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Existence and large time behavior for generalized Kelvin-Voigt equations governing nonhomogeneous and incompressible fluids

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Abstract. Generalized Kelvin-Voigt equations governing nonhomogeneous and incompressible fluids are considered in this work. We assume that, in the momentum equation, the diffusion and relaxation terms are described by two distinct power-laws. Moreover, we assume that the momentum equation is perturbed by an extra term, which, depending on whether its signal is positive or negative, may account for the presence of a source or a sink within the system. For the associated initial-boundary value problem, we study the existence of weak solutions as well as the large time behavior of the solutions.

1. Introduction

The problem we study in this work is motivated by the mathematical modelling of viscoelastic materials. Viscoelasticity is the property of a material that, under stress and deformation, exhibits both viscous and elastic characteristics. Simplest constitutive relations describing the behavior of these materials go back to the works by Kelvin [1] and Maxwell [2], and are usually obtained by combining Hook's law of linear elasticity with Newton's law of viscosity. Departing from the Kelvin stress-strain relation, Voigt [3] has derived a system of equations that govern the behavior of elastic solids with viscous properties, which is known today as the Kelvin-Voigt equations. Later on, Oskolkov [4] derived a similar system of governing equations but for the description of homogeneous and incompressible fluids with elastic properties, to which he has also given the name of Kelvin-Voigt equations. By the same time, Pavlovsky [5] has already used a sort of Kelvin-Voigt equations to model weakly concentrated water-polymer mixtures. The same designation for systems of equations, which although similar, describe different phenomena, can sometimes lead to misinterpretations. To avoid any ambiguity, we must always keep in mind for what Deborah number are the considered Kelvin-Voigt equations associated with. Deborah number is the ratio of the relaxation time of a material to the observation or experimental time, and therefore it is usually used to estimate the memory of materials. Low Deborah numbers always indicate fluid-like behavior, whereas high Deborah numbers means solid-like response. In this work, we consider a very general model that can be used for the description of nonhomogeneous and incompressible fluids with viscoelastic properties. The problem we shall

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study here is the following: given the initial velocity field $\mathbf{v}_0$, the initial density ρ_0 and the forces field $\mathbf{f}$, to find the velocity field $\mathbf{v}$, the pressure π and the density ρ satisfying to

$$\begin{aligned}&\frac{\partial(\rho\,\mathbf{v})}{\partial t}+\operatorname{div}(\rho\,\mathbf{v}\otimes\mathbf{v})=\\&\rho\,\mathbf{f}-\boldsymbol{\nabla}\pi+\operatorname{div}\left(\mu|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})+\varkappa|\mathbf{D}(\mathbf{v})|^{q-2}\frac{\partial\,\mathbf{D}(\mathbf{v})}{\partial t}\right)+\gamma|\mathbf{v}|^{m-2}\mathbf{v},\end{aligned}\tag{1.1}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\,\mathbf{v})=0,\qquad \operatorname{div}\mathbf{v}=0.\tag{1.2}$$

in a general cylinder $Q_T:=\Omega\times(0,T)$, with lateral boundary $\Gamma_T:=\partial\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^d$, with $d\geq 2$, is a bounded domain with its boundary denoted by $\partial\Omega$, and $T>0$. Here $\varkappa$ denotes the relaxation time and μ is the fluid viscosity, both of which are considered to be positive constants. Moreover, the exponents p, q and m are positive constants satisfying to $p,\ q,\ m\in(1,\infty)$ and γ is assumed to be a constant with no predefined sign. We supplement the system (1.1)-(1.2) with the following initial and boundary conditions

$$\rho\mathbf{v}=\rho_0\mathbf{v}_0,\quad \rho=\rho_0\qquad \text{in }\Omega,\quad \text{when } t=0,\tag{1.3}$$

$$\mathbf{v}=\mathbf{0}\qquad \text{on }\Gamma_T.\tag{1.4}$$

In the sequel, we shall refer to (1.1)-(1.2) as the generalized Kelvin-Voigt equations for nonhomogeneous and incompressible fluids, underlying, therefore, that the phenomenon described by these equations, or by any of its simplifications, has an associated low Deborah number. This problem is very general and therefore encompasses many other situations of fluid flows. In particular, the extra term $\gamma|\mathbf{v}|^{m-2}\mathbf{v}$ accounts for a sink or a source within the system, depending if $\gamma<0$ or $\gamma>0$, respectively. If $\gamma=0$ and $p=q=2$ in the momentum equation (1.1), we recover the Kelvin-Voigt model for nonhomogeneous and incompressible fluids, whose particular case of constant ρ was studied by Oskolkov [4] and Ladyzhenskaya [6]. On the other hand, equations (1.1)-(1.2), in the case of $\varkappa=0$, $p=2$ and $\gamma=0$, have been used since the 1960s to describe nonhomogeneous flows of viscous and incompressible fluids (see *e.g.* Antontsev *et al.* [7, 8, 9]). The case of $\varkappa=0$ and $\gamma=0$, but with the possibility of $p\neq 2$ in (1.1), have been studied by Zhikov and Pastukhova [10, 11]. Mathematical questions involving Kelvin-Voigt's equations for homogeneous incompressible viscous fluids, *i.e.* the case of $\varkappa\neq 0$, $\gamma=0$ and $p=q=2$ in (1.1), and constant ρ, were considered by Oskolkov [4] and by Zvyagin and his collaborators [12]. More recently, Antontsev and Khompysh [13, 14] have addressed some mathematical issues for the Kelvin-Voigt equations in the case of homogeneous fluids and for general p. The unique solvability of the homogeneous Kelvin-Voigt equations with anisotropic diffusion, relaxation and damping was considered by Antontsev *et al.* [15]. To the authors best knowledge, Kelvin-Voigt equations for nonhomogeneous incompressible fluids have not yet been considered in previous works.

Definition 1.1. *Let $d\geq 2$, $1<q,\ p,\ m<\infty$ and assume that $\mathbf{f}\in\mathbf{L}^2(Q_T)$. A pair of functions $(\mathbf{v},\rho)$ is a weak solution to the problem (1.1)-(1.4), if:*

(i) $\mathbf{v}\in L^\infty(0,T;\mathbf{H}\cap\mathbf{V}_q)\cap L^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$;

(ii) $\rho>0$ *a.e. in* Q_T, $\rho\in C([0,T];L^\lambda(\Omega))$ *for all* $\lambda\in[1,\infty)$ *and* $\rho|\mathbf{v}|^2\in L^\infty(0,T;L^1(\Omega))$;

(iii) $\mathbf{v}(0)=\mathbf{v}_0$ *and* $\rho(0)=\rho_0$, *with* $\rho_0\geq 0$ *a.e. in* Ω;

(iv) For every $\boldsymbol{\varphi} \in \mathcal{V}$ there holds for a.a. $t \in [0, T]$

$$\frac{d}{dt}\left(\int_\Omega \rho(t)\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x} + \frac{\varkappa}{q-1}\int_\Omega |\mathbf{D}(\mathbf{v}(t))|^{q-2}\mathbf{D}(\mathbf{v}(t)) : \mathbf{D}(\boldsymbol{\varphi})\,d\mathbf{x}\right) +$$
$$\mu\int_\Omega |\mathbf{D}(\mathbf{v}(t))|^{p-2}\mathbf{D}(\mathbf{v}(t)) : \mathbf{D}(\boldsymbol{\varphi})\,d\mathbf{x} + \int_\Omega (\rho(t)\mathbf{v}(t)\cdot\boldsymbol{\nabla})\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}$$
$$= \int_\Omega \rho(t)\mathbf{f}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x} + \gamma\int_\Omega |\mathbf{v}(t)|^{m-2}\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}.$$

(v) For every $\phi \in \mathrm{C}_0^\infty(\Omega)$ there holds for a.a. $t \in [0, T]$

$$\frac{d}{dt}\int_\Omega \rho(t)\,\phi\,d\mathbf{x} + \int_\Omega \rho(t)\mathbf{v}(t)\cdot\boldsymbol{\nabla}\phi\,d\mathbf{x} = 0.$$

We address the reader to the monographs [7, 9] for the definitions and main notations of the function spaces used throughout the paper. We just fix the notations

$$\begin{aligned}
&\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\mathbf{v} = 0\},\\
&\mathbf{H} := \{\text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{L}^2(\Omega)\},\\
&\mathbf{V}_p := \{\text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}^{1,p}(\Omega)\}.
\end{aligned}$$

If $p = 2$, we denote $\mathbf{V}_p$ simply by $\mathbf{V}$.

2. Main results

First we shall consider the case

$$\gamma \leq 0 \quad \text{and} \quad q = 2. \tag{2.1}$$

With respect to assumption $q = 2$, it should be mentioned that existence results for generalized Kelvin-Voigt equations (with $q \neq 2$) are completely open. Despite the fact that existence of solutions is proved only in the case of $q = 2$, many estimates and integral relations are proved for the case $q \neq 2$, since they are used to prove the large time behavior. For the results we aim to establish here, let us define the quantity

$$s := \max\{q, p\},$$

which will resume to $s := \max\{2, p\}$ in the case $q = 2$.

Theorem 2.1 (Global existence: $\gamma \leq 0$)**.** *Let M_1 and M_2, with $M_1 \leq M_2$, be two positive constants such that*

$$0 < M_1 := \inf_{\mathbf{x}\in\Omega}\rho_0(\mathbf{x}) \leq \rho_0(\mathbf{x}) \leq \sup_{\mathbf{x}\in\Omega}\rho_0(\mathbf{x}) =: M_2 < \infty \qquad \forall\,\mathbf{x} \in \overline{\Omega}, \tag{2.2}$$

and let

$$\mathbf{v}_0 \in \mathbf{V} \cap \mathbf{V}_p \cap \mathbf{L}^m(\Omega), \tag{2.3}$$
$$\mathbf{f} \in \mathbf{L}^2(Q_T). \tag{2.4}$$

Assume, in addition to (2.1), (2.2) and (2.3)-(2.4), that one of the following alternatives is fulfilled,

$$2 \leq d \leq 4 \qquad \textit{and} \qquad p > 1, \tag{2.5}$$
$$d \geq 3 \qquad \textit{and} \qquad p \geq \frac{d}{2},$$
$$d \leq m \qquad \textit{and} \qquad \gamma \neq 0. \tag{2.6}$$

If

$$s > \frac{4d}{d+4}, \tag{2.7}$$

then the problem (1.1)-(1.4) has, at least, a weak solution $(\mathbf{v}, \rho)$ *in the sense of Definition 1.1 in the cylinder* Q_T*.*
Moreover, the weak solutions to the problem (1.1)-(1.4) satisfy the following estimates,

$$0 < M_1 \le \rho(\mathbf{x}, t) \le M_2 < \infty \qquad \forall\ (\mathbf{x}, t) \in Q_T \tag{2.8}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\mathbf{v}(t)\|_{2,\Omega}^2 + \|\boldsymbol{\nabla}\mathbf{v}(t)\|_{2,\Omega}^2\right) + \|\boldsymbol{\nabla}\mathbf{v}\|_{p,Q_T}^p + |\gamma|\|\mathbf{v}\|_{m,Q_T}^m \\ &\quad\le C_1\left(\|\mathbf{v}_0\|_{2,\Omega}^2 + \|\boldsymbol{\nabla}\mathbf{v}_0\|_{2,\Omega}^2 + \|\mathbf{f}\|_{2,Q_T}^2\right), \\ &\sup_{t\in[0,T]} \left(\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p + |\gamma|\|\mathbf{v}(t)\|_{m,\Omega}^m\right) + \|\mathbf{v}_t\|_{2,Q_T}^2 + \|\boldsymbol{\nabla}\mathbf{v}_t\|_{2,Q_T}^2 \\ &\quad\le C_2\left(\|\boldsymbol{\nabla}\mathbf{v}_0\|_{p,\Omega}^p + |\gamma|\|\mathbf{v}_0\|_{m,\Omega}^m + \|\mathbf{f}\|_{2,Q_T}^2 + 1\right), \end{aligned} \tag{2.9}$$

where C_1 *and* C_2 *are positive constants.*

We are now interested in existence results to the problem (1.1)-(1.4) in the case of

$$q = 2 \quad \text{and} \quad \gamma > 0. \tag{2.10}$$

Theorem 2.2 (Global existence: $\gamma > 0$)**.** *Assume that (2.2), (2.3)-(2.4) and (2.10) hold, and that one of the alternatives written in (2.5)-(2.6) is fulfilled. In addition, assume that (2.7) holds as well. If one of the following conditions is verified,*

$$\begin{aligned} &m \le 2, \\ &2 < m < p, \end{aligned} \tag{2.11}$$

and if, in the case of (2.11), we additionally have

$$2(m-1) \le p^*, \tag{2.12}$$

then the problem (1.1)-(1.4) has, at least, a weak solution $(\mathbf{v}, \rho)$ *in the sense of Definition 1.1 in the cylinder* Q_T*.*
Moreover, this weak solution satisfies the estimates (2.8)-(2.9) with $\gamma = 0$*.*

Theorem 2.3 (Local existence: $\gamma > 0$)**.** *Assume that (2.2)-(2.4) and (2.10) hold. If one of the following conditions hold,*

$$\begin{aligned} &2 < m \le 2^*, \\ &2 < p \le m < p\left(1 + \frac{2}{d}\right), \end{aligned} \tag{2.13}$$

and if in the case of (2.13) there additionally holds (2.12), then there exists $T_{\max} \in (0, T)$ *such that the problem (1.1)-(1.4) has, at least, a weak solution* $(\mathbf{v}, \rho)$ *in the sense of Definition 1.1 in the cylinder* $Q_{T_{\max}}$*. Moreover, this weak solution also satisfies the estimates (2.8)-(2.9) with* $\gamma = 0$*, but in the cylinder* Q_t*, with* $t < T_{\max}$*.*

The common idea to the proofs of Theorems 2.1, 2.2 and 2.3 is the following.
We construct a solution to the problem (1.1)-(1.3) as a limit of the Galerkin approximations

$$\mathbf{v}_n(\mathbf{x},t)=\sum_{k=1}^{n}c_k^n(t)\boldsymbol{\psi}_k(\mathbf{x}),\quad \boldsymbol{\psi}_k\in\mathbf{V}^n,\qquad \text{and}\quad \rho_n(\mathbf{x},t), \tag{2.14}$$

where $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ is an orthonormal family in $\mathbf{L}^2(\Omega)$ formed by functions of $\mathcal{V}$ whose linear combinations are dense in $\mathbf{V}\cap\mathbf{V}_p\cap\mathbf{L}^m(\Omega)$. The functions $c_1^n(t),\dots,c_n^n(t)$ are obtained from the following Cauchy problem for the system of ordinary differential equations

$$\mathbf{A}\frac{d\mathbf{c}}{dt}=\mathbf{b},\quad \mathbf{c}(0)=\mathbf{c}_0=(c_1^n(0),\dots,c_n^n(0)), \tag{2.15}$$

where $\mathbf{A}=\{A_{jk}^n\}_{j,k=1}^n$, $\mathbf{c}=\{c_k^n\}_{k=1}^n$, $\mathbf{b}=\{b_j^n\}_{j=1}^n$, with

$$\begin{aligned}
A_{jk}^n(t) &:= \int_\Omega \rho_n\boldsymbol{\psi}_j\cdot\boldsymbol{\psi}_k d\mathbf{x}+\varkappa\int_\Omega |\mathbf{D}(\mathbf{v}_n(t))|^{q-2}\mathbf{D}(\boldsymbol{\psi}_k):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x},\\
b_j^n(t) &:= \int_\Omega \rho_n\mathbf{f}(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}+\gamma\int_\Omega |\mathbf{v}_n(t)|^{m-2}\mathbf{v}_n(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}-\int_\Omega \rho_n(t)\left(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}\\
&\quad -\mu\int_\Omega |\mathbf{D}(\mathbf{v}_n(t))|^{p-2}\mathbf{D}(\mathbf{v}_n(t)):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x},
\end{aligned}$$

The density approximations ρ_n satisfy the following Cauchy problem,

$$\frac{\partial\rho_n}{\partial t}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\rho_n=0,\quad \rho_n(0)=\rho_0\quad\text{in }\Omega. \tag{2.16}$$

For simplicity, firstly we assume that $\rho_0\in \mathrm{C}^1(\overline{\Omega})$. Then we have

$$\rho_n(\mathbf{x},t)=\rho_0(y_n(\tau,\mathbf{x},t)_{\big|\tau=0}) \tag{2.17}$$

where y_n is the solution to the Cauchy problem

$$\frac{dy_n}{dt}=\mathbf{v}_n(y_n,\tau),\qquad y_{n\big|\tau=t}=\mathbf{x}.$$

This problem has a unique solution y_n for $\mathbf{v}_n$ given by (2.14) with $c_k^n\in C([0,T])$. Moreover, according to (2.2) and (2.17), one has

$$0<M_1=\inf_{\mathbf{x}\in\overline{\Omega}}\rho_0(\mathbf{x})\le\rho_n(\mathbf{x},t)\le\sup_{\mathbf{x}\in\overline{\Omega}}\rho_0(\mathbf{x})=M_2<\infty\qquad\forall\ (\mathbf{x},t)\in Q_T.$$

We prove that the problem formed by (2.15) and (2.16) has, at least, a solution $\mathbf{c}$ in a neighborhood, say $(0,T_0)$, with $T_0>0$, of the initial condition.
Next, we derive a priori estimates similar to those specified in Theorem 2.1 which do not depend on the number n. Under the conditions of Theorems 2.1 and 2.2, these estimates are valid for any finite interval $[0,T)$. In the case of Theorem 2.3, the estimates are valid only for a local interval $[0,t)$, with $t<T_{\max}$. Then using compactness arguments together with the monotonicity of the operator $\mu|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}$, we can extract a convergent subsequence of approximation solutions. Finally, we realize the passage to the limit as $n\to\infty$. The obtained limit is a solution to the problem (1.1)-(1.4) in the sense of Definition 1.1. In the case of Theorems 2.1 and 2.2, we have a global solution, whereas for Theorem 2.3 we have a local solution.

3. Large time behavior

In this section, we study the large time behavior properties of the weak solution to the problem (1.1)-(1.4). Here, we also consider the open case $q \neq 2$, for which we assume the existence of, at least, a weak solution $(\mathbf{v}, \rho)$ in the sense of Definition 1.1. Throughout this section, we shall assume that

$$\gamma \leq 0, \ \ \mathbf{v}_0 \in \mathbf{H} \cap \mathbf{V}_q. \tag{3.1}$$

Let us define the functions

$$\Phi(t) := \frac{1}{2}\|\sqrt{\rho(t)}\,\mathbf{v}(t)\|^2_{2,\Omega} + \frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}(t)\|^q_{q,\Omega}, \tag{3.2}$$
$$\Psi(t) := \mu\|\boldsymbol{\nabla}\mathbf{v}(t)\|^p_{p,\Omega} + |\gamma|\|\mathbf{v}(t)\|^m_{m,\Omega}.$$

Let us now consider a weak solution $(\mathbf{v}, \rho)$ to the problem (1.1)-(1.4) in the sense of Definition 1.1 and let the function $\Phi(t)$ defined by (3.2) be bounded for all $t \in [0, \infty]$.
We establish the conditions for the solutions to the problem (1.1)-(1.4) to decay in time according to the following power

$$\alpha := \frac{p}{\min\{2, q\}}.$$

Theorem 3.1 (Polynomial decay)**.** *Let $(\mathbf{v}, \rho)$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 1.1 and assume that condition (3.1) holds. In addition, assume that*

$$\frac{2d}{d+2} \leq q \leq p \qquad \text{and} \qquad \alpha > 1.$$

(i) If $\mathbf{f} = 0$ a.e. in Q_T and (2.1) holds, then there exists an independent of t positive constant C such that

$$\Phi(t) \leq C\,(1+t)^{-\frac{1}{\alpha-1}} \quad \forall\ t \geq 0. \tag{3.3}$$

(ii) If $\mathbf{f} \neq 0$, but exist positive constants $C_{\mathbf{f}}$ and σ, with $\sigma \geq \alpha'$, such that

$$\|\mathbf{f}(t)\|^s_{s,\Omega} \leq C_{\mathbf{f}}\,(1+t)^{-\sigma} \quad \forall\ t \in [0, T],$$

for $s = p'$ when (2.1) holds, or $s = m'$ when $\gamma < 0$ is holding, then there exists an independent of t positive constant C such that

$$\Phi(t) \leq C\,(1+t)^{-\frac{\alpha}{\alpha-1}} \quad \forall\ t \geq 0, \tag{3.4}$$

where α', p' and m' denote the Hölder conjugates of α, p and m.

Next we study the limit case of $\alpha = 1$.

Theorem 3.2 (Exponential decay)**.** *Let $(\mathbf{v}, \rho)$ be a weak solution to the problem (1.1)-(1.4) in the sense of Definition 1.1 and assume that condition (3.1) holds. In addition, assume that one of the following conditions hold,*

$$\frac{2d}{d+2} \leq q = p \leq 2 \qquad \text{in the case of (2.1) holding,}$$
$$q = p \quad \text{and} \quad m = 2 \qquad \text{in the case } \gamma < 0 \text{ holding.}$$

(i) If $\mathbf{f} = 0$ a.e. in Q_T and (2.1) holds, then there exists an independent of t positive constant C such that

$$\Phi(t) \leq \Phi(0)e^{-Ct} \quad \forall\ t \geq 0. \tag{3.5}$$

(ii) If $\mathbf{f} \neq 0$ *and*

$$\int_0^\infty \|\mathbf{f}(\tau)\|_{s,\Omega}^s d\tau < \infty,$$

where $s = p'$ *if (2.1) holds together with* $\mathbf{f} \in \mathbf{L}^{p'}(Q_T) \cap \mathbf{L}^2(\Omega)$*, or* $s = m'$ *if* $\gamma < 0$ *holds together with* $\mathbf{f} \in \mathbf{L}^{m'}(Q_T)\mathbf{L}^2(\Omega)$*, then there exist constants* C_1 *and* C_2 *such that*

$$\Phi(t) \leq e^{-C_1 t}\left(\Phi(0) + C_2 \int_0^t e^{C_1\tau}\|\mathbf{f}(\tau)\|_{s,\Omega}^s d\tau\right) \quad \forall\, t \geq 0. \tag{3.6}$$

The proofs of Theorems 3.1 and 3.2 follow by establishing the following nonlinear differential inequality

$$\Phi'(t) + C_1\Phi^\alpha(t) \leq C_2\|\mathbf{f}(t)\|_{s,\Omega}^s \tag{3.7}$$

for independent of t positive constants C_1 and C_2, where $\alpha = \frac{p}{\min\{2,q\}} > 1$ in the case of Theorem 3.1, or $\alpha = 1$ for Theorem 3.2, and where $s = p'$ if (2.1) holds, or $s = m'$ if it is $\gamma < 0$ holding. Then, by using a suitable reasoning, we can show that (3.7) together with the hypotheses of Theorems 3.1 and 3.2 imply the polynomial decays (3.3)-(3.4) and the exponential decays (3.5)-(3.6), respectively.

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