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THE SECOND SYMMETRIC PRODUCT OF FINITE GRAPHS FROM A HOMOTOPICAL VIEWPOINT

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Abstract. This paper describes the classification of the n-fold symmetric product of a finite graph by means of its homotopy type, having as universal models the n-fold symmetric product of the wedge of n-circles; and introduces a CW-complex called *binomial torus*, which is homeomorphic to a space that is a strong deformation retract of the second symmetric products of the wedge of n-circles. Applying the above we calculate the fundamental group, Euler characteristic, homology and cohomology groups of the second symmetric product of finite graphs.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is nonempty and closed}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

We endow at 2^X with the Vietoris topology [6, Theorem 1.2, p. 3], which is generated by the base

$$\beta = \Big\{\langle U_1, \ldots, U_k\rangle : U_i \text{ are open in } X, \text{ for all } i = 1, \ldots, k\Big\},$$

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where

$$\langle U_1, \ldots, U_k \rangle = \left\{ A \in 2^X : A \subseteq \bigcup_{i=1}^{k} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, \ldots, k \right\}.$$

The Vietoris topology matches with the Hausdorff metric [6, Theorem 3.2, p. 18] defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

The hyperspace $F_n(X)$ is called *n-fold symmetric product* of X. The symbols $\approx$, $\cong$ denote homeomorphism and isomorphism respectively. The notion of symmetric product, was first introduced by K. Borsuk and S. Ulam in [1], where they proved that for the interval $I = [0, 1]$ and $n = 1, 2, 3$, $F_n(I) \approx I^n$, but that $F_n(I)$ cannot be embedded in $\mathbb{R}^n$ for $n \geq 4$. R. Molski in [9] shows that $F_2(I^2)$ is homeomorphic to I^4 but that neither $F_n(I^2)$ nor $F_2(I^n)$ can be embedded in $\mathbb{R}^{2n}$ for any $n \geq 3$. Before, R. Bott in [2] shows that $F_3(S^1) \approx S^3$. In [11] R. M. Schori shows that $F_n(I) \approx cone\big(D^{n-2}\big) \times I$, where $D^{n-2} = \{A \in F_n(I) : 0, 1 \in A\}$.

Some results from homotopical viewpoint $F_n(X)$: S. Macías in [7] shows that for any continuum X, the first group of cohomology of Čech $H^1(F_n(X); \mathbb{Z})$ vanishes for $n \geq 3$, and D. Handel in [4] proved that for closed connected n-manifolds M^n (for $n \geq 2$), the singular cohomology group $H^i(F_k(M^n); \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $i = nk$, and 0 for $i > nk$. Also it shows that the inclusion maps $F_k(X, x_0) \hookrightarrow F_{2k-1}(X, x_0)$ and $F_k(X) \hookrightarrow F_{2k+1}(X)$ induce the zero map on all homotopy groups for pathwise connected Hausdorff space X. N. Chinen and A. Koyama in [3] shows that for $n \in \mathbb{N}$, $F_{2n+1}(S^1)$ has the same homotopy type of S^{2n+1} and $F_{2n}(S^1)$ has the same homotopy type of S^{2n-1}.

In this paper we are interested in studying the homotopy of the symmetric products of finite graphs, we will give a classification by means of its homotopy type in Section 3. In Section 4 we will define a new geometric object called *binomial torus*, which is a CW-complex. We will study its fundamental group, homology and cohomology groups. Subsequently in Section 5 we will show that the second symmetric product of the bouquet of n-circles contains a subset homeomorphic to the binomial torus which is a strong deformation retract of the second symmetric product of the bouquet of n-circles. Thus developed machinery of Section 4 will apply to the second symmetric products of a finite graph.

2. Preliminaries

A *map* is a continuous function. Let $f, g : X \to Y$ be maps. We say that f is homotopic to g (in symbols $f \simeq g$) if there exists a homotopy of f to g, that is, a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, in this case the spaces X and Y are said to be *homotopy equivalent* or to have the same *homotopy type*, and the usual notation is $X \simeq Y$. A space having the homotopy type of a point is called *contractible*.

Let X and Y be pointed spaces. Their topological product $X \times Y$ is also pointed with base point (x_0, y_0) if $x_0 \in X$ and $y_0 \in Y$ are the base points of X

and Y, respectively. The *wedge* of X and Y can be considered as a subspace of $X \times Y$,

$$X \vee Y = \{(x, y) \in X \times Y : x = x_0 \text{ or } y = y_0\};$$

that is, $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$. For example $S^1 \vee S^1$ is homeomorphic to the figure eight, two circles touching at a point. In general, let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces. We denote their coproduct or topological sum by $\coprod_{\alpha\in\Lambda} X_\alpha$. If $\{X_\alpha : \alpha \in \Lambda\}$ is a family of pointed spaces, we define the *wedge* as the quotient space

$$\bigvee_{\alpha\in\Lambda} X_\alpha = \coprod_{\alpha\in\Lambda} X_\alpha / \{x_\alpha : \alpha \in \Lambda\},$$

where for each α, $x_\alpha \in X_\alpha$ is the base point. For example, the *bouquet of n-circles* is $\bigvee_n S^1$, which it is the union of n-circles at a single point.

An *arc* is a continuum that is homeomorphic to the interval I. A *graph* G is a topological space which consists of a collection of points $V(G)$, called *vertices*, and a collection of *edges* $E(G)$. Each edge is either homeomorphic to an interval I joining two distinct vertices, or it is homeomorphoric to a circle joining a given vertex to itself. It is assumed that any two distinct edges are either disjoint, or else intersect in a common end point.

A *finite graph* is a continuum that has only a finite number of vertices and edges. If G is a finite graph, let us denote by $|V(G)|$ the number of vertices of G and $|E(G)|$ the number of edges of G. The *Euler characteristic* of a finite graph G is defined by $\chi(G) = |V(G)| - |E(G)|$. The Euler characteristic is a homotopy type invariant, namely.

Lemma 2.1. [8, Corollary 6.3, p. 200] *If two finite graphs G_1 and G_2 have the same homotopy type, then $\chi(G_1) = \chi(G_2)$.*

A *subgraph* of a graph G is a graph whose set of vertices and set of edges are subsets of G. A *tree* is a finite graph that contains no simple closed curve. By a tree in a finite graph G we mean a subgraph that is a tree. We call a tree in a finite graph G *maximal* if it contains all the vertices of G. In fact, every finite graph contains a maximal tree ([5, Proposition 1 A.1, p. 84]). The trees are also characterized by the Euler characteristic.

Lemma 2.2. [8, Proposition 6.1 and 6.4, p. 201] *Let G be a finite graph. Then G is a tree if and only if $\chi(G) = 1$.*

Let G be a finite graph and $T \subseteq G$ a maximal tree with set of edges $E(T) = \{e_1, \ldots, e_s\}$. Let $E(G) - E(T) = \{a_1, \ldots, a_r\}$ the set of edges that are not in T (this set can be empty).

On the other hand, the quotient space G/T is a finite graph with only one vertex. Since T contains all the vertices of G, then the set of edges of G/T is $E(G/T) = \{\overline{a_1}, \ldots, \overline{a_r}\}$, where its elements are loops based in such vertex. Therefore G/T is a bouquet of r-circles and its Euler characteristic is $\chi(G/T) = 1 - r$.

Since T is contractible, then the quotient function $q : G \longrightarrow G/T$ is a homotopy equivalence ([5, Proposition 0.17]). Then by Lemma 2.1 $\chi(G) = \chi(G/T)$. So we have $|V(G)|-|E(G)|=1-r$, hence $r=1-|V(G)|+|E(G)|$. The positive integer r is called *genus* of G. Thus we have the following result.

Theorem 2.3. *Let G be a finite graph. Then G is homotopically equivalent to $\bigvee_{i=1}^{r} C_i$, where $r = 1 - |V(G)| + |E(G)|$ and C_i is homeomorphic to S^1 for all $i = 1, \ldots, r$.*

3. Classification

Let $f : X \to Y$ be a map between continua. For all $n \geq 1$, we define the induced function $F_n(f) : F_n(X) \to F_n(Y)$ by $F_n(f)(A) = f(A)$, which is continuous ([6, Lemma 13.3, p. 106]). If X, Y, Z are continua and $f : X \to Y$ and $g : Y \to Z$ are maps, then the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f}\uparrow & \nearrow {\scriptstyle g\circ f} & \\ X & & \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} F_n(Y) & \xrightarrow{F_n(g)} & F_n(Z) \\ {\scriptstyle F_n(f)}\uparrow & \nearrow {\scriptstyle F_n(g\circ f),} & \\ F_n(X) & & \end{array}$$

where $F_n(g \circ f) = F_n(g) \circ F_n(f)$. Thus $F_n(-)$ defines a homotopic functor.

Theorem 3.1. [4, Proposition 3.2, p. 758] *Let X, Y be continua and $f, g : X \to Y$ maps. If $h : X \times I \to Y$ is a homotopy between f and g. Then for every $n \in \mathbb{N}$, $h_n : F_n(X) \times I \to F_n(Y)$ defined by*

$$h_n(\{x_1, \ldots, x_m\}, t) = \{h(x_1, t), \ldots, h(x_m, t)\} \text{ where } m \leq n$$

is a homotopy between $F_n(f)$ and $F_n(g)$.

If X is homotopically equivalent to Y, then there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. By the Theorem 3.1, $F_n(f) : F_n(X) \to F_n(Y)$ and $F_n(g) : F_n(Y) \to F_n(X)$ are continuous, such that $F_n(g) \circ F_n(f) \simeq F_n(1_X)$ and $F_n(f) \circ F_n(g) \simeq F_n(1_Y)$. So we have the following result.

Theorem 3.2. *Let X and Y be continua, such that X is homotopically equivalent to Y, then $F_n(X)$ is homotopically equivalent to $F_n(Y)$, for all $n \geq 1$.*

Now if G is a finite graph, by Theorem 2.3, G has the same homotopy type of the bouquet of r-circles $\bigvee_r S^1$, where $r = 1-|V(G)|+|E(G)|$. Thus, by Theorem 3.2, $F_n(G)$ has the same homotopy type of the $F_n\big(\bigvee_r S^1\big)$ for all $n \in \mathbb{N}$. So,the following theorem is clear.

Theorem 3.3. *Let G be a finite graph, and let $r = 1 - |V(G)| + |E(G)|$. Then for all $n \in \mathbb{N}$, $F_n(G)$ has the same homotopy type of $F_n\big(\bigvee_{i=1}^{r} C_i\big)$, where C_i is homeomorphic to S^1 for all $i = 1, \ldots, r$.*

Given two finite graphs G_1 and G_2, it is difficult to establish when $F_n(G_1)$ is homeomorphic to $F_n(G_2)$ for all $n \geq 2$. However, for the homotopic case the problem is solved applying the Theorem 3.3, as shown in the following corollary.

Corollary 3.4. *Let G_1, G_2 be two finite graphs such that its genus is the same, then $F_n(G_1)$ and $F_n(G_2)$ have the same homotopy type, for all $n \in \mathbb{N}$.*

In particular, if we take a finite graph G of genus $r = 0$, then G is a tree (Lemma 2.2). Thus G is homotopy equivalent to the point $\{p\}$. So $F_n(G)$ has the same homotopy type of $F_n(\{p\}) = \{p\}$, for all $n \in \mathbb{N}$.

We have the ingredients to make a classification of the n-fold symmetric product of all finite graphs through homotopy, indeed: For all $n \in \mathbb{N}$, consider the set

$$\mathcal{G}F_n = \Big\{ F_n(G) : G \text{ is a finite graph} \Big\}.$$

Let us define a relationship in $\mathcal{G}F_n$, as follows: $F_n(G_1) \sim F_n(G_2)$ if and only if $F_n(G_1)$ has the same homotopy type of $F_n(G_2)$. Notice that the homotopy equivalence is an equivalence relation, then we can consider the set of all equivalence classes

$$\mathcal{G}F_n/\sim = \Big\{ \big[F_n(G)\big] : F_n(G) \in \mathcal{G}F_n \Big\},$$

where $\big[F_n(G)\big]$ denotes the equivalence class of the n-fold symmetric product $F_n(G)$. In consequence of Theorem 3.3, we have

Corollary 3.5. *For all $n \in \mathbb{N}$, the set of equivalence classes $\mathcal{G}F_n/\sim$ can be written as*

$$\Big\{ \big[F_n(\{p\})\big], \big[F_n(C_1)\big], \big[F_n(C_1 \vee C_2)\big], \big[F_n(C_1 \vee C_2 \vee C_3)\big], \dots \Big\},$$

and indeed a bijective function

$$\varphi : \mathbb{Z}^+ \cup \{0\} \longrightarrow \mathcal{G}F_n/\sim$$

defined by

$$\varphi(m) = \begin{cases} \big[F_n(\{p\})\big] & \text{if } m = 0, \\ \big[F_n(C_1 \vee C_2 \vee \cdots \vee C_m)\big] & \text{if } m \neq 0. \end{cases}$$

This result shows that we have *homotopic universal models, i.e.,* we have representatives of equivalence classes distinguished, for $k \geq 1$, namely

$$\Big[F_n\Big(\bigvee_{i=1}^{k} C_i\Big)\Big] \;=\; \Big\{ F_n(G) \in \mathcal{G}F_n : \chi(G) = 1 - k \Big\}.$$

4. Binomial Torus

In this section we define a geometric object called *binomial torus*, which is a CW-complex. We study some of its algebraic invariants, such as: fundamental group, homology and cohomology groups. The binomial torus plays a vital role in the study of homotopical properties of the second symmetric product of a finite graph.

We denote by i the simple closed curve C_i, for all $i = 1, \dots, n$. If

$$1 \cap 2 \cap \cdots \cap n = \{p\},$$

then the bouquet of n-circles is $\bigvee_{i=1}^{n} i$, represented in Figure 1.

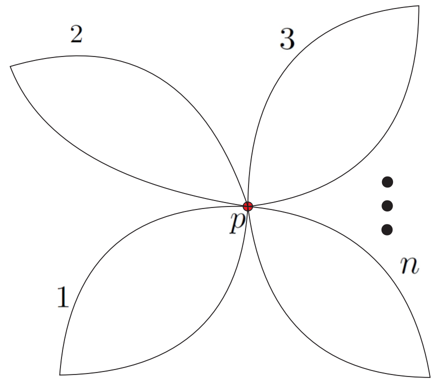

FIGURE 1. bouquet of n-circles.

In the same way, for all $i, j = 1, \ldots, n$ the torus $C_i \times C_j$ will be denoted by ij. The torus ij is illustrated in Figure 2.

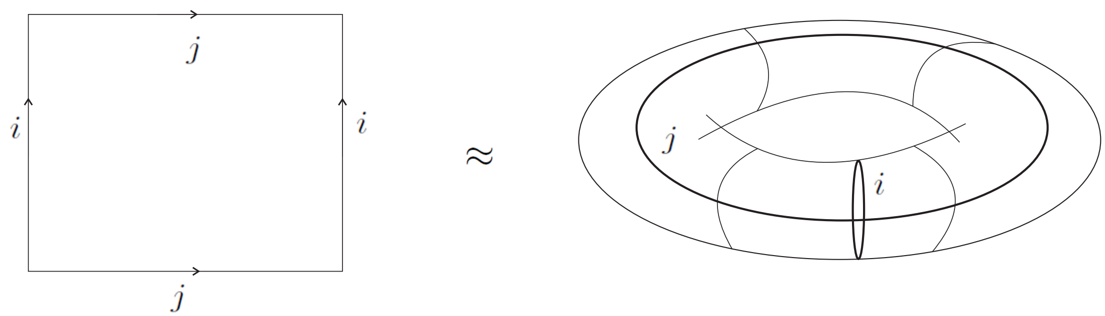

FIGURE 2. Torus ij.

Definition 4.1. Let $\bigvee_{i=1}^{n} i$ a bouquet of n-circles, with $n \geq 2$. We define the binomial torus, and we denote by $T_{\binom{n}{2}}$, the union of $\binom{n}{2}$ torus:

$$\begin{aligned} T_{\binom{n}{2}} &= (12 \cup 13 \cup \cdots \cup 1n) \cup (23 \cup 24 \cup \cdots \cup 2n) \cup \cdots \cup (n-1)n \\ &= \bigcup_{i=1}^{n-1} \Big(\bigcup_{j=i+1}^{n} ij \Big), \end{aligned}$$

with the following intersections

$$\begin{aligned} 12 \cap 13 \cap \cdots \cap 1n &= 1 \times \{p\}, \\ 21 \cap 23 \cap \cdots \cap 2n &= 2 \times \{p\}, \\ &\vdots \\ n1 \cap n2 \cap \cdots \cap n(n-1) &= n \times \{p\}. \end{aligned}$$

Geometrically, the binomial torus can be represented as shown in Figure 3.

To calculate the fundamental group of binomial torus we can make a presentation with generators $1, \ldots, n$ and the following relations

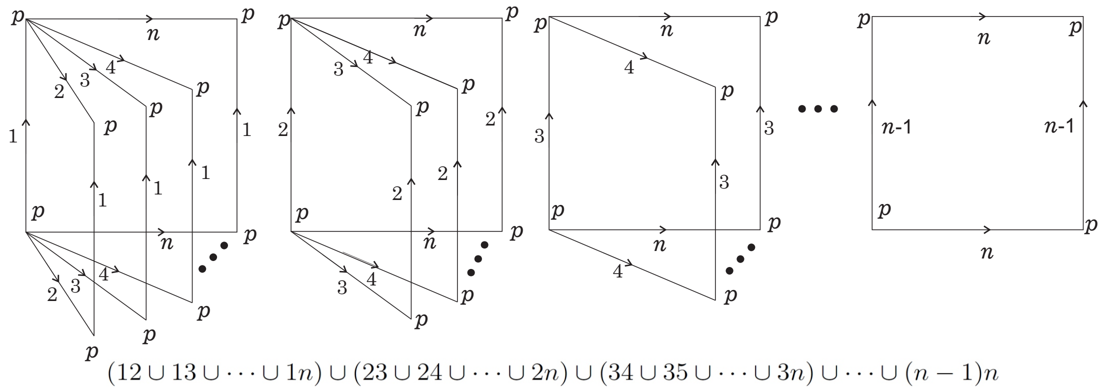

FIGURE 3. Binomial torus.

$$\begin{array}{lll} 121^{-1}2^{-1} = e, & 131^{-1}3^{-1} = e, \dots, & 1n1^{-1}n^{-1} = e \\ 232^{-1}3^{-1} = e, & 242^{-1}4^{-1} = e, \dots, & 2n2^{-1}n^{-1} = e \\ 343^{-1}4^{-1} = e, & 353^{-1}5^{-1} = e, \dots, & 3n3^{-1}n^{-1} = e \\ & \vdots & \\ & (n-1)n(n-1)^{-1}n^{-1} = e. & \end{array}$$

We denote by $[i, j] = iji^{-1}j^{-1}$ the commutator. Therefore the fundamental group of the binomial torus based on the point p is

$$\begin{aligned} \pi_1\big(T_{\binom{n}{2}}, p\big) &= \big\langle 1, \dots, n \mid [i, j], \text{ for all } 1 \le i < j \le n \big\rangle \\ &\cong \mathbb{Z}^n. \end{aligned}$$

So, we have the following theorem.

Theorem 4.2. *The fundamental group of binomial torus $T_{\binom{n}{2}}$ is a free abelian group of rank n.*

On the other hand, we can see $T_{\binom{n}{2}}$ as a CW-complex, as shown in Figure 4. Where $e^0_1 = \{p\}$ is a 0-cell, the simple closed curves $e^1_i = i$, for all $i = 1, \dots, n$ are 1-cell. Finally e^2_{ij} are 2-cell for all $i = 1, \dots, n-1$, $j = 1, \dots n$ and $i < j$. Therefore, the cell decomposition of binomial torus is,

$$T_{\binom{n}{2}} = e^0_1 \cup \left(\bigcup_{i=1}^{n} e^1_i \right) \cup \left(\bigcup_{i=1}^{n-1} \left(\bigcup_{j=i+1}^{n} e^2_{ij} \right) \right).$$

The cell chains of $T_{\binom{n}{2}}$ are

$$\begin{gathered} C_0 = \langle e^0_1 \rangle, \\ C_1 = \langle e^1_1, \dots, e^1_n \rangle, \\ C_2 = \langle e^2_{12}, \dots, e^2_{1n}, e^2_{23}, \dots, e^2_{2n}, e^2_{34}, \dots, e^2_{3n}, \dots, e^2_{(n-1)n} \rangle. \end{gathered}$$

Thus $C_0 \cong \mathbb{Z}$, $C_1 \cong \mathbb{Z}^n$ and $C_2 \cong \mathbb{Z}^{\binom{n}{2}}$. Therefore, the sequence of chain complexes and chain maps are

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

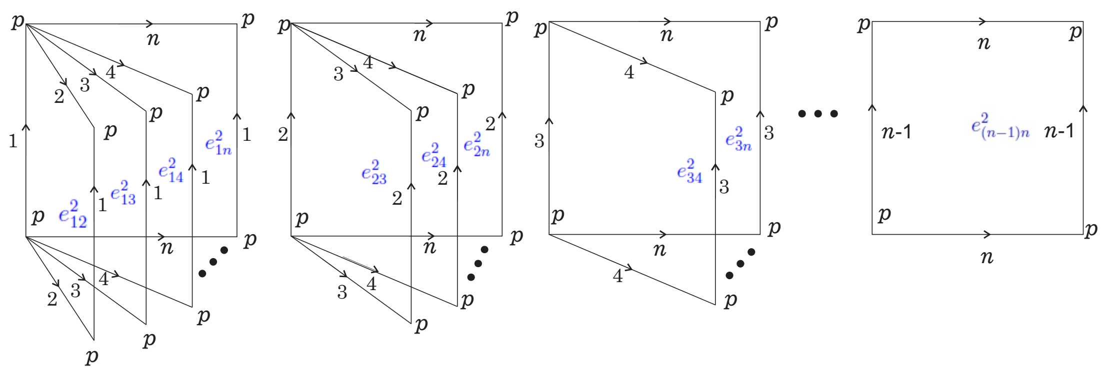

FIGURE 4. $T_{\binom{n}{2}}$ shown as a CW-complex.

Notice that

$$\partial_0(e_1^0) = 0.$$

$$\partial_1(e_i^1) = p - p = 0, \text{ for all } i = 1, \dots, n.$$

$$\partial_2(e_{ij}^2) = i + j - i - j = 0, \text{ for all } i = 1, \dots, n-1,\ j = 1, \dots, n \text{ and } i < j.$$

Which implies that the cycles are

$$Z_0\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \ker(\partial_0) = \mathbb{Z},$$
$$Z_1\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \ker(\partial_1) = \mathbb{Z}^n,$$
$$Z_2\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \ker(\partial_2) = \mathbb{Z}^{\binom{n}{2}}.$$

Also the boundaries are

$$B_0\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \operatorname{im}(\partial_1) = 0,$$
$$B_1\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \operatorname{im}(\partial_2) = 0,$$
$$B_2\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \operatorname{im}(\partial_3) = 0.$$

Therefore we have the following result.

Theorem 4.3. *The homology groups of the binomial torus $T_{\binom{n}{2}}$ with coefficients in $\mathbb{Z}$ are*

$$H_q\big(T_{\binom{n}{2}}; \mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \text{if} \quad q = 0, \\ \mathbb{Z}^n & \text{if} \quad q = 1, \\ \mathbb{Z}^{\binom{n}{2}} & \text{if} \quad q = 2, \\ 0 & \text{if} \quad q \geq 3. \end{cases}$$

Thus the Betti numbers for the binomial torus $T_{\binom{n}{2}}$ are

$$b_0\big(T_{\binom{n}{2}}\big) = 1, \quad b_1\big(T_{\binom{n}{2}}\big) = n, \quad b_2\big(T_{\binom{n}{2}}\big) = \binom{n}{2}, \quad b_i\big(T_{\binom{n}{2}}\big) = 0 \quad \forall i \geq 3.$$

Therefore the Euler characteristic of the binomial torus $T_{\binom{n}{2}}$ is

$$\begin{aligned}\chi\big(T_{\binom{n}{2}}\big) &= b_0\big(T_{\binom{n}{2}}\big) - b_1\big(T_{\binom{n}{2}}\big) + b_2\big(T_{\binom{n}{2}}\big)\\ &= 1 - n + \binom{n}{2}\\ &= \frac{n^2-3n+2}{2}\\ &= \frac{(n-2)(n-1)}{2}.\end{aligned}$$

The first values of the Euler characteristic of the binomial torus are shown in the following table:

n	2	3	4	5	6	7	8	9	10	11
$\chi\big(T_{\binom{n}{2}}\big)$	0	1	3	6	10	15	21	28	36	45

Notice that if $k = n-2$, then $\chi\big(T_{\binom{n}{2}}\big) = k(k+1)/2$. Thus we have the following result.

Theorem 4.4. *For $n \geq 3$, $\chi\big(T_{\binom{n}{2}}\big)$ is a triangular number.*

To calculate the cohomology groups of the binomial torus $T_{\binom{n}{2}}$ we use the universal coefficient theorem for cohomology ([10, Theorem 7.5, p. 66]), namely:

Theorem 4.5. *Let X be a CW-complex. We can calculate cohomology over a general coefficient group G using the corresponding integral homology and the extension product*

$$H^n(X;G) \cong Hom\big(H_n(X;\mathbb{Z}),G\big) \oplus Ext\big(H_{n-1}(X;\mathbb{Z}),G\big).$$

For any abelian group G, we have that $Hom(\mathbb{Z},G) \cong G$ and $Ext(\mathbb{Z},G) = 0$ (see the pages 62 and 63 of [10]). In particular $Hom(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and $Ext(\mathbb{Z},\mathbb{Z}) = 0$. Furthermore, for any two abelian groups A and B,

$$Hom(A \oplus B,\mathbb{Z}) = Hom(A,\mathbb{Z}) \oplus Hom(B,\mathbb{Z})$$

and

$$Ext(A \oplus B,\mathbb{Z}) = Ext(A,\mathbb{Z}) \oplus Ext(B,\mathbb{Z})$$

(see [5] page 195). The following proposition is easy to see.

Proposition 4.6. *For any positive integer r, we have*

$$Hom(\mathbb{Z}^r,\mathbb{Z}) \cong \mathbb{Z}^r, \qquad Ext(\mathbb{Z}^r,\mathbb{Z}) \cong 0.$$

By Theorem 4.3, Theorem 4.5 and Proposition 4.6, we have the following:

$$\begin{aligned}H^0\Big(T_{\binom{n}{2}};\mathbb{Z}\Big) &\cong Hom\Big(H_0\big(T_{\binom{n}{2}};\mathbb{Z}\big),\mathbb{Z}\Big) \oplus Ext\Big(H_{-1}\big(T_{\binom{n}{2}};\mathbb{Z}\big),\mathbb{Z}\Big)\\ &\cong Hom\big(\mathbb{Z},\mathbb{Z}\big) \oplus Ext\big(0,\mathbb{Z}\big)\\ &\cong \mathbb{Z} \oplus 0\\ &\cong \mathbb{Z}.\end{aligned}$$

$$\begin{aligned} H^1\Big(T_{\binom{n}{2}};\mathbb{Z}\Big) &\cong Hom\Big(H_1\big(T_{\binom{n}{2}};\mathbb{Z}\big),\mathbb{Z}\Big) \oplus Ext\Big(H_0\big(T_{\binom{n}{2}},\mathbb{Z}\big),\mathbb{Z}\Big) \\ &\cong Hom\big(\mathbb{Z}^n,\mathbb{Z}\big) \oplus Ext\big(\mathbb{Z},\mathbb{Z}\big) \\ &\cong \mathbb{Z}^n \oplus 0 \\ &\cong \mathbb{Z}^n. \end{aligned}$$

$$\begin{aligned} H^2\Big(T_{\binom{n}{2}};\mathbb{Z}\Big) &\cong Hom\Big(H_2\big(T_{\binom{n}{2}};\mathbb{Z}\big),\mathbb{Z}\Big) \oplus Ext\Big(H_1\big(T_{\binom{n}{2}};\mathbb{Z}\big),\mathbb{Z}\Big) \\ &\cong Hom\big(\mathbb{Z}^{\binom{n}{2}},\mathbb{Z}\big) \oplus Ext\big(\mathbb{Z}^n,\mathbb{Z}\big) \\ &\cong \mathbb{Z}^{\binom{n}{2}} \oplus 0 \\ &\cong \mathbb{Z}^{\binom{n}{2}}. \end{aligned}$$

Thus we have the following result;

Theorem 4.7. *The cohomology groups of the binomial torus $T_{\binom{n}{2}}$ with coefficients in $\mathbb{Z}$ are*

$$H^q\big(T_{\binom{n}{2}};\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & if \quad q=0, \\ \mathbb{Z}^n & if \quad q=1, \\ \mathbb{Z}^{\binom{n}{2}} & if \quad q=2, \\ 0 & if \quad q \geq 3. \end{cases}$$

5. Homology and Cohomology of the second symmetric product of finite graphs

Suppose that $A \subset X$. We say that A is a *strong deformation retract* of X if there exists a homotopy $h : X \times I \to X$ such that

(i) $h(x,0) = x$, if $x \in X$,
(ii) $h(x,1) \in A$, if $x \in X$,
(iii) $h(a,t) = a$, if $a \in A$, $t \in I$.

Proposition 5.1. *Let X,Y be topological spaces such that X,Y are closed in $X \cup Y$ and $X \cap Y = \{p\}$. If $Z \subset X$ and $W \subset Y$ are both strong deformation retracts and $Z \cap W = \{p\}$. Then $Z \cup W$ is a strong deformation retract of $X \cup Y$.*

Proof. If Z is a strong deformation retract of X, then there exists a homotopy $h_1 : X \times I \to X$ such that

$$\begin{aligned} h_1(x,0) &= x, \quad x \in X, \\ h_1(x,1) &\in Z, \quad x \in X, \\ h_1(a,t) &= a, \quad a \in Z,\ t \in I. \end{aligned}$$

On the other hand, if W is a strong deformation retract of Y, then there exists a homotopy $h_2 : X \times I \to X$ such that

$$\begin{aligned} h_2(y,0) &= y, \quad y \in Y, \\ h_2(y,1) &\in W, \quad y \in Y, \\ h_2(b,t) &= b, \quad b \in W,\ t \in I. \end{aligned}$$

Let $h : X \cup Y \times I \to X \cup Y$ defined by

$$h(x,t) = \begin{cases} h_1(x,t) & \text{if } x \in X \\ h_2(x,t) & \text{if } x \in Y. \end{cases}$$

Since $Z \cap W = \{p\}$ and $h_1(p,t) = h_2(p,t) = p$, then h is a continuous function. Hence

$$\begin{aligned} h(x,0) &= x, \quad x \in X \cup Y, \\ h(x,1) &\in Z \cup W, \quad x \in X \cup Y, \\ h(a,t) &= a, \quad a \in Z \cup W,\ t \in I. \end{aligned}$$

Therefore $Z \cup W$ is a strong deformation retract of $X \cup Y$. □

Proposition 5.2. *Let Z be a strong deformation retract of X. Let W be a topological space such that $X \cap W$ is a subspace of Z. Then $Z \cup W$ is a strong deformation retract of $X \cup W$.*

Proof. Since Z is a strong deformation retract of X, then there exists a homotopy $h : X \times I \to X$ such that $h(x,0) = x$, $h(x,1) \in Z$ for all $x \in X$ and $h(a,t) = a$ for all $a \in Z$ and $t \in I$. Let $\overline{h} : (X \cup W) \times I \to X \cup W$ defined by

$$\overline{h}(x,t) = \begin{cases} h(x,t) & \text{if } x \in X, \\ x & \text{if } x \in W. \end{cases}$$

As $X \cap W$ is a subspace of Z, then $h(y,t) = y$ for all $y \in X \cap W$, thus $\overline{h}$ is continuous. Now, observe that

$$\begin{aligned} \overline{h}(x,0) &= x, \quad x \in X \cup W, \\ \overline{h}(x,1) &\in Z \cup W, \quad x \in X \cup W, \\ \overline{h}(a,t) &= a, \quad a \in Z \cup W,\ t \in I. \end{aligned}$$

Therefore $Z \cup W$ is a strong deformation retract of $X \cup W$. □

Theorem 5.3. *For all $n \geq 2$, $F_2\big(\bigvee_{i=1}^n i\big)$ contains a subset T homeomorphic to the binomial torus $T_{\binom{n}{2}}$ which is a strong deformation retract of $F_2\big(\bigvee_{i=1}^n i\big)$.*

Proof. By induction over n. For $n = 2$, let $1, 2$ be two simple closed curves such that $1 \cap 2 = \{p\}$. Each element $\{x,y\} \in F_2\big(1 \vee 2\big)$ satisfies one of the following three possibilities:

(a) $\{x,y\} \subseteq 1$,
(b) $\{x,y\} \subseteq 2$ or
(c) $x \in 1$ and $y \in 2$.

The set of elements of $F_2\big(1 \vee 2\big)$ that satisfies (a) is

$$B_1 = \big\{\{x,y\} \in F_2(1 \vee 2) : x, y \in 1\big\},$$

and it can be represented as a Moebius strip. Analogously, the set of elements of $F_2\big(1 \vee 2\big)$ that satisfies (b) is

$$B_2 = \big\{\{x, y\} \in F_2(1 \vee 2) : x, y \in 2\big\},$$

that can be represented as another Moebius strip. The set of points that satisfies (c) is

$$\overline{12} = \big\{\{x, y\} \in F_2(1 \vee 2) : x \in 1, y \in 2\big\},$$

which is homeomorphic to the torus 12. Hence

$$F_2\big(1 \vee 2\big) = B_1 \cup B_2 \cup \overline{12}.$$

We denote by $\overline{1} = \big\{\{x, p\} \in F_2(1 \vee 2) : x \in 1\big\}$ and $\overline{2} = \big\{\{x, p\} \in F_2(1 \vee 2) : x \in 2\big\}$. Note that $\overline{1}$ is a strong deformation retract of B_1, and $\overline{2}$ is a strong deformation retract of B_2. Observe that $B_1 \cap B_2 = \{p\}$, then by Proposition 5.1, $\overline{1} \cup \overline{2}$ is a strong deformation retract of $B_1 \cup B_2$. On the other hand

$$\big(B_1 \cup B_2\big) \cap \overline{12} = (B_1 \cap \overline{12}) \cup (B_2 \cap \overline{12}) = \overline{1} \cup \overline{2}.$$

We consider $Z = \overline{1} \cup \overline{2}$, $W = \overline{12}$ and $X = B_1 \cup B_2$, then by Proposition 5.2 $Z \cup W = (\overline{1} \cup \overline{2}) \cup \overline{12}$ is a strong deformation retract of $(B_1 \cup B_2) \cup \overline{12}$.

Since $(\overline{1} \cup \overline{2}) \cup \overline{12} = \overline{12}$, thus $\overline{12}$ is a strong deformation retract of $F_2\big(1 \vee 2\big)$. Therefore the binomial torus $T_1 = \overline{12} \approx T_{\binom{2}{2}}$ is a strong deformation retract of $F_2\big(\bigvee_{i=1}^{2} i\big)$.

Suppose that $F_2\big(\bigvee_{i=1}^{n} i\big)$ contains a subset T_2 homeomorphic to the binomial torus $T_{\binom{n}{2}}$ such that T is a strong deformation retract of $F_2\big(\bigvee_{i=1}^{n} i\big)$.

Each element $\{x, y\} \in F_2\big(\bigvee_{i=1}^{n+1} i\big)$ satisfies one of the following three possibilities:

(a) $\{x, y\} \subset \bigvee_{i=1}^{n} i$,
(b) $\{x, y\} \subset (n + 1)$ or
(c) $x \in \bigvee_{i=1}^{n} i$ and $y \in (n + 1)$.

Notice that the set of elements of $F_2\big(\bigvee_{i=1}^{n+1} i\big)$ that satisfies (a) is homeomorphic to $F_2\big(\bigvee_{i=1}^{n} i\big)$. The set of elements of $F_2\big(\bigvee_{i=1}^{n+1} i\big)$ that satisfies (b) can be represented as a Moebius strip, denoted by B_{n+1}. The set of points that satisfies (c) is

$$\big\{\{x, y\} \in F_2\big(\bigvee_{i=1}^{n+1} i\big) : x \in \bigvee_{i=1}^{n} i, y \in (n + 1)\big\},$$

which is homeomorphic to $\big(\bigvee_{i=1}^{n} i\big) \times (n + 1)$. Thus

$$F_2\Big(\bigvee_{i=1}^{n+1} i\Big) \approx F_2\Big(\bigvee_{i=1}^{n} i\Big) \cup B_{n+1} \cup \big(\bigvee_{i=1}^{n} i\big) \times (n + 1).$$

We denote by $\overline{n+1} = \big\{\{x, p\} \in F_2\big(\bigvee_{i=1}^{n+1} i\big) : x \in (n + 1)\big\}$. Notice that $\overline{n+1}$ is a strong deformation retract of B_{n+1}. Since $F_2\big(\bigvee_{i=1}^{n} i\big) \cap B_{n+1} = \{p\}$, then by Proposition 5.1, $T_2 \cup \overline{n+1}$ is a strong deformation retract of $F_2\big(\bigvee_{i=1}^{n} i\big) \cup B_{n+1}$.

Making

$$Z = T_2 \cup \overline{n+1},$$

$$X = F_2\big(\bigvee_{i=1}^{n} i\big) \cup B_{n+1},$$

$$W = \big(\bigvee_{i=1}^{n} i\big) \times (n+1).$$

We have that $X \cap W \approx \bigvee_{i=1}^{n+1} i$, so $X \cap W \subset Z$. By Proposition 5.2, $Z \cup W = T_2 \cup \overline{n+1} \cup \big(\bigvee_{i=1}^{n} i\big) \times (n+1)$ is a strong deformation retract of $X \cup W = F_2\big(\bigvee_{i=1}^{n} i\big) \cup B_{n+1} \cup \big(\bigvee_{i=1}^{n} i\big) \times (n+1)$.

Since $\overline{n+1}$ is homeomorphic to $(n+1)$ and $(n+1) \subseteq \big(\bigvee_{i=1}^{n} i\big) \times (n+1)$, then

$$T_2 \cup \overline{n+1} \cup \big(\bigvee_{i=1}^{n} i\big) \times (n+1) \approx \Big(T_2 \cup \big(\bigvee_{i=1}^{n} i\big)\Big) \times (n+1).$$

So we conclude that $T_3 = \Big(T_2 \cup \big(\bigvee_{i=1}^{n} i\big)\Big) \times (n+1)$ is a strong deformation retract of $F_2\big(\bigvee_{i=1}^{n+1} i\big)$.

On the other hand,

$$\begin{aligned} \big(\bigvee_{i=1}^{n} i\big) \times (n+1) &= (1 \times n+1) \vee \cdots \vee (n \times n+1) \\ &= 1(n+1) \vee \cdots \vee n(n+1). \end{aligned}$$

Observe that we have the following equalities

$$\begin{aligned} T_{\binom{n}{2}} \cup \big(\bigvee_{i=1}^{n} i\big) \times (n+1) &= T_{\binom{n}{2}} \cup \Big(1(n+1) \vee \cdots \vee n(n+1)\Big) \\ &= T_{\binom{n}{2}} \cup \Big(1(n+1) \cup \cdots \cup n(n+1)\Big) \\ &= T_{\binom{n+1}{2}}. \end{aligned}$$

Therefore $T_3 \approx T_{\binom{n+1}{2}}$ is a strong deformation retract of $F_2\big(\bigvee_{i=1}^{n+1} i\big)$.

$\square$

Notice that if A is a deformation retract of X, then A has the same homotopy type that X. Thus, their homotopy, homology and cohomology groups are isomorphic, respectively. Thus we have the results below.

Applying Theorem 3.3, Theorem 4.2 and Theorem 5.3, we have the following theorem.

Theorem 5.4. *Let G be a finite graph, then*

$$\pi_1\big(F_2(G)\big) \cong \mathbb{Z}^r,$$

where $r = 1 - |V(G)| + |E(G)|$.

Directly from the previous result, we have the following corollary.

Corollary 5.5. *Let G be a finite graph. Then G is a tree if and only if*

$$\pi_1\big(F_2(G)\big) = 0.$$

Applying Theorem 4.3 and Theorem 5.3, we have the following.

Theorem 5.6. *Let G be a finite graph. The homology groups of $F_2(G)$ with coefficients in $\mathbb{Z}$ are given by*

$$H_q\big(F_2(G);\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \text{if} \quad q=0, \\ \mathbb{Z}^r & \text{if} \quad q=1, \\ \mathbb{Z}^{\binom{r}{2}} & \text{if} \quad q=2, \\ 0 & \text{if} \quad q\geq 3. \end{cases}$$

where $r = 1 - |V(G)| + |E(G)|$.

Also we have the following consequence of Theorem 5.3.

Theorem 5.7. *Let G be a finite graph, then the Euler characteristic of $F_2(G)$ is*

$$\chi\big(F_2(G)\big) = \frac{r^2 - 3r + 2}{2}$$

where $r = 1 - |V(G)| + |E(G)|$.

Applying Theorem 4.4 and Theorem 5.3, we have the following.

Theorem 5.8. *Let G be a finite graph, let $r = 1 - |V(G)| + |E(G)|$. Then the Euler characteristic of $F_2(G)$ belongs to the set of the triangular numbers, if $r \geq 2$. For the case $r = 1$, $\chi\big(F_2(G)\big) = 0$.*

Finally, applying Theorem 4.7 and Theorem 5.3, we can state.

Theorem 5.9. *Let G be a finite graph. The cohomology groups of $F_2(G)$ with coefficients in $\mathbb{Z}$ are*

$$H^q\big(F_2(G);\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \text{if} \quad q=0, \\ \mathbb{Z}^r & \text{if} \quad q=1, \\ \mathbb{Z}^{\binom{r}{2}} & \text{if} \quad q=2, \\ 0 & \text{if} \quad q\geq 3. \end{cases}$$

where $r = 1 - |V(G)| + |E(G)|$.

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