Distance Measures in Graphs and Subgraphs

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To my family

Abstract

In this thesis we investigate how the modification of a graph affects various distance measures. The questions considered arise in the study of how the efficiency of communications networks is affected by the loss of links or nodes.

In a graph G, the distance between two vertices is the length of a shortest path between them. The eccentricity of a vertex v is the maximum distance from v to any vertex in G. The *radius* of G is the minimum eccentricity of a vertex, and the *diameter* of G is the maximum eccentricity of a vertex. The *distance* of G is defined as the sum of the distances between all unordered pairs of vertices.

We investigate, for each of the parameters radius, diameter and distance of a graph G, the effects on the parameter when a vertex or edge is removed or an edge is added, or G is replaced by a spanning tree in which the parameter is as low as possible. We find the maximum possible change in the parameter due to such modifications. In addition, we consider the cases where the removed vertex or edge is one for which the parameter is minimised after deletion.

We also investigate graphs which are critical with respect to the radius or diameter, in any of the following senses: the parameter increases when any edge is deleted, decreases when any edge is added, increases when any vertex is removed, or decreases when any vertex is removed.

Contents

Chapter 1

Introduction

The distance between vertices is one of the most thoroughly studied concepts in graph theory. In fact, Buckley and Harary [BH90] devoted an entire book to the subject. In this thesis we deal with the three related graphical parameters radius, diameter and distance (or equivalently, average distance) of a graph. Our underlying motivation is the application of these parameters to communications networks.

1.1 Radius, Diameter and Distance

The distance between two vertices is the length of a shortest path between them. The eccentricity of a vertex v in a graph G is the maximum distance from v to any vertex in G. The *radius* of G is the minimum eccentricity of a vertex in G, and the centre is the set of vertices whose eccentricity attains this minimum. The *diameter* of G is the maximum distance between any two vertices in G.

The *distance* of G is defined as the sum of the distances between all unordered pairs of vertices. The *average* or *mean distance* of G is the average distance between two vertices in G, where the average is taken over all unordered pairs of distinct vertices. The distance of a vertex v in G is the sum of the distances from v to all the other vertices in G.

The radius and diameter are probably too well-known to require any further explanation. The distance of a graph, however, is a far more recent concept, and we give a few remarks on its history.

The concept of the distance of a graph was introduced by the chemist Wiener in 1947 (see [Wie47]), and so is often called the Wiener index. Since then there have been numerous papers in chemistry and chemical graph theory dealing with applications of the average distance (see, e.g., [Ran79]). Here the vertices of a graph might represent carbon atoms in a molecule, and the edges represent the chemical bonds between them. The distance of a graph has been correlated with such properties of the associated chemical compound as boiling and melting points, refractive index, surface tension and viscosity.

The analogous concept for digraphs was first investigated by Harary [Har59] in a sociometric framework. Here the vertices of a graph represent people in an organisation, and an arc from u to v exists if v takes orders from u. Hence the out-distance of a vertex is a measure of the corresponding person's status in the organisation.

The concept of mean distance was introduced into graph theory in 1977 by Doyle and Graver [DG77] as a measure of the "compactness" of a graph, but it had been used in other disciplines before then. In architecture, for example, March and Steadman used mean distance as a tool for evaluating floor plans (see [MS71], Chapter 14). Here each room corresponds to a vertex, and two vertices are adjacent if it is possible to move directly between the corresponding rooms. Mean distance has also been used in modelling computer and communications networks. Since this application is of particular interest to us, we will deal with it in more detail below.

Subsequently, numerous papers on distance and average distance have appeared in graph theory journals. Ore [Ore62] suggested the ratio of the distance of a vertex to the order of the graph containing it as a measure of the "centrality" of the vertex. He suggested that this concept be studied in the case of trees, and this was subsequently done by Zelinka in [Zel68, Zel70]. The concept of the distance of a vertex was also studied in some detail by Sabidussi [Sab66]. In 1976 Entringer, Jackson and Snyder [EJS76] summarised and extended several previous results on distance and proved some new results. A survey of the literature before 1984 is given by Plesník [Ple84].

1.2 Communications Networks

The applications of radius, diameter and distance of most interest to us are those connected with communications networks. Such a network (for example, a city's road system, a computer network, or a telephone exchange) can be conveniently modelled by a graph, in which the vertices of the graph represent the nodes of the network, and the edges represent the links between them. For example, each vertex might represent a computer, two vertices being adjacent if there is a direct communication link between the corresponding computers. Or each vertex might represent a street intersection, and the edges the sections of the streets between intersections. There is a good survey by Caccetta [Cac89] on the use of graph theory in network design, although it does not include the applications of distance and radius.

Weights on the edges might represent the cost of using a particular link, or else the time delay encountered along that link, or even the signal degradation expected along it. Hence when a message is sent between two nodes of a network, the distance between the corresponding two vertices in the associated graph is proportional to the time delay, signal degradation or cost, depending on how the graph is weighted.

In cases where messages between all pairs of vertices are equally frequent, the average distance thus corresponds to the mean time delay of a message, or the average cost of a phone call. Mean distance is therefore a good measure of the performance or operating cost of a communications network which is heavily used.

In some cases, however, it might be more important to know the maximum possible time delay, signal degradation or cost of sending a message in the network; in other words, the performance of the network in the worst case. In this case the diameter might be a more suitable measure of network efficiency.

The radius can also be an important measure of network efficiency. For instance, one might have an emergency facility to locate, and be interested in minimising the maximum response time in an emergency. Or one might want to know the minimum power necessary for a transmitter which must reach all parts of the system. In both cases, one would choose a location corresponding to a central vertex.

On the other hand, one might want to place a transmitter where the *average* signal degradation or cost is minimised. Equivalently, one might have a depot from which deliveries are made to all other points in the system, or a computer storing data that is accessed by all other computers in the network. In this case, one would choose a location corresponding to a vertex of minimum distance in the associated graph.

There are other graph-theoretical parameters which arise naturally in the study of communications networks. For example, technical constraints like the number of line connections available at a node can be expressed as bounds on the maximum degree, while the vertex- and edge-connectivity of a graph (corresponding to the minimum number of nodes or links whose failure could disconnect the associated network) are measures of network reliability. Furthermore, the cost of constructing a network between a given set of nodes is often roughly proportional to the number of links, and hence to the number of edges in the corresponding graph.

Whether one chooses to use the radius, the diameter or the distance of a graph as a measure of the efficiency of the corresponding network depends on the circumstances. In any case, it is desirable for the chosen parameter to be low.

1.3 Network Vulnerability

When a node or link fails in a communications network, the network's efficiency may be affected. In this thesis we investigate the maximum possible damage resulting from such a failure. In other words, we consider the maximum possible increase in the radius, diameter or distance of a graph when a vertex or edge is removed. In the case of vertex removal, the efficiency of the remaining network might actually be increased; we therefore also investigate the maximum possible decrease in radius, diameter or distance when a vertex is removed.

We also consider the maximum possible damage when a *most expendable* node or link is removed; in other words, the maximum increase in the radius, diameter or distance of a graph caused by removing a "best" vertex or edge (one which minimises this increase). This might arise, for example, in a planned communications network which has too many links or nodes. In such cases a link or node may have to be removed, and we are interested in knowing to what extent the resulting damage can be limited if the best vertex or edge is chosen.

We emphasise graphs in which the radius or diameter is changed by the removal of *any* vertex, or by the removal of *any* edge, or by the addition of *any* edge. Graphs whose radius or diameter increases if any edge is removed model networks whose efficiency is decreased by the loss of any link.

A related problem is the maximum possible decrease in the radius, diameter or distance when an edge is added to a graph. This might be of interest, for example, when considering adding facilities to a network to upgrade its performance or reduce its operating costs. We consider in particular those graphs whose radius or diameter is increased by the addition of *any* edge.

If the cost of constructing a communications network connecting a given set of nodes is regarded as proportional to the number of links, then the cheapest system to construct between them is, of course, one modelled by a tree. Such a tree is unlikely to be the cheapest or most efficient network to operate, but it might be efficient enough to justify replacing a proposed network with it. Given a network modelled by a graph G, we investigate how the radius, diameter, and distance of G differ from those of a "best" spanning tree of G (i.e., one in which the parameter is minimised), and find ways of finding such a tree.

We consider unweighted graphs only. These model networks in which the time delay or signal degradation encountered by messages sent along a path is roughly proportional to the number of links in the path, or the cost of using each link for message purposes is the same.

1.4 Definitions and Notation

We list below our basic definitions and notation. All graphs considered are undirected, finite, without loops or multiple edges, and are unweighted unless otherwise specified. In what follows, let G be a graph, and let u and v be vertices of G.

- We denote the vertex set of G by $V(G)$, and the edge set by $E(G)$. The order of G is $n = |V(G)|$, and the size is $m = |E(G)|$. If $E(G) = \emptyset$, then G is said to be empty.
- The *degree* $deg_G(v)$ of v is the number of edges incident with it. The *minimum degree* of G is $\delta(G) := \min_{v \in V(G)} deg_G(v)$, and the *maximum degree* is $\Delta(G) := \max_{v \in V(G)} deg_G(v)$. The *neighbourhood* $N_G(v)$ of v is

the set of vertices adjacent to v in G. A vertex of degree 1 is called an *endvertex.* G is said to be *regular* if all vertices have the same degree in G.

- $S \subseteq V(G)$ means that S is a subset of $V(G)$, and $S \subset V(G)$ means that S is a *proper* subset of $V(G)$. $|S|$ denotes the cardinality of S, and $\langle S \rangle_G$ denotes the subgraph induced in G by S. For two subsets S and T of $V(G)$, $[S, T]_G$ denotes the set of all edges which join a vertex in S to a vertex in T. If $S = \{v\}$, then we write simply $[S, T]_G = [v, T]_G$.

- For a subset F of $E(G)$, $G - F$ is the graph obtained from G by deleting the edges in F; if $F = \{e\}$, then we write simply $G - F = G - e$. For a subset S of $V(G)$, $G - S$ is the graph obtained from G by deleting every vertex in S and all edges incident with it; if $S = \{v\}$, then we write $G - S = G - v$.

- A graph is *connected* if, for any two vertices u and v, there is a $u - v$ path in G. A *component* of G is a maximal connected subgraph of G. A set S of vertices is called a *cutset* if its deletion increases the number of components. A vertex v is called a *cut-vertex* if $\{v\}$ is a cutset, and a *non-cut vertex* or *ncv* otherwise. An edge is called a *bridge* if its deletion increases the number of components.

  Let S be a cutset of a connected graph G, and let G_1 be any component of $G - S$. Then $\langle V(G_1) \cup S \rangle_G$ is called an S-*component* of G, or a v-*component* of G if $S = \{v\}$.

- A *block* B of a graph G is a maximal connected subgraph of G which has no cut-vertices. Hence, for *any* cut-vertex v of G, $B - v$ lies entirely in one component of $G - v$.

  A vertex x is said to be *separated* from a vertex y by a vertex v if v lies on every $x - y$ path (i.e., if x and y are in different components of $G - v$).

A graph is said to be κ*-connected* if the deletion of any $\kappa - 1$ vertices yields a nontrivial connected graph. (So the complete graph K_2 is not 2-connected, although it is a block.)

- The *length* of a walk W is the number of edges in W, and is denoted by $\ell(W)$. The *distance* $d_G(u, v)$ between u and v is the length of a shortest $u - v$ path in G. If u and v are in different components of G, then $d_G(u, v) = \infty$. The *eccentricity* of v is $e_G(v) := \max\{d_G(v, u) \mid u \in V(G)\}$.

  If u is a vertex such that $d_G(u, v) = e_G(v)$, then u is called an *eccentric vertex* of v in G. If there is only one such vertex u, then u is called the *unique eccentric point* or *uep* of v.

- The *radius* and *diameter* of G are defined by $rad(G) := \min\{e_G(v) \mid v \in V(G)\}$ and $diam(G) := \max\{e_G(v) \mid v \in V(G)\}$.

  A vertex v of G is called *central* if $e_G(v) = rad(G)$, and *peripheral* if $e_G(v) = diam(G)$. The *centre* $C(G)$ is the set of all central vertices in G, and the *periphery* $P(G)$ is the set of all peripheral vertices. A pair of vertices at distance $diam(G)$ from each other in G is called a diametral pair. G is said to be *self-centred* if all vertices have the same eccentricity in G.

- The *distance of* v in G is defined as

$$\sigma_G(v) := \sum_{u \in V(G)} d_G(v, u),$$

  and the *distance of* G as

$$\begin{aligned} \sigma(G) &:= \sum_{\{u,v\} \subseteq V(G)} d_G(v, u) \\ &= \frac{1}{2} \sum_{v \in V(G)} \sigma_G(v). \end{aligned}$$

If G is not connected, then $\sigma_G(v) = \infty$ and $\sigma(G) = \infty$.

The *average* or *mean distance* $\mu(G)$ of a graph G of order $n \geq 2$ is defined as

$$\mu(G) := \frac{1}{\binom{n}{2}} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{\sigma(G)}{\binom{n}{2}}.$$

Hence $\mu(G) = \infty$ if G is disconnected. It is convenient to define $\mu(K_1) := 1$.

- The *i-neighbourhood* or *ith distance layer* $N_i(v)$ of v is the set of vertices at distance i from v. (So $N_1(v) = N(v)$.)

- The *distance degree sequence* of v is the sequence of cardinalities of the distance layers of v; i.e., $\{1, |N_1(v)|, |N_2(v)|, ..., |N_{e(v)}|\}$. A graph G is called *distance degree regular* if every vertex of G has the same distance degree sequence.

- An edge is said to be *cyclic* if there is a cycle in G containing it. The *girth* $g(e)$ of a cyclic edge e is defined as the length of a shortest cycle containing e; we set $g(e) = \infty$ if e is a bridge. The girth $g(G)$ of G is the length of a shortest cycle in G; we set $g(G) = \infty$ if G is a forest.

- $s_i(G)$ is the number of unordered pairs of vertices distance i apart in a graph G. $W_i(G)$ is the set of all unordered pairs of non-adjacent vertices of G which are at most distance i apart; i.e., $W_i(G) = \{\{u, v\} \subseteq V(G) \mid 2 \leq d_G(u, v) \leq i\}$.

  Hence, if G has order n and diameter d, $s_1(G) = |E(G)|$, $|W_1(G)| = 0$, and $|W_d(G)| = \binom{n}{2} - |E(G)|$. Furthermore, for $i \in \{2, ...d\}$, $|W_i(G)| = |W_{i-1}(G)| + s_i$.

- We denote the complete k-partite graph with partite sets of cardinality $n_1, ..., n_k$ by $K_{n_1,n_2,...,n_k}$. We refer to the particular case $K_{1,n-1}$ as a *star*.

- $G \cong H$ means that the graphs G and H are isomorphic to each other.

- The complement of G is denoted by $\bar{G}$.

- The complete graph, path and cycle of order n are denoted by K_n, P_n, and C_n, respectively. C_3 is sometimes called a *triangle.* Note that $rad(K_1) = diam(K_1) = \sigma(K_1) = 0$.

- Let T be a tree, and v a vertex in T. Then a maximal subtree of T containing v as an endvertex (i.e., a v-component of T) is called a *branch* of T at v. (So the number of branches at v is $deg_T(v)$.)

- A subgraph H of G is said to be *distance-preserving from* v in G if $d_H(v, u) = d_G(v, u)$ for all $u \in V(H)$. If a spanning tree of G is distance-preserving from a vertex v in G, then we will usually denote it by T_v. A vertex w in T_v is said to be *descended* from a vertex u in T_v if u lies on the $v - w$ path in T_v. (So w is descended from itself.)

- Two vertices v and u of a cycle C are said to be *opposite* each other on C if they are eccentric vertices of each other in C. A vertex v and an edge $e = ab$ of a cycle C are said to be opposite each other on C if v maximises $d_C(v, a) + d_C(v, b)$ over all vertices in C.

  Hence in an even cycle, every vertex is opposite exactly one vertex and the two edges incident with it, while every edge is opposite exactly two vertices. In an odd cycle, every vertex is opposite two adjacent vertices and the edge between them, while every edge is opposite exactly one vertex.

- The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

- Their *join* $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex in $V(G_1)$ to every vertex in $V(G_2)$. If $G_1 \cong K_1$, with $V(G_1) = \{v\}$, then we write $G_1 + G_2$ simply as $v + G_2$.

For three or more disjoint graphs $G_1, G_2, ..., G_k$, the *sequential join* $G_1 + G_2 + ... + G_k$ is the graph

$$(G_1 + G_2) \cup (G_2 + G_3) \cup ... \cup (G_{k-1} + G_k);$$

in other words, $G_1 + G_2 + ... + G_k$ is obtained from $G_1 \cup G_2 \cup ... \cup G_k$ by adding all edges uv such that $u \in V(G_i)$ and $v \in V(G_{i+1})$, for $i = 1, ..., k-1$.

- The *cartesian product* $G_1 \times G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$, in which two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or else $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

  For $r \in \mathbf{N}$, the *r-cube* Q_r is obtained by taking the cartesian product of r copies of K_2. (We regard the set of natural numbers $\mathbf{N}$ to be $\{1, 2, 3, ...\}$.)

- For $k \in \mathbf{N}$, the *kth power* G^k of a graph G is the graph with vertex set $V(G)$ in which two vertices u and v are adjacent whenever u and v are at most distance k apart in G.

- For any connected graph G, the graph with n components, each isomorphic to G, is denoted by nG.

- A graph is called *path-complete* if it can be obtained by joining one endvertex of a (possibly trivial) path to at least one vertex of a complete graph (see, for example, figure 1.1). We prove later that, for any $n \in \mathbf{N}$ and $m \in \{n-1, ..., \binom{n}{2}\}$, there exists a unique path-complete graph of order n and size m, which we denote by $PK_{n,m}$.

  Hence the path-complete graphs $PK_{n,m}$ have paths and complete graphs as the extreme cases $m = n-1$ and $m = \binom{n}{2}$. For convenience we define $PK_{1,0} \cong K_1$. Note that a path-complete graph can also be expressed as the sequential join of complete graphs.

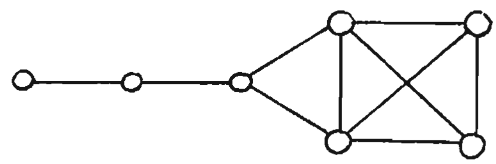


Figure 1.1: The path-complete graph $PK_{7,10}$

If the graph G in question is clear, we may omit the subscript G. Other definitions will be given as needed throughout the chapters. All concepts not defined here will be used in the sense of [CL86].

1.5 Fundamental Results

In this section we provide results which we shall need in subsequent chapters. Most of these can be found in [BH90] or [CL86]. We include proofs of only those results which are neither well-known nor trivial.

The following table summarises the radius, diameter, distance and average distance of some frequently occurring graphs.

	radius	diameter	distance	mean distance
complete graph K_n, $n \geq 2$	1	1	$\binom{n}{2}$	1
path P_n	$\lfloor\frac{1}{2}n\rfloor$	$n-1$	$\frac{1}{6}n(n-1)(n+1)$	$\frac{1}{3}(n+1)$
cycle C_n, n odd	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-1)$	$\frac{1}{8}n(n-1)(n+1)$	$\frac{1}{4}(n+1)$
cycle C_n, n even	$\frac{1}{2}n$	$\frac{1}{2}n$	$\frac{1}{8}n^3$	$\frac{n^2}{4(n-1)}$

It is easy to prove that, for a connected graph G,

$$rad(G) \leq diam(G) \leq 2rad(G).$$

Ostrand showed in [Ost73] that in fact this is the only restriction on the diameter in terms of the radius; in other words, for every two natural numbers r and d with $r \leq d \leq 2r$, there exists a graph with radius r and diameter d.

Furthermore, Plesník showed in [Ple84] that, apart from the obvious restriction

$$1 \leq \mu(G) \leq diam(G),$$

the average distance of a graph G is essentially independent of its radius and diameter (if the order of G is permitted to be arbitrarily large). Specifically, he showed that, for any integers r and d and real number t with $1 \leq r \leq d \leq 2r$ and $1 \leq t \leq d$, and any positive real number ϵ, there exists a graph G with $rad(G) = r$, $diam(G) = d$ and $|\mu(G) - t| < \epsilon$.

Proposition 1.5.1 *Every connected non-trivial graph contains at least two non-cut vertices, and the only graphs containing exactly two non-cut vertices are paths.*

Proposition 1.5.2 *For any connected graph G, the centre $C(G)$ is contained in one block of G.*

The following classic result is due to Whitney, although it is also commonly referred to as Menger's Theorem.

Theorem 1.5.3 *For $\kappa \in \mathbb{N}$, a nontrivial graph G is κ-connected iff every pair of vertices in G is connected by at least κ internally-disjoint paths.*

The next three results deal with trees.

Proposition 1.5.4 *Let T be a tree of radius r. Then either $diam(T) = 2r$ and $C(T)$ contains exactly one vertex, or $diam(T) = 2r-1$ and $C(T)$ consists of two adjacent vertices.*

Proposition 1.5.5 *In a tree, no vertex can be equidistant from two adjacent vertices.*

Proposition 1.5.6 *Let G be any connected graph, and v any vertex in G. Then G contains a spanning tree which is distance-preserving from v.*

Such a tree can be found using the breadth-first-search algorithm with v as root (see, for example, [BH90]). We will usually denote it by T_v.

The next group of results deals with the connection between cycles and distance.

Proposition 1.5.7 *An edge is a bridge iff it is not cyclic.*

Proposition 1.5.8 [EJS76] *Let e be a cyclic edge of a graph G, and let C be a shortest cycle containing e. Then for any vertices u and v on C, $d_G(u, v) = d_C(u, v)$.*

Proposition 1.5.9 *Let e be a cyclic edge of a connected graph G, and let C be a shortest cycle containing e in G. Let v be a vertex opposite e on C. Then removing e from G does not affect the distance from v to any vertex in $V(G)$.*

Proposition 1.5.10 *For any cyclic edge e of a graph G,*

$$g(e) \leq 2diam(G) + 1.$$

Proof:
Let C be a shortest cycle containing e. By Proposition 1.5.8, any two vertices opposite each other on C are distance $\lfloor \frac{1}{2} g(e) \rfloor$ apart in G. It follows that $diam(G) \geq \lfloor \frac{1}{2} g(e) \rfloor$, and hence that $g(e) \leq 2diam(G) + 1$. □

In particular, for any cyclic graph G with girth $g(G)$, we have $g(G) \leq 2diam(G) + 1$.

Proposition 1.5.11 *Let G be a connected graph for which*

$$g(G) = 2diam(G) + 1.$$

Then G is 2-edge connected.

Proof:
Assume, to the contrary, that G has a bridge $e = ab$, and let G_1 and G_2 be the components of $G - e$ containing a and b respectively. Let $diam(G) = d$. Let T be a spanning tree of G which is distance-preserving from a, and note that T is also distance-preserving from b.

Since G is cyclic, $E(G) - E(T)$ is non-empty; let uv be any edge in $E(G) - E(T)$. Assume without loss of generality that $uv \in E(G_1)$. Since u and v are within distance d from b, they must both be within distance $d - 1$ from a. Hence the $a - u$ and $v - a$ paths in T, together with the edge uv, form a closed walk of length at most $2(d - 1) + 1$ in G. But then G contains a cycle of length at most $2d - 1 < g(G)$, which is impossible. It follows that G is 2-edge-connected. □

Our next proposition shows that the path-complete graph $PK_{n,m}$ is unique.

Proposition 1.5.12 *For $n \in \mathbf{N}$ and $m \in \{n-1, ..., \binom{n}{2}\}$, there is exactly one graph $PK_{n,m}$ up to isomorphism.*

Proof:
Note that $PK_{n,m}$ can be obtained from the path $P : v_1, ..., v_n$ by adding $m-(n-1)$ edges in the following way: express the edges in $E(PK_{n,m})-E(P)$ in the form (v_i, v_j) with $i < j$, and order them in such a way that (v_i, v_j) precedes (v_s, v_t) if either $j < t$, or $j = t$ and $i < s$. Then add the first $m-(n-1)$ edges in this ordering to P. It follows that $PK_{n,m}$ is unique. □

Note that for any $n \in \mathbf{N}$ and $m \in \{n-1, ..., \binom{n}{2} - 1\}$, $PK_{n,m+1}$ can be obtained from $PK_{n,m}$ by adding an edge.

Recall that the ith distance layer of a vertex v in a graph G is $N_i(v) := \{u \in V(G) \mid d_G(v,u) = i\}$. We conclude this section by noting some basic properties of distance layers which we shall use frequently.

Let G be a connected graph of diameter d, and let v_0 be any peripheral vertex of G. Then $V(G) = \bigcup_{i=0}^{d} N_i(v_0)$ is a partition of the vertex set into $d+1$ non-empty sets $N_0(v_0), N_1(v_0), ..., N_d(v_0)$, where $N_0(v_0) = \{v_0\}$. Every vertex in $N_i(v_0)$ is adjacent to at least one vertex in $N_{i-1}(v_0)$. Furthermore, no vertex in $N_i(v_0)$ is adjacent to any vertex in $N_j(v_0)$ for $|j-i| \geq 2$.

It follows easily that for any connected graph G of order n, diameter d and maximum degree Δ,

$$\Delta \leq n - d + 1.$$

1.6 Content

In Chapter 2 we consider the radius, in Chapter 3 the diameter, and in Chapter 4 the distance of a graph. In each case we investigate the maximum possible increase or decrease in the parameter when certain changes are made to a graph. In particular, we study the removal of a (best or worst) vertex or edge, the addition of a best edge, and the replacement of the graph by a best spanning tree. In the cases of radius and diameter we also consider graphs for which the parameter is changed by the removal of any vertex, the removal of any edge, or the addition of any edge.

Although we are motivated by the application of radius, diameter and distance to communications networks, we shall usually give our results in graph theoretic terms, without reference to communications networks. We shall, however, continue to use the terms "best" and "worst" to describe the cases where the parameter under discussion is as low (respectively, high) as possible.

Chapter 2

Radius

The radius of a graph G is the minimum eccentricity of a vertex in G. In this chapter we consider how the radius is affected by the removal of an edge or a vertex, the addition of an edge, or the taking of a spanning tree.

2.1 Spanning Trees

In this section we show that every connected graph has a spanning tree of the same radius.

Definition: A spanning tree T of a graph G is said to be *radius-preserving* if $rad(T) = rad(G)$.

Proposition 2.1.1 *If T is a radius-preserving spanning tree of a graph G then $C(T) \subseteq C(G)$.*

Proof:
Let c be any central vertex of T. Since removing edges cannot decrease the eccentricity of any vertex, $e_G(c) \leq e_T(c) = rad(T) = rad(G)$. It follows that $e_G(c) = rad(G)$; i.e., that $c \in C(G)$. □

Note that if a spanning tree T of a graph G is *not* radius-preserving, then $C(T)$ is not necessarily contained in $C(G)$.

Proposition 2.1.2 *Let c be any central vertex of a connected graph G, and let T_c be a spanning tree of G which is distance-preserving from c. Then $c \in C(T_c)$, and $rad(T_c) = rad(G)$.*

Proof:
Since T_c is distance-preserving from c, $rad(T_c) \leq e_{T_c}(c) = e_G(c) = rad(G)$. Since removing edges cannot decrease the eccentricity of any vertex, it follows that $rad(T_c) = rad(G)$ and that $c \in C(T_c)$. □

Not all radius-preserving spanning trees, however, are distance-preserving from some vertex, as shown by the graph in figure 2.1.

Proposition 2.1.2 has another useful consequence:

Proposition 2.1.3 *For any connected graph G of order n and radius r,*

$$r \leq \left\lfloor \frac{1}{2} n \right\rfloor.$$

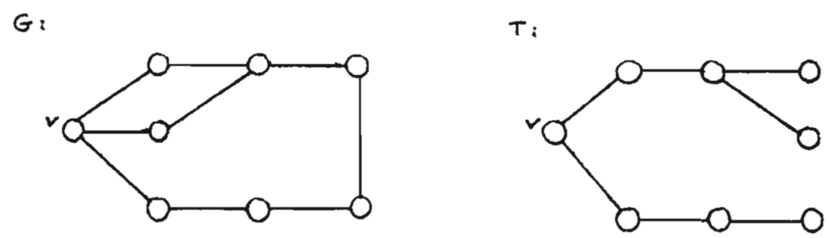


Figure 2.1: T is a radius-preserving spanning tree of G which is not distance-preserving from any vertex.

Proof:

Let c be any central vertex of G, and let T_c be a spanning tree of G which is distance-preserving from c. By Proposition 2.1.2, $rad(T_c) = r$, and hence $diam(T_c) = 2r$ or $2r - 1$. Now let P be any diametral path of T_c, and note that P has $diam(T_c) + 1 \geq 2r$ vertices. It follows that $n \geq 2r$, and hence that $r \leq \lfloor \frac{1}{2} n \rfloor$. $\square$

It is tedious but not difficult to show that equality holds iff

(1) G is a path or cycle, or

(2) n is odd and G consists of a path or cycle of order $2r$, a vertex w, and one, two or three edges joining w to vertices which are at most distance 2 apart in $G - w$.

2.2 Vertex Removal

Removing a vertex from a graph can increase, maintain or decrease the radius. Our first proposition, however, shows that it cannot decrease the radius by more than 1, and also characterises the vertices in a graph whose removal decreases the radius. Recall that ncv stands for non-cut vertex, and uep for unique eccentric point.

Proposition 2.2.1 [Gli75c, Faj88] *Let G be a graph containing a vertex v. Then $rad(G-v) < rad(G)$ iff v is the uep of some central vertex, and in this case $rad(G-v) = rad(G) - 1$.*

Proof:
Let $rad(G-v) < rad(G)$, and let c be any central vertex of $G-v$. So $e_{G-v}(c) = rad(G-v) \leq rad(G) - 1 \leq e_G(c) - 1$. Since removing v cannot decrease the distance between any of the remaining vertices, it follows that v is the uep of c in G. Furthermore, since c is still at distance $e_G(c) - 1$ from the neighbours of v, $rad(G-v) = e_{G-v}(c) \geq e_G(c) - 1 \geq rad(G) - 1$. It follows that $c \in C(G)$ and that $rad(G-v) = rad(G) - 1$.

Conversely, let v be the uep of some central vertex c in G. Then removing v cannot increase the distance between c and any other vertex w (since v cannot lie on a shortest $c-w$ path). It follows that $e_{G-v}(c) < e_G(c)$, and hence that $rad(G-v) < rad(G)$. □

Remark: If $rad(G-v) = rad(G) - 1$ for a vertex v in a graph G, then

$$C(G-v) = \{c \in C(G) \mid v \text{ is the uep of } c \text{ in } G\} \subset C(G).$$

(To prove that $C(G-v) \neq C(G)$: let c be any central vertex of $G-v$, and let c' be the neighbour of c on any shortest $c-v$ path in G. Since every vertex in $V(G) - \{v\}$ is within a distance of $rad(G) - 1$ from c in G, c' is central in G. But v is not an eccentric vertex of c' in G, and hence c' is not central in $G-v$.)

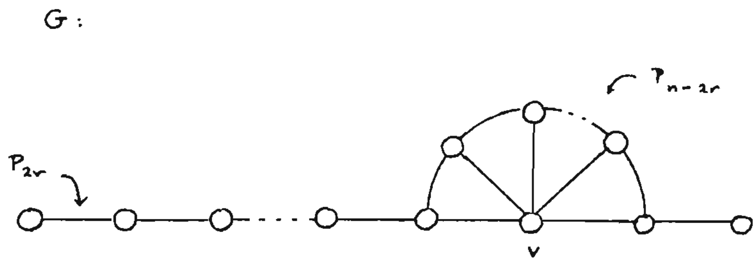


Figure 2.2: $\frac{rad(G-v)}{rad(G)} \to \infty$ as $n \to \infty$

In Section 2.3 we will prove that $rad(G-e) \leq 2rad(G)$ for every cyclic edge e in a connected graph G. The following example, however, shows that one cannot find a similar bound for the removal of ncv's. For given natural numbers r and $n \geq 2r+1$, let G be the graph obtained from two paths $P \cong P_{2r}$ and $P' \cong P_{n-2r}$ by joining some internal vertex v of P to every vertex on P', and joining the neighbours of v on P to the endpoints of P' (see figure 2.2). Then $\frac{rad(G-v)}{rad(G)} = \frac{\lfloor \frac{1}{2}(n-1) \rfloor}{r} \to \infty$ as $n \to \infty$.

In fact, since paths have the maximum possible radius for connected graphs of given order, it follows that the above G and v maximise $\frac{rad(G-v)}{rad(G)}$ over all connected graphs of order n and radius r and all ncv's in these graphs.

Definitions: A *conjugate vertex* v^* of a vertex v is a central vertex with v as its uep. (So a vertex might have more than one conjugate vertex, or none.) A *conjugate pair* is a pair of central vertices, each of which is the uep of the other.

We have the following result:

Proposition 2.2.2 [Faj88] *Let $\{v, v^*\}$ be any conjugate pair in a graph $G \ncong K_2$. Then removing v and v^* from G cannot decrease the radius.*

Proof:
Let c be a central vertex of $G - \{v, v^*\}$, and let w be an eccentric vertex of c in G. Then $d_G(c, w) \geq rad(G)$. Since v and v^* are within distance $rad(G) - 1$ from all vertices in G except each other, w cannot be v or v^*. Since removing v and v^* cannot decrease the distance between c and w, it follows that $e_{G-\{v,v^*\}}(c) \geq d_G(c, w)$, and hence that $rad(G - \{v, v^*\}) \geq rad(G)$. □

2.2.1 Vertex-radius-decreasing graphs

Definition: We define a non-trivial graph G to be *vertex-radius-decreasing* or *vrd* if $rad(G - v) < rad(G)$ for every ncv v of G.

For example, even paths and cycles are vrd, but odd paths and cycles are not. The only disconnected vrd graph is $\bar{K}_2$.

The odd cycles are examples of graphs for which the removal of any vertex leaves the radius unchanged. The following proposition shows that there is no purpose in defining a vertex-radius-*increasing* graph.

Proposition 2.2.3 [Gli75c] *There is no graph G such that $rad(G - v) > rad(G)$ for every vertex v in G.*

Proof:
Let G be any graph, c any central vertex of G and v any eccentric vertex of c. Then v cannot lie on a shortest path between c and another vertex, and hence $e_{G-v}(c) \leq e_G(c)$. It follows that $rad(G - v) \leq rad(G)$. □

The following property of vrd graphs will be useful. It is based on ideas developed by Fajtlowicz in [Faj88].

Lemma 2.2.4 *Let G be a vertex-radius-decreasing graph, and v a ncv of G. If v is not central, then all its conjugate vertices are cut-vertices. If v is central, then it has exactly one conjugate vertex v^*, and v^* is a ncv (so v and v^* form a conjugate pair).*

Proof:
By Proposition 2.2.1, v has a conjugate vertex v^*. If v^* is also a ncv of G, then, since G is vertex-radius-decreasing, v^* must be the uep of some central vertex v^{**}. Hence $d_G(v^*, v^{**}) = rad(G)$ — but the only vertex at distance rad(G) from v^* is v. It follows that v must be central, and v^{**} must be v.

This proves firstly that, if v is not central, then all its conjugate vertices are cut-vertices, and secondly that no vertex v can have two conjugate vertices which are ncv's. (Otherwise both would need to have v as a conjugate vertex; i.e., both would need to be the *unique* eccentric point of v.)

If v^* is a cut-vertex, let w be any vertex separated from v by v^*. Then $e_G(v) \geq d_G(v, w) = d_G(v, v^*) + d_G(v^*, w) \geq rad(G) + 1$; i.e., v is non-central. It follows that if v is central, then it has a unique conjugate vertex v^*, and v^* is a ncv. □

(In fact, it follows from one of our later results, Theorem 2.2.10, that *every* ncv v in a vrd graph G has a unique conjugate vertex v^*, and hence that $|C(G - v)| = 1$.)

As a direct consequence of Proposition 2.2.1 and Lemma 2.2.4, we have the following characterisation of 2-connected vertex-radius-decreasing graphs:

Proposition 2.2.5 [Gli75c, Faj88] *A graph G of order n is a vertex-radius-decreasing block iff G is self-centred, n is even, and $V(G)$ can be partitioned into conjugate pairs.*

Graphs in which every vertex has exactly one eccentric vertex are called *unique eccentric point graphs*, and were studied by Parthasarathy and Nandakumar in [PN83]. It follows from Proposition 2.2.5 that the vrd blocks are precisely the self-centred unique eccentric point graphs. Mulder also refers to them as diametrical graphs in [Mul80]. We have the following existence theorem for such graphs:

Proposition 2.2.6 [Gli75c] *If G and H are vertex-radius-decreasing blocks, then the cartesian product $G \times H$ is a vertex-radius-decreasing block of radius $rad(G) + rad(H)$.*

Proof:
For any two vertices (u, x) and (v, y) in $G \times H$,

$$d_{G\times H}((u,x),(v,y)) = d_G(u,v) + d_H(x,y).$$

Hence, for any vertex (u, x) in $G \times H$, $e_{G\times H}((u,x)) = e_G(u) + e_H(x)$. It follows that $rad(G \times H) = rad(G) + rad(H)$, and that $(u,x)^* = (u^*, x^*)$. Hence, by Proposition 2.2.5, $G \times H$ is also a vrd block. □

For example, since K_2 is a vrd block, Proposition 2.2.6 shows that, for any $r \in \mathbf{N}$, the r-cube (which is isomorphic to the cartesian product of r copies of K_2) must be a vrd block of radius r.

In addition, Gliviak gave a construction in [Gli75c] to prove that, for every graph G and $r \geq 3$, there exists a vrd block of radius r which contains G as an induced subgraph. This shows that the class of vrd blocks is very large.

We mention two more properties of vrd graphs:

Proposition 2.2.7 [Gli75c] *If u and v are any two vertices in a vertex-radius-decreasing block G of radius at least 3, then u must have at least one neighbour that is not v or adjacent to v.*

Proof:
If $u = v^*$, then $d_G(u, v) = r \geq 3$, and u and v can have no neighbours in common.

Otherwise, let w be the neighbour of u on a shortest $u - v^*$ path in G. So $d_G(w, v^*) = d_G(u, v^*) - 1 \leq rad(G) - 2$. Since $d_G(v, v^*) = rad(G)$, it follows that w cannot be v or adjacent to v. □

Proposition 2.2.8 *If G is a vertex-radius-decreasing block of order n and radius $r \geq 3$, then $\Delta(G) \leq \frac{1}{2}(n - r + 1)$.*

Proof:
Let v be a vertex of maximum degree in G, and let u be any neighbour of v. Since $d_G(u, w) \leq r - 1$ for any $w \in \{N_0(v), ..., N_{r-2}(v)\}$ and $N_r(v) = \{v^*\} \neq \{u^*\}$, it follows that $u^* \in N_{r-1}(v)$. Hence $|N_{r-1}(v)| \geq |N_1(v)|$.

Now let P be a shortest v–v^* path, and note that P contains $r+1$ vertices, exactly two of which are in $N_1(v) \cup N_{r-1}(v)$. Therefore

$$\begin{aligned} n &\geq (r+1) + |N_1(v)| + |N_{r-1}(v)| - 2 \\ &\geq 2\Delta(G) + r - 1, \end{aligned}$$

and the result follows. □

For $k \in \mathbf{N}$, the graph C^k_{4k+2} is a vrd block of radius 3 in which every vertex has degree $2k = \frac{1}{2}((4k+2) - 3 + 1)$. This shows that the above bound cannot be improved for $r = 3$. It is easy to prove, however, that the bound is not sharp if $r \geq 4$.

Remark: In [Gli75c], Gliviak states that if u and v are any two adjacent vertices in a vrd block G, then u^* and v^* are also adjacent in G, and uses this to prove several results. In fact, the statement is untrue, as shown by the vrd block G in figure 2.3. We note that u and v are adjacent in G, whereas u^* and v^* are not. In fact, v and v^* do not even have the same degree.

We now consider the structure of vrd graphs with cut-vertices. The following proposition was essentially proved by Fajtlowicz in [Faj88], although he did not use the concept of vrd graphs.

Lemma 2.2.9 [Faj88] *In any vertex-radius-decreasing graph containing at least one cut-vertex, every ncv has degree 1.*

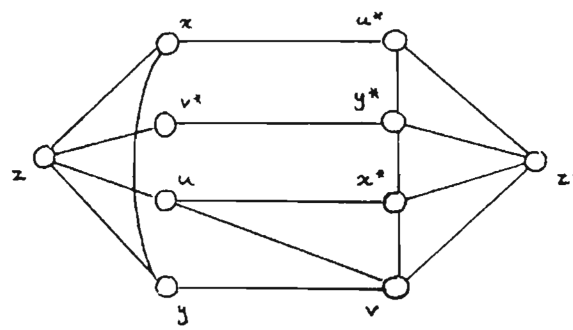


Figure 2.3: A counter-example to a statement of Gliviak

Proof:
Let G be a vrd graph containing a ncv v of degree at least 2, and let x and y be any neighbours of v. We will prove that then G has no cut-vertices.

By Proposition 2.2.1, v has a conjugate vertex v^* such that $d_G(v^*, v) = rad(G)$ and $d_G(v^*, u) \leq rad(G) - 1$ for every $u \in V(G) - \{v\}$. Hence $d_G(v^*, x) = rad(G) - 1$.

It follows that, if u is any vertex in $V(G) - \{v, x\}$, then no shortest $v^* - u$ path can contain x. In particular, $G - x$ contains a $v^* - y$ path and hence a $v^* - v$ path. So every two vertices in $G - x$ are connected to v^*, and hence to each other.

It follows that no neighbour of v is a cut-vertex. Since we also know that every neighbour of v has degree at least 2 (otherwise v would have been a cut-vertex), it follows in the same way that no vertex in $N_2(v)$ is a cut-vertex, and so on. Hence G contains no cut-vertices. □

Definition: We define the *ℓ-corona* $S_\ell(G)$ of a graph G as the graph obtained from G by attaching to each vertex v of G a path of length ℓ with v as endpoint.

(See, for example, figure 2.4.)

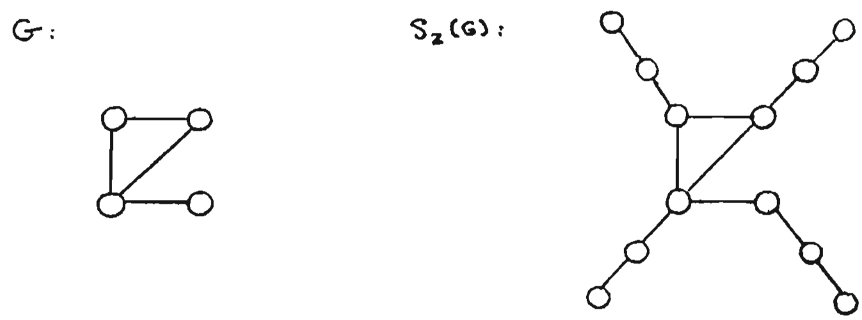


Figure 2.4: A graph and its 2-corona

Our next theorem reduces the study of vrd graphs to that of vrd blocks. It is in fact a consequence of a theorem of Gliviak [Gli76a], but we present our own proof.

Theorem 2.2.10 *G is a vertex-radius-decreasing graph containing a cut-vertex iff $G = S_\ell(B)$ for some $\ell \in \mathrm{N}$ and some vrd block B. Moreover, $\langle C(G)\rangle_G = B$.*

Proof:
Let G be a vertex-radius-decreasing graph containing a cut-vertex v.

(1) $C(G)$ is contained in one block of G and hence in one v-component, G_1. Let $G_2 = G - V(G_1 - v)$. G_2 must contain at least two ncv's; let u be any ncv of G_2 different from v.

Since u is also a ncv of G, by Proposition 2.2.1 it must be the uep of some $c \in C(G)$. Since $c \in V(G_1)$, this means that u must be the *unique* eccentric point of v in G_2. It follows that G_2 must have exactly two ncv's, v and u, and must therefore be a path.

(2) Now let B be the block of G containing $C(G)$, and suppose B contains an endvertex u of G. Then B cannot be a cyclic block, and must be a

K_2. Since u is not central in G, $|C(G)| \leq |V(B)| - 1 = 1$. But G must have at least two ncv's, each of which (by Proposition 2.2.1) must be the uep of some central vertex of G; hence $|C(G)| \geq 2$, and we have a contradiction. So every vertex of B has degree at least 2 in G, and is therefore, by Lemma 2.2.9, a cut-vertex of G. It follows by (1) that G can be obtained from B by attaching to each vertex of B a path with v as endpoint.

(3) We now show that each of these paths attached to vertices in B has the same length, which we shall call ℓ. Assume to the contrary that there exist two paths P_1 and P_2, attached to neighbouring vertices v_1 and v_2 in B, such that P_2 is longer than P_1. Let the other endpoints of P_1 and P_2 be u_1 and u_2 respectively.

Every end-vertex of G is a ncv and hence, by Proposition 2.2.1, the uep of some central vertex; say u_1 is the uep of the central vertex c. It follows that $d(c, u_2) \leq d(c, u_1) - 1$, and hence that $d(c, v_2) + \ell(P_2) \leq d(c, v_1) + \ell(P_1) - 1 \leq d(c, v_1) + \ell(P_2) - 2$. But then $d(c, v_2) \leq d(c, v_1) - 2$, which is impossible since v_1 and v_2 are adjacent. It follows that each path has the same length ℓ, and hence that $G = S_\ell(B)$.

(4) Since G contains $|V(B)|$ end-vertices, each of which is a ncv and must therefore be the uep of some central vertex, $|V(B)| \leq |C(G)|$. Since $C(G)$ is contained in B, it follows that $C(G) = V(B)$.

(5) Finally, we prove that B is vertex-radius-decreasing. Let v be any vertex in B, and let u be the other endpoint of the path attached to v. Note that $e_G(x) = e_B(x) + \ell$ for every $x \in V(B)$, and hence that $C(B) = C(G)$.

It follows that u is the uep of some $c \in C(G) = C(B)$ iff v is the uep of c in B. Hence by Proposition 2.2.1 the fact that G is vrd implies that B is also vrd. This completes the first part of the proof.

For the converse, let B be any vrd block, ℓ any natural number, and $G = S_\ell(B)$.

By Proposition 2.2.5, B is self-centred and $V(B)$ is partitioned into conjugate pairs. Since all paths attached to vertices in B have the same length, for every conjugate pair $\{u, u^*\}$ in B the end-vertex u' of the path attached to u^* is the uep of u in G. Hence for every $u \in V(B)$, $e_G(u) = d_G(u, u^*) + d_G(u^*, u') = rad(B) + \ell$.

Now let w be any vertex in $V(G) - V(B)$; say w is in the path attached to $u \in V(B)$. Then $e_G(w) \geq d_G(w, u') = d_G(w, u) + d_G(u, u') \geq 1 + rad(B) + \ell$. It follows that $V(B) = C(G)$.

Finally, let x be any ncv of G. Then x is an endpoint of the path attached to some vertex $y \in V(B)$. Let y^* be the conjugate vertex of y in B, and note that x is the uep of y^* in G. Since $y^* \in C(G)$, it follows by Proposition 2.2.1 that $rad(G - v) < rad(G)$, and hence that G is vertex-radius-decreasing. □

It follows from the above theorem that all vrd graphs are unique eccentric point graphs.

The only vrd graph of radius 1 is K_2. We can now characterise all vrd graphs of radius 2:

Proposition 2.2.11 [Gli75c] *Let G be a vertex-radius-decreasing graph of order n and radius 2. Then n is even and G is either P_4 or a complete $(\frac{1}{2}n)$-partite graph $K_{2,2,\ldots,2}$.*

Proof:
If G is 2-connected, then every vertex v in G has a conjugate vertex v^* at distance 2 from it, and is adjacent to every other vertex in G. It follows that G is a complete $(\frac{1}{2}n)$-partite graph, each partite set consisting of a conjugate pair.

If G has a cut-vertex, then by Theorem 2.2.10 G can be obtained from a vrd block B of radius 1 by attaching a path of length 1 to every vertex in B.

Since B must be K_2, it follows that G must be P_4. □

Remark: Gliviak [Gli75c] defined a graph G to be v-critical if $rad(G - v) \neq rad(G)$ for every vertex v in G. The class of v-critical graphs might appear to be larger than that of vrd graphs, but in fact Gliviak proved in [Gli76a] that they are the same. In other words, he proved that in a v-critical graph any vertex whose removal increases the radius must be a cut-vertex.

2.2.2 Radius-critical graphs

We now consider a special class of vertex-radius-decreasing graphs, defined and characterised by Fajtlowicz in [Faj88].

Definition: A nontrivial graph G is *r-radius-critical* (or, briefly, *radius-critical*) if $rad(G - S) < rad(G) = r$ for every non-empty proper subset S of $V(G)$ such that $G - S$ is connected.

For example, any even cycle C_{2r} is r-radius-critical, since every proper induced connected subgraph of C_{2r} is a path P_k with $k \leq 2r - 1$. Odd cycles, on the other hand, are not radius-critical, since C_{2r+1} contains P_{2r} as an induced subgraph.

Every graph that is radius-critical is also vertex-radius-decreasing, but the converse is not true. For example, if G is any vrd graph containing a conjugate pair $\{v, v^*\}$ such that $G - \{v, v^*\}$ is connected, then (by Proposition 2.2.2) $rad(G - \{v, v^*\}) \geq rad(G)$, and G is not radius-critical. The r-cubes for $r \geq 3$ are examples of such graphs.

Definition: For $r \in \mathrm{N}$ and $q \in \{1, \ldots, r\}$ we call $S_{r-q}(C_{2q})$ an *r-ciliate* (where we are using the convention $C_2 = K_2$).

(See, for example, figure 2.5.) Note that $rad(S_{r-q}(C_{2q})) = q + (r - q) = r$, and that r-ciliates include even paths and even cycles as the extreme cases

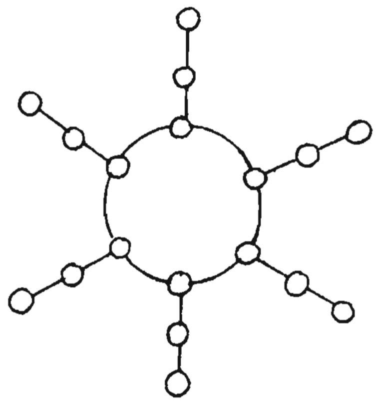


Figure 2.5: A 5-ciliate $S_2(C_6)$

$q = 1$ and $q = r$. So an r-ciliate is any graph of radius r that can be obtained from an even cycle or K_2 by attaching to each of its vertices a path of the same length. For $r \in \mathbf{N}$, there are r different r-ciliates (up to isomorphism), each of which can be obtained from a cycle of length $2q$ for some $q \in \{1, \ldots, r\}$, by attaching to each of its vertices a path of length $r - q$.

It is easily seen that all r-ciliates are r-radius-critical. Fajtlowicz [Faj88] proved that in fact all r-radius-critical graphs are r-ciliates. He first proved the following result:

Lemma 2.2.12 [Faj88] *If G is a radius-critical graph with no cut-vertices, then G is an even cycle or K_2.*

Proof:
If $\mathrm{rad}(G) = 1$, then, since G is radius-critical, $G \cong K_2$ and we are done. So assume $\mathrm{rad}(G) = r \geq 2$. Since G is a vertex-radius-decreasing graph with no cut-vertices, by Proposition 2.2.5 $V(G)$ can be partitioned into conjugate pairs. Let $\{v, v^*\}$ be any such pair. Since G is radius-critical, if $G - \{v, v^*\}$ is connected it must have radius less than r. Hence, by Proposition 2.2.2, $G - \{v, v^*\}$ is disconnected.

Let G_1 and G_2 be any two $\{v, v^*\}$-components of G. Since G has no cut-vertices, G_1 and G_2 are connected. Let P_1 be a shortest $v - v^*$ path in

G_1 and P_2 in G_2.

Then $C = \langle V(P_1) \cup V(P_2) \rangle_G$ is an induced cycle in G with length at least $2d_G(v, v^*) = 2r$, and hence with radius at least r. Hence C cannot be *properly* contained in G without contradicting the fact that G is radius-critical. It follows that $G = C$, and that (since odd cycles are not radius-critical) C is an even cycle. $\square$

Using Lemmas 2.2.9 and 2.2.12, Fajtlowicz proved his main result by induction on the radius. We present an alternative proof, which is a consequence of Theorem 2.2.10 and Lemma 2.2.12.

Theorem 2.2.13 [Faj88] *A graph is r-radius-critical iff it is an r-ciliate.*

Proof:
It is easily seen that all r-ciliates are r-radius-critical. For the converse, let G be an r-radius-critical graph. Since G is also vertex-radius-decreasing, by Theorem 2.2.10 G consists of a block $B = \langle C(G) \rangle_G$, to every vertex of which is attached a path of length ℓ. We want to prove that B is an even cycle or K_2.

We first prove that B is radius-critical. Assume, to the contrary, that B has a proper induced connected subgraph B' with $rad(B') \geq rad(B)$. (So $|V(B')| \geq 2$.) Let G' be the proper connected subgraph of G induced by the vertices in B' and the paths of length ℓ attached to them.

We note that $C(G') \subseteq V(B')$ (since $|V(B')| \geq 2$, and hence for any vertex $w \neq u$ in the path attached to $u \in V(B')$, $e_{G'}(w) > e_{G'}(u)$). Also, for every vertex $v \in V(B')$, $e_{G'}(v) = e_{B'}(v) + \ell$. It follows that $\mathrm{rad}(G') = rad(B') + \ell$.

Similarly, since $\langle C(G) \rangle_G = B$ and $e_G(v) = e_B(v) + \ell$ for every vertex $v \in V(B)$, it follows that $rad(G) = rad(B) + \ell$. Hence $rad(G') \geq rad(G)$, which is impossible. So B must be radius-critical.

Since B is a block, it now follows by Lemma 2.2.12 that B is an even cycle or K_2, and hence that G is an r-ciliate. $\square$

Erdös, Saks and Sós proved in [ESS86] that every connected non-trivial graph of radius r contains an induced P_{2r-1} (and this is best possible, as demonstrated by even cycles). Fajtlowicz proved a more general result, requiring the following lemma:

Lemma 2.2.14 *Any connected non-trivial graph of radius r contains an induced subgraph with radius $r-1$.*

Proof:
Let G be a connected graph of radius r, and let c be any central vertex of G. Recall that, by Proposition 2.2.1, removing one vertex from a graph cannot lower the radius by more than 1, and note that removing all the eccentric vertices of c would certainly lower the radius.

We can therefore construct an induced subgraph of G with radius $r-1$ by removing eccentric vertices of c, one by one, until the radius decreases. □

Theorem 2.2.15 [Faj88] *Every connected non-trivial graph of radius r contains an r-ciliate as an induced subgraph.*

Proof:
Let G be a connected non-trivial graph with radius r. If G is r-critical, then by Theorem 2.2.13 G is itself an r-ciliate, and we are done. Otherwise, G contains a proper induced connected subgraph with radius at least r. Of all such subgraphs, let G' be one of lowest order.

By our choice of G', every proper induced connected subgraph of G' must have radius less than r; hence, by Lemma 2.2.14, $rad(G') = r$. It follows that G' is r-radius-critical, and hence by Theorem 2.2.13 an r-ciliate. □

Since every r-ciliate contains an induced P_{2r-1}, the result of Erdös et al follows from Theorem 2.2.15.

2.3 Edge Removal

In this section we discuss how the removal of edges affects the radius of a graph.

2.3.1 Removing the best edge

Removing an edge from a graph cannot decrease the radius. Proposition 2.1.2, however, shows that every connected cyclic graph G has a radius-preserving spanning tree, and hence certainly contains at least one edge that can be removed without increasing the radius. In other words, the only edge-radius-increasing graphs are trees.

2.3.2 Removing the worst edge

Our next proposition shows that removing a cyclic edge from a connected graph can increase the radius by at most a factor of 2.

Proposition 2.3.1 *Let G be any connected graph of radius r containing a cyclic edge e. Then*

$$rad(G-e) \leq 2r.$$

Proof:
Let C be a shortest cycle containing e in G, and let z be a vertex opposite e on C. Since (by Proposition 1.5.9) $d_{G-e}(z,w) = d_G(z,w)$ for every $w \in V(G)$, we have $e_{G-e}(z) = e_G(z)$, and hence

$$\begin{aligned} rad(G-e) &\leq e_{G-e}(z) \\ &= e_G(z) \\ &\leq diam(G) \\ &\leq 2r. \end{aligned}$$

This completes the proof. $\square$

Our next proposition gives some necessary conditions for a graph G and edge $e \in E(G)$ to satisfy $rad(G-e) = 2rad(G)$.

Proposition 2.3.2 *Let G be a graph containing an edge e such that $rad(G) = r$ and $rad(G-e) = 2r$. Then*
(1) $diam(G) = 2r$, $g(e) = 2r+1$, and e is incident with every central vertex c of G, and
(2) G contains as an induced subgraph either the graph consisting of two cycles of length $2r+1$ with only e in common, or the graph obtained from this by deleting one edge opposite e on either cycle.

Proof:
Let $e = ab$, let c be any central vertex of G, and let T_c be a spanning tree of G which is distance-preserving from c. By Proposition 2.1.2, $rad(T_c) = r$.

Clearly T_c must contain e; we assume without loss of generality that $d_{T_c}(c,a) < d_{T_c}(c,b)$. Let B be the set consisting of the descendants of b in T_c, and let $A = V(T_c) - B$. Let $T' = \langle B \rangle_{T_c}$, and note that $e_{T'}(b) \leq r-1$.

Since $G-e$ is connected, some vertex $v \in B$ must be adjacent in $G-e$ to some vertex $u \in A$. Let x be any eccentric vertex of u in $G-e$; so $d_{G-e}(u,x) \geq rad(G-e) = 2r$.

Note that, if $x \in B$, then $d_{G-e}(u,x) \leq d_{G-e}(u,v) + d_{T'}(v,b) + d_{T'}(b,x) \leq 1+2(r-1) < 2r$, which is impossible. It follows that x is in A, and hence that removing e does not affect the distance between u and x in T_c. Therefore

$$2r \leq d_{G-e}(u,x) \leq d_{T_c-e}(u,x) \leq d_{T_c}(u,c) + d_{T_c}(c,x) \leq 2r.$$

It follows that $d_{G-e}(u,x) = 2r$, and that u and x are both eccentric vertices of c in T_c and hence in G.

Next we let u' be the neighbour of u on the $u-c$ path in T_c. So, in $G-e$, u' is within distance $2r-1$ of every vertex in A. Since $e_{G-e}(u') \geq$

$rad(G - e) = 2r$, it follows that there must be some vertex y in B whose distance from u' in $G - e$ is at least $2r$. We therefore have

$$\begin{aligned} 2r &\leq d_{G-e}(u', y) \\ &\leq d_{T_c-e}(u', u) + d_{G-e}(u, v) + d_{T_c-e}(v, b) + d_{T_c-e}(b, y) \\ &\leq 1 + 1 + (r - 1) + (r - 1) \\ &= 2r. \end{aligned}$$

Hence $d_{G-e}(v, y) = 2(r - 1)$. Furthermore, $d_{T_c}(b, v) = d_{T_c}(b, y) = r - 1$; hence $a = c$ (i.e., e is incident with every central vertex of G) and v and y are eccentric vertices of c in T_c and hence in G.

Recall that $d_G(c, x) = d_G(c, u) = r$, and that $d_{G-e}(u, x) = 2r$. Hence if a shortest $u - x$ path in G contains the edge e (and therefore the vertex $a = c$), then its length is at least $2r$. It follows that $diam(G) = d_G(u, x) = 2r$.

Since $d_{T_c}(c, u) = d_{T_c}(c, v) = r$, if any vertex in B is adjacent in $G - e$ to a vertex in A they must both be eccentric vertices of c in T_c. It follows that any smallest cycle containing e has length $2r + 1$.

Finally, let H be the subgraph of G consisting of the $c - y$, $c - v$, $c - u$ and $c - x$ paths in T_c, together with the edge vu and the edge yx if y and x are adjacent in G. Recall that $d_{G-e}(u, x) = 2r$ and $d_{G-e}(y, v) = 2(r - 1)$ (i.e., that the $u - x$ and $y - v$ paths in T_c are also shortest paths in $G - e$), and that no vertex in A can be adjacent to a vertex in B unless they are both at distance r from c in G. It follows that H is an induced subgraph of G. □

Some graphs G for which $rad(G - e) = 2rad(G)$ are shown in figure 2.6. Note that our necessary conditions are not sufficient, as shown by the graph in figure 2.7. In fact there does not appear to be an elegant characterisation of these extremal graphs.

The above problem was generalised by Segawa [Seg94] to the case where more than one edge is removed. He proved that if G is a connected graph

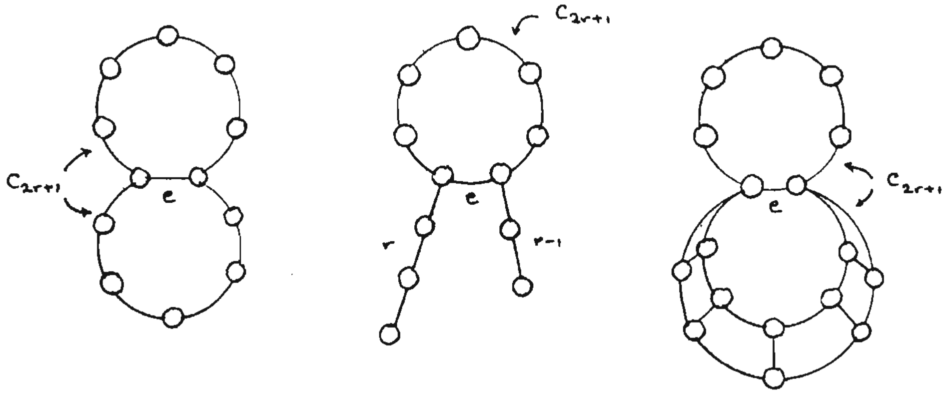


Figure 2.6: Some graphs G for which $rad(G-e) = 2rad(G)$

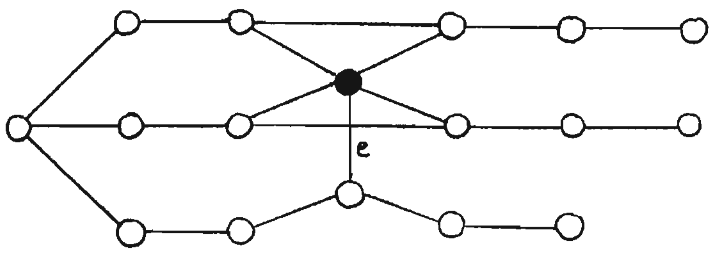


Figure 2.7: G satisfies the conditions of Proposition 2.3.1, but $rad(G-e) < 2rad(G)$.

and F is a set of edges in G such that $G - F$ is connected, then $rad(G-F) \leq (|F|+1)rad(G) - \lfloor \frac{1}{2}|F| \rfloor$.

2.4 Edge Addition

In this section we consider graphs for which the addition of any edge decreases the radius.

Definition: A graph G is called *edge-radius-decreasing* or *erd* if $rad(G+e) < rad(G)$ for every $e \in E(\bar{G})$.

For example, any even cycle is erd, while no path is (since its endpoints can be joined to form a cycle of the same radius). Erd graphs have been studied by Nishanov [Nis73, Nis75], Harary and Thomassen [HT76] and Gliviak, Knor and Šoltés [GKS94], but no simple characterisation is known. Instead we study a special class of edge-radius-increasing graphs, considered by Vizing in [Viz67] — viz., those graphs of given order and radius with the maximum possible number of edges. In the remainder of this section we will establish Vizing's upper bound on the size of a graph of given order and radius. There is a serious gap in the proof of [Viz67], which we have corrected.

We need some preliminary results:

Proposition 2.4.1 [Viz67] *For any connected graph G of order n,*

$$\Delta(G) \leq n - 2rad(G) + 2.$$

Proof:
Let v be a vertex of maximum degree in G, and let T_v be a distance-preserving spanning tree of G with v as root — so $deg_{T_v}(v) = deg_G(v) = \Delta(G)$.

Let P be a diametral path of T_v; then P has length $diam(T_v) \geq 2rad(T_v) - 1 \geq 2rad(G) - 1$. So P contains at least $2rad(G)$ vertices, at most two of which can be neighbours of v (since if P contained three neighbours of v, we would have a cycle in T). Hence there must be at least $\Delta(G) - 2$ neighbours of v which are not on P. It follows that $n \geq 2rad(G) + \Delta(G) - 2$. □

Proposition 2.4.2 [Viz67] *If x and y are vertices of a connected graph G of order n for which $d_G(x, y) \geq 3$, then*

$$deg(x) + deg(y) \leq n - 2rad(G) + 4.$$

Proof:

Let U be the set of edges incident with x or y. Since $d_G(x, y) \geq 3$, x and y have no neighbours in common, and hence no cycle in G consists entirely of edges in U. It is therefore possible to find a spanning tree T of G containing all the edges in U — i.e., in which x and y have the same degrees as in G.

If $deg(x)+deg(y) \leq 4$, then it follows from Proposition 2.1.3 that $deg(x)+deg(y) \leq n-2rad(G)+4$, and we are done. Otherwise, let P be a diametral path of T. Then P contains $diam(T) + 1 \geq 2rad(T) \geq 2rad(G)$ vertices. Since P can contain at most two neighbours of x, and two of y, it follows that there are at least $deg(x) + deg(y) - 4$ neighbours of x and y which are not on P. Hence $n \geq 2rad(G) + deg(x) + deg(y) - 4$. □

Definitions: Let n and r be any natural numbers such that $n \geq 2r \geq 2$. Define $f(n, r)$ to be the maximum possible number of edges in a graph of order n and radius r, and $C(n, r)$ to be the set of all graphs with order n, radius r and $f(n, r)$ edges.

Lemma 2.4.3 [Viz67] *For $n \in \mathbf{N}$ and $r \in \{1, ..., \lfloor \frac{1}{2}n \rfloor\}$,*
(1) $f(n + 1, r) > f(n, r)$, and
(2) $f(n, r + 1) < f(n, r)$;
that is, the function f increases as n increases, decreases as r increases.

Proof:

(1) Let G be a graph with order n, radius r and the maximum possible number of edges — i.e., $|E(G)| = f(n, r)$. Let c be any central vertex

of G, and let G' be the graph obtained from G by adding a vertex x adjacent only to c.

Since x has degree 1, its addition cannot make any new shortest paths between vertices available, and therefore cannot decrease the eccentricity of any other vertex. Furthermore, $e_{G'}(x) = e_G(c) + 1$ and $e_{G'}(c) = e_G(c)$. Hence G' also has radius r.

Since G' has $n+1$ vertices, it follows that $f(n+1, r) \geq |E(G')| = |E(G)| + 1 = f(n, r) + 1$.

(2) Let G be a graph with order n, radius $r + 1 \geq 2$, and $f(n, r + 1)$ edges. Let u and v be any two vertices in G such that $d_G(u, v) = 2$, and let G' be the graph obtained from G by adding the edge $e = uv$.

Note that the addition of e cannot increase the distance between any two vertices, and cannot decrease it by more than 1. Hence $r \leq rad(G') \leq r + 1$. But G' cannot have radius $r + 1$ without contradicting the fact that G has the maximum possible number of edges for a graph of order n and radius $r + 1$. It follows that $rad(G') = r$.

Hence $f(n, r) \geq |E(G')| = |E(G)| + 1 = f(n, r + 1) + 1$. □

Theorem 2.4.4 [Viz67] *For any natural numbers n and r such that $n \geq 2r \geq 2$,*
(1) $f(n, 1) = \frac{1}{2}n(n-1)$
(2) $f(n, 2) = \frac{1}{2}n(n-1) - \lceil \frac{1}{2}n \rceil = \lfloor \frac{1}{2}n(n-2) \rfloor$
(3) $f(n, r) = g(n, r) := \frac{1}{2}(n^2 - 4rn + 5n + 4r^2 - 6r)$ *for* $n \geq 2r \geq 6$.

Proof:

(1) The graph with radius 1 and the maximum possible number of edges is the complete graph, which has $\frac{1}{2}n(n-1)$ edges.

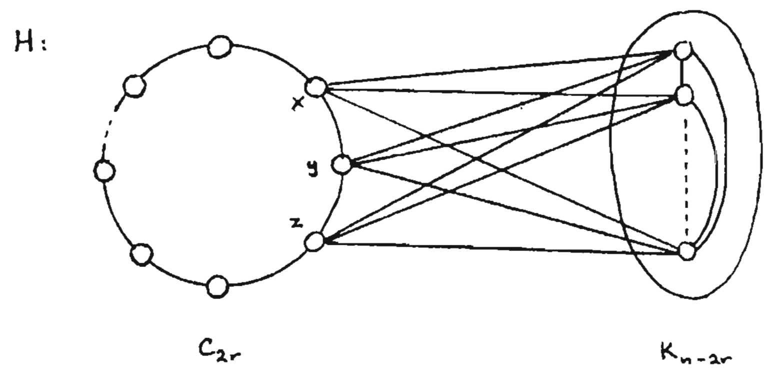


Figure 2.8: A graph with order n, radius r and maximum size

(2) In a graph with radius 2, every vertex must be nonadjacent to at least one other vertex. If n is even, therefore, $C(n, 2)$ consists of all graphs of order n in which each vertex is nonadjacent to exactly one other vertex. If n is odd, this is not possible — some vertex has to be nonadjacent to two other vertices. In either case, $C(n, 2)$ consists of all graphs obtained from K_n by removing $\lceil \frac{1}{2}n \rceil$ edges covering $V(K_n)$.

(3) To prove that the inequality $f(n, r) \geq g(n, r)$ holds for $n \geq 2r \geq 6$, we just need to construct a graph H with order n, radius r and $g(n, r)$ edges.

Let H consist of a complete graph K_{n-2r} and a cycle C_{2r}, where every vertex of the K_{n-2r} is joined to three consecutive vertices x, y, z of the C_{2r} (see figure 2.8). Since $|E(H)| = 2r + 3(n - 2r) + \frac{1}{2}(n - 2r)(n - 2r - 1) = g(n, r)$, it follows that $f(n, r) \geq |E(H)| = g(n, r)$.

(Note that, for given n and r, H is not necessarily the only graph of order n and radius r with $f(n, r)$ edges.)

To prove the remaining inequality $f(n, r) \leq g(n, r)$ for $n \geq 2r \geq 6$, we use double induction on n and r.

(4) We first show that $f(n, 3) \leq g(n, 3)$ for all $n \geq 6$.

Let G be any graph in $C(n,3)$. Recall that (by Proposition 2.4.2), if x and y are a pair of vertices such that $d_G(x,y) \geq 3$, then $deg(x) + deg(y) \leq n - 2r + 4$. Furthermore (by Proposition 2.4.1), for any $u \in V(G)$, $deg(u) \leq n - 2r + 2$. Therefore all we need to prove is that G contains three disjoint pairs of vertices at distance at least 3, since then

$$
\begin{aligned}
f(n,3) &= |E(G)| \\
&= \frac{1}{2}\sum_{u \in V(G)} deg(u) \\
&\leq \frac{1}{2}(3(n-6+4) + (n-6)(n-6+2)) \\
&= g(n,3). \qquad (1)
\end{aligned}
$$

We consider two cases:

Case (1): $diam(G) \geq 4$.

In this case G contains an induced path $P : x_1, \ldots, x_5$. Let y be any eccentric vertex of x_3 — then $d(x_3, y) \geq r = 3$, and y is not on P. Hence $\{x_1, x_4\}$, $\{x_2, x_5\}$ and $\{x_3, y\}$ are three disjoint pairs of vertices at distance at least 3, and so inequality (1) holds.

Case (2): $diam(G) = r = 3$.

Let $P : a, b, c, d$ be a diametral path in G (clearly P is an induced path), and let b' and c' be any eccentric vertices of b and c respectively — note that neither b' nor c' lies on P. If $b' \neq c'$, then $\{a, d\}$, $\{b, b'\}$ and $\{c, c'\}$ are three disjoint pairs of vertices at distance at least 3, and so inequality (1) holds.

Otherwise, let $b' = c' = x$. Since $d(x, a) \leq 3$, and $d(x, d) \leq 3$, there must be some neighbours p of a and q of d at distance at most 2 from x. Note that $p \neq q$ and $p, q \notin \{x, a, b, c, d\}$, since $d(a, d) = d(b, x) = d(c, x) = 3$. Let p' and q' be any eccentric vertices of p and q respectively. Clearly $p' \notin \{p, a, b, x\}$ and $q' \notin \{q, d, c, x\}$.

Now if $p' \neq d$, then $\{a, d\}$, $\{b, x\}$ and $\{p, p'\}$ are three disjoint pairs of vertices at distance at least 3, and so inequality (1) holds. Similarly, if $q' \neq a$, then $\{a, d\}$, $\{b, x\}$ and $\{q, q'\}$ will do. Finally, if $p' = d$ and $q' = a$, then we can use $\{p, d\}$, $\{q, a\}$ and $\{b, x\}$.

(5) Next we show that $f(2r, r) \leq g(2r, r)$ for all $r \geq 3$.

Let G be any graph in $C(n, r)$, where $n = 2r$. By Proposition 2.4.1, $\Delta(G) \leq 2r - 2r + 2 = 2$. It follows that $f(2r, r) = |E(G)| \leq \frac{1}{2}n\Delta(G) = 2r = g(2r, r)$.

(6) Now let n and r be any natural numbers such that $r \geq 4$ and $n \geq 2r+1$, and assume inductively that $f(n', r') \leq g(n', r')$ for all $n', r' \in \mathbf{N}$ such that either $3 \leq r' \leq r - 1$, or else $r' = r$ and $2r \leq n' \leq n - 1$. We want to prove that then $f(n, r) \leq g(n, r)$.

Let G be any vertex in $C(n, r)$. We consider two cases:

Case (1): G contains a ncv v such that $rad(G - v) \geq r$.

In this case,

$$\begin{aligned} |E(G - v)| &\leq f(n - 1, rad(G - v)) \\ &\leq f(n - 1, r) \qquad \text{(by Lemma 2.4.3)} \\ &\leq g(n - 1, r) \qquad \text{(by the induction hypothesis).} \end{aligned}$$

Since $\deg(v) \leq \Delta(G) \leq n - 2r + 2$ by Proposition 2.4.1, it follows that

$$\begin{aligned} f(n, k) &= |E(G)| \\ &= |E(G - v)| + deg(v) \\ &\leq g(n - 1, r) + n - 2r + 2 \\ &= g(n, r), \end{aligned}$$

which is the desired result.

Case (2): For every ncv u in G, $rad(G - u) < r$ — i.e., G is a vertex-radius-decreasing graph.

We consider two subcases:

Subcase (2.1): G contains at least one cut-vertex.

By Lemma 2.2.9 any ncv of G must have degree 1. Hence G contains two endvertices, x_1 and x_2. Let $G' = G - \{x_1, x_2\}$, and note that if $rad(G') \leq r - 2$, then any central vertex c of G' is within distance $r - 2$ from every vertex in $V(G) - \{x_1, x_2\}$, including the neighbours of x_1 and x_2. But then c is within distance $r - 1$ from x_1 and x_2, contradicting the fact that $\text{rad}(G) = r$. Hence $rad(G') \geq r - 1$.

So

$$\begin{aligned}
f(n,r) &= |E(G)| \\
&= 2 + |E(G')| \\
&\leq 2 + f(n-2, rad(G')) \\
&\leq 2 + f(n-2, r-1) \qquad \text{(by Lemma 2.4.3)} \\
&\leq 2 + g(n-2, r-1) \qquad \text{(by induction hypothesis)} \\
&= g(n,r),
\end{aligned}$$

and we are done.

Subcase (2.2): G has no cut-vertices.

By Proposition 2.2.5, n is even and $V(G)$ can be partitioned into $\frac{1}{2}n$ conjugate pairs $\{v, v^*\}$. Note that for any $v \in V(G)$, $d(v, v^*) = r \geq 3$, and so (by Proposition 2.4.2) $deg(v) + deg(v^*) \leq n - 2r + 4$. Hence

$$|E(G)| = \frac{1}{2} \sum_{u \in V(G)} deg(u) \leq \frac{1}{2}(\frac{1}{2}n)(n - 2k + 4).$$

Note that $\frac{1}{4}n(n - 2k + 4) \leq g(n,k) = \frac{1}{2}(n^2 - 4nk + 5n + 4k^2 - 6k)$
iff $\quad 0 \leq (n - 2r)(n - 4r + 6)$
iff $\quad n \geq 4r - 6 \qquad$ (since $n > 2r$).

So, if $n \geq 4r - 6$, then $f(n, r) \leq g(n, r)$, and we are done. We therefore need to consider only the case $2r + 1 \leq n \leq 4r - 7$; i.e. (since n is even) we may assume

$$2r + 2 \leq n \leq 4r - 8.$$

(7) Since G has no cut-vertices, by Menger's Theorem each pair of vertices of G is contained in a cycle. In particular, any pair of conjugate vertices $\{x, x^*\}$ is contained in a cycle of length at least $2d(x, x^*) = 2r$. Let M be a shortest cycle of length at least $2r$ in G, and let its length be ℓ.

(8) We want to find the maximum possible number of edges in G. To this end, we first show that M is an induced cycle of G. Assume, to the contrary, that M has a chord ab, and let M_1 and M_2 be the two $a - b$ sections of M, where say $\ell(M_1) \geq \ell(M_2)$. Then the two cycles $C_1 = M_1 + ab$ and $C_2 = M_2 + ab$ are shorter than M, and must therefore have length at most $2r - 1$.

Let x be a central vertex of M_1, and let $P_1 : x, x_1, \ldots, x_{r-1}$ and $P_2 : x, y_1, \ldots, y_{r-1}$ be the two paths of length $r - 1$ emanating along M from x. Then a and b must lie on P_1 and P_2, or else C_1 would have length at least $2(r - 1) + 2 = 2r$. Say $a = x_i$ and $b = y_j$, where $|i - j| \leq 1$. Assume without loss of generality that $j \geq i$.

Now let $S = \{x, x_1, \ldots, x_{r-2}, y_1, \ldots, y_{r-1}\}$; so $|S| = 2r - 2$. Note that every pair of vertices in S lies on one of the cycles C_1, C_2 (each of which has length at most $2r - 1$), or on one of the paths P_1, P_2, $P_3 : x_1, x_2, \ldots, x_i(= a), y_j(= b), y_{j+1}, \ldots, y_{r-1}$ and $P_4 : y_1, y_2, \ldots, y_j(= b), x_i(= a), \ldots, x_{r-2}$ (each of which has length at most $r - 1$). It follows that no pair of vertices in S can be distance r apart, and hence that S contains at most one vertex of every conjugate pair. Therefore $n \geq 2|S| = 4r - 4$, which contradicts our assumption that $n \leq 4r - 8$. Hence M must be an induced cycle.

(9) Next we prove that no vertex in $V(G) - V(M)$ can have more than three neighbours on M.

Assume, to the contrary, that there exists some vertex $x \in V(G) - V(M)$ which has at least four neighbours on M. Then x must have two neighbours u and v on M such that $d_M(u,v) \geq 3$. Let the two $u - v$ sections of M be M_1 and M_2 respectively, where $\ell(M_1) \geq \ell(M_2)$, and let $C_1(C_2)$ be the cycle bounded by the path u, x, v and $M_1(M_2)$. It follows that C_1 and C_2 are shorter than M, and hence that $\ell(C_2) \leq \ell(C_1) \leq 2r - 1$.

Again we find a set of more than $2r - 4$ vertices all within a distance of $r - 1$ from each other. Let c be a central vertex of M_1, and let $P_1 : c, a_1, \ldots, a_{r-1}$ and $P_2 : c, b_1, \ldots, b_{r-1}$ be the two paths of length $r - 1$ emanating along M from c, where without loss of generality $u = b_j, v = a_i$ and $0 \leq j - i \leq 1$ (since $i + j + 2 = \ell(C_1) \leq 2r - 1$, and hence $j \leq r - 1$ and $i \leq r - 2$).

Let $S = \{x, c, a_1, \ldots, a_{r-3}, b_1, \ldots, b_{r-2}\}$; so $|S| = 2r - 3$. If $j \leq r - 2$ and $i \leq r - 3$, then any two vertices of S are on one of the cycles C_1, C_2 (each of which has length at most $2r - 1$), or on one of the paths $P_1, P_2, P_3 : a_1, \ldots, a_i, x, b_j, \ldots, b_{r-2}$ and $P_4 : b_1, \ldots, b_j, x, a_i, \ldots, a_{r-3}$ (each of which has length at most $r - 1$). Otherwise, if $j = r - 1$ or $i = r - 2$, then P_3 and P_4 are not defined, but in this case all vertices in S are on C_1.

It follows that no two vertices in S can be distance r apart, and hence that for every vertex y in S, y^* is in $V(G) - S$. So $|V(G) - S| \geq |S| = 2r - 3$, and hence $n \geq 2|S| = 4r - 6$. Since this contradicts our assumption that $n \leq 4r - 8$, it follows that no vertex in $V(G - M)$ can have more than three neighbours on M.

(10) All that now remains is to count the maximum number of edges in G, in the case $n \leq 4r - 8$. Since $G \in C(n,r)$,

$$\begin{aligned}
f(n,r) &= |E(G)| = |E(G-M)| + |E(M)| + |[V(G-M), V(M)]| \\
&\leq \binom{n-\ell}{2} + \ell + (n-\ell)(3) \\
&= \frac{1}{2}(n^2 - 2n\ell + 5n + \ell^2 - 3\ell) \\
&\leq g(n,r),
\end{aligned}$$

where we have used $n \geq \ell \geq 2r$.

The result now follows by induction. $\square$

Chapter 3

Diameter

The diameter of a graph G is the maximum distance between two vertices in G. In this chapter we consider how the diameter is affected by the removal of an edge or a vertex, the addition of an edge, or the taking of a spanning tree.

3.1 Preliminaries

Recall that $s_i(G)$ is the number of unordered pairs of vertices distance i apart in a graph G, and that $W_i(G)$ is the set of all unordered pairs of non-adjacent vertices of G which are at most distance i apart. Our first result is a lower bound on $|W_k(G)|$. The bound is due to Šoltés [Sol91], who also showed that it is attained by path-complete graphs. Using an idea of Goddard [God96], we have extended the result by finding *all* graphs attaining the bound. In this chapter we will use only Šoltés's original result, but our new result will allow us to improve some results on distance in Chapter 4.

Theorem 3.1.1 *Let G be a connected graph of order n and diameter $d \geq 3$. Then for any $k \in \{2, ..., d-1\}$,*

$$|W_k(G)| \geq \sum_{i=2}^{k}(n-i). \tag{1}$$

Moreover, equality occurs for all $k \in \{2, ..., d-1\}$ *iff*
(a) G is a path-complete graph, or
(b) $G \cong K_{n_0} + K_1 + ... + K_1 + K_{n_d}$, where $n_0 + n_d = n - d + 1$, or
(c) $d = 3$ and $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$, where $n_1 + n_2 = n - 2$.

Proof:

(1) Let $G_d : v_0, v_1, ..., v_d$ be a diametral path of G. Number the vertices not lying in G_d as $v_{d+1}, ..., v_{n-1}$, in such a way that the graphs $G_j := \langle\{v_0, ..., v_j\}\rangle_G$ are connected for all $j \in \{d+1, ..., n-1\}$.

If $G = G_d$, then G is a path, equality holds in (1) and G is path-complete. Otherwise, for any $j \in \{d+1, ..., n-1\}$, $W_k(G_j)$ contains $W_k(G_{j-1})$, and we now show that there are at least $k-1$ pairs of vertices in the set $W_k(G_j) - W_k(G_{j-1})$ — in other words, that there are at least $k-1$ vertices u in G_j such that $2 \leq d_{G_j}(v_j, u) \leq k$.

If $e_{G_j}(v_j) \geq k$, then let w be an eccentric vertex of v_j in G_j, and let $P : v_j = u_0, ..., u_{e_{G_j}(v_j)} = w$ be a shortest $v_j - w$ path. Then $2 \leq d_{G_j}(v_j, u_i) \leq k$ for each of the $k-1$ vertices $u_2, ..., u_k$.

Otherwise, if $e_{G_j}(v_j) < k$, note that since G_d is a shortest path, v_j can be adjacent to at most three vertices of G_d. Hence there are at least $d - 2 \geq k - 1$ vertices of G_d that are non-adjacent to v_j but within distance $e_{G_j}(v_j) < k$ of it.

In either case, $|W_k(G_j) - W_k(G_{j-1})| \geq k - 1$.

(2) Now

$$\begin{aligned}
|W_k(G)| &= (|W_k(G_{n-1})| - |W_k(G_{n-2})|) + (|W_k(G_{n-2})| - |W_k(G_{n-3})|) \\
&\quad + ... + (|W_k(G_{d+1})| - |W_k(G_d)|) + |W_k(G_d)| \\
&\geq (k-1)(n-d-1) + \sum_{i=2}^{k}(d+1-i) \\
&= \sum_{i=2}^{k}(n-i),
\end{aligned}$$

which is inequality (1).

(3) We now assume that equality holds in inequality (1) for *every* $k \in \{2, ..., d-1\}$. Hence there must be *exactly* $k-1$ vertices in G_j which are between distance 2 and distance k from v_j, for every $j \in \{d+1, ..., n-1\}$ and every $k \in \{2, ..., d-1\}$.

This must hold no matter in what order the vertices $v_{d+1}, ..., v_n$ are labelled, as long as the graph G_j remains connected each time the vertex v_j is added.

(4) In particular, observation (3) must hold for $k = 2$; in other words, for every vertex v_j added, there must be exactly one vertex at distance 2 from v_j in G_j. Call this vertex the *double* of v_j.

Consider any vertex v which is adjacent to at least one vertex on G_d in G. It is a candidate for v_{d+1}. Since $d > 2$, v cannot be adjacent to all the vertices on G_d; hence v has a double on G_d. In order for this double to be unique, v must be adjacent to all vertices in an initial or final segment of G_d. Note further that, since G_d is a shortest $v_0 - v_d$ path, v_j cannot be adjacent to more than three vertices on G_d.

Next we prove that no vertex in G can have exactly one neighbour on G_d. Assume, to the contrary, that there exists a vertex v in G which is adjacent to only v_0 (say) on G_d. Let P be a shortest $v_d - v$ path in G, and note that v_1 is not on P (since otherwise $d_G(v_d, v) = d + 1$). Now let t be the number of vertices in $V(P)$ that are not in $V(G_d)$, and label these vertices as $v_{d+1}, ..., v_{d+t} = v$; in other words, let the first t vertices added after v_d be the vertices of $V(P) - V(G_d)$, ending with v. But then when v is added it has two doubles — v_1 and one on P — which is impossible.

It follows that if a vertex v is adjacent to at least one vertex on G_d, then its neighbour set on G_d is one of $\{v_0, v_1\}$, $\{v_0, v_1, v_2\}$, $\{v_{d-1}, v_d\}$ or $\{v_{d-2}, v_{d-1}, v_d\}$.

(5) We now show that every vertex in G is adjacent to at least one vertex on G_d. Otherwise, there exists some vertex w in G which is at distance 2 from G_d via a vertex v. Let $v_{d+1} = v$ and $v_{d+2} = w$; then w has at least two doubles on G_d (viz., the neighbours of v on G_d), which is impossible.

Furthermore, any two vertices u and v in $V(G) - V(G_d)$ which have a common neighbour on G_d must be adjacent to each other. Otherwise, if we let $v_{d+1} = u$ and $v_{d+2} = v$, then v would have two doubles — u and one on G_d.

Hence by observation (4),

$$G \cong K_{n_0} + K_{n_1} + K_1 + ... + K_1 + K_{n_{d-1}} + K_{n_d},$$

where $n_0 + n_1 + n_{d-1} + n_d = n - d + 3$.

(6) Now, observation (3) must also hold for $k = d - 1$; in other words, there must be exactly $d - 2$ vertices in G_j which are between distance 2 and distance $d - 1$ from each vertex v_j added. Hence if $n_1 \geq 2$, then either $d = 3$ and $n_d = 1$, or $n_{d-1} = n_d = 1$; in other words, $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$ or G is path-complete. A similar conclusion holds if $n_{d-1} \geq 2$. The only other possibility is $n_1 = n_{d-1} = 1$, in which case $G \cong K_{n_0} + K_1 + ... + K_1 + K_{n_d}$.

It remains to show that the above graphs do indeed achieve equality in inequality (1) for all $k \in 2, ..., d - 1$. We give the calculation only for path-complete graphs; the other two cases are easy.

Note that a path-complete graph of order n and diameter d can be obtained from a path $P : v_0, ..., v_{d-2}$ by joining v_{d-2} to t vertices of a K_{n-d+1}. Label the vertices of the K_{n-d-1} which are adjacent to v_{d-2} by $v_{d-1}, ..., v_{d-2+t}$, and label the remaining vertices as $v_{d-1+t}, ..., v_n$. Note that only the $n-d+1-t$ vertices in $\{v_{d-1+t}, ..., v_{n-1}\}$ have a vertex of lower index at distance d from them (viz., v_0), so $s_d = n-d+1-t$. For $i \in \{2, ..., d-1\}$, however, every vertex except those in $\{v_0, ..., v_{i-1}\}$ has exactly one vertex of lower index at distance i from it, and so $s_i = n - i$. It follows that $|W_k(PK_{n,m})| = \sum_{i=2}^{k} s_i = \sum_{i=2}^{k}(n - i)$.

This completes the proof. □

Our next result is an upper bound on the diameter of a graph, given the numbers of vertices and edges. The bound is due to Harary [Har62], but we present our own proof based on Theorem 3.1.1. The related question of the maximum number of edges in a graph of given order and diameter is discussed in Section 3.4.

Theorem 3.1.2 [Har62] *Let G be a connected graph with n vertices and m edges. Then $diam(G) \leq diam(PK_{n,m})$.*

Proof:
Let $diam(G) = d$, and $diam(PK_{n,m}) = D$. Note that $s_D(PK_{n,m})$ is equal to the number of vertices at distance D from a vertex of lowest degree in $PK_{n,m}$, and is therefore at most $n - D$. Now suppose $d > D$.

If $D = 1$, then $m = \binom{n}{2}$, and so $d = 1 = D$, contradicting our assumption. It follows that $D \geq 2$, and so $d \geq 3$. Hence

$$\begin{aligned}
|W_d(G)| &> |W_D(G)| \\
&\geq \sum_{i=2}^{D}(n-i) \qquad \text{(by Theorem 3.1.1)} \\
&= \sum_{i=2}^{D-1}(n-i) + (n-D) \\
&\geq \sum_{i=2}^{D-1}(n-i) + s_D(PK_{n,m}) \\
&= |W_{D-1}(PK_{n,m})| + s_D(PK_{n,m}) \qquad \text{(by Theorem 3.1.1)} \\
&= |W_D(PK_{n,m})|.
\end{aligned}$$

But since G and $PK_{n,m}$ both have n vertices and m edges, $|W_d(G)| = |W_D(PK_{n,m})| = \binom{n}{2} - m$. We therefore have a contradiction, and it follows that $d \leq D$. □

Note that $PK_{n,m}$ is not necessarily the only graph of order n and size m with maximum diameter. For example, the graph G in figure 3.1 has order 8, size 10, and the same diameter as $PK_{8,10}$.

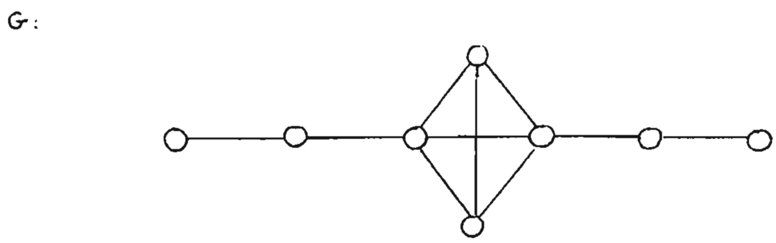


Figure 3.1: Another graph of maximum diameter (here $n = 8,\ m = 10$)

3.2 Spanning Trees

We saw earlier that, for any connected graph G, it is always possible to find a spanning tree of G which has the same radius. This is not always true for diameter. In this section we characterise graphs which have diameter-preserving spanning trees, and in fact find the minimum diameter of a spanning tree of *any* given graph.

We first examine the more restricted problem of determining when a graph has an eccentricity-preserving spanning tree.

3.2.1 Eccentricity-preserving spanning trees

Definition: A spanning tree T of a graph G is said to be *eccentricity-preserving* if $e_T(v) = e_G(v)$ for every vertex $v \in V(G)$.

Nandakumar [Nan86] characterised the graphs that have eccentricity-preserving spanning trees. We present our own proof of his result. This requires some preliminary notation and results.

Recall that, by Propositions 2.1.1 and 2.1.2, if c is any central vertex of a connected graph G and T_c is a spanning tree of G which is distance-preserving from c, then $rad(T_c) = rad(G)$ and $C(T_c) \subseteq C(G)$. We now consider a more specific type of such tree.

Definition: For any adjacent vertices v and u in a graph, we let $T_{v,u}$ denote a spanning tree which is distance-preserving from v, in which as many vertices as possible are descended from u.

For example, the tree found using the breadth-first-search algorithm with v as root and u as the first neighbour of v is such a tree.

Proposition 3.2.1 *Let G be a connected graph containing two adjacent central vertices c_1 and c_2 with no eccentric vertices in common. Then $C(T_{c_1,c_2}) = \{c_1, c_2\}$.*

Proof:
The only vertices in $V(G)$ whose distances from c_1 and c_2 are not the same in T_{c_1,c_2} as in G are those that are equidistant from c_1 and c_2 in G. These vertices are the same distance from c_1 in T_{c_1,c_2} as in G, but distance 1 further from c_2. Since c_1 and c_2 have no eccentric vertices in common, all these vertices are within distance $rad(G) - 1$ from c_2 in G. Hence $e_{T_{c_1,c_2}}(c_2) = rad(G)$, and so $c_2 \in C(T_{c_1,c_2})$. Since c_1 is certainly also in $C(T_{c_1,c_2})$, the result follows. □

Lemma 3.2.2 *If every non-central vertex v of a graph G has a neighbour w with $e_G(w) = e_G(v) - 1$, then no vertex $u \in V(G)$ can be further than $e_G(u) - rad(G)$ from its closest central vertex in G.*

Proof:
Let u be any non-central vertex. Then u has a neighbour u_1 of smaller eccentricity. Similarly, u_1 (if it is not central) has a neighbour u_2 of smaller eccentricity, and so on. It follows that we can reach the centre from u in $e_G(u) - rad(G)$ steps. □

As a direct consequence we have the following result.

Lemma 3.2.3 *If $C(G) = \{c_1, c_2\}$, $diam(G) \leq 2rad(G) - 1$, and every non-central vertex v in G has a neighbour u with $e_G(u) = e_G(v) - 1$, then c_1 and c_2 have no common eccentric vertex.*

Proof:
Let z be any eccentric vertex of c_1; so $d_G(z, c_1) = rad(G)$. By Lemma 3.2.2, z must be within a distance of $e_G(z) - rad(G) \leq diam(G) - rad(G) \leq rad(G) - 1$ from some central vertex of G, which cannot be c_1 and must therefore be c_2. Hence z cannot be an eccentric vertex of c_2. □

We are now ready to present the characterization of eccentricity-preserving spanning trees.

Theorem 3.2.4 [Nan86] *A connected graph G has an eccentricity-preserving spanning tree iff*
(1) either $\langle C(G)\rangle_G \cong K_1$ and $diam(G) = 2rad(G)$, or $\langle C(G)\rangle_G \cong K_2$ and $diam(G) = 2rad(G) - 1$, and
(2) every non-central vertex v in G has a neighbour u with $e_G(u) = e_G(v) - 1$.

Proof:
Let G be a connected graph with an eccentricity-preserving spanning tree T. By Proposition 1.5.4, condition (1) holds for T, and it is not hard to show that condition (2) does too. It follows that the two conditions hold for G.

For the converse, let G be a connected graph satisfying (1) and (2). If $C(G) = \{c_1\}$, let T be the tree T_{c_1}. Then by Propositions 2.1.1 and 2.1.2, $rad(T) = rad(G)$ and $C(T) = C(G)$. Otherwise, if $C(G) = \{c_1, c_2\}$, let T be the tree T_{c_1,c_2}. Again, by Proposition 2.1.2, $rad(T) = rad(G)$. Furthermore, since by Lemma 3.2.3 c_1 and c_2 have no common eccentric vertex, it follows by Proposition 3.2.1 that $C(T) = C(G)$.

Now let v be any vertex in G. If $C(G) = \{c_1, c_2\}$ and v is closer to c_2 than to c_1 in G, let $c = c_2$. Otherwise let $c = c_1$. Then $c \in C(G) = C(T)$, $d_T(v,c) = d_G(v,c)$, and by Lemma 3.2.2, $d_G(v,c) \leq e_G(v) - rad(G)$. Therefore

$$\begin{aligned} e_T(v) &\leq d_T(v,c) + e_T(c) \\ &= d_G(v,c) + rad(T) \\ &= d_G(v,c) + rad(G) \\ &\leq e_G(v). \end{aligned}$$

It follows that $e_T(v) = e_G(v)$ for every $v \in V(G)$. In other words, T is eccentricity-preserving. $\square$

3.2.2 Minimum-diameter spanning trees

Buckley and Lewinter [BL88] characterised all graphs that have a diameter-preserving spanning tree. We generalise their result by determining the minimum diameter of a spanning tree of any given graph.

Proposition 3.2.5 *If T is a minimum-diameter spanning tree of a graph G, then $rad(T) = rad(G)$.*

Proof:
Suppose, to the contrary, that $rad(T) \geq rad(G) + 1$. Let c be any central vertex of G, and let T_c be a spanning tree of G which is distance-preserving from c. Then

$$\begin{aligned} diam(T_c) &\leq 2rad(T_c) \\ &= 2rad(G) \\ &\leq 2(rad(T) - 1) \\ &\leq diam(T) - 1, \end{aligned}$$

which is impossible. □

Theorem 3.2.6 *Let T be a minimum-diameter spanning tree of a graph G. If G contains a pair of adjacent central vertices with no eccentric vertices in common, then $diam(T) = 2rad(G) - 1$. Otherwise, $diam(T) = 2rad(G)$.*

Proof:
By Proposition 3.2.5, $rad(T) = rad(G)$. Since T is a tree, either $diam(T) = 2rad(G)$ or $diam(T) = 2rad(G) - 1$.

If $diam(T) = 2rad(G) - 1$, then let $C(T) = \{c_1, c_2\}$. Since T is radius-preserving, $C(T) \subseteq C(G)$, and any eccentric vertex of c_1 or c_2 in G is also an eccentric vertex in T. Since c_1 and c_2 are adjacent, no vertex in T can be equidistant from them in T. It follows that c_1 and c_2 can have no eccentric vertices in common in T, and hence in G.

For the converse, let G be a graph containing two adjacent central vertices c_1 and c_2 which have no eccentric vertices in common, and consider T_{c_1,c_2}. By Proposition 3.2.1, $|C(T_{c_1,c_2})| = 2$, and hence $diam(T_{c_1,c_2}) = 2rad(G) - 1$. It follows that $diam(T) = 2rad(G) - 1$. □

Since any diameter-preserving spanning tree is clearly also a minimum-diameter spanning tree, Buckley and Lewinter's characterisation follows as a corollary:

Corollary 3.2.7 [BL88] *A graph G has a diameter-preserving spanning tree iff either*
(1) $diam(G) = 2rad(G)$, *or*
(2) $diam(G) = 2rad(G) - 1$ *and G contains a pair of adjacent central vertices with no eccentric vertex in common.*

Some remarks:

(1) Note that any connected graph G contains a spanning tree whose diameter is at most double that of G (for example, any radius-preserving spanning tree of G is such a tree). The odd cycles (among others) show that this bound cannot be improved. This has relevance, for example, when one is considering replacing a proposed communications network by one with the minimum possible building costs, and one needs to know whether this will still yield an acceptable maximum transmission time.

(2) Theorem 3.2.6 provides an easy way to find *one* minimum-diameter spanning tree of a connected graph G. If G contains two adjacent central vertices c_1 and c_2 with no eccentric vertices in common, then T_{c_1,c_2} is a minimum-diameter spanning tree (with diameter $2rad(G) - 1$).

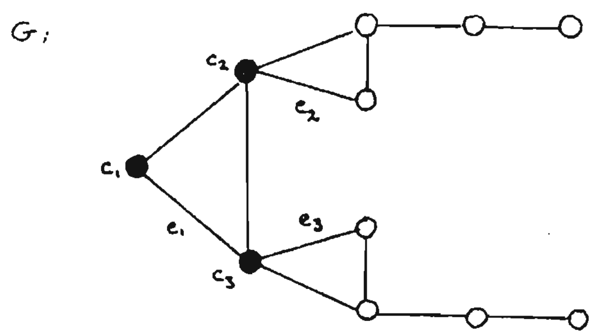


Figure 3.2: A graph illustrating remarks (3) and (4)

Otherwise, for any $c \in C(G)$, T_c is a minimum-diameter spanning tree (with diameter $2rad(G)$).

(3) It is not true that *every* spanning tree which is distance-preserving from a central vertex of a graph G must have minimum diameter, only that at least one of them must. For example, in the graph G in figure 3.2, $diam(T_{c_2}) = diam(T_{c_3}) = 7$, but $diam(T_{c_1}) = 8$.

(4) Nor is it true that every minimum-diameter spanning tree of a graph G must be distance-preserving from some $c \in C(G)$. For example, the minimum-diameter spanning tree obtained by omitting the edges e_1, e_2 and e_3 from G in figure 3.2 is not distance-preserving from any central vertex.

(5) Clearly, any eccentricity-preserving spanning tree also preserves the diameter. One would therefore expect the conditions of Theorem 3.2.4 to imply those of the corollary to Theorem 3.2.6: Lemma 3.2.3 shows that they do.

(6) A graph can, however, have a diameter-preserving spanning tree without having an eccentricity-preserving spanning tree, as shown by the graph $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$ for any $n_1 \geq 2$ and $n_2 \geq 2$. This

example also shows that, unlike in the case of eccentricity-preserving spanning trees, $|C(G)|$ need not be small in order for G to have a diameter-preserving spanning tree. Here $|C(G)| = |V(G)| - 2$, and so $|C(G)|/|V(G)| \to 1$ as $|V(G)| \to \infty$.

(7) The minimum-diameter spanning tree problem for "graphs" whose edges form a continuum of points was studied by Hassin and Tamir [HT95].

(8) Buckley and Lewinter [BL88] also noted that the following are sufficient conditions for a graph to have a diameter-preserving spanning tree:
(a) $C(G) = \{c\}$ and c is a cut-vertex not on a cycle of G, or
(b) G contains two adjacent central vertices c_1 and c_2 such that the edge $e = c_1c_2$ is a bridge.
We omit the straight-forward proofs.

3.3 Edge Removal

In this section we discuss how the removal of edges affects the diameter of a graph.

3.3.1 Removing the worst edge

When an edge is removed from a graph, the diameter cannot decrease. Our first theorem shows that, if the edge is cyclic, then the diameter also cannot increase by more than a factor of 2.

Theorem 3.3.1 [Ple75b, CG84] *Let G be any connected graph and e any cyclic edge of G. Then*

$$diam(G-e) \leq 2diam(G).$$

Moreover, this bound cannot be improved.

Proof:
Let $e = ab$, let C be a shortest cycle of G containing e, and let w be a vertex opposite e on C. Then by Proposition 1.5.9 removing e does not affect the distance from w to any other vertex.

Now let x and y be any diametral pair of $G-e$. Then

$$\begin{aligned} diam(G-e) &= d_{G-e}(x,y) \\ &\leq d_{G-e}(x,w) + d_{G-e}(w,y) \\ &= d_G(x,w) + d_G(w,y) \\ &\leq 2diam(G). \end{aligned}$$

The odd cycle C_{2d+1} shows that the bound is sharp. □

Chung and Garey [CG84] also considered a generalisation of this problem in which $t \geq 2$ edges are removed from a connected graph G of diameter d.

They proved that if the resulting graph G' is still connected, then $diam(G') \leq (t+1)d + t$. Schoone, Bodlaender and van Leeuwen [SBV87] improved this bound to $diam(G') \leq (t+1)d$, and obtained tight bounds for the cases where $t = 2$ or 3, or $d = 2$. Kerjouan [Ker86] further improved the bound to $diam(G') \leq (t+1)d - t + 3$. Peyrat [Pey84] obtained tight upper and lower bounds on $diam(G')$ in the case where $d = 2$ or 3 and G is $(t+1)$-edge-connected, but for general d the maximum possible diameter of G' has not yet been determined exactly.

We have obtained the following upper bound on the increase in diameter when an edge is removed, in terms of the order of the original graph:

Proposition 3.3.2 *Let G be a connected graph of order n, containing a cyclic edge e. Then*

$$diam(G-e) - diam(G) \leq \left\lfloor \frac{1}{2}(n-1) \right\rfloor,$$

and this bound is sharp.

Proof:
If $diam(G) > \frac{1}{2}(n-1)$, then

$$diam(G-e) - diam(G) < (n-1) - \frac{1}{2}(n-1) = \frac{1}{2}(n-1).$$

If $diam(G) \leq \frac{1}{2}(n-1)$, then by Theorem 3.3.1

$$\begin{aligned} diam(G-e) - diam(G) &\leq 2diam(G) - diam(G) \\ &\leq \frac{1}{2}(n-1). \end{aligned}$$

The bound is attained, for example, by the cycle C_n. $\square$

In contrast, there are many graphs whose diameter is unchanged by the removal of any edge. For example, the hypercubes Q_i for $i \geq 3$ are such graphs.

3.3.2 Removing a particular edge

We now consider the maximum change in diameter when a *particular* edge e of girth $g(e)$ is removed. Since the ends of e have their distance increased to $g(e) - 1$, it follows that:

Proposition 3.3.3 [Ple75b] *For any cyclic edge e of a connected graph G,*

$$diam(G - e) \geq g(e) - 1.$$

This bound is attained for example in the cycles. It is in fact sharp for any possible values of $diam(G)$ and $g(e)$ for which $g(e) \geq diam(G) + 1$, as shown by the following construction: for any given $d \in \mathbb{N}$ and $\ell \in \{d+1, ..., 2d+1\}$, let G be the graph obtained from a cycle $C \cong C_\ell$ by attaching a path of length $d - \lfloor \frac{1}{2}\ell \rfloor$ to a vertex v of C, and let e be an edge opposite v on C. Then $diam(G) = d$, $g(e) = \ell$ and $diam(G - e) = \ell - 1$.

Theorem 3.3.4 [Ple75b] *For any cyclic edge e of a connected graph G,*

$$diam(G - e) - diam(G) \leq g(e) - 2,$$

and if $diam(G)$ is even and $diam(G - e) = 2diam(G)$, then $diam(G - e) - diam(G) \leq g(e) - 3$. Moreover, these bounds are sharp for all possible values of $diam(G)$ and $diam(G - e)$.

Proof:
Let $diam(G) = d$. Let $e = ab$, and let C be a shortest cycle containing e. Let x and y be a diametral pair of $G - e$, and let P be a shortest x–y path in G.

If P does not contain e, then $diam(G - e) = d$, and we are done. Otherwise, we may assume without loss of generality that P is of the form

$x, \ldots, a, b, \ldots, y$. Let W be the x–y walk obtained from P by replacing e with $C - e$. Then

$$\begin{aligned} diam(G-e) &= d_{G-e}(x,y) \\ &\leq \ell(W) = d_G(x,y) + g(e) - 2 \\ &\leq d + g(e) - 2, \end{aligned}$$

and the result follows.

Now let d be even and $diam(G-e) = 2d$, and *assume* that equality holds in the above inequality. Hence $g(e) = d+2$, W is a shortest x–y path in $G - e$, and $d_G(x,y) = d$.

Since d is even and $d_G(x,y) = d$, we can assume without loss of generality that $d_G(x,a) \geq \frac{1}{2}d$. Now let w be a vertex opposite a on C; hence (by Proposition 1.5.8) $d_G(w,a) = d_C(w,a) = \frac{1}{2}d + 1$. Since the x–w section of W is a shortest x–w path in $G - e$ and hence (by Proposition 1.5.9) in G, it follows that

$$d_G(x,w) = d_G(x,a) + d_G(a,w) \geq \frac{d}{2} + (\frac{d}{2} + 1) = d + 1,$$

which is impossible. This shows that if d is even and $diam(G-e) = 2d$, then the above inequality is strict.

Finally, we give an example to show the bounds cannot be improved. Let $d \in \mathbf{N}$ and $D \in \{d, \ldots, 2d\}$ be given. Let G be the graph obtained from a cycle C by attaching two disjoint paths P_1 and P_2 to two adjacent vertices a and b on C as follows. If d is even and $D = 2d$, let C have length $d+3$ and let P_1 and P_2 have length $\frac{1}{2}d - 1$. Then $diam(G) = d$, $diam(G-e) = D$, and $g(e) = D - d + 3$. Otherwise, let C have length $D - d + 2$, let P_1 have length $\lfloor \frac{d-1}{2} \rfloor$, and let P_2 have length $\lceil \frac{d-1}{2} \rceil$. Then $diam(G) = d$, $diam(G-e) = D$, and $g(e) = D - d + 2$. □

The following theorem is based on ideas developed by Plesník in [Ple75a].

Theorem 3.3.5 *Let G be a connected graph of diameter d, containing no endvertices, and let e be a cyclic edge of G. Then $diam(G-e) = 2d$ iff $g(e) = 2d+1$.*

Proof:

Let $e = ab$, and $diam(G-e) = 2d$. We establish that $g(G) = 2d+1$ in three steps.

(1) Let C be a shortest cycle containing e in G. Let x and y be any diametral pair of $G-e$, and let P be any shortest x–y path in G. Clearly P contains e; we may therefore assume without loss of generality that P consists of an x–a path P_1 of length ℓ_1, the edge $e = ab$, and a b–y path P_2 of length ℓ_2, where $\ell_1 \geq \ell_2$.

By replacing e by $C-e$ in P, we obtain an $x-y$ walk in $G-e$ of length $d_G(x,y) + g(e) - 2$. It follows that

$$d_G(x,y) + g(e) - 2 \geq d_{G-e}(x,y) = 2d,$$

and hence that

$$g(e) \geq 2d - d_G(x,y) + 2. \tag{1}$$

(2) Since y is not an endvertex it has a neighbour y' not on P. Since y is an eccentric vertex of x in $G-e$, a shortest x–y' path Q in $G-e$ does not contain y. Let w be the vertex at distance $d - d_G(x,y)$ from y' on Q, and let W be the b–w walk consisting of P_2, the edge yy', and the y'–w section of Q (see figure 3.3.)

We now prove that $d_{G-e}(x,w) \leq d$. Suppose, to the contrary, that $d_{G-e}(x,w) > d$. Then any shortest x–w path in G must contain e. It follows that there is a w–b path R in G of length at most $d - \ell_1 - 1$. Note that, since $\ell(W) = d - \ell_1 > \ell(R)$, $V(R) \neq V(W)$, and hence the closed walk consisting of R and W contains a cycle C'. But $\ell(C') \leq 2d - 2\ell_1 - 1 \leq 2d - d_G(x,y)$, which is less than $g(e)$ by inequality (1). Since this is clearly impossible, we conclude that $d_{G-e}(x,w) \leq d$.

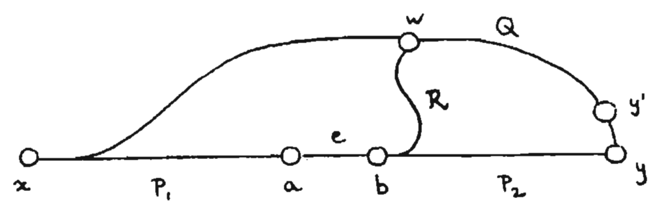


Figure 3.3: Illustration used in the proof of Theorem 3.3.5

(3) It follows that

$$\begin{aligned} 2d &= d_{G-e}(x,y) \\ &\leq d_{G-e}(x,w) + d_{G-e}(w,y') + d_{G-e}(y',y) \\ &\leq d + (d - d_G(x,y)) + 1, \end{aligned}$$

and hence that $d_G(x,y) = 1$.

Therefore, by inequality (1),

$$g(e) \geq 2d - d_G(x,y) + 2 = 2d + 1.$$

Since $g(e) \leq 2d + 1$ by Proposition 1.5.10, the result follows.

For the converse, let $g(e) = 2d + 1$. Then

$$diam(G - e) \geq d_{G-e}(a,b) = g(e) - 1 = 2d.$$

Since $diam(G - e) \leq 2d$ by Theorem 3.3.1, the result follows. □

We will use this result in our discussion of edge-diameter-doubling graphs. Note that the condition that G have no endvertices is necessary, as shown by the graph constructed in Theorem 3.3.4.

3.3.3 Edge-diameter-increasing graphs

In this subsection we consider graphs for which the removal of *any* edge increases the diameter. Such graphs model networks whose maximum message delay time increases if any link fails.

Definition: A connected graph G is called *edge-diameter-increasing* or *edi* if $diam(G - e) > diam(G)$ for every edge e of G.

For example, all trees, complete graphs and cycles are edge-diameter-increasing. Such graphs are also called diameter-minimal graphs, and were studied especially by Gliviak and Plesník (see [Gli68, Gli75a, Gli75b, GKP69a, GKP69b, GP69, GP70, GP71, Ple75a, Ple75b]).

Edi graphs have not been fully characterised, but several results are known. For example, both Gliviak and Plesník [GP70] and Greenwell and Johnson [GJ79] proved that for any graph G and any natural number $d \geq 2$ there exists an edge-diameter-increasing graph of diameter d containing G as an induced subgraph. Also, Plesník gave a construction in [Ple75a] which shows that for any $d \geq 1$ and $\kappa \geq 2$ there exists a κ-regular edi graph with diameter d and connectivity κ. These results show that the class of edi graphs is very large.

By removing edges one at a time until, if any further edge were removed, the diameter would increase, we obtain the following:

Proposition 3.3.6 *Every connected graph has a spanning subgraph of the same diameter which is edge-diameter-increasing.*

We have the following sufficient condition for a graph to be edge-diameter-increasing:

Proposition 3.3.7 [Ple75b] *If G is any graph with $g(G) \geq diam(G) + 2$, then G is edge-diameter-increasing.*

Proof:
By Proposition 3.3.3, for any cyclic edge e of G,

$$diam(G - e) \geq g(e) - 1 \geq diam(G) + 1.$$

It follows that G is edi. $\square$

The edi graph G obtained by attaching a path of length $\ell \geq 1$ to each vertex of a triangle shows that the above condition is not neccesary, since here $g(G) = 3 < 2\ell + 3 = diam(G) + 2$.

In the next few propositions we establish some properties of edi graphs.

Proposition 3.3.8 [Ple75a] *In any edge-diameter-increasing graph, there is at most one cyclic block.*

Proof:
Let G be a connected graph of diameter d containing at least two cyclic blocks. Then for some cut-vertex v of G, there are two v-components G_1 and G_2, each of which contains a cycle. Since $e_{G_1}(v) + e_{G_2}(v) \leq d$, we may assume without loss of generality that $e_{G_1}(v) \leq \frac{1}{2}d$.

Let T be a spanning tree of G which is distance-preserving from v, and let e be any edge in $E(G_1) - E(T)$. Note that removing e from G does not change the distance from v to any other vertex.

Now let x and y be any diametral pair of $G - e$. If x and y are not both in G_1, then removing e does not affect the distance between them. Otherwise, if both x and y are in G_1, then

$$d_{G-e}(x, y) \leq d_{G-e}(x, v) + d_{G-e}(v, y) \leq 2e_G(v) \leq d.$$

In either case, it follows that $diam(G - e) \leq d$, and hence that G is not edi. $\square$

Any graph of order n and diameter 1 (i.e., any complete graph) is edi and has minimum degree $n - 1$, but for edi graphs of diameter greater than 1 the situation is different. Plesník [Ple75b] proved that an edi graph of order n and diameter at least 2 has minimum degree at most $\lfloor \frac{n}{2} \rfloor$. For any $n \geq 3$, the complete bipartite graph $K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$ shows that this bound is sharp for graphs of diameter 2. Our next proposition improves Plesník's bound for graphs of diameter greater than 2.

Proposition 3.3.9 *If G is an edge-diameter-increasing graph of order n and diameter $d \geq 2$, then*

$$\delta(G) \leq \lfloor \frac{1}{2}(n - d + 2) \rfloor.$$

Proof:

Let e be any cyclic edge in G, and let x and y be any diametral pair of $G - e$. Let P be a shortest $x - y$ path in $G - e$.

Since G is edi and $d \geq 2$, $d_{G-e}(x, y) \geq d + 1 \geq 3$. Hence $N_{G-e}(x) \cap N_{G-e}(y) = \emptyset$. Note that $|N_{G-e}(x)| \geq deg_G(x) - 1$, $|N_{G-e}(y)| \geq deg_G(y) - 1$, and that P has exactly one vertex in common with each of $N_{G-e}(x)$ and $N_{G-e}(y)$. Hence

$$\begin{aligned} n &\geq |V(P)| + |N_{G-e}(x)| + |N_{G-e}(y)| - 2 \\ &\geq (d + 2) + (deg_G(x) - 1) + (deg_G(y) - 1) - 2. \end{aligned}$$

It follows that $2\delta(G) \leq deg_G(x) + deg_G(y) \leq n - d + 2$, and hence that $\delta(G) \leq \frac{1}{2}(n - d + 2)$. □

The above bound is attained for $d = 2$ by C_4, and for $d = 3$ by C_6. However, for larger values of d, it seems to be poor. Bermond and Bollobás observed in [BB81] that very little is known about the maximum possible value of $\delta(G)$ for an edi graph G.

Note that, since for any $n \geq 3$ and $d \in \{2, \ldots, n - 1\}$ the graph obtained from the star $K_{1,n-d+1}$ by replacing one edge by a path of length $d - 1$ is edi, the only upper bound on the *maximum* degree of an edi graph G of order n and diameter d is the obvious one of $\Delta(G) \leq n - d + 1$.

Plesník [Ple75a] and, independently, Simon and Murty (see [CH79]) made the following conjecture about edi graphs:

Conjecture 3.3.10 *Let G be an edge-diameter-increasing graph of diameter 2 with n vertices and m edges. Then*

$$m \leq \lfloor \frac{1}{4} n^2 \rfloor,$$

with equality holding iff $G \cong K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$.

In [Ple75a] Plesník proved that $m \leq \frac{3}{4}\binom{n}{2}$. (In fact he proved it for edi graphs of order n and *any* diameter greater than 1.)

Caccetta and Häggkvist [CH79] established the stronger bound of $m \leq 0.27n^2$. Fan [Fan87] proved the conjecture for $n \leq 24$ and $n = 26$, and improved the bound to $0.2532n^2$ for $n \geq 25$. Finally, Füredi [Fur92] proved that the conjecture is true asymptotically.

Caccetta and Häggkvist [CH79] proposed a generalisation of the conjecture to edi graphs of higher diameter. Their conjecture was, however, disproved by Krishnamoorthy and Nandakumar [KN81].

Clearly, since all trees are edi, there is no *lower* bound on the number of edges in an edi graph of order n other than the obvious one of $n - 1$.

3.3.4 Edge-diameter-doubling graphs

We showed above that $diam(G - e) \leq 2diam(G)$ for any cyclic edge e of a connected graph G. We investigate here a special class of edi graphs—viz., those graphs whose diameter doubles when *any* edge is removed.

Definition: A connected graph G is called *edge-diameter-doubling* if $diam(G - e) = 2diam(G)$ for every edge e of G.

Plesník proved in [Ple75b] that any such graph is a block. We prove the following stronger result:

Theorem 3.3.11 *Every edge-diameter-doubling graph is self-centred.*

Proof:
Let G be any graph which is not self-centred. Then G contains a vertex v with eccentricity at most $diam(G)-1$. Let T_v be a spanning tree of G which is distance-preserving from v, and let e be any edge in $E(G)-E(T_v)$. Then $d_{G-e}(v,w) = d_G(v,w)$ for every $w \in V(G)$, and so $rad(G-e) \leq e_{G-e}(v) = e_G(v) \leq diam(G)-1$. Hence

$$diam(G-e) \leq 2rad(G-e) \leq 2(diam(G)-1) < 2diam(G),$$

and G is not edge-diameter-doubling. □

Since the centre of any connected graph must be contained in one block, no self-centred graph can contain a cut-vertex. Hence Plesník's result follows from ours.

The following characterisation of edge-diameter-doubling graphs is due to Plesník, and is a direct consequence of Theorem 3.3.5.

Theorem 3.3.12 [Ple75b] *A connected graph G is edge-diameter-doubling iff $g(G) = 2diam(G)+1$.*

Proof:
Let G be an edge-diameter-doubling graph, and note that every edge of G is cyclic. It therefore follows from Theorem 3.3.5 that $g(G) \geq 2diam(G)+1$.

For the converse, let G be a connected graph with $g(G) = 2diam(G)+1$. Then by Proposition 1.5.11, G is 2-edge-connected. It follows from Theorem 3.3.5 that G is edge-diameter-doubling. □

So the study of graphs whose diameter is doubled by the removal of any edge reduces to the study of graphs with maximum possible girth for given diameter.

We conclude this section with some remarks on Moore graphs.

It is easy to prove (see, for example, [HS60]) that if a connected graph has order n, diameter d and maximum degree Δ, then

$$n \leq 1 + \Delta \sum_{i=0}^{d-1} (\Delta - 1)^i.$$

Those graphs for which equality holds in the above equation are called Moore graphs. We present next our proofs of the results that connect Moore graphs and girth.

Recall that the distance degree sequence of a vertex v is given by

$$\{1, |N_1(v)|, |N_2(v)|, \ldots, |N_{e(v)}(v)|\}.$$

Our first proposition shows that Moore graphs are distance degree regular, a result stated without proof in [BH90].

Proposition 3.3.13 *Let G be a Moore graph of diameter d and maximum degree Δ. Then every vertex in G has the same distance degree sequence*

$$\{1,\ \Delta,\ \Delta(\Delta-1),\ \Delta(\Delta-1)^2,\ \ldots,\ \Delta(\Delta-1)^{d-1}\}.$$

Proof:
Let G have radius r, and let c be any central vertex in G. Note that there are $deg(c) \leq \Delta$ vertices in $N_1(c)$. Furthermore, for $i \in \{2, \ldots, r\}$, $|N_i(c)| \leq (\Delta - 1)|N_{i-1}(c)|$, with equality holding iff every vertex in $N_{i-1}(c)$ has degree Δ and no two vertices in $N_{i-1}(c)$ are adjacent or have a common neighbour in $N_i(c)$. It follows that

$$\begin{aligned} n &\leq 1 + \sum_{i=1}^{r} \Delta(\Delta-1)^{i-1} \\ &\leq 1 + \sum_{i=1}^{d} \Delta(\Delta-1)^{i-1} \qquad \text{(since } r \leq d\text{)} \\ &= n \qquad \text{(since } G \text{ is a Moore graph).} \end{aligned}$$

Hence $r = d$ (i.e., G is self-centred), and the distance degree sequence of c is $\{1, \Delta, \Delta(\Delta-1), \Delta(\Delta-1)^2, \ldots, \Delta(\Delta-1)^{d-1}\}$.

Now note that, since G is self-centred, *any* vertex in G could have been chosen as c. Hence every vertex in G has the same distance degree sequence, and the result follows. □

It follows that if G is a Moore graph of diameter d, then G is self-centred and regular, and for any vertex $v \in V(G)$ the graph induced by $\{v\} \cup N_1(v) \cup N_2(v) \cup \ldots \cup N_{d-1}(v)$ in G is a tree. Hence (since G is Δ-regular) every vertex in $N_d(v)$ must be adjacent to $\Delta - 1$ other vertices in $N_d(v)$.

Our next two results, obtained by Bosak in [Bos70] and stated without proof in [Ple75b], characterise the Moore graphs as those cyclic graphs with the maximum possible girth for given diameter.

Proposition 3.3.14 [Bos70] *If G is a Moore graph with at least three vertices and diameter d, then $g(G) = 2d + 1$.*

Proof:

Since by Proposition 3.3.13 G is regular, and $G \not\cong K_2$, G is not a tree. Let C be a shortest cycle in G, and let v be any vertex on C. Since by the above discussion $\langle\{v\} \cup N_1(v) \cup N_2(v) \cup \ldots \cup N_{d-1}(v)\rangle_G$ is a tree, C must contain some edge uw such that both u and w are in $N_d(v)$.

Hence $g(G) = \ell(C) \geq d + 1 + d$. Since $g(G) \leq 2d + 1$ by Proposition 1.5.10, the result follows. □

Theorem 3.3.15 [Bos70] *If G is a connected graph with diameter d and girth $g(G) = 2d + 1$, then G is a Moore graph.*

Proof:

The proof is in five steps.

(1) Let v be any vertex in G, and let T be a spanning tree of G which is distance-preserving from v. Since by Theorems 3.3.12 and 3.3.11 G is self-centred, v has eccentricity d. Let F be the set of edges in $\langle N_d(v) \rangle_G$. If $T \neq G - F$, then there exists an edge uw in G that is neither in $\langle N_d(v) \rangle_G$ nor in T. Hence at least one of the vertices u and w is within a distance of $d-1$ from v. Combining the v–u and v–w paths in T with the edge uw, we obtain a closed walk in G, containing a cycle of length not exceeding $d_T(v, u) + 1 + d_T(w, v) \leq (d-1) + 1 + d = 2d < g(G)$, which is impossible. Hence it must hold that $T = G - F$.

(2) Since $g(G) = 2d + 1$, G has no bridges and hence no end-vertices. For any neighbour x of v, let S_x be the set of vertices in $N_d(v)$ which are descended from x in T, or $S_x = \{x\}$ if $d = 1$. Note that for any two neighbours x and y of v, S_x and S_y are disjoint. Note further that, since G has no vertices of degree 1, all endvertices of T must be in $N_d(v)$; hence every neighbour of v has at least one descendent in $N_d(v)$; i.e., $|S_x| \geq 1$ for every $x \in N(v)$.

(3) Now let z be any eccentric vertex of v; then $z \in S_x$ for some neighbour x of v. Note that, for *each* $y \in N(v) - \{x\}$, z must be adjacent to at least one vertex in S_y, since otherwise $d_G(z, y) > d$. It follows that z is adjacent to at least $deg(v) - 1$ other vertices in $N_d(v)$. Hence, since z is also adjacent to at least one vertex in $N_{d-1}(v)$, $deg(z) \geq deg(v)$.

(4) Since v was *any* vertex of G (so we could have chosen z instead of v and considered $v \in N_d(z)$), it follows that any two eccentric vertices have the same degree. Now let w and u be any two neighbours of v, and note that w is at distance d from every vertex in S_u. Hence w and v have at least one eccentric vertex in common, and so $deg_G(w) = deg_G(v)$. It follows that any two neighbours in G have the same degree, and hence (since G is connected) that G is regular. Let the degree of the vertices in G be Δ.

(5) Since G is Δ-regular and $G - F$ is a tree, G has order

$$n = 1 + \Delta \sum_{i=0}^{d-1} (\Delta - 1)^i.$$

This means that G is a Moore graph. □

We therefore have the following characterisation of Moore graphs:

Theorem 3.3.16 [Ple75b] *The following three statements are equivalent for any connected graph G with at least 3 vertices:*
(1) $diam(G - e) = 2diam(G)$ for every edge e in G.
(2) $g(G) = 2diam(G) + 1$.
(3) G is a Moore graph.

Moore graphs have been studied by many authors, of whom we name a few. Hoffman and Singleton showed in [HS60] that any Moore graph of diameter 2 must be an odd cycle ($\Delta = 2$), the Petersen graph ($\Delta = 3$), the Hoffman-Singleton graph ($\Delta = 7$), or possibly an (as yet unknown) Moore graph of diameter 2 and degree 57. Bannai and Ito [BI73] and, independently, Damerell [Dam73] proved that there is no Moore graph with diameter $d \geq 3$ and degree $\Delta \geq 3$. It follows that any Moore graph must be a complete graph ($d = 1, \Delta = n - 1$), an odd cycle ($d = \frac{1}{2}(n - 1), \Delta = 2$), or one of the graphs named above.

3.4 Edge Addition

3.4.1 Edge-diameter-decreasing graphs

In this subsection we consider graphs whose diameter is decreased by the addition of *any* edge.

Definition: A graph G is *edge-diameter-decreasing* or *edd* if $diam(G+e) < diam(G)$ for every $e \in E(\bar{G})$.

For example, any path or complete graph is edd. A disconnected graph is edd iff it is the disjoint union of two complete components. Edge-diameter-decreasing graphs are also called diameter-maximal graphs, and were completely characterised by Ore in [Ore68]:

Theorem 3.4.1 [Ore68] *A connected graph G of diameter $d \geq 2$ is edge-diameter-decreasing iff*
(1) G has exactly two peripheral vertices, v_0 and v_d,
(2) $\langle N_i(v_0)\rangle_G$ is complete for each $i \in \{0,\ldots,d\}$, and
(3) every vertex in $N_i(v_0)$ is adjacent to every vertex in $N_{i+1}(v_0)$ for $i \in \{0,\ldots,d-1\}$.

Proof:
It is easy to see that graphs which obey the three conditions are edge-diameter-decreasing. So suppose that G is edd, and let v_0 and v_d be a diametral pair of G.

Adding an edge in G between two vertices $u \in N_i(v_0)$ and $v \in N_j(v_0)$, where $|j-i| \leq 1$, cannot decrease the distance between v_0 and v_d, and therefore cannot decrease the diameter. Hence conditions (2) and (3) hold.

If $N_d(v_0)$ contains a vertex v'_d different from v_d, then v'_d can be joined to every vertex in $N_{d-2}(v_0)$ without decreasing the distance between v_0 and v_d — a contradiction. So $N_d(v_0)$ contains only v_d, and condition (1) follows. □

Note that the structure of a connected edd graph G of diameter $d \geq 2$ can also be described in terms of a sequential join: for some $d-1$ natural numbers n_i, G has the form

$$K_1 + K_{n_1} + K_{n_2} + \ldots + K_{n_{d-1}} + K_1.$$

It is clear that G is κ-connected iff $n_i \geq \kappa$ for every $i \in \{1, \ldots, d-1\}$. Furthermore, Caccetta and Smyth proved in [CS87b] that G is λ-edge connected iff

(1) $n_i n_{i+1} \geq \lambda$ for every $i \in \{1, \ldots, d-1\}$, and

(2) every consecutive triple $K_{n_{i-1}}, K_{n_i}, K_{n_{i+1}}$ contains at least $\lambda + 1$ vertices.

Ore [Ore68] also characterised the edd graphs of order n, diameter d, and connectivity κ with the maximum possible number of edges, as follows. When $d = 2$, $n_1 = n - 2$, and the number of edges is $\binom{n}{2} - 1$. When $d = 3$, $n_1 + n_2 = n - 2$; hence every vertex in $N_1(v_0) \cup N_2(v_0)$ has degree $n-2$, and the number of edges is $\frac{1}{2}(n-1)(n-2)$. The remainder of the characterisation is given by the next theorem.

Theorem 3.4.2 [Ore68] *An edge-diameter-decreasing graph G of order n, diameter $d \geq 4$, connectivity κ and the maximum possible number of edges has the form*

$$K_1 + K_\kappa + K_{n_2} + \ldots + K_{n_{d-2}} + K_\kappa + K_1,$$

where $n_i = \kappa$ for every i, except possibly one or two consecutive values of i for which $n_i > \kappa$.

This leads to the following upper bound on the number of edges in a graph of order n and diameter $d \geq 2$ (which is reminiscent of Vizing's bound on the size of graphs of given order and radius (see Theorem 2.4.4)):

Theorem 3.4.3 [Ore68] *A graph of order n and diameter d has at most*

$$d + \frac{1}{2}(n-d-1)(n-d+4)$$

edges, and this bound is sharp.

Obviously graphs attaining this maximum are edge-diameter-decreasing.

In a series of papers Caccetta and Smyth [CS87a, CS87b, CS87c, CS88a, CS88b] studied edge-diameter-decreasing graphs of given connectivity and edge-connectivity. In [CS87a], using results they developed in [CS87c], they showed that edd graphs of given order, diameter and *edge*-connectivity with the maximum possible number of edges have a structure similar to that of Ore's graphs, but somewhat more complicated:

Theorem 3.4.4 [CS87a] *An edge-diameter-decreasing graph G of order n, diameter $d \geq 6$, edge-connectivity $\lambda \geq 8$ and the maximum possible number of edges has the form*

$$K_1 + K_\lambda + K_{n_2} + \ldots + K_{n_{d-2}} + K_\lambda + K_1,$$

where
(1) $n_i n_{i+1} \geq \lambda$ for every $i \in \{1, \ldots, d-1\}$, and
(2) $n_{i-1} + n_i + n_{i+1} = \lambda + 1$ for all $i \in \{3, \ldots, d-3\}$, except possibly one, which must be $i = 3$ or $i = d-3$.

Note that the above theorem does not specify precise values of n_i. These in fact depend heavily on the values of n, d and λ and are given in [CS88b]. The cases $d \leq 5$ and $\lambda \leq 7$ are considered in [CS87c].

Edge-diameter-decreasing graphs were also studied by Chomenko and Ostroverchij [CO70], among others.

3.4.2 Adding the best edge

It follows from Theorem 3.3.1 that adding an edge to a graph cannot decrease the diameter by more than half. Chung and Garey [CG84] showed that when t edges are added to a graph G of diameter d, the resulting graph G'

always has diameter at least $\frac{d-t}{t+1}$. Schoone, Bodlaender and van Leeuwen [SBV87] improved this bound to $diam(G') \geq \frac{d}{t+1}$, and Kerjouan [Ker86] to $diam(G') \geq \frac{d+t-3}{t+1}$, but a sharp bound has not yet been determined.

It is easily seen that, if only the diameter d and not the order n of G is specified, then the minimum possible value of $diam(G')$ is attained when G is a path.

3.5 Vertex Removal

Unlike in the edge case, removing a vertex from a graph might increase or decrease the diameter, or leave it unchanged.

Our first proposition describes those vertices whose removal decreases the diameter, and also shows that their removal cannot decrease the diameter by more than 1. Recall that ncv stands for non-cut vertex and uep stands for unique eccentric point.

Proposition 3.5.1 [Gli76b] *Let v be a vertex of a connected graph G for which $diam(G - v) < diam(G)$. Then v is the uep of every other peripheral vertex in G, and $diam(G - v) = diam(G) - 1$.*

Proof:
Since removing v cannot decrease the distance between any of the remaining vertices, $d_G(u, w) \leq diam(G) - 1$ for every $u, w \in V(G) - \{v\}$. It follows that v is peripheral and is the uep of every other peripheral vertex u in G. Since u is still at distance $diam(G) - 1$ from the neighbours of v in $G - v$, $diam(G - v) = diam(G) - 1$. □

We note that the above necessary condition (that v is the uep of every other peripheral vertex) is not sufficient: for example, let u be any vertex of a cycle C of length $2d$, where $d \geq 3$, and let G be the graph obtained from C by joining the two neighbours of u. Let v be the vertex opposite u on C. Then v is the uep of u, and u and v are the only peripheral vertices of G, but $diam(G - v) = 2d - 3 \geq d = diam(G)$.

As a direct consequence of Proposition 3.5.1 we have the following result:

Proposition 3.5.2 [Gli76b] *No graph contains more than two vertices whose removal decreases the diameter.*

Proof:
By Proposition 3.5.1, any vertex whose removal decreases the diameter must

be an endpoint of every diametral path in G. The result then follows from the fact that a diametral path has only two endpoints. $\square$

The sequential join of $d+1$ copies of K_2 is an example of a graph whose diameter is unchanged by the removal of any vertex. We now consider graphs for which the removal of any vertex decreases the diameter.

Definition: A nontrivial connected graph G is called *vertex-diameter-decreasing* or *vdd* if $diam(G-v) < diam(G)$ for every ncv v of G.

Proposition 3.5.3 *A nontrivial connected graph is vertex-diameter-decreasing iff it is a path.*

Proof:
Clearly, any nontrivial path is vertex-diameter-decreasing. Now let G be a vdd graph. Since the removal of any ncv of G decreases its diameter, by Proposition 3.5.2 G has at most two ncv's. It follows, by Proposition 1.5.1, that G is a path. $\square$

In Section 3.3 we proved that removing a cyclic edge from a graph cannot increase its diameter by more than a factor of 2. The graphs $K_1 + P_{n-1}$ show that there is no similar bound for the removal of a ncv.

Bounds involving the removal of more than one vertex from a graph of given connectivity were investigated by Chung and Garey in [CG84].

3.5.1 Vertex-diameter-increasing graphs

In this subsection we consider graphs whose diameter increases if any vertex is removed. Such graphs model communication networks which decrease in efficiency if any node fails. They were studied by Boals, Sherwani and Ali [BSA90], Plesník [Ple75a], Gliviak [Gli76b], and Gliviak and Plesník [GP70].

Definition: A nontrivial connected graph G is called *vertex-diameter-increasing* or *vdi* if $diam(G-v) > diam(G)$ for every vertex v in G.

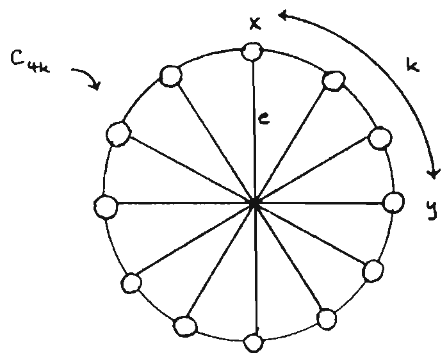


Figure 3.4: A graph which is vdi but not edi

For example, any cycle of length at least 5 is vdi, but no tree or complete graph is.

Remark: The concepts of vertex-diameter-increasing and edge-diameter-increasing graphs are independent, as shown by the following examples. Any cycle C_n, where $n \geq 5$, is both vdi and edi. Any tree is edi but not vdi. Finally, the graph G obtained from an even cycle C_{4k}, where $k \geq 3$, by joining every pair of vertices opposite each other on the cycle is vdi but not edi (see figure 3.4). Here $diam(G-e) = k = diam(G)$, but $diam(G-v) = d_{G-v}(x,y) = k+1$.

Gliviak and Plesník [GP70, Gli76b] and, independently, Boals, Sherwani and Ali [BSA90] proved that for any graph G and natural number $d \geq 2$, there exists a vertex-diameter-increasing graph of diameter d containing G as an induced subgraph. This shows that the class of vdi graphs is very large.

We have the following sufficient condition for a graph to be vdi:

Proposition 3.5.4 [Gli76b] *If G is a cyclic connected graph with $\delta(G) \geq 2$ and $g(G) \geq diam(G) + 3$, then G is vertex-diameter-increasing.*

Proof:

Let v be any ncv of G. Since v is not an end-vertex it must lie on a cycle.

Let C be a shortest cycle containing v, and let x and y be the neighbours of v on C. Then $C - v$ is a shortest x–y path in $G - v$, and so $d_{G-v}(x, y) = \ell(C) - 2 \geq g(G) - 2$. Hence $diam(G - v) \geq diam(G) + 1$, and G is vdi. □

It follows, for example, that any cycle C_n with $n \geq 5$ is vdi.

We note that C_4 has $\delta(C_4) = 2$, but $g(C_4) = diam(C_4) + 2$, and is not vdi. Since clearly no graph containing an endvertex can be vdi, it follows that the conditions of Proposition 3.5.4 cannot be improved. They are not, however, necessary conditions for a graph to be vertex-diameter-increasing: the vdi graph G in figure 3.4, for example, has girth $g(G) = 4 < diam(G) + 3$.

In the remainder of this subsection we establish some properties of vdi graphs.

Proposition 3.5.5 [BSA90, Ple75a] *Every vertex-diameter-increasing graph is a block.*

Proof:
Let G be a graph of diameter d containing a cut-vertex v. Let G_1 be a v-component of G in which v has the smallest eccentricity, and let u be an eccentric vertex of v in G_1. Finally, let the subgraph induced by all the other v-components in G be G_2.

Note that $e_{G_1}(v) + e_{G_2}(v) \leq d$, and hence that $e_{G_1}(v) \leq \frac{1}{2}d$. Note further that removing u does not change the distance from v to any other vertex in G_1.

Now let x and y be any diametral pair of $G - u$. If x and y are not both in G_1, then removing u does not affect the distance between them. Otherwise, if both x and y are in G_1, then

$$\begin{aligned} d_{G-u}(x, y) &\leq d_{G-u}(x, v) + d_{G-u}(v, y) \\ &= d_G(x, v) + d_G(v, y) \\ &\leq 2e_{G_1}(v) \\ &\leq d. \end{aligned}$$

In either case, it follows that $diam(G-u) \leq diam(G)$, and hence that G cannot be vdi. $\square$

Since any cycle C_n with $n \geq 5$ has a 2-cutset and is vdi, the theorem cannot be improved.

Proposition 3.5.6 [BSA90] *If u and v are two distinct vertices of a vertex-diameter-increasing graph, then u must have at least one neighbour that is not v or adjacent to v.*

Proof:
Let G be a graph containing two vertices u and v such that $N_G(u) \subseteq N_G(v) \cup \{v\}$.

Let x and y be any vertices in $V(G) - \{u\}$, and let P be a shortest x–y path in G. If P contains u, then it clearly does not contain v; so let P' be the path obtained from P by replacing u by v. Otherwise, if P does not contain u, let $P' = P$. So P' is also a shortest x–y path in G, and does not contain u. It follows that $d_{G-u}(x,y) = d_G(x,y)$ for all $x, y \in V(G) - \{u\}$, and hence that $diam(G-u) = diam(G)$. Hence G is not vdi. $\square$

Boals, Sherwani and Ali [BSA90] used Proposition 3.5.6 to show that if G is a vdi graph of order n, then $\Delta(G) \leq n-3$. Our next proposition is a small improvement on this result.

Proposition 3.5.7 *If G is a vertex-diameter-increasing graph of order n, diameter d and maximum degree Δ, then*

$$\Delta \leq n - d - 1.$$

Proof:
Let v be a vertex of maximum degree in G, and let u be an eccentric vertex of v. Clearly u is a ncv; let P be a diametral path of $G-u$.

Since G is vdi, v cannot have eccentricity 1 without violating Proposition 3.5.6; hence u is not adjacent to v in G, and $|N_{G-u}(v)| = \Delta$. Note that $|V(P)| = diam(G-u)+1 \geq d+2$. Since P is a shortest path, it can contain at most three vertices of $N_{G-u}(v) \cup \{v\}$. Hence

$$|V(G-u)| \geq |V(P)| + |N(v) \cup \{v\}| - 3;$$

i.e.,

$$n-1 \geq (d+2) + (\Delta+1) - 3,$$

from which it follows that

$$\Delta \leq n-d-1. \qquad \square$$

(Since for vdi graphs $d \geq 2$, this implies Boals, Sherwani and Ali's result $\Delta \leq n-3$.)

Boals, Sherwani and Ali believe that in fact for vdi graphs of diameter 2 a much stronger statement is true:

Conjecture 3.5.8 [BSA90] *If G is a vertex-diameter-increasing graph of order n and diameter 2, then $\Delta(G) \leq \frac{n}{2}$.*

If the conjecture is true, then, for a vdi graph G of order n and diameter 2, $|E(G)| \leq \frac{1}{4}n^2$. This is similar to Simon and Murty's conjecture for edi graphs (see Conjecture 3.3.10). According to Plesník [Ple75a], all known examples of vdi graphs of order n and *any* diameter have at most $\lfloor \frac{1}{4}n^2 \rfloor$ edges.

The following bound on the minimum degree of a vdi graph is due to Gliviak, and is an improvement on a bound found by Plesník in [Ple75a].

Proposition 3.5.9 [Gli76b] *If G is a vertex-diameter-increasing graph of order n and diameter d, then*

$$\delta(G) \leq \lfloor \frac{1}{2}(n-d+1) \rfloor.$$

Proof:
Let v be any vertex in G, and let x and y be any diametral pair of $G - v$. By Proposition 3.5.5, $G - v$ is connected. Let P be a shortest x–y path in $G - v$.

Since G is vdi, $d \geq 2$, and $d_{G-v}(x, y) \geq d + 1 \geq 3$. Hence $N_{G-v}(x) \cap N_{G-v}(y) = \emptyset$. We also note that $|V(P) \cap N_{G-v}(x)| = |V(P) \cap N_{G-v}(y)| = 1$, and that $|N_{G-v}(x)| \geq deg_G(x) - 1$ and $|N_{G-v}(y)| \geq deg_G(y) - 1$. Hence

$$\begin{aligned} n &\geq |V(P)| + |N_{G-v}(x)| + |N_{G-v}(y)| - 2 + |\{v\}| \\ &\geq (d + 2) + (deg_G(x) - 1) + (deg_G(y) - 1) - 2 + 1 \\ &\geq d + 2\delta(G) - 1. \end{aligned}$$

It follows that $\delta(G) \leq \frac{1}{2}(n - d + 1)$. □

Note that C_5 is a vdi graph with diameter 2 and minimum degree $\frac{1}{2}(5 - 2 + 1) = 2$, and C_6 is a vdi graph with diameter 3 and minimum degree $\frac{1}{2}(6 - 3 + 1) = 2$. It follows that the bound cannot be improved for vdi graphs of diameter 2 or 3, although it seems to be poor for vdi graphs of higher diameter.

Our next few results concern vdi graphs of small diameter.

Proposition 3.5.10 [BSA90] *Every vertex-diameter-increasing graph of diameter* 2 *is self-centred.*

Proof:
If some vertex v of a graph G has eccentricity 1, then any vertex other than v can be removed without increasing the diameter, and hence G is not vdi. It follows that no vdi graph can have radius 1, and hence that every vdi graph of diameter 2 is self-centred. □

Figure 3.5, however, shows that vdi graphs of higher diameter are not necessarily self-centred.

Figure 3.5: A vdi graph which is not self-centred

Theorem 3.5.11 [BSA90] *Removing a vertex from a vertex-diameter-increasing graph of diameter* 2 *increases the diameter by exactly* 1.

Proof:
Let G be a vdi graph of diameter 2, and let v be any vertex in G. Let x and y be any diametral pair of $G - v$.

Since $d_G(x, y) \leq 2$ and $d_{G-v}(x, y) = diam(G - v) \geq 3$, x and y must both be adjacent to v. Hence by Proposition 3.5.6, x must have a neighbour u that is not v or adjacent to v. Since $d_G(u, y) \leq 2$, it follows that u and y have a neighbour $w \neq v$ in common. Since x, u, w, y is an x–y path of length 3 in $G - v$, it follows that $diam(G - v) = 3$. □

Theorem 3.5.12 [BSA90] *Removing a vertex from a vertex-diameter-increasing graph of diameter* 3 *can increase the diameter by at most* 2.

Proof:
Let G be a vdi graph of diameter 3, and let v be any vertex in G. Let x and y be any diametral pair in $G - v$.

Since $d_G(x, y) \leq 3$ and $d_{G-v}(x, y) \geq 4$, v must lie on every x–y path of length at most 3 in G. Since $d_G(x, y) \leq 3$, either x or y (or both) must be adjacent to v. Without loss of generality we may assume y is adjacent to v.

Case (1): $d_G(x, y) = 2$.
In this case both x and y are adjacent to v. Hence, by Proposition 3.5.6, x has a neighbour u and y has a neighbour w, neither of which is adjacent to v. Since $d_{G-v}(x, y) \geq 4$, $u \neq w$.

Let P be a shortest u–w path in G, and note that, since $d_G(u, w) \leq 3$ and neither u nor w is adjacent to v, v does not lie on P. Furthermore, neither x nor y lies on P, since otherwise there would be an $x - y$ path of length not exceeding 3 that does not contain v. Hence x, P, y is an x–y walk of length at most 5 which does not contain v. It follows that $diam(G - v) \leq 5$.

Case (2): $d_G(x, y) = 3$.
In this case x is not adjacent to v. By Proposition 3.5.6, y has a neighbour u which is not adjacent to v. Let P be a shortest $x - u$ path in G. Since $d_G(x, u) \leq 3$ and neither x nor u is adjacent to v, v does not lie on P. Hence P, y is an $x - y$ walk of length not exceeding 4 that does not contain v. It follows that $diam(G - v) = 4$. □

C_6 and C_7 are examples of vdi graphs of diameter 3 whose diameters increase by 1 and 2 respectively when any vertex is removed.

After considering many examples, Boals et al made the following conjecture:

Conjecture 3.5.13 [BSA90] *Removing a vertex from a vertex-diameter-increasing graph of diameter d can increase the diameter by at most $d - 1$.*

If this bound is correct, then it is the best possible, as shown by the odd cycles.

Remark: In [Gli76b] Gliviak extended the concept of vdi graphs as follows. A graph G is said to be *$\bar{v}$-critical* if $diam(G - v) \neq diam(G)$ for every $v \in V(G)$. An example is the graph obtained from C_{2d-1} (where $d \geq 4$) by attaching one endvertex. He showed that for $d \leq 3$ the only graph of diameter

d which is $\bar{v}$-critical but not vdi is the path P_{d+1}. For $d \geq 4$, however, there are many $\bar{v}$-critical graphs of diameter d that are not vdi graphs.

Chapter 4

Distance

Recall that the distance of a graph G is the sum of the distances between all unordered pairs of vertices in G. The average or mean distance is the average distance between two vertices, where the average is taken over all pairs of distinct vertices. The distance of a vertex v in G is the sum of the distances from v to all the other vertices in G.

In this chapter we consider how the distance of a graph is affected by the removal of an edge or a vertex, the addition of an edge, or the taking of a spanning tree.

4.1 Preliminaries

In this section we establish a number of bounds on distances in graphs, which will be used in later sections. We first find bounds on the distance of a vertex in a graph.

Proposition 4.1.1 [EJS76] *For any vertex v in a connected graph G,*

$$n - 1 \leq \sigma_G(v) \leq \binom{n}{2},$$

with equality holding in the lower bound iff $e_G(v) = 1$, and in the upper bound iff G is a path with v as endpoint.

Proof:
The proof follows from the fact that

$$\sigma_G(v) = \sum_{i=1}^{n-1} |N_i(v)| i,$$

where $\sum_{i=1}^{n-1} |N_i(v)| = n-1$, $|N_i(v)| \geq 1$ for $i \in \{1, \ldots, e_G(v)\}$ and $|N_i(v)| = 0$ for $i > e_G(v)$. $\square$

Proposition 4.1.2 [EJS76] *For any vertex v in a nontrivial connected graph G,*

$$\sigma(G) \leq \sigma_G(v) + \sigma(G - v),$$

with equality holding iff v is an endvertex or every two neighbours of v are within distance 2 from each other in $G - v$.

Proof:
Note that

$$\sigma(G) = \sum_{\{u,w\} \subseteq V(G)} d_G(u, w)$$

$$
\begin{aligned}
&= \sum_{\{u,w\}\subseteq V(G-v)} d_G(u,w) + \sum_{u\in V(G)} d_G(v,u) \\
&\leq \sum_{\{u,w\}\subseteq V(G-v)} d_{G-v}(u,w) + \sum_{u\in V(G)} d_G(v,u) \\
&= \sigma(G-v) + \sigma_G(v).
\end{aligned}
$$

Equality holds in the above equation iff $d_{G-v}(u,w) = d_G(u,w)$ for every $\{u,w\} \subseteq V(G-v)$ — i.e., iff v is an endvertex, or every two neighbours of v are adjacent or have a neighbour other than v in common. □

Next we find a lower bound on the distance of a graph of given order and size.

Proposition 4.1.3 [EJS76] *If G is any graph of order n and size m, then $\sigma(G) \geq n(n-1) - m$, with equality holding iff $diam(G) \leq 2$.*

Proof:
There are m pairs of vertices at distance 1 from each other in G, and $\binom{n}{2} - m$ at distance at least 2. It follows that $\sigma(G) \geq m + 2(\binom{n}{2} - m)$, with equality holding iff there are no pairs at distance 3 or more. □

Šoltés [Sol91] used Harary's bound on diameter (Theorem 3.1.2) to establish an upper bound on the distance of a connected graph of order n and size m. He showed that it is attained by $PK_{n,m}$ for every $n \in \mathbf{N}$ and $m \in \{n-1, \ldots, \binom{n}{2}\}$. We use our Theorem 3.1.1 to characterise all graphs with maximum distance for given order and size.

Before we give the proof, we need to recall a few concepts from Chapter 1.

Recall that the path-complete graph $PK_{n,m}$ is the unique graph of order n and size m which can be obtained by joining one endvertex of a (possibly trivial) path to at least one vertex of a complete graph (see figure 1.1).

Recall further that, for a graph G and natural number i, $s_i(G)$ is the number of pairs of vertices at distance i from each other in G, and $W_i(G)$ is the set of all pairs of vertices which are non-adjacent, but within distance i from each other in G. (So $|W_1(G)| = 0$, $|W_d(G)| = \binom{n}{2} - m$, and $|W_i(G)| - |W_{i-1}(G)| = s_i$ for every $i \in \{2, \ldots, d\}$.)

Theorem 4.1.4 *Let G be a connected graph with n vertices and m edges. Then $\sigma(G) \leq \sigma(PK_{n,m})$. Moreover, equality holds iff either*
(1) $G \cong PK_{n,m}$, or
(2) $m = \binom{n}{2} - (n-1)$ and $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$, or
(3) $m \geq \binom{n}{2} - (n-2)$.

Proof:
Let D and d be the diameters of the graphs $PK_{n,m}$ and G respectively.

If $m \geq \binom{n}{2} - (n-2)$, then by Theorem 3.1.2 $diam(G) \leq diam(PK_{n,m}) \leq 2$. Hence by Proposition 4.1.3 $\sigma(G) = 2\binom{n}{2} - m = \sigma(PK_{n,m})$. Now assume that $m \leq \binom{n}{2} - (n-1)$; i.e., that $D \geq 3$.

If $d \leq 2$, then by Proposition 4.1.3 $\sigma(G) < \sigma(PK_{n,m})$. If $d \geq 3$, then

$$\begin{aligned}
\sigma(G) &= \sum_{i=1}^{d} i s_i \\
&= s_1 + \sum_{i=2}^{d} i(|W_i(G)| - |W_{i-1}(G)|) \\
&= m + (d|W_d(G)| - |W_1(G)|) - \sum_{j=1}^{d-1} |W_j(G)| \\
&= m + |W_d(G)| + \sum_{j=1}^{d-1} (|W_d(G)| - |W_j(G)|) \\
&= \binom{n}{2} + \sum_{j=1}^{d-1} \left(\binom{n}{2} - m - |W_j(G)| \right). \qquad (1)
\end{aligned}$$

By Theorem 3.1.2, $d \leq D$, and by Theorem 3.1.1, since $d \geq 3$ it holds that $|W_j(G)| \geq |W_j(PK_{n,m})|$ for $j \in \{1, \ldots, d-1\}$.

It follows that

$$\begin{aligned}\sigma(G) &\leq \binom{n}{2} + \sum_{j=1}^{D-1}\left(\binom{n}{2} - m - |W_j(PK_{n,m})|\right) \\ &= \sigma(PK_{n,m}), \qquad (2)\end{aligned}$$

by equation (1).

For equality to hold in (2) we need firstly $D = d$, and secondly $|W_j(G)| = |W_j(PK_{n,m})|$ for *every* $j \in \{1, \ldots, d-1\}$.

By Theorem 3.1.1 the second requirement is met iff $G \cong PK_{n,m}$, $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$, or $G \cong K_{n_0} + K_1 + ... + K_1 + K_{n_d}$.

If $G \cong K_{n_0} + K_1 + ... + K_1 + K_{n_d}$, where $n_0 > 1$ and $n_d > 1$, then let $P : v_0, ..., v_d$ be a diametral path of G. Now let G' be the graph obtained from G by removing the edge v_0v_1 and adding the edge $v_{d-2}v_d$, and note that $diam(G') > d$. Since by Theorem 3.1.2 $diam(G') \leq D$, it follows that $d < D$. Hence G does not meet the first requirement $d = D$, and equality does not hold in (2).

Finally, we note that if $G \cong K_1 + K_{n_1} + K_{n_2} + K_1$, then $d = 3$ and $m = \binom{n}{2} - (n-1)$. Since $PK_{n,\binom{n}{2}-(n-1)}$ also has diameter 3, it follows that in this case $d = D$, and so equality holds in (2).

The result follows. □

As a direct consequence we have the following result of Entringer, Jackson and Snyder:

Proposition 4.1.5 [EJS76] *If G is any connected graph of order n and size m, then*

$$\sigma(G) \leq \sigma(P_n) - m + n - 1,$$

with equality holding iff $G \cong P_n$ or C_3.

Proof:

The result follows from the fact that $PK_{n,m}$ can be obtained from a path P_n

by adding $m - n + 1$ edges, each of which decreases the distance by at least 1.

For equality to hold, every edge added must decrease the distance by *exactly* 1, which is possible iff $G \cong P_n$ or C_3. $\square$

This leads to the following bounds on the distance of a connected graph:

Proposition 4.1.6 [EJS76, DG77] *For any connected graph G,*

$$\binom{n}{2} \leq \sigma(G) \leq \frac{1}{6}n(n-1)(n+1),$$

with equality holding in the lower bound iff G is a complete graph, and in the upper bound iff G is a path.

Our final result in this section is based on ideas developed by Šoltés in [Sol91].

Proposition 4.1.7 *Given $n \geq 2$, let the connected graph G of order n and the vertex $v \in V(G)$ be chosen to maximise $\frac{\sigma_G(v)}{\sigma(G)}$. Then G must be a path-complete graph $PK_{n,m}$ for some $m \in \{n-1, ..., \binom{n}{2} - (n-2)\}$, and v must be an endvertex.*

Proof:
Let $e_G(v) = s$ and $|E(G)| = m$. We first note that $\langle N_i(v) \cup N_{i+1}(v)\rangle_G$ is complete for every $i \in \{0, \ldots, s-1\}$, since otherwise we could add an edge to G that would decrease $\sigma(G)$ without affecting $\sigma_G(v)$, thus contradicting our choice of G and v.

Next we show that v is an endvertex. Assume, to the contrary, that $deg_G(v) \geq 2$. Let w be any neighbour of v, and let $G' = G - vw$. Note that removing the edge vw from G increases the distance between v and w by 1, and does not affect any other distance; hence $\sigma(G') = \sigma(G) + 1$, and

$\sigma_{G'}(v) = \sigma_G(v) + 1$. It follows, since $\frac{\sigma_G(v)}{\sigma(G)} < 1$, that

$$\frac{\sigma_{G'}(v)}{\sigma(G')} = \frac{\sigma_G(v) + 1}{\sigma(G) + 1} > \frac{\sigma_G(v)}{\sigma(G)},$$

which contradicts our choice of G and v. Hence $deg_G(v) = 1$, and so $m \leq \binom{n}{2} - (n-2)$.

Finally, we show that if i is the lowest index for which $N_i(v)$ has more than one element, then $i = s-1$ or s. Assume to the contrary that $i \leq s-2$, and let v_i and v_s be any vertices in $N_i(v)$ and $N_s(v)$, respectively. Now "move v_i into $N_{i+1}(v)$ and v_s into $N_{s-1}(v)$" — i.e., remove the edge joining v_i to the vertex in $N_{i-1}(v)$, join v_i to every vertex in $N_{i+2}(v)$ and join v_s to every vertex in $N_{s-2}(v)$. If $s - i = 2$ or 3, then join v_i and v_s to each other. Note that this leaves $\sigma_G(v)$ unchanged, but decreases $\sigma(G)$, which contradicts our choice of G and v.

It follows that G is a path-complete graph with v as endvertex. □

4.2 Spanning Trees

Unlike in the cases of radius and diameter, a cyclic graph cannot have a spanning tree of the same distance (since every edge removed from a graph must increase its distance by at least 1). However, Entringer, Kleitman and Székely proved in [EKS95] that every connected graph G contains a spanning tree whose distance is less than twice that of G, and showed how to find such a tree.

Theorem 4.2.1 [EKS95] *Let T be a minimum-distance spanning tree of a connected graph G. Then*

$$\sigma(T) \leq 2(1 - \frac{1}{n})\sigma(G),$$

with equality holding iff $G \cong K_n$, in which case $T \cong K_{1,n-1}$.

Proof:

For every vertex v in G, choose a spanning tree T_v of G which is distance-preserving from v. Then for each T_v and each pair of vertices $\{u, w\} \subseteq V(G)$ we have

$$\begin{aligned} d_{T_v}(u, w) &\leq d_{T_v}(u, v) + d_{T_v}(v, w) \\ &= d_G(u, v) + d_G(v, w). \end{aligned}$$

Summing over all pairs $\{u, w\} \subseteq V(G)$, we obtain

$$\sigma(T_v) \leq (n-1)\sigma_G(v).$$

Hence

$$\begin{aligned} \sum_{v \in V(G)} \sigma(T_v) &\leq (n-1) \sum_{v \in V(G)} \sigma_G(v) \\ &= 2(n-1)\sigma(G). \end{aligned} \tag{1}$$

Now let z be a vertex in G for which $\sigma(T_z)$ is minimum. Then

$$\sigma(T) \leq \sigma(T_z) \leq \frac{1}{n} \sum_{v \in V(G)} \sigma(T_v) \leq 2(1 - \frac{1}{n})\sigma(G). \tag{2}$$

Equality holds in (1) iff $d_{T_v}(u,w) = d_{T_v}(u,v) + d_{T_v}(v,w)$ for every $v \in V(G)$ and $\{u,w\} \subseteq V(G)$. That is, iff $T_v \cong K_{1,n-1}$ for every $v \in V(G)$; i.e., iff $G \cong K_n$. Since $K_{1,n-1}$ is a minimum-distance spanning tree of K_n, it follows that equality holds in (2) iff $G \cong K_n$. □

A practical consequence of this theorem is that, while a communications network might be many times cheaper to build if modelled by a best spanning tree of a graph G instead of by G itself — in fact, as much as $\frac{|V(G)|}{2}$ times cheaper, if the building cost is taken as proportional to $|E(G)|$ — it will cost less than twice as much to operate (if operating costs are regarded as proportional to the distance).

The above proof leads to an algorithm for finding a spanning tree of a graph G whose distance is less than twice that of G: for every vertex v of G, use the breadth-first-search algorithm to construct a spanning tree T_v of G which is distance-preserving from v, and find $\sigma(T_v)$. Select the tree that gives the smallest value.

In fact, Entringer et al. give a second proof of Theorem 4.2.1 in which they show that if v is a vertex of minimum distance in a connected graph G, and T_v is a spanning tree of G which is distance-preserving from v, then

$$\sigma(T_v) \leq 2(1 - \frac{1}{n})\sigma(G).$$

This stronger result leads to a faster algorithm for finding a spanning tree of a connected graph G whose distance is less than twice that of G: simply find the distance of every vertex in G, choose a vertex v of minimum distance, and use the breadth-first-search algorithm to find a spanning tree of G which is distance-preserving from v. The second algorithm might, however, find

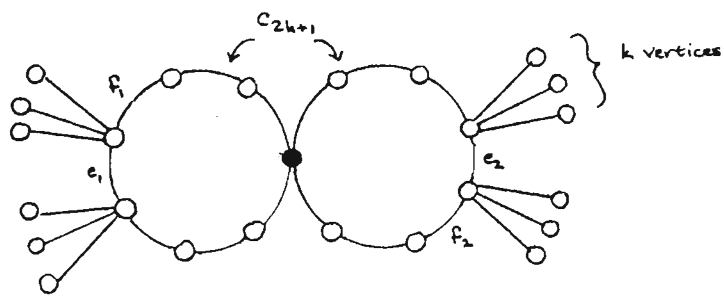


Figure 4.1: Dankelmann's graph G_k

a tree whose distance is greater than that of the trees found using the first algorithm.

The above results led Entringer et al. to pose the following two questions:

(1) Does every connected graph have a minimum-distance spanning tree which is distance-preserving from some vertex?

(2) If so, does every connected graph have a minimum-distance spanning tree which is distance-preserving from a vertex of minimum distance?

Both questions were recently answered in the negative by Dankelmann [Dan96], who gave the example G_k shown in figure 4.1. Here let T be the spanning tree of G_k obtained by removing the edges f_1 and f_2. Let v be any vertex in G_k, and note that T_v contains at most one of the edges e_1 and e_2. It is easily seen that if k is large enough, then T has smaller distance than T_v. Hence no minimum-distance spanning tree of G_k is distance-preserving from a vertex in G_k.

4.3 Edge Removal

Unlike in the radius and diameter cases, removing an edge from a connected graph *always* increases the distance (in other words, all connected graphs are distance-minimal). One can easily characterise the connected graphs G and edges $e \in E(G)$ for which removing e from G increases the distance by exactly 1:

Proposition 4.3.1 *If $e = ab$ is an edge of a connected graph G, then $\sigma(G - e) = \sigma(G) + 1$ iff a and b have at least one neighbour in common, and every neighbour of a (respectively, b) is adjacent in $G - e$ to b (a) or to some neighbour of b (a).*

Any graph of order n and minimum degree at least $\frac{1}{2}n$ is an example of a graph for which the deletion of *any* edge increases the distance by exactly 1.

4.3.1 Removing the best edge — the four-thirds conjecture

Winkler made the following conjecture in [Win89, Win86]:

Conjecture 4.3.2 *Every 2-connected graph G contains an edge e such that*

$$\frac{\mu(G-e)}{\mu(G)} \leq \frac{4}{3}.$$

This bound is attained by the odd cycles.

In [BG88], Bienstock and Györi proved the stronger statement that every connected graph containing no endvertices contains an edge whose removal increases the average distance by at most a factor of $\frac{4}{3}$. Their proof is rather ingenious, but contains some errors in the details. In the remainder of this section we give a corrected version of the proof.

Bienstock and Györi's approach is to consider the cases $m \leq n+1$ and $m \geq n+2$ separately. The former is straightforward if tedious. For the latter they introduce a weighted graph $H(G,e)$ such that the weight of the edge uv in $H(G,e)$ represents the increase in distance between the vertices u and v when e is removed from G. They then establish certain properties of $H(G,e)$, and hence a bound on its total weight. It is here that the details are often wrong.

We need some preliminary results.

Our first four propositions concern the increase in distance between two vertices u and v, when an edge e is removed. We denote this increase by $\delta_e(u,v)$. More formally,

Definition: For any edge e and vertices u, v of a graph G, we define

$$\delta_e(u,v) := d_{G-e}(u,v) - d_G(u,v).$$

Proposition 4.3.3 *If G is any connected graph, and $e = ab$ any cyclic edge of G, then*

$$\max_{\{u,v\}\subseteq V(G)} \delta_e(u,v) = \delta_e(a,b) = g(e) - 2.$$

That is, when e is removed from G, the maximum increase in distance between two vertices occurs for the endpoints of e.

Proof:

Let u and v be any two vertices of G.

If there exists a shortest $u-v$ path not containing e, then $\delta_e(u,v) = 0 \leq \delta_e(a,b)$, and we are done.

Otherwise, let P be any shortest $u-v$ path in G; say P can be partitioned into a $u-a$ section P_1, the edge e and a $b-v$ section P_2. Let C be a shortest cycle containing e in G. So $C-e$ is a shortest $a-b$ path in $G-e$, and hence $\delta_e(a,b) = g(e) - 2$.

Note that P_1, $C-e$ and P_2 together form a $u-v$ walk in $G-e$, and therefore

$$\begin{aligned} d_{G-e}(u,v) &\leq \ell(P_1)+\ell(P_2)+\ell(C-e) \\ &= d_G(u,v)-1+d_{G-e}(a,b). \end{aligned}$$

It follows that, for any $\{u,v\}\subseteq V(G)$, $\delta_e(u,v)\leq\delta_e(a,b)$. □

Proposition 4.3.4 *If $e=ab$ is any cyclic edge of a graph G, and*

$$C: a=u_0,u_1,\ldots,u_{g(e)-1}=b,a$$

is a shortest cycle containing e in G, then for $0\leq i<j\leq g(e)-1$,

$$\delta_e(u_i,u_j)=\begin{cases} 0 & \text{if } \ j-i\leq\frac{1}{2}g(e) \\ 2j-2i-g(e) & \text{if } \ j-i\geq\frac{1}{2}g(e). \end{cases}$$

Proof:
Let P_1 be the u_i-u_j section of C containing e, and P_2 the u_i-u_j section of C not containing e. Recall that by Proposition 1.5.8, either P_1 or P_2 must be a shortest u_i-u_j path in G. It follows that if $j-i\leq\frac{1}{2}g(e)$, then P_2 is a shortest u_i-u_j path in G. Since P_2 is also a path in $G-e$, this means that $\delta_e(u_i,u_j)=0$. Otherwise, if $j-i\geq\frac{1}{2}g(e)$, then P_1 is a shortest u_i-u_j path in G. Now note that since $C-e$ is a shortest $a-b$ path in $G-e$, P_2 is a shortest u_i-u_j path in $G-e$. It follows that

$$\begin{aligned} \delta_e(u_i,u_j) &= d_{G-e}(u_i,u_j)-d_G(u_i,u_j) \\ &= \ell(P_2)-\ell(P_1) \\ &= 2j-2i-g(e). \end{aligned}$$

This completes the proof. □

Proposition 4.3.5 *Let $e = ab$ be any cyclic edge of a graph G. Then for any vertex v in G, either*

$$\delta_e(v, a) = \max_{u \in V(G)} \delta_e(v, u) \text{ and } \delta_e(v, b) = 0,$$

or

$$\delta_e(v, b) = \max_{u \in V(G)} \delta_e(v, u) \text{ and } d_e(v, a) = 0.$$

Proof:

If v is equidistant from a and b, then $e = ab$ cannot lie on a shortest path from v to any other vertex, and so $\delta_e(v, u) = 0$ for *all* vertices $u \in V(G)$, including a and b, and we are done.

Otherwise, assume without loss of generality that v is closer to a than to b, and let u be any vertex such that $e = ab$ lies on a shortest $v - u$ path P (we know that u exists because b is such a vertex). Now let P' be a shortest $v - b$ path in $G - e$. Since P' followed by the $b - u$ section P'' of P is a $v - u$ walk in $G - e$,

$$\begin{aligned} d_{G-e}(v, u) &\leq \ell(P') + \ell(P'') \\ &= d_{G-e}(v, b) + d_G(b, u). \end{aligned}$$

But $d_G(v, u) = d_G(v, b) + d_G(b, u)$, since b lies on P, and hence

$$\delta_e(v, u) = d_{G-e}(v, u) - d_G(v, u) \leq d_{G-e}(v, b) - d_G(v, b) = \delta_e(v, b).$$

Since u was *any* vertex for which e lies on a shortest $v - u$ path, this means that

$$\max_{u \in V(G)} \delta_e(v, u) = \delta_e(v, b).$$

Finally, we note that the $v - a$ section of P is a shortest $v - a$ path in both G and $G - e$, and so $\delta_e(v, a) = 0$. □

Proposition 4.3.6 *Let $e = ab$ be any cyclic edge and v any vertex of a graph G. Then for any* adjacent *vertices x and y in $G - e$ (so $\{x, y\} \neq \{a, b\}$),*

$$|\delta_e(v, x) - \delta_e(v, y)| \leq 2.$$

Proof:
Since x and y are adjacent in both G and $G - e$, $|d_G(v,x) - d_G(v,y)| \leq 1$, and $|d_{G-e}(v,x) - d_{G-e}(v,y)| \leq 1$. It follows that

$$\begin{aligned}
|\delta_e(v,x) - \delta_e(v,y)| &= |(d_{G-e}(v,x) - d_{G-e}(v,y)) + (d_G(v,y) - d_G(v,x))| \\
&\leq 2.
\end{aligned}$$

This completes the proof. □

We now define some concepts which will be useful later on.

Definitions: Let G be a weighted graph in which each edge has positive integer weight. Then we define the generalised weight function $w : E(G) \cup E(\bar{G}) \to \mathbf{N} \cup \{0\}$ by letting $w(e)$ be the weight of the edge e for every $e \in E(G)$, and $w(e) = 0$ iff $e \in E(\bar{G})$. Further, we define

$$\begin{aligned}
m(v) &:= \max_{u \in V(G)} w(vu) \qquad \text{for every vertex } v \text{ in } G, \\
k &:= \max_{\{u,v\} \subseteq V(G)} w(uv), \qquad \text{and} \\
S(G) &:= \sum_{e \in E(G)} w(e).
\end{aligned}$$

Definition: For any connected unweighted graph G and cyclic edge e in G we define an associated weighted graph $H(G,e)$ as follows:

(1) The vertex set of $H(G,e)$ is the vertex set of G.

(2) If $\delta_e(u,v) > 0$ for $u, v \in V(G)$ (i.e., if removing e from G increases the distance between u and v), then the vertices u and v are joined in $H(G,e)$ by an edge of weight $w(uv) = \delta_e(u,v)$.

(3) Otherwise, if $\delta_e(u,v) = 0$, then u and v are non-adjacent in $H(G,e)$, and so $w(uv) = 0$.

The usefulness of $H(G,e)$ lies in the fact that the weight sum of $H(G,e)$ is equal to the increase in the distance of G when e is removed:

$$S(H(G,e)) = \sum_{\{u,v\}\subseteq V(G)} (d_{G-e}(u,v) - d_G(u,v)) = \sigma(G-e) - \sigma(G).$$

Definition: A weighted graph G of order n and maximum edge weight k is called a $B_{n,k}$-graph if its vertices can be 3-coloured in such a way that the following properties hold:

(1) $1 \leq k \leq n-2$.

(2) $V(G)$ is partitioned into a set R of red vertices, a set B of blue vertices and a set Y of yellow vertices, in such a way that no two vertices of the same colour are adjacent, and a vertex is yellow iff it is isolated. (In other words, G is bipartite.)

(3) R and B each contains a subset of $\left\lfloor \frac{k+1}{2} \right\rfloor$ distinguished vertices, respectively

$$B_u = \{u_0, \ldots, u_{\lfloor \frac{k-1}{2} \rfloor}\} \text{ and } R_u = \{u_{\lfloor \frac{k+4}{2} \rfloor}, \ldots, u_{k+1}\}.$$

Y contains a subset

$$Y_u = \{u_{\frac{k}{2}}, u_{\frac{k+2}{2}}\} \text{ if } k \text{ is even, or } Y_u = \{u_{\frac{k+1}{2}}\} \text{ if } k \text{ is odd.}$$

(4) The weights on the edges in $[R_u, B_u]$ are given by:
For $0 \leq i < j \leq k+1$,

$$w(u_i, u_j) = \begin{cases} 2j - 2i - k - 2 & \text{if } \; j - i > \frac{k+2}{2} \\ 0 & \text{if } \; j - i \leq \frac{k+2}{2}. \end{cases}$$

(It is easy to check that this does not violate the condition that no two vertices of the same colour be adjacent.)

(5) Denote $B_0 := B - B_u$, $R_0 := R - R_u$ and $Y_0 := Y - Y_u$.
The weights on the edges in $[R_u, B_0] \cup [R_0, B_u]$ satisfy the following condition:
For any $v \in R_0 \cup B_0$ and any integer $j \in \{2, 3, \ldots, m(v) + 1\}$ there is some vertex $u \in R_u \cup B_u$ such that $w(uv) = j$ or $j - 1$.

Remark: It might be helpful at this point to discuss in more detail the structure of $\langle R_u \cup B_u \cup Y_u \rangle_G$, where G is a $B_{n,k}$-graph.

For odd k, $\langle R_u \cup B_u \cup Y_u \rangle_G$ is shown in figure 4.2a. Each vertex u_i in B_u $(0 \leq i \leq \frac{k-1}{2})$ is joined to the $\frac{k+1}{2} - i$ vertices $u_{i+\frac{k+3}{2}}, \ldots, u_{k+1}$ in R_u, by edges of weights $1, 3, \ldots, k - 2i$ respectively. The vertex $u_{\frac{k+1}{2}}$ is isolated.

For even k, $\langle R_u \cup B_u \cup Y_u \rangle_G$ is shown in figure 4.2b. Each vertex u_i in B_u $(0 \leq i \leq \frac{k}{2} - 1)$ is joined to the $\frac{k}{2} - i$ vertices $u_{\frac{k}{2}+2+i}, \ldots, u_{k+1}$ in R_u, by edges of weights $2, 4, \ldots, k - 2i$ respectively. The vertices $u_{\frac{k}{2}}$ and $u_{\frac{k}{2}+1}$ are isolated.

From the above, it follows that the number of edges in $\langle R_u \cup B_u \rangle_G$ is

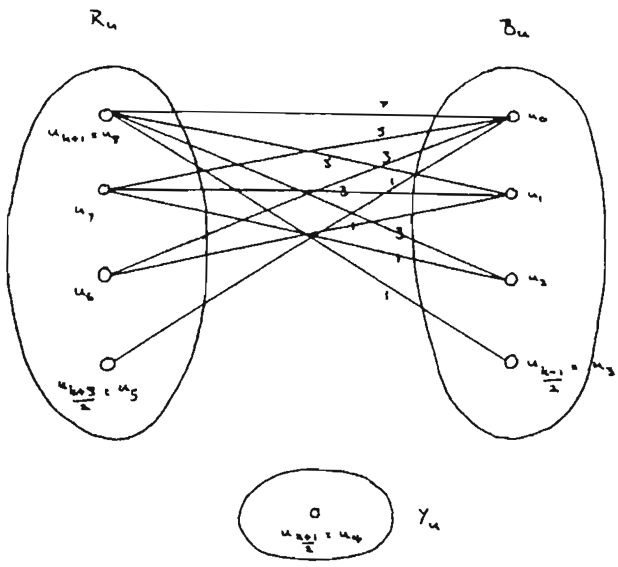
$$1 + 2 + \ldots + \left\lfloor \frac{k+1}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{k+1}{2} \right\rfloor \left\lfloor \frac{k+3}{2} \right\rfloor .$$

Moreover, the weight sum of the edges in $[R_u, B_u]$ is easily calculated:
If k is odd,

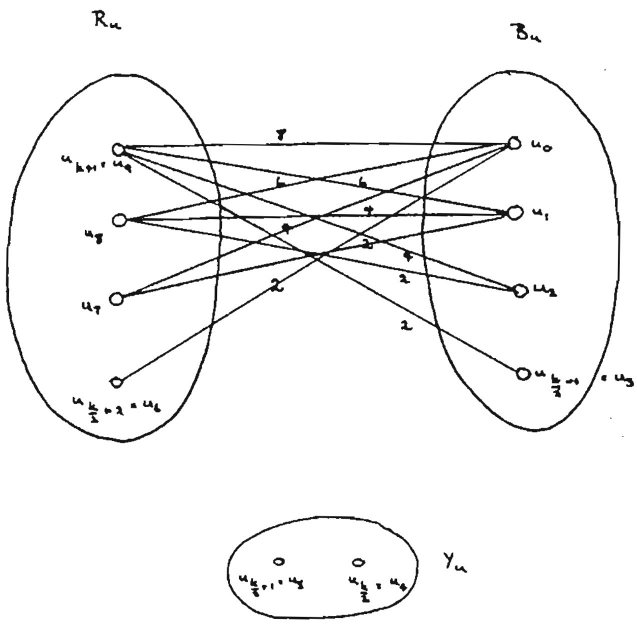
$$\begin{aligned} \sum_{0 \leq i < j \leq k+1} w(u_i u_j) &= \sum_{i=0}^{\frac{k-1}{2}} (1 + 3 + \ldots + (k - 2i)) \\ &= \frac{1}{24}(k+1)(k+2)(k+3). \end{aligned}$$

If k is even,

$$\sum_{0 \leq i < j \leq k+1} w(u_i u_j) = \sum_{i=0}^{\frac{k}{2}-1} (2 + 4 + \ldots + (k - 2i))$$
$$\frac{1}{24}k(k+2)(k+4).$$



(a) k odd (here k=7)



(b) k even (here k=8)

Figure 4.2: $\langle R_u \cup B_u \cup Y_u \rangle$ in a $B_{n,k}$ graph

Proposition 4.3.7 *Let G be any connected graph of order n containing a cyclic edge e. Let $g(e)$ be the length of a shortest cycle containing e in G. Then the associated weighted graph $H(G, e)$ is a $B_{n,k}$-graph with $k = g(e) - 2$.*

Proof:

We will establish a partition of $V(H(G, e))$ into three colour classes, and then prove that it has the defining properties of a $B_{n,k}$-graph.

(1) Let $e = ab$. Note that, by Proposition 4.3.5, any vertex v in G for which $m(v) = \max_{u \in V(G)}\{\delta_e(v, u)\} > 0$ must satisfy either $\delta_e(v, a) > 0$ or $\delta_e(v, b) > 0$, but not both inequalities. It follows that any non-isolated vertex in $H(G, e)$ must be adjacent to either a or b in $H(G, e)$, but cannot be adjacent to both.

    We can therefore assign a unique colour to every vertex in $H(G, e)$ by colouring every neighbour of a in $H(G, e)$ red, every neighbour of b blue, and every isolated vertex yellow.

(2) We now prove that no two adjacent vertices in $H(G, e)$ have the same colour.

    Let u and v be any adjacent vertices in $H(G, e)$. Then $\delta_e(u, v) > 0$; i.e., removing e from G increases the distance between u and v, and so $e = ab$ must lie on a shortest $u - v$ path P in G.

    Assume without loss of generality that a precedes b on P. Then the $u - a$ section of P is a shortest $u - a$ path in G and does not contain e. Hence $w(ua) = \delta_e(u, a) = 0$, and u is blue. Similarly, the $b - v$ section of P is a shortest $b - v$ path in G and does not contain e, and so $w(vb) = \delta_e(v, b) = 0$, and v is red. Hence u and v have different colours.

(3) Note that by Proposition 4.3.3, $k := \max_{\{u,v\} \subseteq V(H(G,e))}\{w(uv)\} = w(ab) = g(e) - 2$, and so $1 \leq k \leq n - 2$. Let

$$C : a = u_0, u_1, \ldots, u_{k+1} = b, a$$

be a shortest cycle containing e in G.

Then it follows from Proposition 4.3.4 that, for $0 \le i < j \le k+1$,

$$w(u_iu_j) = \delta_e(u_i, u_j) = \begin{cases} 2j - 2i - k - 2 & \text{if } \; j - i > \frac{k+2}{2} \\ 0 & \text{if } \; j - i \le \frac{k+2}{2}. \end{cases}$$

Substituting first $i = 0$ and then $j = k+1$ in this expression shows that the vertices in $\{u_{\lfloor \frac{k+4}{2} \rfloor}, \ldots, u_{k+1}\}$ are adjacent to a in $H(G,e)$ and hence red, while the vertices in $\{u_0, \ldots, u_{\lfloor \frac{k-1}{2} \rfloor}\}$ are adjacent to b in $H(G,e)$ and hence blue. The vertices in $\{u_{\frac{k}{2}}, u_{\frac{k+2}{2}}\}$ (if k is even) or $\{u_{\frac{k+1}{2}}\}$ (if k is odd) are adjacent to neither a nor b in $H(G,e)$ and are therefore yellow. Call these three sets respectively R_u, B_u and Y_u, and let $B_0 := B - B_u, R_0 := R - R_u$ and $Y_0 := Y - Y_u$.

(4) Finally, let v be any vertex in $R_0 \cup B_0$; suppose without loss of generality that v is red. Hence, by Proposition 4.3.5, $w(va) = m(v)$ and $w(vb) = 0$.

Note that for any $i \in \{0, \ldots, k\}$, u_i and u_{i+1} are adjacent in $G - e_0$, and hence, by Proposition 4.3.6, $|w(vu_i) - w(vu_{i+1})| \le 2$.

It follows that the weights $w(vu_i)$ for $i = 0, \ldots, k+1$ must run from 0 to $m(v)$, never missing out more than one consecutive integer — in other words, for any $j \in \{2, \ldots, m(v)+1\}$, v has some neighbour u in $R_u \cup B_u$ such that $w(vu) = j$ or $j - 1$.

This completes the proof. □

We now consider a particular example of a $B_{n,k}$-graph.

Definition: For any $n \in \mathbf{N}$, $k \in \{1, \ldots, n-2\}$ and $m \in \{1, \ldots, \binom{n}{2}\}$, we define $A_n(k,m)$ to be the (unique) $B_{n,k}$-graph with m edges that satisfies the following conditions (if such a graph exists for this value of m):

(1) $0 \le |B_0| - |R_0| \le 1$ (i.e., $|B_0|$ and $|R_0|$ differ by as little as possible, and $|B_0| \ge |R_0|$).

(2) B_0 either is empty or contains a distinguished vertex v_0.

(3) Every vertex v in R_0 is adjacent to all the blue vertices except possibly v_0. The weights of the edges in $[v, B_u]$ are given by $w(vu_0) = k$ and $w(vu_i) = 2\left\lfloor\frac{k+1}{2}\right\rfloor - 2i$ for $i = 1, \ldots, \left\lfloor\frac{k-1}{2}\right\rfloor$, while the edges in $[v, B_0 - \{v_0\}]$ all have weight k.

(4) Similarly, every vertex u in $B_0 - \{v_0\}$ is adjacent to all the red vertices. The weights of the edges in $[u, R_u]$ are given by $w(uu_i) = 2i - 2\left\lfloor\frac{k+4}{2}\right\rfloor$ for $i = \left\lfloor\frac{k+2}{2}\right\rfloor, \ldots, k$, and $w(uu_{k+1}) = k$, while the edges in $[u, R_0]$ all have weight k.

(5) The degree of v_0, on the other hand, can be any integer from 1 to $|R_u| + |R_0|$. If $deg(v_0) < |R_u| = \left\lfloor\frac{k+1}{2}\right\rfloor$, then v_0 is joined to the vertices $u_{k+2-deg(v_0)}, u_{k+1-deg(v_0)}, \ldots, u_{k+1}$ in R_u, by edges of weights $2, 4, \ldots, 2\,deg(v_0)$ respectively. Otherwise, if $deg(v_0) \geq \left\lfloor\frac{k+1}{2}\right\rfloor$, then v_0 is adjacent to all $\left\lfloor\frac{k+1}{2}\right\rfloor$ of the vertices in R_u, and to $deg(v_0) - \left\lfloor\frac{k+1}{2}\right\rfloor$ vertices in R_0. In this case, the weights of the edges in $[v_0, R_u]$ are given by $w(v_0u_i) = 2i - 2\left\lfloor\frac{k+2}{2}\right\rfloor$ for $i = \left\lfloor\frac{k+4}{2}\right\rfloor, \ldots, k$, and $w(v_0u_{k+1}) = k$, while the edges in $[v_0, R_0]$ all have weight k.

From the definition of $A_n(k, m)$ it is clear that the values of n, k and m determine both $N(v_0)$ and $|Y|$; hence $A_n(k, m)$ is unique.

Proposition 4.3.8 *If G is any $B_{n,k}$-graph with m edges, then $A_n(k, m)$ exists and*

$$S(G) \leq S(A_n(k, m)).$$

Proof:
Of all $B_{n,k}$-graphs with m edges, let H be one for which $S(H)$ is as large as possible.

(1) We first note that if v is any vertex in $B_0 \cup R_0$ and v is joined to p vertices in $B_u \cup R_u$ by edges with weights $w_1 \leq w_2 \leq \ldots \leq w_p$, then, for $i = 1, \ldots, p$,

$$w_i = \begin{cases} 2i & \text{if} \quad i < \left\lfloor \frac{k+1}{2} \right\rfloor \\ k & \text{if} \quad i = \left\lfloor \frac{k+1}{2} \right\rfloor . \end{cases} \tag{1}$$

[Otherwise, let i_0 be the minimum value of i for which equation (1) does not hold. By property (5) of $B_{n,k}$-graphs it follows that w_{i_0} must be *less* than the value predicted by (1). But then w_{i_0} can be increased by 1 without violating property (5), which contradicts our choice of H.

(Note that by property (5) $w_p = m(v)$, so if $w_{i_0} = w_p$ this procedure involves increasing $m(v)$ — this is permissible, since $m(v)$ is not being increased beyond k.)]

(2) We next note that if a vertex in R_0 (respectively, B_0) is adjacent to a vertex in B_0 (R_0), then it is already adjacent to every vertex in B_u (R_u).

[Otherwise, let v be a vertex in R_0 (say) which is adjacent to some vertex z in B_0 and non-adjacent to some vertex u in B_u. Then by observation (1) the edges in $[v, B_u]$ have weights $2, 4, \ldots, m(v)-2, m(v)$ (where $m(v) \leq k-1$), and of course $w(vz) \leq m(v)$. So deleting the edge vz and adding an edge of weight $m(v)+1 \leq k$ between v and u produces another $B_{n,k}$-graph on m edges whose weight sum is greater than that of H, which is impossible.]

It follows from observations (1) and (2) that for any vertex v in $R_0 \cup B_0$, the edges in $[v, (R_u \cup B_u)]$ have weights $2, 4, \ldots, 2\left\lfloor \frac{k-1}{2} \right\rfloor, k$ if $m(v) = k$, or $2, 4, \ldots, 2deg(v)$ if $m(v) < k$.

(3) It now follows that every edge in $[R_0, B_0]$ has weight k.

[Otherwise, let uv be an edge in $[R_0, B_0]$ such that $w(uv) < k$. Then by observations (1) and (2), $m(u) = m(v) = k$, and so $w(uv)$ can be increased to k without violating any of the properties of $B_{n,k}$-graphs.]

(4) If $R_0 \cup B_0$ is not empty, let v_0 be a vertex in $R_0 \cup B_0$ such that $m(v_0)$ is as small as possible; assume without loss of generality that v_0 is blue. Then $m(v) = k$ for every vertex v in $(R_0 \cup B_0) - \{v_0\}$.

[Suppose, to the contrary, that $m(u) < k$ for some vertex u in $(R_0 \cup B_0) - \{v_0\}$. (So of course $m(v_0) \leq m(u) < k$.) Hence all neighbours of u or v_0 are in $R_u \cup B_u$, and there is some vertex w in $R_u \cup B_u$ which is a different colour from u, but is not adjacent to u. But then deleting the edge of weight $m(v_0)$ incident to v_0 and adding an edge of weight $m(u)+1$ between u and w produces a $B_{n,k}$-graph with m edges whose weight sum is greater than that of H, which is impossible.]

(5) $|B_0|$ might be less than $|R_0|$ in H, or greater than $|R_0|$ by more than 1 — but we now show that we can construct from H a $B_{n,k}$-graph H' with the same weight sum and number of edges, which has in addition $|B_0| = |R_0|$ or $|R_0| + 1$.

If $|B_0| < |R_0|$, we move some vertex v from R_0 to B_0, replace the $|B_u|$ edges in $[v, B_u]$ with $|R_u| = |B_u|$ edges of the same weights in $[v, R_u]$, and replace the edges in $[v, B_0]$ (of which there are at most $|B_0| < |R_0|$) with the same number of edges of the same weights in $[v, R_0]$. If $|B_0| > |R_0| + 1$, then we move some vertex v from $B_0 - \{v_0\}$ to R_0, replace the $|R_u|$ edges in $[v, R_u]$ with $|B_u| = |R_u|$ edges of the same weights in $[v, B_u]$, and replace the edges in $[v, R_0]$ (of which there are at most $|R_0| < |B_0| - 1$) with the same number of edges of the same weights in $[v, B_0]$.

Repeating this procedure, we eventually obtain a $B_{n,k}$-graph H' such that $|E(H')| = m$, $S(H') = S(H)$ and $|B_0| = |R_0|$ or $|R_0| + 1$ in H'.

(6) We now show that, although H' might contain two vertices of different colour in $(R_0 \cup B_0) - \{v_0\}$ which are not adjacent, we can construct from

H' a $B_{n,k}$-graph H'', with the same weight sum and number of edges, in which $(R_0 \cup B_0) - \{v_0\}$ does induce a complete bipartite graph.

Let q be the number of pairs $\{u, v\}$ with $u \in R_0$, $v \in B_0 - \{v_0\}$ such that u and v are not adjacent (q might be 0). Note that at least q of the edges incident to v_0 must have weight k.

[Otherwise we could delete all the edges of weight k incident to v_0 (if there are any, i.e., if $m(v_0) = k$) *and* one with next-highest weight (which will be $2\left\lfloor\frac{k-1}{2}\right\rfloor$ if $m(v_0) = k$, or $m(v)$ if $m(v) < k$), and add the same number of edges to $\langle(R_0 \cup B_0) - \{v_0\}\rangle_H$, giving them all weight k and thus increasing the weight sum.]

We can therefore delete q edges of weight k incident to v_0, and add q edges of weight k to $\langle(R_0 \cup B_0) - \{v_0\}\rangle_H$ to make it complete bipartite. The graph H'' we obtain is a $B_{n,k}$-graph with m edges and the same weight sum as H.

(7) Finally, note that $A_n(k, m)$ can be obtained from H'' by rearranging the weights of the edges in $[(R_0 \cup B_0), (R_u \cup B_u)]$. It follows that $S(H'') = S(A_n(k, m))$, and hence that $A_n(k, m)$ is a $B_{n,k}$-graph with maximum weight sum for given m. $\square$

We need one more preliminary result.

Proposition 4.3.9 *Given $n \in \mathbf{N}$ and $m \leq \frac{n(n-1)(n+1)}{8(n+2)}$, let G be a B_{n,k_G}-graph of size m, for some $k_G \in \{1, 2, \ldots, n-2\}$. Then*

$$S(G) \leq \frac{m(n+2)}{3},$$

with equality holding iff n is even and $k_G = n - 2$, in which case $m = \frac{1}{8}n(n-2)$.

Proof:

Let k be chosen to maximise $S(A_n(k, m))$ for given n and m. Then by

Proposition 4.3.8 no $B_{n,k'}$-graph of size m, for *any* $k' \in \{1, 2, \ldots, n-2\}$, can have a greater weight sum than $A_n(k, m)$.

(1) We first note that if $|R_0 \cup B_0| \geq 2$, then either $A_n(k, m)$ contains exactly two isolated vertices and $deg(v_0) = 1$, or else $A_n(k, m)$ contains at least three isolated vertices.

[The proof involves a simple but tedious calculation, using the restriction $m \leq \frac{n(n-1)(n+1)}{8(n+2)}$; we give only a sketch. We assume, to the contrary, that either $|Y| = 2$ and $deg(v_0) \geq 2$, or $|Y| = 1$. From

$$n = 2\left\lfloor \frac{k+1}{2} \right\rfloor + |R_0| + |B_0| + |Y|$$

we obtain an expression for k in terms of n, $|R_0|$, $|B_0|$ and $|Y|$, which we substitute into

$$m = \frac{1}{2}\left\lfloor \frac{k+1}{2} \right\rfloor \left\lfloor \frac{k+3}{2} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor (|R_0| + |B_0| - 1) + |R_0|(|B_0| - 1) + deg(v_0).$$

We then set $m \leq \frac{n(n-1)(n+1)}{8(n+2)}$, and obtain an inequality involving n, $|R_0|$, $|B_0|$, $|Y|$ and $deg(v_0)$.

We now consider four cases, depending on whether $|B_0| = |R_0|$ or $|R_0| + 1$, and on whether $|Y| = 1$ and $deg(v_0) \geq 1$, or $|Y| = 2$ and $deg(v_0) \geq 2$. This, together with the fact that since $k \geq 1$, $n \geq |R_0| + |B_0| + |Y| + 3$, allows us in each case to write our inequality in terms of $|R_0|$ and $deg(v_0)$.

Finally we use the facts that $|R_0| \geq 1$ and that $deg(v_0) \geq 1$ or 2, depending on the case, to obtain a contradiction.]

(2) Secondly, we note that if k is odd, then $|Y| = 1$.

[Let k be odd, and suppose Y_0 is non-empty. Then we can construct $A_n(k+1, m)$ from $A_n(k, m)$ in the following way:

Add 1 to the weight of every edge in $[R_u, B_u] \cup [R_0, B_0]$, and to every edge of weight k in $[(R_0 \cup B_0), (R_u \cup B_u)]$. Then relabel the vertices of R_u, adding 1 to each index so that $R_u = \{u_{\frac{k+1}{2}}, \ldots, u_{k+2}\}$. Finally, choose any vertex in Y_0 and relabel it $u_{\frac{k+1}{2}+1}$ (i.e., move it into Y_u).

It is easy to check that now

$$w(u_i u_j) = \begin{cases} 2j - 2i - (k+1) - 2 & \text{if } \; j - i \geq \frac{(k+1)+4}{2} \\ 0 & \text{if } \; j - i \leq \frac{(k+1)+2}{2}, \end{cases}$$

so that the graph we have constructed is indeed $A_n(k+1, m)$. Hence $S(A_n(k+1, m)) > S(A_n(k, m))$, which contradicts our choice of k.]

(3) Furthermore, $|R_0 \cup B_0| \leq 1$.

[Assume, to the contrary, that $|R_0 \cup B_0| \geq 2$. Then by observation (1) $|Y| \geq 2$, and so by observation (2) k is even. We will show that this allows us to construct a graph G' from $A_n(k, m)$ which will turn out to be a $B_{n,k+2}$-graph of size m with a greater weight sum than that of $A_n(k, m)$, thus producing a contradiction.

Add a new vertex x in R_u, and join x to all the vertices in B_u by edges with weights $4, 6, \ldots, k+2$. Add a new vertex y in B_u, and join y only to x, by an edge of weight 2. Label x as $u_{(k+2)+1}$ and y as $u_{\frac{k+2}{2}-1}$, and relabel $u_{\frac{k}{2}}$ and $u_{\frac{k}{2}+1}$ in Y_0 as $u_{\frac{k+2}{2}}$ and $u_{\frac{k+2}{2}+1}$.

Now join x to every vertex in $B_0 - \{v_0\}$, and y to every vertex in R_0, by an edge of weight $k+2$. Increase the weight of every edge in $[R_0, B_0 - \{v_0\}]$ from k to $k+2$.

At this stage we have increased the number of vertices by 2, the number of edges by $\frac{k}{2} + 1 + |R_0| + |B_0 - \{v_0\}|$, and the weight sum by $\alpha = (2+4+\ldots+(k+2)) + (k+2)(|R_0| + |B_0 - \{v_0\}|) + 2|R_0|\,|B_0 - \{v_0\}|$. We now proceed in one of two ways, depending on the number of vertices in $R_0 \cup B_0$.

Case (1): $|R_0 \cup B_0| \geq 3$.

Let u be any vertex in R_0, non-adjacent to v_0 if possible, and let w be any vertex in B_0. Delete u and w and all their incident edges, thus restoring the number of vertices to n. Then the number of edges decreases by

$$|B_u \cup \{y\}| + |R_u \cup \{x\}| + |B_0 - \{v_0\}| + |R_0| - 1 - q,$$

where $q = 1$ if u is adjacent to v_0 (i.e., if $deg(v_0) = |R_0 \cup R_u|$) and $q = 0$ otherwise, and the weight sum decreases by

$$\beta = 2(2 + 4 + \ldots + (k+2)) + (k+2)(|R_0| + |B_0 - \{v_0\}| - 1 + q).$$

Note that at this stage we have decreased the number of edges by

$$(k + 1 + q + |R_0| + |B_0 - \{v_0\}|) - (\frac{k}{2} + 1 + |R_0| + |B_0 - \{v_0\}|) = \frac{k}{2} + q.$$

We want the graph we are constructing to have m edges, so we add $\frac{k}{2} + q$ edges in one of the following ways:

If $deg(v_0) = 1$, then $q = 0$ and we add $\frac{k}{2}$ edges of weights $4, 6, \ldots, k+2$ between v_0 and R_u.

Otherwise, if $deg(v_0) \geq 2$, then $|Y| \geq 3$ and hence Y_0 is not empty. Let v_1 be any vertex in Y_0, and move it to R_0, joining it to $\frac{k}{2} + q$ vertices in B_u by edges with weights $2, 4, \ldots, 2(\frac{k}{2} + q)$.

Either way, the weight sum increases by at least

$$\gamma = 2 + 4 + \ldots + 2\left(\frac{k}{2} + q\right) = \frac{1}{4}(k + 2q)(k + 2q + 2).$$

It is easy to check that the graph G we have constructed is a $B_{n,k+2}$-graph of size m (although not necessarily $A_n(k+2, m)$). But

$$\begin{aligned} S(G) &\geq S(A_n(k,m)) + \alpha - \beta + \gamma, \\ &= S(A_n(k,m)) + 2|R_0|\,|B_0 - \{v_0\}| \\ &\geq S(A_n(k,m)) + 2, \end{aligned}$$

since we are assuming $|R_0| \geq 1$ and $|B_0 - \{v_0\}| \geq 1$.

This contradicts our choice of k, and hence proves that the case $|R_0 \cup B_0| \geq 3$ cannot occur.

Case (2): $|R_0 \cup B_0| = 2$.
Let $B_0 = \{v_0\}$, $R_0 = \{v_1\}$. Then v_1 is joined to all $\frac{k+2}{2}$ vertices in $B_u \cup \{y\}$ by edges with weights $2, 4, \ldots, (k+2)$, and possibly joined to v_0 by an edge of weight $k+2$.

Recall that so far we have increased the number of vertices by 2, the number of edges by $\frac{k}{2}+1+|R_0|+|B_0-\{v_0\}| = \frac{k}{2}+2$, and the weight sum by $(2+4+\ldots+(k+2))+(k+2)(|R_0|+|B_0-\{v_0\}|)+2|R_0|\,|B_0-\{v_0\}| = (2+4+\ldots+(k+2))+(k+2)$.

If $deg(v_0) = 1$, then delete v_0 and v_1, together with all incident edges. Then the number of edges decreases by $\frac{k+2}{2}+1$, and the weight sum by $(2+4+\ldots+(k+2))+2$.

Otherwise, if $deg(v_0) \geq 2$, then recall that $|Y| \geq 3$; i.e., that Y_0 is not empty. Delete v_1 and some vertex in Y_0. Note that the edges incident with v_1 might or might not include an edge v_0v_1 of weight k — if not, then also delete the edge incident to v_0 which has weight $m(v_0) \leq k$. This decreases the number of edges by $\frac{k+2}{2}+1$, and the weight sum by at most $(2+4+\ldots+(k+2))+k$.

The graph G we have constructed is a $B_{n,k+2}$-graph of size m, but has weight sum

$$\begin{aligned} S(G) &\geq S(A_n(k,m)) + ((2+4+\ldots+(k+2))+(k+2)) \\ &\qquad\qquad -((2+4+\ldots+(k+2))+k) \\ &= S(A_n(k,m))+2, \end{aligned}$$

which is impossible.

So the case $|R_0 \cup B_0| = 2$ can also not occur, and we must have $|R_0 \cup B_0| \leq 1$.]

(4) We now show that $S(A_n(k,m)) \leq \frac{1}{3}m(n+2)$, with equality being attained iff n is even and $k = n-2$.

[Recall that, by (3), $|R_0 \cup B_0| = 1$ or 0.

Case (1): $|R_0 \cup B_0| = 1$, $n \geq k+4$.
In this case Y_0 is non-empty, and so by observation (2) k must be even.

$$\begin{aligned} S(A_n(k,m)) &= \frac{1}{24}k(k+2)(k+4) + (2+4+\ldots+2\,deg(v_0)) \\ &\leq \frac{1}{24}k(k+2)(n) + deg(v_0)(\frac{k}{2}+1), \end{aligned}$$

where we have used $n \geq k+4$ and $deg(v_0) \leq \frac{k}{2}$. Now we cunningly add two zero terms and rearrange to obtain

$$\begin{aligned} S(A_n(k,m)) &\leq \frac{(n+2)}{3}\frac{1}{8}k(k+2) + deg(v_0)\left(\frac{k}{2}+1-\frac{(n+2)}{3}\right) \\ &\qquad -\frac{1}{12}k(k+2) + \frac{(n+2)}{3}\,deg(v_0) \\ &\leq \frac{(n+2)}{3}m + deg(v_0)\left(\frac{1}{6}k-1\right) - \frac{1}{12}k(k+2), \end{aligned}$$

where we have used $m = \frac{1}{8}k(k+2) + deg(v_0)$ and $n \geq k+4$. Hence

$$\begin{aligned} S(A_n(k,m)) &\leq \frac{(n+2)}{3}m - \frac{1}{6}k\left(\frac{1}{2}(k+2) - deg(v_0)\right) - deg(v_0) \\ &< \frac{n+2}{3}m, \end{aligned}$$

since $1 \leq deg(v_0) \leq \frac{k}{2}$.

Case (2): $|R_0 \cup B_0| = 1$, $n = k+3$.
In this case Y_0 is empty, and so k can be even or odd.

If k is even, then

$$\begin{aligned} deg(v_0) &= m - \frac{1}{8}k(k+2) \\ &\leq \frac{(k+3)(k+2)(k+4)}{8(k+5)} - \frac{1}{8}k(k+2), \end{aligned}$$

by our condition on m and the fact that $n = k+3$. Simplifying, we find

$$\begin{aligned} deg(v_0) &\leq \frac{(k+2)(k+6)}{4(k+5)} \\ &< \frac{k+2}{3}. \end{aligned}$$

Again we add a zero term to our expression for $S(A_n(k,m))$ and rearrange:

$$\begin{aligned} S(A_n(k,m)) &= \frac{1}{24}k(k+2)(k+4) + \ deg(v_0)(deg(v_0)+1) \\ &= \left(\frac{1}{8}k(k+2)\frac{(k+5)}{3} - \frac{1}{24}k(k+2)\right) \\ &\quad +deg(v_0)\left(\ deg(v_0) + 1 - \frac{(k+5)}{3}\right) + \frac{(k+5)}{3}\ deg(v_0) \\ &= \frac{(k+5)}{3}\left(\frac{1}{8}k(k+2) + \ deg(v_0)\right) - \frac{1}{24}k(k+2) \\ &\quad -deg(v_0)\left(\frac{(k+2)}{3} - \ deg(v_0)\right) \\ &< \frac{(k+5)}{3}m \\ &= \frac{(n+2)}{3}m. \end{aligned}$$

The proof for odd k is similar.

Case (3): $|R_0 \cup B_0| = 0$, k is odd.
Then $m = \frac{1}{8}(k+1)(k+3)$, and

$$S(A_n(k,m)) = \frac{1}{24}(k+1)(k+2)(k+3).$$

So

$$\begin{aligned} S(A_n(k,m)) &= \frac{(k+2)}{3}m \\ &< \frac{1}{3}m(n+2). \end{aligned}$$

Case (4): $|R_0 \cup B_0| = 0$, k is even.
Then $m = \frac{1}{8}k(k+2)$, and

$$S(A_n(k,m)) = \frac{1}{24}k(k+2)(k+4).$$

So

$$\begin{aligned} S(A_n(k,m)) &= \frac{(k+4)}{3}m \\ &\leq \left(\frac{n+2}{3}\right)m, \end{aligned}$$

with equality holding iff $k = n-2$ (i.e., iff $R_0 \cup B_0 \cup Y_0$ is empty, in which case $m = \frac{1}{8}n(n-2)$).]

(5) It now follows from Proposition 4.3.8 that

$$S(G) \leq S(A_n(k_G, m)) \leq S(A_n(k,m)) \leq \frac{1}{3}(n+2)m,$$

with equality holding iff n is even and $k_G = n-2$, in which case $m = |[R_u, B_u]| = \frac{1}{8}n(n-2)$. This completes the proof. □

We are finally in a position to prove the edge case of Winkler's four-thirds conjecture.

Theorem 4.3.10 [BG88] *If G is a connected graph containing no endvertices, then G contains an edge e such that*

$$\frac{\sigma(G-e)}{\sigma(G)} \leq \frac{4}{3}.$$

Moreover, the inequality is strict iff G is not an odd cycle.

Proof:
Let $|V(G)| = n$ and $|E(G)| = m$. Since G is connected but contains no

endvertices we must have $m \geq n$. We consider three cases, each of which has a completely different proof.

Case (1): $m = n$.

Since G has no vertices of degree 1,

$$2m = \sum_{v \in V(G)} deg(v) \geq 2n = 2m.$$

It follows that G is 2-regular, and hence a cycle. Let e be any edge in G.

If n is even, then

$$\frac{\sigma(G-e)}{\sigma(G)} = \frac{\sigma(P_n)}{\sigma(C_n)} = \frac{\frac{1}{6}n(n-1)(n+1)}{\frac{1}{8}n^3} < \frac{4}{3}.$$

If n is odd, then

$$\frac{\sigma(G-e)}{\sigma(G)} = \frac{\sigma(P_n)}{\sigma(C_n)} = \frac{\frac{1}{6}n(n-1)(n+1)}{\frac{1}{8}n(n-1)(n+1)} = \frac{4}{3}.$$

So equality is attained by the odd cycles.

Case (2): $m = n + 1$.

From the fact that G has no endvertices and can be reduced to a tree by the removal of two edges, it follows that either

(1) G consists of two cycles joined by a path (which could be a P_1), or

(2) G consists of two vertices joined by three internally disjoint paths.

We consider the two cases separately:

Subcase (2.1): G consists of two cycles C_1 and C_2, which either intersect in a unique vertex v, or else are vertex disjoint and are joined by a non-trivial $v-u$ path P, with $v \in V(C_1)$ and $u \in V(C_2)$.

Let e be an edge opposite v on C_1. Note that removing e does not affect the distance from v to any other vertex, and therefore does not affect the distance between any two vertices x and y for which v lies on a shortest $x-y$ path in G. In particular,

$$d_{G-e}(x,y) = d_G(x,y) \text{ for all } x \in V(C_1) \text{ and } y \in V(P) \cup V(C_2).$$

Since of course

$$d_{G-e}(x,y) = d_G(x,y) \text{ for all } \{x,y\} \subset V(P) \cup V(C_2),$$

this means that the only distances affected by the removal of e are those between vertices in C_1. Thus

$$\begin{aligned} \sigma(G-e) - \sigma(G) &= \sum_{\{x,y\}\subset V(C_1)} (d_{G-e}(x,y) - d_G(x,y)) \\ &= \sum_{\{x,y\}\subset V(C_1)} d_{C_1-e}(x,y) - \sum_{\{x,y\}\subset V(C_1)} d_{C_1}(x,y) \\ &= \sigma(C_1 - e) - \sigma(C_1), \end{aligned}$$

and hence (since $\sigma(C_1) < \sigma(G)$),

$$\begin{aligned} \frac{\sigma(G-e)}{\sigma(G)} &= \frac{\sigma(G-e)-\sigma(G)}{\sigma(G)} + 1 \\ &< \frac{\sigma(C_1-e)-\sigma(C_1)}{\sigma(C_1)} + 1 = \frac{\sigma(C_1-e)}{\sigma(C_1)} \\ &\leq \frac{4}{3} \quad \text{(by Case (1))}. \end{aligned}$$

Subcase (2.2): G consists of two vertices, v and u, connected by three internally disjoint paths, P_1, P_2 and P_3, with $\ell(P_1) \leq \ell(P_2) \leq \ell(P_3)$.

Let $S_i = V(P_i) - \{v,u\}$ and $\ell(P_i) = \ell_i$ ($i = 1,2,3$). Further, let $C_1 = P_1 \cup P_2$, $C_2 = P_1 \cup P_3$ and $C_3 = P_2 \cup P_3$. Choose $e = ab \in E(P_3)$, where a and b are the central vertices of P_3 if ℓ_3 is odd, and a is the central vertex of P_3 if ℓ_3 is even.

Now let x and y be any vertices in $V(G)$ such that $\delta_e(x,y) > 0$. Note that, as distances between vertices of C_1 are unaffected by the removal of e, at least one of the vertices x and y must be in S_3; assume without loss of generality that y is in S_3, with $d_{P_3}(v, y) \leq d_{P_3}(u, y)$.

If $x \in S_1 \cup S_3 \cup \{u, v\}$, then clearly $d_G(x, y) = d_{C_2}(x, y)$ and $d_{G-e}(x, y) = d_{C_2-e}(x, y)$. Now consider $x \in S_2$.

Since $\delta_e(x, y) > 0$, e must lie on all shortest $x - y$ paths in G. It follows that

$$d_{C_1}(x, v) + d_{P_3}(v, y) > d_{C_1}(x, u) + d_{P_3}(u, y) = d_G(x, y),$$

and hence that $d_{C_1}(u, x) < d_{C_1}(v, x)$. So v does not lie on a shortest $u - x$ path in C_1, and hence $d_{C_1}(u, x) = d_{P_2}(u, x)$. It follows that

$$d_G(x, y) = d_{P_2}(x, u) + d_{P_3}(u, y) = d_{C_3}(x, y).$$

Finally, we note that if $x \in S_2$ then of course

$$d_{G-e}(x, y) \leq d_{C_3-e}(x, y).$$

We now have

$$\begin{aligned}
&\sigma(G - e) - \sigma(G) \\
&= \sum_{\substack{y \in S_3 \\ x \in S_1 \cup S_3 \cup \{u,v\}}} (d_{G-e}(x, y) - d_G(x, y)) + \sum_{\substack{y \in S_3 \\ x \in S_2}} (d_{G-e}(x, y) - d_G(x, y)) \\
&\leq \sum_{\substack{y \in S_3 \\ x \in V(C_2)}} (d_{C_2-e}(x, y) - d_{C_2}(x, y)) + \sum_{\substack{y \in S_3 \\ x \in S_2}} (d_{C_3-e}(x, y) - d_{C_3}(x, y)) \\
&= \sigma(C_2 - e) - \sigma(C_2) + \sum_{\substack{y \in S_3 \\ x \in S_2}} (d_{C_3-e}(x, y) - d_{C_3}(x, y)) \\
&\leq \frac{1}{3}\sigma(C_2) + \sum_{\substack{y \in S_3 \\ x \in S_2}} (d_{C_3-e}(x, y) - d_{C_3}(x, y)),
\end{aligned}$$

where we have used Case (1) and the fact that $d_{C_2-e}(x, y) - d_{C_2}(x, y) = 0$ if x and y are both in $V(C_2) - S_3$.

One may now consider four cases, depending on whether ℓ_2 and ℓ_3 are even or odd. We provide the details for only the case where ℓ_2 is even and ℓ_3 is odd.

Say $\ell_2 = 2k_2$ and $\ell_3 = 2k_3 + 1$ (so $k_2 \leq k_3$). Let C_3 be

$$u_0(= a), u_1, \ldots, u_{k_3}(= v), \ldots, u_{k_3+2k_2}(= w), \ldots, u_{2k_2+2k_3}(= b).$$

Then

$$\begin{aligned}
&\sum_{\substack{y\in S_2\\ x\in S_3}} (d_{C_3-e}(x,y) - d_{C_3}(x,y)) \\
&= 2\sum_{i=k_3+1}^{k_3+k_2-1}\left(\sum_{j=k_3+2k_2+1}^{2k_2+2k_3} (d_{C_{3-e}}(u_i,u_j) - d_{C_3}(u_i,u_j)) + \sum_{j=0}^{k_3-1}(d_{C_{3-e}}(u_i,u_j) - d_{C_3}(u_i,u_j))\right) \\
&= 2\sum_{i=k_3+1}^{k_3+k_2-1}\ \sum_{j=i+(k_2+k_3+1)}^{2k_2+2k_3} (2j - 2i - 2k_2 - 2k_3 - 1) \qquad \text{(by Proposition 4.3.4)} \\
&= \frac{2}{3}k_2(k_2-1)(k_2-\frac{1}{2}).
\end{aligned}$$

Now certainly

$$\begin{aligned}
\sum_{\substack{y\in S_3\\ x\in S_2}} d_G(x,y) &> 2\sum_{i=0}^{k_3-1}\ \sum_{j=k_3+1}^{k_3+k_2} d_G(u_i,u_j) \\
&= 2\sum_{i=0}^{k_3-1}\ \sum_{j=k_3+1}^{k_3+k_2} (j-i) \\
&= 2k_2k_3\left(\frac{1}{2}k_2 + \frac{1}{2}k_3 + 1\right) \\
&\geq 2k_2^2(k_2+1) \qquad \text{(since } k_3 \geq k_2\text{)}.
\end{aligned}$$

It follows that

$$\frac{\sum_{\substack{y\in S_3\\ x\in S_2}} [d_{C_3-e}(x,y) - d_{C_3}(x,y)]}{\sum_{\substack{y\in S_3\\ x\in S_2}} d_G(x,y)} \leq \frac{\frac{2}{3}k_2(k_2-1)(k_2-\frac{1}{2})}{2k_2^2(k_2+1)} < \frac{1}{3}.$$

Hence finally

$$
\begin{aligned}
\frac{\sigma(G-e)-\sigma(G)}{\sigma(G)} &\leq \frac{\frac{1}{3}\sigma(C_2)+\sum_{\substack{y\in S_3\\ x\in S_2}}(d_{C_2-e}(x,y)-d_{C_3}(x,y))}{(\sigma(C_1)-\sigma(P_1))+\sigma(C_2)+\sum_{\substack{y\in S_3\\ x\in S_2}} d_G(x,y)} \\
&< \frac{\frac{1}{3}\left(\sigma(C_2)+\sum_{\substack{y\in S_3\\ x\in S_2}} d_G(x,y)\right)}{(\sigma(C_1)-\sigma(P_1))+\left(\sigma(C_2)+\sum_{\substack{y\in S_3\\ x\in S_2}} d_G(x,y)\right)} \\
&< \frac{1}{3} \quad (\text{since } \sigma(C_1)>\sigma(P_1)).
\end{aligned}
$$

Thus

$$\frac{\sigma(G-e)}{\sigma(G)} < \frac{4}{3}.$$

The proof in the other three subcases differs very little.

Case (3): $m \geq n+2$.

(1) First note that if $\sigma(G) > \frac{1}{8}n(n-1)(n+1)$, then (since $\sigma(G-e) \leq \sigma(P_n)$ by Proposition 1.5.8)

$$
\begin{aligned}
\frac{\sigma(G-e)}{\sigma(G)} &\leq \frac{\sigma(P_n)}{\sigma(G)} \\
&< \frac{\frac{1}{6}n(n-1)(n+1)}{\frac{1}{8}n(n-1)(n+1)} \\
&= \frac{4}{3},
\end{aligned}
$$

and we are done. In what follows we will therefore assume that $\sigma(G) \leq \frac{1}{8}n(n-1)(n+1)$.

(2) For every pair $\{u,v\}$ of vertices in G, choose a shortest $u-v$ path $P(u,v)$. For any edge e in G, let $p(e)$ denote the number of pairs $\{u,v\}$ for which e lies on $P(u,v)$.

(Note: For a pair $\{u, v\}$ to contribute to $p(e)$, e does not have to lie on *all* shortest $u - v$ paths, just on the one chosen as $P(u, v)$ — so the removal of e need not increase the distance between u and v.)

Let $e_0 = ab$ be an edge of G for which $p(e)$ attains a minimum value.

(3) We now show that e_0 must be a cyclic edge of G. Assume, to the contrary, that $e_0 = ab$ is a bridge, and let G_1 and G_2 be the components of $G - e_0$, with $a \in V(G_1)$ and $b \in V(G_2)$. Let $|V(G_1)| = r$, $|V(G_2)| = n - r$, and suppose without loss of generality that $r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Then e_0 is contained in all paths $P(u, v)$ where u and v are in different components of $G - e_0$, and none where u and v are in the same component. So

$$\begin{aligned} p(e_0) &= |\{\{u, v\} \mid u \in V(G_1) \text{ and } v \in V(G_2)\}| \\ &= |V(G_1)|\,|V(G_2)| \\ &= r(n - r). \end{aligned}$$

Since G contains no endvertices, a must have at least one neighbour in G_1. If a is adjacent to exactly one vertex x in G_1, then the edge ax is a bridge of G, and is contained in $(r-1)(n-r+1)$ paths $P(u, v)$. But then (since $2r < n + 1$)

$$\begin{aligned} p(ax) &= r(n - r) + 2r - n - 1 \\ &< r(n - r) \\ &= p(e_0), \end{aligned}$$

which contradicts our choice of e_0. So a must have at least 2 neighbours in G_1.

Let e_1 and e_2 be two edges in G_1 incident with a. Note that none of the $r(n - r)$ paths $P(u, v)$ for which $u \in V(G_1)$ and $v \in V(G_2)$ can contain *both* e_1 and e_2. It follows that either e_1 or e_2 — say without loss of generality e_1 — is contained in at most $\frac{1}{2}r(n-r)$ paths $P(u,v)$ of this type.

Since of course e_1 is not contained in any path $P(u,v)$ for which u and v are both in G_2, and there are $\binom{r}{2}$ pairs $\{u,v\}$ for which both u and v are in G_1, we have (since $n-r \geq \frac{n}{2}$)

$$\begin{aligned} p(e_1) &\leq \frac{r(n-r)}{2} + \binom{r}{2} \\ &= r\left(\frac{n-1}{2}\right) \\ &< r(n-r) \\ &= p(e_0), \end{aligned}$$

which again contradicts our choice of e_0. It follows that e_0 must be a cyclic edge of G.

(4) Next, we note that

$$\begin{aligned} \sigma(G) &= \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) \\ &= \sum_{\{u,v\}\subseteq V(G)} (\text{number of edges in } P(u,v)) \\ &= \sum_{e\in E(G)} p(e) \\ &\geq m\, p(e_0) \\ &\geq (n+2)p(e_0). \end{aligned}$$

This gives us a restriction on the number of edges in $H(G,e_0)$, thus allowing us to use Proposition 4.3.9:

$|E(H(G,e_0))|$ is the number of pairs $\{u,v\}$ of vertices in G such that $d_{G-e}(u,v) > d_G(u,v)$ — i.e., such that e_0 lies on all shortest $u-v$ paths. Therefore $|E(H(G,e_0))|$ is at most the number of pairs $\{u,v\}$ of vertices in G such that e_0 lies on the specific $u-v$ path we have called

$P(u,v)$ — in other words, $|E(H(G,e_0))| \leq p(e_0)$. It follows that

$$|E(H(G,e_0))| \leq \frac{\sigma(G)}{n+2}.$$

So by Proposition 4.3.7, $H(G,e_0)$ is a $B_{n,k}$-graph with $k = g(e_0) - 2$, in which $|E(H(G,e_0))| \leq \frac{\sigma(G)}{n+2} \leq \frac{n(n-1)(n+1)}{8(n+2)}$. Hence by Proposition 4.3.9,

$$\begin{aligned} S(H(G,e_0)) &\leq \frac{1}{3}|E(H(G,e_0))|(n+2) \\ &\leq \frac{\sigma(G)}{3}, \end{aligned} \tag{1}$$

with equality holding iff n is even, $n = k+2 = g(e_0)$, and $\sigma(G) = |E(H(G,e_0))|(n+2) = \frac{1}{8}n(n-2)(n+2)$.

But note that if n is even and $n = g(e_0)$ then G must be an even cycle, which implies that $\sigma(G) = \frac{1}{8}n^3 > \frac{1}{8}n(n-2)(n+2)$. Hence equality cannot hold in (1).

(5) It now follows that

$$\sigma(G - e_0) - \sigma(G) = S(H(G,e_0)) < \frac{\sigma(G)}{3},$$

and hence that

$$\frac{\sigma(G-e_0)}{\sigma(G)} < \frac{4}{3}.$$

Note that equality can occur only in Case (1); specifically, iff G is an odd cycle. This completes the proof. □

Remark: Györi [Gyo88] extended the four-thirds conjecture as follows:

Conjecture 4.3.11 *Every connected graph which is not a tree contains an edge whose removal increases the average distance by at most a factor of $\frac{4}{3}$.*

In [Gyo88] he gave an example of a connected graph G with $\delta(G) = 1$ in which, if e_0 is defined as in the proof of Theorem 4.3.10, $\frac{\mu(G-e_0)}{\mu(G)} > \frac{4}{3}$ (although this G still contained an edge whose removal increased the average distance by less than a factor of $\frac{4}{3}$). In other words, if one is to attempt to prove this stronger conjecture, one has to proceed in an entirely different way.

4.3.2 Removing the worst edge

In [Ple84], Plesník posed the following problem: Given $n \in \mathbf{N}$, find the maximum possible value of $\sigma(G-e) - \sigma(G)$, where G is a 2-edge connected graph of order n and e is an edge of G. This is related to the maximum damage caused in a communications network by the failure of one link.

Favaron, Kouider and Mahéo answered this question in [FKM89], and found an infinite class of graphs attaining the bound. We give an altered version of their proof, which uses some of the results of the previous subsection.

Definition: Given $n \in \mathbf{N}$ and $k \in \{1, ..., n-2\}$, we define $F_{n,k}$ to be a $B_{n,k}$-graph with the following additional properties:

(1) Y_0 is empty, $|R_0| = \lfloor \frac{n-k-2}{2} \rfloor$, and $|B_0| = \lceil \frac{n-k-2}{2} \rceil$.

(2) Every vertex $u \in R_0$ is joined to every vertex $v \in B_0$ by an edge of weight $w(uv) = k$.

(3) For every vertex $v \in R_0 \cup B_0$,

$$w(vu_i) = \begin{cases} k - 2i & \text{if } \; 0 \le i \le \lfloor \frac{k-1}{2} \rfloor \\ 2i - k - 2 & \text{if } \; \lfloor \frac{k+4}{2} \rfloor \le i \le k+1. \end{cases}$$

We will find it useful to know the value(s) of k for which $F_{n,k}$ has its maximum possible weight sum.

Definition: Given $n \in \mathbf{N}$, we define k_n to be any value of k in $\{1, ..., n-2\}$ that maximises $S(F_{n,k})$. (For certain values of n, k_n may have more than one possible value.)

Proposition 4.3.12 *For any $n \in \mathbf{N}$,*

$$k_n = (2 - \sqrt{2})n + O(1), \qquad \text{and}$$

$$S(F_{n,k_n}) = \frac{1}{6}(\sqrt{2} - 1)n^3 + O(n^2).$$

Proof:
We seek to maximise $S(F_{n,k})$ over all $k \in \{1, ..., n-2\}$.

For any $k \in \{1, ..., n-2\}$, the weight sum of the edges in $F_{n,k}$ follows from the definitions of a $B_{n,k}$-graph and of $F_{n,k}$. The weight sum of the edges in $[R_u, B_u]$ is $\frac{1}{24}k(k+2)(k+4)$ if k is even, or $\frac{1}{24}(k+1)(k+2)(k+3)$ if k is odd. The weight sum of the edges in $[(R_0 \cup B_0), (R_u \cup B_u)]$ is $(n-k-2)(2+4+...+k)$ if k is even, or $(n-k-2)((2+4+...(k-1))+k)$ if k is odd. Finally, the weight sum of the edges in $[R_0, B_0]$ is $\lfloor\frac{n-k-2}{2}\rfloor\lceil\frac{n-k-2}{2}\rceil(k)$.

We therefore need to consider four cases, depending on whether n and k are even or odd. For example, if n and k are both odd, then

$$S(F_{n,k_n}) = \frac{1}{24}(k+1)(k+2)(k+3)+(n-k-2)((2+4+...+(k-1))+k)+\frac{1}{4}(n-k-2)^2k.$$

This is a cubic function in k, which is easily maximised to find k_n and $S(F_{n,k_n})$. The proof in the other cases is similar. $\square$

We now define a class of graphs which we will later show to attain Favaron, Kouider and Mahéo's bound.

Definition: Given $n \in \mathbf{N}$, we define $\mathscr{F}_n$ to be the set of ordered pairs (G, e), where G is a connected graph of order n and $e = ab$ a cyclic edge in G, such that G has the following form:

$V(G)$ consists of three disjoint sets A, B, and C such that

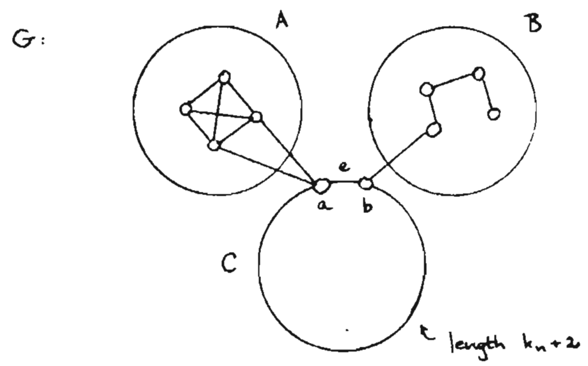


Figure 4.3: $(G, e) \in \mathscr{F}_n$

(1) $||A| - |B|| \leq 1$,

(2) $\langle C \rangle_G$ is a cycle of length $k_n + 2$ containing e,

(3) $\langle A \cup \{a\} \rangle_G$ and $\langle B \cup \{b\} \rangle_G$ are connected (but possibly trivial), and

(4) no vertex in A is adjacent to any vertex in $B \cup (C - \{a\})$, and no vertex in B is adjacent to any vertex in $A \cup (C - \{b\})$.

(See, for example, figure 4.3.) If there is more than one possible value for k_n, the definition of $\mathscr{F}_n$ allows all of them.

The connection between $F_{n,k}$ and $\mathscr{F}_n$ is shown in the next proposition.

Proposition 4.3.13 *Let G be a connected graph of order n, containing a cyclic edge e. Then $(G, e) \in \mathscr{F}_n$ iff $H(G, e) = F_{n,k_n}$.*

Proof:

It is easily seen that if $(G, e) \in \mathscr{F}_n$, then $H(G, e) \cong F_{n,k_n}$. We now prove the converse. Let $H(G, e) \cong F_{n,k_n}$, and let $e = ab$.

By Proposition 4.3.7, $H(G, e)$ is a B_{n,k_n}-graph in which the $k_n + 2 = g(e)$ vertices in $R_u \cup B_u \cup Y_u$ are the vertices inducing a shortest cycle containing

e in G. Furthermore, B (respectively, R) consists of vertices whose distance from b (respectively, a) increases when e is removed from G.

Let $C = R_u \cup B_u \cup Y_u$, $A = B_0$ and $B = R_0$. Since $H(G,e) \cong F_{n,k_n}$, Y_0 is empty and $|B_0| - |R_0| \leq 1$. Hence $A \cup B \cup C = V(G)$, and $||A| - |B|| \leq 1$. Furthermore, for every $u \in A$ and $v \in B$, $w(uv) = k_n$. In other words, removing e from G increases the distance between u and v by $k_n = g(e) - 2$. Hence e must lie on a shortest $u - v$ path P in G, and the walk obtained from P by replacing e by $\langle C \rangle_G - e$ must be a shortest $u - v$ path in $G - e$. It follows that u (respectively, v) cannot be adjacent to any vertex in C, except possibly a (respectively, b), and that u and v cannot be adjacent to each other.

Finally, we note that $\langle A \cup \{a\} \rangle_G$ and $\langle B \cup \{b\} \rangle_G$ are connected, since G is connected. It now follows that $(G,e) \in \mathscr{F}_n$. □

We are now in a position to prove Favaron, Kouider and Mahéo's upper bound on $\sigma(G-e) - \sigma(G)$. We strengthen their result slightly by proving that equality is attained *only* by the pairs (G,e) in $\mathscr{F}_n$.

Theorem 4.3.14 [FKM89] *Given $n \in \mathbb{N}$, $\sigma(G-e)-\sigma(G)$ is maximised over all connected graphs G of order n and cyclic edges $e \in E(G)$ iff $(G,e) \in \mathscr{F}_n$. The value of the maximum difference is $\frac{1}{6}(\sqrt{2}-1)n^3 + O(n^2)$.*

Proof:

Let G be any connected cyclic graph of order n, and let $e = ab$ be any cyclic edge in G.

By Proposition 4.3.5, for every vertex v in G, $\max_{u \in V(G)} w(vu)$ is either $w(va)$ or $w(vb)$. It follows, by Proposition 4.3.7, that $H(G,e)$ is a $B_{n,k}$-graph for $k = g(e) - 2$, with the additional property that for every $v \in V(H(G,e))$, $m(v) = w(va)$ if v is red, or $w(vb)$ if v is blue.

Recall that $a = u_0$ and $b = u_{k+1}$. Hence, by property (4) of $B_{n,k}$-graphs, for $i \in \{1, ..., k+1\}$,

$$m(u_i) = \begin{cases} w(uu_{k+1}) = 2(k+1) - 2i - k - 2 = k - 2i & \text{if } 0 \leq i \leq \lfloor\frac{k-1}{2}\rfloor \\ w(uu_0) = 2i - 0 - k - 2 = 2i - k - 2 & \text{if } \lfloor\frac{k+4}{2}\rfloor \leq i \leq k+1. \end{cases}$$

It follows that, for any vertex v in $R_0 \cup B_0$,

$$\begin{aligned} \sum_{i=0}^{k+1} w(vu_i) &\leq \sum_{i=0}^{k+1} m(u_i) \\ &= \begin{cases} 1 + 3 + ... + k & \text{if } k \text{ is odd} \\ 2 + 4 + ... + k & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Recall that in a $B_{n,k}$-graph, $|R_0| \leq \lfloor\frac{n-k-2}{2}\rfloor$, $|B_0| \leq \lceil\frac{n-k-2}{2}\rceil$, and

$$\sum_{0 \leq i \leq j \leq k+1} w(u_i u_j) = \begin{cases} \frac{1}{24}(k+1)(k+2)(k+3) & \text{if } k \text{ is odd} \\ \frac{1}{24}k(k+2)(k+4) & \text{if } k \text{ is even.} \end{cases}$$

Hence, if k is odd,

$$\begin{aligned} \sigma(G-e) - \sigma(G) &= S(H(G, e_0)) \\ &\leq \lfloor\frac{n-k-2}{2}\rfloor\lceil\frac{n-k-2}{2}\rceil k + (n-k-2)(1+3+...+k) \\ &\quad + \frac{1}{24}(k+1)(k+2)(k+3) \\ &= S(F_{n,k}) \\ &\leq S(F_{n,k_n}) \end{aligned} \tag{1}$$

with equality holding iff $H(G,e) \cong F_{n,k_n}$ (for one of possibly more values of k_n).

Similarly, if k is even,

$$\sigma(G-e) - \sigma(G) \leq S(F_{n,k_n}), \tag{2}$$

with equality holding iff $H(G,e) \cong F_{n,k_n}$.

It now follows from Proposition 4.3.13 that the maximum value of $\sigma(G - e) - \sigma(G)$ is attained iff $(G,e) \in \mathscr{F}_n$. By Proposition 4.3.12, the value of this maximum is

$$\frac{1}{6}(\sqrt{2}-1)n^3 + O(n^2).$$

This completes the proof. □

Some remarks:

(1) Plesník posed his question for 2-edge-connected graphs, but Favaron et al only demanded that G be cyclic. Note, however, that for $(G,e) \in \mathscr{F}_n$ $\sigma(G-e) - \sigma(G)$ is unaffected by the structure of $\langle A \rangle_G$ and $\langle B \rangle_G$, as long as they are both connected. It follows that requiring G to be 2-edge-connected would not give a better bound.

(2) Since all graphs attaining the upper bound have cut-vertices, demanding higher vertex-connectivity does decrease the upper bound. In [FKM89], however, an example is given which shows that it does not reduce the *order* $O(n^3)$ of the upper bound.

(3) In [CM80] Cockayne and Miller determined the edge e which, when added to the path P_n, minimises $\sigma(P_n + e)$. Their results agree with Theorem 4.3.14 in this particular case — i.e., the edge e will be added to the path in such a way as to make the resultant cycle have length k_n, and the trailing "ends" have roughly the same length.

(4) Note that Theorem 4.3.14 improves the trivial bound

$$\sigma(G-e) - \sigma(G) \leq \frac{1}{6}n(n-1)(n-2) = \frac{1}{6}n^3 + O(n^2),$$

obtained from the inequalities

$$\binom{n}{2} \leq \sigma(G) \leq \frac{1}{6}n(n-1)(n+1).$$

(See Proposition 4.1.6.)

(5) Favaron et al also found in [FKM89] an upper bound (in terms of n) for the ratio $\frac{\sigma(G-e)}{\sigma(G)}$, where G is a connected graph of given order n and e is a cyclic edge of G:

$$\frac{\sigma(G-e)}{\sigma(G)} \leq \frac{\sqrt{n}}{2\sqrt{3}} + O(1).$$

The coefficient $\frac{1}{2\sqrt{3}}$ does not seem to be the best possible, but an example is given to show that the order $O(\sqrt{n})$ is exact.

(6) Favaron et al also found a connected graph G of order n containing a cyclic edge e for which $\frac{\sigma(G-e)}{\sigma(G)} \to \infty$ as $n \to \infty$. In other words, the failure of a particular link can cause an arbitrarily large amount of damage to a network for which the average distance is an important measure of performance — unlike in the cases of radius and diameter, where the failure of one link can at most double the parameter concerned.

4.4 Vertex Removal

Unlike in the case of edge removal, when a vertex is removed from a graph the distance can increase, stay the same, or decrease. For example, removing any vertex from an odd cycle C_n decreases the distance if $n < 11$, leaves the distance unchanged if $n = 11$, and increases the distance if $n > 11$.

4.4.1 Removing the worst vertex

In this subsection we consider the maximum possible increase in the distance of a graph caused by the removal of a ncv (non-cut vertex). This problem was explored by Šoltés in [Sol91]. We consider both the absolute and the relative change in distance, and find that they lead to the same extremal graphs.

The following upper bounds were given by Šoltés [Sol91], who also showed that the bound is attained by $G = v + PK_{n-1,m-n+1}$. We have improved his result by characterising all graphs attaining the bound.

Theorem 4.4.1 [Sol91] *Let the connected graph G of order $n \geq 2$ and size $m \geq 2n-3$ and the ncv v in G be chosen to maximise $\sigma(G-v) - \sigma(G)$ or $\frac{\sigma(G-v)}{\sigma(G)}$.*
If $m \leq \binom{n-1}{2}$, then $G = v + PK_{n-1,m-n+1}$. If $m = \binom{n-1}{2} + 1$, then $G = v + PK_{n-1,m-n+1}$, or $G = v + (K_1 + K_{n_1} + K_{n_2} + K_1)$, where $n_1 + n_2 = n-3$. If $m \geq \binom{n-1}{2} + 2$, then G can be any graph of order n and size m in which v has eccentricity 1. (In this case, $diam(G-v) \leq 2$.)

Proof:
By Proposition 4.1.3, since the graph $w + PK_{n-1,m-n+1}$ has diameter at most 2,

$$\sigma(G) \geq n(n-1) - m = \sigma(w + PK_{n-1,m-n+1}),$$

with equality holding iff $diam(G) \leq 2$.

The graph $G-v$ has $n-1$ vertices and $m-deg_G(v)$ edges, where $deg_G(v) \leq n-1$. Hence, from Theorem 4.1.4 and the fact that $PK_{n-1,m-n+1}$ can be obtained from $PK_{n-1,m-deg_G(v)}$ by removing edges (if necessary), it follows that

$$\begin{aligned}\sigma(G-v) &\leq \sigma(PK_{n-1,m-deg_G(v)}) \\ &\leq \sigma(PK_{n-1,m-n+1}),\end{aligned}$$

with equality holding iff $deg_G(v) = n-1$ and either $G = v + PK_{n-1,m-n+1}$, or $G = v + (K_1 + K_{n_1} + K_{n_2} + K_1)$, or $m \geq \binom{n-1}{2} + 2$.

Combining these results, we obtain

$$\sigma(G-v) - \sigma(G) \leq \sigma(PK_{n-1,m-n+1}) - \sigma(w + PK_{n-1,m-n+1}),$$

and

$$\frac{\sigma(G-v)}{\sigma(G)} \leq \frac{\sigma(PK_{n-1,m-n+1})}{\sigma(w + PK_{n-1,m-n+1})},$$

where equality holds iff G is one of the graphs listed in the statement of the theorem. Note finally that if $m \geq \binom{n-1}{2} + 2$, then by Theorem 3.1.2 $diam(G-v) \leq diam(PK_{n,m}) \leq 2$. This completes the proof. □

Corollary 4.4.2 *The expressions $\sigma(G-v)-\sigma(G)$ and $\frac{\sigma(G-v)}{\sigma(G)}$ are maximised over all connected graphs G and ncv's $v \in G$ iff $G = v + P_{n-1}$.*

The proof is lengthy, though not difficult, and we omit it.

4.4.2 Removing the best vertex, in the best case

Removing a node from a network might actually increase the efficiency of the remaining network. Here we investigate the maximum possible decrease in operating costs — i.e. in distance — caused by the removal of one vertex.

Again we consider both the absolute and the relative change in distance; these lead to different extremal graphs.

We first find a lower bound on $\sigma(G-v)-\sigma(G)$.

Proposition 4.4.3 *The expression $\sigma(G-v)-\sigma(G)$ is minimised over all connected graphs G of order $n \geq 2$ and ncv's $v \in V(G)$ iff $G \cong P_n$ and v is an endvertex.*

Proof:
Let the connected graph G of order n and the ncv $v \in V(G)$ be chosen to minimise $\sigma(G-v)-\sigma(G)$.

We first show that v is an endvertex. Assume, to the contrary, that $deg_G(v) \geq 2$, and let e be any edge incident with v. Since v is a ncv in G, $G-e$ is a connected graph of order n which also contains v as a ncv. Since $\sigma(G-e) > \sigma(G)$, while $\sigma((G-e)-v) = \sigma(G-v)$, this contradicts our choice of G. It follows that $deg_G(v) = 1$.

Hence $\sigma(G) = \sigma(G-v) + \sigma_G(v)$, and $\sigma(G-v)-\sigma(G)$ is minimised iff $\sigma_G(v)$ is maximised. It now follows from Proposition 4.1.1 that G is a path with v as endpoint. □

The following lower bound for $\frac{\sigma(G-v)}{\sigma(G)}$ was given by Šoltés [Sol91], who also showed that there exists a path-complete graph attaining the bound. We have improved the result slightly by proving that, in fact, only this path-complete graph attains the bound.

Theorem 4.4.4 *Given $n \geq 2$, let the connected graph G of order n and the ncv $v \in V(G)$ be chosen to minimise $\frac{\sigma(G-v)}{\sigma(G)}$. Then G is a path-complete graph and v is an end-vertex (i.e., $G \cong PK_{n,m}$ for some $m \in \{n-1, \ldots, \binom{n}{2} - (n-2)\}$).*

Proof:
We first note that v must be an endvertex, since otherwise removing any

edge incident with v does not change $\sigma(G-v)$, but increases $\sigma(G)$. Hence $\sigma(G) = \sigma(G-v) + \sigma_G(v)$, and so

$$\frac{\sigma(G-v)}{\sigma(G)} = 1 - \frac{\sigma_G(v)}{\sigma(G)}.$$

Therefore $\frac{\sigma(G-v)}{\sigma(G)}$ is minimised over all connected graphs G of order n and ncv's $v \in V(G)$ iff $\frac{\sigma_G(v)}{\sigma(G)}$ is maximised over all connected graphs G of order n and endvertices $v \in V(G)$.

Hence by Proposition 4.1.7, G must be a path-complete graph with v as endvertex. So, since there are $n-2$ vertices not adjacent to v in G, $G \cong PK_{n,m}$ for some $m \in \{n-1, \ldots, \binom{n}{2} - (n-2)\}$. □

Remark: We have now proved that, if G is a connected graph of order $n \geq 2$ and v is a ncv of G, then

$$\min_{n-1 \leq m \leq \binom{n}{2}-(n-2)} \frac{\sigma(PK_{n-1,m-1})}{\sigma(PK_{n,m})} \leq \frac{\sigma(G-v)}{\sigma(G)}.$$

It is of course possible to find an expression for $\sigma(PK_{n,m})$ and then minimise over m in the given range to find the bound in terms of n only. If $n \geq 5$, then the extremal graph is not a path; hence considering the absolute and the relative change in distance leads to different extremal graphs.

4.4.3 Removing the best vertex, in the worst case — the four-thirds conjecture

In the worst case, removing even the *best* node from a network increases the average cost of sending a message. Here we investigate the maximum possible extent of the damage.

Winkler [Win89, Win86] stated a vertex case of the four-thirds conjecture:

Conjecture 4.4.5 *Every connected graph G contains a vertex v such that*

$$\frac{\mu(G-v)}{\mu(G)} < \frac{4}{3}.$$

If the conjecture is true, then the cycle C_n shows that the bound cannot be improved, since $\frac{\mu(P_{n-1})}{\mu(C_n)} = \frac{4\lfloor\frac{1}{2}n\rfloor}{3\lceil\frac{1}{2}n\rceil} \to \frac{4}{3}$ as $n \to \infty$.

Bienstock and Györi proved in [BG88] that the conjecture is true asymptotically; in fact they proved that every connected graph G of order n has a vertex whose removal increases the average distance by at most a factor of $\frac{4}{3} + O(n^{-\frac{1}{5}})$. The proof has much in common with their proof of the edge case (described in section 4.3.1), and we do not include it.

Althöfer [Alt90] proved the vertex case of the four-thirds conjecture completely for 4-connected graphs, and in fact strengthened it for more highly connected graphs:

Theorem 4.4.6 [Alt90] *For $\ell \geq 2$, every ℓ-connected graph has a vertex whose removal increases the average distance by less than a factor of $\frac{\ell}{\ell-1}$.*

Proof:
The proof is based on Menger's Theorem, and works by counting.

Let G be an ℓ-connected graph of order n (where $n \geq 3$, since $\ell \geq 2$), and let u and w be any two vertices in $V(G)$. By Menger's Theorem, there exist ℓ internally-disjoint paths $P_1, \ldots, P_\ell$ between u and w, each of which has length at least $d_G(u,w)$.

Now let P be any shortest $u-w$ path, and let v be any vertex in $V(G) - \{u,w\}$. If v does not lie on P, then removing v does not affect the distance between u and w. Otherwise note that, since the paths $P_1, \ldots, P_\ell$ are internally disjoint, removing v can destroy at most one of them — we may therefore assume, without loss of generality, that $P_2, \ldots, P_\ell$ are not destroyed. Hence

$$d_{G-v}(u,w) \leq \min_{2 \leq i \leq \ell} \ell(P_i)$$

$$
\begin{aligned}
&\le \frac{(\text{number of internal vertices in } P_2, \ldots, P_\ell)}{\ell - 1} + 1 \\
&\le \frac{n - \ (\text{number of vertices in } P_1)}{\ell - 1} + 1 \\
&\le \frac{n - d_G(u,w) - 1}{\ell - 1} + 1.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{v \in V(G) - \{u,w\}} d_{G-v}(u,w) \\
&= \sum_{v \text{ not on } P} d_{G-v}(u,w) + \sum_{v \text{ internal on } P} d_{G-v}(u,w) \\
&\le (n - d_G(u,w) - 1) d_G(u,w) + (d_G(u,w) - 1)\left(\frac{n - d_G(u,w) - 1}{\ell - 1} + 1\right) \\
&= (n-2) d_G(u,w)\left(1 + \frac{1}{\ell - 1}\right) - \\
&\qquad (d_G(u,w) - 1)^2 - \frac{(n-2)}{\ell - 1} - \frac{(d_G(u,w) - 1)^2}{\ell - 1} \\
&\le (n-2)\frac{\ell}{\ell - 1} d_G(u,w) - \frac{(n-2)}{\ell - 1} \\
&< (n-2)\frac{\ell}{\ell - 1} d_G(u,w), \qquad (1)
\end{aligned}
$$

since $n \ge 3$. So finally,

$$
\begin{aligned}
\min_{v \in V(G)} \mu(G - v) &\le \frac{1}{n} \sum_{v \in V(G)} \mu(G - v) \\
&= \frac{1}{n} \sum_{v \in V(G)} \frac{1}{\binom{n-1}{2}} \left(\sum_{\{u,w\} \subseteq V(G) - \{v\}} d_{G-v}(u,w) \right) \\
&= \frac{1}{(n-2)} \frac{1}{\binom{n}{2}} \sum_{\{u,w\} \subseteq V(G)} \left(\sum_{v \in V(G) - \{u,w\}} d_{G-v}(u,w) \right) \\
&< \frac{1}{(n-2)} \frac{1}{\binom{n}{2}} \sum_{\{u,w\} \subseteq V(G)} (n-2)\frac{\ell}{\ell - 1} d_G(u,w) \qquad \text{(by (1))} \\
&= \frac{\ell}{\ell - 1}\, \mu(G).
\end{aligned}
$$

This completes the proof. □

4.4.4 Vertex-distance-stable graphs

An interesting open question is posed by Šoltés in [Sol91]: Determine all graphs for which the removal of any vertex leaves the distance unchanged. The only such graph of which Šoltés was aware is C_{11}.

Chapter 5

Conclusion

In this thesis we have explored in depth three measures of distance in a graph. Several open questions remain. Of these, the characterisation of edge-radius-decreasing graphs and Conjecture 3.5.13 on vertex-diameter-increasing graphs seem particularly interesting. Also, it would be nice to have a complete and simple proof of the four-thirds conjecture.

Finally, if these results are to be useful in the study of communication networks, one needs to consider designing graphs where the removal of even the worst edge is not unduly disruptive.

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