



Cosmology

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 - verbal averaging: can we do better?

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 - scalar (GR) averaging: statistically homogeneous spatial slices

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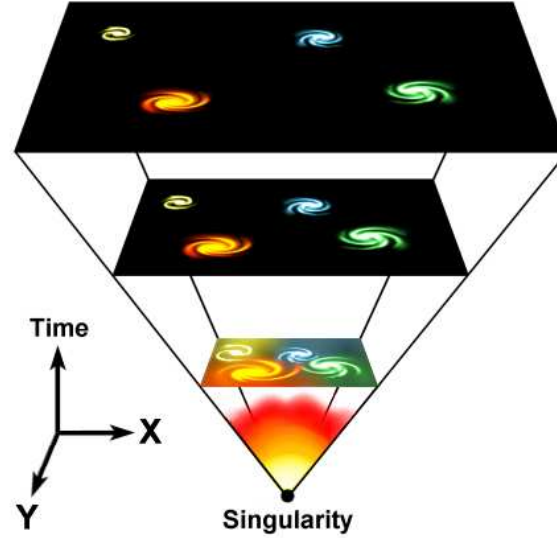
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 3. assume that (M, \mathbf{g}) remains unchanged if we add density perturbations to an early time slice

verbal averaging

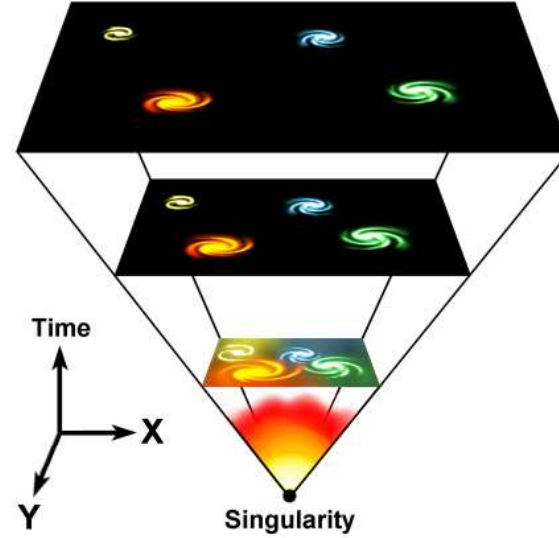
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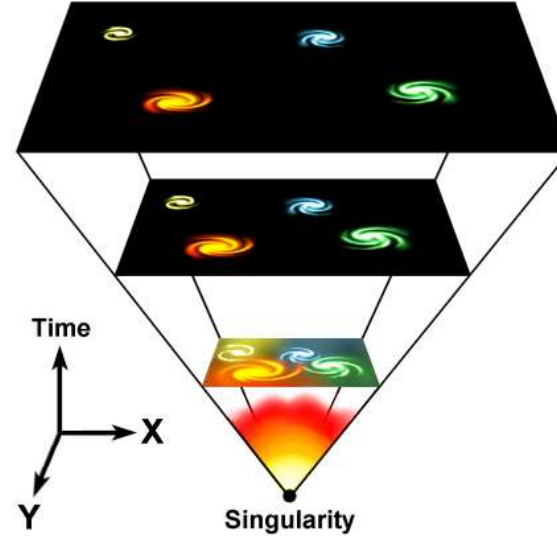
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$$\Delta x(t) = a(t) \Delta r$$

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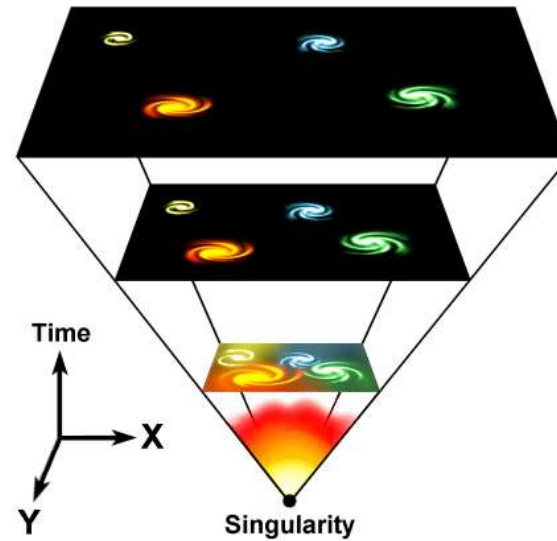


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- spherical coordinates for spatial slice

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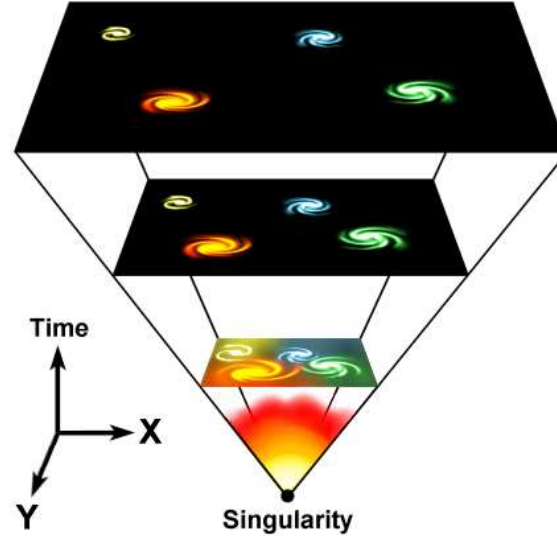


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$$\int_{(t, r_1, \theta, \phi)}^{(t, r_2, \theta, \phi)} ds = a(t) \Delta r = a(t) |r_2 - r_1|$$

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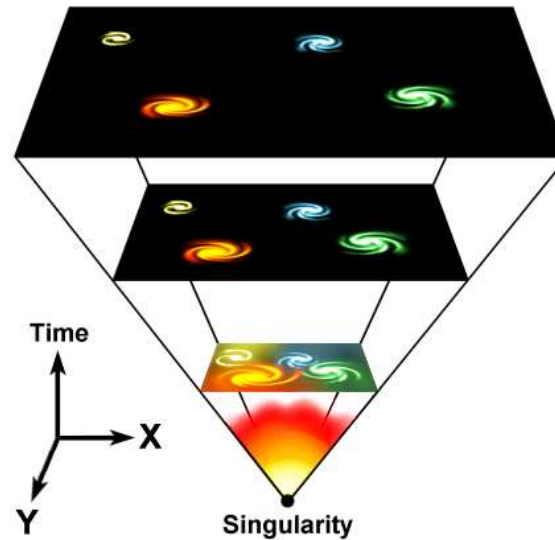


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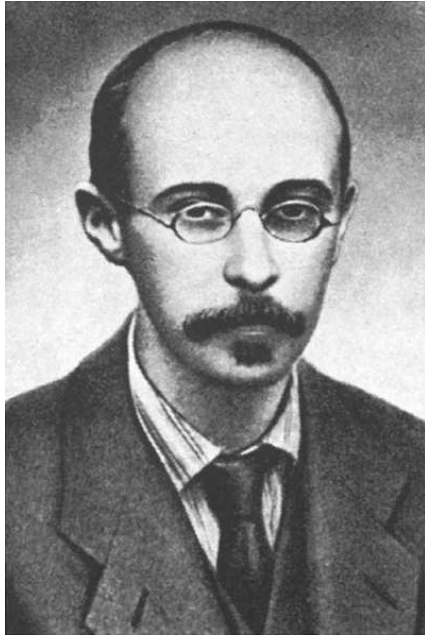
- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

- w:Friedmann–Lemaître–Robertson–Walker metric

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- w:Howard Percy Robertson
w:Arthur Geoffrey Walker

FLRW metric

$$ds^2 = -dt^2 + \dots$$

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$$u := \int \frac{dt}{a(t)}$$

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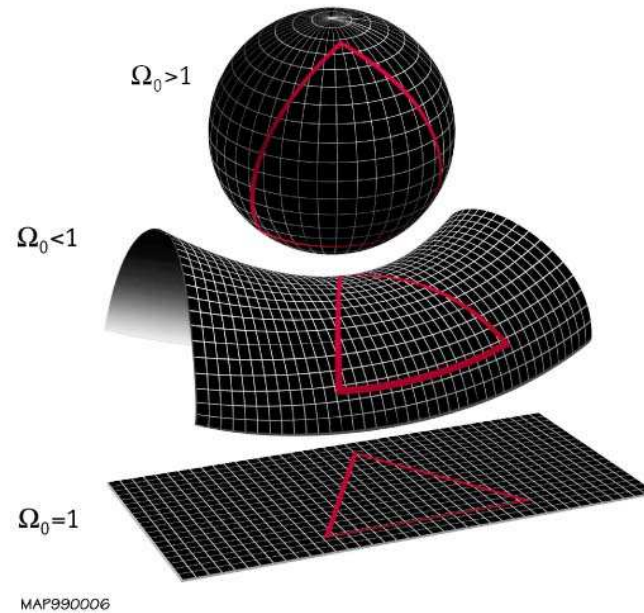
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$$ds^2 = a^2(u) [-du^2 + dr^2 + r_{\perp}^2 (d\theta^2 + \cos^2 \theta d\phi^2)]$$

- but $\int du \neq$ proper time; *more:* [arXiv:astro-ph/0707.2106](https://arxiv.org/abs/0707.2106)

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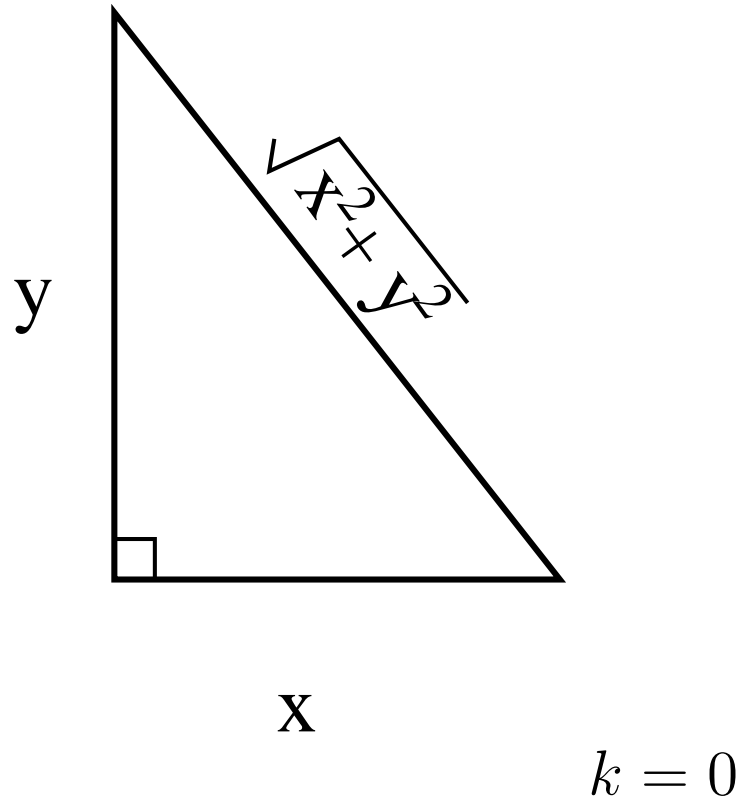


where $r_{\perp} := \begin{cases} R_C \sinh \frac{r}{R_C} & k < 0 \\ r & k = 0 \\ R_C \sin \frac{r}{R_C} & k > 0 \end{cases}$

for a comoving radius of curvature R_C and curvature of sign k

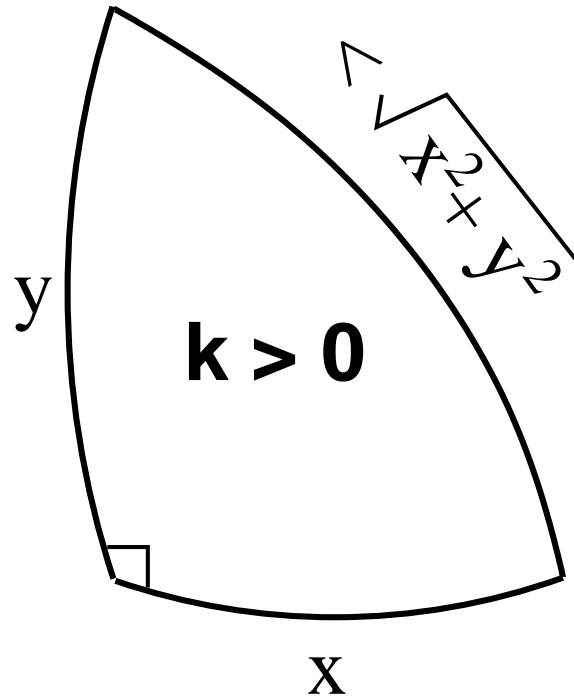
curvature

- on a spatial slice (fixed value of t):



curvature

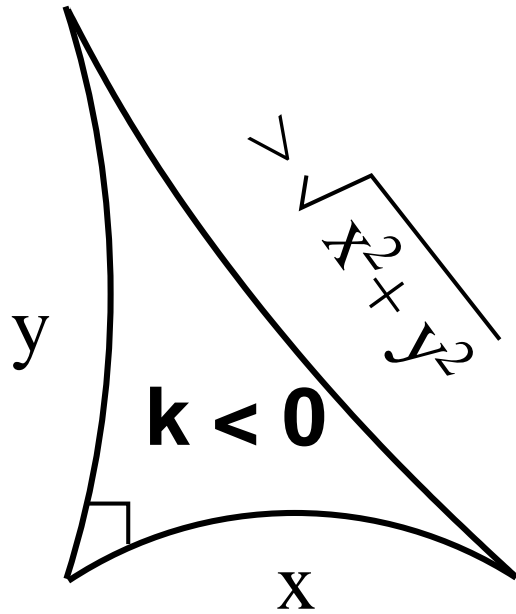
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$$k > 0$$

curvature

- on a spatial slice (fixed value of t):



$$k < 0$$

2D curvature intuition: $k > 0$

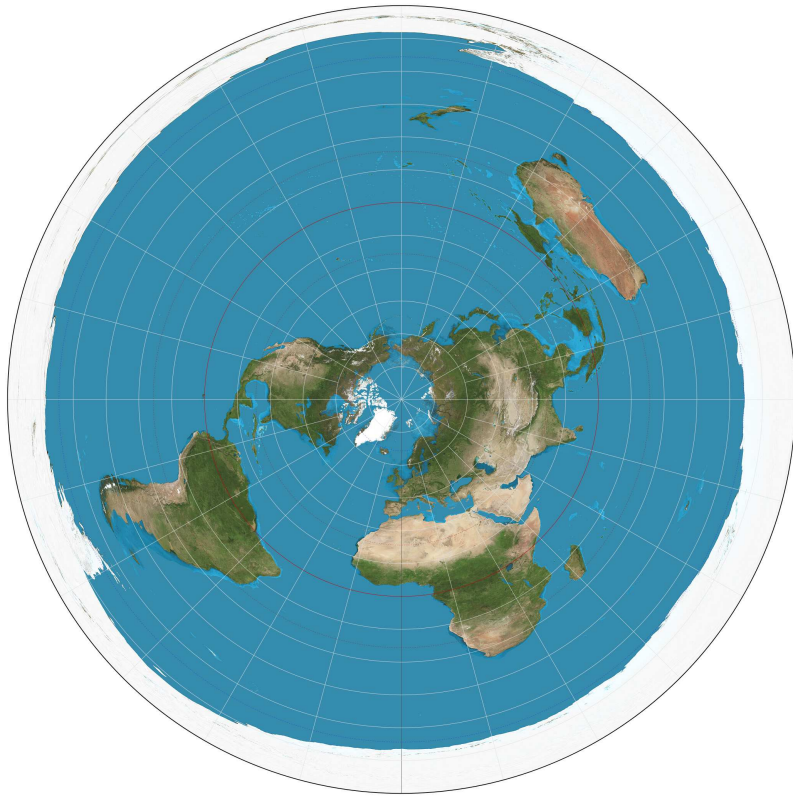
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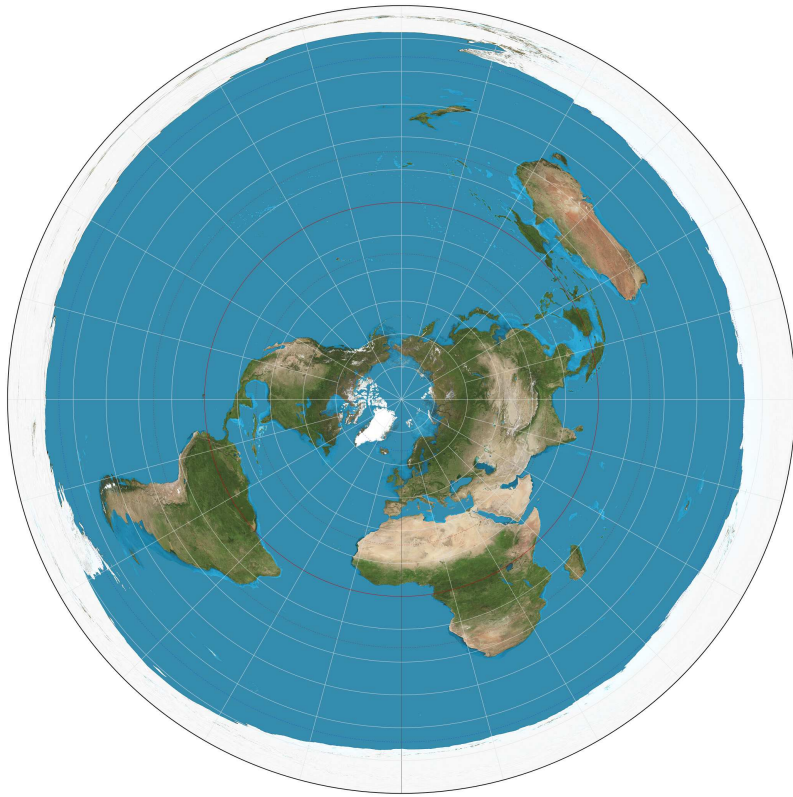


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(al-Biruni, c. 1000 CE)

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- intuition switch: S^2 easier vs S^3 more physical

2D topology intuition ($k = 0$)



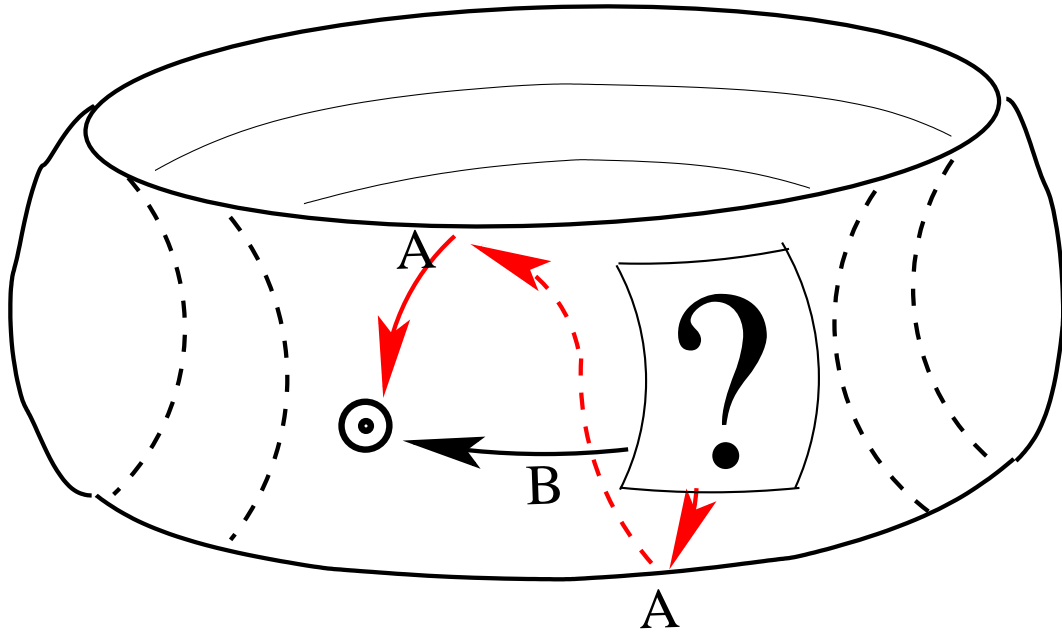
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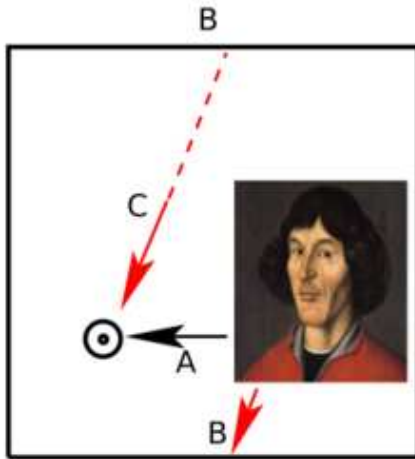


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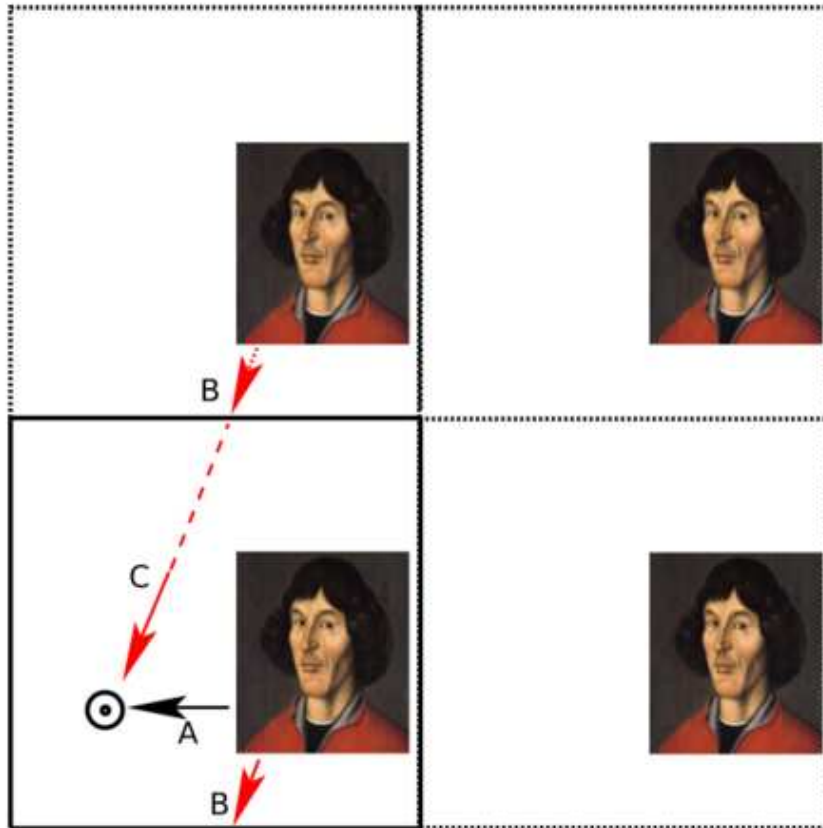


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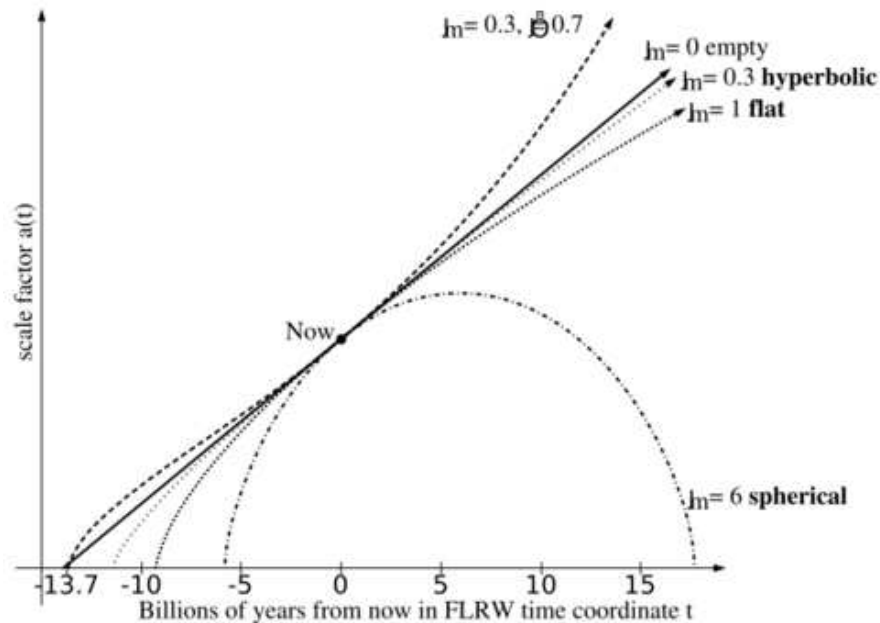
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(Defn: $a_0 := 1$)

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$$1 + z = \frac{1}{a_{\text{em}}}$$

(Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_r \propto a^{-4} = (1 + z)^4$

Black body

■ Planck's Law:
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- \Rightarrow black body + primordial nucleosynthesis

CMB discovery: McKellar 1941

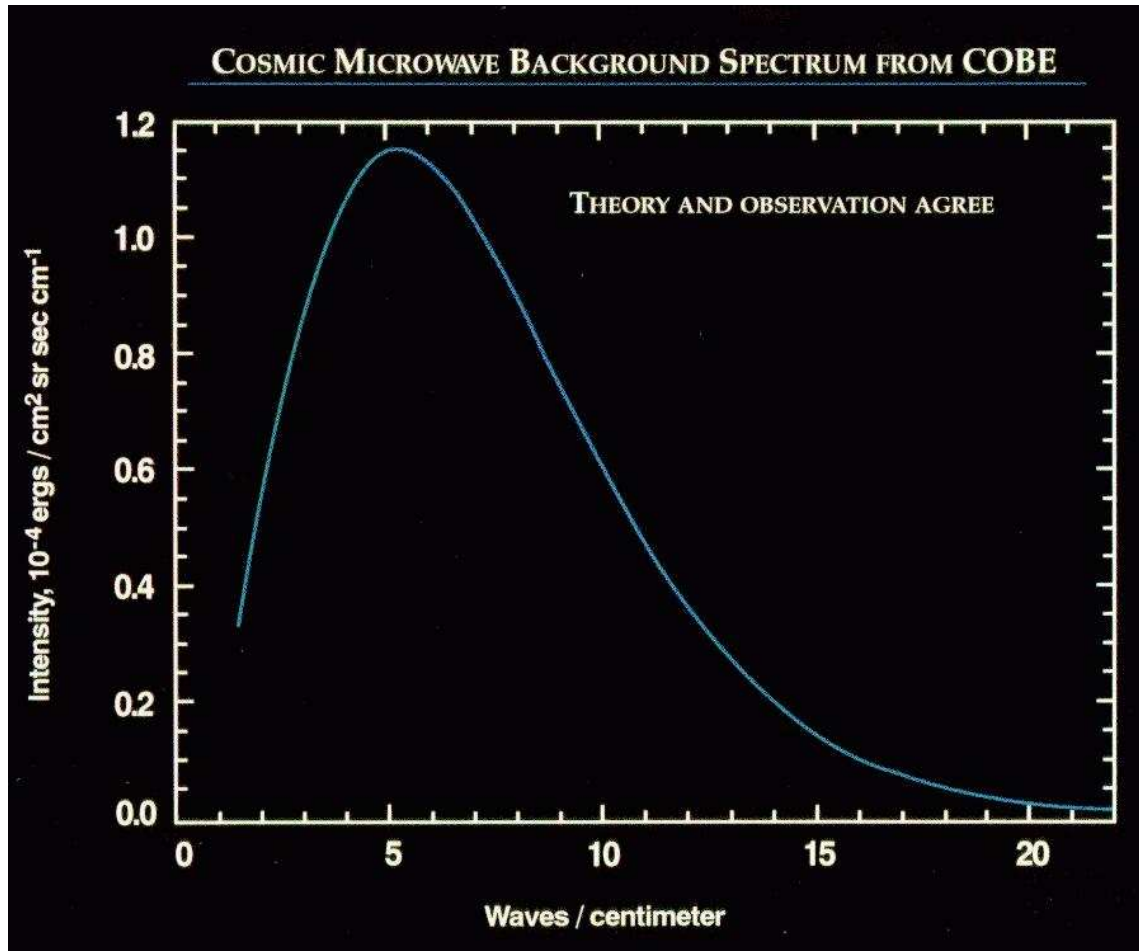
- $T \approx 2.3$ K — Andrew McKellar (1941; [ADS:1941PDAO....7..251M](#)) from observations by Walter S. Adams (1941; [ADS:1941ApJ....93...11A](#))
- Penzias & Wilson rediscovery (1965 + Nobel prize)

Black body: COBE (~ 1992)

- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)

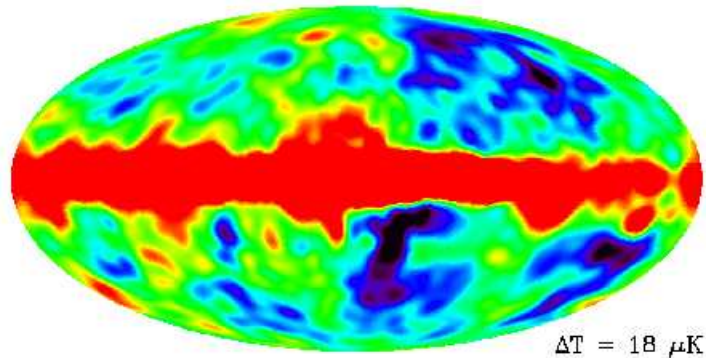
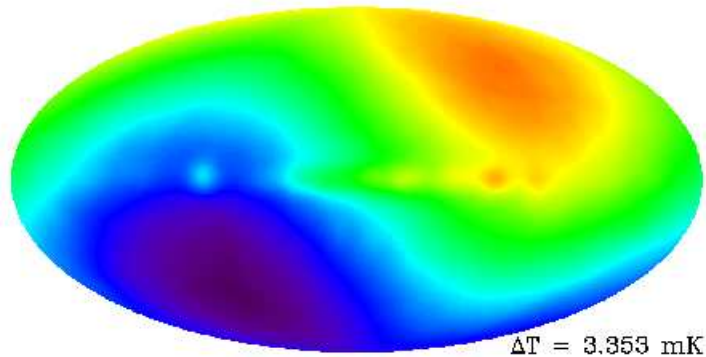
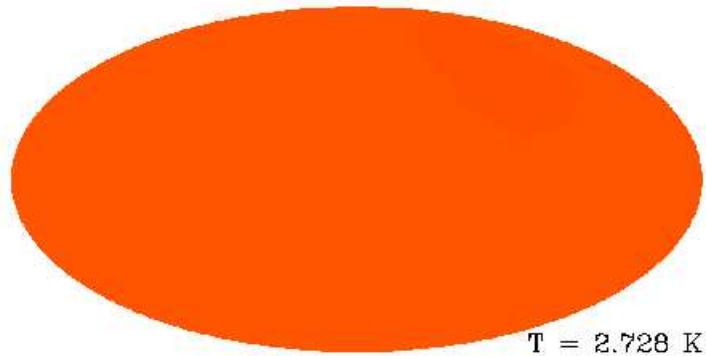
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- COBE /DMR (Differential Microwave Radiometer)



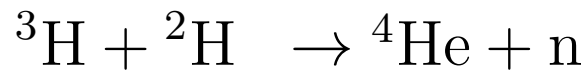
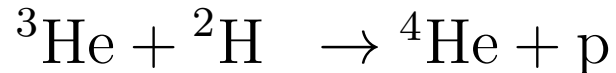
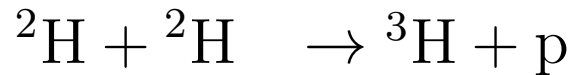
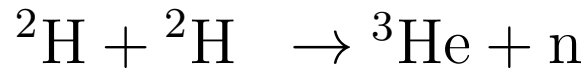
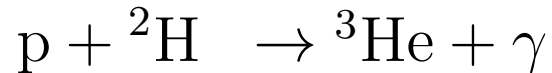
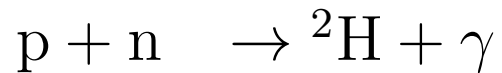
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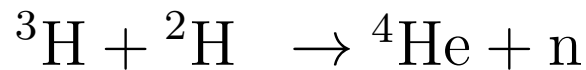
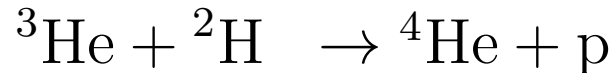
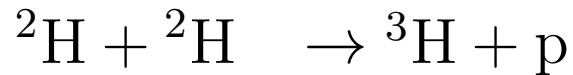
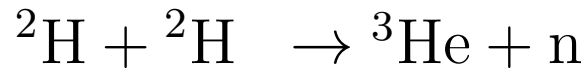
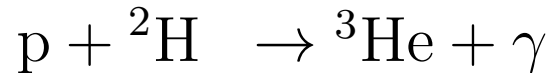
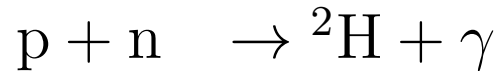
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- MAXIMA: calculate \mathbf{G} and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify:
<https://cosmo.torun.pl/Cosmo/FLRWEquationsGR>

FLRW: $a(t) = ?$

- $$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \cos^2 \theta d\phi^2) \right]$$

- universe content: $\text{diag}(\mathbf{T}) = (-\rho, p, p, p)$

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$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3}$$

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- acceleration Eqn:
$$\frac{\ddot{a}}{a} = -\frac{4\pi G (\rho + 3p/c^2)}{3}$$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

- matter-dominated epoch: $\rho = \rho_m$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho_{m0}}{3 a^3} - \frac{c^2 k}{a^2}$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $\dot{a} \propto a^{-1/2}$ for $a > 0$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $da \propto a^{-1/2} dt$ for $a > 0$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $a^{1/2} da \propto dt$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $(2/3)a^{3/2} \propto t$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $a \propto t^{2/3}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

- $k = 0$ case:

$$a = \left(\frac{t}{t_0}\right)^{2/3}$$

Einstein–de Sitter model (EdS)

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

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Einstein–de Sitter model (EdS)

Defn:

Hubble parameter $H := \dot{a}/a$

FLRW matter-dominated epoch

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$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

- $k = 0$ case: $a = \left(\frac{t}{t_0}\right)^{2/3}$ Einstein–de Sitter model (EdS)

Defn: $H := \dot{a}/a$

\Rightarrow Hubble constant $H_0 := H(z = 0)$

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- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

- $k = 0$ case: $a = \left(\frac{t}{t_0}\right)^{2/3}$ Einstein–de Sitter model (EdS)

Defn: $H := \dot{a}/a$

\Rightarrow Hubble constant $H_0 := H(z = 0)$

\Rightarrow $k = 0$ case: $\frac{\dot{a}}{a} = \frac{2}{3t}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

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\Rightarrow $H(t) = \frac{2}{3t}$;

FLRW matter-dominated epoch

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- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

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Defn: $H := \dot{a}/a$

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\Rightarrow $k = 0$ case: $\frac{\dot{a}}{a} = \frac{2}{3t}$

\Rightarrow $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

FLRW matter-dominated epoch

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$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

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$$\Rightarrow H(t) = \frac{2}{3t}; H_0 = H(t_0) = \frac{2}{3t_0}$$

- convenient conversion: $1 \text{ km/s} \approx 1.04 \text{ kpc/Gyr} \approx 1 \text{ kpc/Gyr}$

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$$\Rightarrow H(t) = \frac{2}{3t}; H_0 = H(t_0) = \frac{2}{3t_0}$$

- Lemaître (1927) [ADS:1927ASSB...47...49L](#): $H_0 \approx 600$ km/s/Mpc

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- Lemaître (1927) [ADS:1927ASSB...47...49L](#): $H_0 \approx 0.6 \text{ Gyr}^{-1}$

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- Hubble (1929) [ADS:1929PNAS...15..168H](#): $H_0 \approx 500 \text{ km/s/Mpc}$

FLRW matter-dominated epoch

- Friedmann Eqn:

$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

- $k = 0$ case: $a = \left(\frac{t}{t_0}\right)^{2/3}$ Einstein–de Sitter model (EdS)

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kpc/Gyr/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn:

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FLRW matter-dominated epoch

- Friedmann Eqn:

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0}$$

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr}$$

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr} < t_{\text{Earth}} \approx 4.5 \text{ Gyr}$$

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- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

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- Friedmann Eqn:

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr} < t_{\text{Earth}} \approx 4.5 \text{ Gyr}$$

- 1980's: $H_0 \approx 0.05$ or $0.1 \text{ Gyr}^{-1} \Rightarrow t_0(\text{EdS}) \approx 13.0$ or 6.5 Gyr , resp.

FLRW: ρ_{crit}

- Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3} - \frac{c^2 k}{a^2}$$

FLRW: ρ_{crit}

- Friedmann Eqn:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

FLRW: ρ_{crit}

- Friedmann Eqn:
$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$
- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

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- Friedmann Eqn:
$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$
- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$
 - ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$
 - ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0 \text{ flat}$

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$$H^2 = \frac{8\pi G \rho}{3} - \frac{c^2 k}{a^2}$$
- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$
 - ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0 \text{ flat}$
 - ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

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 - ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0 \text{ spherical}$

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- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$
 - ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
 - ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical
 - ◆ $\rho_{\text{m}0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$

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 - ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
 - ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical
 - ◆ $\rho_{\text{m}0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$ hyperbolic

FLRW: ρ_{crit}

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FLRW: ρ_{crit}

- Friedmann Eqn:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

Defn:

$$\rho_{\text{crit}} := \frac{3H^2}{8\pi G} \text{ critical density}$$

FLRW: ρ_{crit}

- Friedmann Eqn:

$$H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{c^2 k}{a^2}$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

FLRW: ρ_{crit}

■ Friedmann Eqn:
$$H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{c^2 k}{a^2}$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

FLRW: ρ_{crit}

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$$H^2 = \Omega_m H^2 - \frac{c^2 k}{a^2}$$

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Defn:
$$\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$$
 critical density

Defn:
$$\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$$
 matter density parameter

Defn:
$$\Omega_k := -\frac{c^2 k}{a^2 H^2}$$
 curvature density parameter

FLRW: ρ_{crit}

■ Friedmann Eqn:
$$H^2 = \Omega_m H^2 - \frac{c^2 k}{a^2}$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

■ Friedmann Eqn: $H^2 = \Omega_m H^2 + \Omega_k H^2$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

- Friedmann Eqn:

$$1 = \Omega_m + \Omega_k$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

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Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

FLRW: ρ_{crit}

- Friedmann Eqn:

$$1 = \Omega_m + \Omega_k$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$

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■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat

FLRW: ρ_{crit}

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■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat

◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$

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- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$
 - ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
 - ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical

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- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$

FLRW: ρ_{crit}

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- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
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FLRW: ρ_{crit}

■ Friedmann Eqn: $1 = \Omega_{\text{tot}} + \Omega_k$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

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- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} :=$

FLRW: ρ_{crit}

■ Friedmann Eqn: $1 = \Omega_{\text{tot}} + \Omega_k$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

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■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat

◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical

◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} := \Omega_m +$

FLRW: ρ_{crit}

■ Friedmann Eqn: $1 = \Omega_{\text{tot}} + \Omega_k$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} := \Omega_m + \Omega_r +$

FLRW: ρ_{crit}

■ Friedmann Eqn: $1 = \Omega_{\text{tot}} + \Omega_k$

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Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat

◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical

◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} := \Omega_m + \Omega_r + \Omega_\Lambda$

FLRW: ρ_{crit}

■ Friedmann Eqn: $1 = \Omega_{\text{tot}} + \Omega_k$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} := \Omega_b + \Omega_{\text{nbDM}} + \Omega_r + \Omega_\Lambda$

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

FLRW curvature constant

■ metric in

- ◆ azimuthal equidistant coords: R_C
- ◆ orthographic coords: k

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1$

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$

FLRW curvature constant

- metric in
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 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \Rightarrow \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \Rightarrow k = -\frac{\Omega_{k0} H_0^2}{c^2}$

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \Rightarrow R_C^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
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FLRW curvature constant

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \Rightarrow R_C^2 = -\frac{c^2}{H_0^2} \frac{1}{1 - \Omega_{\text{tot}0}}$

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow $R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$

FLRW curvature constant

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow $R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$
- $\Omega_{\text{tot}0} > 1$

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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$$R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$$
- $\Omega_{\text{tot}0} > 1$ *spherical*

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow
$$R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$$
- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow $R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$
- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow $R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$
- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
 - ◆ azimuthal equidistant coords: R_C
 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ \Rightarrow $R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$
- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\text{tot}0} < 1$

FLRW curvature constant

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 - ◆ orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\text{tot}0} < 1$ *hyperbolic*

FLRW curvature constant

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 - ◆ azimuthal equidistant coords: R_C
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$$R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\text{tot}0} - 1}}$$
- $\Omega_{\text{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\text{tot}0} < 1$ *hyperbolic* R_C imaginary (or use $|R_C|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
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Einstein's free parameter: Λ

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- MAXIMA: calculate G and $G + g\Lambda = 8\pi T$ and simplify:
<https://cosmo.torun.pl/Cosmo/FLRWEquationsGR>

Einstein's free parameter: Λ

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[ADS:1917SPAW.....142E](#)
- MAXIMA: calculate G and $G + g\Lambda = 8\pi T$ and simplify:
<https://cosmo.torun.pl/Cosmo/FLRWEquationsGR>
- *hint*: mixed index form of g is easy

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$):

$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{4 \pi G (\rho + 3 p/c^2)}{3} + \frac{c^2 \Lambda}{3}$$

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Defn: “dust solution”: $p(t) = 0$

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Defn: “dust solution”: $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad \frac{\ddot{a}}{a} = -\frac{H^2}{2} \frac{\rho}{\rho_{\text{crit}}} + \Omega_\Lambda H^2$$

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad \frac{\ddot{a}}{a} = -\frac{H^2 \Omega_m}{2} + \Omega_\Lambda H^2$$

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$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad \frac{\ddot{a} a^2}{a \dot{a}^2} = -\frac{\Omega_m}{2} + \Omega_\Lambda$$

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$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

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$$\text{Defn: } q := -\frac{\ddot{a} a}{\dot{a}^2}$$

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Defn: “dust solution”: $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad q = \frac{\Omega_m}{2} - \Omega_\Lambda$$

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Defn: “dust solution”: $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

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- $q = \frac{\Omega_m}{2} - \Omega_\Lambda$

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Defn: “dust solution”: $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

- $q = \frac{\Omega_m}{2} - \Omega_\Lambda$ acceleration equation
- if $\Lambda = 0$ and $\Omega_m > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

Defn: “dust solution”: $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

- $q = \frac{\Omega_m}{2} - \Omega_\Lambda$ acceleration equation
- if $\Lambda = 0$ and $\Omega_m > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$ *deceleration*

Einstein's free parameter: Λ

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

- $q = \frac{\Omega_m}{2} - \Omega_\Lambda$ acceleration equation
- if $\Omega_\Lambda > \Omega_m/2$ then $\frac{\ddot{a}}{a} > 0$, i.e. $q < 0$ *acceleration*

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- high-level frontends (e.g. python) should be easy to write

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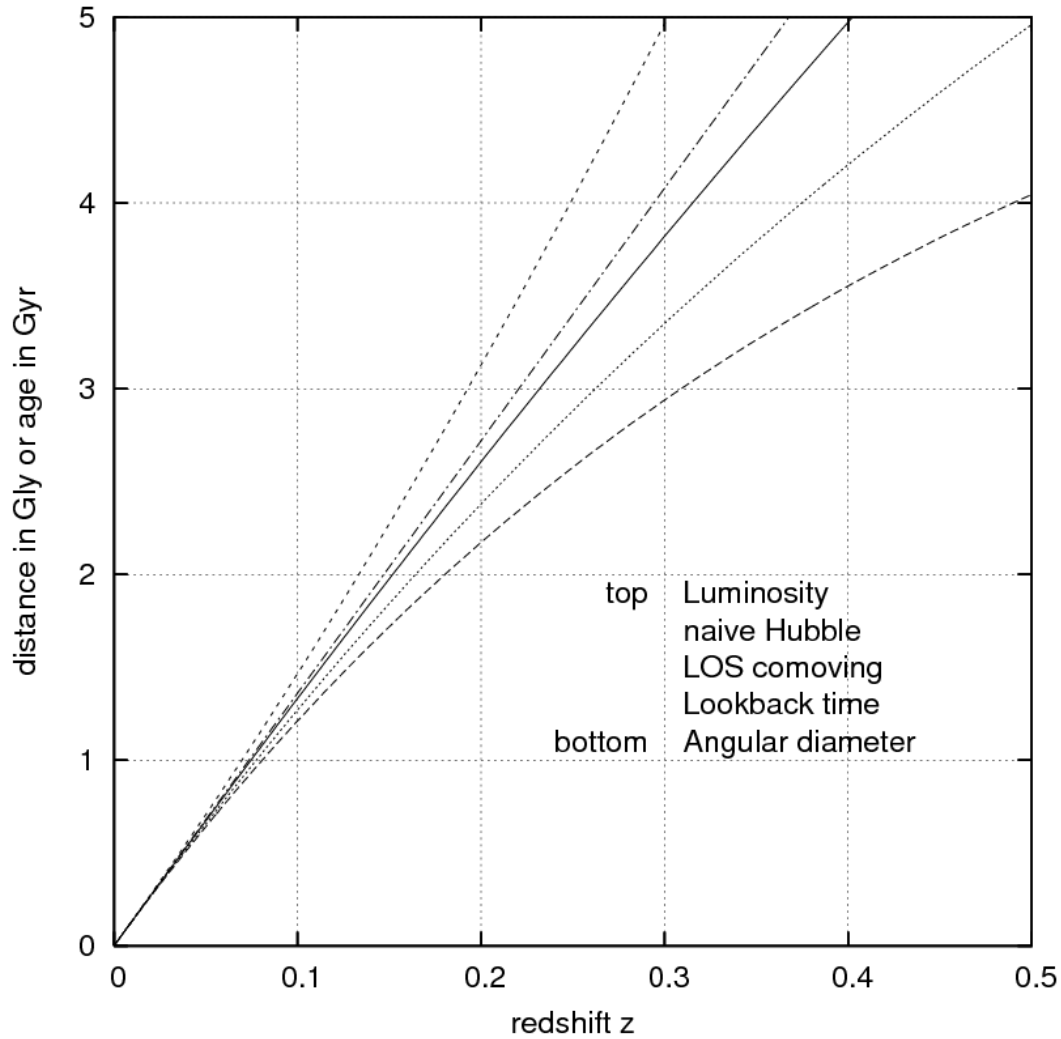
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- w:Distance measures (cosmology)



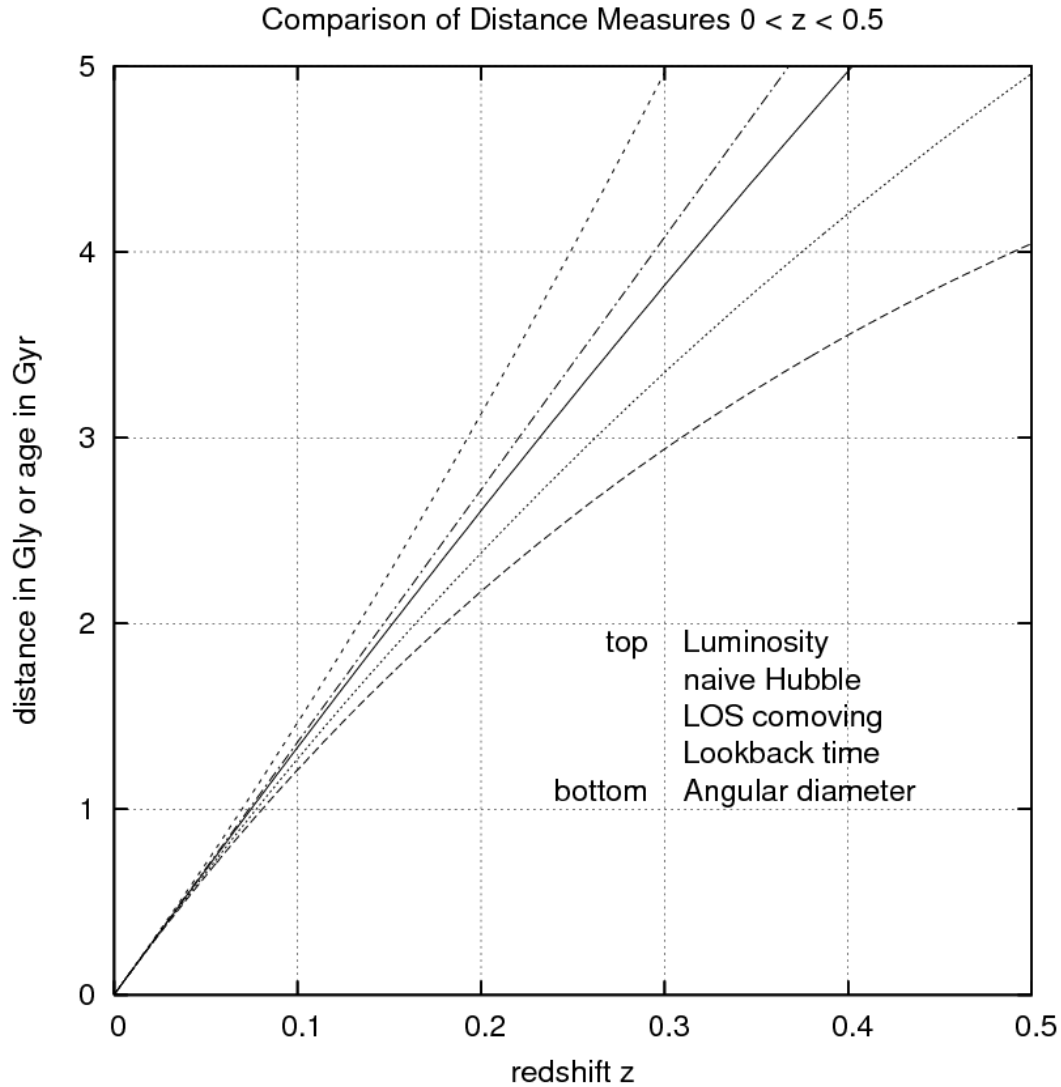
FLRW distances e.g. Λ CDM



Comparison of Distance Measures $0 < z < 0.5$

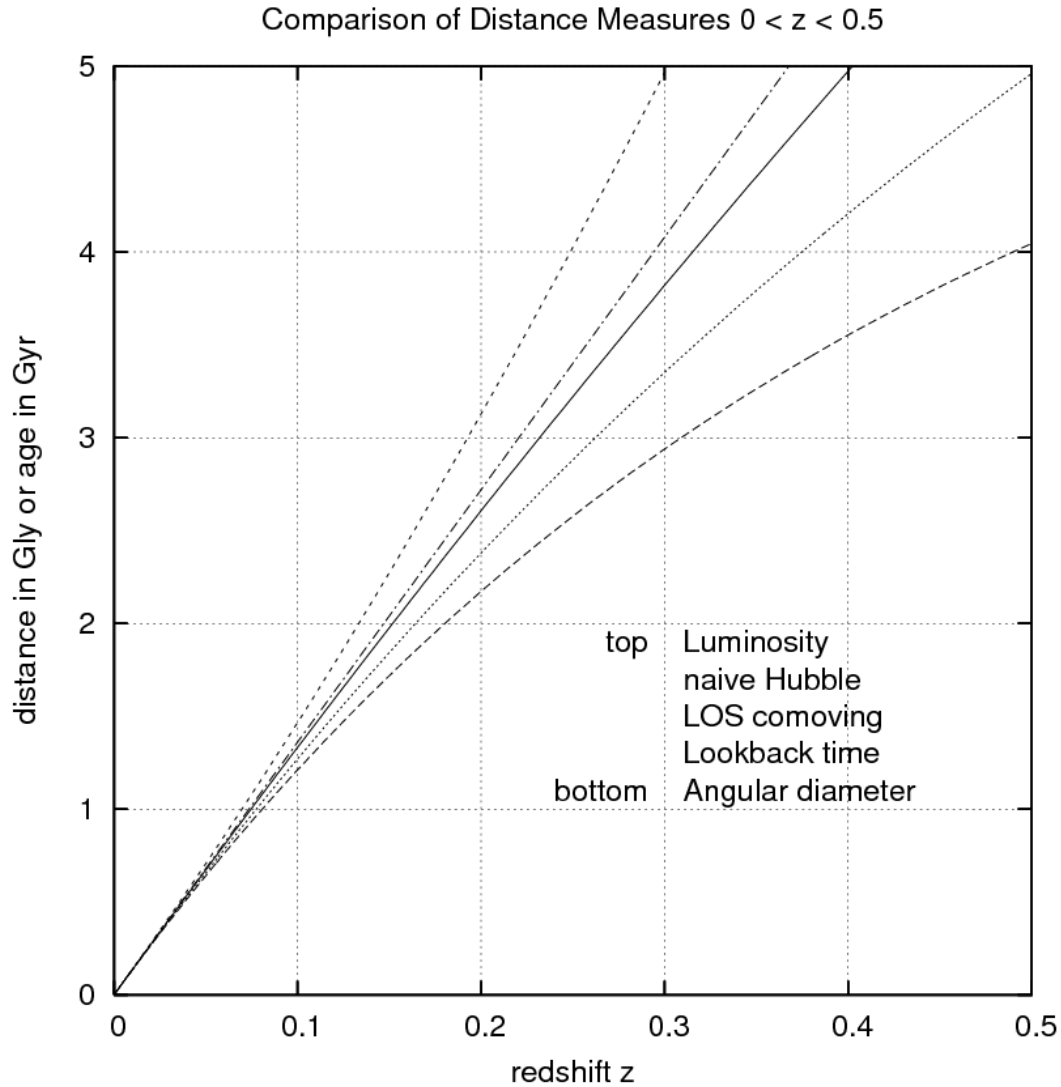


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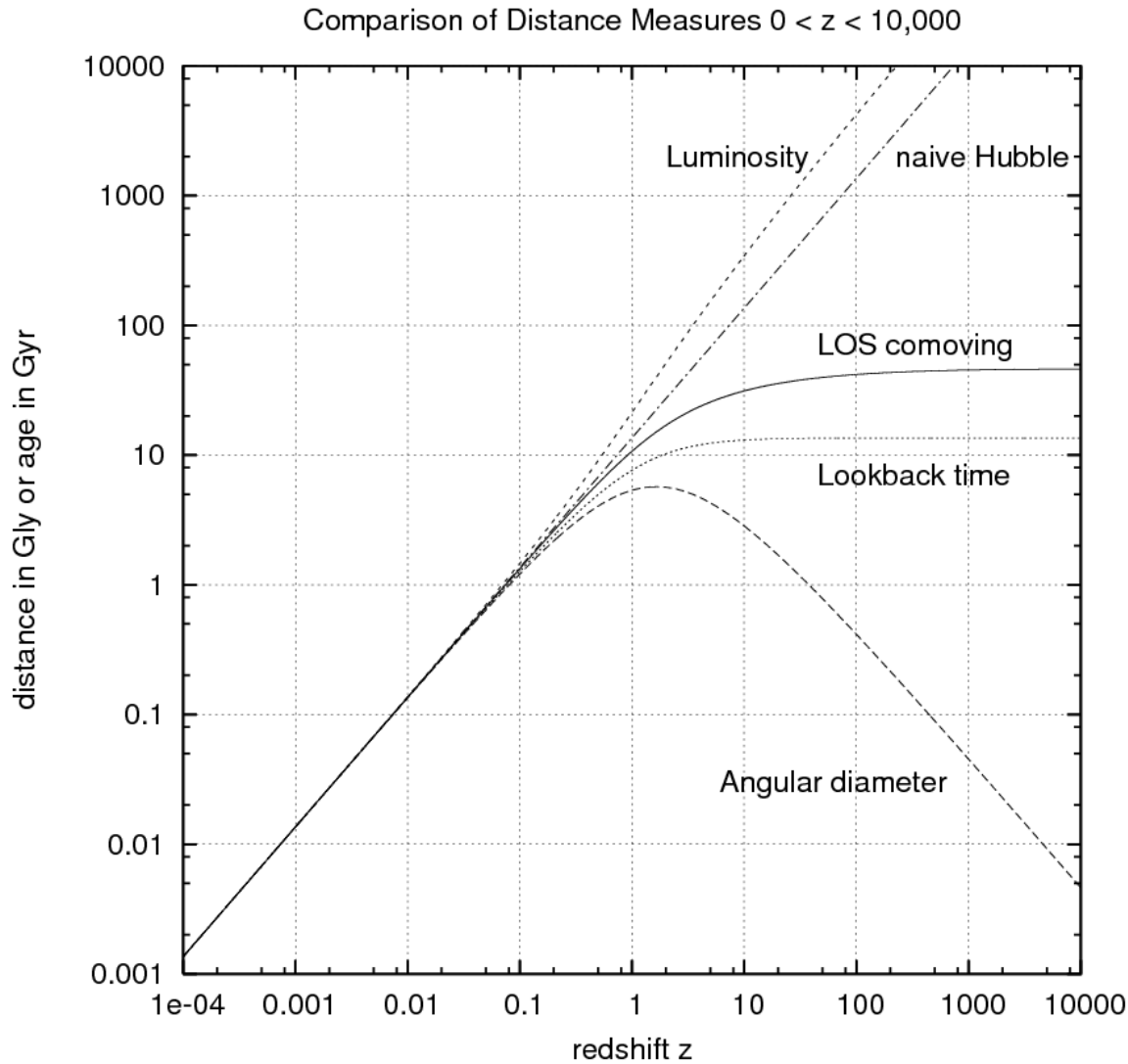
Defn: $h := H_0/100 \text{ km/s/Mpc}$ (without a "0" subscript on h)

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$$\Omega_{m0} = 0.3, \Omega_{r0} = 10^{-4}, \Omega_{\Lambda 0} = 1.0 - (\Omega_{m0} + \Omega_{r0}), h = 0.7, \Omega_{k0} = 0$$

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- \Rightarrow no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature (+, 0, -)?

Non-radial spatial geodesics

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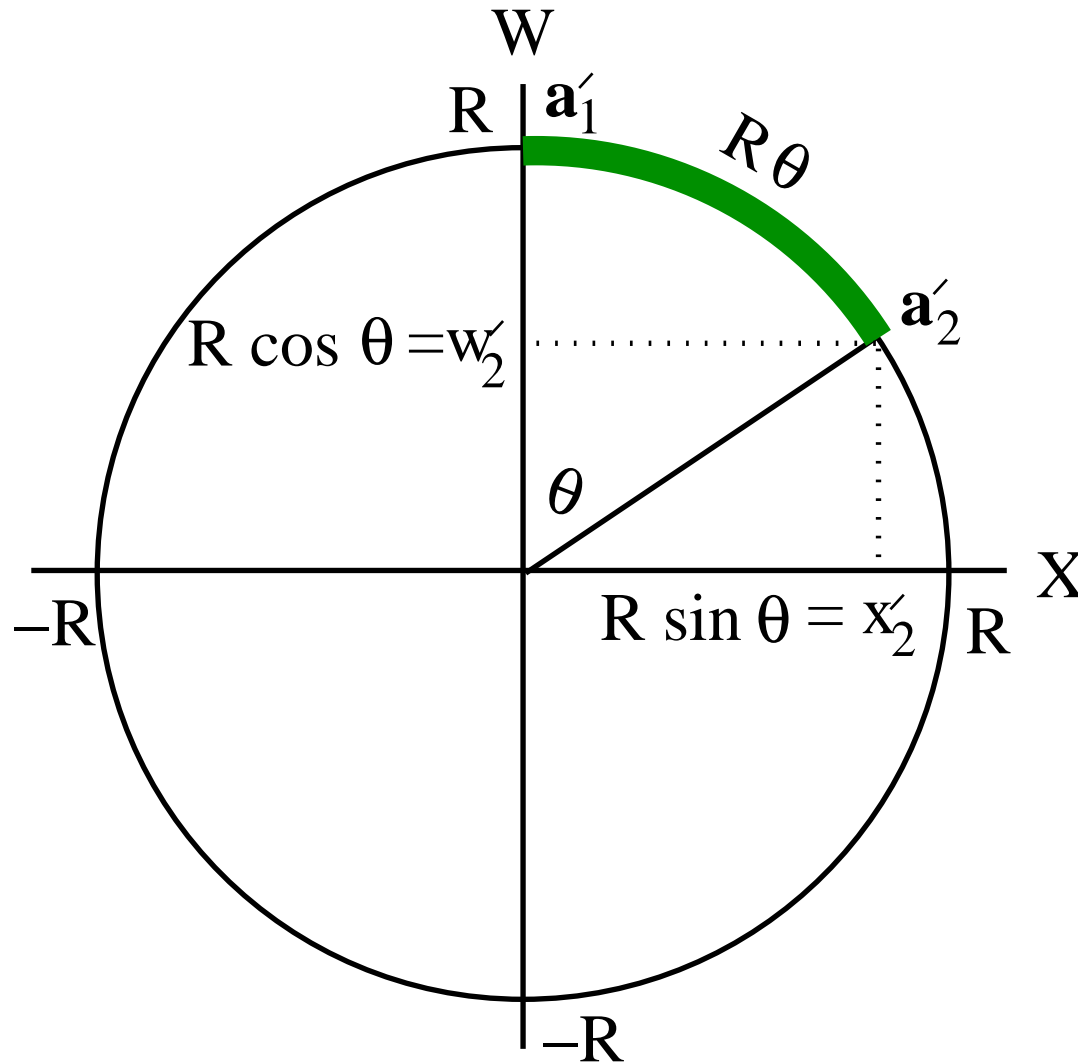
$$\chi_{12} = R_C \theta_{12} = R_C \cos^{-1} [\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_C^2]$$

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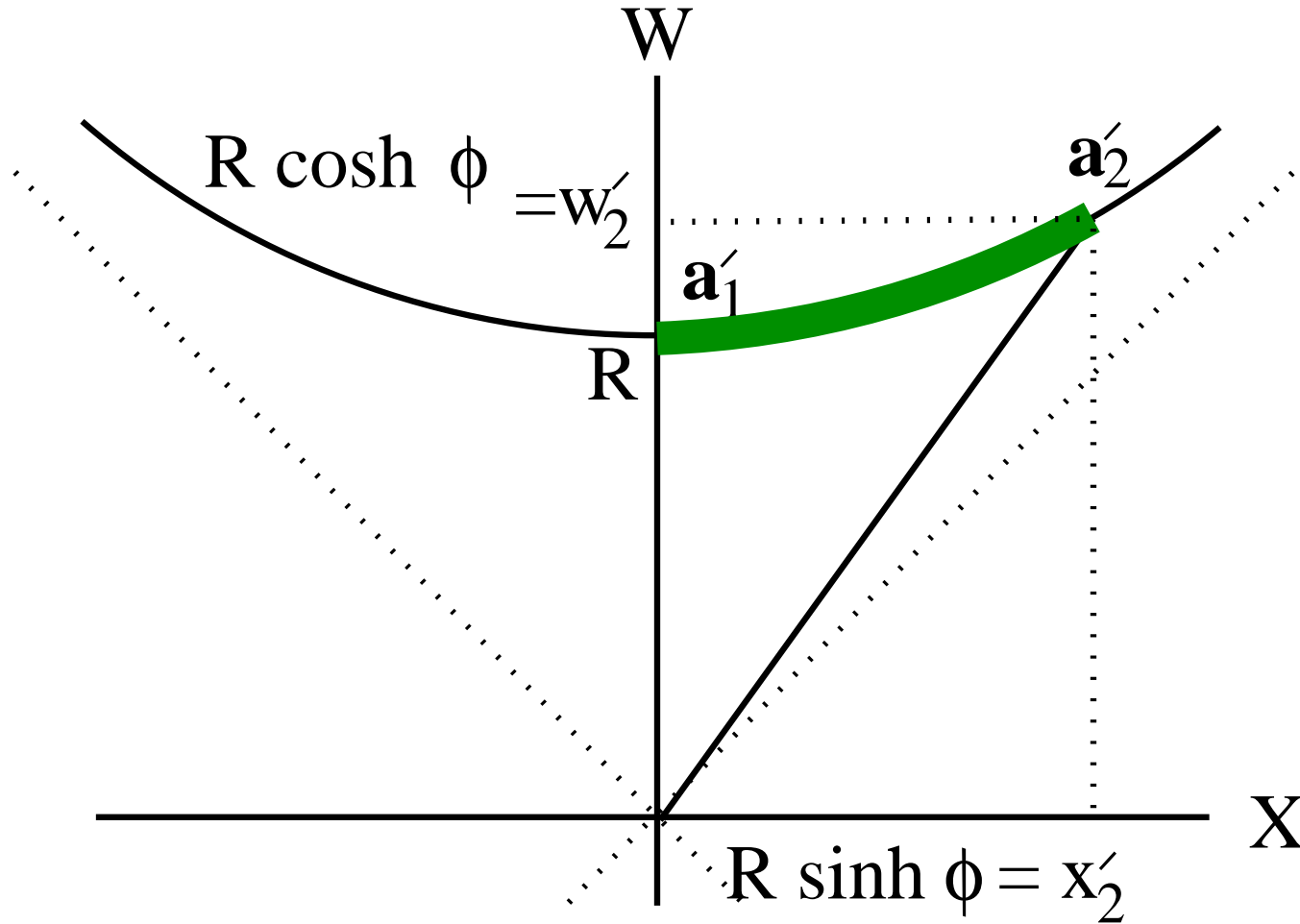


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distances on $S^3 \subset \mathbb{R}^4$ or $H^3 \subset M^4$

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metric on S^3 (or \mathbb{R}^3 or H^3):

$$ds^2 = \begin{cases} (k/|k|) (dx^2 + dy^2 + dz^2) + dw^2 & k \neq 0 \\ dx^2 + dy^2 + dz^2 & k = 0 \end{cases}$$

distances on $S^3 \subset \mathbb{R}^4$ or $H^3 \subset M^4$

$$\Sigma(\chi_i) := \begin{cases} R_C \sinh(\chi_i/R_C) & k < 0 \\ \chi_i & k = 0 \\ R_C \sin(\chi_i/R_C) & k > 0 \end{cases}$$

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inner product:

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \equiv \begin{cases} (k/|k|) (x_1 x_2 + y_1 y_2 + z_1 z_2) + w_1 w_2 & k \neq 0 \\ x_1 x_2 + y_1 y_2 + z_1 z_2 & k = 0 \end{cases}$$

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[arXiv:astro-ph/0102099](https://arxiv.org/abs/astro-ph/0102099)

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