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Consistencia en Juegos sin Utilidad Transferible

Memoria presentada por ***Eulalia Romero Palacios*** para optar al grado de Doctora por la Universidad Pablo de Olavide, de Sevilla.

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A quienes pelean por mis besos cada mañana,

algún día entenderán que mi reparto siempre es consistente.

A mis padres que me han enseñado a repartir aun cuando no se tiene.

Índice general

INTRODUCCIÓN

La *Teoría de Juegos* modeliza interacciones estratégicas entre individuos (jugadores) racionales, esto es, individuos que toman decisiones de acuerdo a sus preferencias y tratando de alcanzar sus objetivos. Básicamente, esta es la razón por la que esta teoría es cada vez mas relevante en numerosas disciplinas.

La importancia que la *Teoría de Juegos* alcanza en las últimas décadas queda avalada por los numerosos teóricos de juegos que, desde que se instauró el Nobel de Economía en 1968, han obtenido tal reconocimiento. Comenzando por P. A. Samuelson, sólo dos años mas tarde, y terminando por L. S. Shapley, que lo compartió en 2012 con Alvin Roth, son 13 los teóricos de juegos galardonados.

En esta memoria se analiza únicamente una clase especial de juegos: *los juegos cooperativos*. Estos juegos formalizan situaciones en las que los jugadores pueden comunicarse libremente entre sí y tomar acuerdos vinculantes.

Los individuos que se enfrentan a un conflicto con opción de cooperar, tratarán de alcanzar mediante el consenso, o a través de negociaciones -en ocasiones mediadas-, un acuerdo. La *Teoría de Juegos Cooperativos* trata de predecir el resultado final de estas interacciones a través de mecanismos que, aplicados a cada situación, marquen de forma automática un resultado. Tales mecanismos (reglas de asignación) son conocidos como *soluciones*.

Desde 1944, año en que J. von Neumann y O. Morgenstern publicaron *Theory of Games and Economic Behaviour* -obra que se considera el origen moderno de esta teoría-, hasta hoy se han propuesto básicamente tres tipos de juegos cooperativos para modelizar distintas situaciones en función de su complejidad: los *juegos de negociación*, los *juegos de utilidad transferible* (*juegos TU*) y los *juegos de utilidad no transferible* (*juegos NTU*).

Fundamentalmente la diferencia entre *juegos de negociación* y *juegos de utilidad -transferible* o *no transferible*- es la consideración del papel que pueden jugar las coaliciones en el desarrollo y posterior resultado del juego. En los primeros, sólo se considera el conjunto de posibles resultados que pueden alcanzar los jugadores si todos cooperan y el resultado que obtendrían en caso contrario; no así en los segundos, en los cuales las coaliciones desempeñan un papel esencial.

Al hablar de la redistribución de ganancias alcanzadas por un grupo de jugadores, la disponibilidad de un bien de consumo perfectamente divisible es crucial. La existencia de dicho bien permite que los jugadores puedan compensar entre ellos los sacrificios realizados para conseguir una meta común. Si esto es factible, la situación se modeliza con los denominados *juegos cooperativos de utilidad transferible*, si no se contempla tal posibilidad, o el bien con el que se negocia no cumple dicha condición, la situación se modeliza con *juegos cooperativos de utilidad no transferible*.

En los *juegos de negociación* y los *juegos de utilidad transferible*, se entiende como *solución* una correspondencia (en la mayoría de los casos una función) cuyo dominio es la clase de todos los problemas posibles, o bien un subconjunto suficientemente amplio de dicha clase. Para el estudio de las soluciones de este tipo de juegos se ha seguido tradicionalmente una *metodología axiomática*. Esto es, se establece una colección de propiedades relativamente simples, denominadas *axiomas*, que se consideran deseables y que determinan la solución de manera unívoca. La caracterización axiomática de las soluciones brinda, además de la posibilidad de identificar la solución más apropiada para cada situación en función de los axiomas que satisface, la oportunidad de comparar la soluciones en términos de la propiedades que las caracterizan.

Esta metodología ha resultado ser muy fructífera, habiéndose podido caracterizar, con un conjunto no extenso de axiomas, numerosas soluciones. Podríamos destacar entre otras, la *solución de negociación* de Nash (1950) y las *soluciones igualitarias* de Kalai (1977) en los *juegos de negociación* y el *valor de Shapley* (1953), en los *juegos de utilidad transferible*.

En los *juegos sin utilidad transferible*, sin embargo, el procedimiento adoptado para encontrar soluciones ha sido diferente. Teniendo en cuenta que los *juegos de negociación* y los *juegos TU* son casos particulares de *juegos NTU*, se ha pretendido extender los conceptos de solución ya conocidos para los primeros juegos al entorno más genérico de utilidad no transferible. Concretamente, Harsanyi (1959, 1963 y 1977), Shapley (1969) y Owen (1972) elaboraron diferentes soluciones[1] que coinciden con la solución de negociación de Nash en los *juegos de negociación* y con el valor de Shapley en los *juegos TU*. Años más tarde las soluciones propuestas por Harsanyi y Shapley fueron caracterizadas por Hart (1985) y Aumann (1985) respectivamente con sistemas de axiomas muy similares. Otras soluciones a destacar en *juegos NTU* son las denominadas soluciones igualitarias definidas y caracterizadas[2] por Kalai y Samet (1985), que coinciden con las soluciones proporcionales de Kalai en los *juegos de negociación* y con el valor de Shapley en los *juegos TU*.

Esta memoria se centra en el análisis axiomático de algunas de las soluciones para *juegos NTU* más referenciadas en la literatura, como son la *solución de Harsanyi*, la *solución Shapley NTU* y *las soluciones igualitarias*. En concreto se caracterizan con diferentes sistemas de axiomas que incluyen el *Axioma de Consistencia*, determinando dichas soluciones como *soluciones consistentes*.

Requerir consistencia es requerir estabilidad: una solución se dice *consistente* si el reparto que determina para cada problema concuerda con los que determina para cada uno de sus problemas reducidos asociados, obtenidos al suponer que algunos jugadores dejan el juego con su respectiva asignación.

Es notable que, si bien la propiedad de *Consistencia* no es una de las que constituyen invariablemente los sistemas que caracterizan diferentes soluciones, son muy valorados los trabajos en los que se ha conseguido encontrar un nuevo sistema de axiomas que caracterice dichas soluciones como *soluciones consistentes*.

Así, se encuentran en la literatura reconocidos trabajos como el de Lensberg (1988), que caracteriza la solución de negociación de Nash sustituyendo el *Axioma de Independencia de Alternativas Irrelevantes* por el *Axioma de Consitencia*; o el de Hart y Mas-Collel (1989), que incluye el *Axioma de Consistencia* en la caracterización del valor de Shapley, para *juegos TU* y de las soluciones igualitarias, para *juegos NTU*.

No obstante, hasta el momento, no había sido posible la inclusión de este axioma en

[1] Estas soluciones desafortunadamente no determinan un único reparto como solución.

[2] Nótese que, a diferencia de las anteriores, éstas son caracterizadas por los autores en el mismo artículo en el cual se presentan.

la caracterización de la solución propuesta por Harsanyi (1963) para *juegos NTU*. Es más, Mashler y Owen (1989) investigaron la existencia de una solución eficiente, simétrica y consistente que, a diferencia de las igualitarias, fuese además covariante[3] bajo transformaciones afines de utilidad. Pero esta búsqueda resultó infructusa, ya que encontraron un juego de hiperplanos[4] de tres jugadores cuyo conjunto de soluciones eficientes, simétricas, consistentes y covariantes es el conjunto vacío. En su empeño por encontrar soluciones consistentes Mashler y Owen (1989) proponen una propiedad de consistencia más débil, que denominaron *consistencia bilateral*, con la que caracterizaron la solución que lleva su nombre en la familia de los juegos de hiperplanos.

Como Capítulo 2 de esta memoria se incluye el artículo *Consistency of the Harsanyi NTU configuration value* publicado en la revista GAMES AND ECONOMIC BEHAVIOUR cuyo principal resultado es la demostración de la existencia de una solución eficiente, simétrica, consistente y covariante (la *solución de Harsanyi*) en una gran familia juegos, aquellos para los que el conjunto de asignaciones factibles de la gran coalición viene dado por un semi-espacio[5], que se ha denominado juegos de G-hiperplanos[6].

Para la consecución de este resultado, se considera la noción de *configuraciones de pagos* como concepto de solución. Introducido por Hart (1985) en la primera caracterización que se conoce de la solución de Harsanyi, este concepto de solución no presenta la solución a un juego como pagos a individuos recogidos en un vector de coordenadas reales, sino como un vector de pagos coalicionales; i.e. un vector de vectores cuyas componentes son los pagos a los jugadores en cada posible subcoalición de la gran coalición. Este concepto de solución sigue siendo utilizado en la literatura, De Clippel et al. (2004), por ejemplo, utilizan este concepto para comparar y caracterizar diferentes soluciones en la familia de juegos NTU y Hart (2005) lo vuelve a utilizar para caracterizar la solución para juegos NTU de Mashler y Owen.

El sistema de axiomas que caracteriza la solución de Harsanyi como solución consis-

[3] La propiedad de covarianza es bastante aceptada si se asume que las preferencias de los jugadores están representadas por funciones de utilidad de von-Neuman-Morgenstern.

[4] Véase Section 6 de Mashler y Owen (1989).

[5] Aunque por lo general la solución de Harsanyi no determina un único resultado, en esta familia sí define un único reparto.

[6] Nótese que esta clase de juegos contiene a los juegos de hiperplanos.

tente incluye eficiencia, simetría, covarianza, jugador nulo y un axioma adicional que se ha denominado *independencia de amenazas óptimas*, este último requiere coherencia entre los pagos asignados por la configuración de pagos a las diferentes coaliciones.

El Capítulo 3 de esta tesis recoge el artículo *Consistency of the Shapley NTU value on G-Hyperplane Games* aceptado en la revista REVIEW OF ECONOMIC DESING.

En este trabajo, siguiendo la metodología de Hart se introduce una definición formal de *juego reducido* adaptado a configuraciones de pagos. Se considera que cuando los jugadores de la coalición T (los que no han dejado el juego) renegocian sus pagos, las alternativas factibles para cualquier subcoalición S de T están relacionadas con sus alternativas factibles cuando cooperaba con todos los jugadores de $(N \setminus T)$, que es el enfoque de Moulin(1985) y Hart y MasColell (1989)[7] para definir sus juegos reducidos[8]. Sin embargo, la principal diferencia con estos trabajos previos estriba es que la coalición S compensa a los miembros de $(N \setminus T)$ de acuerdo con la (única) configuración de pagos determinada por la solución, que coincide con el vector de pagos de $S \cup (N \setminus T)$ en la solución.

A posteriori, con la correspondiente propiedad de consistencia, la solución de Shapley NTU se caracteriza como consistente para los juegos de G-hiperplanos. El sistema de axiomas propuestos incluye los axiomas de maximalidad, covarianza, simetría, jugador nulo y un axioma adicional que se ha denominado *independencia de pagos intermedios* que requiere algo de coherencia en las componentes de las configuraciones de pagos asignadas a coaliciones intermedias.

Es destacable el hecho de que, con este segundo artículo, otra conocida solución covariante para juegos NTU, la solución de Shapley NTU, ha sido también caracterizada como solución consistente.

Por último, el Capítulo 4 recoge el artículo *The Egalitarian Configuration Value.*

Las soluciones igualitarias fueron formalmente introducidas y axiomatizadas por Kalai (1977) para los juegos de negociación. Años más tarde, Kalai y Samet (1985) extienden este concepto de solución a juegos NTU manteniendo su denominación, definiendo unas

[7] Este enfoque difiere del de Davis y Maschler (1965), que entiende que la coalición S puede elegir cooperar con cualquier subcoalición Q de $(N \setminus T)$.

[8] Los juegos reducidos definidos en estos trabajos se diferencian en cómo S compensa a los jugadores de $(N \setminus T)$, mientras Moulin determina pagar a dichos jugadores de acuerdo a lo que recibían en el juego original, Hart y MasCollel deciden pagar de acuerdo a lo que la solución asigna a la coalición $S \cup (N \setminus T)$.

soluciones que en juegos TU coincide con el valor de Shapley, estas soluciones determinan que jugadores que cooperan en una misma coalición deben recibir la misma compensación. Dicha compensación se materializa en utilidades comparadas interpersonalmente. El axioma principal en las caracterizaciones de ambos trabajos es el axioma de monotonía: este principio establece que si el conjunto de asignaciones factibles de determinada coalición se incrementa y no varían los conjuntos factibles el resto de las coaliciones, ningún jugador debe recibir menos en la reasignación.

Continuando la línea de investigación seguida en el artículo que constituye el capítulos 2 de esta memoria, se caracterizan las soluciones igualitarias definidas por Kalai y Samet (1985), estudiadas ahora como configuraciones de pagos, mediante la generalización del axioma de consistencia de Hart y MasCollel (1989) propuesta para la caracterización de la solución de Harsanyi.

En este trabajo los juegos NTU son considerados una generalización de los juegos de negociación y se extienden los resultados que obtiene Kalai (1977) siguiendo el enfoque de Nash (1950) para juegos de negociación . Kalai (1977) considera dos principios del proceso de negociación: monotonía y negociación -paso a paso- y hace ver que cada uno de esos principios es suficiente para implicar que los jugadores deben hacer comparaciones interpersonales de utilidad cuando intentan maximizar sus utilidades bajo el supuesto de que todos ganan por igual. En esta línea, se prueba que estos principios junto con los axiomas habituales caracterizan las configuraciones igualitarias en la clase de los problemas de *elección con puntos de referencia*[9], que formalmente son una generalización de los juegos de negociación al caso en el que el punto de desacuerdo pueda no ser una asignación factible. Incluye además este artículo una caracterización de las soluciones igualitarias para la clase completa de los juegos NTU.

[9] Aunque estos juegos fueron definidos en origen por Rubinstein y Zhou (1999) se utilizan en este trabajo bajo el enfoque de Sudhölter y Zarzuelo (2013).

Al lo largo de toda la memoria se utilizarán numerosos tecnicismos matemáticos, a continuación se enuncian las definiciones y notaciones de los conceptos más generales.

Considerado un conjunto potencial de *jugadores* I, de cardinal $|I| \geq 2$, finito, una *coalición* $N \subset I$ es un conjunto no vacío de elementos de I. Dada una coalición N se denota por $\mathbb{R}^N$ el espacio $|N|$-dimensional euclídeo cuyos ejes se etiquetan con los miembros de N. Dados $x = (x_i)_{i \in N} \in \mathbb{R}^N$ y $S \subseteq N$, se denota por x_S la proyección de x en $\mathbb{R}^S$, esto es, $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$.

Dados $x, y \in \mathbb{R}^N$, se entiende por $x \geq y$ $(x > y)$ que $x_i \geq y_i$ $(x_i > y_i)$ para todo $i \in N$. Se denota por $\mathbb{R}^N_+, \mathbb{R}^N_{++}$ y $\mathbb{R}^N_-$ los subconjuntos de $\mathbb{R}^N$ de los vectores $x \geq 0, x > 0$ y $x \leq 0$ respectivamente. Además $x \cdot y$ denota el número real $\sum_{i \in N} x_i y_i$ (*producto escalar*) y $x * y$ el vector $(x_i y_i)_{i \in N} \in \mathbb{R}^N$.

Dados $A, B \in \mathbb{R}^N$, por $A \subset B$ se entiende que $A \subseteq B$ y $A \neq B$ y por $A \pm B$ la clausura del conjunto $\{a \pm b;\, a \in A \wedge b \in B\}$. Y si $x \in \mathbb{R}^N$, se define $x + A := \{x + a : a \in A\}$ y $x * A := \{x * a : a \in A\}$.

Se simplifica la notación de $N \setminus \{i\}$ y $N \cup \{i\}$ utilizando respectivamente $N \setminus i$ y $N \cup i$.

Si $A \in \mathbb{R}^N$, por ∂A se denota la frontera de A.

1. MODELOS DE JUEGOS COOPERATIVOS

1.1. Juegos de Negociación

1.1.1. Preliminares

Los *Juegos de Negociación* constituyen el paradigma más simple de la Teoría de Juegos. El análisis de este tipo de problemas se inició con los trabajos del matemático A. Cournot [1] y el economista F. Zeuthen [2]. Sin embargo, fue Nash , en 1950, quien en su célebre artículo *The Bargaining Problem* aplicó por primera vez las funciones de utilidad de von Neumann-Morgenstern[3] al estudio de este tipo de situaciones. El interés del modelo presentado por Nash en dicho trabajo, queda contrastado al seguir siendo aplicado hoy en día por los estudiosos de este tema, sin sufrir apenas modificaciones.

Un juego de negociación concierne a un grupo de individuos (jugadores), que tienen la capacidad de, entre varias opciones posibles, seleccionar una de ellas de manera unánime. A diferencia de otras teorías de elección social, en las situaciones de negociación existe una única alternativa factible cuando no se se alcanza una aprobación por unanimidad, denominada *alternativa de desacuerdo*. La capacidad de cada jugador de vetar cualquier alternativa distinta a la que se alcanzaría en caso de desacuerdo implica que el estudio del papel de posibles coaliciones de jugadores, diferentes de la coalición total, pierda todo sentido.

Suponiendo que cada individuo posee una relación de preferencias sobre las opciones

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[1] Cournot, en 1838, fue pionero en proponer la sistematización formal de la *Ciencia Económica*.

[2] Zeuthen, en 1930, sienta las bases de la *Teoría de la Negociación* en su trabajo *Problems of Monopoly and Economic Warfare*.

[3] Von Neumann y Morgenstern son considerados los padres de la *Teoría de Juegos* al identificarla como un campo de estudio diferente y autónomo, describiendo los juegos como una clase, delimitando la estructura de información de un juego, dibujando un árbol de juego y definiendo una solución de un juego en su conocida obra *Theory of Games and Economic Behaviour*, 1944.

de una negociación, cuando todos los jugadores prefieren alguna opción factible diferente a la de desacuerdo llegarán a un trato, ahora bien, si uno o más jugadores discrepan sobre la alternativa a elegir, surge el problema de determinar qué es lo mejor para el grupo. Aparece la necesidad de negociar cuál será la opción escogida, o de apelar a un árbitro que decida el resultado.

Se entiende por solución de un problema de negociación cualquier regla que indique una única alternativa de entre todas las alternativas posibles en cada situación que se plantee. El estudio de las soluciones de este tipo de problemas persigue, bien predecir el comportamiento de los agentes implicados, bien predecir la determinación que tomaría un árbitro al que se encomendase la elección de una de las opciones posibles.

1.1.2. Formalización

Definición 1.1.1. *Un Problema de Negociación es un par (S, d) tal que:*

- *$S \subseteq \mathbb{R}^N$ es convexo, cerrado y comprehensivo (i.e. $x \in S \wedge y \leq x \Rightarrow y \in S$).*
- *$d \in S$ y existe $x \in S$ tal que $x > d$.*
- *El conjunto $S_d = \{x \in S;\ x \geq d\}$ es compacto.*

El conjunto S representa las *asignaciones factibles* para los jugadores de la coalición N y $d = (d_i)_{i \in N}$ es la *asignación de desacuerdo* denominada también *punto de desacuerdo.*

Se denota por $\mathcal{B}_N$ el conjunto de todos los problemas de negociación en N.

La definición anterior se puede interpretar como sigue, N representa un grupo de n individuos cuyas preferencias sobre un conjunto de alternativas posibles, S, viene determinado por funciones de utilidad individuales, de manera que a cada una de las posibles alternativas tiene asociado un vector $x \in \mathbb{R}^N$ en el que cada x_i es la utilidad que reporta al jugador i dicha alternativa. Así, $S \in \mathbb{R}^N$ representa, a través de las utilidades, el conjunto de asignaciones factibles y d representa las utilidades de cada uno de los jugadores si no se llegase a un acuerdo.

La posibilidad de que los jugadores puedan optar a loterías entre las alternativas posibles y que las preferencias sobre estas se puedan expresar mediante funciones de utilidad de von Neumann-Mogernstern da lugar a un conjunto de alternativas factibles S convexo. La necesidad de que el conjunto S sea un conjunto cerrado es un tecnicismo, pero que el conjunto sea además comprehensivo[4] es consecuencia de la *Hipótesis de libre disponibilidad de utilidad* de los agentes. Según esta hipótesis cada jugador puede, voluntariamente, disminuir en cualquier cantidad su propia utilidad.

El requerimiento de que $d \in S$ se deriva de la factibilidad de la alternativa de desacuerdo. La existencia de $x \in S$, tal que $x > d$, nos asegura un proceso de negociación.

Al suponer que los jugadores siempre actúan de manera racional, la selección de alternativas se limitará a las denominadas *alternativas racionales* representadas por el conjunto S_d, ya que cualquier otra asignación será vetada por algún jugador. Que el conjunto de alternativas racionales, S_d, sea compacto implica que la utilidad de los jugadores en la opción finalmente escogida están acotados.

A continuación se formaliza cómo seleccionar alguna asignación, entre las asignaciones factible y la asignación de desacuerdo, resultado de un acuerdo entre los agente implicados o resultado de un arbitraje, para cualquier problema de negociación.

Definición 1.1.2. *Una Solución de Negociación en B es una aplicación ϕ de B en $I\!R^N$ que verifica:*

i) $\phi(S,d) \in S$, $\forall\, (S,d) \in \mathcal{B}_N$. *(Factibilidad)*

ii) $\phi(S,d) \geq d$, $\forall\, (S,d) \in \mathcal{B}_N$. *(Racionalidad Individual)*

Una solución de Negociación es pues un criterio para seleccionar una única alternativa en cada situación de negociación. La *Factibilidad* exige que esta esté al alcance del grupo y la *Racionalidad Individual* especifica que debe ser preferida a la obtenida en caso de desacuerdo.

Además de la solución propuesta por Nash en 1950, se presentan la *Solución Igualitaria* debida a Kalai, en 1977 y las *Soluciones Proporcionales* propuestas por Kalai y Samet en 1985. La selección de estas tres soluciones viene motivada por el importante papel que éstas juegan en la definición de las soluciones para la clase de los *juegos de utilidad* no transferible que se estudian con detalle en esta memoria.

[4] Este supuesto de comprehensividad no se encuentra el el modelo original de Nash, la mayoría de los resultados que se enuncian a continuación se pueden obtener sin este supuesto.

La Solución de Nash

Nash formaliza por primera vez un problema básico de negociación, entendiéndolo como un conjunto de posibles asignaciones de utilidad (von Neumann- Morgenstern) resultante de todos los posibles acuerdos que pueden alcanzar las partes negociantes, y una asignación correspondiente al pago que obtiene cada uno de los jugadores en caso de que no logren llegar a un acuerdo. Para buscar una solución al problema de negociación, recurre a establecer una serie de propiedades deseables (axiomas) que debería satisfacer tal solución y posteriormente procede a definirla. A continuación se relacionan dichos axiomas. Se considera ϕ una solución en $\mathcal{B}_N$.

El primero de los axiomas razonables para Nash es el de *Eficiencia*. Esta propiedad requiere que no exista ninguna alternativa considerada preferida por los jugadores a la que selecciona la solución.

[Eficiencia Fuerte]: (EFF) Dados $(S,d) \in \mathcal{B}_N$ y $x \in S$

$$x \geq \phi(S,d) \Rightarrow x = \phi(S,d).$$

Dado un subconjunto S de $\mathbb{R}^N$ se denomina conjunto de *Óptimos Fuertes de Pareto* de S al conjunto:

$$PF(S) = \{x \in S : y \geq x \Rightarrow y = x, \forall y \in S\} \tag{1.1}$$

Así pues, el axioma EFF exige que a cada juego (S,d) la solución le asigne un punto de $PF(S)$. Por ello, esta propiedad también se denomina *Optimalidad Fuerte de Pareto*. El conjunto $PF(S)$ se puede considerar como el conjunto de los elementos maximales de la *preferencia social* determinada por la conjunción de las preferencias individuales. Por lo que este axioma determina que la solución seleccione uno de estos elementos maximales.

El segundo de los axiomas deseables para Nash tiene su origen en el supuesto implícito de que los puntos de S representan utilidades de von Neumann-Morgenstern de las alternativas, y a que éstas están determinadas de manera unívoca salvo transformaciones afines. El cumplimiento de este axioma indica que la solución no depende de las funciones de utilidad de von Neumann-Morgenstern elegidas para representar cada preferencia individual[5].

[5] Por ello este axioma también se denomina *Independencia de Representaciones Equivalentes de Utilidad*.

[Covarianza]: (COV) Para todo $(S,d) \in \mathcal{B}_N$ y para todo $\alpha \in \mathbb{R}^N_{++}$ y $\beta \in \mathbb{R}^N$,

$$\phi(\alpha * S + \beta, \alpha * d + \beta) = \alpha * \phi(S,d) + \beta.$$

El axioma de simetría establece que si la posición de las partes en la negociación es idéntica (en cuanto a su aversión al riesgo, información disponible, etc.) y en el desacuerdo son tratados de la misma manera, entonces en la solución deben recibir lo mismo.

Se dice que $(S,d) \in \mathcal{B}_N$ es un *problema de negociación simétrico* si verifica[6]:

i) $x \in S \Rightarrow \pi(x) \in S$.

ii) $\pi(d) = d$.

[Simetría]: (SIM) Para cada juego simétrico $(S,d) \in B_N$

$$\phi_i(S,d) = \phi_j(S,d) \ \ \forall i,j \in N.$$

El último axioma se refiere a cómo debe comportarse la solución cuando en un problema se recorta el número de asignaciones factibles, manteniéndose la que se determinaba en principio como solución del mismo. En este caso, Nash propone que la solución quede invariante, esto es, establece que la elección de una asignación de utilidades no debe depender de asignaciones que, siendo factibles, no fueron elegidas.

[Independencia de Alternativas Irrelevantes]: (IAI) Dados (S,d) y $(S',d) \in \mathcal{B}_N$ tales que $S' \subseteq S$, si $\phi(S,d) \in S'$, entonces:

$$\phi(S,d) = \phi(S',d).$$

Esta última condición ha resultado ser bastante controvertida, habiéndose generado gran número de discusiones y de trabajos en torno a la conveniencia o no del mismo. Si se plantea la solución a una situación de negociación como un tipo de media entre todas las asignaciones factibles, este axioma no sería deseable. Pero si se entiende como solución un compromiso entre los agentes, esta propiedad no sólo es deseable, sino aconsejable. En 1985, Aumann disipa cualquier duda narrando una divertida anécdota, en la que grandes detractores del uso de este axioma, ante una situación real, lo aplican sin dudar sobre su conveniencia.

[6] Se denota por Π_N el conjunto de las permutaciones de N. Si $x \in \mathbb{R}^N$ y $\pi \in \Pi_N$, se designa por $\pi(x)$ al punto $(x_{\pi(i)})_{i \in N}$.

Nash demuestra que aceptar estos axiomas equivale a resolver los problemas de negociación escogiendo del conjunto S_d el único punto que maximiza el producto de las ganancias de utilidad de todos los agentes con respecto al punto de desacuerdo.

Definición 1.1.3. *Se llama Solución de Nash a la aplicación* $\mathcal{N} : \mathcal{B}_N \to I\!R^N$ *que asigna a cada juego* $(S, d) \in \mathcal{B}_N$ *el punto* $\mathcal{N}(S, d) \in I\!R_N$ *definido por* $\mathcal{N}(S, d) = arg\ máx\{P_d(x) : x \in S_d\}$, *siendo* $P_d(x) = \Pi_{i=1}^{n}(x_i - d_i)$.

La aplicación $\mathcal{N}$ está bien definida. El punto donde se alcanza el máximo existe al ser el P_d una función continua y S_d un conjunto compacto. Además es único, pues si hubiera dos puntos distintos donde se alcanzara, es fácil comprobar que en el punto medio de ambos, que pertenecería a S, la función P_d tomaría un valor estrictamente mayor.

Teorema 1.1.4 (Nash, 1950)**.** *Existe una única solución en* $\mathcal{B}_N$ *que satisface EFF, COV, SIM e IAI y es la solución de Nash*[7].

Demostración 1.1.5. *En primer lugar se prueba que* $\mathcal{N}$*, satisface los cuatro axiomas.*

(EFF) Si $\mathcal{N}$ *no fuese eficiente existirían* $(S, d) \in \mathcal{B}_N$ *y* $x \in S$ *tales que* $x \geq \mathcal{N}(S, d)$ *y* $x \neq \mathcal{N}(S, d)$. *Por lo tanto* $P_d(x) > P_d(\mathcal{N}(S, d))$ *y* $x \in S_d$, *lo que no concuerda con la definición de* $\mathcal{N}$.

(COV) Sean $(S, d) \in \mathcal{B}_N$, $\nu = \mathcal{N}(S, d)$, $\alpha \in I\!R^n_{++}$ *y* $\beta \in I\!R^N$. *Para cada* $x \in S$ *se cumple* $P_d(\nu) \geq P_d(x)$, *luego* $P_{\alpha*d+\beta}(\alpha * \nu + \beta) \geq P_{\alpha*d+\beta}(\alpha * x + \beta)$, *de donde se sigue el resultado.*

(SIM) Sea $(S, d) \in \mathcal{B}_N$ *simétrico. Entonces para cada* $\pi \in \Pi_N$, $P_d(\nu) = P_d(\pi(\nu))$ *y la función* P_d *alcanza el máximo en* S_d, *también el punto* $\pi(\nu)$, *como el punto donde se alcanza dicho máximo es único,* $\nu = \pi(\nu)$, *para cada* $\pi \in \Pi_N$.

(IAI) Sean (S, d) *y* (T, d) *dos juegos de* $\mathcal{B}_N$, *tales que* $\nu = \mathcal{N}(S, d)$, $T \subseteq S$ *y* $\nu \in T$. *entonces* $\nu \in T_d \subseteq S_d$, *y* P_d *alcanza su máximo en* T_d *en el punto* ν, *luego* $\mathcal{N}(T, d) = \nu$.

Queda ver que $\mathcal{N}$ *es la única solución que satisface los axiomas. Para ello se considera una solución* ϕ *que cumpla las propiedades anteriores.*

[7] Si en la definición de *Solución de Negociación* se utiliza la propiedad de *Racionalidad Individual Fuerte* ($\phi(S, d) > d$, $\forall\, (S, d) \in \mathcal{B}_N$) en lugar de la mencionada *Racionalidad Individual*, el axioma EFF resulta innecesario, ver Roth (1979).

Sea el conjunto $\overline{A} = \left\{ x \in I\!R^N : \sum_{i=1}^{n} x_i \leq n \right\}$. *Por cumplir* ϕ *los axiomas EFF y SIM,* $\phi(\overline{A}, \mathbf{0}) = \mathbf{1}$. *Sea ahora* $(S,d) \in \mathcal{B}_N$ *arbitrario tal que* $\nu = \mathcal{N}(S,d)$ *y sea* L *la transformación lineal de* $I\!R^N$ *tal que* $L(d) = \mathbf{0}$ *y* $L(\nu) = \mathbf{1}$.

Es claro que $L(S) \subseteq \overline{A}$ *y , por IAI,* $\phi(L(S), \mathbf{0}) = \mathbf{1}$.

Aplicando COV, $\phi(S,d) = L^{-1}(\phi(L(S)), \mathbf{0}) = \nu$.

Luego $\phi = \mathcal{N}$. □

Se mencionó, al enunciar el axioma *Independencia de Alternativas Irrelevantes* (IAI), que esta propiedad no está libre de controversia. Lensberg (1988) presenta una axiomatización de la solución de negociación de Nash en la que excluye este axioma y lo sustituye por un axioma de consistencia que denomina *Estabilidad Multilateral* (EM). Para ello necesita sustituir el axioma de *Simetría* (SIM) por una versión más fuerte del mismo, el axioma de *Anonimidad* (AN). A continuación se definen estos axiomas y se enuncia el teorema del que sólo incluimos un esquema de la demostración.

La condición de anonimidad establece que dos jugadores en igualdad de condiciones deben recibir el mismo pago. Esto es, lo que la solución asigne al jugador no dependerá de la denominación con que se represente a éste al abstraer el juego.

[Anonimidad]: (AN) Para todo $(S,d) \in \mathcal{B}_N$ y para cualquier permutación π de N,

$$\pi\phi(S,d) = \phi(\pi S, \pi d).$$

Lensberg define el axioma de *Estabilidad Multilateral* como una versión más fuerte del axioma *Estabilidad Bilateral* introducido por Harsanyi (1963):

[Estabilidad Bilateral]: (EB) Sean $M \subset N \subseteq I$ con $|M| = 2$, $S \subseteq \mathbb{R}^N$, $d \in S$, $(S,d) \in \mathcal{B}_N$,y $\phi(S,d) = x = (x_i)_{i \in N}$. Entonces:

$$T = \{y \in \mathbb{R}^M; (y, x_{N \setminus M}) \in S\} \Rightarrow (T, d_M) \in \mathcal{B}_M \ \text{ y } \ \phi(T, d_M) = x_M.$$

La idea que subraya Harsanyi para introducir esta nueva propiedad es que un jugador racional no aceptará un acuerdo si tiene razón para creer que puede amenazar con éxito algún otro jugador para que éste haga una concesión a su favor. Si un jugador no puede retar simultáneamente a cualquier otro jugador, éste basará sus creencias sobre la disposición a conceder del otro jugador en lo que ambos jugadores conocen de las soluciones de los juegos de negociación bipersonales. Como la situación es similar para todos los miembros de N entonces x puede ser solución de (S, d) sólo si coincide con la solución de cada uno de los problemas de negociación de dos personas resultantes de considerar que el resto de los jugadores recibe el pago indicado por x.

Lensberg se basa en esta idea para definir un axioma de consistencia más fuerte, la nueva condición requiere que la solución a un problema de negociación coincida con la solución a todos los subproblemas que se puedan generar a partir de él, no únicamente los bipersonales.

[Estabilidad Multilateral]: (EM) Sean $M \subset N \subseteq I$, $S \subseteq \mathbb{R}^N$, $d \in S$ y sean $(S, d) \in \mathcal{B}_N$ y $\phi(S, d) = x$. Entonces:

$$T = \{y \in \mathbb{R}^M; (y, x_{N \setminus M}) \in S\} \Rightarrow (T, d_M) \in \mathcal{B}_M \ \text{ y } \ \phi(T, d_M) = (x_M).$$

Teorema 1.1.6 (Lensberg, 1988). *Existe una única solución en $\mathcal{B}_N$ que cumple EFF, COV, AN y EM. Dicha solución es la solución de Nash.*

Demostración 1.1.7. *La prueba de Lensberg sigue el sigueinte esquema: se comprueba que la solución de Nash verifica cada uno de los axiomas y, a posteriori, se demuestra que la solución de Nash es la única que los satisface en dos pasos. En primer lugar se prueba que si una solución satisface EFF, AN y EB entonces dicha solución define para problemas de negociación de dos jugadores la solución de Nash. En segundo lugar, se demuestra que si una solución satisface los axiomas de EFF, COV, EM y coincide con la solución de Nash para juegos de negociación de dos personas entonces es la solución de Nash. De ambos resultados se concluye la demostración del teorema.* □

La Solución Igualitaria

En estudios sucesivos al de Nash otros autores, siguiendo su ejemplo, idearon nuevas propiedades deseables para una solución de negociación. Kalai (1977), por ejemplo argumenta que, en un proceso de negociación, los individuos realizan persistentes comparaciones interpersonales de utilidad, razón por la cual las soluciones propuestas deben ser consistentes con este este hecho. Así pues, propone una solución en la que las ganancias de la negociación deben repartirse de forma igualitaria, independientemente del poder de negociación de los agentes, de su actitud frente al riesgo, o de cualquier otro aspecto que pudiera afectar los resultados en la solución de Nash.

Hay un aspecto que es importante señalar respecto a la solución igualitaria y es que, para su construcción, Kalai recurre a la condición de invarianza ante descomposiciones del proceso de negociación en etapas; es decir, el resultado de la negociación debe ser el mismo, independientemente de que se negocie de una sola vez el objeto total, o de si después de que este haya sido fraccionado, se realiza la negociación por etapas y cada parte del objeto se negocia en una etapa diferente del proceso, en donde cada acuerdo alcanzado representa el status quo (punto de desacuerdo) de la siguiente etapa.

A continuación se presentan los axiomas que caracterizan esta solución.

[EFICIENCIA DÉBIL]: (EFD) Diremos que la solución ϕ en $\mathcal{B}_N$ satisface el axioma de Eficiencia Débil si para cada (S, d) de $\mathcal{B}_N$ se cumple:

$$x > \phi(S, d) \Rightarrow x \notin S, \, \forall x \in \mathbb{R}^N.$$

Esta propiedad, como la Eficiencia Fuerte, exige que no haya opciones estrictamente preferidas por todos los jugadores, a la que determina la solución, pero se diferencia en que admite la posibilidad de que existan opciones preferidas estrictamente a la *alternativa-solución* por algunos jugadores y que resultan indiferentes a los restantes.

Dado un subconjunto S de $\mathbb{R}^N$ denominamos conjunto de *Óptimos Débiles de Pareto* de S al conjunto:

$$PD(S) = \{x \in S : y > x \Rightarrow y \notin S\}.$$

La propiedad EFD se puede reescribir diciendo que una solución ϕ es *débilmente eficiente* si para cada (S, d) de $\mathcal{B}_N$ se cumple $\phi(S, d) \in PD(S)$, por lo también se denomina Optimalidad Débil de Pareto.

[COVARIANZA DÉBIL]: (COV*) Diremos que la solución ϕ en $\mathcal{B}_N$ satisface el axioma de Covarianza Débil si para cada (S, d) de $\mathcal{B}_N$ se cumple:

$$\phi(\alpha S + \beta, \, \alpha d + \beta) = \alpha \phi(S, d) + \beta, \, \forall \alpha \in \mathbb{R}_{++}, \, \forall \beta \in \mathbb{R}^N.$$

Este axioma debilita es más débil que COV y viene motivado porque el axioma COV requiere que la solución sea insensible a información que no está reflejada en las funciones de utilidad de los jugadores[8].

[Monotonía]: (MON) Diremos que la solución ϕ en $\mathcal{B}_N$ satisface el axioma de Monotonía si cumple:

$$T \subseteq S \Rightarrow \phi(T,d) \leq \phi(S,d),\ \forall (T,d),(S,d) \in \mathcal{B}_N.$$

La Monotonía requiere que ningún jugador resulte perjudicado si el conjunto de alternativas posible aumenta. Un argumento que justifica este axioma es que si la aparición de nuevas opciones perjudicara a algún jugador, éste podría vetarlas para que no se modificara la situación original.

Estos axiomas determinan una nueva solución de negociación.

Definición 1.1.8. *Llamamos solución Igualitaria a la aplicación $\mathcal{I} : \mathcal{B}_N \to I\!R^N$ que asigna a cada juego $(S,d) \in \mathcal{B}_N$ el punto $\mathcal{I}(S,d) \in I\!R^N$ definido por $\mathcal{I}(S,d) = d + \lambda(S,d) \cdot \mathbf{1}$, siendo $\lambda(S,d) = \max\{t \in I\!R : d + t\mathbf{1} \in S_d\}$ y $\mathbf{1}$ el vector de $I\!R^N$ cuyas componentes son 1.*

Teorema 1.1.9 (Kalai, 1977)**.** *Existe una única solución en $\mathcal{B}_N$ que cumple EFD, COV*, SIM y MO. Dicha solución es la solución igualitaria $\mathcal{I}$.*[9]

Demostración 1.1.10. *Es fácil comprobar que $\mathcal{I}$ cumple los axiomas del enunciado. Sean, pues, ϕ una solución en $\mathcal{B}_N$ que los verifique, $(S,\mathbf{0})$ un juego de $\mathcal{B}_N$ y $\lambda(S,\mathbf{0})$ como en la definición previa. Se probará que se cumple:*

(1) $\phi(S,\mathbf{0}) \geq \lambda(S,\mathbf{0})\mathbf{1}$, *para* $(S,\mathbf{0})$ *de* $\mathcal{B}_N$.

(2) *Si* $\lambda(S,\mathbf{0})\mathbf{1} \in PF(S)$ *entonces* $\phi(S,\mathbf{0}) = \lambda(S,\mathbf{0})\mathbf{1}$.

(3) $\phi(S,\mathbf{0}) \leq \lambda(S,\mathbf{0})\mathbf{1}$.

Como (1) *y* (3) *implican que ϕ coincide con $\mathcal{I}$ en los juegos en que $d = \mathbf{0}$, la prueba quedará completa aplicando COV*.*

[8] Ver Roth (1979).

[9] En Kalai (1977) aparecen diferentes caracterizaciones de esta solución.

Para cada $\varepsilon \in [1, n]$ *y cada* $i \in N$*, sea* $q^i(\varepsilon) \in I\!R^N$ *tal que* $q^i_j(\varepsilon) = 0$ *si* $i \neq j$ *y* $q^i_i(\varepsilon) = \varepsilon$. *Sea* V_ε *la envolvente convexa y comprehensiva de* $\{\mathbf{0}, \mathbf{1}, q^1(\varepsilon), ..., q^n(\varepsilon)\}$*. Por EFD y SIM,* $\phi(V_\varepsilon, \mathbf{0}) = \mathbf{1}$ *para cada* $\varepsilon \in [1, n]$.

Obsérvese que, por la comprehensividad de S*, para cada* $\delta \in (0, 1)$ *existe* $\varepsilon \in [1, n]$*, de modo que* $\delta \cdot \lambda(S, \mathbf{0}) V_\varepsilon \subseteq S$*. Por MON e COV*, se tiene que* $\phi(S, \mathbf{0}) \geq \phi(\delta \cdot \lambda(S, \mathbf{0}) V_\varepsilon, \mathbf{0}) = \delta \cdot \lambda(S, \mathbf{0})\mathbf{1}$*, y se sigue* (1) *dada la arbitrariedad de* δ*.* (2) *se sigue de* (1) *por definición de* $PF(S)$*.*

Queda por probar (3)*. Para cada* $\vartheta > 1$*, sea* S^ϑ *la envolvente convexa y comprehensiva del conjunto* $S \cup \{\vartheta \cdot \lambda(S, \mathbf{0})\mathbf{1}\}$*. Entonces,* $\lambda(S^\vartheta, \mathbf{0}) = \vartheta \cdot \lambda(S, \mathbf{0})$ *y además,* $\vartheta \cdot \lambda(S, \mathbf{0})\mathbf{1} \in PF(S^\vartheta)$*. Luego por (2),* $\phi(S^\vartheta, \mathbf{0}) = \vartheta \cdot \lambda(S, \mathbf{0})\mathbf{1}$*, y por MON,* $\phi(S, \mathbf{0}) \leq \vartheta \cdot \lambda(S, \mathbf{0})\mathbf{1}$*. Al ser* ϑ *es arbitrario,* $\phi(S, \mathbf{0}) \leq \lambda(S, \mathbf{0})\mathbf{1}$*.*

□

Las Soluciones Proporcionales

Otra solución destacable para juegos de negociación es la familia de soluciones proporcionales, éstas son una generalización de las soluciones igualitarias que permiten que el vector director de la solución sea cualquier vector de $\mathbb{R}^N$. En el teorema de caracterización que sigue se aprecia cómo esta familia de soluciones surge de prescindir del axioma SIM en la caracterización de la solución Igualitaria.

Definición 1.1.11. *Dado* $\mathbf{p} \in I\!R^N_+$ *llamamos Solución Proporcional de peso* $\mathbf{p}$ *a la aplicación* $\mathcal{I}^{\mathbf{p}} : \mathcal{B}_N \to I\!R^N$ *que asigna a cada juego* $(S, d) \in \mathcal{B}_N$ *el punto* $\mathcal{I}^{\mathbf{p}}(S, d) \in I\!R^N$*, definido por:* $\mathcal{I}^{\mathbf{p}}(S, d) = d + \lambda(S, d)\mathbf{p}$*, donde* $\lambda(S, d) = \text{máx}\{t \in I\!R : d + t\mathbf{p} \in S_d\}$*.*

Teorema 1.1.12 (Kalai, 1977)**.** *Una solución* ϕ *en* $\mathcal{B}_N$ *cumple EFD, COV* y MON si y sólo si existe* $p \in I\!R^N$ *tal que* $\phi = \mathcal{I}^p$*.*

Demostración 1.1.13. *Es inmediato comprobar que* $\mathcal{I}^{\mathbf{p}}$ *satisface los axiomas de enunciado para cada* $\mathbf{p} \in I\!R^N_{++}$*. Sea entonces* ϕ *una solución que cumpla dichos axiomas. Consideremos el conjunto:* $\overline{A} = \left\{ x \in I\!R^N : \sum_{i \in N} x_i \leq n \right\}$ *y* $\mathbf{p} = \phi(\overline{A}, \mathbf{0})$*. Por ser* ϕ *una solución* *de negociación,* $\boldsymbol{p} > \boldsymbol{0}$. *Ha de probarse que* $\phi = \mathcal{I}^{\boldsymbol{p}}$, *lo cual se logra sustituyendo en la prueba del teorema anterior* $\boldsymbol{1}$ *por* $\boldsymbol{p}$. □

1.2. Juegos con Utilidad Transferible

1.2.1. Preliminares

Los *juegos de negociación* analizan situaciones en las que cada individuo tiene la capacidad de vetar cualquier alternativa diferente de la alternativa de desacuerdo, esto implica que cualquier posible acuerdo entre unos pocos jugadores carece de sentido. Sin embargo, en situaciones con más de dos individuos implicados, el hecho de que uno de ellos no quiera llegar a un acuerdo, no impide generalmente que el resto, o algunos entre éstos, colaboren con objeto de conseguir más de lo que separadamente se pueden asegurar, entendiendo que la cooperación nunca reduce las posibilidades de cada individuo.

Los *juegos de utilidad transferible*, *juegos TU*, surgen de la necesidad de estudiar situaciones que posibilitan la formación de *coaliciones* con capacidad de alcanzar alternativas factibles sin el consentimiento unánime de todos los jugadores. En estas circunstancias se entiende que al (re)distribuir las ganancias obtenidas entre los miembros de una coalición es posible que unos jugadores decidan compensar a otros por posibles renuncias de éstos en su ganancia individual en pos de la obtención de un mayor beneficio común. Estas compensaciones, conocidas como *pagos laterales*, se efectúan mediante transferencias de utilidad, de ahí la denominación de estos juegos. Esta posibilidad de redistribución del beneficio obtenido implica, además, que un único número real pueda describir todas la alternativas posibles para una coalición.

Se entiende por *solución* de un juego con utilidad transferible a toda regla de asignación que determine un reparto del total.

1.2.2. Formalización

Definición 1.2.1. *Un juego TU es un par (N, v), donde N es el conjunto de jugadores y v una función, denominada función característica del juego, que asigna a cada coalición $S \subseteq N$ un número real $v(S) \in I\!R$ y satisface $v(\emptyset) = 0$.*

Se denota por $\mathcal{G}^{TU}$ la clase de juegos de utilidad transferible y por $\mathcal{G}_N^{TU}$ la clase de juegos de utilidad transferible de N jugadores.

Dado (N, v) y una coalición de jugadores $S \subseteq N$ el número real $v(S)$ representa el *valor* de la coalición S. Este valor puede admitir dos interpretaciones que dependen del contexto que se trate de formalizar. La primera es que $v(S)$ representa la máxima utilidad que los jugadores de S pueden garantizarse independientemente de las acciones de los jugadores que no están en S. La segunda es que representa la mínima utilidad que los jugadores que no están en S pueden evitar que obtengan los miembros de S.

Con las operaciones *adición* y *producto por un escalar*[10] el conjunto $\mathcal{G}_N^{TU}$ tiene estructura de espacio vectorial $2^{|N|-1}$-dimensional sobre $\mathbb{R}$. Una base de este espacio vectorial viene dada por la familia de los *Juegos de Unanimidad* $\{u_T; T \in \mathcal{P}(N)\}$, definidos a continuación.

Dado $T \in \mathcal{P}(N)$, se llama *Juego de Unanimidad* de T al juego u_T, definido para cada $S \in N$ como sigue:

$$u_T(S) = \begin{cases} 1 & \text{si } T \subseteq S \\ \\ 0 & \text{en otro caso.} \end{cases}$$

La familia de los denominados *Juegos de Identidad* $\{\delta_T; T \in \mathcal{P}(N)\}$, definidos a continuación, también conforma una base de $\mathcal{G}_N^{TU}$.

Dado $T \in \mathcal{P}(N)$, se llama *Juego de Identidad* de T al juego δ_T, definido para cada

[10] Dados $v, w \in \mathcal{G}_N^{TU}$ y $\alpha \in \mathbb{R}$, se definen $(v + w) \in \mathcal{G}_N^{TU}$ y $(\alpha v) \in \mathcal{G}_N^{TU}$ como $(v + w)(S) = v(S) + w(S)$ y $(\alpha v)(S) = \alpha v(S)$, para cada $S \subseteq N$.

$S \in N$ como sigue:

$$\delta_T(S) = \begin{cases} 1 & \text{si } T = S \\ \\ 0 & \text{en otro caso.} \end{cases}$$

En las situaciones de cooperación que se estudian en esta memoria, se entiende que si se integran más jugadores en cualquier coalición, el valor de la misma no se reduce. Esto es, no existen jugadores que resten valor a una coalición. Los juegos que cumplen esta propiedad se denominan *Juegos Monótonos*.

Un juego cooperativo (N, v) es *Monótono* si para todo $S, T \subseteq N$, tales que $T \subseteq S$, se cumple que:

$$v(T) \leq v(S).$$

Una propiedad relacionada con ésta es la denominada *superaditividad*, según la cual si se unen dos coaliciones disjuntas el valor de la nueva coalición será mayor o igual que la suma de los valores de las coaliciones originales, los juego que cumplen esta propiedad se denominan *Juegos Superaditivos*, formalmente:

Se dice que un juego cooperativo (N, v) es *Superaditivo* si para cualesquiera $S, T \subseteq N$, tales que $T \cap S = \emptyset$, se cumple que:

$$v(T \cup S) \geq v(T) + v(S).$$

En la búsqueda de solución de un juego de utilidad transferible, pueden adoptarse dos enfoques. El *enfoque descriptivo* trataría de predecir el resultado que seguiría a las acciones de los jugadores. Pero, es el *enfoque normativo o axiomático*, el seguido por los autores que introducen las soluciones que se estudian en esta memoria. Este enfoque trata de especificar criterios de distribución de utilidad entre los jugadores que cumplan determinados principios estimados aceptables a priori. De acuerdo con esta filosofía se tiene el siguiente concepto general de solución.

Definición 1.2.2. *Se llama Solución o Valor sobre* $\mathcal{G}_N^{TU}$ *a toda aplicación* ϕ *de* $\mathcal{G}_N^{TU}$ *en* $I\!R^N$.

Una de las de soluciones más conocidas entre las propuestas para los juegos de utilidad transferible: el *valor de Shapley*, Shapley (1953). A continuación se presentan los axiomas que Shapley utilizó para caracterizar su solución y una axiomatización alternativa propuesta por Hart and Mas-Collel (1989) que determina esta solución como *solución consistente.*

El valor de Shapley

Para enunciar cada uno de los axiomas que siguen se considera ϕ una solución en $\mathcal{G}_N^{TU}$.

[Eficiencia]: (EF) Se dice que una solución ϕ cumple el axioma de *eficiencia* si:

$$\sum_{i=1}^{n} \phi_i(v) = v(N), \;\; \forall v \in \mathcal{G}_N^{TU}.$$

Este axioma exige que la suma de lo obtenido por cada uno de los jugadores coincida con el total obtenido por la coalición de todos los jugadores, N, denominada en lo que sigue *gran coalición.*

Sean $\pi \in \Pi_N$ una permutación de N y $S \subseteq N$ una coalición. πS representa la coalición que resulta de aplicar π a S y πv denota al juego TU definido por: $(\pi v)(S) = v(\pi S)$, para cada $S \subseteq N$.

[Anonimidad](AN) Se dice que una solución ϕ cumple el axioma de *anonimidad* si:

$$\phi_i(\pi v) = \phi_{\pi i}(v), \;\; \forall \pi \in \Pi_N, \; \forall i \in N.$$

Esta condición establece que el valor que la solución asigna a un jugador no dependa de qué jugador sea.

[Aditividad]: (AD) Se dice que una solución ϕ cumple el axioma de *aditividad* si:

$$\phi(v+w) = \phi(v) + \phi(w), \;\; \forall v, w \in \mathcal{G}_N^{TU}.$$

Esto es, el valor del juego suma de dos juegos debe ser la suma de los valores de dichos juegos.

Se entiende por *jugador nulo* de un juego aquél que nunca aporta nada a ninguna coalición. Formalmente, dados $v \in \mathcal{G}_N^{TU}$ e $i \in N$, se dice que el jugador i es un jugador nulo si

$$v\left(S \cup i\right) = v(S).$$

[Jugador Nulo]: (JN) Se dice que una solución ϕ cumple el axioma de *jugador nulo* si:

$$i \in N \text{ es jugador nulo de } v \Rightarrow \phi_i(v) = 0.$$

Shapley demuestra que estos axiomas determinan una única solución en $\mathcal{G}_N^{TU}$.

Definición 1.2.3. *Se llama Valor de Shapley a la función Sh de $\mathcal{G}_N^{TU}$ en $I\!R$ definida por:*

$$Sh_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \cdot \big(v(S \cup i) - v(S)\big)\,,$$

para cada $v \in \mathcal{G}_N^{TU}$ y cada $i \in N$, siendo $s = |S|$ y $n = |N|$.

El valor de Shapley tiene una interpretación probabilista: si se escoge de manera aleatoria un jugador y, a partir de él, hasta completar la coalición N, se van añadiendo, siempre de modo aleatorio, jugadores de uno en uno, recibiendo cada jugador su contribución marginal al valor de la coalición que se forma al entrar él; es decir si se une i a la coalición S forma la coalición $S \cup i$ e i recibiría $\big(v(S \cup i) - v(S)\big)$. Si $s = |S|$, entonces $s!(n-s-1)!$ es el número de formas posibles en que este proceso puede darse de modo que S sea la coalición formada inmediatamente antes de incorporarse i, recibiendo por tanto $\big(v(S \cup i) - v(S)\big)$ y siendo $n!$ el número total de formas posibles, $\frac{s!(n-s-1)!}{n!}$ es la probabilidad de que el pago de i sea $\big(v(S \cup i) - v(S)\big)$ y así, $Sh_i(v)$ es el pago esperado del jugador i si los pagos se realizan como se ha descrito.

El valor de Shapley admite también una formulación diferente siguiendo la interpretación anterior. Sea $\mathcal{R}$ el conjunto de todos los órdenes totales en N y, para cada $R \in \mathcal{R}$ e $i \in N$, sea $R[i]$ el conjunto de todos los elementos anteriores a i de acuerdo con el orden R. Si se considera sobre $\mathcal{R}$ la distribución de probabilidad $I\!P$ definida por $I\!P(R) = 1/n!$, para cada $R \in \mathcal{R}$ y si se define para cada $v \in \mathcal{G}_N^{TU}$ y cada $i \in N$, la siguiente variable aleatoria sobre $\mathcal{R}$:

$$\mathcal{C}_i(v, R) = v\left(R[i] \cup i\right) - v\left(R[i]\right)$$

(valor conocido como *Contribución Marginal* del jugador i en el orden R), entonces el valor de Shapley puede expresarse como sigue:

$$Sh_i(v) = E_{\mathbb{P}}\left(\mathcal{C}_i(v,\cdot)\right) = \sum_{R\in\mathcal{R}} \frac{1}{n!}\cdot \mathcal{C}_i(v,R).$$

Teorema 1.2.4 (Shapley, 1953). *Existe una única solución sobre $\mathcal{G}_N^{TU}$ que satisface EFF, AN, AD y JN. Dicha solución es el valor de Shapley.*

Demostración 1.2.5. *Para probar la existencia basta mostrar que el valor de Shapley cumple los axiomas:*

$$EFF: \sum_{i\in N} Sh_i(v) = \sum_{i\in N}\frac{1}{n!}\sum_{R\in\mathcal{R}}(v(R[i]\cup i) - v(R[i])) = \frac{1}{n!}\sum_{R\in\mathcal{R}}\sum_{i\in N}(v(R[i]\cup i) - v(R[i])).$$

Si se prueba que para cada $R\in\mathcal{R}$ se cumple:

$$\sum_{i\in N}(v(R[i]\cup i) - v(R[i])) = v(N),$$

quedará demostrada la eficiencia del valor de Shapley.

Sea $R\in\mathcal{R}$ tal que $i_1\,R\,i_2\,R\ldots R\,i_n$, siendo $N=\{i_1,i_2,\ldots,i_n\}$, entonces:

$$\sum_{i\in N}\Big(v(R[i]\cup i) - v(R[i])\Big) = \sum_{j=1}^{n}\Big(v(R[i_j]\cup i_j) - v(R[i_j])\Big) =$$

$$\big(v(i_1)-v(\emptyset)\big) + \big(v(\{i_1,i_2\}) - v(i_1)\big) + \ldots + \big(v(\{i_1,i_2,\ldots,i_n\}) - v(\{i_1,i_2,\ldots,i_{n-1}\})\big) = v(N).$$

AN: Sean $v\in\mathcal{G}_N^{TU}$ y $\pi\in\Pi_N$,

$$Sh_i(\pi(v)) = \sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}\cdot\big(\pi(v(S\cup i)) - \pi(v(S))\big) =$$

$$= \sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}\cdot\big(v(\pi(S)\cup\pi(i)) - v(\pi(S))\big) =$$

$$= \sum_{T\subseteq N\setminus \pi(i)}\frac{t!(n-t-1)!}{n!}\cdot\big(v(T\cup\pi(i)) - v(T)\big) = Sh_{\pi(i)}(v).$$

AD: Se sigue de la definición del valor de Shapley.

JN: Sea $v\in\mathcal{G}_N^{TU}$ y sea i un jugador nulo del juego v. Como $\big(v(S\cup i) - v(S)\big) = 0$ para cada $S\in N\setminus i$ es claro que Sh cumple este axioma.

Queda probar por tanto la unicidad: Por el axioma AD y por el hecho de que la familia $\{u_T : T \in \mathcal{P}(N)\}$ *es una base de* $\mathcal{G}_N^{TU}$*, bastará probar que si una solución cumple los axiomas anteriores coincide con el valor de shapley en cada juego de la forma* αu_T*, para cada* $\alpha \in I\!R$ *y cada* $T \in \mathcal{P}(N)$*. Sea pues* ϕ *una solución que cumpla los axiomas anteriores.*

Si $i \notin T$*, por el axioma JN,* $\phi_i(\alpha u_T) = 0 = Sh_i(\alpha u_T)$*.*

Por otra parte si $i, j \in T$ *sea* $\pi \in \Pi_N$ *tal que* $\pi(i) = j$ *y* $\pi(T) = T$*, en ese caso* $\pi(\alpha u_T) = \alpha u_T$*. Al cumplir* ϕ *el axioma AN* $\phi_i(\alpha u_T) = \phi_i(\pi(\alpha u_T)) = \phi_{\pi(i)}(\alpha u_T) = \phi_j(\alpha u_T)$*.*

Finalmente por EFF $\phi_i(\alpha u_T) = \frac{\alpha}{t} = Sh_i(\alpha u_T)$*, para todo* $i \in T$*.*

En conclusión, $\phi = Sh$*.* □

Hasta ahora se ha definido el valor de Shapley a través de las contribuciones marginales de cada jugador, de manera que el valor de un juego puede verse como un vector de *pagos esperados* por los jugadores, como su contribución marginal a las coaliciones. Esta idea llevó a Hart y Mas-Colell a exponer otro enfoque basado en el *potencial*[11] de un juego. La definición del valor de Shapley a través del *potencial del juego* permite la caracterización de esta solución como *solución consistente*, determinándola como la única solución consistente y estándar para juegos de dos personas.

Una solución estándar para juegos de dos jugadores es aquella que reparte por igual el beneficio que estos obtienen al cooperar, formalmente:

Una solución ϕ en $\mathcal{G}^{TU}$ es *estándar para juegos de dos personas* si, para cada juego de dos personas (N, v) y para cada $i \in N$, se verifica:

$$\phi_i\big(N, v\big) = v\big(\{i\}\big) + \frac{1}{2}\Big[v\big(N\big) - v\big(\{i\}\big) - v\big(N \setminus i\big)\Big]. \tag{2.2}$$

Sea (N, v) un juego TU y sea $T \subseteq N$. Se llama *juego reducido* con respecto a ϕ en T, que se notará $(T, v_{T,\phi})$, al juego TU definido para cada $S \subseteq T$ como sigue[12]:

$$v_{T,\phi}(S) = \sum_{i \in S} \phi_i\big(S \cup (N \backslash T), v\big). \tag{2.3}$$

[11] Se denomina función potencial a una función $P : \mathcal{G}_N^{TU} \to I\!R$ tal que $P(\emptyset, v) = 0$ y satisface la siguiente condición: $\sum_{i \in N} D^i P(N, v) = v(N)$, donde $D^i P(N, v) = P(N, v) - P(N \setminus i, v)$.

[12] Al considerar soluciones eficientes, se usa la siguiente expresión, que es la expresión (4.3) en Hart y Mas-Colell (1989).

[Consistencia]: (CONS) Se dice que una solución ϕ cumple el axioma de *consistencia* si para cada coalición $T \in N$ se cumple:

$$\phi(T, v_{T,\phi}) = \phi_T(N, v).$$

Teorema 1.2.6 (Hart y Mas-Colell, 1989)**.** *Una solución en $\mathcal{G}^{TU}$ es consistente y estándar para juegos de dos personas si y sólo si es el valor de Shapley.*

Demostración 1.2.7. [13] *La prueba del teorema sigue el siguiente esquema: una vez probada la consistencia del valor de Shapley se demuestra por inducción que una solución, consistente y estándar para dos personas, cumple eficiencia. Hart y Mas-Colell justifican que la propiedad de consistencia es esencialmente equivalente a la existencia de un* potencial*, lo que implica que la solución se trata de la solución de Shapley.*

[13] Así como la demostración de Lensberg, vista en el apartado anterior, y otros resultados de consistencia necesitan un numero de jugadores no acotado en las pruebas, Hart y Mas-Colell consideran siempre un numero fijo finito de jugadores, como máximo de n jugadores

1.3. Juegos sin Utilidad Transferible

1.3.1. Preliminares

Los *juegos de utilidad transferible* se introducen como respuesta a la necesidad de generalizar los *problemas de negociación* a situaciones en las que un grupo individuos puedan obtener alguna alternativa factible sin requerir un acuerdo con el resto del grupo. Pero de manera inherente se considera la posibilidad de que los jugadores efectúen pagos laterales que pretenden compensar los posibles sacrificios que algunos miembros de una coalición hubieran podido realizar para obtener un beneficio común. Se plantea por tanto un nuevo horizonte ¿qué ocurriría si estos pagos laterales estuviesen de alguna manera vetados?

Los *juegos de utilidad no transferible*, *juegos NTU* tratan situaciones en las que una coalición puede obtener una de sus asignaciones factibles sin la aprobación de todos los jugadores, como los *juegos TU*, no obstante, la posibilidad de que esos pagos laterales estén prohibidos, bien por imposición, bien porque el bien común no sea divisible, implica que el pago disponible para una coalición no pueda determinarse con un único número real, sino que venga determinado por un conjunto de vectores de utilidades factibles. Una parte sustancial de la literatura sobre juegos cooperativos se centra en el análisis de este tipo de juegos por su gran versatilidad.

Al ser los *juegos TU* y los *juegos de negociación* casos particulares de los *juegos NTU* las soluciones para éstos se han concebido fundamentalmente como generalizaciones de las estudiadas para aquellos más sencillos. Así, Harsanyi en 1963 y Shapley en 1969, proponen soluciones formalmente parecidas que coinciden con la solución de Nash para juegos de negociación y con el valor de Shapley de juegos TU. Y Kalai y Samet en 1985, definen las *soluciones igualitarias* que coinciden con las soluciones proporcionales de Kalai en los juegos de negociación y con el valor de Shapley en los juegos TU. Las soluciones igualitarias son caracterizadas en el mismo artículo que se presentan, pero la solución de Shapley y la solución Harsanyi no son caracterizadas hasta 1985 por Aumann y Hart respectivamente, curiosamente estos tres artículos aparecen en el mismo volumen (53) de la revista *Econometrica* .

En la caracterización de la solución de Harsanyi, Hart, expone un nuevo concepto de solución al considerar que ésta debe especificar un vector de pagos para cada coalición, en lugar de un vector de pagos únicos para la gran coalición, introduce el concepto de *configuración de pagos* y presenta la solución a un juego no como pagos a individuos recogidos en un vector de coordenadas reales, sino como un vector de pagos coalicionales, i.e. un vector de vectores cuyas componentes son los pagos a los jugadores en cada posible subcoalición de la gran coalición.

Esta forma de concebir la solución de un juego constituye un pilar fundamental en el desarrollo de los trabajos que conforman esta memoria. Ya que el uso de *configuraciones de pagos*, así como leves modificaciones en la definición de *juego reducido* permiten la caracterización de las soluciones enunciadas previamente como ***soluciones consistentes***.

1.3.2. Formalización

Definición 1.3.1. *Un juego NTU es un par (N, V) donde N es una coalición y V es una función que asigna a cada coalición $S \subseteq N$ un subconjunto $V(S)$ de R^S que satisface:*

(A.1) *$V(S)$ es no vacío, cerrado,comprensivo (i.e. $x \in V(S)$* y *$x \geq y$* implica *$y \in V(S)$) y acotado superiormente.*

(A.2) *$V(S)$ es uniformemente no-nivelado, i.e. existe un número real $\delta(S, V) > 0$ tal que, para cada vector normalizado $\alpha \in I\!R^N_{++}$ (i.e. $\sum_{i \in N} \alpha_i = 1$) se cumplen las siguientes condiciones:*

$$\sup_{x \in V(S)} \alpha x < \infty \text{ implica } \alpha_i \geq \delta(S, V) \text{ para cada } i \in S;$$

Denotaremos por $\mathcal{G}$ al conjunto de todos los *juegos NTU* sobre I.

Dado un juego (N, V) y una coalición $S \subseteq N$, se denota por (S, V) al *subjuego* obtenido al restringir V únicamente a las subcoaliciones de S.

Los tipos de juegos estudiados previamente en esta introducción son subfamilias de los juegos NTU.

Uno de los casos más simples de los *juegos NTU* ocurre cuando para cada coalición $S \subseteq N$ existe un número real $v(S)$ tal que:

$$V(S) = \Big\{x \in \mathbb{R}^S : \sum_{i\in S} x_i \leq v(S)\Big\} \tag{3.4}$$

La familia de estos juegos se denota por $\mathcal{G}^{TU}$.

Una familia de juegos más general es la de los juegos para los que el conjunto $V(S)$ viene limitado por un hiperplano, son conocidos como *juegos de Hiperplano* y son aquellos juegos NTU para los que $V(S)$ está definido como:

$$V(S) = \Big\{x \in \mathbb{R}^S : \sum_{i\in S} p_i^S x_i \leq r_S\Big\}, \tag{3.5}$$

donde $p_i^S > 0 \; \forall\, i \in S$ y $r_S \in \mathbb{R}$.

Esta familia se denota por $\mathcal{G}^{HIP}$.

Una familia que contiene a la anterior es la de los juegos que NTU que cumplen que la frontera del conjunto de asignaciones para la gran coalición, $V(N)$, es un hiperplano, y no impone ninguna condición para los conjuntos $V(S)$, $S \neq N$. Estos juegos son conocidos como juegos G-hiperplanos (aludiendo G a la gran coalición. Son juegos NTU que verifican:

$$V(N) = \Big\{x \in \mathbb{R}^N : \sum_{i\in S} x_i \leq v(N)\Big\} \tag{3.6}$$

La familia de juegos g-hiperplanos se denota por $\mathcal{G}^G$.

Nótese que $\mathcal{G}^{TU} \subseteq \mathcal{G}^{HIP} \subseteq \mathcal{G}^G$.

Por último, los *Juegos de Negociación* son *juegos NTU* para los que existe un vector $d \in V(N)$ tal que, $d_S \in \delta V(S)$ para cada coalición $S \neq N$.

Definición 1.3.2. *Una configuración de pagos es un elemento* $\mathbf{x} = \left(x^S\right)_{S\subset N} \in \mathbf{x}^N$, $\mathbf{x}^N = \Pi_{S\subset N} \mathbb{R}^S$, *que asigna un vector de pagos* $\left(x_i^S\right)_{i\in S} \in \mathbb{R}^S$ *a cada coalición* S.

El valor de Shapley NTU

Podría decirse que una de las razones que hace de la solución de Shapley (para juegos TU) una de las más conocidas y aplicadas es su sencilla y útil caracterización, por lo que sorprende que, tras la definición por Shapley en 1969 de la extensión a juegos NTU de esta solución[14], no exista una axiomatización de la misma hasta que en 1985 Aumann la caracteriza con un conjunto de axiomas que combinan propiedades de los axiomas que caracterizan la solución de Shapley para juegos TU y de aquellos que caracterizan la solución de Nash en problemas de negociación.

A continuación se redacta una definic La definición del valor de Shapley NTU puede aproximarse como sigue: dado un juego NTU (N, V) y un vector $\lambda \in \mathbb{R}^N$ para los jugadores, se define un juego TU asociado (N, v_λ) tal que $v_\lambda(S) = \sup\left\{\lambda^S x; x \in V(S)\right\}$. El valor de Shapley NTU de (N, V) será un punto $i \in \delta(V(N))$ tal que para algún $\lambda \in \mathbb{R}^N$ positivo el juego (N, v_λ) está bien definido[15] y además $\lambda y = Sh(v_\lambda)$.

El valor de Harsanyi

La extensión del valor de Shapley a juegos NTU que se presenta en la sección anterior no es la única posible. Existen diferentes formas de extender el concepto de Shapley en juegos TU al contexto más genérico de juegos NTU.

Harsanyi propone en 1963 una *solución generalizada de negociación.* En principio estaba concebida como una extension de la solución de negociación de Nash a juegos NTU con subcoaliciones no triviales lo que luego resultó coincidir el valor de Shapley para juegos TU. Sin embargo, hoy día son más frecuentas las referencias a la misma como una extension del valor de Shapley a juegos NTU.

La intención de Harsanyi era encontrar una solución que generalizara la solución de Nash para los juegos de negociación. Para ello descompone un juego de la clase más general en varios juegos de negociación bipersonales, y aplica a éstos los resultados obtenidos al analizar el comportamiento de la solución de Nash en juegos compuestos.

[14] Esta solución definida por Shapley en 1969 recibe el nombre de λ-*transfer solution*, pero hoy día es conocida como la solución de Shapley para juegos NTU.

[15] Se dice que el juego (N, v_λ) está bien definido si dicho supremo es finito para cualquier coalición S de N.

Igualitarias

Kalai y Samet introducen en 1953 la soluciones igualitarias como una familia de soluciones monótonas para juegos con utilidad no transferible. Estas soluciones generalizan la idea de que los jugadores que forman una misma coalición deben obtener igual compensación por pertenecer a la misma, donde esa igual compensación se realiza en términos de utilidad comparada de manera interpersonal. Las soluciones igualitarias generalizan los valores ponderados de Shapley definidos para juegos con utilidad transferible y las soluciones proporcionales de Kalai definidos para problemas de negociación. Se demuestra que en presencia de otros axiomas débiles las soluciones igualitarios son la única monotónica queridos. La condición de monotonicidad se demuestra que es necesario y suficiente para lograr plena cooperación si asumimos que los jugadores son maximizadores de utilidad individuales y pueden controlar sus niveles de cooperación.

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2. CONSISTENCY OF THE HARSANYI NTU CONFIGURATION VALUE

2.1. Introduction.

In a remarkable paper Hart and Mas-Colell (1989) characterized the Shapley value as the only single-valued solution for TU games that coincides with the 2-person standard solution and is consistent with respect to a notion of reduced game proposed in the same work. This interesting result parallels other classical characterizations of cooperative solutions: the prenucleolus (Sobolev, 1975); the core (Peleg, 1985 and 1986); the Nash bargaining solution (Lensberg, 1988). Consistency has also played a prominent role in some other contexts: for instance in bankruptcy problems (Aumann and Maschler, 1985; Thomson, 2003) and other allocation problems (for a survey see Thomson, 2006).

Hart and Mas-Colell (1989) extended simultaneously their result to the wider class of non-transferable utility (NTU) games, and characterized a generalization of the Shapley value to NTU games: namely, the egalitarian solution (Kalai and Samet, 1985). On the other hand Maschler and Owen (1989) investigated whether there exists an efficient, symmetric, and consistent solution that, unlike the egalitarian solution, were also covariant under affine transformations of utility. Covariance is a quite compelling property if it is assumed that the preferences of the players are represented by von Neuman-Morgenstern utility functions. This search turned out to be fruitless since Maschler and Owen (1989) found a simple 3-person hyperplane game (see Section 6) for which such a solution would prescribe the empty set.[1]

In this paper we will consider payoff configurations instead of payoff vectors as solution outcomes. The notion of payoff configuration was introduced by Hart (1985) in his characterization of the Harsanyi NTU solution. More recently, De Clippel et al. (2004) compared

[1] Accordingly Maschler and Owen (1989) proposed a weaker consistency requirement —bilateral consistency—, and subsequently characterized the Maschler-Owen solution on the class of hyperplane games.

and characterized several NTU solution concepts by working with payoff configurations as well.

Adopting Hart's methodology permit us to show that an efficient, symmetric, consistent solution does exist that is also covariant, namely the Harsanyi NTU solution.[2] Although the Harsanyi NTU solution is, in general, a multi-valued solution, it turns out to be single-valued on the class of games for which the feasible set of the grand coalition is given by a half-space. We have characterized this solution precisely on this class —where hyperplane games are obviously included—. The axiom system includes consistency plus other plausible axioms: efficiency, covariance, symmetry, the null-player axiom, and an additional axiom, that we have called Optimal Threats Independence, requiring some coherence in the components of the payoff configurations assigned to the intermediate coalitions.

The paper is organized as follows. Section 2 and 3 contain some preliminaries and definitions. In Section 4, the Harsanyi NTU solution and the axiom system are introduced, and the main result is stated. Proofs are postponed to Section 5. In Section 6, the logical independence of the axioms is proved, and some final remarks can be found in Section 7 about the characterization of the Harsanyi NTU solution on a wider class of NTU games.

2.2. Preliminaries.

Most of the definitions and notation here follow those in Hart (1985).

Let I be a finite set of potential *players*, with cardinality $|I| \geq 3$. A *coalition* is any non-empty subset of I. For each coalition $N \subset I$, the $|N|$-dimensional Euclidean space whose axes are labeled with the members of N is denoted by $\mathbb{R}^N$. If $x = (x_i)_{i \in N} \in \mathbb{R}^N$ and $S \subset N$, then the projection of x onto $\mathbb{R}^S$ is denoted x_S, i. e., $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$.

Given $x, y \in \mathbb{R}^N$, then $x \geq y$ ($x > y$) means $x_i \geq y_i$ ($x_i > y_i$) for all $i \in N$. The subsets of $\mathbb{R}^N$ formed by vectors $x \geq 0$, and $x > 0$ are denoted by $\mathbb{R}^N_+$, and $\mathbb{R}^N_{++}$ respectively. Moreover, $x \cdot y$ denotes the real number $\sum_{i \in N} x_i y_i$ (scalar product), and $x * y$ the vector $(x_i y_i)_{i \in N}$. If $A,\, B \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $A + B$, $x + A$ and $x * A$ are defined by $A + B := \{a + b :\, a \in A,\, b \in B\}$, $x + A := \{x + a : a \in A\}$ and $x * A := \{x * a : a \in A\}$ respectively. The boundary of A is denoted by ∂A.

[2] In this work we will call it the Harsanyi configuration solution to avoid confusion when assigning payoff vectors as solution outcomes.

A *non-transferable utility (NTU) game* — a game, for short — is a pair (N, V) where N is a coalition and V is a set-valued function (the characteristic function) that assigns a subset $V(S)$ of $\mathbb{R}^S$ to each coalition $S \subset N$ such that

(A.1) $V(S)$ is non-empty, closed, comprehensive, and bounded from above.

(A.2) $V(S)$ is 'uniformly non-leveled'. That is, there exists a real number $\delta(S, V) > 0$ such that for every normalized vector $\lambda \in \mathbb{R}^N$ (i.e. $\sum_{i \in N} \lambda_i = 1$) the following condition holds

$$\sup_{x \in V(S)} \lambda \cdot x < \infty \text{ implies } \lambda_i \geq \delta(S, V) \text{ for every } i \in S.$$

The set of all NTU games will be denoted by $\mathcal{G}_I$.

Given a game (N, V) and a coalition $S \subset N$, then (S, V) denotes the subgame obtained by restricting V to subcoalitions of S only.

One of the simplest cases of an NTU game occurs when every coalition $S \subset N$ is assigned a real number $v(S)$ such that

$$V(S) = \Big\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \Big\}. \tag{2.1}$$

Such games are known as *transferable utility (TU) games*. We say that (N, V) corresponds to v, and V and v are denoted interchangeably and no confusion will appear. The set of TU games will be denoted $\mathcal{G}_I^{TU}$.

In a somewhat more general case, each $V(S)$ is a half-space given by a linear inequality, that is

$$V(S) = \Big\{ x \in \mathbb{R}^S : \sum_{i \in S} \lambda_i^S x_i \leq r^S \Big\}, \tag{2.2}$$

where $\lambda_i^S \in \mathbb{R}_{++}$ is a positive number for each $i \in S$, and $r^S \in \mathbb{R}$. Such games are known as *hyperplane games*. The set of hyperplane games is represented by $\mathcal{G}_I^{HYP}$.

We shall also be interested in the class of games (N, V) for which only $V(N)$ is a half-space. These games are called *G-hyperplane games* (G stands for *grand* coalition). The set of G-hyperplane games will be denoted by $\mathcal{G}_I^G$. Obviously $\mathcal{G}_I^{TU} \subset \mathcal{G}_I^{HYP} \subset \mathcal{G}_I^G$.

Another simple class of games is the class of *pure bargaining games*. These are games (N, V) for which there exists a vector $a \in V(N)$ such that $a_S \in \partial V(S)$ for every proper coalition S.

Given $(N, V) \in \mathcal{G}_I$, $\lambda \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}^N$, the game $(N, \lambda * V + a)$ is defined by $(\lambda * V + a)(S) = \lambda_S * V(S) + a_S$.

Let $\boldsymbol{X}^N$ denote the product $\prod_{S \subset N} \mathbb{R}^S$; an element $\mathbf{x} = (x^S)_{S \subset N} \in \boldsymbol{X}^N$ is called a *payoff configuration*. It assigns a payoff vector, $x^S = (x^S_i)_{i \in S} \in \mathbb{R}^S$, to every coalition S.[3]

2.3. *Consistent solutions on NTU games.*

Given a family of games $\mathcal{F} \subset \mathcal{G}_I$, a *value* ϕ on $\mathcal{F}$ is a function that assigns to each game $(N, V) \in \mathcal{F}$ a payoff vector $\phi(N, V) \in \mathbb{R}^N$.

On the class of TU games the *Shapley value*, denoted Sh, assigns to every game (N, v), the payoff vector defined for each $i \in N$ by

$$Sh_i(N, v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \big(v(S \cup i) - v(S)\big).^4$$

A leitmotif in this work is the characterization of the Shapley value due to Hart and Mas-Colell (1989) by means of consistency. It is stated below after some definitions.

Let ϕ be a value on the class of TU games, (N, v) a TU game, and $T \subset N$ a coalition. The *reduced game* $(T, v_{T,\phi})$ is the TU game given for every coalition $S \subset T$ by

$$v_{T,\phi}(S) = v\big(S \cup (N \setminus T)\big) - \sum_{i \in N \setminus T} \phi_i\big(S \cup (N \setminus T), v\big). \tag{3.3}$$

A value ϕ on the class $\mathcal{G}^{TU}_I$ is said to be *consistent* if, for every TU game (N, v) and every coalition $T \subset N$, it holds $\phi(T, v_{T,\phi}) = \phi_T(N, v)$.

Moreover, a value ϕ on $\mathcal{G}^{TU}_I$ is said to be *standard for two-person games*, if for every 2-person game $\big(N, v\big)$, and for each $i \in N$

$$\phi_i\big(N, v\big) = v\big(\{i\}\big) + \frac{1}{2}\Big[v\big(N\big) - v\big(\{i\}\big) - v\big(N \setminus i\big)\Big]. \tag{3.4}$$

[3] Notice the difference between x^S and x_S. By x^S we denote the payoff vector of the payoff configuration $\mathbf{x}$ corresponding to coalition S. In contrast, x_S is the projection of a vector $x \in \mathbb{R}^N$ to $\mathbb{R}^S$.

Theorem 2.3.1. *[Hart and Mas-Colell, 1989] A value on $\mathcal{G}_I^{TU}$ is consistent and standard for two-person games if and only if it is the Shapley value.*[5]

Hart and Mas-Colell (1989) generalized this result to the whole class of NTU games, and characterize an extension of the Shapley value: the egalitarian solution (Kalai and Samet, 1985).

The *egalitarian solution* assigns to every NTU game (N, V) the unique payoff vector $\varepsilon(N, V)$ for which there exists a family of real numbers $\left(d^T\right)_{T \subset N}$ such that, if we denote $x_i^S = \sum_{\substack{T \subseteq S \\ i \in T}} d^T$ for every coalition $S \subset N$ and every $i \in S$, then

$$x^N = \varepsilon(N, V), \tag{3.5}$$

$$x^S \in \partial V(S), \quad \text{for each coalition } S \subset N. \tag{3.6}$$

The egalitarian solution combines the efficiency and fairness principles in the payoff vector of every coalition. Indeed, condition (3.3) states that every intermediate payoff vector x^S is efficient. Moreover, the payoff x_i^S of each member of any coalition S is the sum of the 'dividends' d^T from all the subcoalitions T of S to which player i belongs. Since the dividends are the same for all members of T, we can say that the payoff x^S is fair.

The definition of a reduced NTU game is the natural extension of (3.1) (see Hart and Mas-Colell, 1989; also Maschler and Owen, 1989).

Let (N, V) be a NTU game, $T \subset N$ a coalition, and ϕ a value on $\mathcal{G}_I$. The *reduced game* $(T, V_{T,\phi})$ is defined for every coalition $S \subset T$ as follows:

$$V_{T,\phi}(S) = \left\{ y \in \mathbb{R}^S \,:\, \Big(y, \phi_{N \setminus T}\big(S \cup (N \setminus T), V\big) \Big) \in V\big(S \cup (N \setminus T)\big) \right\}. \tag{3.7}$$

A value ϕ on $\mathcal{G}_I$ is said to be *consistent* if for every NTU game (N, V) and every coalition $T \subset N$, it holds $\phi(T, V_{T,\phi}) = \phi_T(N, V)$.

Theorem 2.3.2. *[Hart and Mas-Colell,1989] The egalitarian solution is the only consistent value on $\mathcal{G}_I$ such that its restriction to $\mathcal{G}_I^{TU}$ is standard for two-person games.*

On the other hand Maschler and Owen (1989) were interested in finding values that, besides being consistent, were also covariant. A value, ϕ, defined on $\mathcal{F} \subseteq \mathcal{G}_I$ is said to

[5] Actually Hart and Mas-Colell (1989) considered an infinite set I, but can obviously adapted for the case of a finite set such that $|I| \geq 2$.

be *covariant under linear changes of utility* when we find $x = \phi(N,V)$ if and only if $\lambda * x + a = \phi(N, \lambda * V + a)$ $(a, \lambda \in \mathbb{R}^N,\ \lambda > 0)$.

Since the egalitarian solution is not covariant, from Theorem 4.3.2 we cannot expect to find any value on the whole class of NTU games that is (i) consistent, (ii) covariant and, moreover, (iii) the standard solution for two-person TU games.

One might ask if by reducing conveniently the domain we could find a covariant and consistent value for a wider family of NTU games.[6] However, Maschler and Owen (1989) showed that if we want to include the class of hyperplane games in this wider family the answer is negative with the following simple example. Let $N = \{1,2,3\}$, $V\big(\{i\}\big) = 0 - \mathbb{R}_+$ for $i = 1,2,3$, $V\big(\{1,2\}\big) = \big\{(x_1,x_2) \,:\, 2x_1 + 3x_2 \leq 180\big\}$, $V\big(\{i,j\}\big) = (0,0) - \mathbb{R}_+^{\{i,j\}}$ for $\{i,j\} = \{1,3\}, \{2,3\}$, and $V\big(\{1,2,3\}\big) = \big\{(x_1,x_2,x_3) \,:\, x_1 + x_2 + x_3 \leq 120\big\}$. The reader is referred to Maschler and Owen (1989) for further details.

Notwithstanding, in the following sections, it will be shown that, by adopting a different notion of solution concept, the answer becomes positive on the family of games (N,V) for which $V(N)$ is determined by a hyperplane.

2.4. *The Harsanyi NTU configuration value: a characterization.*

As mentioned earlier, a value customarily specifies a payoff vector for the grand coalition. In contrast, Hart (1985) considered that a solution outcome specifies a payoff vector for each coalition, that is, a payoff configuration. Accordingly we will distinguish value from configuration value.

Given a class of games $\mathcal{F} \subset \mathcal{G}_I$, a *configuration value* Ψ on $\mathcal{F}$ is a function that assigns to each game $(N,V) \in \mathcal{F}$ a payoff configuration $\Psi(N,V) \in \boldsymbol{X}^N$.

A payoff configuration $\mathbf{x} = (x^S)_{S \subset N}$ is a *Harsanyi payoff configuration*[7] of the game

[6] Actually, on the class of pure bargaining games such a value does exist: the Nash bargaining solution (Lensberg, 1988).

[7] Harsanyi (1959, 1963) originally defined this solution for games in strategic form.

(N,V) if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that

$$x^S \in \partial V(S), \quad \text{for each coalition } S \subset N, \tag{4.8}$$

$$\lambda \cdot x^N \geq \lambda \cdot y \quad \text{for all } y \in V(N), \text{ and} \tag{4.9}$$

$$\lambda_S * x^S = Sh(S, v_{\lambda,\mathbf{x}}) \text{ for each coalition } S \subset N, \tag{4.10}$$

where $(S, v_{\lambda,\mathbf{x}})$ is the TU game defined by $v_{\lambda,\mathbf{x}}(T) = \sum_{i\in T} \lambda_i x_i^T$ for every coalition $T \subset S$.

The following propositions are Proposition 4.9 and 4.10 respectively in Hart (1985), and will be used later on.

Proposition 2.4.1. *If (N,V) is a game in $\mathcal{G}_I^G$, then it has a unique Harsanyi payoff configuration.*

Proposition 2.4.2. *If (N,v) is a game in $\mathcal{G}_I^{TU}$ then it has a unique Harsanyi payoff configuration, namely:* $\mathcal{H}(N,v) = \Big\{ \big(Sh(S,v) \big)_{S \subset N} \Big\}$.

The first one states that the function $\mathcal{H} : \mathcal{G}_I^G \to \boldsymbol{X}$ that assigns to every game $(N,V) \in \mathcal{G}_I^G$ the corresponding Harsanyi payoff configuration is actually a configuration value. It will be called the Harsanyi NTU configuration value. Our main goal is to characterize the Harsanyi NTU configuration value in $\mathcal{G}_I^G$ with a consistency property together with some additional axioms. Further notation and definitions are needed in advance to state these axioms.

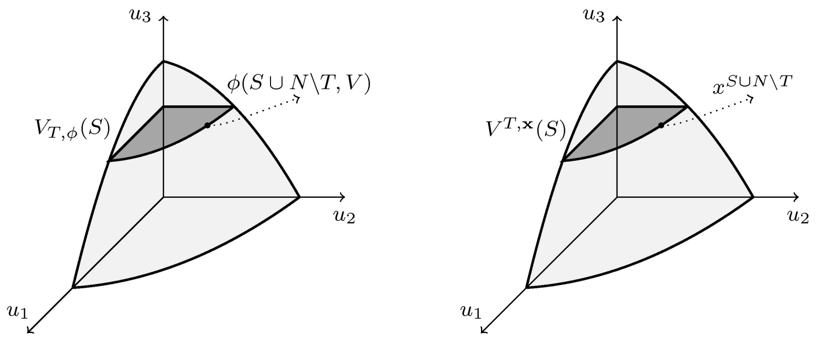
Every permutation π of I induces a linear mapping π_* from $\mathcal{G}_I$ onto itself, defined by $\pi_*(N,V) = (\pi N, \pi V)$, where $\pi V(\pi S) = V(S)$ for every coalition $S \subset N$. A subspace $\mathcal{F}$ of $\mathcal{G}_I$ is called *symmetric* if $\pi_* \mathcal{F} = \mathcal{F}$, for every permutation π of I.

We say that a player $i \in N$ is a *null player* in the game $(N,V) \in \mathcal{G}_I$ if $V(S) = \big\{ x \in \mathbb{R}^S : (x,0) \in V(S \cup i) \big\}$ for every coalition $S \subset N \backslash i$.

Let (N,V) be a game, $T \subset N$ a coalition, and $\mathbf{x} \in \boldsymbol{X}^N$ a payoff configuration. The *reduced game* $(T, V^{T,\mathbf{x}})$ is defined for each coalition $S \subset T$ by

$$V^{T,\mathbf{x}}(S) = \Big\{ y \in \mathbb{R}^S : \Big(y, x_{N\backslash T}^{S\cup(N\backslash T)} \Big) \in V\big(S \cup (N \backslash T) \big) \Big\}. \tag{4.11}$$

Notice that by assumptions (A.1) and (A.2) in the definition of an NTU game, $(T, V^{T,\mathbf{x}})$ is also an NTU game in $\mathcal{G}_I$.



$S = \{1,2\},\ N\backslash T = \{3\}$

Remark 2.4.3. *Let $\mathcal{F} \subset \mathcal{G}_I$ be a family of games such that $(N,V) \in \mathcal{F}$ implies $(S,V) \in \mathcal{F}$, for every $S \subset N$. With every value ϕ on $\mathcal{F}$ we can associate a configuration value Ψ^ϕ defined by $\Psi^\phi(N,V) = \big(\phi(S,V)\big)_{S\subset N}$ for each $(N,V) \in \mathcal{F}$. By doing so, it turns out that the reduced games defined in expressions (3.4) and (4.6) coincide, that is, $V_{T,\phi} = V^{T,\Psi^\phi(N,V)}$. Hence, the reduced game of (4.6) may be regarded as the natural extension of the one proposed by Hart and Mas-Colell with respect to payoff configurations.*

The following axioms are now imposed, where Ψ denotes a configuration value on a symmetric family of NTU games, $\mathcal{F} \in \mathcal{G}_I$, and (N,V), (N,W) are arbitrary games of $\mathcal{F}$.

Efficiency: (EFF)

$\Psi(N,V) \in \partial V = \prod_{S\subset N} \partial V(S)$.

Covariance under Linear Changes of Utility: (COV)

$\Psi(N, \lambda * V + a) = \lambda * \Psi(N,V) + a$, for all $\lambda \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}^N$.

Anonymity: (ANO)

$\Psi\pi_* = \pi\Psi$ for each permutation π of I.

Null Player: (NP)

If $i \in N$ is a null player in (N,V) and $\Psi(N,V) = \mathbf{x}$ then

$$x_i^{S\cup i} = 0 \text{ for all coalition } S \subset N\backslash i.$$

Consistency: (CONS)

If $\Psi(N, V) = \mathbf{x}$ then

$$\Psi\left(T, V^{T,\mathbf{x}}\right) = \left(x_S^{S\cup N\setminus T}\right)_{S\subset T} \text{ for every coalition } T \subset N,\ T \neq N.$$

Optimal Threats Independence: (OTI)

Let $\Psi(N, V) = \mathbf{x}$ and $\Psi(N, W) = \mathbf{y}$, and let $S \subset N$ be a coalition. If $V(N) = W(N)$, and $V(R) = W(R)$ for every coalition $R \subset S$, then

$$x^R = y^R \text{ for every coalition } R \subset S.$$

The first five axioms are standard in the literature. According to Harsanyi (1963), for every coalition $S \subset N$, one may interpret the component x^S of the payoff configuration $\mathbf{x}$ as the payoff vector that the members of S agree upon, as an optimal threat of coalition S against its complement, in the bargaining problem of the grand coalition N. In this regard, OTI requires that these optimal threats remain unchanged, whenever the opportunities for the grand coalition and for all the subcoalitions of S are the same.

Now we state our main result, that is the characterization of the Harsanyi configuration value on the class of games (N, V) for which $V(N)$ is determined by a hyperplane.

Theorem 2.4.4. *On the class $\mathcal{G}_I^G$ the Harsanyi NTU configuration value, $\mathcal{H}$, is the unique configuration value which satisfies EFF, COV, ANO, NP, CONS, and OTI.*

2.5. Proof of the Main Theorem

Proposition 2.5.1. *On the class $\mathcal{G}_I^G$, the Harsanyi NTU configuration value $\mathcal{H}$ satisfies EFF, COV, ANO, NP, CONS, and OTI.*

Proof 2.5.2. *EFF is symply expression (4.8). COV is also immediate. ANO, NP, and CONS can be logically concluded from (4.6), since the Shapley value satisfies anonymity, the null-player property and consistency. Finally, OTI is a consequence of (4.6) and the definition of the Shapley value.*

Remark 2.5.3. *From the non-levelness assumption (A.2), if $\mathbf{x}$ is efficient, then $\left(x_S^{S\cup(N\setminus T)}\right)_{S\subset T}$ is also efficient in the reduced game defined in expression (4.6). Notice that without this assumption, the Harsanyi NTU configuration value would fail to satisfy CONS.*

Now we turn to prove the uniqueness part. Firstly we will prove in Proposition 3.5.14 that this axiom system uniquely determines the configuration value on the class of TU games. Later we will extend this result to the wider class $\mathcal{G}_I^G$ in Proposition 4.5.11

For the remaining of this section, let Ψ represent a configuration value on the class $\mathcal{G}_I^G$

.

Associated with the configuration value Ψ, define the value ϕ^Ψ on $\mathcal{G}_I^G$ by

$$\phi^\Psi(N,V) = x^N \quad \text{whenever } \Psi(N,V) = \mathbf{x}. \tag{5.12}$$

Proposition 2.5.4. *Let Ψ satisfy EFF, COV, ANO, and CONS. If, in addition, Ψ satisfies for every TU game (N,v) the following property,*

$$\Psi(N,v) = \mathbf{x} \textit{ implies } \phi^\Psi(T,v) = x^T \textit{ for all coalition } T \subset N. \tag{5.13}$$

then $\Psi(N,v) = \mathcal{H}(N,v)$ for every TU game (N,v),[8]

Proof 2.5.5. *Let (N,v) be a TU game, such that $\Psi(N,v) = \mathbf{x}$, and $T \subset N$ be a coalition. Let ϕ^Ψ be the value defined in (5.7), and let v_{T,ϕ^Ψ} be the TU reduced game defined according to (3.1). In addition let $v^{T,\mathbf{x}}$ be the reduced game defined according to (4.6). Since Ψ satisfies (5.8), we have $\phi^\Psi(R,v) = x^R$ for every coalition $R \subset N$. Thus for every coalition $S \subset T$ we have*

$$\begin{aligned} v_{T,\phi^\Psi}(S) &= v\big(S \cup (N\backslash T)\big) - \sum_{i \in N\backslash T} \phi_i^\Psi\big(S \cup (N\backslash T), v\big) \\ &= v\big(S \cup (N\backslash T)\big) - \sum_{i \in N\backslash T} x_i^{S\cup(N\backslash T)} = v^{T,\mathbf{x}}(S). \end{aligned}$$

Consequently $v^{T,\mathbf{x}} = v_{T,\phi^\Psi}$, and hence

$$\phi^\Psi(T, v_{T,\phi^\Psi}) = \phi^\Psi(T, v^{T,\mathbf{x}}) = x_T^N = \phi_T^\Psi(N,v),$$

where the second equality follows on from CONS, since $\Psi(N,v) = \mathbf{x}$. Therefore the restriction of the value ϕ^Ψ to TU games is consistent.

[8] Notice that Property (5.8) is equivalent to: $\Psi(N,v) = \{(x^S)_{S\subset N}\}$ implies $\Psi(T,v) = \{(x^S)_{S\subset T}\}$ for every coalition $T \subset N$; i.e., the payoff configuration of the solution outcome of a subgame is precisely the restriction of the payoff configuration of the solution outcome of the whole game.

Furthermore, ϕ^{Ψ} *is standard on 2-person TU games (since* Ψ *satisfies EFF, COV, and ANO). Then Theorem 4.3.1 provides* $\phi^{\Psi} = Sh$*, and hence* $x^T = Sh(T, v)$ *for every coalition* $T \subset N$*. From Proposition 3.4.3,* $\Psi(N, v) = \mathcal{H}(N, v)$*.*

Lemma 2.5.6. *Let* Ψ *satisfy COV and OTI, and let* (N, v) *and* (N, w) *be TU games such that* $\Psi(N, v) = \mathbf{x}$ *and* $\Psi(N, w) = \mathbf{y}$*. If* $T \subset N$ *is a coalition such that* $v(R) = w(R)$ *for all coalition* $R \subset T$*, then* $x^R = y^R$ *for all coalition* $R \subset T$*.*

Proof 2.5.7. *Let* $i \in N \backslash T$ *be fixed and consider the auxiliary TU game* $(N, \tilde{v})$ *defined for each coalition* $S \subset N \backslash i$ *by* $\tilde{v}(S) = v(S)$*, and* $\tilde{v}(S \cup i) = v(S \cup i) + w(N) - v(N)$*. If* $\Psi(N, \tilde{v}) = \tilde{\mathbf{x}}$*, then COV implies* $\tilde{x}^R = x^R$ *for all* $R \subset T$*. Moreover, by applying OTI,* $\tilde{x}^R = y^R$ *is obtained for all coalition* $R \subset T$ *and the conclusion can be drawn.*

In order to facilitate the proof of Proposition 3.5.14, the following property (that can be viewed as a stronger version of the Null Player Axiom) will be considered:

Null Player*: (NP*)

If $i \in N$ is a null player in (N, V) and $\Psi(N, V) = \mathbf{x}$, then

$$x_i^{S \cup i} = 0 \quad \text{and} \quad x_S^{S \cup i} = x^S \qquad \text{for all coalition } S \subset N \backslash i.$$

Proposition 2.5.8. *If* Ψ *satisfies EFF, COV, ANO, NP*, CONS, and OTI, then* $\Psi(N, v) = \mathcal{H}(N, v)$ *for every TU game* (N, v)*.*

Proof 2.5.9. *We shall prove that* Ψ *satisfies condition (5.8), and the result will follow from Proposition 4.5.4.*

Let (N, v) *be a TU game and* $T \subset N$ *a proper coalition. Define the TU game* (N, w) *for all* $S \subset N$ *by* $w(S) = v(S \cap T)$*. Let* $\Psi(N, v) = \mathbf{x}$ *and* $\Psi(N, w) = \mathbf{y}$*.*

On the one hand every player in $N \backslash T$ *is null in* $\big(N, w\big)$*, which, through NP*, implies that* $y_{N \backslash T}^{S \cup N \backslash T} = 0$*, for every coalition* $S \subset T$*. Therefore*

$$w^{T,\mathbf{y}}(S) = w(S \cup N \backslash T) - \sum_{i \in N \backslash T} y_i^{S \cup N \backslash T} = v(S) - 0 = v(S),$$

that is $\big(T, w^{T,\mathbf{y}}\big) = (T, v)$*. Hence by CONS*

$$\Psi(T, v) = \Psi(T, w^{T,\mathbf{y}}) = \big(y_S^{S \cup N \backslash T}\big)_{S \subset T},$$

and consequently $\phi^{\Psi}(T,v) = y^N_T$.

Furthermore if $S \subset T$*, through Lemma 3.5.6, we get* $y^S = x^S$*, and NP* yields* $y_S^{S\cup N\setminus T} = y^S$*. In particular by choosing* $S = T$*, we obtain* $y^N_T = x^T$.

So we can conclude that $\phi^{\Psi}(T,v) = x^T$*, and* Ψ *satisfies condition (5.8) as claimed.*

However, under the weaker NP Axiom, more work is required since the equality $y_S^{S\cup N\setminus T} = y^S$ used in the proof above must also be satisfied.

Lemma 2.5.10. *Let* Ψ *satisfy EFF, COV, and ANO. If* $\big(N,v\big)$*, with* $N = \{i,j\}$*, is a 2-person TU game, then* $\Psi\big(N,v\big) = \mathcal{H}\big(N,v\big)$*. That is,* $\Psi\big(N,v\big) = \mathbf{x}$ *if and only if*

$$x_i^{\{i\}} = v(i),\ x_j^{\{j\}} = v(j), \tag{5.14}$$

$$x_i^N - x_j^N = x_i^{\{i\}} - x_j^{\{j\}} \quad \text{and} \quad x_i^N + x_j^N = v(N). \tag{5.15}$$

Proof 2.5.11. *It is straightforward.*

Lemma 2.5.12. *Let* Ψ *satisfy EFF, COV, ANO, and CONS, and* $(N,V) \in \mathcal{G}_I^G$*, with* $|N| \geq 2$*, such that* $V(N)$ *is a half-space whose normal vector is* $\mathbf{1}$*. If* $\Psi(N,V) = \mathbf{x}$*, then*

$$x_i^N - x_j^N = x_i^{N\setminus j} - x_j^{N\setminus i} \quad \text{for all } i,j \in N,\ i \neq j. \tag{5.16}$$

Proof 2.5.13. *Notice that the reduced game* $\big(\{i,j\}, V^{\{i,j\},\mathbf{x}}\big)$ *is the TU game that corresponds to the characteristic function* v *defined by* $v\big(\{i\}\big) = x_i^{N\setminus j}$*,* $v\big(\{j\}\big) = x_j^{N\setminus i}$ *and* $v\big(\{i,j\}\big) = x_i^N + x_j^N$*. The result therefore follows from Lemma 4.5.3.*

Proposition 2.5.14. *If* Ψ *satisfies EFF, COV, ANO, NP, CONS, and OTI, then* $\Psi(N,v) = \mathcal{H}(N,v)$ *for every TU game* (N,v).

Proof 2.5.15. *We shall prove that* Ψ *satisfies condition (5.8), and the result will follow from Proposition 4.5.4.*

Let (N,v) *be a TU game. If* $|N| = 1$ *the result is obviously true, and for* $|N| = 2$ *it follows from Lemma 4.5.3.*

Now assume that $|N| \geq 3$*. Let* $T \subset N$ *be any proper coalition. Consider the TU game* (N,w) *defined for all* $S \subset N$ *by* $w(S) = v(S \cap T)$*. Let* $\Psi(N,v) = \mathbf{x}$ *and* $\Psi(N,w) = \mathbf{y}$*. The steps in the proof of Proposition 4.5.5 above can be repeated in order to obtain* $\phi^{\Psi}(T,v) =$

y^N_T, and in addition $y^S = x^S$ for every $S \subset T$, and hence $y^T = x^T$. Therefore, if it is proved that $y^N_T = y^T$, then we will have that Ψ satisfies condition (5.8) as claimed.

So we turn to prove that $y^N_T = y^T$ for every coalition $T \subset N$. The case $|T| = 1$ it is straightforward by EFF. For the case in which T is not a singleton we will proceed by induction on $|N\backslash T|$.

Assume first that $N\backslash T = \{i\}$, for some $i \in N$. Through Lemma 4.5.6, if $j \in N\backslash i$, then $y^N_i - y^N_j = y^{N\backslash j}_i - y^{N\backslash i}_j$. Moreover, since i is a null player in (N, w), NP Axiom yields $y^N_i = y^{N\backslash j}_i = 0$. Consequently $y^N_j = y^{N\backslash i}_j$ for all $j \in N\backslash i$, i.e. $y^N_{N\backslash i} = y^{N\backslash i}$, as required.

Observe that for the case $|N| = 3$, the proof that $\Psi(N, V) = \mathcal{H}(N, v)$ is already completed.

Assume now that $y^N_T = y^T$ is true when $|N\backslash T| \leq k - 1$. Notice that this implies that $\Psi(N, v) = \mathcal{H}(N, v)$ is already proved for the case in which $|N| \leq k + 1$, and by Proposition 3.4.3, this yields

$$\phi^\Psi(N, v) = Sh(N, v) \qquad \textit{whenever } |N| \leq k + 1, \tag{5.17}$$

Now let us suppose that $|N\backslash T| = k$. Let $j \in T$ and consider the reduced TU game $\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big)$ defined according to (4.6). Notice that players in $N\backslash T$ are null players in (N, w), and the induction argument implies $y^N_R = y^R$, for every $R \supset T$, $R \neq T$. Therefore, through EFF,

$$w^{(N\backslash T)\cup j,\mathbf{y}}(S) = \begin{cases} 0 & \text{if } j \notin S, \\ y^T_j & \text{if } S = \{j\}, \\ y^N_j & \text{otherwise.} \end{cases} \tag{5.18}$$

Hence $Sh_i\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \dfrac{(k-1)!}{(k+1)!}(y^N_j - y^T_j)$, whenever $i \in N\backslash T$.

Now if $i \in N\backslash T$ then i is a null player in (N, w), and NP and CONS together with (5.10), since $|(N\backslash T) \cup j| = k + 1$, provide

$$0 = y^N_i = \phi^\Psi_i\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \frac{(k-1)!}{(k+1)!}(y^N_j - y^T_j). \tag{5.19}$$

Thus, $y^N_j = y^T_j$ holds true for every $j \in T$, as required.

Lemma 2.5.16. *Let* Ψ *satisfy EFF, COV, ANO, NP, CONS, and OTI. Also let* (N,V) *be a NTU game in* $\mathcal{G}_I^G$ *such that* $V(N)$ *is a half-space whose normal vector is* $\mathbf{1}$ *and for which* $\Psi(N,V)=\mathbf{x}$. *Let* $\bar{k}\in N$ *be a fixed player. Consider the NTU game* (N,W) *defined by*

$$W(S)=\begin{cases}\{x\in I\!R^{N\setminus\bar{k}}\ :\ \sum_{i\neq\bar{k}}x_i\leq\sum_{i\neq\bar{k}}x_i^{N\setminus\bar{k}}\}, & \textit{if } S=N\setminus\bar{k};\\ V(S), & \textit{otherwise.}\end{cases}$$

Then $\Psi(N,W)=\Psi(N,V)$.

Proof 2.5.17. *Let* $\Psi(N,W)=\mathbf{y}$. *By OTI, to prove that* $\mathbf{y}=\mathbf{x}$ *it is enough to show that* $y^N=x^N$ *and* $y^{N\setminus\bar{k}}=x^{N\setminus\bar{k}}$.

First, through OTI,

$$y^{N\setminus j}=x^{N\setminus j}\qquad \textit{for all } j\in N\setminus\bar{k}. \tag{5.20}$$

And by Lemma 4.5.6,

$$x_i^N-x_j^N=x_i^{N\setminus j}-x_j^{N\setminus i}\qquad \textit{for all } i,j\in N,\ i\neq j, \tag{5.21}$$

$$y_i^N-y_j^N=y_i^{N\setminus j}-y_j^{N\setminus i}\qquad \textit{for all } i,j\in N,\ i\neq j. \tag{5.22}$$

Combining (5.17) with (5.18) and (5.19) for the case $i=\bar{k}$ *yields*

$$x_{\bar{k}}^N-x_j^N+x_j^{N\setminus\bar{k}}=y_{\bar{k}}^N-y_j^N+y_j^{N\setminus\bar{k}}\quad \textit{for all } j\in N\setminus\bar{k}, \tag{5.23}$$

and for the case $i\neq\bar{k}$, *yields*

$$x_i^N-x_j^N=y_i^N-y_j^N\qquad \textit{for all } i,j\in N\setminus\bar{k}. \tag{5.24}$$

From (5.20), it follows that

$$\big(|N|-1\big)x_{\bar{k}}^N-\sum_{j\in N\setminus\bar{k}}x_j^N+\sum_{j\in N\setminus\bar{k}}x_j^{N\setminus\bar{k}}=\big(|N|-1\big)y_{\bar{k}}^N-\sum_{j\in N\setminus\bar{k}}y_j^N+\sum_{j\in N\setminus\bar{k}}y_j^{N\setminus\bar{k}},$$

or equivalently,

$$|N|x_{\bar{k}}^N-\sum_{j\in N}x_j^N+\sum_{j\in N\setminus\bar{k}}x_j^{N\setminus\bar{k}}=|N|y_{\bar{k}}^N-\sum_{j\in N}y_j^N+\sum_{j\in N\setminus\bar{k}}y_j^{N\setminus\bar{k}}. \tag{5.25}$$

Now EFF yields $\sum_{j\in N} x_j^N = \sum_{j\in N} y_j^N$ *and* $\sum_{j\in N\backslash\bar{k}} x_j^{N\backslash\bar{k}} = \sum_{j\in N\backslash\bar{k}} y_j^{N\backslash\bar{k}}$. *Hence (5.22) yields*

$$y_{\bar{k}}^N = x_{\bar{k}}^N. \tag{5.26}$$

Furthermore, EFF together with equalities (5.21) and (5.23) yield $y_i^N = x_i^N$ *for all* $i \in N$, *that is*

$$y^N = x^N. \tag{5.27}$$

as claimed.

Finally this last equality (5.24) together with (5.20) imply that $y_j^{N\backslash\bar{k}} = x_j^{N\backslash\bar{k}}$ *for all* $j \in N\backslash\bar{k}$, *i.e.,* $y^{N\backslash\bar{k}} = x^{N\backslash\bar{k}}$, *and the proof is complete.*

Proposition 2.5.18. *Let* Ψ *satisfy EFF, COV, ANO, NP, CONS, and OTI. Let* (N,V) *be a game in* $\mathcal{G}_I^G$ *such that* $V(N)$ *is a half-space whose normal vector is* $\mathbf{1}$ *and for which* $\Psi(N,V) = \mathbf{x}$. *Let* $T \subset N$ *be a fixed coalition. Consider the NTU game* (N,W) *defined by*

$$W(S) = \begin{cases} \{x \in I\!R^T : \sum_{i\in T} x_i \le \sum_{i\in T} x_i^T\}, & \text{if } S = T; \\ V(S), & \text{otherwise.} \end{cases}$$

Then $\Psi(N,W) = \Psi(N,V)$.

Proof 2.5.19. *Induction is now used on* $|N\backslash T|$, *where the case* $|N\backslash T| = 1$ *is Proposition 4.5.9 above.*

Let T *be a fixed coalition such that* $|N\backslash T| > 1$. *In view of the induction hypothesis it can be assumed (to avoid further notation), that* $V(S)$ *is already TU for every coalition* S *such that* $|S| > |T|$; *that is, it can be assumed that* $V(S) = \{x \in I\!R^S : \sum_{j\in S} x_j \le \sum_{j\in S} x_j^S\}$ *whenever* $|S| > |T|$.

Let us denote $\Psi(N,W) = \mathbf{y}$. *According to OTI, in order to prove that that* $\mathbf{y} = \mathbf{x}$ *it is sufficient to show that* $y^{T\cup R} = x^{T\cup R}$ *for every* $R \subset N\backslash T$. *This will be proved in three steps.*

Step 1. $y_R^{T\cup R} = x_R^{T\cup R}$ *for every* $R \subset N\backslash T$, $R \neq \emptyset$.

By induction on the cardinality of R. So let us assume first that $R = \{\bar{k}\}$ *for certain* $\bar{k} \in N\backslash T$.

For any $j \in T$, let us consider the reduced games $\big((N\backslash T)\cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ and $\big((N\backslash T)\cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$. Through our assumption if $|S| > |T|$ then $V(S)$ is a half-space whose normal vector is $\mathbf{1}$, it turns out that these reduced games are also TU games. By Proposition 3.5.14 and CONS we get

$$\Psi\big((N\backslash T)\cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \mathcal{H}\big((N\backslash T)\cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \Big(x^{S\cup(T\backslash j)}\Big)_{S\subset(N\backslash T)\cup j}$$
$$\Psi\big((N\backslash T)\cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \mathcal{H}\big((N\backslash T)\cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \Big(y^{S\cup(T\backslash j)}\Big)_{S\subset(N\backslash T)\cup j}.$$

Now consider the respective subgames $\big(\{\bar{k},j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ and $\big(\{\bar{k},j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$, and assume that

$$\Psi\big(\{\bar{k},j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \{\boldsymbol{a}\} \quad \text{and} \quad \Psi\big(\{\bar{k},j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \{\boldsymbol{b}\}.$$

By Proposition 3.4.3

$$\begin{aligned} a^{\{\bar{k},j\}} &= (x_{\bar{k}}^{T\cup\bar{k}}, x_j^{T\cup\bar{k}}) & b^{\{\bar{k},j\}} &= (y_{\bar{k}}^{T\cup\bar{k}}, y_j^{T\cup\bar{k}}) & \\ a^{\{\bar{k}\}} &= x_{\bar{k}}^{(T\cup\bar{k})\backslash j} & b^{\{\bar{k}\}} &= y_{\bar{k}}^{(T\cup\bar{k})\backslash j} & \textit{for all } j\in T. \\ a^{\{j\}} &= x_j^{T} & b^{\{j\}} &= y_j^{T} & \end{aligned}$$

Consequently, it follows on from Lemma 4.5.3 that

$$x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}} = x_{\bar{k}}^{(T\cup\bar{k})\backslash j} - x_j^{T} \qquad \textit{for all } j\in T, \tag{5.28}$$
$$y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j} - y_j^{T} \qquad \textit{for all } j\in T. \tag{5.29}$$

Furthermore, OTI implies $x^S = y^S$ for any coalition $S \subset N\backslash j$. In particular $x_{\bar{k}}^{(T\cup\bar{k})\backslash j} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j}$, and hence, from (5.29) and (5.30), it can be concluded that

$$(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - (y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = y_j^{T} - x_j^{T} \qquad \textit{for all } j\in T. \tag{5.30}$$

Therefore, $\sum_{j\in T}(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - \sum_{j\in T}(y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = \sum_{j\in T} y_j^T - \sum_{j\in T} x_j^T$. Now EFF yields $\sum_{j\in T} y_j^T = \sum_{j\in T} x_j^T$ and $\sum_{j\in T\cup\bar{k}} x_j^{T\cup\bar{k}} = \sum_{j\in T\cup\bar{k}} y_j^{T\cup\bar{k}}$, and thus $\big(|T|+1\big)(x_{\bar{k}}^{T\cup\bar{k}} - y_{\bar{k}}^{T\cup\bar{k}}) = 0$, which implies

$$x_{\bar{k}}^{T\cup\bar{k}} = y_{\bar{k}}^{T\cup\bar{k}}. \tag{5.31}$$

Now let us prove $y_R^{T\cup R} = x_R^{T\cup R}$ when $R \subset N\backslash T$ has more than one element.

W.l.o.g. assume that $\bar{k} \in R$ *and let us prove that* $x_{\bar{k}}^{T\cup R} = y_{\bar{k}}^{T\cup R}$.

For each $j \in (T \cup R)\backslash\bar{k}$ *consider the subgames* $\big(\{\bar{k}, j\}, V^{(N\backslash(T\cup R))\cup\{\bar{k},j\},\mathbf{x}}\big)$ *and* $\big(\{\bar{k}, j\}, W^{(N\backslash(T\cup R))\cup\{\bar{k},j\},\mathbf{y}}\big)$.

Similarly to expressions (5.29) and (5.30) we can get

$$x_{\bar{k}}^{T\cup R} - x_j^{T\cup R} = x_{\bar{k}}^{(T\cup R)\backslash j} - x_j^{(T\cup R)\backslash\bar{k}} \qquad \text{for all } j \in (T \cup R)\backslash\bar{k}, \tag{5.32}$$

$$y_{\bar{k}}^{T\cup R} - y_j^{T\cup R} = y_{\bar{k}}^{(T\cup R)\backslash j} - y_j^{(T\cup R)\backslash\bar{k}} \qquad \text{for all } j \in (T \cup R)\backslash\bar{k}. \tag{5.33}$$

It can be shown that $x_{\bar{k}}^{(T\cup R)\backslash j} = y_{\bar{k}}^{(T\cup R)\backslash j}$ *for every* $j \in (T \cup R)\backslash\bar{k}$*; indeed, when* $j \in T$ *Axiom OTI yields this equality, and when* $j \in R$*, then the induction process on the cardinality of* R *provides this equality, because* $|R\backslash j| < |R|$.

By taking this equality into account, it can be concluded from (5.33) and (5.34) that

$$\sum_{j\in(T\cup R)\backslash\bar{k}} (x_{\bar{k}}^{T\cup R} - x_j^{T\cup R}) - \sum_{j\in(T\cup R)\backslash\bar{k}} (y_{\bar{k}}^{T\cup R} - y_j^{T\cup R}) = \sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} - \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}.$$

From EFF we can deduce
$\sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} = \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}$ *and* $\sum_{j\in(T\cup R)} y_j^{(T\cup R)} = \sum_{j\in(T\cup R)} x_j^{(T\cup R)}$.
Hence $\big(|T\cup R|\big)x_{\bar{k}}^{T\cup R} - \big(|T\cup R|\big)y_{\bar{k}}^{T\cup R} = 0$*, which implies* $x_{\bar{k}}^{T\cup R} = y_{\bar{k}}^{T\cup R}$. *So* $x_R^{T\cup R} = y_R^{T\cup R}$ *for every coalition* $R \subset N\backslash T$.

Step 2. $y_T^{T\cup R} = x_T^{T\cup R}$ *for every* $R \subset N\backslash T$, $R \neq \emptyset$.

If $R \subset N\backslash T$, $R \neq \emptyset$*, from Step 1 we can easily conclude that* $x_R^{S\cup R} = y_R^{S\cup R}$ *whenever* $T \subset S$*, and by Axiom OTI* $x_R^{S\cup R} = y_R^{S\cup R}$ *when* $T\backslash S \neq \emptyset$*. So we can deduce that that* $x_R^{S\cup R} = y_R^{S\cup R}$ *for every* $S \subset N$*, and consequently* $(N\backslash R, V^{N\backslash R,\mathbf{x}}) = (N\backslash R, W^{N\backslash R,\mathbf{y}})$ *for every* $R \subset N\backslash T$, $R \neq \emptyset$*. Through CONS* $x_S^{S\cup R} = y_S^{S\cup R}$ *for every* $S \subset N\backslash R$*. In particular* $x_T^{T\cup R} = y_T^{T\cup R}$ *as required.*

Step 3. $x^T = y^T$.

It follows on from expression (5.31) since $x_{\bar{k}}^{T\cup\bar{k}} = y_{\bar{k}}^{T\cup\bar{k}}$ *for* $\bar{k} \notin T$.

Proposition 2.5.20. *If* Ψ *satisfies EFF, COV, ANO, NP, CONS, and OTI, then* $\Psi = \mathcal{H}$ *on* $\mathcal{G}_I^G$.

Proof 2.5.21. *We have to show that* $\Psi(N,V) = \mathcal{H}(N,V)$ *for each* $(N,V) \in \mathcal{G}_I^G$. *Since* Ψ *satisfies COV, it can be assumed* $V(N)$ *is a half-space whose normal vector is* $\mathbf{1}$. *Now suppose that* $\Psi(N,V) = \mathbf{x}$, *and consider the TU game* $(N, v_{\mathbf{x}})$, *whose characteristic function is defined by* $v_{\mathbf{x}}(T) = \sum_{i\in T} x_i^T$ *for every coalition* $T \subset N$. *Therefore, Proposition 3.5.14 and the definition of the Harsanyi NTU configuration solution give* $\Psi(N, v_{\mathbf{x}}) = \mathcal{H}(N, v_{\mathbf{x}}) = \mathcal{H}(N,V)$, *and Proposition 4.5.10, applied recursively, yields* $\Psi(N, v_{\mathbf{x}}) = \Psi(N,V)$, *and the result follows.*

Proof of Theorem 4.4.5. This is consequence of Propositions 4.5.1 and 4.5.11.

2.6. *Independence of the axioms*

In this section we will give some examples of configuration values that serve to show the logical independence of the axiom system in Theorem 4.4.5.

• COV is independent: The egalitarian solution of Kalai and Samet (1985) can be translated into the payoff configurations terminology as follows: a payoff configuration $\mathbf{x} = (x^S)_{S\subset N}$ is the *egalitarian payoff configuration* of (N,V) if (i) $x^S \in \partial V(S)$ and (ii) $x^S = Sh(S, v_{\mathbf{x}})$, for each coalition $S \subset N$, where $(S, v_{\mathbf{x}})$ is the TU game defined by $v_{\mathbf{x}}(T) = \sum_{i\in T} x_i^T$ for every coalition $T \subset N$.

The configuration value that assigns the unique egalitarian payoff configuration to every game (N,V) is called the *egalitarian configuration value*, and satisfies all the axioms except Covariance.

In order to show the logical independence of the remaining axioms, it is sufficient to define a configuration value for 0-normalized games, that is, games such that $V(\{i\}) = \{x \in \mathbb{R}^{\{i\}} : x \leq 0\}$, and $V(N) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i \leq c\}$ for any $c \in \mathbb{R}$. By using COV, the value can be uniquely extended to the whole space $\mathcal{G}_I^G$.

• EFF is independent: Define for every 0-normalized game $(N,V) \in \mathcal{G}_I^G$, the configuration value $\Psi^1(N,V) = \mathbf{0}$, the payoff configuration that associates the payoff vector 0_S to every coalition $S \subset N$. This configuration solution satisfies all the Axioms except Efficiency.

• ANO is independent: Let $w \in R_{++}^I$ be a vector of weights, and let Sh^w be the weighted

Shapley value for TU games (Shapley, 1953). A payoff configuration $\mathbf{x} = (x^S)_{S \subset N}$ is said to be a *w-Harsanyi payoff configuration* of the game $(N, V) \in \mathcal{G}_I$ if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that (i) $x^S \in \partial V(S)$, for each coalition $S \subset N$, (ii) $\lambda \cdot x^N \geq \lambda \cdot y$, for all $y \in V(N)$, and (iii) $\lambda_S * x^S = Sh^w(S, v_{\lambda,\mathbf{x}})$, for each coalition $S \subset N$ and each $i \in S$, where $v_{\lambda,\mathbf{x}}(T) = \sum_{i \in T} \lambda_i x_i^T$ for every coalition $T \subset N$.

A fixed $w \in R^I_{++}$ define the configuration value, $\mathcal{H}^w$, on $\mathcal{G}_I^G$ that associates the unique w-Harsanyi payoff configuration to every game $(N, V) \in \mathcal{G}_I$. It can be shown that this configuration solution satisfies all the Axioms but Anonymity.

• NP is independent: Consider the configuration value, Ψ^4, that assigns to each $(N, V) \in \mathcal{G}_I^G$ a payoff configuration, $\Psi^4(N, V) = \mathbf{x}$, where x^S is the only efficient point in $V(S)$ verifying $x_i^S = x_j^S$ for every $i, j \in S$. Then Ψ^4 satisfies all the axioms except Null Player Axiom.

• CONS is independent: Define the solution function Ψ^5 for each $(N, V) \in \mathcal{G}_I^G$ as follows. Assume that the egalitarian configuration value of the game is $\mathfrak{E}(N, V) = \mathbf{y}$, then let $\Psi^5(N, V) = \mathbf{x}$, where

$$x^S = \begin{cases} y^S, & \text{if } S \neq N; \\ \nu(N, v_{\mathbf{y}}), & \text{if } S = N. \end{cases}$$

Here $\nu(N, v_{\mathbf{y}})$ represents the *prenucleolus* (Sobolev, 1975) of the TU game $(N, v_{\mathbf{y}})$ defined by $v_{\mathbf{y}}(T) = \sum_{i \in T} y_i^T$ for every coalition $T \subset N$. Therefore, Ψ^5 satisfies all the axioms apart from Consistency.

• OTI is independent: For each real number $c \in \mathbb{R}$ consider the games (I, V^c) and $(I, \tilde{V}^c)$ given by:

$$V^c(S) = \begin{cases} \{x \in \mathbb{R}^I : \sum_{i \in I} x_i \leq c\}, & \text{if } S = I; \\ \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq 0\}, & \text{otherwise.} \end{cases}$$

$$\tilde{V}^c(S) = \begin{cases} \{x \in \mathbb{R}^I \,:\, \sum_{i\in I} x_i \leq c\}, & \text{if } S = I; \\ \{x \in \mathbb{R} \,:\, x \leq 1\}, & \text{if } |S| = 1; \\ \{x \in \mathbb{R}^S \,:\, \sum_{i\in S} x_i \leq 0\}, & \text{otherwise.} \end{cases}$$

Assume that $\mathcal{H}(N, \tilde{V}^c) = \{\mathbf{x}(c)\} = \left(x^S(c)\right)_{S\subset N}$, and define the payoff configuration $\mathbf{y}(c)$ as follows,

$$y^S(c) = \begin{cases} x^S(c), & \text{if } |S| > 1, \\ 0, & \text{if } |S| = 1. \end{cases}$$

Now let Ψ^6 the only configuration value on $\mathcal{G}_I^G$ that is defined for every game 0-normalized game (N, V) by

$$\Psi^6(N, V) = \begin{cases} \{\mathbf{y}(c)\}, & \text{if } (N, V) = (I, V^c) \\ \mathcal{H}(N, V), & \text{otherwise.} \end{cases}$$

The configuration solution Ψ^6 satisfies all the axioms except Optimal Threats Independence.

2.7. Further Remarks

The main theorem (Theorem 4.4.5) refers to the class of games for which the feasible set of the grand coalition is a half-space. However, it is easy to see that this characterization remains valid if we consider the more restricted class of hyperplane games. Indeed, the corresponding restrictions of the solutions defined in the former section show that the axioms remain logically independent.

In the characterization given in our main theorem the postulate of single-valuedness of the configuration value Ψ plays a crucial role repeatedly in the proof of uniqueness throughout section 5. Yet it is possible to consider solution concepts for which the set of possible outcomes is not a singleton. To do this we will impose the following restrictions on a game (N, V)

(A.3) The surface $\partial V(N)$ is smooth; i.e. at each point $x \in \partial V(N)$ this surface has a unique normal vector, which is normalized in such way that the sum of its coordinates is 1.

(A.4) The set $K(N,V) = \{\lambda \in \mathbb{R}^N_{++} : \sup_{x \in V(N)} \lambda \cdot x < \infty\}$ is closed.

The class of games in $\mathcal{G}_I$ that satisfy also (A.3) and (A.4), in addition to (A.1) and (A.2), is denoted by $\mathcal{G}_I^{SMO}$. Obviously $\mathcal{G}_I^G \subset \mathcal{G}_I^{SMO}$, and this inclusion is strict.

Given a class of games $\mathcal{F} \subset \mathcal{G}_I$, a *configuration solution* Ψ on $\mathcal{F}$ is a set-valued function that assigns to each game $(N,V) \in \mathcal{F}$ a set of payoff configurations $\Psi(N,V) \subset \boldsymbol{X}^N$.

By abusing notation we will denote by $\mathcal{H}$ the Harsanyi NTU configuration solution, i.e. the correspondence that associates with every $(N,V) \in \mathcal{G}_I^{SMO}$ the set of Harsanyi payoff configurations of (N,V).

We consider the following axioms where Ψ denotes a configuration solution on $\mathcal{G}_I^{SMO}$, and (N,V), (N,W) are arbitrary games.

Non-Emptiness : (NE)

$\Psi(N,V)$ is not empty.

Efficiency: (EFF)

$\Psi(N,V) \subset \partial V = \prod_{S \subset N} \partial V(S)$.

Covariance under Linear Changes of Utility: (COV)

$\Psi(N, \lambda * V + a) = \lambda * \Psi(N,V) + a$, for all $\lambda \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}^N$.

Anonymity: (ANO)

$\Psi\pi_* = \pi\Psi$ for each permutation π of I.

Null Player: (NP)

If $i \in N$ is a null player in (N,V) and $\mathbf{x} \in \Psi(N,V)$ then

$$x_i^{S \cup i} = 0 \text{ for all coalition } S \subset N \backslash i.$$

Consistency: (CONS)

If $\mathbf{x} \in \Psi(N,V)$ then

$$\left(x_S^{S \cup N \backslash T}\right)_{S \subset T} \in \Psi\left(T, V^{T,\mathbf{x}}\right) \text{ for every coalition } T \subset N,\ T \neq N.$$

Optimal Threats Independence*: (OTI*)

> Let $\mathbf{x} \in \Psi(N,V)$ and $\mathbf{y} \in \Psi(N,W)$, and $S \subset N$ a coalition. If $\lambda \in \mathbb{R}^N$ is such that $\sup_{x \in V(N)} \lambda \cdot x = \lambda \cdot x^N$ and $\sup_{x \in W(N)} \lambda \cdot x = \lambda \cdot y^N$, and $V(R) = W(R)$ for all $R \subset S$, then
>
> $$x^R = y^R \text{ for all coalition } R \subset S.$$

The first axiom is just a non-emptiness condition. Except OTI* the rest of the axioms are adaptations of those presented in Section 4 to the present context of multi-valued solutions. Notice that the Harsanyi NTU configuration solution does not satisfy OTI on $\mathcal{G}_I^{SMO}$, so we will consider OTI* instead, that is just a slight—and stronger on $\mathcal{G}_I^{G}$—adaptation of OTI.

Next theorem shows that the Harsanyi NTU configuration solution is maximal among the solutions that satisfy the axioms above.

Theorem 2.7.1. *If Ψ is a configuration solution on $\mathcal{G}_I^{SMO}$ satisfying NE, EFF, COV, ANO, NP, CONS, and OTI*, then $\Psi(N,V) \subset \mathcal{H}(N,V)$ for every $(N,V) \in \mathcal{G}_I^{SMO}$.*

Proof 2.7.2. *It is straightforward that the Harsanyi configuration solution satisfies the new axioms (for NE one can use a fixed point argument taking into account condition (A.4)).*

Let $(N,V) \in \mathcal{G}_I^{SMO}$, and $\mathbf{x} \in \Psi(N,V)$. Consider the game (N,W) where the boundary of $W(N)$ is the set determined by the unique supporting hyperplane of $V(N)$ at x^N (notice that here we use condition (A.3); and moreover, notice also that that by COV we can assume that this normal vector is $\mathbf{1}$); and $W(S) = V(S)$ otherwise.

By NE, EFF, CONS and OTI the configuration solution Ψ is single-valued on (N,W), since it is a game whose feasible set for the grand coalition is a hyperplane. Indeed, by NE $\Psi(N,V) \neq \emptyset$ holds, and if $\mathbf{y},\mathbf{z} \in \Psi(N,W)$, then by OTI* it holds $y^S = z^S$ for all proper coalition $S \subset N$. Now by CONS (applied to two-person reduced games) we have: $y_i^N - y_j^N = y_i^{N\setminus j} - y_j^{N\setminus i} = z_i^{N\setminus j} - z_j^{N\setminus i} = z_i^N - z_j^N$. So by EFF we can conclude $y^N = z^N$, and consequently $\mathbf{y} = \mathbf{z}$.*

Let $\{\mathbf{y}\} = \Psi(N,W)$. By OTI we have $x^S = y^S$ for all coalition $S \subset N$. And by CONS (applied to two-person reduced games) we have: $x_i^N - x_j^N = x_i^{N\setminus j} - x_j^{N\setminus i} = y_i^{N\setminus j} - y_j^{N\setminus i} = y_i^N - y_j^N$. So by EFF necessarily $\mathbf{x} = \mathbf{y}$.*

Consequently $\{\mathbf{x}\} = \Psi(N, W)$. *Now applying Theorem 4.4.5 to the restriction of* Ψ *to* $\mathcal{G}_I^G$ *(actually an immediate corollary of this theorem since OTI* is stronger than OTI), this means* $\{\mathbf{x}\} = \mathcal{H}(N, W)$, *which in turn implies* $\{\mathbf{x}\} \in \mathcal{H}(N, V)$ *as was to be proved.*

To obtain a characterization without requiring maximality we still can use the well-known Independence of Irrelevant Alternatives property (see Hart, 2005).

INDEPENDENCE OF IRRELEVANT ALTERNATIVES: (IIA)

> Let (N, V) and (N, W) two games, and $\mathbf{x} \in \Psi(N, W)$. If $V(S) \subset W(S)$ and $x^S \in V(S)$ for all coalition $S \subset N$, then $\mathbf{x} \in \Psi(N, V)$.

THEOREM 2.7.3. *On the class* $\mathcal{G}_I^{SMO}$ *the Harsanyi configuration solution,* $\mathcal{H}$, *is the unique configuration solution which satisfies EFF, COV, ANO, NP, CONS, OTI*, and IIA.*

Proof 2.7.4. *It is straightforward that the Harsanyi configuration solution satisfies IIA.*

Now let Ψ *be a configuration solution that satisfies the above axiom system. By Theorem 2.7.1 if* $(N, V) \in \mathcal{G}_I^{SMO}$ *we have* $\Psi(N, V) \subset \mathcal{H}(N, V)$.

Now let $(N, V) \in \mathcal{G}_I^{SMO}$, *and* $\mathbf{x} \in \mathcal{H}(N, V)$. *Consider as in the proof above the game* (N, W) *such that the boundary of* $W(N)$ *is the half-space determined by the supporting hyperplane of* $V(N)$ *at* x^N, *and let* $W(S) = V(S)$ *otherwise. By Covariance we can assume again that the normal vector of the half-space is* $\mathbf{1}$. *As in the proof of the theorem above* Ψ *is single-valued on the game* $(N, W) \in \mathcal{G}_I^G$, *and consequently* $\Psi(N, W) = \mathcal{H}(N, W) = \{\mathbf{x}\}$. *Applying IIA we can conclude* $\mathbf{x} \in \Psi(N, V)$. *Then* $\mathcal{H}(N, V) = \Psi(N, V)$ *as was to be proved.*

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3. CONSISTENCY OF THE SHAPLEY NTU VALUE ON G-HYPERPLANE GAMES

3.1. Introduction.

The Shapley (1969)[1] NTU (non-transferable utility) value is a solution concept for the class of games in which the utility is not transferable among the players. Many economic and political contexts are more appropriately modeled by NTU games than by TU (transferable utility) games, and the Shapley NTU value has been successfully applied to this models—for an excellent set of references, although not recent, see Aumann (1985b)—. Moreover the Shapley NTU value generalizes the Shapley (1953) value on the subclass of TU games and coincides with the Nash's (1950) solution for pure bargaining games. In addition Aumann (1985a) axiomatized the Shapley NTU value throughout a set of axioms that was a mixture of those used by Nash (1950) and Shapley (1953) to characterize their respective solution concepts.

Our aim in the present paper is to offer an alternative axiomatization of the Shapley NTU solution by means of the consistency axiom. The consistency principle can be roughly described as follows, if a subgroup of players receive their share and leave the others in a renegotiation, then the shares of the remaining players do not change in the subsequent reduced situation. However, there is not a canonical way of modeling the reduced situation, and consequently several formal definitions of a reduced game have been proposed in the literature. In fact different versions of this axiom had been formerly used by Lensberg (1988) to characterize the Nash bargaining solution, and by Hart and Mas-Colell (1989) to characterize the Shapley value of TU games. The axiom system proposed in this paper, except some technicalities, can be seen as an amalgamation of those in these two works.

It is noteworthy to mention here that these results parallel other classic characterizations of cooperative solutions throughout consistency: the prenucleolus (Sobolev, 1975);

[1] Shapley (1988) is a more accesible version.

the core (Peleg, 1985 and 1986; Tadenuma, 1992). Consistency has also played a prominent role in other contexts: for instance, in bankruptcy problems (Aumann and Maschler, 1985; Thomson, 2003) and other allocation problems (Thomson, 2006).

Concerning the variety of the formal versions of consistency, some remarks are in order. In the multiplayer bargaining situations we are considering, the crucial issue is to identify the available alternatives for intermediate coalitions in a reduced situation. When a coalition T is renegotiating the payoff distribution, it is natural to consider that the set of alternatives available for any subgroup S of T, in this reduced situation, are related to the alternatives already feasible when S was cooperating with some members in $(N\backslash T)$. For instance, in the reduced game proposed by Davis and Maschler (1965), the subcoalition S can choose to cooperate any subcoalition Q in $N\backslash T$, and has to guarantee the original solution payoff to the members of Q. Moulin (1985) suggests that the coalition S cooperates with all the members in $N\backslash T$ and pays them the original solution payoff as well. On the other hand, in the reduced game of Hart and Mas-Colell (1989), the coalition S also cooperates with all the members in $N\backslash T$; but it differs from Moulin (1985) in that S pays to $N\backslash T$ according to the solution prescribed in the subgame in which only the members in $S \cup (N\backslash T)$ participate, instead of the solution of the original game.

In this work we propose a reduced game that is similar to the ones of Moulin (1985) and Hart and Mas-Colell (1989), but it refers to payoff configurations instead of payoff vectors. On the class of cooperative games, a solution specifies customarily a payoff vector only for the grand coalition. In contrast, Hart (1985a) considered that a solution outcome should specify a payoff vector also for each intermediate coalition, that is, a payoff configuration. Payoff configurations have been used to compare and characterize several NTU solution concepts by Hart (1985a and 1985b) and more recently by De Clippel et al. (2004) and Hart (2005). Here we take this approach, and accordingly we introduce a formal definition of a reduced game adapted to payoff configurations. In this reduced game a subcoalition S cooperates with all the members that have left, that is with $N\backslash T$ as in Moulin (1985) and Hart and Mas-Colell (1989). However, the main difference is that coalition S rewards to the members of $N\backslash T$ according to the (unique) payoff configuration prescribed by the solution, that is according to the payoff vector specified for coalition $S \cup (N\backslash T)$ in the solution.

Subsequently with the corresponding consistency property at hand, the Shapley NTU solution is characterized on a family of games whose main feature is that the feasible set of the grand coalition is given by a half-space. The proposed axiom system includes other plausible axioms, namely: maximality, covariance, symmetry, the null-player axiom, and an additional axiom, that we have called Intermediate Payoffs Independence, which requires some coherence in the components of the payoff configurations assigned to the intermediate coalitions. This characterization of the Shapley NTU solution is closely related to that of the Harsanyi NTU solution of Hinojosa et al. (2012). The main difference, besides the maximality axiom that replaces the efficiency axiom, is the new definition of the reduced game.

Finally it is worthwhile to mention the following fact: In their remarkable work Hart and Mas-Colell (1989) extended their characterization of the Shapley value based on consistency, to the wider class of non-transferable utility (NTU) games, and characterized a generalization of the Shapley value to NTU games, namely the egalitarian solution (Kalai and Samet, 1985). On the other hand, Maschler and Owen (1989) investigated whether there exists an efficient, symmetric, and consistent solution that, unlike the egalitarian solution, was also covariant under affine transformations of utility. Unfortunately, they found a simple 3-person game (see Section 6 in Maschler and Owen, 1989) for which such a solution would prescribe the empty set.[2] Yet, by considering payoff configurations instead of payoff vectors as solution outcomes, Hinojosa et al. (2012) managed to characterize a covariant solution, namely the Harsanyi NTU solution with, a consistency axiom that is similar to the one of Hart and Mas-Colell (1989) on the subclass of TU games. The main result in this paper is that, by taking a similar approach, another prominent covariant solution of NTU games is also characterized with consistency, namely the Shapley NTU value.

The paper is organized as follows. Sections 2 and 3 contain some preliminaries and definitions. In Section 4 we propose the axiom system and the main result is stated, and subsequently it is proved in Section 5. In Section 6, the logical independence of the axioms is shown.

[2] Subsequently Maschler and Owen (1989) proposed a weaker consistency requirement—called bilateral consistency—, to characterize the Maschler-Owen solution on the class of hyperplane games.

3.2. Preliminaries.

Most of the definitions and notation here follow those in Hart (1985a).

Let I be a finite set of potential *players*, with cardinality $|I| \geq 3$. A *coalition* is any non-empty subset of I. For each coalition $N \subset I$, the $|N|$-dimensional Euclidean space whose axes are labeled with the members of N is denoted by $\mathbb{R}^N$. If $x = (x_i)_{i \in N} \in \mathbb{R}^N$ and $S \subset N$, then the projection of x onto $\mathbb{R}^S$ is denoted x_S, i. e. $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$.

Given $x, y \in \mathbb{R}^N$, then $x \geq y$ ($x > y$) means $x_i \geq y_i$ ($x_i > y_i$) for every $i \in N$. The subsets of $\mathbb{R}^N$ formed by vectors $x \geq 0$, and $x > 0$ are denoted by $\mathbb{R}^N_+$ and $\mathbb{R}^N_{++}$, respectively. Moreover, $x \cdot y$ denotes the real number $\sum_{i \in N} x_i y_i$ (scalar product), and $x * y$, the vector $(x_i y_i)_{i \in N}$. If $A, B \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $A+B$, $x+A$ and $x*A$ are defined by $A + B := \{a + b \,:\, a \in A,\, b \in B\}$, $x + A := \{x + a \,:\, a \in A\}$ and $x * A := \{x * a \,:\, a \in A\}$, respectively. The boundary of A is denoted by ∂A.

A *non-transferable utility (NTU) game*—a game, for short—is a pair (N, V) where N is a coalition and V is a set-valued function (the characteristic function) that assigns a subset $V(S)$ of $\mathbb{R}^S$ to each coalition $S \subset N$ such that:

(A.1) $V(S)$ is non-empty, closed, and comprehensive (i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$);

(A.2) $V(S)$ is 'uniformly non-levelled', i.e. there exists a real number $\delta(S, V) > 0$ such that for every normalized vector $\alpha \in \mathbb{R}^N_{++}$ (i.e. $\sum_{i \in N} \alpha_i = 1$), the following condition holds

$$\sup_{x \in V(S)} \alpha \cdot x < \infty \text{ implies } \alpha_i \geq \delta(S, V) \text{ for every } i \in S;$$

(A.3) for each coalition S there is a payoff vector x such that

$$V(S) \times \{0^{N \setminus S}\} \subset x + V(N).$$

Condition (A.1) is a familiar regularity condition. Condition (A.2) is slightly stronger than the usual non-levelness condition, and it was already used by Maschler and Owen (1992). With respect to (A.3), it can be seen as a extremely weak kind of monotonicity already used by Aumann (1985a) in his characterization of the Shapley NTU value.

The set of all NTU games is denoted by $\mathcal{G}$.

Let $\boldsymbol{X}^N$ denote the product $\prod_{S\subset N} \mathbb{R}^S$; an element $\mathbf{x} = (x^S)_{S\subset N} \in \boldsymbol{X}^N$ is called a *payoff configuration.* It assigns a payoff vector, $x^S = (x_i^S)_{i\in S} \in \mathbb{R}^S$, to every coalition S.[3] Let $\mathbf{X}$ denote the set $\mathbf{X} = \bigcup_{N\subset I} \mathbf{X}^N$

Given a game (N, V) and a coalition $S \subset N$, then (S, V) denotes the subgame obtained by restricting V to subcoalitions of S only.

One of the simplest cases of an NTU game occurs when every coalition $S \subset N$ is assigned a real number $v(S)$ such that

$$V(S) = \Big\{x \in \mathbb{R}^S \,:\, \sum_{i\in S} x_i \leq v(S)\Big\}. \tag{2.1}$$

Such games are known as *transferable utility (TU) games.* We say that (N, V) corresponds to v, and that V and v are denoted interchangeably and no confusion will appear. The set of TU games is denoted $\mathcal{G}^{TU}$.

We shall mainly be interested in the class of games (N, V) for which $V(N)$ is a half-space given by a linear inequality, that is

$$V(N) = \Big\{x \in \mathbb{R}^N \,:\, \lambda^V \cdot x \leq r\Big\}, \tag{2.2}$$

where $\lambda^V \in \mathbb{R}_{++}$, and $r \in \mathbb{R}$. These games are called *G-hyperplane games* (G stands for *grand* coalition). The set of G-hyperplane games is denoted by $\mathcal{G}^G$. Obviously $\mathcal{G}^{TU} \subset \mathcal{G}^G$.

Remark 3.2.1. *Maschler and Owen (1989) considered what they called "hyperplane games". These are pairs (N, V) for which $V(S)$ is a half-space for every coalition S. That is $V(S) = \{x \in \mathbb{R}^N : \lambda^S \cdot x \leq r^S\}$ for some $\lambda^S \in \mathbb{R}^S_{++}$ and $r^S \in \mathbb{R}$. Unless a "hyperplane game" is also a TU game, it fails to satisfy condition (A.3), and so these are not NTU games according to the definition of a NTU game given in this paper. Moreover, a "hyperplane game" has not any Shapley NTU value,[4] except if it is a TU game.*

[3] Notice the difference between x^S and x_S. By x^S we denote the payoff vector of the payoff configuration $\mathbf{x}$ corresponding to coalition S. In contrast, x_S is the projection of a vector $x \in \mathbb{R}^N$ to $\mathbb{R}^S$.

[4] See Section 3.4 for a formal definition of the Shapley NTU value.

3.3. The Shapley value of TU games.

A *value* ϕ on a subclass $\mathcal{G}' \subset \mathcal{G}^G$ is a mapping that assigns a payoff vector to each game $(N, V) \in \mathcal{G}'$ such that $\phi(N, V) \in V(N)$.

On the class of TU games, the *Shapley value*, denoted Sh, assigns the payoff vector to every game (N, v) defined by[5]

$$Sh_i(N, v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \big(v(S \cup i) - v(S)\big).$$

A leitmotif in this work is the characterization of the Shapley value due to Hart and Mas-Colell (1989) by means of consistency. This characterization is stated below after some definitions.

Let ϕ be a value on the class of TU games, (N, v) a TU game, and $T \subset N$ a coalition. The *reduced game* $(T, v_{T,\phi})$ is the TU game given for every coalition $S \subset T$ by

$$v_{T,\phi}(S) = v\big(S \cup (N \setminus T)\big) - \sum_{i \in N \setminus T} \phi_i\big(S \cup (N \setminus T), v\big). \tag{3.3}$$

A value ϕ on the class $\mathcal{G}^{TU}$ is said to be *consistent* if, for every TU game (N, v) and every coalition $T \subset N$, the equality $\phi(T, v_{T,\phi}) = \phi_T(N, v)$ holds.

Moreover, a value ϕ on $\mathcal{G}^{TU}$ is said to be *standard for two-person games* if, for every 2-person game $\big(N, v\big)$, and for each $i \in N$,

$$\phi_i\big(N, v\big) = v\big(\{i\}\big) + \frac{1}{2}\Big[v\big(N\big) - v\big(\{i\}\big) - v\big(N \setminus i\big)\Big]. \tag{3.4}$$

Theorem 3.3.1. *[Hart and Mas-Colell, 1989] A value on $\mathcal{G}^{TU}$ is consistent and standard for two-person games if and only if it is the Shapley value.*[6]

3.4. The Shapley NTU configuration value: A characterization on $\mathcal{G}^G$.

Shapley (1969) proposed the following extension of the Shapley value to the whole class of NTU games.

---

[5] For simplicity, we use $N \setminus i$ and $S \cup i$ instead of $N \setminus \{i\}$ and $S \cup \{i\}$, respectively. Moreover $v(\emptyset) = 0$.

[6] Actually Hart and Mas-Colell (1989) considered an infinite set I of potential players, but this can be obviously adapted for the case of a finite set with $|I| \geq 3$.

Let (N, V) be an NTU game. For each vector $\lambda \in \mathbb{R}^N_{++}$ and coalition $T \subset N$, write

$$v_\lambda(T) = \sup\big\{\lambda_T \cdot x : x \in V(T)\big\}. \tag{4.5}$$

We say that the TU game v_λ is *defined* if the right side of expression (4.5) is finite for all S.

A *Shapley NTU value* of a game $(N, V) \in \mathcal{G}$ is a payoff vector $x \in V(N)$ such that for some positive vector $\lambda \in \mathbb{R}^N_{++}$, the TU game v_λ is defined, and $\lambda * x = Sh(N, v_\lambda)$.

A value customarily specifies a payoff vector only for the grand coalition. In contrast, Hart (1985a) considered that a solution outcome should specify a payoff vector for each coalition, that is, a payoff configuration. In this paper we will take Hart's (1985) approach and consider solution concepts formed by payoff configurations. The reason for doing so is the following one: As it was mentioned in the Introduction our aim is to characterize the Shapley NTU solution by means of consistency. However Maschler and Owen (1989) found a simple example of a 3-person NTU game for which a covariant and consistent (w.r.t Hart and Mas-Colell (1989) reduced game) solution formed by payoff vectors specifies the empty set. Nonetheless, by considering solutions consisting of payoff configurations, we characterize in our main theorem (Theorem 4.4.5) the Shapley NTU solution that is covariant and consistent, and so we are able to overcome the impossibility result of Maschler and Owen (1989).

Firstly we will distinguish the Shapley NTU value from the Shapley NTU payoff configuration.

A *Shapley NTU payoff configuration* of the game $(N, V) \in \mathcal{G}$ is a payoff configuration $\mathbf{x} = (x^S)_{S \subset N}$ such that for some positive vector $\lambda \in \mathbb{R}^N_{++}$, the TU game v_λ is defined, and

$$\lambda_S * x^S = Sh(S, v_\lambda) \quad \text{for each } S \subset N. \tag{4.6}$$

We denote by $\mathcal{S}(N, V)$ the set of Shapley NTU payoff configurations for the game $(N, V) \in \mathcal{G}$.

A remark concerning interpretation is in order. A solution of a game $(N, V) \in \mathcal{G}$ is a payoff configuration $\mathbf{x}$ specifying an outcome x^S for every coalition S, reflecting the bargaining inside this coalition. Several interpretations have been suggested for these intermediate payoff vectors. Hart (1985a) interprets that x^S assigns the amounts that players in S would receive if S will forms. Yet according to Harsanyi (1963)[7], x^S could be considered as an optimal threat of coalition S against $N \backslash S$. An alternative interpretation is to view x^S as a kind of reference point. More specifically, following Shapley's (1969) procedure to propose the Shapley NTU solution, the players are looking for *efficient* outcomes, that are simultaneously *equitable*. Shapley suggests that efficient outcomes are simply those that maximize the sum of of utilities, with respect to a given set of comparison weights. It will remain to fix which payoffs would be considered equitable, among the efficient ones. Shapley (1969) proposes as candidates those given by the Shapley value in the TU game resulting from the sum of the weighted utilities, whenever they are feasible (see Figure 4.1 for an illustration.)

The following propositions are straightforward.

Proposition 3.4.1. *If (N, V) is a game in $\mathcal{G}^G$, then it has a unique Shapley NTU payoff configuration.*

Proof 3.4.2. *It is due to the fact that if $(N, V) \in \mathcal{G}^G$, then the vector λ^V is the unique normal vector at every $x \in \partial V(N)$.*

Proposition 3.4.3. *If (N, v) is a game in $\mathcal{G}^{TU}$ then its unique Shapley NTU payoff configuration is:* $\subset (N, v) = \left\{ \left(Sh(S, v) \right)_{S \subset N} \right\}$.

Proof 3.4.4. *If $(N, v) \in \mathcal{G}^{TU}$, then the vector $\lambda^V = \mathbf{1}^N$ and the result follows.*

The first proposition states that the function $\subset : \mathcal{G}^G \to \boldsymbol{X}$ that assigns to every game $(N, V) \in \mathcal{G}^G$ the corresponding Shapley NTU payoff configuration is actually single-valued. It will be called the Shapley NTU configuration value. The second result establishes that for TU games the Shapley NTU configuration value consists of the Shapley values for each

[7] Harsanyi (1963) originally considered games in strategic form.

subgame.

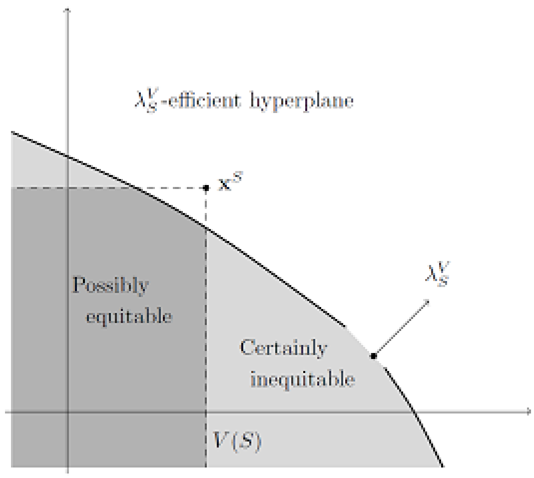

Fig. 4.1: Interpretation of $\mathbf{x}^S$

As stated in the Introduction, our main goal is to characterize the Shapley NTU configuration value ⊂ on $\mathcal{G}^G$ with a consistency property together with some additional axioms. Further notation and definitions are needed in advance to state these axioms.

Given $(N, V) \in \mathcal{G}$, $\lambda \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}^N$, the game $(N, \lambda * V + a)$ is defined by $(\lambda * V + a)(S) = \lambda_S * V(S) + a_S$.

Every permutation π of I induces a linear mapping π_* from $\mathcal{G}$ onto itself, defined by $\pi_*(N, V) = (\pi N, \pi V)$, where $\pi V(\pi S) = V(S)$ for every coalition $S \subset N$. Notice that $\pi_* \mathcal{G}^G = \mathcal{G}^G$, for every permutation π of I.

We say that a player $i \in N$ is a *null player* in the game $(N, V) \in \mathcal{G}$ if $V(S) = \big\{x \in \mathbb{R}^S \,:\, (x, 0) \in V(S \cup i)\big\}$ for every coalition $S \subset N \backslash i$.

Let (N, V) be a game, $T \subset N$ a proper coalition, and $\mathbf{x} \in \mathbf{X}^N$ be a payoff configuration. Denote

$$m_S^{T,\mathbf{x}} = \text{máx}\Big\{\lambda^V_{N\backslash T} \cdot z_{N\backslash T} : \, z \in V\big(S \cup (N \setminus T)\big),\, \lambda^V_S \cdot z_S = \lambda^V_S \cdot x_S^{S\cup N\backslash T}\Big\}, \tag{4.7}$$

$$A_S^{T,\mathbf{x}} = \big\{z \in V\big(S \cup (N \setminus T)\big) \,:\, \lambda^V_{N\backslash T} \cdot z_{N\backslash T} = m_S^{T,\mathbf{x}}\big\}. \tag{4.8}$$

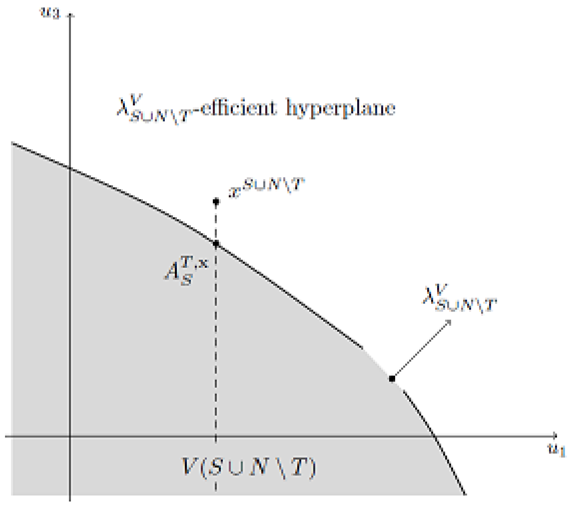

Fig. 4.2: Geometric representation of $m^{T,\mathbf{x}}_S$ for the case $S=\{1\}$, $T=\{1,2\}$, and $N=\{1,2,3\}$.

The *reduced game* $(T, V^{T,\mathbf{x}})$ is defined for each coalition $S \subset T$ as

$$V^{T,\mathbf{x}}(S) = \Big\{ y \in \mathbb{R}^S : \, (y, z_{N\setminus T}) \in V\big(S \cup (N \setminus T)\big), \text{ for some } z \in A^{T,\mathbf{x}}_S \Big\} \tag{4.9}$$

The interpretation is as follows. Assuming that the utility is transferable with respect to the weights determined by λ^V, the real number $m^{T,\mathbf{x}}_S$ represents the maximum λ^V-weighted utility that players in $N\setminus T$ could obtain in coalition $S \cup (N \setminus T)$, when players in S are assured the original λ^V-weighted utility $\lambda^V_S \cdot x^{S\cup N\setminus T}_S$. Then $A^{T,\mathbf{x}}_S$ is the set of payoff vectors where this maximum is attained (Figure 4.2 depicts these concepts for a special case, in which the set $A^{T,\mathbf{x}}_S$ consists precisely of a unique element. Consequently $V^{T,\mathbf{x}}(S)$ describes the possible payoffs that the members of S can attain by cooperating with all the members in $N\setminus T$, but they pay to these members in such a way that they can guarantee for themselves the original λ-weighted utility according to the payoff configuration $\mathbf{x}$.

Although the reduced game in expression 4.6 refers to payoff configurations, its definition reminds the reduced games proposed by Moulin (1985) (see also Tadenuma, 1992) and the one of Hart and Mas-Colell (1989). In the reduced games of these authors it is also considered that coalition S joins $N\setminus T$ to determine their reduced feasible set. However they propose to reward to coalition $N\setminus T$ in a different way. While Moulin (1985) suggests that coalition S can guarantee the payoff of the original solution, Hart and Mas-Colell (1999) suggest the payoff of the original solution of the corresponding subgame.

Remark 3.4.5. *Given a value ϕ on the class of TU games $\mathcal{G}^{TU}$, we can associate the configuration value Ψ^ϕ on $\mathcal{G}^{TU}$ defined by*

$$\Psi^\phi(N, v) = \big\{(\phi(S, v)_{S \subset N}\big\},$$

that is, the payoff vectors of the subcoalitions are the values of the subgames.

If ϕ is an efficient value (i.e. $\phi(N, v) = v(N)$), and $\mathbf{x} = \Psi^\phi(N, V)$, it is straightforward to prove that $m_S^{T,\mathbf{x}} = v(S \cup N \setminus T) - \sum_{i \in N \setminus T} \phi_i(S \cup N \setminus T, v)$, for every coalition $S \subset T$, and consequently the reduced game $(T, v^{T,\mathbf{x}})$ is a TU game defined for every $S \subset T$ by

$$v^{T,\mathbf{x}}(S) = v\big(S \cup (N \backslash T)\big) - \sum_{i \in N \backslash T} \phi_i\big(S \cup (N \backslash T), v\big).$$

That is $v^{T,\mathbf{x}}(S) = v_{T,\phi}(S)$. Hence the definition given in expression (4.6) *can be seen as a generalization of the reduced game of Hart and Mas-Colell (1989).*

Lemma 3.4.6. *Let (N, V) be an NTU game in $\mathcal{G}^G$, let $\mathbf{x} \in V(N)$, and a coalition $T \subset N$. Then the reduced game $(T, V^{T,\mathbf{x}})$ is a game in $\mathcal{G}^G$.*

Proof 3.4.7. *Owing to the closeness of $V(S \cup N \setminus T)$, and the fact that this set satisfies condition (A.2), it follows that $m_S^{T,\mathbf{x}}$ is always a finite real number, and for the same reasons $A_S^{T,\mathbf{x}}$ is not empty. Now taking into account that $V\big(S \cup (N \backslash T)\big)$ satisfies assumptions (A.1), (A.2) and (A.3), it is straightforward to check that the sets $V^{T,\mathbf{x}}(S)$ also satisfy these conditions for every $S \subset T$. Moreover, it turns out that $V^{T,\mathbf{x}}(T) = \big\{y \in I\!R^T : (y, x^N_{N \backslash T}) \in V(N)\big\}$ that is clearly a hyperplane, its normal vector being λ_T^V and consequently, $(T, V^{T,\mathbf{x}})$ is also an NTU game in $\mathcal{G}^G$.*

The following axioms are now imposed, where Ψ denotes a single-valued mapping from $\mathcal{G}^G$ to $\boldsymbol{X}$, and (N, V), (N, W) are NTU games in $\mathcal{G}^G$.

MAXIMALITY: (MAX)

$$\Psi(N, V) = \mathbf{x} \text{ implies } \lambda_S^V \cdot x^S = \sup\{\lambda_S^V \cdot y : \, y \in V(S)\} \text{ for every } S \subset N.$$

Covariance under Linear Changes of Utility: (COV)

$\Psi(N, \lambda * V + a) = \lambda * \Psi(N, V) + a$, for every $\lambda \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}^N$,

where $\lambda * \mathbf{x} = (\lambda_S * s^S)_{S \subset N}$ if we denote $\mathbf{x} = \Psi(N, V)$.

Anonymity: (AN)

$\Psi\pi_* = \pi\Psi$ for each permutation π of I.

Null Player: (NP)

If $i \in N$ is a null player in (N, V) and $\Psi(N, V) = \mathbf{x}$ then

$$x_i^{S \cup i} = 0 \text{ for every coalition } S \subset N \backslash i.$$

Consistency: (CONS)

If $\Psi(N, V) = \mathbf{x}$ then

$$\Psi\left(T, V^{T,\mathbf{x}}\right) = \left(x_S^{S \cup N \backslash T}\right)_{S \subset T}, \text{ for every proper coalition } T \subset N.$$

Intermediate Payoffs Independence: (IPI)

Let $\Psi(N, V) = \mathbf{x}$ and $\Psi(N, W) = \mathbf{y}$, and a coalition $S \subset N$. If $V(N) = W(N)$, and $V(R) = W(R)$ for every coalition $R \subset S$, then

$$x^R = y^R \text{ for every coalition } R \subset S.$$

The MAX axiom requires that $\mathbf{x}$ has to maximize the weighted utility sum in the case in which side-payments are admitted inside any intermediate coalition. With respect to IPI, whenever the opportunities for the grand coalition and for all the subcoalitions of S are the same, this axiom requires the intermediate payoffs x^R to remain unchanged for every $R \subset S$. The other axioms are standard in the literature.

Now we state our main result: the characterization of the Shapley configuration value on the class of games (N, V) for which $V(N)$ is determined by a hyperplane.

Theorem 3.4.8. *A mapping $\Psi : \mathcal{G}^G \to \mathbf{X}$ satisfies MAX, COV, AN, NP, CONS, and IPI if and only if $\Psi = \subset$.*

3.5. Proof of the Main Theorem

Proposition 3.5.1. *On the class $\mathcal{G}^G$, the Shapley configuration value, $\subset$, satisfies MAX, COV, AN, NP, CONS, and IPI.*

Proof 3.5.2. *The MAX axiom is consequence of the efficiency of the Shapley TU value. The COV axiom is also immediate. The AN, NP and CONS axioms can be logically concluded from (4.6), since the Shapley value satisfies anonymity, the null-player property, and consistency. Finally, the IPI axiom is a consequence of (4.6) and the definition of the Shapley value.*

Remark 3.5.3. *Note that without assumption (A.2), the Shapley configuration value would fail to satisfy the CONS axiom.*

The uniqueness is now proved. In Proposition 3.5.14, it is first shown that this axiom system uniquely determines the Shapley configuration value on the class of TU games. This result is then extended to the wider class $\mathcal{G}^G$ in Proposition 4.5.11.

For the remaining part of this section, let Ψ represent a configuration value on the class $\mathcal{G}^G$. Associated with the configuration value Ψ, the value ϕ^Ψ is defined on $\mathcal{G}^G$ as

$$\phi^\Psi(N,V) = x^N \quad \text{whenever } \Psi(N,V) = \mathbf{x}. \tag{5.10}$$

Proposition 3.5.4. *Let Ψ satisfy MAX, COV, AN, and CONS. If, in addition, for every TU game (N,v), Ψ also satisfies the following property,*

$$\Psi(N,v) = \mathbf{x} \textit{ implies } \phi^\Psi(T,v) = x^T \textit{ for every coalition } T \subset N, \tag{5.11}$$

then $\Psi(N,v) = \subset (N,v)$, for every TU game (N,v).[8]

Proof 3.5.5. *Let (N,v) be a TU game, such that $\Psi(N,v) = \mathbf{x}$, and let $T \subset N$ be a coalition. Let ϕ^Ψ be the value defined in (5.7), and let v_{T,ϕ^Ψ} be the TU reduced game defined according to (3.1). In addition, let $v^{T,\mathbf{x}}$ be the reduced game defined according to (4.6). Since Ψ*

[8] Property (5.8) is equivalent to: $\Psi(N,v) = (x^S)_{S\subset N}$ implies $\Psi(T,v) = (x^S)_{S\subset T}$ for every coalition $T \subset S$, i.e., the payoff configuration of the solution outcome of a subgame is precisely the restriction of the payoff configuration of the solution outcome of the whole game.

satisfies (5.8), we have $\phi^{\Psi}(R,v)=x^R$ *for every coalition* $R\subset N$*. Thus for every coalition* $S\subset T$ *we have*

$$v_{T,\phi^{\Psi}}(S)=\sum_{i\in S}\phi_i^{\Psi}\big(S\cup(N\backslash T),v\big)=\sum_{i\in S}x_i^{S\cup(N\backslash T)}=v^{T,\mathbf{x}}(S).$$

Consequently $v^{T,\mathbf{x}}=v_{T,\phi^{\Psi}}$*, and hence*

$$\phi^{\Psi}(T,v_{T,\phi^{\Psi}})=\phi^{\Psi}(T,v^{T,\mathbf{x}})=x_T^N=\phi_T^{\Psi}(N,v),$$

where the second equality follows on from CONS, since $\Psi(T,V)=(x^S)_{S\subset T}$*. Therefore the restriction of the value* ϕ^{Ψ} *to TU games is consistent.*

Furthermore, ϕ^{Ψ} *is standard on 2-person TU games (since* Ψ *satisfies MAX, COV, and AN). Therefore Theorem 4.3.1 provides* $\phi^{\Psi}=Sh$*, and hence* $x^T=Sh(T,v)$ *for every coalition* $T\subset N$*. From Proposition 3.4.3,* $\Psi(N,v)=\subset(N,v)$*.*

Lemma 3.5.6. *Let* Ψ *satisfy COV and IPI, and let* (N,v) *and* (N,w) *be TU games such that* $\Psi(N,v)=\mathbf{x}$ *and* $\Psi(N,w)=\mathbf{y}$*. If* $T\subset N$ *is a coalition such that* $v(R)=w(R)$ *for every coalition* $R\subset T$*, then* $x^R=y^R$ *for every coalition* $R\subset T$*.*

Proof 3.5.7. *Let* $i\in N\backslash T$ *be fixed and consider the auxiliary TU game* $(N,\tilde{v})$ *defined for each coalition* $S\subset N\backslash i$ *by* $\tilde{v}(S)=v(S)$*, and* $\tilde{v}(S\cup i)=v(S\cup i)+w(N)-v(N)$*. If* $\Psi(N,\tilde{v})=\tilde{\mathbf{x}}$*, then COV implies* $\tilde{x}^R=x^R$ *for every* $R\subset T$*. Moreover, by applying IPI,* $\tilde{x}^R=y^R$ *is obtained for every coalition* $R\subset T$ *and the conclusion can be drawn.*

In order to facilitate the proofs, the following property (which can be viewed as a stronger version of the Null Player Axiom) will be considered:[9]

Null Player*: (NP*)

If $i\in N$ is a null player in (N,V) and $\mathbf{x}=\Psi(N,V)$, then

$$x_i^{S\cup i}=0 \quad\text{and}\quad x_S^{S\cup i}=x^S \qquad \text{for every coalition } S\subset N\backslash i.$$

[9] Actually, if we replace in Theorem 4.4.5 the property NP by this new one, the proof of the new result is considerably shorter.

Proposition 3.5.8. *If Ψ satisfies MAX, COV, AN, NP*, CONS, and IPI, then $\Psi(N,v) =\subset (N,v)$ for every TU game (N,v).*

Proof 3.5.9. *We shall prove that Ψ satisfies condition (5.8), and the result will follow from Proposition 4.5.4.*

Let (N,v) be a TU game and $T \subset N$ a proper coalition. Define the TU game (N,w) for every $S \subset N$ as $w(S) = v(S \cap T)$. Let $\Psi(N,v) = \mathbf{x}$ and $\Psi(N,w) = \mathbf{y}$.

On the one hand, every player in $N\backslash T$ is null in $\big(N,w\big)$, which, through NP, implies that $y_{N\backslash T}^{S\cup N\backslash T} = 0$, for every coalition $S \subset T$. Therefore*

$$w^{T,\mathbf{y}}(S) = \sum_{i\in S} y_i^{S\cup N\backslash T} = v(S),$$

i.e. $\big(T, w^{T,\mathbf{y}}\big) = (T,v)$. Hence from CONS,

$$\Psi(T,v) = \Psi(T,w^{T,\mathbf{y}}) = \big(y_S^{S\cup N\backslash T}\big)_{S\subset T}$$

and consequently $\phi^{\Psi}(T,v) = y_T^N$.

Furthermore, if $S \subset T$, then through Lemma 3.5.6, $y^S = x^S$, and NP yields $y_S^{S\cup N\backslash T} = y^S$. In particular, by choosing $S = T$, the equality $y_T^N = x^T$ is obtained. It can therefore be concluded that $\phi^{\Psi}(T,v) = x^T$, and Ψ satisfies condition (5.8) as claimed.*

However, under the weaker NP axiom, more work is required since the equality $y_S^{S\cup N\backslash T} = y^S$ used in the proof above must also be satisfied.

Lemma 3.5.10. *Let Ψ satisfy MAX, COV, and AN. If $\big(N,v\big)$ is a 2-person TU game, with $N = \{i,j\}$, then $\Psi\big(N,v\big) =\subset \big(N,v\big)$. That is, $\Psi\big(N,v\big) = \mathbf{x}$ if and only if*

$$x_i^{\{i\}} = v(i),\ x_j^{\{j\}} = v(j), \tag{5.12}$$

$$x_i^N - x_j^N = x_i^{\{i\}} - x_j^{\{j\}} \quad \text{and} \quad x_i^N + x_j^N = v(N). \tag{5.13}$$

Proof 3.5.11. *It is straightforward.*

Lemma 3.5.12. *Let Ψ satisfy MAX, COV, AN, and CONS, and $(N,V) \in \mathcal{G}^G$, with $|N| \geq 2$, such that $V(N)$ is a half-space whose normal vector is $\mathbf{1}$. If $\Psi(N,V) = \mathbf{x}$, then*

$$x_i^N - x_j^N = x_i^{N\backslash j} - x_j^{N\backslash i} \quad \text{for every } i,j \in N,\ i \neq j. \tag{5.14}$$

Proof 3.5.13. *Notice that the reduced game* $\big(\{i,j\}, V^{\{i,j\},\mathbf{x}}\big)$ *is the TU game that corresponds to the function* v *defined by* $v\big(\{i\}\big) = x_i^{N\backslash j}$, $v\big(\{j\}\big) = x_j^{N\backslash i}$ *and* $v\big(\{i,j\}\big) = x_i^N + x_j^N$. *The result therefore follows from Lemma 4.5.3.*

Proposition 3.5.14. *If* Ψ *satisfies MAX, COV, AN, NP, CONS, and IPI, then* $\Psi(N,v) =\subset (N,v)$ *for every TU game* (N,v).

Proof 3.5.15. *We shall prove that* Ψ *satisfies condition (5.8), and the result will follow from Proposition 4.5.4.*

Let (N,v) *be a TU game. If* $|N| = 1$, *then the result is obviously true, and for* $|N| = 2$ *the result follows from Lemma 4.5.3.*

Now assume that $|N| \geq 3$. *Let* $T \subset N$ *be any proper coalition. Consider the TU game* (N,w) *defined for every* $S \subset N$ *by* $w(S) = v(S \cap T)$. *Let* $\Psi(N,v) = \mathbf{x}$ *and* $\Psi(N,w) = \mathbf{y}$. *The steps in the proof of Proposition 4.5.5 above can be repeated in order to obtain* $\phi^\Psi(T,v) = y_T^N$. *Furthermore,* $y^S = x^S$ *for every* $S \subset T$, *and hence* $y^T = x^T$. *Therefore, if it is proved that* $y_T^N = y^T$, *then it will be ascertained that* Ψ *satisfies condition (5.8) as claimed.*

It must now be proved that $y_T^N = y^T$ *for every coalition* $T \subset N$. *In the case* $|T| = 1$, *this proof is straightforward. For the case in which* T *is not a singleton, we will proceed by induction on* $|N\backslash T|$.

Assume first that $N\backslash T = \{i\}$, *for some* $i \in N$. *Through Lemma 4.5.6, if* $j \in N\backslash i$, *then* $y_i^N - y_j^N = y_i^{N\backslash j} - y_j^{N\backslash i}$. *Moreover, since* i *is a null player in* (N,w), *the NP axiom yields* $y_i^N = y_i^{N\backslash j} = 0$. *Consequently* $y_j^N = y_j^{N\backslash i}$ *for every* $j \in N\backslash i$, *i.e.* $y_{N\backslash i}^N = y^{N\backslash i}$, *as required.*

Observe that for the case $|N| = 3$, *the proof that* $\Psi(N,V) =\subset (N,v)$ *has already been completed.*

Assume now that $y_T^N = y^T$ *is true when* $|N\backslash T| \leq k-1$. *Notice that this implies that* $\Psi(N,v) =\subset (N,v)$ *has already been proved for the case in which* $|N| \leq k+1$, *and by Proposition 3.4.3, this yields*

$$\phi^\Psi(N,v) = Sh(N,v), \qquad \text{whenever } |N| \leq k+1. \tag{5.15}$$

Now let us suppose that $|N\backslash T| = k$. *Let* $j \in T$ *and consider the reduced TU game* $\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big)$ *defined according to (4.6). Notice that players in* $N\backslash T$ *are null* *players in (N,w), and the induction argument implies $y^N_R = y^R$, for every $R \supset T$, $R \neq T$. Therefore, through MAX,*

$$w^{(N\backslash T)\cup j,\mathbf{y}}(S) = \begin{cases} 0 & \text{if } j \notin S, \\ y^T_j & \text{if } S = \{j\}, \\ y^N_j & \text{otherwise.} \end{cases} \tag{5.16}$$

Hence $Sh_i\big((N\backslash T)\cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \frac{(k-1)!}{(k+1)!}(y^N_j - y^T_j)$, whenever $i \in N\backslash T$.

Now if $i \in N\backslash T$, then i is a null player in (N,w). Then, since $|(N\backslash T)\cup j| = k+1$, NP axiom and CONS axiom together with (5.10) provide

$$0 = y^N_i = \phi^\Psi_i\big((N\backslash T)\cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \frac{(k-1)!}{(k+1)!}(y^N_j - y^T_j). \tag{5.17}$$

Thus, $y^N_j = y^T_j$ holds true for every $j \in T$, as required.

Lemma 3.5.16. *Let Ψ satisfy MAX, COV, AN, NP, CONS, and IPI. Furthermore, let $(N,V) \in \mathcal{G}^G$ with $\lambda^V = \mathbf{1}$ for which $\Psi(N,V) = \mathbf{x}$. Let $\bar{k} \in N$ be a fixed player, and consider the NTU game (N,W) defined by*

$$W(S) = \begin{cases} \{x \in I\!R^{N\backslash \bar{k}} : \sum_{i\neq \bar{k}} x_i \leq \sum_{i \neq \bar{k}} x^{N\backslash \bar{k}}_i\}, & \text{if } S = N\backslash \bar{k}; \\ V(S), & \text{otherwise.} \end{cases}$$

Then $\Psi(N,W) = \Psi(N,V)$.

Proof 3.5.17. *Let $\Psi(N,W) = \mathbf{y}$. According to IPI axiom, in order to prove that $\mathbf{y} = \mathbf{x}$, it is enough to show that $y^N = x^N$ and $y^{N\backslash \bar{k}} = x^{N\backslash \bar{k}}$.*

First, through IPI,

$$y^{N\backslash j} = x^{N\backslash j} \qquad \text{for every } j \in N\backslash \bar{k}, \tag{5.18}$$

and from Lemma 4.5.6,

$$x^N_i - x^N_j = x^{N\backslash j}_i - x^{N\backslash i}_j, \qquad \text{for every } i,j \in N,\ i \neq j, \tag{5.19}$$

$$y^N_i - y^N_j = y^{N\backslash j}_i - y^{N\backslash i}_j, \qquad \text{for every } i,j \in N,\ i \neq j. \tag{5.20}$$

The combination of (5.17) with (5.18) and (5.19) for the case $i = \bar{k}$ *yields*

$$x_{\bar{k}}^N - x_j^N + x_j^{N\backslash\bar{k}} = y_{\bar{k}}^N - y_j^N + y_j^{N\backslash\bar{k}}, \quad \textit{for every } j \in N\backslash\bar{k}, \tag{5.21}$$

and for the case $i \neq \bar{k}$*, yields*

$$x_i^N - x_j^N = y_i^N - y_j^N, \qquad \textit{for every } i, j \in N\backslash\bar{k}. \tag{5.22}$$

From (5.20), it follows that

$$\big(|N|-1\big)x_{\bar{k}}^N - \sum_{j\in N\backslash\bar{k}} x_j^N + \sum_{j\in N\backslash\bar{k}} x_j^{N\backslash\bar{k}} = \big(|N|-1\big)y_{\bar{k}}^N - \sum_{j\in N\backslash\bar{k}} y_j^N + \sum_{j\in N\backslash\bar{k}} y_j^{N\backslash\bar{k}},$$

or equivalently,

$$|N|x_{\bar{k}}^N - \sum_{j\in N} x_j^N + \sum_{j\in N\backslash\bar{k}} x_j^{N\backslash\bar{k}} = |N|y_{\bar{k}}^N - \sum_{j\in N} y_j^N + \sum_{j\in N\backslash\bar{k}} y_j^{N\backslash\bar{k}}. \tag{5.23}$$

Now MAX axiom yields $\sum_{j\in N} x_j^N = \sum_{j\in N} y_j^N$ *and* $\sum_{j\in N\backslash\bar{k}} x_j^{N\backslash\bar{k}} = \sum_{j\in N\backslash\bar{k}} y_j^{N\backslash\bar{k}}$*. Hence (5.22) yields*

$$y_{\bar{k}}^N = x_{\bar{k}}^N. \tag{5.24}$$

Furthermore, MAX axiom, together with equalities (5.21) and (5.23) yield $y_i^N = x_i^N$ *for every* $i \in N$*, that is*

$$y^N = x^N \qquad \textit{as claimed.} \tag{5.25}$$

Finally, this last equality (5.24) together with (5.20) imply that $y_j^{N\backslash\bar{k}} = x_j^{N\backslash\bar{k}}$ *for every* $j \in N\backslash\bar{k}$*, i.e.,* $y^{N\backslash\bar{k}} = x^{N\backslash\bar{k}}$*, and the proof is complete.*

Proposition 3.5.18. *Let* Ψ *satisfy MAX, COV, AN, NP, CONS, and IPI. Let* $(N,V) \in \mathcal{G}^G$*, with* $\lambda^V = \mathbf{1}$*, for which* $\Psi(N,V) = \mathbf{x}$*. Let* $T \subset N$ *be a fixed coalition, and consider the NTU game* (N,W) *defined by*

$$W(S) = \begin{cases} \{x \in I\!R^T \, : \, \sum_{i\in T} x_i \leq \sum_{i\in T} x_i^T\}, & \textit{if } S = T; \\ V(S), & \text{otherwise.} \end{cases} \tag{5.26}$$

Then $\Psi(N,W) = \Psi(N,V)$*.*

Proof 3.5.19. *Induction is now used on* $|N\backslash T|$*, where the case* $|N\backslash T| = 1$ *is covered by Proposition 4.5.9 above.*

Let T *be a fixed coalition such that* $|N\backslash T| = m > 1$*. In view of the induction hypothesis it can be assumed (to prevent the need for further notation), that* $V(S)$ *is already TU for every coalition* S *such that* $|S| > |T|$*; that is, it can be assumed that* $V(S) = \{x \in I\!R^S : \sum_{j\in S} x_j \leq \sum_{j\in S} x_j^S\}$ *whenever* $|S| > |T|$*.*

Let us denote $\Psi(N, W) = \mathbf{y}$*. According to IPI, in order to prove that* $\mathbf{y} = \mathbf{x}$*, it is sufficient to show that* $y^{T\cup R} = x^{T\cup R}$ *for every* $R \subset N\backslash T$*. This equality will first be proved for the case in which* $R \neq \emptyset$*, by induction on the cardinality of* R*, and later on we will prove that* $y^T = x^T$*.*

Therefore, let us assume first that $R = \{\bar{k}\}$ *for certain* $\bar{k} \in N\backslash T$*.*

For any $j \in T$*, let us consider the reduced games* $\big((N\backslash T)\cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ *and* $\big((N\backslash T)\cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$*. Through our assumption that if* $|R| > |T|$ *then* $V(R)$ *is a half-space whose normal vector is* $\mathbf{1}$*, it can be concluded that these reduced games are also TU games.*

Now consider the subgames $\big(\{\bar{k}, j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ *and* $\big(\{\bar{k}, j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$*, and assume that*

$$\Psi\big(\{\bar{k}, j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \{\boldsymbol{a}\} \quad \text{and} \quad \Psi\big(\{\bar{k}, j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \{\boldsymbol{b}\}.$$

The CONS axiom and Lemma 4.5.3 yield

$$\begin{array}{lll} a^{\{\bar{k},j\}} = (x_{\bar{k}}^{T\cup\bar{k}}, x_j^{T\cup\bar{k}}) & b^{\{\bar{k},j\}} = (y_{\bar{k}}^{T\cup\bar{k}}, y_j^{T\cup\bar{k}}) & \\ a^{\{\bar{k}\}} = x_{\bar{k}}^{(T\cup\bar{k})\backslash j} & b^{\{\bar{k}\}} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j} & \textit{for every } j \in T. \\ a^{\{j\}} = x_j^T & b^{\{j\}} = y_j^T & \end{array}$$

Consequently, it follows on from Lemma 4.5.6 that

$$x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}} = x_{\bar{k}}^{(T\cup\bar{k})\backslash j} - x_j^T, \qquad \textit{for every } j \in T, \tag{5.27}$$

$$y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j} - y_j^T, \qquad \textit{for every } j \in T. \tag{5.28}$$

Furthermore, IPI implies $x^S = y^S$ *for any coalition* $S \subset N\backslash j$*. In particular* $x_{\bar{k}}^{(T\cup\bar{k})\backslash j} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j}$*, and hence, from (5.29) and (5.30), it can be concluded that*

$$(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - (y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = y_j^T - x_j^T, \qquad \textit{for every } j \in T. \tag{5.29}$$

Therefore, $\sum_{j\in T}(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - \sum_{j\in T}(y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = \sum_{j\in T} y_j^T - \sum_{j\in T} x_j^T$. *Now MAX yields* $\sum_{j\in T} y_j^T = \sum_{j\in T} x_j^T$ *and* $\sum_{j\in T\cup\bar{k}} x_j^{T\cup\bar{k}} = \sum_{j\in T\cup\bar{k}} y_j^{T\cup\bar{k}}$, *and thus* $\big(|T|+1\big)(x_{\bar{k}}^{T\cup\bar{k}} - y_{\bar{k}}^{T\cup\bar{k}}) = 0$, *which implies*

$$x_{\bar{k}}^{T\cup\bar{k}} = y_{\bar{k}}^{T\cup\bar{k}}. \tag{5.30}$$

Moreover, from equality (5.32) above, we have $(N\backslash\bar{k}, V^{N\backslash\bar{k},\mathbf{x}}) = (N\backslash\bar{k}, W^{N\backslash\bar{k},\mathbf{y}})$. *Through CONS,* $x_S^{S\cup\bar{k}} = y_S^{S\cup\bar{k}}$, *for every* $S\subset N\backslash\bar{k}$, *and in particular* $x_T^{T\cup\bar{k}} = y_T^{T\cup\bar{k}}$. *Summing up,* $x^{T\cup\bar{k}} = y^{T\cup\bar{k}}$

Now let us prove $y^{T\cup R} = x^{T\cup R}$ *when* $R\subset N\backslash T$ *has more than one element.*

Without any loss of generality, assume that $\bar{k}\in R$ *and let us prove first that* $x_{\bar{k}}^{T\cup R} = y_{\bar{k}}^{T\cup R}$. *For each* $j\in N\backslash\bar{k}$, *consider the subgames* $\big(\{\bar{k},j\}, V^{(N\backslash(T\cup R))\cup\bar{k},\mathbf{x}}\big)$ *and* $\big(\{\bar{k},j\}, W^{(N\backslash(T\cup R))\cup\bar{k},\mathbf{y}}\big)$. *Similarly from expressions (5.29) and (5.30):*

$$x_{\bar{k}}^{T\cup R} - x_j^{T\cup R} = x_{\bar{k}}^{(T\cup R)\backslash j} - x_j^{(T\cup R)\backslash\bar{k}}, \qquad \text{for every } j\in N\backslash\bar{k}, \tag{5.31}$$

$$y_{\bar{k}}^{T\cup R} - y_j^{T\cup R} = y_{\bar{k}}^{(T\cup R)\backslash j} - y_j^{(T\cup R)\backslash\bar{k}}, \qquad \text{for every } j\in N\backslash\bar{k}. \tag{5.32}$$

It can be shown that $x_{\bar{k}}^{(T\cup R)\backslash j} = y_{\bar{k}}^{(T\cup R)\backslash j}$ *for every* $j\in(T\cup R)\backslash\bar{k}$; *indeed, when* $j\in T$, *IPI yields this equality, and when* $j\notin T$, *then the induction process provides this equality, since* $|R\backslash j| < |R|$.

By taking this equality into account, it can be concluded from (5.33) and (5.34) that

$$\sum_{j\in(T\cup R)\backslash\bar{k}} (x_{\bar{k}}^{T\cup R} - x_j^{T\cup R}) - \sum_{j\in(T\cup R)\backslash\bar{k}} (y_{\bar{k}}^{T\cup R} - y_j^{T\cup R})$$
$$= \sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} - \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}.$$

From MAX we can deduce $\sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} = \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}$ *and* $\sum_{j\in(T\cup R)} y_j^{T\cup R} = \sum_{j\in(T\cup R)} x_j^{T\cup R}$. *Hence* $\big(|T\cup R|\big)x_{\bar{k}}^{T\cup R} - \big(|T\cup R|\big)y_{\bar{k}}^{T\cup R} = 0$, *which implies* $x_{\bar{k}}^{T\cup R} = y_{\bar{k}}^{T\cup R}$. *Therefore,* $x_R^{T\cup R} = y_R^{T\cup R}$ *for every coalition* $R\subset N\backslash T$.

To prove $y^{T\cup R} = x^{T\cup R}$, *it remains to be shown that* $x_T^{T\cup R} = y_T^{T\cup R}$. *Since it has benn ascertained that if* $R\subset N\backslash T$ *is not empty, then* $x_R^{T\cup R} = y_R^{T\cup R}$, *and therefore* $(N\backslash R, V^{N\backslash R,\mathbf{x}}) = (N\backslash R, W^{N\backslash R,\mathbf{y}})$. *Through CONS,* $x_S^{S\cup R} = y_S^{S\cup R}$ *for every* $S\subset N\backslash R$. *In particular,* $x_T^{T\cup R} = y_T^{T\cup R}$ *as required.*

Finally $x^T = y^T$ *follows on from expression (5.31) since* $x_{\bar{k}}^{T\cup\bar{k}} = y_{\bar{k}}^{T\cup\bar{k}}$ *for* $\bar{k}\notin T$ *holds.*

Proposition 3.5.20. *If Ψ satisfies MAX, COV, AN, NP, CONS, and IPI, then $\Psi = \subset$ on $\mathcal{G}^G$.*

Proof 3.5.21. *It needs to be shown that $\Psi(N,V) = \subset (N,V)$ for each $(N,V) \in \mathcal{G}^G$. Since Ψ satisfies COV, it can be assumed $V(N)$ is a half-space whose normal vector is $\mathbf{1}$. Now suppose that $\Psi(N,V) = \mathbf{x}$, and consider the TU game $\Psi(N, v_{\mathbf{x}})$, whose characteristic function is defined by $v_{\mathbf{x}}(T) = \sum_{i\in T} x_i^T$ for every coalition $T \subset N$. Therefore, Proposition 3.5.14 and the definition of the Shapley configuration solution give $\Psi(N, v_{\mathbf{x}}) = \subset (N, v_{\mathbf{x}}) = \subset (N, v)$, and Proposition 4.5.10 yields $\Psi(N, v_{\mathbf{x}}) = \Psi(N,V)$. Then, the result follows.*

Proof of Theorem 4.4.5: Propositions 4.5.1 and 4.5.11 complete the proof of Theorem 4.4.5. □

Remark 3.5.22. *The domain of games considered in Theorem 4.4.5 is $\mathcal{G}^G$, but a close look at the proof of propositions 5.10 and 5.11, permits us to conclude that we can replace this domain by any subfamily of G-hyperplane games Γ satisfying*

a) $\mathcal{G}^{TU} \subset \Gamma$, and

b) If $(N,V) \in \Gamma$, and λ is a positive vector in $I\!R^N$, then $(N, \lambda * V) \in \Gamma$.

c) If $(N,V) \in \Gamma$, $T \subset N$ and $x \in \partial V(N)$, then $(T, V^{T,\mathbf{x}}) \in \Gamma$.

c) If $(N,V) \in \Gamma$, and $T \subset N$, then $(N,W) \in \Gamma$, where W is defined in expression (5.26).

In particular $\mathcal{G}^{TU}$ satisfies these properties, and hence Theorem 4.4.5 is an axiomatization on the family of TU games of the Shapley NTU configuration value, whose payoff vector to the grand coalition coincide with those of the Shapley value.

3.6. Independence of the axioms

In this section, examples of configuration solutions are given that serve to show the logical independence of the axiom system in Theorem 4.4.5.

• COV is independent: The egalitarian solution of Kalai and Samet (1985) can be translated into the payoff configurations terminology as follows: a payoff configuration $\mathbf{x} = (x^S)_{S\subset N}$ is the *egalitarian payoff configuration* of (N,V) if (i) $x^S \in \partial V(S)$ and

(ii) $x^S = Sh(S, v_{\mathbf{x}})$, for each coalition $S \subset N$, where $(S, v_{\mathbf{x}})$ is the TU game defined by $v_{\mathbf{x}}(T) = \sum_{i \in T} x_i^T$ for every coalition $T \subset S$.

To every game (N, V) in $\mathcal{G}^G$ let us associate the G-hyperplane game (N, V^λ) defined by $V^\lambda(S) = \text{máx}\left\{\lambda_S^V \cdot x \,:\, x \in V(S)\right\}$. Define the configuration value, Ψ^0 that associates to every game (N, V) the unique egalitarian payoff configuration of the game (N, V^λ). Then Ψ^0 satisfies all the axioms except Covariance.

In order to show the logical independence of some of the remaining axioms, it is sufficient to define a solution for 0-normalized games, that is, games such that $V\big(\{i\}\big) = \{x \in \mathbb{R}^{\{i\}} : x \leq 0\}$, and $V(N) = \{x \in \mathbb{R}^N \,:\, \sum_{i \in N} x_i \leq c\}$, for any $c \in \mathbb{R}$. By using COV, the solution can be uniquely extended to the whole space $\mathcal{G}^G$.

• MAX is independent: Define the configuration value, Ψ^1, that associates to every 0-normalized game (N, V), the payoff configuration $\Psi^1(N, V) = \mathbf{0}$, that associates the payoff vector 0^S to every coalition $S \subset N$. Then Ψ^1 satisfies all the axioms except Maximality.

• AN is independent: Let $w \in R^I_{++}$ be a vector of weights, and let Sh^w be the weighted Shapley value for TU games (Shapley, 1953). A payoff configuration $\mathbf{x} = (x^S)_{S \subset N}$ is said to be a *w-Shapley payoff configuration* of the game $(N, V) \in \mathcal{G}$ if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that (i) $x^S \in \partial V(S)$, for each coalition $S \subset N$, (ii) $\lambda \cdot x^N \geq \lambda \cdot y$, for every $y \in V(N)$, and (iii) $\lambda_S * x^S = Sh^w(S, v_{\lambda,\mathbf{x}})$, for each coalition $S \subset N$ and each $i \in S$, where $v_{\lambda,\mathbf{x}}(T) = \sum_{i \in T} \lambda_i x_i^T$ for every coalition $T \subset N$.

A fixed $w \in R^I_{++}$ defines the configuration value, $\subset^w$, on $\mathcal{G}^G$ that associates the unique w-Shapley payoff configuration to every game $(N, V) \in \mathcal{G}$. It can be shown that $\subset^w$ satisfies all the axioms except Anonymity.

• NP is independent: Consider the mapping Ψ^4 that assigns a payoff configuration, $\Psi^4(N, V) = \mathbf{x}$, to each $(N, V) \in \mathcal{G}^G$, where x^S is the only λ_S^V-efficient point in $V(S)$ which verifies $x_i^S = x_j^S$ for every $i, j \in S$. Then Ψ^4 satisfies all the axioms except Null Player Axiom.

• CONS is independent: Define Ψ^5 for each $(N, V) \in \mathcal{G}^G$ as follows. Assume that the egalitarian configuration solution of the game is $\mathbf{y}$, then let $\Psi^5(N,V) = \mathbf{x}$, where

$$x^S = \begin{cases} y^S, & \text{if } S \neq N; \\ \nu(N, v_{\mathbf{y}}), & \text{if } S = N. \end{cases}$$

Here $\nu(N, v_{\mathbf{y}})$ represents the *prenucleolus* (Sobolev, 1975) of the TU game $(N, v_{\mathbf{y}})$ defined by $v_{\mathbf{y}}(T) = \sum_{i \in T} y_i^T$, for every coalition $T \subset N$. Therefore, Ψ^5 satisfies all the axioms apart from Consistency.

• IPI is independent: For each real number $c \in \mathbb{R}$, consider the games (I, V^c) and $(I, \tilde{V}^c)$ given by:

$$V^c(S) = \begin{cases} \{x \in \mathbb{R}^I : \sum_{i \in I} x_i \leq c\}, & \text{if } S = I; \\ \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq 0\}, & \text{otherwise.} \end{cases}$$

$$\tilde{V}^c(S) = \begin{cases} \{x \in \mathbb{R}^I : \sum_{i \in I} x_i \leq c\}, & \text{if } S = I; \\ \{x \in \mathbb{R} : x \leq 1\}, & \text{if } |S| = 1; \\ \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq 0\}, & \text{otherwise.} \end{cases}$$

Assume that $\subset (I, \tilde{V}^c) = \mathbf{x}(c) = \left(x^S(c)\right)_{S \subset N}$, and define the payoff configuration $\mathbf{y}(c)$ as follows,

$$y^S(c) = \begin{cases} x^S(c), & \text{if } |S| > 1, \\ 0, & \text{if } |S| = 1. \end{cases}$$

Now let Ψ^6 be the only mapping on $\mathcal{G}^G$ that is defined for every game 0-normalized game (N,V) as

$$\Psi^6(N,V) = \begin{cases} \mathbf{y}(c), & \text{if } (N,V) = (I, V^c) \\ \subset (N,V), & \text{otherwise.} \end{cases}$$

Then Ψ^6 satisfies all the axioms except IPI.

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4. THE EGALITARIAN CONFIGURATION VALUE

4.1. Introduction

The Shapley (1953) value for transferable utility (TU) cooperative games was characterized by Hart and Mas-Colell (1989) as de only consistent single-valued solution which is standard for two-person games. In the same paper, they generalized their result to the whole class of non-transferable utility (NTU) games, and characterized a generalization of the Shapley value to NTU games, namely the egalitarian solution (Kalai and Samet, 1985).

On the other hand Maschler and Owen (1989) demonstrate that there not exists an efficient, symmetric, and consistent solution that, unlike the egalitarian solution, were also covariant under affine transformations of utility.

Nevertheless, the authors of the present paper have recently shown that, if payoff configurations[1] are considered instead of payoff vectors as solution outcomes, and if we refer to the class of games for which the feasible set of the grand coalition is a half-space (containing the class of hyperplane games), both the Harsanyi (1963) NTU solution (Hinojosa et al. (2012)) and the Shapley (1969)[2] NTU solution (Hinojosa et al. (2015)) can be characterized by means of a consistency axiom together with some plausible axioms.

The consistency principle can be roughly described as follows, if a subgroup of players receive their share and leave the others in a renegotiation, then the shares of the remaining players do not change in the subsequent reduced situation. However, there is not a canonical way of modeling the reduced situation, and consequently several formal definitions of a reduced game have been proposed in the literature. In fact different versions of this axiom

[1] The notion of payoff configuration was introduced by Hart (1985) in his characterization of the Harsanyi NTU solution. More recently, De Clippel et al. (2004) compared and characterized several NTU solution concepts by working with payoff configurations as well.

[2] Shapley (1988) is a more accesible version.

had been formerly used by Lensberg (1988) to characterize the Nash (1950) bargaining solution, and by Hart and Mas-Colell (1989) to characterize the Shapley value of TU games. The reduced games considered in Hinojosa et al. (2012) and Hinojosa et al. (2015) may be both regarded as natural extensions of the one proposed by Hart and Mas-Colell (1989) with respect to payoffs configurations.

In this paper we follow the same line as that in the above-mentioned papers to characterize the egalitarian configuration value by means of a consistency axiom which consists of a generalization of that in Hart and Mas-Colell (1989).

Nevertheless, in this paper NTU games are regarded as a generalization of bargaining games. In fact we extend the results obtained in Kalai (1977) for pure bargaining games by following the same approach as that in Nash (1950). In this result two principles about the outcome of the bargaining process are considered, namely monotonicity and step-by-step negotiations. It is shown that each of these principles is sufficient to imply that the players must be doing interpersonal comparison of utility among themselves, when they maximize their utilities subject to the restriction that they all gain equally in a given situation.

In the present paper it is shown the above principles, together with some usual axioms characterizes the egalitarian configuration value on the class of choice problems with reference points (see Sudhölter and Zarzuelo, 2013), that can be formally seen as a generalization of the class of pure bargaining games by permitting the disagreement point to be unfeasible. Moreover, a characterization of this solution involving consistency is provided on the class of general NTU games.

The paper is organized as follows. Section 2 and 3 contain some preliminaries and definitions. In Section 4, the egalitarian configuration value is introduced, and the main results are stated. Firstly we consider choice problems with reference points, and then we extend the result to general NTU games. Proofs are postponed to Section 5.

4.2. *Preliminaries*

Let I be a finite set of potential *players*, with cardinality $|I| \geq 3$. A *coalition* is any non-empty subset of I. For each coalition $N \subset I$, the $|N|$-dimensional Euclidean space whose axes are labeled with the members of N is denoted by $\mathbb{R}^N$. If $x = (x_i)_{i\in N} \in \mathbb{R}^N$ and $S \subset N$, then the projection of x onto $\mathbb{R}^S$ is denoted x_S, i. e., $x_S = (x_i)_{i\in S} \in \mathbb{R}^S$.

Given $x, y \in \mathbb{R}^N$, then $x \geq y$ ($x > y$) means $x_i \geq y_i$ ($x_i > y_i$) for all $i \in N$. The subsets of $\mathbb{R}^N$ formed by vectors $x \geq 0$, and $x > 0$ are denoted by $\mathbb{R}^N_+$, and $\mathbb{R}^N_{++}$ respectively. Moreover, $x \cdot y$ denotes the real number $\sum_{i\in N} x_i y_i$ (scalar product). If A, $B \subset \mathbb{R}^N$, $c \in \mathbb{R}_+$, and $x \in \mathbb{R}^N$, then $A + B$, $x + A$ and cA are defined by $A + B := \{a + b : a \in A, b \in B\}$, $x+A := \{x + a : a \in A\}$ and $cA := \{ca : a \in A\}$ respectively. The boundary of A is denoted by ∂A.

A *non-transferable utility (NTU) game* — a game, for short — is a pair (N, V) where N is a coalition and V is a set-valued function (the characteristic function) that assigns a subset $V(S)$ of $\mathbb{R}^S$ to each coalition $S \subset N$ such that

(A.1) $V(S)$ is non-empty, closed, comprehensive, and bounded from above.

(A.2) $V(S)$ is 'uniformly non-leveled'. That is, there exists a real number $\delta(S, V) > 0$ such that for every normalized vector $\lambda \in \mathbb{R}^N$ (i.e. $\sum_{i\in N} \lambda_i = 1$) the following condition holds

$$\sup_{x\in V(S)} \lambda \cdot x < \infty \text{ implies } \lambda_i \geq \delta(S, V) \text{ for every } i \in S.$$

The set of all NTU games will be denoted by $\mathcal{G}$.

Given a game (N, V) and a coalition $S \subset N$, then (S, V) denotes the subgame obtained by restricting V to subcoalitions of S only.

One of the simplest cases of an NTU game occurs when every coalition $S \subset N$ is assigned a real number $v(S)$ such that

$$V(S) = \Big\{x \in \mathbb{R}^S : \sum_{i\in S} x_i \leq v(S)\Big\}.$$

Such games are known as *transferable utility (TU) games*. We say that (N, V) corresponds to v, and V and v are denoted interchangeably and no confusion will appear. The set of TU games will be denoted $\mathcal{G}^{TU}$.

Another simple class of NTU games studied in the literature is the class of *pure bargaining games*. A bargaining situation is described by a set of alternatives which are feasible for a set of individuals when they cooperate, and an alternative which comes about when they do not cooperate[3]. These games are NTU games, (N,V), for which $0 \in \partial V(S)$ for every proper coalition $S \subset N$, but $0 \in V(N) \setminus \partial V(N)$[4]. In these games agents in N bargain to agree on a feasible or achievable outcomes in $V(N) \subset \mathbb{R}^N$ for them when they cooperate, and they obtain all a null outcome (disagreement point) when they do not cooperate.

In this paper we consider a class of NTU games, denoted by $\mathcal{G}^\beta$, containing the class of pure bargaining games. These games are NTU games, (N,V), for which $0 \in \partial V(S)$ for every proper coalition $S \subset N$, but $0 \notin \partial V(N)$. Notice that we do not impose $0 \in V(N)$. We call these games *choice problems with reference points*.

4.3. Consistent solutions on NTU games.

Given a family of games $\mathcal{F} \subset \mathcal{G}$, a *value* ϕ on $\mathcal{F}$ is a function that assigns to each game $(N,V) \in \mathcal{F}$ a payoff vector $\phi(N,V) \in \mathbb{R}^N$.

On the class of TU games the *Shapley value*, denoted Sh, assigns to every game (N,v), the payoff vector defined for each $i \in N$ by

$$Sh_i(N,v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \big(v(S \cup i) - v(S)\big).^5$$

Let ϕ be a value on the class of TU games, (N,v) a TU game, and $T \subset N$ a coalition. The *reduced game* $(T, v_{T,\phi})$ is the TU game given for every coalition $S \subset T$ by

$$v_{T,\phi}(S) = v\big(S \cup (N \setminus T)\big) - \sum_{i \in N \setminus T} \phi_i\big(S \cup (N \setminus T), v\big). \tag{3.1}$$

A value ϕ on the class $\mathcal{G}^{TU}$ is said to be *consistent* if, for every TU game (N,v) and every coalition $T \subset N$, it holds $\phi(T, v_{T,\phi}) = \phi_T(N,v)$.

Moreover, a value ϕ on $\mathcal{G}^{TU}$ is said to be *standard for two-person games*, if for every 2-person game $\big(N,v\big)$, and for each $i \in N$

$$\phi_i\big(N,v\big) = v\big(\{i\}\big) + \frac{1}{2}\Big[v\big(N\big) - v\big(\{i\}\big) - v\big(N \setminus i\big)\Big].$$

[3] It is considered that the game is 0-normalized, which means that the disagreement point is 0

[4] We abuse of notation by denoting simply 0 to vectors of different dimension whose components are all equal to 0.

Theorem 4.3.1. *[Hart and Mas-Colell, 1989] A value on $\mathcal{G}^{TU}$ is consistent and standard for two-person games if and only if it is the Shapley value.*[6]

Hart and Mas-Colell (1989) generalized this result to the whole class of NTU games, and characterize an extension of the Shapley value: the egalitarian solution (Kalai and Samet, 1985).

The *egalitarian solution* assigns to every NTU game (N, V) the unique payoff vector $\varepsilon(N, V)$ for which there exists a family of real numbers $\left(d^T\right)_{T \subset N}$ such that, if we denote $x_i^S = \sum_{\substack{T \subseteq S \\ i \in T}} d^T$ for every coalition $S \subset N$ and every $i \in S$, then

$$x^N = \varepsilon(N, V), \tag{3.2}$$

$$x^S \in \partial V(S), \quad \text{for each coalition } S \subset N. \tag{3.3}$$

The egalitarian solution (3.2) combines the efficiency and fairness principles in the payoff vector of every coalition. Indeed, condition (3.3) states that every intermediate payoff vector x^S is efficient. Moreover, the payoff x_i^S of each member of any coalition S is the sum of the 'dividends' d^T from all the subcoalitions T of S to which player i belongs. Since the dividends are the same for all members of T, we can say that the payoff x^S is fair.

The definition of a reduced NTU game is the natural extension of (3.1) (see Hart and Mas-Colell, 1989; also Maschler and Owen, 1989).

Let (N, V) be a NTU game, $T \subset N$ a coalition, and ϕ a value on $\mathcal{G}$. The *reduced game* $(T, V_{T,\phi})$ is defined for every coalition $S \subset T$ as follows:

$$V_{T,\phi}(S) = \Big\{ y \in \mathbb{R}^S \, : \, \Big(y, \phi_{N \backslash T}\big(S \cup (N \backslash T), V\big)\Big) \in V\big(S \cup (N \backslash T)\big) \Big\}. \tag{3.4}$$

A value ϕ on $\mathcal{G}$ is said to be *consistent* if for every NTU game (N, V) and every coalition $T \subset N$, it holds $\phi(T, V_{T,\phi}) = \phi_T(N, V)$.

Theorem 4.3.2. *[Hart and Mas-Colell,1989] The egalitarian solution is the only consistent value on $\mathcal{G}$ such that its restriction to $\mathcal{G}^{TU}$ is standard for two-person games.*

[6] Actually Hart and Mas-Colell (1989) considered an infinite set I, but can obviously adapted for the case of a finite set such that $|I| \geq 2$.

In this paper we consider a different notion of solution that consists of specifying a payoff vector for each coalition instead of only a vector for the grand coalition (value), and adopt a 'naturaléxtension of the consistency property by means of which we provide different characterizations.

4.4. *The egalitarian configuration value*

As mentioned earlier, a value customarily specifies a payoff vector for the grand coalition. In contrast, Hart (1985) considered that a solution outcome specifies a payoff vector for each coalition, that is, a payoff configuration. Accordingly we will distinguish value from configuration value.

Let $\boldsymbol{X}^N$ denote the product $\prod_{S\subset N}\mathbb{R}^S$; an element $\mathbf{x} = (x^S)_{S\subset N} \in \boldsymbol{X}^N$ is called a *payoff configuration.* It assigns a payoff vector, $x^S = (x_i^S)_{i\in S} \in \mathbb{R}^S$, to every coalition S.[7]

Given a class of games $\mathcal{F} \subset \mathcal{G}$, a *configuration value* Ψ on $\mathcal{F}$ is a function that assigns to each game $(N,V) \in \mathcal{F}$ a payoff configuration $\Psi(N,V) \in \boldsymbol{X}^N$.

The *egalitarian configuration value* assigns to every NTU game (N,V) the unique payoff configuration $\mathcal{E}(N,V) = \mathbf{x} \in \mathbf{X}^N$ for which there exists a family of real numbers $\left(d^T\right)_{T\subset N}$ such that, if $x_i^S = \sum_{\substack{T\subset S\\ i\in T}} d^T$ for every coalition $S \subset N$ and every $i \in S$, then $\mathbf{x} \in \partial V = \prod_{S\subset N}\partial V(S)$.

Notice that a payoff configuration $\mathbf{x} = (x^S)_{S\subset N}$ is the egalitarian configuration value of de game (N,V) if and only if $\mathbf{x} \in \partial V$ and

$$x^S = Sh(S, v_{\mathbf{x}}), \tag{4.5}$$

for each coalition $S \subset N$, where $(S, v_{\mathbf{x}})$ is the TU game defined by $v_{\mathbf{x}}(T) = \sum_{i\in T} x_i^T$ for every coalition $T \subset S$.

As stated in the Introduction, our main goal is to characterize the Egalitarian NTU configuration value $\mathcal{E}$ with a consistency property together with some additional axioms. Firstly we characterize the configuration value in the class of choice problems with reference

[7] Notice the difference between x^S and x_S. By x^S we denote the payoff vector of the payoff configuration $\mathbf{x}$ corresponding to coalition S. In contrast, x_S is the projection of a vector $x \in \mathbb{R}^N$ to $\mathbb{R}^S$.

points by means of the principles considered in Kalai (1977), namely monotonicity and step-by-step negotiations.

4.4.1. Choice Problems with Reference Points

Let Ψ denote a configuration value on $\mathcal{G}^\beta \in \mathcal{G}$, and (N,V), (N,W) are arbitrary games of $\mathcal{F}$. Denote $\mathbf{x} = \Psi(N,V)$ and $\mathbf{y} = \Psi(N,W)$. The following axioms are imposed in order to characterize the egalitarian configuration value on the class $\mathcal{G}^\beta$.

EFFICIENCY: (EFF)

$\mathbf{x} = \Psi(N,V) \in \partial V = \prod_{S \subset N} \partial V(S)$.

RATIONALITY: (RATIO)

$x^S = 0$ for every $S \neq N$, $x_i^N < 0$, for each $i \in N$, if $0 \notin V(N)$, and $x^N \geq 0$ otherwise[8].

MONOTONICITY: (MON) $W(N) \subset V(N)$ implies $y_i^N \leq x_i^N$ for every $i \in N$.

ANONYMITY: (AN)

$\Psi\pi_* = \pi\Psi$ for each permutation π of I, where π_* is a linear mapping from $\mathcal{G}$ onto itself, defined by $\pi_*(N,V) = (\pi N, \pi V)$, and $\pi V(\pi S) = V(S)$ for every coalition $S \subset N$

THEOREM 4.4.1. *In the class $\mathcal{G}^\beta \subset \mathcal{G}$, Ψ verifies EFF, RATIO, MON and AN if and only if it is the egalitarian configuration value.*

Consider the game $(N,W^*) \in \mathcal{G}^\beta$ defined as $W^*(N) = V(N) - y^N$. A *step by step condition* impose that the outcome of the game (N,V) can be obtained by stages playing first (N,W) and afterwords the game (N,W^*). Formally:

STEP BY STEP CONDITION: (SbS)

If $W^*(N) = V(N) - y^N$ and $\mathbf{z} = \Psi(N,W^*)$, then $x^N = y^N + z^N$.

It is easy to see that the egalitarian configuration value verifies SbS, and moreover, if Ψ satisfies the SbS axiom, then Ψ satisfies the MON axiom. Therefore the MON axiom can be replaced by a step by step condition.

[8] Notice that under this rationality assumption, $\mathbf{0} \in \partial V$ implies $x^S = 0$ for every $S \subset N$.

Corollary 4.4.2. *The egalitarian configuration value, $\mathcal{E}$, is the unique configuration value on $\mathcal{G}^\beta \subset \mathcal{G}$ which satisfies EFF, RATIO, SbS and AN.*

Now we are going to see that under continuity the MON axiom can be replaced by independence of irrelevant alternatives and a individual monotonicity property.

Independence of Irrelevant Alternatives: (IIA)

$V(N) \subset W(N)$ and $y^N \in V(N)$, implies $\mathbf{y} = \Psi(N, V)$.

Individual Monotonicity: (IM)

$$\left.\begin{array}{l} \left.\begin{array}{l} 0 \in V(N) \\ V(N) \cap \mathbb{R}_+^{N\setminus i} = W(N) \cap \mathbb{R}_+^{N\setminus i} \\ V(N) \subset W(N) \end{array}\right\} \text{ or} \\ \\ \left.\begin{array}{l} 0 \notin W(N) \\ V(N) \cap \mathbb{R}_-^{N\setminus i} = W(N) \cap \mathbb{R}_-^{N\setminus i} \\ V(N) \subset W(N) \end{array}\right\} \end{array}\right\} \text{imply } x_i^T \leq y_i^T.$$

Continuity: (CON)

If $\{(N, V_i)\}_{i=1}^{\infty}$ is a sequence of games in $\mathcal{G}^\beta$, and $(N, V) \in \mathcal{G}^\beta$ is such that, $V_i(N) \to V(N)$ (in the Hausdorff topology) then $\Psi(N, V_i) \to \Psi(N, V)$.[9]

Theorem 4.4.3. *The egalitarian configuration value, $\mathcal{E}$, is the unique configuration value on $\mathcal{G}^\beta$ which satisfies EFF, RATIO, IIA, IM, CON and AN.*

4.4.2. General NTU games

Our main goal is to characterize the egalitarian configuration value on $\mathcal{G}$ with a consistency property together with some additional axioms. Further notation and definitions are needed in advance to state these axioms.

[9] $\Psi(N, V_i) \to \Psi(N, V)$ means $x(i)^N \to x^N$, where $\mathbf{x(i)} = \Psi(N, V_i)$ and $\mathbf{x} = \Psi(N, V)$.

Given $(N,V) \in \mathcal{G}$, the game $(N, a+V) \in \mathcal{G}$, where $a \in \mathbb{R}^N$, is defined for each $S \subset N$ by $(a+V)(S) = a_S + V(S)$.

For every payoff configuration $\mathbf{x} \in \mathbf{X}^N$, the payoff configuration $\mathbf{y} = a + \mathbf{x} \in \mathbf{X}^N$, where $a \in \mathbb{R}^N$, is defined for each $S \subset N$ by $\mathbf{y}^S = a_S + \mathbf{x}^S$.

We say that a player $i \in N$ is a *null player* in the game $(N,V) \in \mathcal{G}$ if $V(S) = \big\{x \in \mathbb{R}^S : (x,0) \in V(S \cup i)\big\}$ for every coalition $S \subset N \backslash i$.

Let (N,V) be a game, $T \subset N$ a coalition, and $\mathbf{x} \in \boldsymbol{X}^N$ a payoff configuration. The *reduced game* $(T, V^{T,\mathbf{x}})$ is defined for each coalition $S \subset T$ by

$$V^{T,\mathbf{x}}(S) = \Big\{ y \in \mathbb{R}^S : \Big(y, x_{N\backslash T}^{S \cup (N \backslash T)}\Big) \in V\big(S \cup (N\backslash T)\big)\Big\}. \tag{4.6}$$

Notice that by assumptions (A.1) and (A.2) in the definition of an NTU game, $(T, V^{T,\mathbf{x}})$ is also an NTU game in $\mathcal{G}$.

Remark 4.4.4. *Let $\mathcal{F} \subset \mathcal{G}$ be a family of games such that $(N,V) \in \mathcal{F}$ implies $(S,V) \in \mathcal{F}$, for every $S \subset N$. With every value ϕ on $\mathcal{F}$ we can associate a configuration value Ψ^ϕ defined by $\Psi^\phi(N,V) = \big(\phi(S,V)\big)_{S \subset N}$ for each $(N,V) \in \mathcal{F}$. By doing so, it turns out that the reduced games defined in expressions (3.4) and (4.6) coincide, that is, $V_{T,\phi} = V^{T,\Psi^\phi(N,V)}$. Hence, the reduced game of (4.6) may be regarded as the natural extension of the one proposed by Hart and Mas-Colell with respect to payoff configurations.*

The following axioms are now imposed, where Ψ denotes a configuration value on a symmetric family of NTU games, $\mathcal{F} \in \mathcal{G}$, and (N,V), (N,W) are arbitrary games of $\mathcal{F}$. Denote $\mathbf{x} = \Psi(N,V)$ and $\mathbf{y} = \Psi(N,W)$.

Rationality*: (RATIO*)

If $0 \in \partial V(S)$ for every $S \neq T$, then

(i) for every $R \not\supset T$, $x_T^R = 0$ and

(ii) for every $R \supset T$, $x_T^R < 0$ if $0 \notin V(T)$, and $x_T^R \geq 0$ otherwise

Notice that under this rationality assumption, $\mathbf{0} \in \partial V$ implies $x^S = 0$ for every $S \subset N$.

Translation Invariance: (TINV)

$\Psi(N, a+V) = a + \Psi(N, V)$.

Independence of Coalitional Opportunities differences: (ICOD)

If $V(S) = W(S)$ for each $S \neq T$, then $x^R = y^R$ for every $R \not\supset T$.

Monotonicity*: (MON*)

$$\left.\begin{array}{l} V(S) = W(S) \text{ for each } S \neq T \text{ and} \\ \\ V(T) \subset W(T) \end{array}\right\} \text{ imply } x_T^R \leq y_T^R \text{ for every } R \supset T.$$

Null Player: (NP)

If $i \in N$ is a null player in (N, V), then $x_i^{S \cup i} = 0$ for every $S \subset N \backslash i$.

Consistency: (CONS)

$\Psi\left(T, V^{T,\mathbf{x}}\right) = \left(x_S^{S \cup N \backslash T}\right)_{S \subset T}$ for every coalition $T \subset N,\ T \neq N$.

Now we state our main result, that is the characterization of the egalitarian configuration value.

Theorem 4.4.5. *The egalitarian configuration value, $\mathcal{E}$, is the unique configuration value which satisfies EFF, RATIO*, TINV, ICOD, MON*, AN, NP and CONS.*

4.5. Proofs

Proof of Theorem 4.4.1

Let (N, V) be an NTU game in the class $\mathcal{G}^\beta$, and denote $\Psi(N, V) = \mathbf{x}$. By RATIO, $x^S = 0$ for every $S \neq N$.

Consider the $|N|$-dimensional simplex $\Delta^{|N|-1} = \left\{\lambda \in \mathbb{R}_+^{|N|};\ \sum_{i \in N} \lambda_i = 1\right\}$. For each ε, $0 < \varepsilon < 1/|N|$, let Λ_ε be the subset of $\Delta^{|N|-1}$ described as follows:

$$\Lambda_\varepsilon = \left\{\lambda \in \Delta^{|N|-1};\ \lambda_i \geq \varepsilon,\ \forall\, i = 1, ..., |N|.\right\}$$

.

Let e^+ and e^- be two $|N|$-dimensional vectors with all their components equal to 1 and -1 respectively and consider the two NTU games $(N, V_\varepsilon^{e^+})$ and $(N, V_\varepsilon^{e^-})$ such that $V_\varepsilon^{e^+}(S) = V_\varepsilon^{e^-}(S) = V(S)$ for every $S \neq N$, and

$$V_\varepsilon^{e^+}(N) = \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot e^+,\ \forall\, \lambda \in \Lambda_\varepsilon\right\} \text{ and}$$

$$V_\varepsilon^{e^-}(N) = \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot e^-,\ \forall\, \lambda \in \Lambda_\varepsilon\right\}.$$

If we denote by $\mathbf{y}$ and $\mathbf{y}'$ to $\Psi(N, V_\varepsilon^{e^+})$ and $\Psi(N, V_\varepsilon^{e^-})$, respectively, then, by RATIO*, $y^S = y'^S = 0$ for every $S \neq N$ and, by EFF, $y^N \in \partial V_\varepsilon^{e^+}(N)$ and $y'^N \in \partial V_\varepsilon^{e^-}(N)$ respectively. Moreover, by AN, it follows that $y^N = e^+$ and $y'^N = e^-$ respectively.

Consider $t^* = \text{máx}\{t \in \mathbb{R} :\ te^+ \in V(N)\}$ and $\tau(N, V) = |t^*| \in \mathbb{R}_+$.

We are going to see that $x^N = \tau(N, V)e^+$ if $0 \in V(N)$ and $x^N = \tau(N, V)e^-$ if $0 \notin V(N)$.

- Case 1: $0 \in V(N)$.

In this case there exists $\varepsilon^* > 0$, $0 < \varepsilon^* < 1/|N|$, such that $\tau(N, V)V_{\varepsilon^*}^{e^+} \subset V(N)$. To see that, consider $\Lambda^* = \{\lambda \in \Delta^{|N|-1};\ \max_{x \in V(N)} \lambda \cdot x < \infty\}$. Since $V(N)$ is uniformly non-leveled, then Λ^* is a polyhedron of strictly positive vectors in Δ^{n-1}. Consider $0 < \varepsilon^* < \delta = \text{mín}\{\delta_1, \delta_2, \ldots, \delta_{|N|}\}$, where $\delta_i = \min_{\lambda \in \Lambda^*} \lambda_i$, $i = 1, 2, \ldots, |N|$. Notice that $\Lambda^* \subset \Lambda_{\varepsilon^*}$. Therefore,

$$\tau(N, V)V_{\varepsilon^*}^{e^+} = \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \tau(N, V)e^+,\ \forall\, \lambda \in \Lambda_{\varepsilon^*}\right\} \subset$$

$$\subset \left\{x \in \mathbb{R}^{|N|};\ \lambda \cdot x \leq \lambda \cdot \tau(N, V)e^+,\ \forall\, \lambda \in \Lambda^*\right\},$$

and, since $V(N)$ is comprehensive and uniformly non-leveled, it is easy to see that this last set is a subset of $V(N)$.

Notice that $(N, \tau(N, V)V_{\varepsilon^*}^{e^+}) = (N, V_{\varepsilon^*}^{\tau(N,V)e^+})$, $0 \in \partial V_{\varepsilon^*}^{\tau(N,V)e^+}(S)$ for every $S \neq N$, and $V_{\varepsilon^*}^{\tau(N,V)e^+}(N)$ is a symmetric set. Then, if we denote $\Psi(N, V_{\varepsilon^*}^{\tau(N,V)e^+}) = \mathbf{z}$, then, by RATIO, $z^S = 0$, for every $S \neq N$ and by AN $z^N = \tau(N, V)e^+$.

Consequently, through MON* axiom $x^N \geq \tau(N, V)e^+$. Finally, since $V(N)$ is comprehensive and non-leveled, $\tau(N, V)e^+$ is a strongly Pareto-optimal point of $V(N)$, and therefore $x^N = \tau(N, V)e^+$.

• Case 2: $0 \notin V(N)$. In this case the proof is analogous since there exists $\varepsilon^* > 0$, $0 < \varepsilon^* < 1/|N|$, such that $\tau(N,V)V_{\varepsilon^*}^{e^-} \subset V(N)$. Therefore, by MON*, $x^N \geq \tau(N,V)e^-$, and by EFF we conclude $x^N = \tau(N,V)e^-$. □

Proof of Theorem 4.4.3

It is easy to prove that the egalitarian configuration value satisfies the conditions of the theorem.

Let (N,V) be an NTU game in the class $\mathcal{G}^\beta$, and denote $\Psi(N,V) = \mathbf{x}$. The RATIO axiom imposes $x^S = 0$ for every $S \subset N$, $S \neq N$. Consider

$$\tau(N,V) = \text{máx}\{t \in \mathbb{R} :\, te^+ \in V(N)\} \text{ where } e^+\mathbb{R}^N,\ e_i^+ = 1,\ \forall\, i \in N$$

and the NTU games $(N, W_{\tau(N,V)})$ and $(N, V_{\tau(N,V)})$ defined by $W_{\tau(N,V)}(S) = V_{\tau(N,V)}(S) = V(S)$ for every $S \neq N$, and

$$W_{\tau(N,V)}(N) = \left\{ x \in \mathbb{R}^N :\, \sum_{i \in N} x_i \leq \tau(N,V)|N| \right\} \text{ and}$$

$$V_{\tau(N,V)}(N) = W_{\tau(N,V)}(N) \cap V(N).$$

If we denote $\mathbf{y} = \Psi(N, W_{\tau(N,V)})$, then through the EFF, RATIO and AN axioms, $y^S = 0$ for each $S \neq N$ and $y_i^N = \tau(N,V)$ for every $i \in N$, that is Ψ provides the egalitarian configuration value. Moreover, by IIA $\Psi(N, V_{\tau(N,V)})$ is also $\mathbf{y}$.

In what follows, it is shown that $\mathbf{x} = \mathbf{y}$.

Consider a vector $a(N,V) \in \mathbb{R}^N_{++}$, which will be denoted simply a in what follows (we suppose that a is normalized so that $\sum_{i \in N} a_i = 1$) such that $\sup_{x \in V(N)} a \cdot x < \infty$. Consider also $\tau_a(N,V) = \text{máx}\{t \in \mathbb{R} :\, ta \in \partial W_{\tau(N,V)}(N)\}$

Let $\Delta^{|N|-1} = \left\{\lambda \in \mathbb{R}_+^{|N|};\, \sum_{i \in N} \lambda_i = 1\right\}$ be the $|N|$-dimensional simplex.

- Case 1: $\tau(N,V) > 0$ (and therefore $\tau_a(N,V) > 0$).

  Let $\delta > 1$ be a real number such that $\delta \tau_a(N,V) a_1 \leq \tau(N,V)|N|$ for every $i \in N$. A polyhedron with $|N|$ positive extreme points, $\Lambda^a \subset \Delta^{|N|-1}$, exists such that, for every $i \in N$,

$$\left\{x \in \mathbb{R}^N;\, \lambda \cdot x \leq \lambda \cdot \delta\tau_a(N,V)a,\ \forall\, \lambda \in \Lambda^a\right\} \cap \mathbb{R}_+^{N\setminus i} = W_{\tau(N,V)}(N) \cap \mathbb{R}_+^{N\setminus i}.$$

Notice that a is an interior point of Λ^a.

Consider the NTU game (N, V^a) defined by $V^a(S) = V(S)$ for all $S \neq N$ and

$$V^a(N) = V(N) \cap \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau_a(N,V)a,\ \forall\, \lambda \in \Lambda^a\right\}.$$

If we denote $\mathbf{z} = \Psi(N, V^a)$, then, by RATIO*, $z^S = 0$ for every $S \neq N$ and, by IM, $y^N \leq z^N$. Therefore, by EFF $\mathbf{z} = \mathbf{y}$.

We are going to construct a succession of games $\{(N, V_k^a)\}_{k=1,2,\ldots\infty}$ such that

- $V_k^a(S) = V(S)$ for every $S \neq N$ and every k,
- $V_k^a(N) \subset V_{k+1}^a(N)$ for every k,
- $\Psi(N, V_k^a) = \mathbf{y}$ for every k, and
- $(N, V_k^a) \to (N, V)$.

Therefore, through CON axiom, $\mathbf{y} = \mathbf{x}$.

To construct such a succession of games, we proceed by induction.

- Define $V_1^a(S) = V(S)$ for all $S \neq N$, and

  $$V_1^a(N) = V(N) \cap \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau_a(N,V)a,\ \forall\, \lambda \in \Lambda_1^a\right\},$$

  where $\lambda(1)^i = a - (1-\varepsilon_1)(a - \lambda^i)$, $i = 1, 2, \ldots, |N|$, are the extreme point of Λ_1^a; here λ^i, $i = 1, 2, \ldots, |N|$ are the extreme points of Λ^a and $\varepsilon_1 > 0$ is small enough so that, if $\mathbf{y}(1) = \Psi(N, V_1^a)$, then $y(1)^N \in V^a(N)$. By CON, such that ε_1 exists because y^N is an interior point of $\left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau(N,V)a,\ \forall\, \lambda \in \Lambda^a\right\}$.

  Since $V_1^a(S) = V^a(S)$ for every $S \neq N$, $V^a(N) \subset V_1^a(N)$ because $\Lambda_1^a \subset \Lambda^a$, and $y(1)^N \in V^a(N)$ then, by IIA, $\mathbf{y}(1) = \Psi(N, V_1^a) = \mathbf{y}$.

- By hypothesis of induction let (N, V_k^a) such that $V_k^a(S) = V(S)$ for all $S \neq N$,

  $$V_k^a(N) = V(N) \cap \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau(N,V)a,\ \forall\, \lambda \in \Lambda_k^a\right\},$$

  and $\Psi(N, V_k^a) = \mathbf{y}$

Define $V^a_{k+1}(S) = V(S)$ for all $S \neq N$, and

$$V^a_{k+1}(N) = V(N) \cap \left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau(N,V)a,\ \forall\, \lambda \in \Lambda^a_{k+1}\right\},$$

where $\lambda(k+1)^i = a - (1-\varepsilon_{k+1})(a - \lambda^i(k))$, $i = 1,2,\ldots,|N|$, are the extreme point of Λ^a_{k+1}; here $\lambda(k)^i$, $i = 1,2,\ldots,|N|$, are the extreme points of Λ_k and $\varepsilon_{k+1} > 0$ is small enough so that, if $\mathbf{y}(k+1) = \Psi(N, V^a_{k+1})$, then $y(k+1)^N \in V^a(N)$. By CON, such that ε_{k+1} exists because y^N is an interior point of the set $\left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau(N,V)a,\ \forall\, \lambda \in \Lambda^a_k\right\}$.

Since $V^a_{k+1}(S) = V^a_k(S)$ for every $S \neq N$, $V^a_k(N) \subset V^a_{k+1}(N)$ because $\Lambda^a_{k+1} \subset \Lambda^a_k$, and $y(k+1)^N \in V^a_k(N)$, then, by IIA, $\Psi(N, V^a_{k+1}) = \Psi(N, V^a_k) = \mathbf{y}$.

- Finally, since $\Lambda^a_k \to \{a\}$, then $V^a_k(N) \to V(N)$. Through the CON axiom, it can be concluded that $\mathbf{y} = \mathbf{x}$, as required

- Case 2: $\tau(N,V) < 0$ (and therefore $\tau_a(N,V) < 0$).

In this case, consider $0 < \delta < 1$ and a polyhedron with $|N|$ positive extreme points, $\Lambda^a \subset \Delta^{|N|-1}$ such that, for every $i \in N$,

$$\left\{x \in \mathbb{R}^N;\ \lambda \cdot x \leq \lambda \cdot \delta\tau_a(N,V)a,\ \forall\, \lambda \in \Lambda^a\right\} \cap \mathbb{R}^{N\setminus i}_{-} = W_{\tau(N,V)}(N) \cap \mathbb{R}^{N\setminus i}_{-}.$$

The rest of the proof is analogous to that shown in Case 1. □

Proof of Theorem 4.4.5

Proposition 4.5.1. *The egalitarian configuration value, $\mathcal{E}$, satisfies EFF, RATIO*, ICOD, MON*, AN, NP and CONS.*

Demostración. EFF is imposed in the definition. RATIO* ICOD and MON* are immediate. AN, NP, and CONS can be logically concluded from (4.5) as a consequence of the fact that the Shapley value satisfies anonymity, the null-player property and consistency. □

Remark 4.5.2. *From the non-levelness assumption (A.2), if $\mathbf{x}$ is efficient, then $\left(x_S^{S\cup(N\setminus T)}\right)_{S\subset T}$ is also efficient in the reduced game defined in expression (4.6). Notice that without this assumption, the egalitarian configuration value would fail to satisfy CONS.*

Now we turn to prove the uniqueness part. Firstly notice that the result in Theorem 4.4.1 can be used to establish that EFF, RATIO*, ICOD, MON* and AN uniquely determines the egalitarian configuration value on the class $\mathcal{G}^{\beta}$. Secondly we will prove in Proposition 4.5.7 that the axiom system involved in Theorem 4.4.5 uniquely determines the egalitarian configuration value on the class of TU games, $\mathcal{G}^{TU}$. Finally we will extend this result to the class $\mathcal{G}$ in Proposition 4.5.11.

For the remaining of this section, let Ψ represent a configuration value on the class $\mathcal{G}$.

Associated with the configuration value Ψ, define the value ϕ^{Ψ} on $\mathcal{G}$ by

$$\phi^{\Psi}(N,V) = x^N \quad \text{whenever } \Psi(N,V) = \mathbf{x}. \tag{5.7}$$

Lemma 4.5.3. *In the class of two person NTU games, Ψ satisfies EFF, RATIO*, TINV, ICOD, MON* and AN if and only if it is the egalitarian configuration value.*

Demostración. Let $(N,V) \in \mathcal{G}$ be a two person game ($N = \{i,j\}$), and let $a \in \mathbb{R}^2$ be a vector whose components are $a_i = \partial V(\{i\})$ and $a_j = \partial V(\{j\})$. Consider $(N,W) \in \mathcal{G}$ such that $a_S + W(S) = V(S)$ for every $S \subset N$. Since $(N,W) \in \mathcal{G}^{\beta}$, by Theorem 4.4.1, $\Psi(N,W) = \mathcal{E}(N,W)$. Moreover, by TINV, $\Psi(N,V) = a + \Psi(N,W)$, that is, if we denote $\mathbf{x} = \Psi(N,V)$, then

$$x_i^{\{i\}} = a_i,\ x_j^{\{j\}} = a_j,$$
$$x_i^N - a_i = x_j^N - a_j.$$

and this means $\Psi(N,V) = \mathcal{E}(N,V)$. □

Proposition 4.5.4. *Let Ψ satisfy EFF, RATIO*, TINV, ICOD, MON*, AN, and CONS. If, in addition, Ψ satisfies for every TU game (N,v) the following property,*

$$\Psi(N,v) = \mathbf{x} \textit{ implies } \phi^{\Psi}(T,v) = x^T \textit{ for all coalition } T \subset N. \tag{5.8}$$

then $\Psi(N,v) = \mathcal{E}(N,v)$ for every TU game (N,v),[10]

[10] Notice that Property (5.8) is equivalent to: $\Psi(N,v) = \{(x^S)_{S\subset N}\}$ implies $\Psi(T,v) = \{(x^S)_{S\subset T}\}$ for every coalition $T \subset N$; i.e., the payoff configuration of the solution outcome of a subgame is precisely the restriction of the payoff configuration of the solution outcome of the whole game.

Demostración. Let (N, v) be a TU game, such that $\Psi(N, v) = \mathbf{x}$, and $T \subset N$ be a coalition. Let ϕ^{Ψ} be the value defined in (5.7), and let $v_{T,\phi^{\Psi}}$ be the TU reduced game defined according to (3.1). In addition let $v^{T,\mathbf{x}}$ be the reduced game defined according to (4.6). Since Ψ satisfies (5.8), we have $\phi^{\Psi}(R, v) = x^R$ for every coalition $R \subset N$. Thus for every coalition $S \subset T$ we have

$$v_{T,\phi^{\Psi}}(S) = v\big(S \cup (N\backslash T)\big) - \sum_{i\in N\backslash T} \phi_i^{\Psi}\big(S \cup (N\backslash T), v\big)$$
$$= v\big(S \cup (N\backslash T)\big) - \sum_{i\in N\backslash T} x_i^{S\cup(N\backslash T)} = v^{T,\mathbf{x}}(S).$$

Consequently $v^{T,\mathbf{x}} = v_{T,\phi^{\Psi}}$, and hence

$$\phi^{\Psi}(T, v_{T,\phi^{\Psi}}) = \phi^{\Psi}(T, v^{T,\mathbf{x}}) = x_T^N = \phi_T^{\Psi}(N, v),$$

where the second equality follows on from CONS, since $\Psi(N, v) = \mathbf{x}$. Therefore the restriction of the value ϕ^{Ψ} to TU games is consistent.

Furthermore, by Lemma 4.5.3, ϕ^{Ψ} is standard on 2-person TU games. Then Theorem 4.3.1 provides $\phi^{\Psi} = Sh$, and hence $x^T = Sh(T, v)$ for every coalition $T \subset N$. Then, taking into account (4.5) it follows $\Psi(N, v) = \mathcal{E}(N, v)$. □

In order to facilitate the proof of Proposition 4.5.7, the following property (that can be viewed as a stronger version of the Null Player Axiom) will be considered:

NULL PLAYER*: (NP*)

If $i \in N$ is a null player in (N, V) and $\Psi(N, V) = \mathbf{x}$, then

$$x_i^{S\cup i} = 0 \quad \text{and} \quad x_S^{S\cup i} = x^S \qquad \text{for all coalition } S \subset N\backslash i.$$

Proposition 4.5.5. *If Ψ satisfies EFF, RATIO*, ICOD, MON*, AN, NP* and CONS, then $\Psi(N, v) = \mathcal{E}(N, v)$ for every TU game (N, v).*

Demostración. We shall prove that Ψ satisfies condition (5.8), and the result will follow from Proposition 4.5.4.

Let (N, v) be a TU game and $T \subset N$ a proper coalition. Define the TU game (N, w) for all $S \subset N$ by $w(S) = v(S \cap T)$. Let $\Psi(N, v) = \mathbf{x}$ and $\Psi(N, w) = \mathbf{y}$.

On the one hand every player in $N\backslash T$ is null in $\big(N, w\big)$, which, through NP*, implies that $y_{N\backslash T}^{S\cup N\backslash T} = 0$, for every coalition $S \subset T$. Therefore

$$w^{T,\mathbf{y}}(S) = w(S \cup N\backslash T) - \sum_{i \in N\backslash T} y_i^{S\cup N\backslash T} = v(S) - 0 = v(S),$$

that is $\big(T, w^{T,\mathbf{y}}\big) = (T, v)$. Hence by CONS

$$\Psi(T, v) = \Psi(T, w^{T,\mathbf{y}}) = \big(y_S^{S\cup N\backslash T}\big)_{S\subset T},$$

and consequently $\phi^{\Psi}(T, v) = y_T^N$. Moreover, NP* yields $y_S^{S\cup N\backslash T} = y^S$. In particular by choosing $S = T$, we obtain $y_T^N = y^T$.

On the other hand, we are going to see that if $S \subset T$, then $y^S = x^S$. To prove it, consider (N, w^*) defined by $w^*(S) = \text{máx}\{v(S), w(S)\}$, and for $l = 0, 1, 2, \ldots 2^N - 2^T$, the TU games (N, v^l) and (N, w^l), and the set of coalitions $\mathcal{C}^l$, defined by $v^0 = w^0 = w^*$ and $\mathcal{C}^0 = \{S \subset N : \; S \not\subset T\}$ and for $l = 1, 2, \ldots 2^N - 2^T$, chosen a coalition $R_{l-1} \in \mathcal{C}^{l-1}$,

$$v^l(S) = v^{l-1}(S) \quad \forall\, S \neq R_{l-1}; \;\; v^l(R_{l-1}) = v(R_{l-1})$$

$$w^l(S) = w^{l-1}(S) \quad \forall\, S \neq R_{l-1}; \;\; w^l(R_{l-1}) = w(R_{l-1})$$

$$\mathcal{C}^l = \mathcal{C}^{l-1} \setminus R_{l-1}.$$

The ICOD axiom can be applied recursively to (N, v^l) and (N, w^l) to conclude that $x^S = y^S$ for all $S \subset T$. In particular, $x^T = y^T$.

Therefore, we can conclude that $\phi^{\Psi}(T, v) = y_T^N = y^T = x^T$. Since T is any arbitrary proper coalition in N, Ψ satisfies condition (5.8) as claimed. □

However, under the weaker NP Axiom, more work is required since the equality $y_S^{S\cup N\backslash T} = y^S$ used in the proof above must also be satisfied.

Lemma 4.5.6. *Let* Ψ *satisfy EFF, RATIO*, TINV, ICOD, MON*, AN, and CONS, and* $(N, V) \in \mathcal{G}$ *with* $|N| \geq 2$. *If* $\Psi(N, V) = \mathbf{x}$, *then*

$$x_i^N - x_j^N = x_i^{N\backslash j} - x_j^{N\backslash i} \quad \textit{for all } i, j \in N, \; i \neq j. \tag{5.9}$$

Demostración. Since the reduced game $\big(\{i, j\}, V^{\{i,j\},\mathbf{x}}\big) \in \mathcal{G}$, then the result follows through CONS and Lemma 4.5.3. □

Proposition 4.5.7. *If Ψ satisfies EFF, RATIO*, TINV, ICOD, MON*, AN, NP and CONS, then $\Psi(N,v) = \mathcal{E}(N,v)$ for every TU game (N,v).*

Demostración. We shall prove that Ψ satisfies condition (5.8). Then, the result follows from Proposition 4.5.4.

Let (N,v) be a TU game. If $|N| = 1$, then the result is obviously true, and for $|N| = 2$, then the result follows through Lemma 4.5.3.

Assume $|N| \geq 3$. Let $T \subset N$ be any proper coalition, and consider the TU game (N,w) defined for every $S \subset N$ as $w(S) = v(S \cap T)$. Let $\Psi(N,v) = \mathbf{x}$ and $\Psi(N,w) = \mathbf{y}$. The reasonings in the proof of Proposition 4.5.5 above can be repeated in order to obtain $\phi^{\Psi}(T,v) = y^N_T$, and $y^S = x^S$ for every $S \subset T$ (in particular $y^T = x^T$). Therefore, if it is proved that $y^N_T = y^T$, then condition (5.8) holds as claimed.

Therefore, it is needed to prove that $y^N_T = y^T$ for every coalition $T \subset N$. The case $|T| = 1$ is straightforward through EFF axiom. For the case in which T is not a singleton we will proceed by induction on $|N \backslash T|$.

Assume first that $N \backslash T = \{i\}$, for some $i \in N$. Through Lemma 4.5.6, if $j \in N \backslash i$, then $y^N_i - y^N_j = y^{N\backslash j}_i - y^{N\backslash i}_j$. Moreover, since i is a null player in (N,w), NP Axiom yields $y^N_i = y^{N\backslash j}_i = 0$. Consequently $y^N_j = y^{N\backslash i}_j$ for all $j \in N\backslash i$, i.e., $y^N_{N\backslash i} = y^{N\backslash i}$, as required.

Observe that for the case $|N| = 3$, the proof that $\Psi(N,V) = \mathcal{E}(N,v)$ is already completed.

Assume now that $y^N_T = y^T$ is true when $|N\backslash T| \leq k-1$. Notice that this implies that $\Psi(N,v) = \mathcal{E}(N,v)$ is already proved for the case in which $|N| \leq k+1$, and by (4.5), this yields

$$\phi^{\Psi}(N,v) = Sh(N,v) \qquad \text{whenever } |N| \leq k+1, \tag{5.10}$$

Now let us suppose that $|N\backslash T| = k$. Let $j \in T$ and consider the reduced TU game $\left((N\backslash T) \cup j, w^{(N\backslash T)\cup j, \mathbf{y}}\right)$ defined according to (4.6). Notice that players in $N\backslash T$ are null players in (N,w), and the induction argument implies $y^N_R = y^R$, for every $R \supset T$, $R \neq T$.

Therefore, through EFF,

$$w^{(N\backslash T)\cup j,\mathbf{y}}(S) = \begin{cases} 0 & \text{if } j \notin S, \\ y_j^T & \text{if } S = \{j\}, \\ y_j^N & \text{otherwise.} \end{cases} \tag{5.11}$$

Hence $Sh_i\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \dfrac{(k-1)!}{(k+1)!}(y_j^N - y_j^T)$, whenever $i \in N\backslash T$.

Now if $i \in N\backslash T$ then i is a null player in (N, w), and NP and CONS axioms, together with (5.10) ($|(N\backslash T) \cup j| = k + 1$), provide

$$0 = y_i^N = \phi_i^\Psi\big((N\backslash T) \cup j, w^{(N\backslash T)\cup j,\mathbf{y}}\big) = \frac{(k-1)!}{(k+1)!}(y_j^N - y_j^T). \tag{5.12}$$

Thus, $y_j^N = y_j^T$ holds true for every $j \in T$, as required. □

Lemma 4.5.8. *Let* Ψ *satisfy EFF, ICOD and MON*. Also let* (N, V) *be a NTU game in* $\mathcal{G}$ *for which* $\Psi(N, V) = \mathbf{x}$. *Consider the NTU game* (N, W) *defined by*

$$W(S) = \begin{cases} \left\{x \in I\!R^N \,:\, \sum_{i\in N} x_i \le \sum_{i\in N} x_i^N\right\}, & \textit{if } S = N; \\ V(S), & \text{otherwise.} \end{cases}$$

Then $\Psi(N, W) = \Psi(N, V)$.

Demostración. Consider the $|N|$-dimensional simplex $\Delta^{|N|-1} = \left\{\lambda \in \mathbb{R}_+^N;\, \sum_{i=1}^{|N|} \lambda_i = 1\right\}$. For each ε, $0 < \varepsilon < 1/|N|$, let Λ_ε be the subset of $\Delta^{|N|-1}$ described as follows

$$\Lambda_\varepsilon = \left\{\lambda \in \Delta^{|N|-1};\, \lambda_i \ge \varepsilon,\, \forall\, i = 1, ..., |N|.\right\},$$

and consider the NTU game (N, V_ε) defined by

$$V_\varepsilon(S) = \begin{cases} \left\{x \in \mathbb{R}^N;\, \lambda \cdot x \le \lambda \cdot x^N,\ \forall\, \lambda \in \Lambda_\varepsilon\right\}, & \text{if } S = N; \\ V(S), & \text{otherwise.} \end{cases}$$

Since $V(N)$ is uniformly non-leveled, there exists $\varepsilon^* > 0$, $0 < \varepsilon^* < 1/n$, such that $V_{\varepsilon^*}(N) \subset V(N)$ (see the proof of Theorem 4.4.1). If we denote $\Psi(N, V_{\varepsilon^*}) = \mathbf{z}$, then through MON* and ICOD axioms, $z^N \leq x^N$ and $z^S = x^S$ for each $S \neq N$ respectively. Since by EFF $z^N \in \partial V_{\varepsilon^*}(N)$, then $z^N = x^N$ and $\mathbf{z} = \mathbf{x}$.

On the other hand, $V_{\varepsilon^*}(N) \subset W(N)$. Then, if we denote $\Psi(N, W) = \mathbf{y}$, through MON* and ICOD axioms, $z^N \leq y^N$ and $z^S = y^S$ for each $S \neq N$. Since by EFF $y^N \in \partial W(N)$, then $y^N = z^N$ and $\mathbf{y} = \mathbf{z}$. Therefore $\mathbf{y} = \mathbf{x}$, as required. □

Lemma 4.5.9. *Let Ψ satisfy EFF, RATIO*, TINV, ICOD, MON* AN, NP and CONS. Also let (N, V) be a NTU game in $\mathcal{G}$ for which $\Psi(N, V) = \mathbf{x}$. Let $\bar{k} \in N$ be a fixed player. Consider the NTU game (N, W) defined by*

$$W(S) = \begin{cases} \left\{ x \in \mathbb{R}^{N \backslash \bar{k}} : \sum_{i \neq \bar{k}} x_i \leq \sum_{i \neq \bar{k}} x_i^{N \backslash \bar{k}} \right\}, & \text{if } S = N \backslash \bar{k}; \\ V(S), & \text{otherwise.} \end{cases}$$

Then $\Psi(N, W) = \Psi(N, V)$.

Demostración. Firstly, notice that, by Lemma 4.5.8, $\Psi(N, V)$ coincides with the payoff configuration value assigned by Ψ to the game obtained when only $V(N)$ is changed for the set $\left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq \sum_{i \in N} x_i^N \right\}$. Therefore, to avoid further notation, we assume $V(N) = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq \sum_{i \in N} x_i^N \right\}$.

Consider the $(n-1)$-dimensional simplex $\Delta^{n-2} = \left\{ \lambda \in \mathbb{R}^n_+ ; \sum_{i=1}^{n-1} \lambda_i = 1 \right\}$. For each ε, $0 < \varepsilon < 1/(n-1)$, let Λ_ε be the subset of Δ^{n-2} described as follows

$\Lambda_\varepsilon = \left\{ \lambda \in \Delta^{n-2} ; \lambda_i \geq \varepsilon, \forall\, i = 1, ..., n-1. \right\}$, and consider the NTU game (N, V_ε) defined by

$$V_\varepsilon(S) = \begin{cases} \left\{ x \in \mathbb{R}^{N \backslash \bar{k}} ; \lambda \cdot x \leq \lambda \cdot x^{N \backslash \bar{k}}, \ \forall\, \lambda \in \Lambda_\varepsilon \right\}, & \text{if } S = N \backslash \bar{k}; \\ V(S), & \text{otherwise.} \end{cases}$$

Since $V(N \backslash \bar{k})$ is uniformly non-leveled, there exists $\varepsilon^* > 0$, $0 < \varepsilon^* < 1/n$, such that $V_{\varepsilon^*}(N \backslash \bar{k}) \subset V(N \backslash \bar{k})$ (see the proof of Theorem 4.4.1). Then, if we denote $\Psi(N, V_{\varepsilon^*}) = \mathbf{z}$, through EFF, ICOD and MON*,

$$z^S = x^S \qquad \text{for all } S \neq N, \text{ and} \tag{5.13}$$

$$z^N_{N \backslash \bar{k}} \leq x^N_{N \backslash \bar{k}} \qquad \text{being } \sum_{i \in N} z_i^N = \sum_{i \in N} x_i^N. \tag{5.14}$$

Notice also that $V_{\varepsilon^*}(N\backslash\bar{k}) \subset W(N\backslash\bar{k})$ and $V_{\varepsilon^*}(S) = W(S)$ for all $S \neq N\backslash\bar{k}$. Then, if we denote $\Psi(N, W) = \mathbf{y}$, through EFF, ICOD and MON*, (5.13) and (5.14),

$$y^S = x^S \qquad \text{for all } S \neq N, \text{ and} \tag{5.15}$$

$$z^N_{N\backslash\bar{k}} \leq y^N_{N\backslash\bar{k}} \qquad \text{being } \sum_{i\in N} x^N_i = \sum_{i\in N} y^N_i = \sum_{i\in N} z^N_i. \tag{5.16}$$

To prove that $\mathbf{y} = \mathbf{x}$ it is sufficient to show that $y^N = x^N$.

First, from (5.15),

$$y^{N\backslash j} = x^{N\backslash j} \qquad \text{for all } j \in N\backslash\bar{k}. \tag{5.17}$$

And by Lemma 4.5.6,

$$x^N_i - x^N_j = x^{N\backslash j}_i - x^{N\backslash i}_j \qquad \text{for all } i, j \in N,\ i \neq j, \tag{5.18}$$

$$y^N_i - y^N_j = y^{N\backslash j}_i - y^{N\backslash i}_j \qquad \text{for all } i, j \in N,\ i \neq j. \tag{5.19}$$

Combining (5.17) with (5.18) and (5.19) for the case $i = \bar{k}$ yields

$$x^N_{\bar{k}} - x^N_j + x^{N\backslash\bar{k}}_j = y^N_{\bar{k}} - y^N_j + y^{N\backslash\bar{k}}_j \quad \text{for all } j \in N\backslash\bar{k}, \tag{5.20}$$

and for the case $i \neq \bar{k}$, yields

$$x^N_i - x^N_j = y^N_i - y^N_j \qquad \text{for all } i, j \in N\backslash\bar{k}. \tag{5.21}$$

From (5.20), it follows that

$$\big(|N| - 1\big)x^N_{\bar{k}} - \sum_{j\in N\backslash\bar{k}} x^N_j + \sum_{j\in N\backslash\bar{k}} x^{N\backslash\bar{k}}_j = \big(|N| - 1\big)y^N_{\bar{k}} - \sum_{j\in N\backslash\bar{k}} y^N_j + \sum_{j\in N\backslash\bar{k}} y^{N\backslash\bar{k}}_j,$$

or equivalently,

$$|N|x^N_{\bar{k}} - \sum_{j\in N} x^N_j + \sum_{j\in N\backslash\bar{k}} x^{N\backslash\bar{k}}_j = |N|y^N_{\bar{k}} - \sum_{j\in N} y^N_j + \sum_{j\in N\backslash\bar{k}} y^{N\backslash\bar{k}}_j. \tag{5.22}$$

Now EFF yields $\sum_{j\in N} x^N_j = \sum_{j\in N} y^N_j$ and $\sum_{j\in N\backslash\bar{k}} x^{N\backslash\bar{k}}_j = \sum_{j\in N\backslash\bar{k}} y^{N\backslash\bar{k}}_j$. Hence (5.22) yields

$$y^N_{\bar{k}} = x^N_{\bar{k}}. \tag{5.23}$$

Furthermore, EFF together with equalities (5.21) and (5.23) yield $y_i^N = x_i^N$ for all $i \in N$, that is

$$y^N = x^N. \tag{5.24}$$

as claimed. □

Proposition 4.5.10. *Let* Ψ *satisfy EFF, RATIO*, TINV, ICOD, MON*, AN, NP and CONS. Let* (N,V) *be a game in* $\mathcal{G}$ *for which* $\Psi(N,V) = \mathbf{x}$. *Let* $T \subset N$ *be a fixed coalition. Consider the NTU game* (N,W) *defined by*

$$W(S) = \begin{cases} \{x \in I\!R^T : \sum_{i \in T} x_i \le \sum_{i \in T} x_i^T\}, & \textit{if } S = T; \\ V(S), & \textit{otherwise.} \end{cases}$$

Then $\Psi(N,W) = \Psi(N,V)$.

Demostración. Induction is now used on $|N \backslash T|$, where the case $|N \backslash T| = 1$ is Proposition 4.5.9 above.

Let T be a fixed coalition such that $|N \backslash T| > 1$. In view of the induction hypothesis it can be assumed (to avoid further notation), that $V(S)$ is already TU for every coalition S such that $|S| > |T|$; that is, it can be assumed that $V(S) = \left\{x \in \mathrm{I\!R}^S : \sum_{j \in S} x_j \le \sum_{j \in S} x_j^S\right\}$ whenever $|S| > |T|$.

Consider the $(|T|)$-dimensional simplex $\Delta^{|T|-1} = \left\{\lambda \in \mathrm{I\!R}^n_+; \sum_{i=1}^{|T|} \lambda_i = 1\right\}$. For each ε, $0 < \varepsilon < 1/|T|$, let Λ_ε be the subset of $\Delta^{|T|-1}$ described as follows

$\Lambda_\varepsilon = \left\{\lambda \in \Delta^{|T|-1}; \lambda_i \ge \varepsilon, \forall\, i = 1, ..., |T|.\right\}$, and consider the NTU game (N, V_ε) defined by

$$V_\varepsilon(S) = \begin{cases} \left\{x \in \mathrm{I\!R}^T; \lambda \cdot x \le \lambda \cdot x^T, \ \forall\, \lambda \in \Lambda_\varepsilon\right\}, & \text{if } S = T; \\ V(S), & \text{otherwise.} \end{cases}$$

Since $V(T)$ is uniformly non-leveled, there exists $\varepsilon^* > 0$, $0 < \varepsilon^* < 1/|T|$, such that $V_{\varepsilon^*}(T) \subset V(T)$ (see the proof of Theorem 4.4.1). Then, if we denote $\Psi(N, V_{\varepsilon^*}) = \mathbf{z}$, through EFF ICOD and MON*,

$$z^S = x^S \qquad \text{if } T \subset S,\, T \ne S \text{ does not hold, and} \tag{5.25}$$

$$z_T^S \leq x_T^S, \quad \text{being} \sum_{i\in S} z_i^S = \sum_{i\in S} x_i^S, \text{ if } T \subset S. \tag{5.26}$$

Notice also that $V_{\varepsilon^*}(T) \subset W(T)$ and $V_{\varepsilon^*}(S) = W(S)$ for all $S \neq T$. Then, if we denote $\Psi(N,W) = \mathbf{y}$, through EFF, ICOD and MON*, (5.25) and (5.26),

$$y^S = x^S \qquad \text{if } T \subset S,\, T \neq S \text{ does not hold, and} \tag{5.27}$$

$$z_T^S \leq y_T^S, \text{ being} \sum_{i\in S} x_i^S = \sum_{i\in S} y_i^S = \sum_{i\in S} z_i^S, \text{ if } T \subset S. \tag{5.28}$$

To prove that that $\mathbf{y} = \mathbf{x}$ it is sufficient to show that $y^{T\cup R} = x^{T\cup R}$ for every $R \subset N\backslash T$. This will be proved in two steps.

Step 1. $y_R^{T\cup R} = x_R^{T\cup R}$ for every $R \subset N\backslash T$, $R \neq \emptyset$:

The proof proceeds by induction on the cardinality of R. Assume first that $R = \{\bar{k}\}$ for certain $\bar{k} \in N\backslash T$.

For any $j \in T$, consider the reduced games $\big((N\backslash T) \cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ and $\big((N\backslash T) \cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$.

Since $V(S) = \big\{x \in \mathbb{R}^S : \sum_{j\in S} x_j \leq \sum_{j\in S} x_j^S\big\}$ whenever $|S| > |T|$, the above reduced games are both TU games. Then, by Proposition 4.5.7 and CONS axiom,

$$\Psi\big((N\backslash T) \cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \mathcal{E}\big((N\backslash T) \cup j, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \Big(x^{S\cup(T\backslash j)}\Big)_{S\subset(N\backslash T)\cup j}$$

$$\Psi\big((N\backslash T) \cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \mathcal{E}\big((N\backslash T) \cup j, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \Big(y^{S\cup(T\backslash j)}\Big)_{S\subset(N\backslash T)\cup j}.$$

Now consider the subgames $\big(\{\bar{k},j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big)$ and $\big(\{\bar{k},j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big)$, and assume that

$$\Psi\big(\{\bar{k},j\}, V^{(N\backslash T)\cup j,\mathbf{x}}\big) = \boldsymbol{a} \quad \text{and} \quad \Psi\big(\{\bar{k},j\}, W^{(N\backslash T)\cup j,\mathbf{y}}\big) = \boldsymbol{b}.$$

By (4.5),

$$\begin{array}{lll} a^{\{\bar{k},j\}} = (x_{\bar{k}}^{T\cup\bar{k}}, x_j^{T\cup\bar{k}}) & b^{\{\bar{k},j\}} = (y_{\bar{k}}^{T\cup\bar{k}}, y_j^{T\cup\bar{k}}) & \\ a^{\{\bar{k}\}} = x_{\bar{k}}^{(T\cup\bar{k})\backslash j} & b^{\{\bar{k}\}} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j} & \text{for all } j \in T. \\ a^{\{j\}} = x_j^T & b^{\{j\}} = y_j^T & \end{array}$$

Consequently, it follows on from Lemma 4.5.3 that

$$x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}} = x_{\bar{k}}^{(T\cup\bar{k})\backslash j} - x_j^T \qquad \text{for all } j \in T, \tag{5.29}$$

$$y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j} - y_j^T \qquad \text{for all } j \in T. \tag{5.30}$$

Furthermore, ICOD implies $x^S = y^S$ for any coalition $S \subset N\backslash j$. In particular $x_{\bar{k}}^{(T\cup\bar{k})\backslash j} = y_{\bar{k}}^{(T\cup\bar{k})\backslash j}$, and hence, from (5.29) and (5.30), it can be concluded that

$$(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - (y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = y_j^T - x_j^T \qquad \text{for all } j \in T. \tag{5.31}$$

Therefore, $\sum_{j\in T}(x_{\bar{k}}^{T\cup\bar{k}} - x_j^{T\cup\bar{k}}) - \sum_{j\in T}(y_{\bar{k}}^{T\cup\bar{k}} - y_j^{T\cup\bar{k}}) = \sum_{j\in T} y_j^T - \sum_{j\in T} x_j^T$.

Through EFF axiom, $\sum_{j\in T} y_j^T = \sum_{j\in T} x_j^T$ and $\sum_{j\in T\cup\bar{k}} x_j^{T\cup\bar{k}} = \sum_{j\in T\cup\bar{k}} y_j^{T\cup\bar{k}}$. Hence, $\big(|T|+1\big)(x_{\bar{k}}^{T\cup\bar{k}} - y_{\bar{k}}^{T\cup\bar{k}}) = 0$, which implies

$$x_{\bar{k}}^{T\cup\bar{k}} = y_{\bar{k}}^{T\cup\bar{k}}. \tag{5.32}$$

Suppose, by hypothesis of induction, that $y_R^{T\cup R} = x_R^{T\cup R}$ whenever $|R| < l$. Consider $R \subset N\backslash T$, $|R| = l \geq 2$, and let $\bar{k}$ be an arbitrary agent in R.

It is needed to prove $y_{\bar{k}}^{T\cup R} = x_{\bar{k}}^{T\cup R}$. To do that, consider, for any $j \in (T\cup R)\backslash\bar{k}$, the reduced games

$$\big(N\backslash(T\cup R)\cup\{\bar{k},j\}, V^{N\backslash(T\cup R)\cup\{\bar{k},j\},\mathbf{x}}\big), \text{ and}$$

$$\big(N\backslash(T\cup R)\cup\{\bar{k},j\}, W^{N\backslash(T\cup R)\cup\{\bar{k},j\},\mathbf{y}}\big)$$

Again, since $V(S) = \big\{x \in \mathbb{R}^S : \sum_{j\in S} x_j \leq \sum_{j\in S} x_j^S\big\}$ whenever $|S| > |T|$, the above reduced games are both TU games. Then, by Proposition 4.5.7, Ψ provides the egalitarian configuration value, and CONS axiom yields

$$\Psi\big(N\backslash(T\cup R)\cup\{\bar{k},j\}, V^{N\backslash(T\cup R)\cup\{\bar{k},j\},\mathbf{x}}\big) = \Big(x^{S\cup(T\cup R)\backslash\{\bar{k},j\}}\Big)_{S\subset N\backslash(T\cup R)\cup\{\bar{k},j\}}$$
$$\Psi\big(N\backslash(T\cup R)\cup\{\bar{k},j\}, W^{N\backslash(T\cup R)\cup\{\bar{k},j\},\mathbf{y}}\big) = \Big(y^{S\cup(T\cup R)\backslash\{\bar{k},j\}}\Big)_{S\subset N\backslash(T\cup R)\cup\{\bar{k},j\}}.$$

Therefore, by considering the two person subgames $\big(\{\bar{k},j\}, V^{(N\backslash(T\cup R))\cup\{\bar{k},j\},\mathbf{x}}\big)$ and $\big(\{\bar{k},j\}, W^{(N\backslash(T\cup R))\cup\{\bar{k},j\},\mathbf{y}}\big)$, and taking into account (4.5) and Lemma 4.5.3 analogous expressions to that in (5.29) and (5.30) we can be obtained:

$$x_{\bar{k}}^{T\cup R} - x_j^{T\cup R} = x_{\bar{k}}^{(T\cup R)\backslash j} - x_j^{(T\cup R)\backslash\bar{k}} \qquad \text{for all } j \in (T\cup R)\backslash\bar{k}, \tag{5.33}$$
$$y_{\bar{k}}^{T\cup R} - y_j^{T\cup R} = y_{\bar{k}}^{(T\cup R)\backslash j} - y_j^{(T\cup R)\backslash\bar{k}} \qquad \text{for all } j \in (T\cup R)\backslash\bar{k}. \tag{5.34}$$

.

Notice that $x_{\bar{k}}^{(T\cup R)\backslash j} = y_{\bar{k}}^{(T\cup R)\backslash j}$ for every $j \in (T\cup R)\backslash\bar{k}$ because, when $j \in T$, the equality holds through the ICOD axiom, and when $j \in R$, $x_{R\backslash j}^{T\cup(R\backslash j)} = y_{R\backslash j}^{T\cup(R\backslash j)}$ by hipothesis of induction ($|R\backslash j| < l = |R|$).

By taking this equality into account, it can be concluded from (5.33) and (5.34) that

$$\sum_{j\in(T\cup R)\backslash\bar{k}} (x_{\bar{k}}^{T\cup R} - x_j^{T\cup R}) - \sum_{j\in(T\cup R)\backslash\bar{k}} (y_{\bar{k}}^{T\cup R} - y_j^{T\cup R}) = \sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} - \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}.$$

Through EFF axiom, the equalities $\sum_{j\in(T\cup R)\backslash\bar{k}} y_j^{(T\cup R)\backslash\bar{k}} = \sum_{j\in(T\cup R)\backslash\bar{k}} x_j^{(T\cup R)\backslash\bar{k}}$ and $\sum_{j\in(T\cup R)} y_j^{(T\cup R)} = \sum_{j\in(T\cup R)} x_j^{(T\cup R)}$ holds. Hence $|T\cup R|\left(x_{\bar{k}}^{T\cup R} - y_{\bar{k}}^{T\cup R}\right) = 0$, which implies $x_{\bar{k}}^{T\cup R} = y_{\bar{k}}^{T\cup R}$. Therefore, $x_R^{T\cup R} = y_R^{T\cup R}$ for every coalition $R \subset N\backslash T$.

Step 2. $y_T^{T\cup R} = x_T^{T\cup R}$ for every $R \subset N\backslash T$, $R \neq \emptyset$:

Chosen $R \subset N\backslash T$, $R \neq \emptyset$, notice that $x_R^{S\cup R} = y_R^{S\cup R}$ holds for every $S \subset N$: if $T \subset S$, then the equality is an immediate consequence of Step 1 above, and if $T\backslash S \neq \emptyset$, then the equality is deduced through the ICOD axiom.

As a consequence, the equality $(N\backslash R, V^{N\backslash R,\mathbf{x}}) = (N\backslash R, W^{N\backslash R,\mathbf{y}})$ holds. Then, through CONS axiom, $x_S^{S\cup R} = y_S^{S\cup R}$ for every $S \subset N\backslash R$. In particular $x_T^{T\cup R} = y_T^{T\cup R}$ as required. □

Proposition 4.5.11. *If Ψ satisfies EFF, RATIO*, TINV, ICOD, MON*, AN, NP and CONS, then $\Psi = \mathcal{E}$ on $\mathcal{G}$.*

Demostración. Let (N,V) be an NTU game in $\mathcal{G}$. Denote $\Psi(N,V) = \mathbf{x}$, and consider the TU game $(N, v_{\mathbf{x}})$, whose characteristic function is defined by $v_{\mathbf{x}}(T) = \sum_{i\in T} x_i^T$ for every coalition $T \subset N$. Therefore, Proposition 4.5.7 and (4.5) give $\Psi(N, v_{\mathbf{x}}) = \mathcal{E}(N, v_{\mathbf{x}}) = \mathcal{E}(N,V)$. On the other hand, Proposition 4.5.10, applied recursively, yields $\Psi(N, v_{\mathbf{x}}) = \Psi(N,V)$, and the result follows. □

The proof of Theorem 4.4.5 is consequence of Propositions 4.5.1 and 4.5.11. □

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