure 2.3 for an example of one such hexagon on a conic and its associated collinear points.

Proof. Consider 2 triples of lines:

$$L_1 : PAF, L_2 : QDE, L_3 : RBC$$

and

$$M_1 : PCD, M_2 : QAB, M_3 : REF.$$

These define two cubics - $C_1 = L_1 + L_2 + L_3$ and $C_2 = M_1 + M_2 + M_3$. C_1 and C_2 intersect in the nine named points of the diagram. Of the nine points, no four are collinear, and no seven are conconic, so we can apply Proposition 2.3.3.

First, we show the converse. Suppose that PQR are collinear, on line N. We may also suppose that there is a conic Γ through $ABCDE$, since through any five points we can construct a conic by Proposition 2.3.1. Then $N + \Gamma$ is a cubic.

Then, by Proposition 2.3.3, since $N + \Gamma$ goes through the eight points A, B, C, D, E, P, Q, R, it must also contain F. Note that F cannot belong on N by assumption – since REF is assumed to be distinct from PQR, if RF belongs to both REF and PQR, those lines are no longer distinct, which is a contradiction. Hence, F must belong to Γ, and $ABCDEF$ are conconic.

Now, we show the implication. Suppose that $ABCDEF$ are conconic on the conic Γ, and let $N = PQ$. Then $N+\Gamma$ is a cubic through A, B, C, D, E, F, P, Q, and we can apply Proposition 2.3.3 again. Hence, it must also go through R. We show that R is not on the conic Γ by contradiction. If R were on Γ, then Γ must contain the lines RBC and REF, by the original construction of R. Then either REF and RBC must coincide, which is a contradiction, since we assumed that these were distinct lines, or Γ is a line pair, which is also leads us to a contradiction. Note that if Γ is a line pair, with BC on one line and EF on the other, then A must live on one of these lines. If A lives on BC, then RBC coincides with QAB, which is again a contradiction. Similarly, if A is on EF, REF coincides with PAF, also a contradiction. Thus, R must be on PQ, and PQR are collinear. □

Chapter 3

Background, Tropical Geometry

In this chapter, we will walk through the basics of tropical algebra, consider what polynomials look like in tropical algebra, and then move on to the concepts of lines and conics in tropical geometry. We follow Maclagan and Sturmfels (2015).

3.1 The Tropical Semiring

Definition 3.1.1. The *tropical semiring*, $(\mathbb{R} \cup \infty, \oplus, \odot)$, is the set of real numbers with infinity under tropical addition and tropical multiplication. Tropical addition, denoted $\oplus$, is the minimum, while tropical multiplication, denoted $\odot$, is the usual addition.

We will illustrate these operations with some examples.

Example 3.1.2. Consider $24 \oplus 2$. Since $\oplus$ is the minimum, we get

$$24 \oplus 2 = \min(24, 2) = 2.$$

Note also that for any finite x, $\min(x, \infty)$ is always x, so ∞ is the additive identity. However, there are no additive inverses, in general.

Example 3.1.3. Consider $3 \odot 7$. Since $\odot$ is addition, this is $3 \odot 7 = 3 + 7 = 10$.

Since $x + 0$ is always x, the multiplicative identity is zero. We also have multiplicative inverses, and it is left to the reader to determine them.

We can also consider tropical polynomials:

Example 3.1.4. Let $f(x) = x^4 \oplus 2 \odot x^3 \oplus -4 \odot x^2 \oplus 3$, where x^2 means $x \odot x$. Then we can rewrite this as

$$\min(4x, 2 + 3x, 2x - 4, 3).$$

Thus, for any classical polynomial, we can *tropicalize* it by interpreting each multiplication as tropical multiplication and each addition as tropical addition. Note that for any polynomial f, its tropicalization will be the minimum of a series of terms that are linear in each variable. The next section discusses how we formally handle this process.

3.2 Tropicalization of Polynomials

Definition 3.2.1. The *tropicalization* of a polynomial $f = \sum_{n \in \mathbb{Z}} a_n x^n$, with $a_n \in \mathbb{R}$, denoted $\text{trop}(f)$, is

$$\text{trop}(f) = \min_{u \in \mathbb{Z}} (a_u + ux).$$

However, in order to take minima when we have multiple variables and coefficients from rings other than the real numbers, we'll have to get more creative. The first ingredient that we'll need is valuations – in short, these are maps that take elements of some field to an abelian group; for our purposes, that abelian group will be $\mathbb{R} \cup \{\infty\}$ – exactly the set we've defined tropical algebra over.

Definition 3.2.2. Suppose that K is a field, and let K^* be the nonzero elements of K. Then a *valuation* is a map $\text{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying the following:

(i) $\text{val}(a) = \infty$ if and only if $a = 0$,

(ii) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$, and

(iii) $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$ for all $a, b \in K^*$.

The trivial valuation of a field takes every element of that field to 0.

A more interesting (and relevant) example from Maclagan and Sturmfels (2015) concerns Puiseux series. Let K be the field of Puiseux series with coefficients. Then an element of K is given by

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \cdots,$$

where the c_i are all nonzero complex numbers and $a_1 < a_2 < a_3 < \cdots$ are rational numbers with a common denominator. This field K has a natural valuation val : $K \to \mathbb{R}$ where $\mathrm{val}(c(t))$ is the lowest exponent of t that appears in the series expansion of $c(t)$.

Example 3.2.3. The polynomial $c(t) = t^3 + 4t^18$ has valuation $\mathrm{val}(c(t)) = 3$.

Given a valuation, we may also define a residue field. The general definition is beyond the scope of this thesis. However, for Puiseux series and their natural valuation as defined above, we may map any element

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \cdots$$

with $a_1 \geq 0$ to the residue field by taking $c(t)$ to 0 if $a_1 > 0$ and by taking $c(t)$ to c_1 if $a_1 = 0$.

The concept of valuation allows us to consider the tropicalization of polynomials in multiple variables and with coefficients from K, the field of Puiseux series.

Let $f \in K[x_1, \ldots, x_n]$. That is, f is a polynomial in the variables $x_1, x_2, \ldots, x_n$ with coefficients from K. Then we can express f by

$$f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{N}^n} c_u x^u.$$

Here, $u = (u_1, \ldots, u_n)$ with $u_i \in \mathbb{N}$, $c_u \in K$, and $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$.

Definition 3.2.4. Let f be defined by $f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{N}^n} c_u x^u$. Then we define the *tropicalization of* f, $\mathrm{trop}(f)$, by

$$\begin{aligned} \mathrm{trop}(f)(w) &= \min_{u \in \mathbb{N}^n}(\mathrm{val}(c_u) + \sum_{i=1}^{n} u_i w_i) \\ &= \min_{u \in \mathbb{N}^n}(\mathrm{val}(c_u) + u \cdot w) \end{aligned}$$

for $w \in \mathbb{R}^n$.

However, $\mathrm{trop}(f)(w)$ is in $\mathbb{R}$, so graphing $\mathrm{trop}(f)(w)$ against w gives us a series of hyperplanes in $\mathbb{R}^{n+1}$. This is rather unwieldy to work with, and we're interested in a subset of $\mathbb{R}^n$, not an embedding of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$. To solve a similar issue in algebraic geometry, we considered the zero set of a polynomial – in the next section we examine what a good analogue of the zero set is for a tropical polynomial.

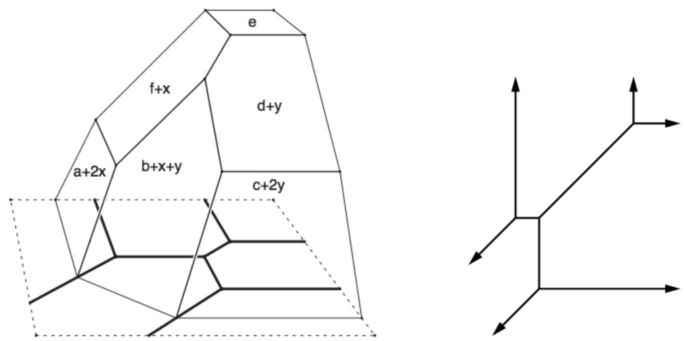

Figure 3.1 The graph of a tropical quadratic and its corner locus, from Maclagan and Sturmfels (2015)

3.3 Tropical Varieties

Classically, when we define a curve corresponding to some form, we consider the set of points where the form is zero. However, for tropicalized polynomials, this choice doesn't make as much sense. By definition, a tropical polynomial is zero when the minimum of its linear terms is zero. This isn't particularly noteworthy, especially considering that the zeroes of traditional forms correspond to roots of the polynomial. Recall that the roots of a polynomial allow us to factor the polynomial, so the roots of a polynomial tell us something fundamental about the structure of the polynomial itself.

So, what aspect of a tropical polynomial might we choose to highlight when defining a variety for them? We note that for any given tropical polynomial, we can graph it as a continuous piecewise linear function.

Since this function is continous, but only piecewise linear, we note that there must be some corners between these linear sections. Since each linear section corresponds to one of the terms we take the minimum of, the corners occur where the minimum is generated by at least two of the terms.

For polynomials in a single variable, the location of the corners corresponds to roots of the polynomial, and we can use this to factor the polynomial into linear terms. While multivariate polynomials cannot be uniquely factored, this object still contains interesting information about the polynomial, so we'll choose this as our variety.

We call this collection of points the *corner locus*, since these points live on the corners of our surface. We will also call varieties corresponding to polynomials of degree 1 lines, those corresponding to degree 2 polynomials conics, and those corresponding to degree 3 polynomials cubics.

In order to work with the concept more concretely later, we'll define it here:

Definition 3.3.1. The corner locus of a tropicalized polynomial $f \in K[x_1, \ldots, x_n]$, or the *tropical hypersurface* $\text{trop}(V(f))$, is the set

$$\{w \in \mathbb{R}^n : \text{the minimum in trop}(f) \text{ is achieved at least twice}\}.$$

This concept may also be expressed through initial forms, which we will define and discuss now.

Definition 3.3.2. Given a polynomial $f = \sum_{u \in \mathbb{N}^n} c_u x^u$ and a vector $w \in \mathbb{R}^n$, with $W = \text{trop}(f)(w)$, we define the *initial form* of f with respect to w as follows:

$$\text{in}_w(f) = \sum_{u \in \mathbb{N}^{n+1} : (\text{val}(c_u) + u \cdot w) = W} \overline{c_u t^{-val(c_u)}} x^u,$$

where the bar denotes the image in the residue field.

For any given w and f, note that $\text{in}_w(f)$ is a collection of terms such that each term indicates where and how $\text{trop}(f)(w) = \min_{u \in \mathbb{N}^n}(\text{val}(c_u) + u \cdot w)$. That is, if the initial form $\text{in}_w(f)$ is a monomial, the minimum is achieved only once. Otherwise, w belongs to the corner locus of the tropicalization of f.

We will finish this section with a theorem that connects the two conceptions of the corner locus we discussed. Kapranov's Theorem lets us move between classical polynomials, their tropicalizations, and associated tropical curves.

Theorem 3.3.1 (Kapranov's Theorem, after Thm 3.1.3 in Maclagan and Sturmfels (2015))**.** *Fix a Laurent polynomial* $f = \sum_{u \in c_u x^u}$ *in* $K[x_1^{\pm}1, \ldots, x_n^{\pm}1]$. *The following three sets coincide*:

1. *the tropical hypersurface trop*$(V(f))$ *in* $\mathbb{R}^n$;
2. *the closure in* $\mathbb{R}^n$ *of the set* $\{w \in \Gamma_{val}^n : in_w(f) \text{is not a monomial}\}$;
3. *the closure in* $\mathbb{R}^n$ *of* $\{val(y_1), \ldots, val(y_n) : (y_1, \ldots, y_n) \in V(f)\}$.

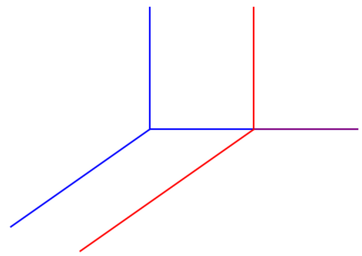

Figure 3.2 Two two tropical lines intersecting in special position.

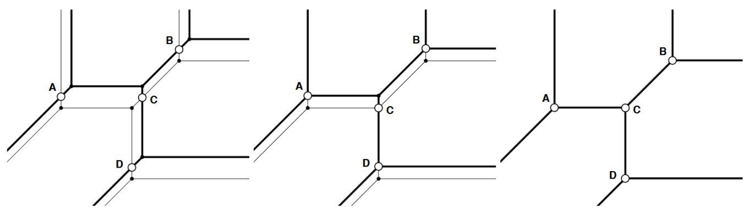

Figure 3.3 The self-intersection of a tropical conic, from Maclagan and Sturmfels (2015)

3.4 Intersections of Tropical Curves

Given two tropical curves, we would like to know where and how they intersect. For two curves in general position, the natural choice is the points that they have in common. Sometimes, however, two tropical curves share a line segment or half-ray, as in Figure 3.2. Instead of this infinite collection of points, we want a well-defined and finite set of points of intersection for pair of tropical curves.

Definition 3.4.1. Let C and D be two tropical curves. Let C_ϵ be the curve C shifted by ϵ in some direction, and define D_ϵ similarly. Then the *stable intersection* of C and D consists of the limits of the intersection of all curves C_ϵ and D_ϵ as ϵ approaches zero.

With this definition, we can even define the intersection of a tropical curve with itself. This ends up being the vertex or vertices of the curve in question, as in Figure 3.3.

3.5 Some Other Parallels

In order to prove the tropicalized version of Pascal's Theorem, we will need some more machinery from tropical geometry. First, we will walk through a result similar to Proposition 2.2.6. We will then discuss some implications of Kapranov's Theorem (3.3.1) that let us lift tropical problems back up to classical algebraic geometry.

Recall that Proposition 2.2.6 allows us to "add" curves together to get curves of a higher degree. An identical statment holds for curves in tropical geometry, although here we will only consider two particular cases.

Proposition 3.5.1. Suppose we have two tropical lines K and L. Then $K \cup L$ is a tropical conic. Similarly, given a tropical line L and a tropical conic Q, $L \cup Q$ is a tropical cubic.

Proof. We can express K as the corner locus of $g(x, y) = \min(a + x, b + y, c)$ and similarly express L as the corner locus of $h(x, y) = \min(d + x, e + y, f)$. Now consider the tropical polynomial $g(x, y) \otimes h(x, y) = p(x, y)$. We can express this polynomial as

$$\begin{aligned} p(x, y) = \min(&a + x + d + x, a + x + e + y, a + x + f, b + y + d + x, \\ &b + y + e + y, b + y + f, c + d + x, c + e + y, c + f). \end{aligned}$$

Notice that every point in the corner locus of g (i.e., K) is in the corner locus of p: If g achieves a minimum in multiple terms, then at least two terms of p will achieve the minimum. There are several cases to show; we will show one and leave the rest to the reader to check. Suppose that $(x_0, y_0) \in K$, with $b + y_0 = c = g(x_0, y_0)$, and suppose that $h(x_0, y_0) = f$. Then

$$b + y_0 + f = c + f = p(x_0, y_0),$$

and (x_0, y_0) belongs to the corner locus of p. Similarly, L is also contained in the corner locus of p.

Now it only remains to show that the corner locus of p does not contain any other points. That is, we want to show that if any pair of terms achieves the minimum $p(x_0, y_0)$, $(x_0, y_0) \in K \cup L$. Note that if a pair of terms has any coefficients in common, it corresponds to a point in K or L, as we saw above. So any points in the corner locus of p that do not belong to K or L must achieve the minimum in two terms that do not have any common coefficients. Again, there are several cases, of which we will show only one. Suppose $b + y_0 + e + y_0 = c + f = p(x_0, y_0)$. Then if $b + y_0 < c$, we know $b + y_0 + f < c + f$, which is a contradiction, since $c + f$ is a minimum. Similarly, if $b + y_0 > c$, we know $b + y_0 + e + y_0 > c + e + y_0$, which is also a contradiction. Thus, $b + y_0$ must equal c. But then (x_0, y_0) must be in K. Thus, the corner locus of p is exactly $K \cup L$, as desired.

The proof of the second statement follows similarly and is left to the reader. □

Throughout this chapter, we've approached (and defined) tropical polynomials and curves as objects that can be constructed from classical polynomials. That is, every tropical polynomial and curve we handle has a classical counterpart. However, given two tropical curves that intersect, we are not necessarily guaranteed that their classical counterparts will also intersect, and if they do, that they will intersect in a similar way.

All of the tropical curves that arise in our tropicalization of Pascal's theorem have codimension 1 (i.e., they occupy all but one of the dimensions of the space we're working in – our curves are lines in a plane, as opposed to lines in a three-dimensional space). This means that they have transverse intersections, so that we may, by work done in Osserman and Payne (2013)[1], lift every tropical intersection to a classical one. Additionally, given two curves that intersect transversely in a number of places, we are guaranteed that their corresponding classical counterparts will intersect in the same way. This is exactly sufficient for a proof of a tropicalization of a key lemma (2.3.3) in Reid's proof of Pascal's theorem.

3.6 Stating Pascal's Theorem

Having defined lines and conics in tropical geometry, we now want to translate Pascal's Theorem from algebraic geometry to tropical geometry. As in algebraic geometry, two general points define a line, and five general points define a conic. So, the original statement that, given some generality constraints, six points are conconic if and only if a related set of three points is collinear also makes sense - we expect any five points to be conconic, while adding a sixth is something special, and we expect any two points to be collinear, and adding a third is again a special occurrence.

While Bézout's theorem does not hold classically in $\mathbb{R}^2$ or even in $\mathbb{P}^2_{\mathbb{R}}$, it holds in $\mathbb{R}^2$ under tropical geometry. This is the tropical projective space. However, we do not need to use forms, since the tropical projective space

[1]Theorem 1.1, for the interested reader

does not have the same equivalence relation, and so the properties of forms are no longer necessary.

We end by formally stating our conjecture.

Theorem 3.6.1. *Given six points A, B, C, D, E, and F in tropical $\mathbb{R}^2$, let $ABCDEF$ be a hexagon through them. Then let P, Q, and R be the three points of intersection of the three pairs of lines going through opposite sides of the hexagon. If the six lines and nine points are distinct, then $ABCDEF$ are conconic exactly when PQR are collinear.*

Chapter 4

A Proof of the Tropical Hexagon

In order to prove the tropicalized version of Pascal's theorem, we follow Reid's proof idea. However, instead of developing analoguous statements about the dimensions of spaces of curves, we skip straight to a tropicalization of the last lemma proven about cubic curves. Armed with this and the fact that the union of three tropical lines is a tropical cubic, we may closely follow Reid's proof.

4.1 A Simple Example

Before we get to the heart of the proof, let's first examine an example. That is, we start with a tropical conic, select six points on that conic, and connect them in order with six tropical lines. We then find the intersections of opposite sides and show that they indeed lie on a tropical line. While this is certainly not a rigorous proof, the reader may find it helpful or at least entertaining to have a picture handy.

We begin with the left side of Figure 4.1, which is a tropical conic. This conic is the tropicalization of the polynomial[1]

$$f(x, y) = t^2x^2 + xy + (t^2 + t^3)y^2 + (1 + t^3)x + t^{-1}y + t^3.$$

It is left to the reader to check that the tropicalization of this polynomial is

$$\mathrm{trop}(f)(x, y) = \min(2 + 2x, x + y, 2 + 2y, x, -1 + y, 3).$$

[1]This polynomial and its tropicalization are from Example 3.1.2 in Maclagan and Sturmfels (2015).

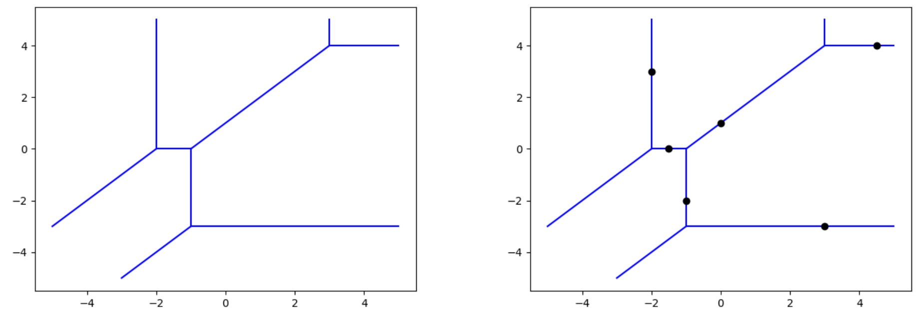

Figure 4.1 A tropical conic and six starting points

We have chosen the following six points that lie on the tropical conic:

$$A : (4.5, 4),\ B : (-2, 3),\ C : (-1, -2),\ D : (0, 1),\ E : (-1.5, 0),\ \text{and } F : (3, -3).$$

The right side of Figure 4.1 shows these six points.

Through any two points we may construct a tropical line, and in the left side of Figure 4.2 we connect two of the points with a tropical line. Since this tropical line has a vertex at $(-1, 4)$, one tropical polynomial that has this line as its corner locus is

$$\min(1 + x, -4 + y, 0).$$

The right side of Figure 4.2 shows all six lines AB, BC, CD, DE, EF, and FA.

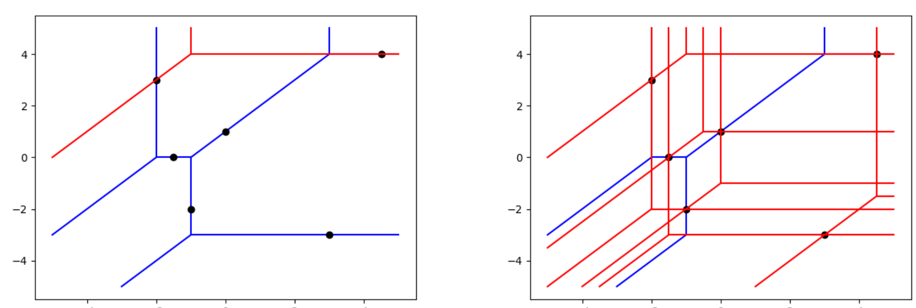

Figure 4.2 The development of a tropical "hexagon"

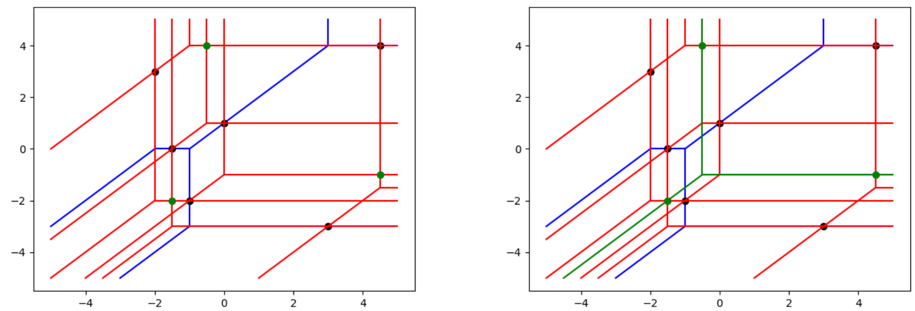

Figure 4.3 Intersections of opposite sides and the line through them

Finally, we may intersect opposite sides. There are three intersections, occurring at the points

$$P : (-0.5, 4),\ Q : (-1.5, -2),\ \text{and}\ R : (4.5, -1),$$

which have been marked in green. Luckily for us, we find that these three points lie on the tropical line given by

$$\min(0.5 + x, 1 + y, 0),$$

exactly as desired.

4.2 Tropical Intersections of Cubic Curves

What we want to have is a tropical analogue of Proposition 2.3.3. We'll restate the original here for convenience.

Proposition 2.3.3. Let C_1, C_2 be two cubic curves whose intersection consists of nine distinct points, $C_1 \cap C_2 = \{P_1, \ldots, P_9\}$. Then a cubic D that goes through $P_1, \ldots, P_8$ also goes through P_9.

We now state the tropical version of this theorem.

Proposition 4.2.1. Let C_1, C_2 be two tropical cubic curves whose intersection consists of nine distinct points, $C_1 \cap C_2 = \{P_1, \ldots, P_9\}$. Then a tropical cubic D that goes through $P_1, \ldots, P_8$ also goes through P_9.

To prove this proposition, we will lift the conditions to classical algebraic geometry. Since every tropical intersection lifts to a classical intersection under the assumptions in Osserman and Payne (2013), the lifted conditions are identical to the conditions of the original proposition. We can then apply the original proposition and tropicalize the result to obtain the desired tropical conclusion.

Proof. We may lift C_1 and C_2 to cubic curves $\tilde{C_1}$ and $\tilde{C_2}$ in $\mathbb{P}^2$ that intersect in exactly nine points. We may also lift any tropical cubic D that goes through exactly eight of those points to a cubic curve $\tilde{D}$ that goes through eight of the nine intersection points of $\tilde{C_1}$ and $\tilde{C_2}$. Then we have exactly the conditions of Proposition 2.3.3, so $\tilde{D}$ goes through the ninth intersection of $\tilde{C_1}$ and $\tilde{C_2}$. We may then tropicalize $\tilde{D}$ to D, and the ninth intersection tropicalizes to the desired last tropical intersection. □

We now have everything we need to prove the tropical version of Pascal's Theorem.

4.3 Proving Pascal's Theorem

We first restate the tropical version:

Theorem 4.3.1. *Given six points A, B, C, D, E, and F in $\mathbb{R}^2$, let $ABCDEF$ be a hexagon through them. Then let P, Q, and R be the three points of intersection of the three pairs of lines going through opposite sides of the hexagon. If the six lines and nine points are distinct (we might need a different generality condition), then $ABCDEF$ are conconic exactly when PQR are collinear.*

Proof. Consider 2 triples of lines:

$$L_1 : PAF, L_2 : QDE, L_3 : RBC$$

and

$$M_1 : PCD, M_2 : QAB, M_3 : REF.$$

These define two cubics - $C_1 = L_1 + L_2 + L_3$ and $C_2 = M_1 + M_2 + M_3$. C_1 and C_2 intersect in the nine named points of the diagram. Of the nine points, no four are collinear, and no seven are conconic, so we can apply Proposition 4.2.1.

First, we show the converse. Suppose that PQR are collinear, on line N. We may also suppose that there is a conic Γ through $ABCDE$, since through any five points we can construct a tropical conic. Then $N + \Gamma$ is a cubic by Proposition 4.2.1.

Then, by Proposition 4.2.1, since $N + \Gamma$ goes through the eight points A, B, C, D, E, P, Q, R, it must also contain F. Note that F cannot belong on N by assumption – since REF is assumed to be distinct from PQR, if RF belongs to both REF and PQR, those lines are no longer distinct, which is a contradiction. Hence, F must belong to Γ, and $ABCDEF$ are conconic.

Now, we show the implication. Suppose that $ABCDEF$ are conconic on the conic Γ, and let $N = PQ$. Then $N+\Gamma$ is a cubic through A, B, C, D, E, F, P, Q, and we can apply Proposition 4.2.1 again. Hence, it must also go through R. We show that R is not on the conic Γ by contradiction. If R were on Γ, then Γ must contain the lines RBC and REF, by the original construction of R. Then either REF and RBC must coincide, which is a contradiction, since we assumed that these were distinct lines, or Γ is a line pair, which is also leads us to a contradiction. Note that if Γ is a line pair, with BC on one line and EF on the other, then A must live on one of these lines. If A lives on BC, then RBC coincides with QAB, which is again a contradiction. Similarly, if A is on EF, REF coincides with PAF, also a contradiction. Thus, R must be on PQ, and PQR are collinear. □

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