Factorization of Matrix Functions and the Resolvents of Certain Operators

Ana C. Conceição, Viktor G. Kravchenko, and Francisco S. Teixeira

The explicit factorization of matrix functions of the form

$$A_\gamma(b) = \begin{pmatrix} e & b \\ b^* & b^*b+\gamma e \end{pmatrix},$$

where b is an $n \times n$ matrix function, e represents the identity matrix, and γ is a complex constant, is studied. To this purpose some relations between a factorization of A_γ and the resolvents of the self-adjoint operators

$$N_+(b) = P_+bP_-b^*P_+ \quad \text{and} \quad N_-(b) = P_-b^*P_+bP_-$$

are analyzed. The main idea is to show that, if b is a matrix function that can be represented through the decomposition $b = b_- + b_+$ where at least one of the summands is a rational matrix, then it is possible to construct an algorithm that allows us to determine an effective canonical factorization of the matrix function A_γ.

1. Introduction

The explicit factorization problem for matrix functions has applications in different areas, such as the theory of singular integral operators, contour problems, the theory of non-linear differential equations (see, for instance, [CG, LS, FT]). However, only for some restricted classes of matrix functions (such as rational matrix functions) there exists an algorithm that allows us to determine their explicit factorization.

In this work we intend to obtain an explicit factorization of matrix functions of the type

$$A_\gamma(b) = \begin{pmatrix} e & b \\ b^* & b^*b+\gamma e \end{pmatrix},$$

where γ is an arbitrary complex constant, e represents the identity matrix function, b is a matrix function which belongs to $[L_\infty(\mathbb{T})]_{n,n}$, where $\mathbb{T}$ denotes the unit circle, and b^* is the Hermitian adjoint of b.

The inherent idea of our work (see [KM]) is to disclose a relation between a factorization of $A_\gamma(b)$ and the resolvents of the self-adjoint operators

$$N_+(b) = P_+bP_-b^*P_+ \quad \text{and} \quad N_-(b) = P_-b^*P_+bP_-,$$

$$N_\pm(b) : [L_2(\mathbb{T})]_{n,n} \to [L_2(\mathbb{T})]_{n,n}.$$

In the first part, the resolvent of the operator $N_-(b)$ ($N_+(b)$) is obtained when the matrix function $A_\gamma(b)$ admits a left (right) canonical generalized factorization. In the second part, perhaps the most important, an explicit left canonical generalized factorization of $A_\gamma(b)$ is constructed using the resolvent of the operator $N_+(b)$ (the same factorization could be obtained through the operator $N_-(b)$). If a right factorization is wanted, then a similar procedure can be developed by making use of the resolvents of the operators $N_-(b^*)$ or $N_+(b^*)$.

In the case where $b \in [C(\mathbb{T})+L^+_\infty(\mathbb{T})]_{n,n}$, the self-adjoint operators N_+ and N_- are compact operators and the corresponding resolvent operators can be represented through their eigenvalues and respective eigenfunctions. Furthermore, in some particular cases, when b belongs to a decomposing subalgebra of $[L_\infty(\mathbb{T})]_{n,n}$, a canonical generalized factorization can be obtained in terms of the eigenfunctions of the operator considered. Moreover, if a decomposition $b = b_- + b_+$ is given, then the construction of a left (right) canonical factorization does not depend on the knowledge of b_- (b_+, respectively). This fact is very important, because it leads to the conclusion that there exists a quite general class of matrix functions for which the problem of the existence of an explicit left (right) canonical factorization depends only on the matrix function b_+ (b_-, respectively). In particular, if b_+ (b_-) is a rational matrix function then $N_\pm(b)$ ($N_\pm(b^*)$) are finite dimensional operators and, consequently, it is possible to determine their eigenvalues and eigenfunctions. These can in turn be used to design an algorithm for obtaining a left (right) factorization of $A_\gamma(b)$ which is similar to the known algorithm for the factorization of rational matrix functions (see [P]).

Finally, some examples are given which clarify the results obtained.

2. Preliminaries

Let $\mathbb{T}$ denote the unit circle. We will consider the space $L_2(\mathbb{T})$. As usual, $P_\pm = (I \pm S)/2$ denote the Cauchy projectors associated with the singular integral operator S,

$$S\varphi(t) = \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t}\, d\tau, \ t \in \mathbb{T},$$

where the integral is understood in the sense of its principal value. We introduce the spaces

$$L_2^+(\mathbb{T}) = \mathrm{im}P_+,\ L_2^{-,0}(\mathbb{T}) = \mathrm{im}P_-,\ L_2^-(\mathbb{T}) = L_2^{-,0}(\mathbb{T}) \oplus \mathbb{C}.$$

Further, we need some standard definitions (see, for instance, [CG, GK, K, KL, LS]). Let us consider a matrix function $A \in G[L_\infty(\mathbb{T})]_{n,n}$. We say that A admits a left (right) canonical generalized factorization in $L_2(\mathbb{T})$ if it can be represented as

$$A = A_+A_- \quad (A = A_-A_+),$$

where

(i) $A_+^{\pm 1} \in [L_2^+(\mathbb{T})]_{n,n}$ and $A_-^{\pm 1} \in [L_2^-(\mathbb{T})]_{n,n}$

(ii) The operator $A_-^{-1}P_-A_+^{-1}I \quad (A_+^{-1}P_+A_-^{-1}))$ is a bounded linear operator in $[L_2(\mathbb{T})]_n$.

Moreover, if condition (i) holds and condition (ii) is dropped, we simply say that A admits a canonical factorization.

In the following sections we shall be dealing with the matrix function

$$A_\gamma(b) = \begin{pmatrix} e & b \\ b^* & b^*b + \gamma e \end{pmatrix}$$

with $b \in [L_\infty(\mathbb{T})]_{n,n}$. It is easy to see that the matrix function can be expressed in the form

$$A_\gamma(b) = \begin{pmatrix} e & 0 \\ b^* & \gamma e \end{pmatrix} \begin{pmatrix} e & b \\ 0 & e \end{pmatrix},$$

from which it follows that $\det A_\gamma(b) = \gamma^n$. Consequently, A_γ is an invertible matrix function if and only if γ is a non-zero complex constant and in that case its inverse is given by the matrix function

$$A_\gamma^{-1}(b) = \tfrac{1}{\gamma} \begin{pmatrix} \gamma e + bb^* & -b \\ -b^* & e \end{pmatrix}.$$

From now on, we will assume $\gamma \neq 0$.

3. Resolvents of N_- and N_+ through the factorization of A_γ

Let $b \in [L_\infty(\mathbb{T})]_{n,n}$. Considering the equation

$$\gamma\varphi + N_-(b)\varphi = f,$$

where

$$\varphi = \varphi_+ + \varphi_-, \quad \varphi_\pm \in [L_2^\pm(\mathbb{T})]_{n,n}, \quad f \in [L_2(\mathbb{T})]_{n,n},$$

we obtain immediately that $\varphi_+ = \gamma^{-1} P_+ f$. It remains to solve the equation

$$\gamma\varphi_- + N_-(b)\varphi_- = f_-,$$

where $f_- = P_- f$. Writing $b\varphi_- = \omega_+ - \omega_-$, it follows that $\gamma\varphi_- + b^*\omega_+ - \chi_+ = f_-$, where $\omega_\pm \in [L_2^\pm(\mathbb{T})]_{n,n}$ and $\chi_+ \in [L_2^+(\mathbb{T})]_{n,n}$. It is easy to prove that

$$\begin{pmatrix} e & 0 \\ b^* & -e \end{pmatrix} \Phi_+ = \begin{pmatrix} e & b \\ 0 & -\gamma e \end{pmatrix} \Phi_- + \Psi_-,$$

where

$$\Phi_+ = \begin{pmatrix} \omega_+ \\ \chi_+ \end{pmatrix}, \quad \Phi_- = \begin{pmatrix} \omega_- \\ \varphi_- \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} 0 \\ f_- \end{pmatrix}.$$

Therefore, we associate the Riemann-Hilbert problem

$$\Phi_+ = A_\gamma(b)\Phi_- - \Psi_-$$

with the matrix function $A_\gamma(b)$ (see, for instance, [KL]). Let us suppose that the matrix $A_\gamma(b)$ admits a left canonical generalized factorization of the form $A_\gamma(b) = A_\gamma^+ A_\gamma^-$. If we consider the operators

$$\hat{\pi}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \quad \text{and} \quad \check{\pi}x = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

the consideration of the above Riemann-Hilbert problem shows that

$$\varphi_- = \hat{\pi}\ (A_\gamma^-)^{-1} P_- (A_\gamma^+)^{-1}\ \check{\pi}\ f_-.$$

Using a similar procedure for the operator $N_+(b)$ and denoting the resolvent of $N_\pm(b)$ by $R(N_\pm(b), -\gamma I) = (N_\pm(b) + \gamma I)^{-1}$ and the resolvent set of $N_\pm$ by $\rho(N_\pm)$, we obtain the following result.

Theorem 3.1.

(i) *If $A_\gamma(b)$ admits a canonical generalized factorization of the form $A_\gamma(b) = A_\gamma^+ A_\gamma^-$, then $-\gamma \in \rho(N_-(b))$. In that case,*

$$R(N_-(b), -\gamma I) = \hat{\pi}\ (A_\gamma^-)^{-1} P_- (A_\gamma^+)^{-1}\ \check{\pi}\ P_- + \tfrac{1}{\gamma} P_+.$$

(ii) *If $A_\gamma(b^*)$ admits a canonical generalized factorization of the form $A_\gamma(b^*) = A_\gamma^- A_\gamma^+$, then $-\gamma \in \rho(N_+(b))$. In that case,*

$$R(N_+(b), -\gamma I) = \hat{\pi}\ (A_\gamma^+)^{-1} P_+ (A_\gamma^-)^{-1}\ \check{\pi}\ P_+ + \tfrac{1}{\gamma} P_-.$$

4. Factorization of A_γ through the resolvent of N_- and N_+

4.1. The cases $b \in [L_\infty(\mathbb{T})]_{n,n}$ and $b \in [C(\mathbb{T}) + L_\infty^+(\mathbb{T})]_{n,n}$

Let us study the Riemann-Hilbert problem

$$\begin{cases} \Psi_+ = A_\gamma(b)(E + \Psi_-) \\ \\ \Psi_-(\infty) = 0, \end{cases}$$

where $b \in [L_\infty(\mathbb{T})]_{n,n}$ and $E = \mathrm{diag}(e, e)$. The initial objective is to determine matrix functions $\Psi_\pm \in [L_2^\pm(\mathbb{T})]_{2n,2n}$ satisfying the above equations.

Assuming that $-\gamma$ does not belong to the spectrum of the operator $N_+(b)$, that is, $-\gamma \in \rho(N_+(b))$, and writing $\Psi_\pm$ as

$$\Psi_\pm = \begin{pmatrix} x_\pm & y_\pm \\ z_\pm & v_\pm \end{pmatrix},$$

it can be shown, after some computations, that the components of the matrices $\Psi_\pm$ can be represented through the resolvent of the operator $N_+(b)$ by

$$\begin{cases} x_+ = \gamma R(N_+(b), -\gamma I)e \\ y_+ = \gamma R(N_+(b), -\gamma I)P_+ b \\ z_- = -P_- b^* R(N_+(b), -\gamma I)e \\ v_- = -P_- b^* R(N_+(b), -\gamma I)P_+ b, \end{cases}$$

which yields the following result.

Lemma 4.1. *If $-\gamma \in \rho(N_+(b))$, then a solution to the Riemann-Hilbert problem can be represented by*

$$\Psi_+ = \begin{pmatrix} \gamma R(N_+(b), -\gamma I)e & \gamma R(N_+(b), -\gamma I)P_+ b \\ \\ \gamma P_+ b^* R(N_+(b), -\gamma I)e & \gamma e + \gamma P_+ b^* R(N_+(b), -\gamma I)P_+ b \end{pmatrix},$$

and

$$\Psi_- = \begin{pmatrix} P_- b P_- b^* R(N_+(b), -\gamma I)e & -P_- b + P_- b P_- b^* R(N_+(b), -\gamma I)P_+ b \\ \\ -P_- b^* R(N_+(b), -\gamma I)e & -P_- b^* R(N_+(b), -\gamma I)P_+ b \end{pmatrix}.$$

Let us remember that our goal is to determine a generalized factorization of $A_\gamma(b)$. If we introduce the notation

$$B_\gamma^+(b) = \Psi_+ \text{ and } B_\gamma^-(b) = E + \Psi_-,$$

then the essential question is to verify if $A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1}$ is a canonical generalized factorization of the matrix function $A_\gamma(b)$.

It is obvious that

$$\det(B_\gamma^\pm(b))(t) \neq 0 \ \forall t \in \mathbb{T} \quad \text{and} \quad B_\gamma^\pm(b) \in [L_2^\pm(\mathbb{T})]_{2n,2n}.$$

Therefore, it remains to check that the following conditions hold:

i) $\det(B_\gamma^\pm(b))(z) \neq 0 \ \forall z \in \mathbb{T}_\pm$

ii) $(B_\gamma^\pm(b))^{-1} \in [L_2^\pm(\mathbb{T})]_{2n,2n}$

iii) $B_\gamma^-(b)P_-(B_\gamma^+(b))^{-1}I$ is a bounded operator on $[L_2(\mathbb{T})]_{2n}$

In general, we cannot give a complete answer. But in some particular cases it can be shown that

$$A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1}$$

is really a canonical generalized factorization of $A_\gamma(b)$.

Let us now look at the case where b belongs to the algebra of all matrices whose components can be represented as a sum of a continuous function and an essentially bounded function which admits a bounded analytic extension to the interior of $\mathbb{T}$, that is, $b \in [C(\mathbb{T}) + L_\infty^+(\mathbb{T})]_{n,n}$. In that case, N_+ is a self-adjoint and compact operator (see [LS]). Hence, the resolvent of N_+ can be represented in terms of the eigenfunctions and eigenvalues. Thus, using the spectral theorem for compact self-adjoint operators and introducing the operator

$$K(\psi) = \sum_k \frac{\lambda_k(\psi, \nu_k^+)}{\lambda_k + \gamma}\nu_k^+,$$

where $\{\nu_k^+\}$ is the orthonormal system in $[L_2(\mathbb{T})]_n$ formed by the eigenfunctions of the operator $N_+(b)$ and where λ_k are the corresponding eigenvalues, we get

$$R(N_+(b), -\gamma I)\Psi = \frac{1}{\gamma}\Psi - \frac{1}{\gamma}K(\Psi),$$

where $\Psi = [\psi_1, \ldots, \psi_n]$ and $K(\Psi) = [K(\psi_1), \ldots, K(\psi_n)]$.

We obtain the following lemma.

Lemma 4.2. *If $-\gamma \in \rho(N_+(b))$, then a solution to the Riemann-Hilbert problem*

$$\begin{cases} B_\gamma^+(b) = A_\gamma(b)B_\gamma^-(b) \\ B_\gamma^-(b)(\infty) = E \end{cases}$$

is given by

$$B_\gamma^+(b) = \begin{pmatrix} e - K(e) & P_+b - K(b) \\ P_+b^*(e-K(e)) & \gamma e + P_+b^*(P_+b - K(b)) \end{pmatrix}$$

and

$$B_\gamma^-(b) = \begin{pmatrix} e + \frac{1}{\gamma}P_-bP_-b^*(e-K(e)) & -P_-b + \frac{1}{\gamma}P_-bP_-b^*(P_+b-K(b)) \\ -\frac{1}{\gamma}P_-b^*(e-K(e)) & e - \frac{1}{\gamma}P_-b^*(P_+b-K(b)) \end{pmatrix},$$

where $e = [e_1, \dots, e_n]$ *and*

$$K(e_j) = \sum_k \frac{2\pi\lambda_k \overline{\nu_{kj}^+(0)}}{\lambda_k + \gamma}\nu_k^+.$$

This lemma shows that $B_\gamma^\pm(b)$ can be obtained through the eigenvalues and eigenfunctions of $N_+(b)$. However, our question remains open, i.e., we do not yet know if $A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1}$ is or is not a canonical generalized factorization of $A_\gamma(b)$. This motivates the next subsection.

4.2. The case $b \in [\mathcal{A}(\mathbb{T})]_{n,n}$

Let us consider a decomposing algebra $\mathcal{A}(\mathbb{T})$ of continuous functions and let

$$\mathcal{A}^\pm(\mathbb{T}) = \mathcal{A}(\mathbb{T}) \cap \mathcal{C}^\pm(\mathbb{T}),$$

where $\mathcal{C}^\pm(\mathbb{T})$ refer to the classes of continuous functions on $\mathbb{T}$ which have a holomorphic extension to $\mathbb{T}_\pm$ and are continuous on $\mathbb{T}_\pm \cup \mathbb{T}$. Here $\mathbb{T}_+$ is the open unit disk and $\mathbb{T}_-$ is the exterior region of the unit circle (∞ included).

It is well known that the following result holds (see, for instance, [LS, GK]).

Lemma 4.3. *If* $A(z) \in [\mathcal{A}^\pm(\mathbb{T})]_{n,n}$ *and* $\det A(z) \neq 0$ *for all* $z \in \mathbb{T}_\pm$*, then* $A^{-1}(z) \in [\mathcal{A}^\pm(\mathbb{T})]_{n,n}$.

Let $b \in [\mathcal{A}(\mathbb{T})]_{n,n}$. In this case, if we can prove that i) is satisfied, then our problem is solved, i.e.,

$$A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1},$$

is a canonical generalized factorization of $A_\gamma(b)$.

Thus, we will establish necessary and sufficient conditions for the factorization $A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1}$ to be a canonical generalized factorization of $A_\gamma(b)$ in terms of the resolvent of $N_+(b)$. These conditions amount to the linear independence of the columns of the matrix functions $B_\gamma^+(b)$ and $(B_\gamma^-(b))^{-1}$ at every point of $\mathbb{T}_+$ and $\mathbb{T}_-$, respectively. After some straightforward computations, we can prove the following result.

Theorem 4.4. *Let* $-\gamma \in \rho(N_+(b))$, $b_- + b_+ = b \in [\mathcal{A}(\mathbb{T})]_{n,n}$,

$$c_{ik} = \frac{2\pi\lambda_k \overline{\nu_{ki}^+(0)}}{\lambda_k + \gamma}, \quad d_{ik} = \frac{\lambda_k (P_+ b_i, \nu_k^+)}{\lambda_k + \gamma}, \quad f_i(z) = e_i - \sum_k c_{ik}\nu_k^+(z),$$

and $g_i(z) = P_+ b_i(z) - \sum_k d_{ik}\nu_k^+(z)$. *Then* $A_\gamma(b) = B_\gamma^+(b)(B_\gamma^-(b))^{-1}$ *is a left canonical generalized factorization of the matrix function* $A_\gamma(b)$ *if and only if the vector systems*

$$\left\{ \begin{pmatrix} f_i(z) \\ P_+ b_+^*(z) f_i(z) \end{pmatrix}, \begin{pmatrix} g_i(z) \\ \gamma e_i + P_+ b_+^*(z) g_i(z) \end{pmatrix} \right\}_{i=\overline{1,n}}$$

and

$$\left\{ \begin{pmatrix} e_i + \frac{1}{\gamma} P_- b_+(z) P_- b_+^*(z) f_i(z) \\ -\frac{1}{\gamma} P_- b_+^*(z) f_i(z) \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma} P_- b_+(z) P_- b_+^*(z) g_i(z) \\ e_i - \frac{1}{\gamma} P_- b_+^*(z) g_i(z) \end{pmatrix} \right\}_{i=\overline{1,n}}$$

are linearly independent at each point of $\mathbb{T}_\pm$.

In general, it is hard to verify these conditions. However, we will present some concrete examples in which these conditions can be checked and thus yield an explicit left canonical generalized factorization of $A_\gamma(b)$.

Remark 4.5. Instead of $N_+(b)$, we could consider the operator $N_-(b)$ from the very beginning of this section, obtaining similar results. Furthermore, it is possible to prove that $\sigma(N_+(b)) = \sigma(N_-(b))$ and to show that the factorizations obtained in either way are equal for the case where $b \in [C(\mathbb{T}) + L_\infty^+(\mathbb{T})]_{n,n}$.

It should be noted that if a decomposition $b = b_- + b_+$ is considered, then the construction of an explicit left generalized factorization of the matrix function $A_\gamma(b)$ (where the resolvent of $N_+(b)$ or $N_-(b)$ is used) does not depend on the knowledge of b_-. This fact is very important, because it reveals that we have a quite general class of matrix functions where we only have to worry about the term b_+. In particular, if b_+ is a rational matrix function, then $N_+(b)$ and $N_-(b)$ are finite dimensional operators and, consequently, it is possible to determine their eigenvalues and eigenfunctions, which makes the conditions of Theorem 4.4 easier to verify. Therefore, this result can lead us to an algorithm similar to the well known algorithm for the factorization of rational matrix functions (see [P]). Roughly speaking, if some "problems" arise at points of $\mathbb{T}_+$, it should be possible to remove them separately, one by one, until a generalized (although perhaps not canonical) factorization is obtained. In fact, let $b = b_- + b_+$ where $b_\pm \in \mathcal{A}^\pm(\mathbb{T})$. Then

$$\begin{pmatrix} e & b \\ b^* & b^*b + \gamma e \end{pmatrix} = \begin{pmatrix} e & 0 \\ b_-^* & e \end{pmatrix} \begin{pmatrix} e & b_+ \\ b_+^* & b_+^*b_+ + \gamma e \end{pmatrix} \begin{pmatrix} e & b_- \\ 0 & e \end{pmatrix}.$$

Thus, if b_+ is a rational matrix function with its poles in $\mathbb{T}_-$ and vanishing at infinity, then one can always factorize the middle factor on the right-hand side, since this involves the factorization of a rational matrix function. Then, no matter what b_- is (rational or non-rational), one can factorize $A_\gamma(b)$. In other words, a factorization of $A_\gamma(b_+)$ always leads to a factorization of $A_\gamma(b)$, independently of how complicated b_- is.

Finally, we note that the study of the Riemann-Hilbert problem

$$\begin{cases} \Psi_+ = A_\gamma^{-1}(b)(E + \Psi_-) \\ \Psi_-(\infty) = 0 \end{cases}$$

and the use of the operator $N_+(b^*)$ or $N_-(b^*)$ allows us to state a theorem analogous to Theorem 4.4, which provides us with a necessary and sufficient condition for the existence of a right canonical generalized factorization of $A_\gamma(b)$, written explicitly in the form

$$A_\gamma(b) = C_\gamma^-(b^*)(C_\gamma^+(b^*))^{-1}$$

where the factors $C_\gamma^\pm(b^*)$ are given by

$$C_\gamma^-(b^*) = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} B_\gamma^-(b^*) \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}$$

and

$$C_\gamma^+(b^*) = \frac{1}{\gamma} \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} B_\gamma^+(b^*) \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}.$$

Here K is the operator defined by

$$\mathrm{K}(\Psi) = \Psi - \gamma R(N_+(b^*), -\gamma I)\Psi,$$

which can be expressed, as before, in terms of the eigenvalues and eigenfunctions of the operator $N_+(b^*)$. In this case, the construction of an explicit right canonical generalized factorization depends only on the knowledge of the function b_- and comments similar to those presented after Remark 4.5 still hold.

4.3. An example

We will now present an example to clarify the results obtained (for the case of left factorization). Let us consider the case where

$$b_- + \operatorname{diag}\left[\beta_i \frac{1}{t-a_i}\right]_{i=\overline{1,n}} = b \in [\mathcal{A}(\mathbb{T})]_{n,n}$$

with $a_i, \beta_i \in \mathbb{C}$ and $|a_i| > 1$ for all i.

With the matrix functions

$A_{11}^+ = \operatorname{diag}\left[1 - \frac{c_{ii}}{k_i(t-a_i)}\right]_{i=\overline{1,n}}$,
$A_{12}^+ = \operatorname{diag}\left[\frac{\gamma\alpha_i}{k_i(\lambda_i+\gamma)(t-a_i)}\right]_{i=\overline{1,n}}$,
$A_{21}^+ = b_-^* A_{11}^+ + \operatorname{diag}\left[-\frac{\overline{\alpha_i}}{k_i\overline{a_i}} + \frac{\overline{\alpha_i} a_i c_{ii}}{k_i^2(|a_i|^2-1)(t-a_i)}\right]_{i=\overline{1,n}}$,
$A_{22}^+ = b_-^* A_{12}^+ + \gamma e + \operatorname{diag}\left[-\frac{\gamma|\alpha_i|^2 a_i}{k_i^2(\lambda_i+\gamma)(|a_i|^2-1)(t-a_i)}\right]_{i=\overline{1,n}}$,
$A_{11}^- = -b_- A_{21}^- + e + \operatorname{diag}\left[\left(1 + \frac{c_{ii}\overline{a_i}}{k_i(|a_i|^2-1)}\right) \frac{|\alpha_i|^2}{\gamma k_i^2 \overline{a_i}(|a_i|^2-1)\left(t - \frac{1}{\overline{a_i}}\right)}\right]_{i=\overline{1,n}}$,
$A_{12}^- = -b_- A_{22}^- + \operatorname{diag}\left[-\frac{|\alpha_i|^2\alpha_i}{k_i^3(|a_i|^2-1)^2(\lambda_i+\gamma)\left(t-\frac{1}{\overline{a_i}}\right)}\right]_{i=\overline{1,n}}$,
$A_{21}^- = \operatorname{diag}\left[\left(1 + \frac{c_{ii}\overline{a_i}}{k_i(|a_i|^2-1)}\right) \frac{\overline{\alpha_i}}{\gamma k_i \overline{a_i}^2\left(t-\frac{1}{\overline{a_i}}\right)}\right]_{i=\overline{1,n}}$,
$A_{22}^- = \operatorname{diag}\left[1 - \frac{|\alpha_i|^2}{k_i^2\overline{a_i}(|a_i|^2-1)(\lambda_i+\gamma)\left(t-\frac{1}{\overline{a_i}}\right)}\right]_{i=\overline{1,n}}$,
$k_i = \sqrt{\frac{2\pi}{|a_i|^2-1}}$, $\alpha_i = k_i\beta_i$, $\lambda_i = \frac{|\alpha_i|^2}{k_i^2(|a_i|^2-1)^2}$, and $c_{ii} = -\frac{k_i\lambda_i(|a_i|^2-1)}{\overline{a_i}(\lambda_i+\gamma)}$, $\forall i = \overline{1,n}$,
we have

$$A_\gamma(b) = A_\gamma^+ A_\gamma^-,$$

where

$$A_\gamma^+ = \begin{pmatrix} A_{11}^+ & A_{12}^+ \\ A_{21}^+ & A_{22}^+ \end{pmatrix} \quad \text{and} \quad A_\gamma^- = \begin{pmatrix} A_{11}^- & A_{12}^- \\ A_{21}^- & A_{22}^- \end{pmatrix}^{-1}.$$

For any $\gamma \neq -\lambda_i, \forall i = \overline{1,n}$, this represents a left canonical generalized factorization of $A_\gamma(b)$, since the conditions of Theorem 4.4 are satisfied. Besides that, it is easy to see that A_{21}^- and A_{22}^- commute and that $A_{11}^- A_{22}^- - A_{12}^- A_{21}^- = e$. Thus, if we consider, for example, the case where $\beta_i \neq 0, \forall i = \overline{1,n}$, then A_{21}^- is invertible on $\mathbb{T}_-$ and we get (see [H])

$$A_\gamma^- = \begin{pmatrix} (A_{21}^-)^{-1} A_{22}^- A_{21}^- & -(A_{21}^-)^{-1}(A_{22}^- A_{21}^- A_{11}^- (A_{21}^-)^{-1} - e) \\ -A_{21}^- & A_{21}^- A_{11}^- (A_{21}^-)^{-1} \end{pmatrix}.$$

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Área Dep. de Matemática
Universidade do Algarve
Campus de Gambelas
8000-810 Faro, Portugal

Área Dep. de Matemática
Universidade do Algarve
Campus de Gambelas
8000-810 Faro, Portugal

Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa, Portugal

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Received