

Article

Displacement Calculus

Ignacio Márquez Albés  and F. Adrián F. Tojo * 

Instituto de Matemáticas, Universidade de Santiago de Compostela, 15705 Santiago de Compostela, Spain; ignacio.marquez@usc.es

* Correspondence: fernandoadrian.fernandez@usc.es

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Abstract: In this work, we establish a theory of Calculus based on the new concept of *displacement*. We develop all the concepts and results necessary to go from the definition to differential equations, starting with topology and measure and moving on to differentiation and integration. We find interesting notions on the way, such as the integral with respect to a path of measures or the displacement derivative. We relate both of these two concepts by a Fundamental Theorem of Calculus. Finally, we develop the necessary framework in order to study displacement equations by relating them to Stieltjes differential equations.

Keywords: displacement; ordinary differential equation; fundamental theorem of calculus; Stieltjes differentiation

MSC: 28A; 34A; 54E

1. Introduction

Derivatives are, in the classical sense of Newton [1], infinitesimal rates of change of one (dependent) variable with respect to another (independent) variable. Formally, the derivative of f with respect to x is

$$f'(x) := \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}.$$

The symbol Δ represents what we call the *variation*, that is, the change of magnitude underwent by a given variable in the *calculus of variations*, the study variable is a function f and the variation of f is noted by δf . This variation is, in the classical setting, defined in the most simple possible way as $\Delta x = \tilde{x} - x$, where x is the point at which we want to compute the derivative (the point of departure) and \tilde{x} another point which we assume close enough to x . From this, it follows naturally that the variation of the dependent variable has to be expressed as $\Delta f = f(\tilde{x}) - f(x)$. This way, when \tilde{x} tends to x , that is, when Δx tends to zero, we have

$$f'(x) := \lim_{\tilde{x} \rightarrow x} \frac{f(\tilde{x}) - f(x)}{\tilde{x} - x}.$$

Of course, this naïve way of defining the variation is by no means the unique way of giving meaning to such expression. The intuitive idea of variation is naturally linked to the mathematical concept of distance. After all, in order to measure how much a quantity has varied, it is enough to see *how far apart* the new point \tilde{x} is from the first x , that is, we have to measure, in some sense, the distance between them. This manner of extending the notion of variation—and thus of derivative—has been accomplished in different ways. The most crude of these is what is called the *absolute derivative*.

Definition 1 ([2] expression (1)). Let (X, d_X) and (Y, d_Y) be two metric spaces and consider $f : X \rightarrow Y$ and $x \in X$. We say f is absolutely differentiable at x if and only if the following limit—called absolute derivative of f at x —exists:

$$f^{| \cdot |}(x) := \lim_{\tilde{x} \rightarrow x} \frac{d_Y(f(x), f(\tilde{x}))}{d_X(x, \tilde{x})}.$$

In the case of differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, we have that, as expected, $f^{| \cdot |} = |f'|$ [2] (Proposition 3.1). Hence, this result conveys the true meaning of the absolute derivative—it is the absolute value of the derivative—and it extends the notion of derivative to the broader setting of metric spaces. Even so, this definition may seem somewhat unfulfilling as a generalization. For instance, in the case of the real line, it does not preserve the spirit of the intuitive notion of ‘infinitesimal rates of change’: changes of rate have, of necessity, to be allowed to be *negative*.

A more subtle extension of differentiability to the realm of metric spaces can be achieved through *mutational analysis* where the affine structure of differentials is changed by a family of functions, called *mutations* that mimic the properties and behavior of derivatives. We refer the reader to [3] for more information on the subject.

The considerations above bring us to another possible extension of the notion of derivative: that of the *Stieltjes derivative*, also known as *g-derivative*. Here, we present the definition used in [4]. However, a lot of previous work exists on the topic of differentiation with respect to a function, such as the work Averna and Preiss, [5], Daniell [6,7], or even more classical references like [8].

Definition 2 ([4]). Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone nondecreasing function which is continuous from the left. The Stieltjes derivative with respect to g —or *g-derivative*—of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ at a point $x \in \mathbb{R}$ is defined as follows, provided that the corresponding limits exist:

$$f'_g(x) = \lim_{\tilde{x} \rightarrow x} \frac{f(\tilde{x}) - f(x)}{g(\tilde{x}) - g(x)} \text{ if } g \text{ is continuous at } x, \text{ or}$$

$$f'_g(x) = \lim_{\tilde{x} \rightarrow x^+} \frac{f(\tilde{x}) - f(x)}{g(\tilde{x}) - g(x)} \text{ if } g \text{ is discontinuous at } x.$$

Clearly, we have defined Δx through a rescaling of the abscissae axis by g . Observe that, although $d(x, \tilde{x}) = |g(\tilde{x}) - g(x)|$ is a pseudometric [9], $\Delta x = g(\tilde{x}) - g(x)$ is allowed to change sign.

The aim of this paper is to take this generalization one step further in the following sense. The definition of Δx does not have to depend on a rescaling, but its absolute value definitely has to suggest, in a broad sense, the notion, if not of distance, of being *far apart* or *close* as well as the *direction*—change of sign. This is why we introduce the notion of *displacement* (Definition 3). This definition takes to full generality the ideas and results in [4,9].

This work is structured as follows. In Section 2, we define the basic concept the rest of the paper revolves around: the notion of *displacement space*. Specifically, in Section 2.1, we develop the definition and basic properties of displacements, linking them to previously known concepts and illustrating their diversity with several examples. On the other hand, in Section 2.2, we endow the displacement space with a natural topology and prove various useful properties.

Section 3 deals with the construction of a measure associated with displacement spaces. We restrict ourselves to the real line. Then, we construct a theory of integration for displacement spaces. Here, we define the concept of *integral with respect to a path of measures* which will be the key to defining an integral associated with a displacement.

Section 4 is devoted to the definition and properties of a displacement derivative which will be later be proven to be compatible with the displacement measure in that we can provide a Fundamental Theorem of Calculus relating both of them (Theorems 1 and 2). Later, in Section 5, we study the connection existing between this type of derivatives and Stieltjes derivatives and, in Section 6, we propose a diffusion model on smart surfaces based on displacements.

The last section is devoted to the conclusions of this work and the open problems lying ahead.

2. Displacement Spaces

In this section, we focus on the definition of displacement spaces. This new framework is then illustrated with some examples, which show, for example, that every set equipped with a metric map is a displacement space. We also study a topological structure that displacement spaces can be endowed with.

2.1. Definitions and Properties

Let us make explicit the basic definition of this paper.

Definition 3. Let $X \neq \emptyset$ be a set. A displacement is a function $\Delta : X^2 \rightarrow \mathbb{R}$ such that the following properties hold:

(H1) $\Delta(x, x) = 0, x \in X.$

(H2) For all $x, y \in X,$

$$\lim_{z \rightarrow y} |\Delta(x, z)| = |\Delta(x, y)|,$$

where

$$\lim_{z \rightarrow y} |\Delta(x, z)| := \sup \left\{ \liminf_{n \rightarrow \infty} |\Delta(x, z_n)| : (z_n)_{n \in \mathbb{N}} \subset X, \Delta(y, z_n) \xrightarrow{n \rightarrow \infty} 0 \right\}.$$

All limits occurring in this work will be considered with the usual topology of \mathbb{R} . A pair (X, Δ) is called a displacement space.

Remark 1. Why (H1) and (H2)? These two hypotheses are of prominent topological flavor. (H1) will guarantee that open balls are nonempty in the to-be-defined non-necessarily-metric topology related to Δ . On the other hand, (H2) will be sufficient (and indeed necessary) to show that open balls are, indeed, open (Lemma 3) and that the Δ -topology is second countable (Lemma 5). We will later discuss (Remark 4) whether or not we can forestall (H2) when we restrict ourselves to displacement calculus.

Remark 2. Note that, for (H2) to be satisfied, it is enough to show that $\lim_{z \rightarrow y} |\Delta(x, z)| \leq |\Delta(x, y)|$ for all $x, y \in X,$ as the reverse inequality always holds.

The following lemma gives a useful sufficient condition for (H2) to be satisfied.

Lemma 1. Let X be a set and $\Delta : X^2 \rightarrow \mathbb{R}.$ Assume that the following property holds:

(H2') There exists a strictly increasing left-continuous map $\varphi : [0, +\infty) \rightarrow [0, +\infty),$ continuous at 0, satisfying $\varphi(0) = 0$ and such that, for $\psi(x, y) := \varphi(|\Delta(x, y)|),$

$$\psi(x, z) \leq \psi(x, y) + \psi(y, z); \quad x, y, z \in X. \tag{1}$$

Then, Δ satisfies (H2).

Proof. Fix $x, y \in X$ and let $(z_n)_{n \in \mathbb{N}} \subset X$ such that $\lim_{n \rightarrow \infty} |\Delta(y, z_n)| = 0.$ Then, condition (1) yields $\psi(x, z_n) - \psi(y, z_n) \leq \psi(x, y), n \in \mathbb{N}$ Hence,

$$\psi(x, y) \geq \liminf_{n \rightarrow \infty} (\psi(x, z_n) - \psi(y, z_n)) \geq \liminf_{n \rightarrow \infty} \psi(x, z_n) - \limsup_{n \rightarrow \infty} \psi(y, z_n).$$

Since $\varphi(0) = 0, \varphi$ is continuous at 0 and $\lim_{n \rightarrow \infty} |\Delta(y, z_n)| = 0, \limsup_{n \rightarrow \infty} \psi(y, z_n) = 0,$ so

$$\psi(x, y) \geq \liminf_{n \rightarrow \infty} \psi(x, z_n). \tag{2}$$

Let us show that

$$\liminf_{n \rightarrow \infty} \varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n \rightarrow \infty} |\Delta(x, z_n)|\right). \tag{3}$$

Indeed, by definition of \liminf , we have that, for any $\varepsilon \in \mathbb{R}^+$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $|\Delta(x, z_n)| \geq \liminf_{n \rightarrow \infty} |\Delta(x, z_n)| - \varepsilon$. Since φ is strictly increasing, for each $\varepsilon \in \mathbb{R}^+$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n \rightarrow \infty} |\Delta(x, z_n)| - \varepsilon\right).$$

Thus, for any $\varepsilon > 0$, we have that

$$\liminf_{n \rightarrow \infty} \varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n \rightarrow \infty} |\Delta(x, z_n)| - \varepsilon\right),$$

which, using the left-continuity of φ , leads to inequality (3). Hence, it follows from inequalities (2) and (3) that

$$\varphi(|\Delta(x, y)|) = \psi(x, y) \geq \liminf_{n \rightarrow \infty} \psi(x, z_n) = \liminf_{n \rightarrow \infty} \varphi(|\Delta(x, z_n)|) \geq \varphi\left(\liminf_{n \rightarrow \infty} |\Delta(x, z_n)|\right),$$

which, together with the fact that φ is strictly increasing, yields that

$$|\Delta(x, y)| \geq \liminf_{n \rightarrow \infty} |\Delta(x, z_n)|.$$

Since this holds for any $(z_n)_{n \in \mathbb{N}}$ in X such that $\lim_{n \rightarrow \infty} |\Delta(y, z_n)| = 0$, we have the desired result. \square

Lemma 1 illustrates that condition (H2) is a way of avoiding the triangle inequality—or more general versions of it—which is common to metrics and analogous objects. We can find similar conditions in the literature. For instance, in [10] (Definition 3.1), they use, while defining an *RS-generalized metric space* (X, Δ) , the condition

(D₃') There exists $C > 0$ such that if $x, y \in X$ and

$$\lim_{n \rightarrow \infty} \Delta(x_n, x) = \lim_{n \rightarrow \infty} \Delta(x, x_n) = \lim_{n, m \rightarrow +\infty} \Delta(x_n, x_m) = 0,$$

then

$$\Delta(x, y) \leq C \limsup \Delta(x_n, y).$$

More complicated conditions can be found in [3] ((H3) Section 3.1, (H3') Section 4.1).

Finally, we remark that the same statement as (H2'), but dropping the left-continuity, is actually sufficient to prove the results in this work.

In the next examples, we use the sufficient condition provided by Lemma 1.

Example 1. Consider the sphere \mathbb{S}^1 and define the following map:

$$\begin{aligned} \mathbb{S}^1 \times \mathbb{S}^1 &\xrightarrow{\Delta} [0, 2\pi) \\ (x, y) &\longmapsto \min\{\theta \in [0, +\infty) : xe^{i\theta} = y\}. \end{aligned}$$

$\Delta(x, y)$ is a displacement that measures the minimum counter-clockwise angle necessary to move from x to y . From a real-life point of view, this map describes the way cars move in a roundabout. Suppose that a car enters the roundabout at a point x and wants to exit at a point y . In that case, circulation rules force the car to move in a given direction, which happens to be counterclockwise in most of the countries around the world. In this case, drivers are assumed to take the exit y as soon as they reach it.

It is clear that (H1) holds. For (H2'), take $\varphi(r) = r$. Then, for $x, y, z \in \mathbb{S}^1$, if $\Delta(x, y) + \Delta(y, z) \geq 2\pi$, then (H2') clearly holds. Otherwise, $\Delta(x, z) = \Delta(x, y) + \Delta(y, z)$, so (H2') holds.

Example 2. Let (X, E) be a complete weighted directed graph, that is, $X = \{x_1, \dots, x_n\}$ is a finite set of $n \in \mathbb{N}$ vertices and $E \in \mathcal{M}_n(\mathbb{R})$ is a matrix with zeros in the diagonal and positive numbers everywhere else. The element $e_{j,k}$ of the matrix E denotes the weight of the directed edge from vertex x_j to vertex x_k . This kind of graph can represent, for instance, the time it takes to get from one point in a city to another by car, as Figure 1 illustrates.

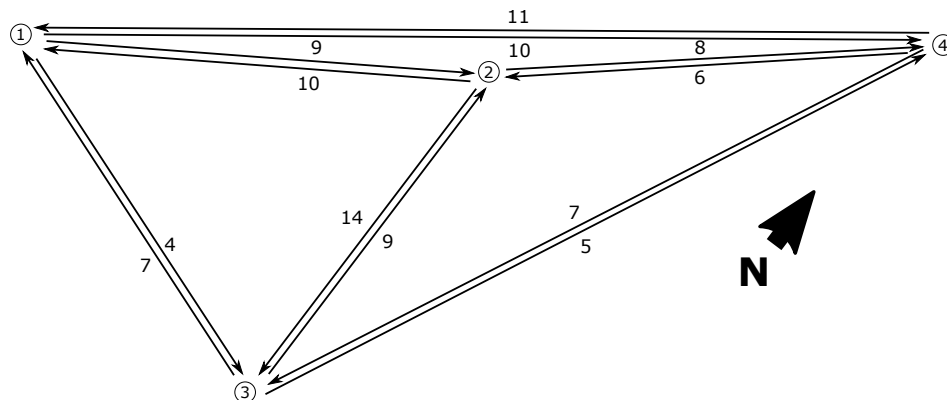


Figure 1. Graph indicating the time in minutes it takes to go from one place to another in Santiago de Compostela by car (using the least time consuming path) according to Google Maps—good traffic conditions assumed. The points are placed in their actual relative geometric positions, being 1: Faculty of Mathematics (USC), 2: Cathedral, 3: Train station, 4: Bus station. Most of the streets in Santiago are one way, which accounts for the differences in time depending on the direction of the displacement.

Now, consider the set $\{x_1, \dots, x_4\}$ and the matrix E as given in Figure 1, that is,

$$E \equiv (e_{j,k})_{j,k=1}^4 := \begin{pmatrix} 0 & 9 & 4 & 10 \\ 10 & 0 & 14 & 8 \\ 7 & 9 & 0 & 5 \\ 11 & 6 & 7 & 0 \end{pmatrix},$$

and the map $\Delta(x_j, x_k) := e_{j,k}$. It can be checked that Δ is subadditive—which is to be expected since, if we could get faster from a point to another through a third one, Google Maps would have chosen that option. Hence, (H2') holds for $\varphi(r) = r$, and so Δ is a displacement.

Example 3 (Zermelo’s navigation problem). In 1931, Zermelo solved the following navigation problem [11]. Let $F = (u, v) \in C(\mathbb{R}^2, \mathbb{R}^2)$ be a vector field, for instance, the velocity field of the wind on top of a body of water, or the velocity field of the water itself. Assume an object that moves with constant celerity V on that body of water wants to go from a point A (which we can assume at the origin) to a point B . Which is the least time consuming path to take?

We are going to assume that $V > W := \max_{x \in \mathbb{R}^2} \sqrt{u(x)^2 + v(x)^2}$, that is, the object can navigate against wind. Zermelo proved, using variational methods, that the solution of the problem satisfies the following system of partial differential equations:

$$\begin{aligned} x' &= V \cos \theta + u, \\ y' &= V \sin \theta + v, \\ \theta' &= \sin^2 \theta \frac{\partial v}{\partial x} + \sin \theta \cos \theta \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - \cos^2 \theta \frac{\partial u}{\partial y}, \end{aligned}$$

being the last equation known as Zermelo’s equation. Observe that, if $u, v \in C^2(\mathbb{R}^2)$, there exists a unique solution of the system. Through the change of variables $(\tilde{x}, \tilde{y}) = B - (x, y)$, instead of going from the origin to the point B , we go from B to the origin, and the equations will provide a different time. This illustrates the fact that, when measuring how far apart something is in terms of time, symmetry is not generally satisfied. For instance, if we measure the distance between two points of a river by the time it takes to get from one point to another, it is not the same to go upstream than downstream.

Let $A, B \in \mathbb{R}^2$. If $\Delta(A, B)$ is the smallest time necessary to arrive from A to B in Zermelo’s navigation problem, $\Delta \geq 0$ is a displacement on \mathbb{R}^2 , for Δ is subadditive and (H1)–(H2’) are clearly satisfied.

In the symmetric setting—that is, $\Delta(x, y) = \Delta(y, x)$ —this problem is a paradigmatic example of Finslerian length space. The theory regarding these spaces has been thoroughly developed, but, as stated in [12], although “one could modify the definitions to allow non-symmetric length structures and metrics”, this case has not been studied yet. What we present in this paper might be an starting point for a theory of non-symmetric length spaces.

Example 4. The map $\Delta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined as $\Delta(x, y) = g(y) - g(x)$ for a nondecreasing and left-continuous function g (cf.[4]) is a displacement as it satisfies (H2’) for $\varphi = \text{Id}$. In what follows, we will refer to these displacements as Stieltjes displacements. Furthermore, the following lemma shows a way to ensure that a displacement is a Stieltjes displacement.

Lemma 2. Let (X, Δ) be a displacement space. Then, there exists $g : X \rightarrow \mathbb{R}$ such that $\Delta(x, y) = g(y) - g(x)$ for every $x, y \in X$ if and only if, for every $x, y, z \in X$,

1. $\Delta(x, y) = -\Delta(y, x)$,
2. $\Delta(x, z) = \Delta(x, y) + \Delta(y, z)$.

Proof. Necessity is straightforward. In order to prove sufficiency, take $x_0 \in X$ and define $g(x) = \Delta(x_0, x)$ for $x \in X$. Then,

$$\Delta(x, y) = \Delta(x, x_0) + \Delta(x_0, y) = -\Delta(x_0, x) + g(y) = g(y) - g(x). \quad \square$$

For the interest of the work ahead, we include the following example of a non-Stieltjes displacement.

Example 5. Let $X = [0, 1]$ and $\Delta : X^2 \times X^2 \rightarrow [0, +\infty)$ given by

$$\Delta(x, y) = e^{y^2 - x^2} - e^{x - y}.$$

Clearly, condition (H1) is satisfied. Now, for condition (H2), fix $x, y \in X$. Since $\Delta(x, \cdot)$ is a continuous function, it is enough to show that

$$|\Delta(x, y)| = \sup \left\{ \lim_{n \rightarrow \infty} |\Delta(x, z_n)| : (z_n)_{n \in \mathbb{N}} \subset X, \lim_{n \rightarrow \infty} |\Delta(y, z_n)| = 0 \right\}. \quad (4)$$

Let $(z_n)_{n \in \mathbb{N}} \subset X$ be a sequence such that $|\Delta(y, z_n)| \xrightarrow{n \rightarrow \infty} 0$. Then, we have that $e^{z_n^2 + z_n} \xrightarrow{n \rightarrow \infty} e^{y^2 - y}$, from which we get that

$$\lim_{n \rightarrow \infty} z_n^2 + z_n = y^2 + y. \quad (5)$$

Define

$$h(t) = \frac{-1 + \sqrt{1 + 4t}}{2}, \quad t \in [0, +\infty).$$

For any $a \in [0, 1]$, we have that $h(a^2 + a) = a$. Therefore, applying h to both sides of expression (5) and noting that h is a continuous function, we obtain

$$y = h(y^2 + y) = h\left(\lim_{n \rightarrow \infty} z_n^2 + z_n\right) = \lim_{n \rightarrow \infty} h(z_n^2 + z_n) = \lim_{n \rightarrow \infty} z_n.$$

That is, if $(z_n)_{n \in \mathbb{N}} \subset X$ is such that $|\Delta(y, z_n)| \xrightarrow{n \rightarrow \infty} 0$, it follows that $(z_n) \xrightarrow{n \rightarrow \infty} y$. Hence, equation (4) is trivially satisfied, and thus Δ is a displacement. However, Δ is not a Stieltjes displacement as

$$\Delta(1/2, 0) = e^{-1/4} - e^{1/2} \neq e^{-1/2} - e^{1/4} = -\Delta(0, 1/2).$$

2.2. Displacement Topologies

It is a well-known result that a set equipped with a metric map generates a topology through the definition of open balls. The same thing happens with displacement spaces. However, fewer nice properties can be obtained from just the definition.

Definition 4. Given a displacement space $\Delta : X^2 \rightarrow \mathbb{R}$, $x \in X$ and $r \in \mathbb{R}^+$, we define the Δ -ball or simply ball) of center x and radius r as

$$B_\Delta(x, r) := \{y \in X : |\Delta(x, y)| < r\}.$$

In addition, we define the Δ -topology in the following way:

$$\tau_\Delta := \{U \subset X : \forall x \in U \exists r \in \mathbb{R}^+, B_\Delta(x, r) \subset U\}.$$

Clearly, τ_Δ is a topology. We denote by \mathcal{E}_Δ the set of Δ -balls in X and by τ_u the usual euclidean topology of \mathbb{R}^n for any $n \in \mathbb{N}$.

We recall the following definition.

Definition 5. Let τ_s be the topology generated by the intervals $(-\infty, r) \subset \mathbb{R}$ and (X, τ) a topological space. We say $f : X \rightarrow \mathbb{R}$ is upper-semicontinuous if $f : (X, \tau) \rightarrow (\mathbb{R}, \tau_s)$ is continuous.

In what follows, we will write $\Delta_x(y) := \Delta(x, y)$.

Lemma 3. Let $\Delta : X \times X \rightarrow \mathbb{R}$. Then, the following are equivalent:

1. Δ satisfies (H2).
2. $B_\Delta(x, r) \in \tau_\Delta$ for all $x \in X$ and $r \in \mathbb{R}^+$.
3. For all $x \in X$, $|\Delta_x| : X \rightarrow \mathbb{R}$ is upper-semicontinuous.

Proof. 1 \Rightarrow 2. Assume first that (X, Δ) satisfies (H2). Let $x \in X$ and $r \in \mathbb{R}^+$ be fixed. If $B_\Delta(x, r) = \emptyset$, then $B_\Delta(x, r) \in \tau_\Delta$ trivially. Assume that $B_\Delta(x, r) \neq \emptyset$. Let us show that, for every $y \in B_\Delta(x, r)$, there exists $\varepsilon \in \mathbb{R}^+$ such that $B_\Delta(y, \varepsilon) \subset B_\Delta(x, r)$. Assume this is not the case. Then, there exists $y \in B_\Delta(x, r)$ and $(z_n)_{n \in \mathbb{N}} \subset X$ such that, for all $n \in \mathbb{N}$,

$$|\Delta(y, z_n)| < 1/n, \quad |\Delta(x, z_n)| \geq r.$$

Hence, we have a sequence $(z_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} |\Delta(y, z_n)| = 0$ and, by (H2),

$$\liminf_{n \rightarrow \infty} |\Delta(x, z_n)| \geq r > |\Delta(x, y)| = \lim_{z \rightarrow y} |\Delta(x, z)|,$$

which contradicts the definition of supremum.

2 \Rightarrow 1. Now, if $B_\Delta(x, r) \in \tau_\Delta$ for all $x \in X$ and $r \in \mathbb{R}^+$, fix $x, y \in X$ and let $r := |\Delta(x, y)|$, $\varepsilon \in \mathbb{R}^+$. Clearly, $y \in B_\Delta(x, r + \varepsilon)$, so there exists $\delta_\varepsilon \in \mathbb{R}^+$ such that $B_\Delta(y, \delta_\varepsilon) \subset B_\Delta(x, r + \varepsilon)$. Hence,

if $(z_n)_{n \in \mathbb{N}} \subset X$ is such that $|\Delta(y, z_n)| \rightarrow 0$ as $n \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that $|\Delta(y, z_n)| < \delta_\varepsilon$ for every $n \geq N$, so $|\Delta(x, z_n)| < r + \varepsilon$ for every $n \geq N$. Hence, $\liminf_{n \rightarrow \infty} |\Delta(x, z_n)| \leq r + \varepsilon$. Since ε was arbitrarily fixed, we get that $\liminf_{n \rightarrow \infty} |\Delta(x, z_n)| \leq r$, which ends the result.

$2 \Leftrightarrow 3$. Just observe that $|\Delta_x|^{-1}((-\infty, r)) = B_\Delta(x, r)$. \square

Remark 3. Note that hypothesis (H1) is not necessary for the previous result or the definition of the topology itself. In fact, it has only been used so far to show that open balls are nonempty. Allowing the open balls to be the empty set changes nothing as it always belongs to the topology, making the result true in any case. However, hypothesis (H1) will be key in the definition of the displacement derivative in Section 4.

Lemma 4. Let $\Delta : X \times X \rightarrow \mathbb{R}$. If (H1) and (H2) hold then:

1. Every element of \mathcal{E}_Δ is nonempty.
2. Every element of τ_Δ is union of elements in \mathcal{E}_Δ .
3. \mathcal{E}_Δ is a basis of τ_Δ .

Proof. 1. Since $\Delta(x, x) = 0$, $x \in B(x, r)$ for any $r \in \mathbb{R}^+$.

2. Fix $U \in \tau_\Delta$. By definition of τ_Δ , we know that, for every $x \in U$, there exists $r_x \in \mathbb{R}^+$ such that $B(x, r_x) \subset U$. Since $x \in B(x, r_x)$, we have that $X = \bigcup_{x \in X} B(x, r_x)$.

3. By (H2), we have that $\mathcal{E}_\Delta \subset \tau_\Delta$. If $x \in U \cap V$ for $U, V \in \mathcal{E}_\Delta$, since U and V are open, so is $U \cap V$ and hence, using 1, there exists $W \in \mathcal{E}_\Delta$ such that $x \in W \subset U \cap V$. \square

Example 6. The conditions obtained in Lemma 4 do not suffice to obtain both (H1) and (H2). Consider the space $X = \{0, 1\}$ together with the function Δ given by $\Delta(0, 1) = \Delta(1, 1) = 0$, $\Delta(1, 0) = \Delta(0, 0) = 1$. In this case, $\mathcal{E}_\Delta = \tau_\Delta$ and the topology coincides with that of the Sierpiński space. Observe that Δ is not a displacement, although it satisfies theses 1–3 of Lemma 4.

It is also worth observing that the map $\tilde{\Delta}(0, 1) = \tilde{\Delta}(1, 1) = \tilde{\Delta}(0, 0) = 0$, $\tilde{\Delta}(1, 0) = 1$ is a displacement and $\tau_{\tilde{\Delta}} = \tau_\Delta$. This means that, if we want to find sufficient conditions in order for a Δ to be a displacement, those conditions cannot be purely topological. Furthermore, since the Sierpiński space is not regular, we deduce that it is not uniformizable, and thus we conclude that not every displacement space is uniformizable.

In the particular context of the real line, some further results can be obtained. In order to achieve them, we ask for the following hypothesis for $\Delta : X^2 \subset \mathbb{R}^2 \rightarrow \mathbb{R}$.

(H3) $\Delta(x, y) \leq \Delta(x, z)$ for every $x, y, z \in X$ such that $y \leq z$.

Lemma 5. Let $\Delta : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy (H1)–(H3). Then, $(\mathbb{R}, \tau_\Delta)$ is a second-countable topological space.

Proof. First of all, given $x \in \mathbb{R}$ and $r \in \mathbb{R}^+$, we can express $B_\Delta(x, r)$ as follows:

$$B_\Delta(x, r) = \{y \in \mathbb{R} : |\Delta(x, y)| < r\} = \{y \in \mathbb{R} : -r < \Delta_x(y) < r\} = \Delta_x^{-1}((-r, r)).$$

Moreover, since Δ_x is non-decreasing, due to the bounded completeness of (\mathbb{R}, \leq) , $B_\Delta(x, r)$ is an interval (not necessarily open) with extremal points

$$a = \inf\{t \in \mathbb{R} : -r < \Delta_x(t)\}, \quad b = \sup\{t \in \mathbb{R} : \Delta_x(t) < r\}.$$

Let $U \in \tau_\Delta$. Then, $U = \bigcup_{x \in U} B_\Delta(x, r_x)$ by definition of open set, and so, since each $B_\Delta(x, r_x)$ is a interval, we can write

$$U = \bigcup_{i \in \mathcal{I}} (a_i, b_i) \cup \bigcup_{j \in \mathcal{J}} [a_j, b_j] \cup \bigcup_{k \in \mathcal{K}} (a_k, b_k] \cup \bigcup_{l \in \mathcal{L}} [a_l, b_l],$$

for some sets of indices $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$, where each of those intervals is an open ball of τ_Δ .

The set $A = \bigcup_{i \in \mathcal{I}} (a_i, b_i)$ is an open set in (\mathbb{R}, τ_u) and therefore second countable, which implies that A is Lindelöf [13] (p. 182) and, hence, there exists a countable subcover of A , i.e., $A = \bigcup_{n \in \mathbb{N}} (a_{i_n}, b_{i_n})$ for some set of indices $\{i_n\}_{n \in \mathbb{N}}$. Similarly, the set $B = \bigcup_{j \in \mathcal{J}} [a_j, b_j]$ is an open set in the Sorgenfrey line, which is hereditarily Lindelöf [13] (p. 79), and so $B = \bigcup_{n \in \mathbb{N}} [a_{j_n}, b_{j_n}]$ for some set of indices $\{j_n\}_{n \in \mathbb{N}}$. Analogously, the set $C = \bigcup_{k \in \mathcal{K}} (a_k, b_k]$ can be expressed as $\bigcup_{n \in \mathbb{N}} (a_{k_n}, b_{k_n}]$. Finally, the set $D = \bigcup_{l \in \mathcal{L}} [a_l, b_l]$ can be decomposed as $D = \bigcup_{l \in \mathcal{L}} [a_l, b_l] \cup \bigcup_{l \in \mathcal{L}} (a_l, b_l]$, and once again, arguing as for the sets B and C , we obtain that

$$D = \bigcup_{l_n \in \mathbb{N}} [a_{l_n}, b_{l_n}] \cup \bigcup_{l'_n \in \mathbb{N}} (a_{l'_n}, b_{l'_n}],$$

for some sets of indices $\{l_n\}_{n \in \mathbb{N}}$, $\{l'_n\}_{n \in \mathbb{N}}$. However, by the definition of D , we have that $a_l, b_l \in D$ for all $l \in \mathcal{L}$, so

$$D = \bigcup_{l_n \in \mathbb{N}} [a_{l_n}, b_{l_n}] \cup \bigcup_{l'_n \in \mathbb{N}} [a_{l'_n}, b_{l'_n}],$$

which is clearly countable. Therefore, U is the countable union of open balls, i.e., τ_Δ is a second-countable topology. \square

Remark 4. This last proof relies heavily on the fact that the real number system, with its usual order, is bounded complete; that is, that every bounded (in the order sense) set has an infimum and a supremum. Observe also that the interaction between the topologies τ_u and τ_Δ plays a mayor role in the proof. Finally, hypothesis (H2) is necessary in this result through Lemma 3, which implies that open Δ -balls are, indeed, open.

Related to this last point, the authors would like to comment on the fact that hypothesis (H2) will not be necessary in the particular setting of the displacement calculus. However, it provides—as illustrated before with Lemmas 3 and 5—some information about the relation between τ_Δ and the displacement calculus we have yet to develop. In particular, Lemma 5 shows that, for the real line, every τ_u -Borel σ -algebra is, in particular, a τ_Δ -Borel σ -algebra so the integration theory that will follow, when considering (H2), will be valid for the open sets of τ_Δ . Nevertheless, while studying specific problems—like differential equations—we will deal, in general, with intervals or other elements of the τ_u -Borel σ -algebra without worrying about the specifics of the τ_Δ topology, which, as said before, makes (H2) unneeded.

Definition 6. Given displacement spaces (X, Δ_1) and (Y, Δ_2) , a function $f : X \rightarrow Y$ is said to be Δ_1^2 -continuous if $f : (X, \tau_{\Delta_1}) \rightarrow (Y, \tau_{\Delta_2})$ is continuous.

We say that a map $f : X \rightarrow \mathbb{R}^n$ is Δ_1 -continuous if $f : (X, \Delta_1) \rightarrow (\mathbb{R}^n, \tau_u)$ is continuous.

As usual, continuity can be characterized using open balls, as it is shown in the following result.

Lemma 6. Let (X, Δ_1) and (Y, Δ_2) be displacement spaces. A map $f : X \rightarrow Y$ is Δ_1^2 -continuous if and only if

$$\forall x \in X, \forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \text{ such that } f(y) \in B_{\Delta_2}(f(x), \varepsilon) \quad \forall y \in B_{\Delta_1}(x, \delta). \tag{6}$$

Proof. First, assume that f is Δ_1^2 -continuous and fix $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^+$. Since $U = B_{\Delta_2}(f(x), \varepsilon) \in \tau_{\Delta_2}$, we have that $f^{-1}(U) \in \tau_{\Delta_1}$. Moreover, $x \in f^{-1}(U)$ and so there exists $\delta \in \mathbb{R}^+$ such that $B_{\Delta_1}(x, \delta) \subset f^{-1}(U)$. Hence, $f(B_{\Delta_1}(x, \delta)) \subset f(f^{-1}(U)) \subset U$; that is, there exists $\delta \in \mathbb{R}^+$ such that

$$|\Delta_1(x, y)| < \delta \implies |\Delta_2(f(x), f(y))| < \varepsilon.$$

Conversely, let $U \in \tau_{\Delta_2}$, $y \in f^{-1}(U)$ and $x = f(y)$. Since $U \in \tau_{\Delta_2}$, there exists $\varepsilon_x \in \mathbb{R}^+$ such that $B_{\Delta_2}(x, \varepsilon_x) \subset U$. Now, condition (6) guarantees the existence of $\delta_y \in \mathbb{R}^+$ such that

$$f(z) \in B_{\Delta_2}(x, \varepsilon_x), \quad \forall z \in B_{\Delta_1}(y, \delta_y).$$

Note that $B_{\Delta_1}(y, \delta_y) \subset f^{-1}(U)$ as for any $z \in B_{\Delta_1}(y, \delta_y)$ we have that $f(z) \in U$. Since $y \in f^{-1}(U)$ was arbitrary, $f^{-1}(U)$ is open and so f is Δ_1^2 -continuous. \square

3. Displacement Measure Theory on the Real Line

In this section, we aim to define a measure over a non-degenerate interval $[a, b] \subset \mathbb{R}$. To do so, we will use “local” measures μ_z , for $z \in [a, b]$, to construct a measure μ which does not depend on a specific point z . In order to achieve that, we will consider $([a, b], \Delta)$ satisfying hypotheses (H1)–(H3), and two extra conditions:

(H4) There exists $\gamma : [a, b]^2 \rightarrow [1, +\infty)$ such that

(i) For all $x, y, z, \bar{z} \in [a, b]$, we have

$$|\Delta(z, x) - \Delta(z, y)| \leq \gamma(z, \bar{z})|\Delta(\bar{z}, x) - \Delta(\bar{z}, y)|.$$

(ii) For all $z \in [a, b]$,

$$\lim_{\bar{z} \rightarrow z} \gamma(z, \bar{z}) = \lim_{\bar{z} \rightarrow z} \gamma(\bar{z}, z) = 1.$$

(iii) For all $z \in [a, b]$, the maps $\gamma(z, \cdot), \gamma(\cdot, z) : [a, b] \rightarrow [1, +\infty)$ are bounded.

(H5) For every $x \in [a, b]$, $\Delta_x(\cdot)$ is left-continuous (with the usual topology of \mathbb{R}) at x .

Remark 5. Note that, under hypothesis (H3), it is enough to check that there exists $\gamma : [a, b]^2 \rightarrow [1, +\infty)$ such that, for all $x, y \in [a, b]$, $x < y$, we have

$$\Delta(z, y) - \Delta(z, x) \leq \gamma(z, \bar{z})(\Delta(\bar{z}, y) - \Delta(\bar{z}, x)),$$

to confirm that (H4) holds.

First of all, note that the set of maps $\Delta : [a, b]^2 \rightarrow \mathbb{R}$ that satisfy hypotheses (H1)–(H5) is not empty, as any Stieltjes displacement satisfies all of them. Moreover, there exist non-Stieltjes displacements that also satisfy all of the hypotheses. To show that this is the case, we will need the following result.

Proposition 1. Let $\Delta : [a, b]^2 \rightarrow \mathbb{R}$ be a given map and let us denote by $D_2\Delta$ its partial derivative with respect to its second variable. If $D_2\Delta$ exists and is continuous on $[a, b]^2$, and there exists $r > 0$ such that

$$D_2\Delta(x, y) \geq r \quad \text{for all } (x, y) \in [a, b]^2,$$

then Δ satisfies (H3)–(H5).

Proof. The assumptions imply that for each $x \in [a, b]$, the mapping $\Delta(x, \cdot)$ is increasing and continuous, which is more than ((H3) and (H5). Now, fix $z, \bar{z} \in [a, b]$. For $a \leq x < y \leq b$, the generalized mean value theorem guarantees the existence of $\xi \in (x, y)$ such that

$$\frac{\Delta(z, y) - \Delta(z, x)}{\Delta(\bar{z}, y) - \Delta(\bar{z}, x)} = \frac{D_2\Delta(z, \xi)}{D_2\Delta(\bar{z}, \xi)},$$

which implies (H4, i) for

$$\gamma(z, \bar{z}) = \max \left\{ 1, \max_{a \leq \xi \leq b} \frac{D_2\Delta(z, \xi)}{D_2\Delta(\bar{z}, \xi)} \right\}. \tag{7}$$

The function γ is well-defined and bounded because the three variable mapping

$$(z, \bar{z}, \xi) \in [a, b]^3 \longmapsto \frac{D_2\Delta(z, \xi)}{D_2\Delta(\bar{z}, \xi)}$$

is continuous on a compact domain. In particular, (H4, iii) holds.

Finally, (H4, ii) is a consequence of the fact that $D_2\Delta$ is continuous on $[a, b]^2$, and, therefore, uniformly continuous on $[a, b]^2$. Indeed, let $z \in [a, b]$ be fixed; for $\varepsilon > 0$, we can find $\delta > 0$ such that, for each $\bar{z} \in [a, b]$, $|z - \bar{z}| < \delta$, we have that

$$|D_2\Delta(z, \zeta) - D_2\Delta(\bar{z}, \zeta)| < r\varepsilon \quad \text{for every } \zeta \in [a, b].$$

Therefore, if $|z - \bar{z}| < \delta$, we have that

$$\left| \frac{D_2\Delta(z, \zeta)}{D_2\Delta(\bar{z}, \zeta)} - 1 \right| = \frac{|D_2\Delta(z, \zeta) - D_2\Delta(\bar{z}, \zeta)|}{D_2\Delta(\bar{z}, \zeta)} < \varepsilon \quad \text{for every } \zeta \in [a, b].$$

We have just proven that

$$\lim_{\bar{z} \rightarrow z} \frac{D_2\Delta(z, \zeta)}{D_2\Delta(\bar{z}, \zeta)} = 1 \quad \text{uniformly in } \zeta \in [a, b].$$

Now, for each $\bar{z} \in [a, b]$, there exists $\zeta_{\bar{z}}$ such that

$$\gamma(z, \bar{z}) = \max \left\{ 1, \frac{D_2\Delta(z, \zeta_{\bar{z}})}{D_2\Delta(\bar{z}, \zeta_{\bar{z}})} \right\},$$

hence $\gamma(z, \bar{z}) \rightarrow 1$ as $\bar{z} \rightarrow z$. Similarly, $\gamma(z, \bar{z}) \rightarrow 1$ as $z \rightarrow \bar{z}$. \square

Example 7. Consider the non-Stieltjes displacement in Example 5, namely $\Delta : [0, 1]^2 \times [0, 1]^2 \rightarrow \mathbb{R}$ where

$$\Delta(x, y) = e^{y^2-x^2} - e^{x-y}, \quad x, y \in [0, 1].$$

It clearly has continuous partial derivatives, and

$$D_2\Delta(x, y) = 2ye^{y^2-x^2} + e^{x-y} \geq e^{x-y} \geq e^{-1} \quad \text{on } [0, 1]^2. \tag{8}$$

Hence, Δ satisfies (H1)–(H5) for γ defined as in equation (7).

Although hypothesis (H5) might seem harmless, when combined with (H4), we obtain left-continuity everywhere.

Proposition 2. Consider $([a, b], \Delta)$ satisfying hypotheses (H4) and (H5). Then, for each $x \in [a, b]$, the map $\Delta_x : [a, b] \rightarrow \mathbb{R}$ is left-continuous everywhere (with the usual topology of \mathbb{R}).

Proof. Let $\varepsilon > 0$, $x, y \in [a, b]$ and γ be the map on (H4). Let us show that Δ_x is left-continuous at y . Since, by (H5), Δ_y is left-continuous at y , there exists $\delta > 0$ such that, for $0 < y - s < \delta$, we have $|\Delta_y(y) - \Delta_y(s)| < \varepsilon/\gamma(x, y)$. Then, for $0 < y - s < \delta$, hypothesis (H4) implies that

$$|\Delta_x(y) - \Delta_x(s)| \leq \gamma(x, y)|\Delta_y(y) - \Delta_y(s)| < \varepsilon. \quad \square$$

With the previous result in mind, we can define the “local” measures μ_z as the Lebesgue–Stieltjes measure associated with the non-decreasing left-continuous map Δ_z . We shall denote by \mathcal{M}_z the σ -algebra over which μ_z is defined.

Let us denote by \mathcal{B} the Borel σ -algebra (for τ_u) and by $\mathcal{M} := \bigcap_{z \in [a, b]} \mathcal{M}_z$. Note that $\mathcal{B} \subset \mathcal{M}$ as $\mathcal{B} \subset \mathcal{M}_z$ for all $z \in [a, b]$. Moreover, \mathcal{M} is a σ -algebra as it is an arbitrary intersection of σ -algebras. Hence, we can consider the restriction of $\mu_z, z \in [a, b]$, to \mathcal{M} . We will still denote it by μ_z . A set $A \in \mathcal{M}$ is said to be Δ -measurable.

Recall that a function $f : ([a, b], \mathcal{M}) \rightarrow (\mathbb{R}, \mathcal{B})$ is measurable if and only if $f^{-1}(U) \in \mathcal{M}$ for all $U \in \mathcal{B}$. We will say in that case that f is Δ -measurable. This notation will be consistent with the Δ -measure that we will introduce later. Observe that f is Δ -measurable if and only if $f : ([a, b], \mathcal{M}_z) \rightarrow (\mathbb{R}, \mathcal{B})$ is measurable for all $z \in [a, b]$.

Hypothesis (H4) allows us to understand the relationship between the different possible measures on \mathcal{M} depending on $z \in [a, b]$. In particular, given $z, \bar{z} \in [a, b]$, and an interval $I \subset [a, b]$, it is clear that $\mu_z(I) \leq \gamma(z, \bar{z})\mu_{\bar{z}}(I)$ and, as a consequence of the definition of the Lebesgue–Stieltjes measures,

$$\mu_z(A) \leq \gamma(z, \bar{z})\mu_{\bar{z}}(A), \quad \text{for all } A \in \mathcal{M}.$$

Thus, we have that $\mu_{\bar{z}} \ll \mu_z \ll \mu_{\bar{z}}$ for all $z, \bar{z} \in X$. Hence, if a property holds μ_z -everywhere, it holds μ_x -everywhere for all $x \in [a, b]$. Again, in order to simplify the notation, we will say that such property holds Δ -everywhere. Analogously, this expression will be consistent with the Δ -measure presented later in this paper.

Then, given $z, \bar{z} \in [a, b]$, we can apply the Radon–Nikodým Theorem [14] to the measures $\mu_z, \mu_{\bar{z}}$, so there exist two Δ -measurable functions $h_{\bar{z},z}, h_{z,\bar{z}} : [a, b] \rightarrow [0, \infty)$ such that

$$\mu_{\bar{z}}(A) = \int_A h_{\bar{z},z} \, d\mu_z, \quad \mu_z(A) = \int_A h_{z,\bar{z}} \, d\mu_{\bar{z}}, \quad \text{for all } A \in \mathcal{M}. \tag{9}$$

From these expressions, it is clear that $h_{z,z} = 1$ and $h_{z,\bar{z}}(t) = h_{\bar{z},z}(t)h_{z,z}(t)$ for Δ -almost all (or simply Δ -a.a.) $t \in [a, b]$ and $z, \bar{z}, \tilde{z} \in [a, b]$. Hence, it follows that $h_{z,\bar{z}}(t) = 1/h_{\bar{z},z}(t)$ for Δ -a.a. $t \in [a, b]$. In addition, note that, for $z, \bar{z} \in [a, b]$, $h_{z,\bar{z}} \neq 0$ Δ -everywhere in $[a, b]$.

Further properties are shown in the next results.

Proposition 3. *Given $z, \bar{z} \in [a, b]$, we have that*

$$\frac{1}{\gamma(\bar{z}, z)} \leq h_{z,\bar{z}}(t) \leq \gamma(z, \bar{z}), \quad \Delta\text{-a.a. } t \in [a, b]. \tag{10}$$

Proof. First, assume that $h_{z,\bar{z}}(t) \leq \gamma(z, \bar{z})$, does not hold for Δ -a.a. $t \in [a, b]$. Then, there would exist $A \in \mathcal{M}$ such that $\mu_{\bar{z}}(A) > 0$ and

$$h_{z,\bar{z}}(t) > \gamma(z, \bar{z}), \quad \text{for all } t \in A.$$

Hence,

$$\mu_z(A) = \int_A h_{z,\bar{z}}(s) \, d\mu_{\bar{z}}(s) > \int_A \gamma(z, \bar{z}) \, d\mu_{\bar{z}}(s) = \gamma(z, \bar{z})\mu_{\bar{z}}(A),$$

which is a contradiction. Therefore,

$$h_{z,\bar{z}}(t) \leq \gamma(z, \bar{z}), \quad \Delta\text{-a.a. } t \in [a, b]. \tag{11}$$

For the other inequality, take $h_{\bar{z},z}$ as in expression (9). Using inequality (11), we have that

$$\gamma(\bar{z}, z) \geq h_{\bar{z},z}(t) = \frac{1}{h_{z,\bar{z}}(t)}, \quad \Delta\text{-a.a. } t \in [a, b],$$

from which the result follows. \square

Note that this result yields that, for $z, \bar{z} \in [a, b]$ fixed, we have that

$$\frac{1}{\gamma(\bar{z}, z)} \leq h_{z,\bar{z}}(t) \leq \gamma(z, \bar{z}), \quad \text{for all } t \in [a, b] \setminus A_z,$$

with $\mu_z(A_z) = 0$. Let us define $\tilde{h}_{z,\bar{z}} : [a, b] \rightarrow [0, +\infty)$ as

$$\tilde{h}_{z,\bar{z}}(t) = \begin{cases} h_{z,\bar{z}}(t), & \text{if } t \in [a, b] \setminus A_z, \\ 1, & \text{if } t \in A_z. \end{cases}$$

Then, it follows from both inequalities in (10) that

$$\frac{1}{\gamma(\bar{z}, z)} \leq \tilde{h}_{z,\bar{z}}(t) \leq \gamma(z, \bar{z}), \quad \text{for all } t \in [a, b]. \tag{12}$$

Moreover, $\tilde{h}_{z,\bar{z}} = h_{z,\bar{z}}$ Δ -a.e. in $[a, b]$, and in particular, μ_z -a.e. in $[a, b]$. Hence, we have that

$$\mu_z(A) = \int_A \tilde{h}_{z,\bar{z}} \, d\mu_z, \quad \text{for all } A \in \mathcal{M}.$$

Thus, we can assume without loss of generality that the functions in expression (9) satisfy the inequalities (12). Given this consideration, we can obtain the following result.

Proposition 4. *For all $t \in [a, b]$, we have that*

$$\lim_{z \rightarrow \bar{z}} h_{z,\bar{z}}(t) = 1. \tag{13}$$

Proof. Fix $\bar{z}, t \in [a, b]$. Then, from the inequalities in (12), we obtain that

$$\frac{1}{\gamma(\bar{z}, z)} \leq h_{z,\bar{z}}(t) \leq \gamma(z, \bar{z}), \quad \text{for all } z \in [a, b]. \tag{14}$$

Hence, it is enough to consider the limit when $z \rightarrow \bar{z}$ in the previous inequalities, together with hypothesis (H3, ii), to obtain the result. \square

Remark 6. *Note that, given $\bar{z} \in [a, b]$, we also have that there exist $m_{\bar{z}}, M_{\bar{z}} > 0$ such that*

$$m_{\bar{z}} \leq h_{t,\bar{z}}(t) \leq M_{\bar{z}}, \quad \text{for all } t \in [a, b].$$

Indeed, fix $\bar{z}, t \in [a, b]$. Then, the inequalities (14) with $z = t$ yield

$$\frac{1}{\gamma(\bar{z}, t)} \leq h_{t,\bar{z}}(t) \leq \gamma(t, \bar{z}).$$

Now, the result follows from (H3, iii).

We will now focus on the definition of the Δ -measure which is based on the integrals defined by the measures $\mu_z, z \in [a, b]$. We first will show that a bigger family of maps is well-defined.

Proposition 5. *Let $\alpha : ([a, b], \mathcal{B}) \rightarrow ([a, b], \mathcal{B})$ be a measurable map and $z \in [a, b]$. Then, the map $\mu_\alpha : \mathcal{M} \rightarrow [0, +\infty]$ given by*

$$\mu_\alpha(A) = \int_A h_{\alpha(t),z}(t) \, d\mu_z(t), \quad A \in \mathcal{M},$$

is well-defined; that is, $h(\cdot, \alpha(\cdot)) := h_{\alpha(\cdot),z}(\cdot)$ is Δ -measurable.

Proof. In order to show that $h(\cdot, \alpha(\cdot))$ is μ_z -measurable, let us define the map $h_z : [a, b]^2 \rightarrow [0, +\infty)$ given by $h_z(t, x) = h_{x,z}(t)$. We will first show that h_z is a Δ_z -Carathéodory in the sense of [9] (Definition 7.1) adapted to our notation, that is:

- (i) for every $x \in [a, b]$, $h_z(\cdot, x)$ is Δ -measurable;

- (ii) for Δ -a.a. $t \in [a, b]$, $h_z(t, \cdot)$ is continuous on $[a, b]$;
- (iii) for every $r > 0$, there exists $f_r \in \mathcal{L}^1_{\Delta_z}([a, b])$ such that

$$|h_z(t, x)| \leq f_r(t) \quad \text{for } \Delta\text{-a.a. } t \in [a, b], \text{ and for all } x \in X, |x| \leq r.$$

Note that condition (i) is trivial as, by definition, $h_z(\cdot, x) = h_{x,z}(\cdot)$ is Δ -measurable. As for condition (ii), let $x \in [a, b]$. It follows directly from the equality (13) that

$$\lim_{\bar{z} \rightarrow x} h_{\bar{z},z}(t) = \lim_{\bar{z} \rightarrow x} h_{\bar{z},x}(t)h_{x,z}(t) = h_{x,z}(t), \quad \text{for } \Delta\text{-a.a. } t \in [a, b],$$

that is, $h_z(t, \cdot)$ is continuous on $[a, b]$ for Δ -a.a. $t \in [a, b]$. Finally, (H4, iii) guarantees the existence of $M > 0$ such that $|\gamma(x, z)| < M$ for all $x \in [a, b]$. Hence, it follows from expression (12) that

$$|h_z(t, x)| \leq M, \quad \text{for } \Delta\text{-a.a. } t \in [a, b], \text{ for all } x \in X, |x| \leq r.$$

Thus, (iii) holds, i.e., the map h_z is Δ_z -Carathéodory. Now, as it is shown in [9] (Lemma 7.2), the composition of a Δ_z -Carathéodory function with a Borel measurable function is Δ -measurable, and so the result follows. \square

Definition 7. Let $\alpha : ([a, b], \mathcal{B}) \rightarrow ([a, b], \mathcal{B})$ be a measurable map and $z \in [a, b]$. Consider the map $\mu_\alpha : \mathcal{M} \rightarrow [0, +\infty)$ given by

$$\mu_\alpha(A) = \int_A h_{\alpha(t),z}(t) \, d\mu_z(t), \quad A \in \mathcal{M}.$$

The map μ_α is a measure, [15] (Theorem 1.29), and it will receive the name of Δ_α -measure. In particular, when α is the identity map, it will be called the Δ -measure, and it will be denoted by $\mu \equiv \mu_{\text{Id}}$.

The following result shows that μ_α does not depend on a specific point of $[a, b]$, as we intended.

Proposition 6. Let $\alpha : ([a, b], \mathcal{B}) \rightarrow ([a, b], \mathcal{B})$ be a measurable map. Then, the map $\mu_\alpha : \mathcal{M} \rightarrow [0, +\infty)$ in Definition 7 is independent of the choice of $z \in [a, b]$.

Proof. Let $z, \bar{z} \in [a, b]$, $z \neq \bar{z}$. Then, we have that $h_{\alpha(s),\bar{z}}(s) = h_{\alpha(s),z}(s)h_{z,\bar{z}}(s)$ for Δ -a.a. $s \in [a, b]$ and so

$$\int_A h_{\alpha(s),z}(s) \, d\mu_z(s) = \int_A h_{\alpha(s),z}(s)h_{z,\bar{z}}(s) \, d\mu_z(s) = \int_A h_{\alpha(s),\bar{z}}(s) \, d\mu_{\bar{z}}(s). \quad \square$$

Observe that the notation we have used so far is consistent with the definition of the Δ -measure μ . Indeed, for example, f is Δ -measurable if and only if it is μ -measurable; as μ and μ_z , $z \in [a, b]$ are both defined, after due restriction of μ_z to \mathcal{M} , over the same σ -algebra. In addition, by definition, we have that $\mu \ll \mu_z$. The converse is also true thanks to (H4, iii). Indeed, for $z \in [a, b]$, there exists $K > 0$ such that $|\gamma(z, x)| < K$ for all $x \in [a, b]$. Hence, the first inequality in (12) implies that

$$\mu(A) = \int_A h_{\alpha(s),z}(s) \, d\mu_z(s) \geq \int_A \frac{1}{\gamma(z, \alpha(s))} \, d\mu_z(s) \geq \int_A \frac{1}{K} \, d\mu_z(s) = \frac{\mu_z(A)}{K} \geq 0.$$

Thus, if $\mu(A) = 0$, then $\mu_z(A) = 0$, i.e., $\mu_z \ll \mu$ for all $z \in [a, b]$. Therefore, a property holds Δ -everywhere if and only if it holds μ -everywhere.

As a final comment, note that $\mu : \mathcal{M} \rightarrow [0, +\infty)$ is a Borel measure that assigns finite measure to bounded sets. As it can be seen in [16] (This is in the book as 1.3.2 Lebesgue-Stieltjes measures on \mathbb{R}), this means that it can be thought of as a Lebesgue-Stieltjes measure, μ_g , given by

$$\mu_g([c, d]) = g(d) - g(c), \quad c, d \in [a, b], c < d, \tag{15}$$

for the nondecreasing and left-continuous function $g : [a, b] \rightarrow \mathbb{R}$ defined as

$$g(a) = 0, \quad g(t) = \mu([a, t]). \tag{16}$$

Definition 8. Let $X \subset [a, b]$, $X \in \mathcal{M}$. We define the integral of a μ_α -measurable function f over X with respect to the path of Δ -measures α as

$$\int_X f \, d\mu_\alpha := \int_X f(t)h_{\alpha(t),z}(t) \, d\mu_z(t),$$

provided the integral exists. This definition does not depend on the z chosen. As usual, we define the set of μ_α -integrable functions on X as

$$\mathcal{L}_{\mu_\alpha}^1(X) := \left\{ f : X \rightarrow \mathbb{R} : f \text{ is } \mu_\alpha\text{-measurable, } \int_X |f| \, d\mu_\alpha < +\infty \right\}.$$

Now, if we consider the restrictions of μ_z , $z \in [a, b]$, to \mathcal{M} , we can define the set of Δ -integrable functions over $X \in \mathcal{M}$ as

$$\mathcal{L}_\Delta^1(X) := \left\{ f : X \rightarrow \mathbb{R} : f \text{ is } \Delta\text{-measurable, } \int_X |f| \, d\mu_z < +\infty, \text{ for all } z \in [a, b] \right\}.$$

We now study the relationship between $\mathcal{L}_\Delta^1(X)$ and $\mathcal{L}_\mu^1(X)$. First of all, recall that, in this framework, μ and μ_z are defined over the same σ -algebra \mathcal{M} , so the concepts of μ -measurable and Δ -measurable are equivalent. Let $f \in \mathcal{L}_\Delta^1(X)$ and $z \in [a, b]$. Hypothesis (H4, iii) implies that there exist $M > 0$ such that $|\gamma(x, z)| < M$ for all $x \in [a, b]$. Thus, using the second inequality in (12), we have

$$\int_X |f| \, d\mu = \int_X |f(s)h_{s,z}(s)| \, d\mu_z(s) \leq \int_X |f(s)\gamma(s, z)| \, d\mu_z(s) \leq M \int_X |f(s)| \, d\mu_z(s) < +\infty,$$

that is, $f \in \mathcal{L}_\mu^1(X)$. Conversely, let $f \in \mathcal{L}_\mu^1(X)$ and $z \in [a, b]$. Again, hypothesis (H4, iii) implies that there exists $K > 0$ such that $|\gamma(z, x)| < K$ for all $x \in [a, b]$. Hence,

$$\int_X |f| \, d\mu = \int_X |f(s)h_{s,z}(s)| \, d\mu_z(s) \geq \int_X \left| \frac{f(s)}{\gamma(z, s)} \right| \, d\mu_z(s) \geq \frac{1}{K} \int_X |f(s)| \, d\mu_z(s),$$

so $\int_X |f| \, d\mu_z < +\infty$. Since $z \in [a, b]$ was arbitrary, we have that $f \in \mathcal{L}_\Delta^1(X)$. That is, $\mathcal{L}_\mu^1(X) = \mathcal{L}_\Delta^1(X)$.

Finally, we study the behavior of μ over some interesting sets related to the map Δ . These sets will be fundamental in the definition of the Δ -derivative. Let us define the sets C_Δ and D_Δ as

$$\begin{aligned} C_\Delta &:= \{x \in [a, b] : \Delta(x, \cdot) = 0 \text{ in } (x - \varepsilon, x + \varepsilon) \text{ for some } \varepsilon \in \mathbb{R}^+\}, \\ D_\Delta &:= \{x \in [a, b] : \Delta(x, x^+) \neq 0\}. \end{aligned} \tag{17}$$

Note that C_Δ is, by definition, an open set in the usual topology of $[a, b]$. Therefore, it can be rewritten uniquely as the disjoint countable union of open intervals, say $C_\Delta = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$. We define N_Δ as

$$N_\Delta := \{a_n, b_n : n \in \mathbb{N}\} \setminus D_\Delta. \tag{18}$$

Proposition 7. Let g be as in expression (16) and let C_g and D_g be as in [4] that is,

$$\begin{aligned} C_g &:= \{x \in [a, b] : g \text{ is constant on } (x - \varepsilon, x + \varepsilon) \text{ for some } \varepsilon \in \mathbb{R}^+\}, \\ D_g &:= \{x \in [a, b] : g(x^+) - g(x) > 0\}. \end{aligned}$$

Then, $C_\Delta = C_g$ and $D_\Delta = D_g$.

Proof. For the equality $D_\Delta = D_g$, it is enough to note that, for any $t \in [a, b]$, we have that

$$g(t^+) - g(t) = \lim_{s \rightarrow t^+} \int_{[t,s)} h_{r,t}(r) \, d\mu_t(r) = \int_{\{t\}} h_{r,t}(r) \, d\mu_t(r) = h_{t,t}(t)(\Delta_t(t^+) - \Delta_t(t)) = \Delta_t(t^+).$$

Now, in order to see that $C_\Delta = C_g$, let $t \in C_\Delta$. Then, $\Delta_t(\cdot) = 0$ on $(t - \varepsilon, t + \varepsilon)$ for some $\varepsilon \in \mathbb{R}^+$. Let $r, s \in (t - \varepsilon, t + \varepsilon)$, $r < s$. Then, by Remark 6, we have that

$$0 \leq g(s) - g(r) = \mu([r, s]) = \int_{[r,s)} h_{x,t}(x) \, d\mu_t(x) \leq M_t \mu_t([r, s]) = 0,$$

since $[r, s] \subset (t - \varepsilon, t + \varepsilon)$. Thus, g is constant on $(t - \varepsilon, t + \varepsilon)$. Conversely, if $t \in C_g$, then g is constant on $(t - \varepsilon, t + \varepsilon)$ for some $\varepsilon \in \mathbb{R}^+$. Let $r, s \in (t - \varepsilon, t + \varepsilon)$, $r < s$. Then, Remark 6 implies that

$$0 \leq m_t(\Delta_t(s) - \Delta_t(r)) = m_t \mu_t([r, s]) \leq \int_{[r,s)} h_{x,t}(x) \, d\mu_t(x) = g(s) - g(r) = 0.$$

That is, Δ_t is constant on $(t - \varepsilon, t + \varepsilon)$, and, since $\Delta_t(t) = 0$, it follows that $t \in D_\Delta$. \square

The first consequence of Proposition 7 is that D_Δ is at most countable since it is the set of discontinuities of a monotone function. Further properties can be obtained from Propositions 2.5 and 2.6 in [4].

Corollary 1. Let C_Δ and N_Δ be as in expressions (17) and (18), respectively. Then, $\mu(C_\Delta) = \mu(N_\Delta) = 0$.

Remark 7. As a consequence of Corollary 1, a property holds μ -a.e. in E if it holds on $E \setminus O_\Delta$ with

$$O_\Delta := C_\Delta \cup N_\Delta.$$

Moreover, note that, if $x \notin O_\Delta$, then $\Delta_z(y) \neq \Delta_z(x)$, for all $y, z \in [a, b]$, $x \neq y$.

4. Displacement Derivatives

We now introduce the concept of displacement derivative of a function defined over a compact interval (of the real line with the usual topology) endowed with a displacement structure. We chose this setting because some nice properties, such as the linearity of the derivative, are quite helpful in order to study the relationship between the displacement derivative and its integral.

Definition 9. Let $([a, b], \Delta)$ satisfy (H1)–(H5). The derivative with respect to the displacement Δ (or Δ -derivative) of a function $f : [a, b] \rightarrow \mathbb{R}$ at a point $x \in [a, b] \setminus O_\Delta$ is defined as follows, provided that the corresponding limits exist:

$$f^\Delta(x) = \begin{cases} \lim_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta(x, y)}, & x \notin D_\Delta, \\ \lim_{y \rightarrow x^+} \frac{f(y) - f(x)}{\Delta(x, y)}, & x \in D_\Delta. \end{cases}$$

Observe that this definition does not require Δ to be symmetric. Furthermore, this definition is a more general setting than g -derivatives (and therefore time-scales, as pointed out in [4]).

Finally, one might think that the natural choice for the definition of the derivative would be by taking the limit in the τ_Δ topology. However, if $x \notin D_\Delta$, x is a continuity point of Δ_x , and it is easy to see that such limit can be translated into a limit in the usual topology, which is far more convenient for the theory that follows. It is at this point that the importance of (H1) arises as commented in Remark 3. Without this hypothesis, we would not be able to assure that the balls of center x and any radii are nonempty, so considering the τ_Δ limit might not be well-defined.

Fundamental Theorem of Calculus

In this section, we will make explicit the relationship between the Δ -derivative of a function f and its integral with respect to the Δ -measure. In particular, our first goal now is to show that, for $f \in \mathcal{L}^1_\mu([a, b])$ and $F(x) = \int_{[a, x]} f(s) \, d\mu$, the equality $F^\Delta(x) = f(x)$ holds for μ -a.a. $x \in [a, b]$. In the particular setting of Stieltjes derivatives, this result has been proven in different ways and can be found in [6] and, more extensively, in [17].

In order to do so, we will follow an approach similar to that of [4], starting by guaranteeing the differentiability of monotone functions. For that matter, we will use the following two results that are direct consequences of Lemmas 4.2 and 4.3 in [4] adapted to our framework.

Proposition 8. *Let $([a, b], \Delta)$ satisfy (H1)–(H5), $\alpha > 0, z \in [a, b], f : [a, b] \rightarrow \mathbb{R}, P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ and S be a nonempty subset of $\{1, 2, \dots, n\}$. If $f(a) \leq f(b)$ and*

$$\frac{f(x_k) - f(x_{k-1})}{\Delta_z(x_k) - \Delta_z(x_{k-1})} < -\alpha \quad \text{for each } k \in S,$$

then

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})| > |f(b) - f(a)| + \alpha L$$

where $L = \sum_{k \in S} (\Delta_z(x_k) - \Delta_z(x_{k-1}))$. The same result is true if $f(a) \geq f(b)$ and

$$\frac{f(x_k) - f(x_{k-1})}{\Delta_z(x_k) - \Delta_z(x_{k-1})} > \alpha \quad \text{for each } k \in S.$$

Proposition 9. *Let $([a, b], \Delta)$ satisfy (H1)–(H5) and $H \subset (a, b)$ be such that for a given $z \in [a, b]$, there is $\varepsilon_z \in \mathbb{R}^+$ such that $\mu_z^*(H) = \varepsilon_z$. Then,*

1. *If \mathcal{I} is any collection of open subintervals of $[a, b]$ that covers H , then there exists a finite disjoint collection $\{I_1, I_2, \dots, I_N\}$ of \mathcal{I} such that*

$$\sum_{k=1}^N \mu_z(I_k) > \frac{\varepsilon_z}{3}.$$

2. *If P is a finite subset of $[a, b] \setminus D_\Delta$ and \mathcal{I} is any collection of open subintervals of $[a, b]$ that covers $H \setminus P$, then there exists a finite disjoint collection $\{I_1, I_2, \dots, I_N\}$ of \mathcal{I} such that*

$$\sum_{k=1}^N \mu_z(I_k) > \frac{\varepsilon_z}{4}.$$

We can now prove the Δ -differentiability of monotone functions. To do so, we will follow the ideas of [18].

Proposition 10. *Let $([a, b], \Delta)$ satisfy (H1)–(H5) and let $f : [a, b] \rightarrow \mathbb{R}$ be a nondecreasing function. Then, there exists $N \subset [a, b]$ such that $\mu(N) = 0$ and*

$$f^\Delta(x) \text{ exists for all } x \in [a, b] \setminus N.$$

Proof. First of all, note that, since f is nondecreasing, f is regulated so if $x \in [a, b] \cap D_\Delta$, $f^\Delta(x)$ exists as

$$f^\Delta(x) = \frac{f(x^+) - f(x)}{\Delta(x, x^+)}.$$

Moreover, since either $a \in D_\Delta$ or $\mu(\{a\}) = 0$, it is enough to show that $f^\Delta(x)$ exists for all $x \in (a, b) \setminus (D_\Delta \cup O_\Delta)$ according to Remark 7.

Let $x \in (a, b) \setminus (D_\Delta \cup O_\Delta)$. Since $\Delta_x(y) \neq \Delta_x(x)$ for any $y \neq x$, we can define the Dini upper and lower Δ -derivatives as

$$\overline{f^\Delta}(x) := \limsup_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta(x, y)}, \quad \underline{f^\Delta}(x) := \liminf_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta(x, y)}.$$

Furthermore, since f is monotone, it has a countable number of discontinuity points, so

$$\mu(\{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ is discontinuous at } x\}) = 0.$$

Thus, it is enough to show that the sets

$$F := \{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ continuous at } x, \overline{f^\Delta}(x) > \underline{f^\Delta}(x)\},$$

$$E := \{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : \overline{f^\Delta}(x) = +\infty\}.$$

Both have Δ -measure zero (and therefore μ -measure zero).

We first show that F is a null Δ -measure set. Fix $z \in (a, b) \setminus (D_\Delta \cup O_\Delta)$ and define the Dini upper and lower Δ_z -derivatives as

$$\overline{f_z^\Delta}(x) := \limsup_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta_z(y) - \Delta_z(x)}, \quad \underline{f_z^\Delta}(x) := \liminf_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta_z(y) - \Delta_z(x)}.$$

Note that (H4) implies that

$$\begin{aligned} \overline{f^\Delta}(x) &= \limsup_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta_x(y) - \Delta_x(x)} = \limsup_{y \rightarrow x} \frac{f(y) - f(x)}{\Delta_x(y) - \Delta_x(x)} \frac{\Delta_z(y) - \Delta_z(x)}{\Delta_z(y) - \Delta_z(x)} \\ &= \overline{f_z^\Delta}(x) \limsup_{y \rightarrow x} \frac{\Delta_x(y) - \Delta_x(x)}{\Delta_z(y) - \Delta_z(x)} \leq \overline{f_z^\Delta}(x) \gamma(z, x), \end{aligned}$$

and, analogously, $\underline{f^\Delta}(x) \geq (\gamma(x, z))^{-1} \underline{f_z^\Delta}(x)$. Hence, F is a subset of

$$F_z := \{(a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ continuous at } x, \overline{f_z^\Delta}(x) \gamma(z, x) \gamma(x, z) > \underline{f_z^\Delta}(x)\}.$$

Now, since $\gamma : [a, b]^2 \rightarrow [1, +\infty)$, it is clear that $F_z \subset \bigcup_{n \in \mathbb{N}} F_n$ with

$$F_n := \{(a, b) \setminus (D_\Delta \cup O_\Delta) : f \text{ continuous at } x, \overline{f_z^\Delta}(x) n > \underline{f_z^\Delta}(x)\},$$

so it suffices to show that $\mu_z(F_n) = 0$ for all $n \in \mathbb{N}$. By contradiction, assume that there exists $n_0 \in \mathbb{N}$ such that $\mu_z(F_{n_0}) > 0$. In that case, we rewrite F_{n_0} as the countable union of sets $F_{n_0, r, s}$ with $r, s \in \mathbb{Q}$, $r > s > 0$, and

$$F_{n_0, r, s} := \left\{ x \in F_{n_0} : \overline{f_z^\Delta}(x) n_0 > r > s > \underline{f_z^\Delta}(x) \right\}.$$

Thus, there exist $r_0, s_0 \in \mathbb{Q}$, $r_0 > s_0 > 0$, and $\varepsilon \in \mathbb{R}^+$ such that $\mu_z(F_{n_0, r_0, s_0}) = \varepsilon$. Now let $\alpha = \frac{r_0 - s_0}{2n_0}$, $\beta = \frac{r_0 + s_0}{2n_0}$ and $h(x) = f(x) - \beta \Delta_z(x)$. Then, $F_{n_0, r_0, s_0} = H$ with

$$H := \left\{ x \in F_{n_0} : \overline{h_z^\Delta}(x) > \alpha, \underline{h_z^\Delta}(x) < -\alpha \right\}.$$

Note that h is of bounded variation as it is the difference of two nondecreasing functions. Therefore, the set

$$V(h) := \left\{ \sum_P |h(x_k) - h(x_{k-1})| : P \text{ is a partition of } [a, b], P \cap D_\Delta \subset \{a, b\} \right\}$$

is bounded from above. Let $T := \sup V(h)$. Since $\alpha, \varepsilon \in \mathbb{R}^+$, there exists a partition $P = \{x_0, x_1, \dots, x_{n-1}\}$ of $[a, b]$ such that $x_k \notin D_\Delta$ for any $k \in \{1, 2, \dots, n-1\}$ and

$$\sum_{k=1}^n |h(x_k) - h(x_{k-1})| > T - \frac{\alpha\varepsilon}{4}.$$

Let $x \in H \setminus P$. Then, $x \in (x_{k-1}, x_k)$ for some $k \in \{1, 2, \dots, n\}$. Since $\overline{h_z^\Delta}(x) > \alpha, h_z^\Delta(x) < -\alpha$ and both Δ_z and h are continuous at x , we can choose $a_x, b_x \in (x_{k-1}, x_k) \setminus D_\Delta$ such that $a_x < x < b_x$ and

$$\frac{h(b_x) - h(a_x)}{\Delta_z(b_x) - \Delta_z(a_x)} < -\alpha \quad \text{or} \quad > \alpha,$$

depending on whether $h(x_{k-1}) \geq h(x_k)$ or $h(x_{k-1}) < h(x_k)$. Note that $\mu_z(a_x, b_x) = \Delta_z(b_x) - \Delta_z(a_x)$. By doing this, we obtain a collection of open subintervals of (a, b) , $\mathcal{I} = \{(a_x, b_x) : x \in H \setminus \{x_1, x_2, \dots, x_{n-1}\}\}$ that covers $H \setminus \{x_1, x_2, \dots, x_{n-1}\}$ and $\{x_1, x_2, \dots, x_{n-1}\} \cap D_\Delta = \emptyset$. Then, Proposition 9 ensures the existence of a finite disjoint subcollection $\{I_1, I_2, \dots, I_N\}$ of \mathcal{I} such that

$$\sum_{k=1}^N \mu_z(I_k) > \frac{\varepsilon}{4}.$$

Now, let $Q = \{y_0, y_1, \dots, y_q\}$ be the partition of $[a, b]$ determined by the points of P and the endpoints of the intervals I_1, I_2, \dots, I_N . For each $[x_{k-1}, x_k]$ containing at least one of the intervals in $\{I_1, I_2, \dots, I_N\}$, Proposition 8 yields that

$$\sum_{[y_{i-1}, y_i] \subset [x_{k-1}, x_k]} |h(y_i) - h(y_{i-1})| > |h(x_k) - h(x_{k-1})| + \alpha L_k,$$

where the summation is taken over the closed intervals determined by Q contained in $[x_{k-1}, x_k]$ and L_k is the sum of the Δ_z -measures of those intervals I_1, I_2, \dots, I_N contained in $[x_{k-1}, x_k]$. By taking the previous inequality and summing over k , we obtain

$$\sum_{k=1}^q |h(y_k) - h(y_{k-1})| > \sum_{k=1}^n |h(x_k) - h(x_{k-1})| + \alpha \sum_{k=1}^N \mu_z(I_k) > T,$$

which contradicts the definition of T .

Hence, all that is left to do is to show that the set E has Δ -measure zero. If we fix $z \in [a, b]$, then for all $x \in (a, b) \setminus (D_\Delta \cup O_\Delta)$, we have the inequality $f^\Delta(x) \leq \overline{f_z^\Delta}(x)\gamma(z, x)$ and so $E \subset E_z$ with

$$E_z := \{x \in (a, b) \setminus (D_\Delta \cup O_\Delta) : \overline{f_z^\Delta}(x) = +\infty\}.$$

Thus, it is enough to show that $\mu_z(E_z) = 0$. Suppose this is not the case. Then, there is $\varepsilon \in \mathbb{R}^+$ such that $\mu_z(E_z) = \varepsilon$. Let $M \in \mathbb{R}^+$ be such that $M > 3(f(b) - f(a))/\varepsilon$. If $x \in E_z$, then $\overline{f_z^\Delta}(x) > M$ and there exist $a_x, b_x \in (a, b) \setminus D_\Delta$ such that $a_x < x < b_x$ and

$$\frac{f(b_x) - f(a_x)}{\Delta_z(b_x) - \Delta_z(a_x)} > M.$$

Therefore, $\{(a_x, b_x) : x \in E_z\}$ covers E_z . Proposition 9 guarantees the existence of a finite disjoint subcollection $\{I_1, I_2, \dots, I_N\}$ such that

$$\sum_{k=1}^N \mu_z(I_k) > \frac{\varepsilon}{3}.$$

Let $I_k = (a_k, b_k)$ for each k . Then, $\mu_z(I_k) = \Delta_z(b_k) - \Delta_z(a_k)$ as each $I_k \subset (a, b) \setminus (D_\Delta \cup O_\Delta)$. Now, since f is nondecreasing, we have

$$f(b) - f(a) \geq \sum_{k=1}^N (f(b_k) - f(a_k)) > M \sum_{k=1}^N (\Delta_z(b_k) - \Delta_z(a_k)) > f(b) - f(a),$$

which is a contradiction. \square

Finally, a key result for the proof of the Fundamental Theorem of Calculus is Fubini’s Theorem on almost everywhere differentiation of series for Δ -derivatives. We now state such result, but we omit its proof as it is essentially the one provided in [19] but using Proposition 10 instead of the classical Lebesgue Differentiation Theorem.

Proposition 11. Let $([a, b], \Delta)$ satisfy (H1)–(H5) and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued nondecreasing functions on $[a, b]$. If the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{converges for all } x \in [a, b],$$

then

$$f^\Delta(x) = \sum_{n=1}^{\infty} f_n^\Delta(x) \quad \text{for } \mu\text{-a.a. } x \in [a, b].$$

We now have all the necessary tools to state and prove the first part of the Fundamental Theorem of Calculus for Δ -derivatives.

Theorem 1 (Fundamental Theorem of Calculus). Let $f \in \mathcal{L}^1_\mu([a, b])$ and $F(x) = \int_{[a, x]} f(s) \, d\mu$. Then, $F^\Delta(x) = f(x)$ for μ -a.a. $x \in [a, b]$.

Proof. Without loss of generality, we can assume that $f \geq 0$, as the general case can be reduced to the difference of two such functions. Since $f \geq 0$, the function F is nondecreasing and therefore Δ -differentiable. We consider several cases separately:

Case 1: If $f = \chi_{(\alpha, \beta)}$, where $(\alpha, \beta) \subset (a, b)$, $\alpha, \beta \notin D_\Delta$, then it is obvious that $F^\Delta(x) = 0 = \chi_{(\alpha, \beta)}(x)$ for $x \notin (\alpha, \beta)$. Now, if $x \in (\alpha, \beta)$, let $H : [a, b] \rightarrow \mathbb{R}$ be defined as

$$H(t) = \int_{[a, t]} h_{s,x}(s) \, d\mu_x(s),$$

where the integral is to be understood as a Lebesgue–Stieltjes integral. Note that H is well-defined. Then,

$$\begin{aligned} F^\Delta(x) &= \lim_{y \rightarrow x^+} \frac{F(y) - F(x)}{\Delta(x, y)} = \lim_{y \rightarrow x^+} \frac{\int_{[a, y]} f \, d\mu - \int_{[a, x]} f \, d\mu}{\Delta(x, y)} \\ &= \lim_{y \rightarrow x^+} \frac{\int_{[a, y]} h_{s,x}(s) f(s) \, d\mu_x(s) - \int_{[a, x]} h_{s,x}(s) f(s) \, d\mu_x(s)}{\Delta(x, y)} \\ &= \lim_{y \rightarrow x^+} \frac{\int_{[x, y]} h_{s,x}(s) f(s) \, d\mu_x(s)}{\Delta(x, y)} = \lim_{y \rightarrow x^+} \frac{H(y) - H(x)}{\Delta_x(y) - \Delta_x(x)} \\ &= H'_{\Delta_x}(x) = h_{x,x}(x) = 1 = \chi_{(\alpha, \beta)}(x) \end{aligned}$$

where the equality $H'_{\Delta_x}(x) = h_{x,x}(x)$ follows from the Fundamental Theorem of Calculus for the Stieltjes derivative (see [4] (Theorem 2.4)).

Case 2: Let $M_0(\Delta)$ be the set of all step functions whose discontinuities are not in D_Δ . If $f \in M_0(\Delta)$, we deduce that $F^\Delta = f$ μ -a.e. from Case 1.

Case 3: There exists a nondecreasing sequence $(f_n)_{n=1}^\infty$ in $M_0(\Delta)$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for μ -a.a. $x \in [a, b)$. We define

$$F_n(x) = \int_{[a,x)} f_n \, d\mu,$$

and then it follows from the Lebesgue's Monotone Convergence Theorem for measures that

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = F_1(x) + \sum_{k=2}^\infty (F_k(x) - F_{k-1}(x))$$

for all $x \in [a, b]$. Since each summand is a nondecreasing step function of x , we can apply Case 2 and Proposition 11 to deduce that for μ -a.a. $x \in [a, b)$, we have

$$\begin{aligned} F^\Delta(x) &= F_1^\Delta(x) + \sum_{k=2}^\infty (F_k^\Delta(x) - F_{k-1}^\Delta(x)) \\ &= f_1(x) + \sum_{k=2}^\infty (f_k(x) - f_{k-1}(x)) = \lim_{n \rightarrow \infty} f_n(x) = f(x). \end{aligned}$$

General Case. For any $f \in \mathcal{L}_\Delta^1([a, b))$, we have $f = f_1 - f_2$, where each of the f_i 's is the limit of a nondecreasing sequence of step functions in the conditions of Case 3. \square

Definition 10. Let $x \in \mathbb{R}$ and $F : [a, b] \rightarrow \mathbb{R}$. We shall say that F is Δ_x -absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ verifying

$$\sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a_n)) < \delta$$

implies that

$$\sum_{n=1}^m |F(b_n) - F(a_n)| < \varepsilon.$$

A map $F : [a, b] \rightarrow \mathbb{R}^n$ is Δ_x -absolutely continuous if each of its components is a Δ_x -absolutely continuous function.

Remark 8. Note that, as a consequence of (H4), if F is Δ_x -absolutely continuous, it is Δ_y -absolutely continuous for all $y \in \mathbb{R}$. Hence, we will just say that F is Δ -absolutely continuous.

In the following results, we present some of the properties that Δ -absolutely continuous functions share.

Proposition 12. Let $f : [a, b] \rightarrow [c, d]$ be a Δ -absolutely continuous function and let $f_2 : [c, d] \rightarrow \mathbb{R}$ satisfy a Lipschitz condition on $[c, d]$. The composition $f_2 \circ f_1$ is Δ -absolutely continuous on $[a, b]$.

Proof. Let $L > 0$ be a Lipschitz constant for f_2 on $[c, d]$. Fix $x \in [a, b]$. For each $\varepsilon > 0$, take $\delta > 0$ in Definition 10 with ε replaced by ε/L . Now, for an open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ such that

$$\sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a_n)) < \delta$$

we have that

$$\sum_{n=1}^m |f_2(f_1(b_n)) - f_2(f_1(a_n))| \leq L \sum_{n=1}^m |f_1(b_n) - f_1(a_n)| < \varepsilon,$$

that is, $f_2 \circ f_1$ is Δ_x -absolutely continuous. \square

Proposition 13. *Let $F : [a, b] \rightarrow \mathbb{R}$ be a Δ -absolutely continuous function. Then, F is of bounded variation.*

Proof. To prove this result, we will use the following remark: if for any $[\alpha, \beta] \subset (a, b)$ there exists $c > 0$ such that the total variation of F on $[\alpha, \beta]$ is bounded from above by c , then F has bounded variation on $[a, b]$. Indeed, assume that, for any $[\alpha, \beta] \subset (a, b)$, there exists $c > 0$ such that the total variation of F on $[\alpha, \beta]$ is bounded from above by c . Then, for each $x \in (a, b)$,

$$|F(x)| \leq \left| F(x) - F\left(\frac{a+b}{2}\right) \right| + \left| F\left(\frac{a+b}{2}\right) \right| \leq c + \left| F\left(\frac{a+b}{2}\right) \right|.$$

Hence, $|F|$ is bounded on $[a, b]$. Let $K > 0$ be one of its bounds. For any partition $\{x_0, x_1, \dots, x_n\}$ of $[a, b]$, we have that

$$\sum_{k=1}^n |F(x_k) - F(x_{k-1})| = |F(x_1) - F(a)| + \sum_{k=2}^{n-1} |F(x_k) - F(x_{k-1})| + |F(b) - F(x_{n-1})| \leq 4K + c,$$

and so our claim holds.

Now, to prove that F has bounded variation on $[a, b]$, fix $x \in [a, b]$ and take $\varepsilon = 1$ in the definition of Δ_x -absolute continuity. Then, there exists $\delta > 0$ such that for any family $\{(a_n, b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a, b]$,

$$\sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a_n)) < \delta \implies \sum_{n=1}^m |F(b_n) - F(a_n)| < 1.$$

Consider a partition $\{y_0, y_1, \dots, y_n\}$ of $[\Delta_x(a), \Delta_x(b)]$ such that $0 < y_k - y_{k-1} < \delta, k = 1, 2, \dots, n$. Define $I_k = \Delta_x^{-1}([y_{k-1}, y_k]), k = 1, 2, \dots, n$. Since Δ_x is nondecreasing, the sets I_k are empty or they are intervals not necessarily open nor close. Anyway, $[a, b] = \cup I_k$, and so it is enough to show that F has bounded variation on the closure of each I_k . We assume the nontrivial case that is, $\bar{I}_k = [a_k, b_k], a_k < b_k$. If $[\alpha, \beta] \subset (a_k, b_k)$ and $\{t_0, t_1, \dots, t_m\}$ is a partition of $[\alpha, \beta]$, then

$$\sum_{i=1}^m (\Delta_x(t_i) - \Delta_x(t_{i-1})) = \Delta_x(\beta) - \Delta_x(\alpha) \leq y_k - y_{k-1} < \delta \implies \sum_{i=1}^m |F(x_i) - F(x_{i-1})| < 1.$$

Now, our previous claim implies that F has bounded variation on each \bar{I}_k , and therefore F has bounded variation on $[a, b]$. \square

Proposition 14. *Let $F : [a, b] \rightarrow \mathbb{R}$ be a Δ -absolutely continuous function. Then, F is left-continuous everywhere. Moreover, F is continuous where Δ is continuous.*

Proof. Fix $x \in [a, b]$ and $\varepsilon > 0$ and let $\delta > 0$ be given by the definition of Δ_x -absolute continuity of F . Since $\Delta_x(\cdot)$ is left-continuous at x , there exists $\delta' > 0$ such that, if $0 < x - t < \delta'$, then

$$\Delta_x(x) - \Delta_x(t) < \delta \implies |F(x) - F(t)| < \varepsilon.$$

The proof in the case Δ_x is right-continuous at $x \in [a, b]$ is analogous, and we omit it. \square

As a consequence of these two previous propositions, given F , a Δ -absolutely continuous function, there exist two nondecreasing and left-continuous functions, F_1, F_2 , such that $F = F_1 - F_2$. We denote by $\mu_i : \mathcal{B}([a, b]) \rightarrow \mathbb{R}$ the Lebesgue–Stieltjes measure defined by $F_i, i = 1, 2$. Recall that Lebesgue–Stieltjes measures are positive measures that are also outer regular; that is, for every $E \in \mathcal{B}([a, b])$, we have

$$\mu_i(E) = \inf\{\mu_i(V) : E \subset V, V \text{ open}\}.$$

A natural definition for a signed measure for the function F is given by

$$\mu_F(E) = \mu_1(E) - \mu_2(E), \quad E \in \mathcal{B}([a, b]).$$

Lemma 7. Let $F : [a, b] \rightarrow \mathbb{R}$ be a Δ -absolutely continuous function. Then, for every $x \in [a, b]$, we have $\mu_F \ll \mu_x$.

Proof. Let $x \in [a, b]$, $\varepsilon > 0$ and $\delta > 0$ given by the definition of Δ_x -absolute continuity with ε replaced by $\varepsilon/2$. Fix an open set $V \subset (a, b)$ such that $\mu_x(V) < \delta$. Without loss of generality, we can assume that $V = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ for a pairwise disjoint family of open intervals. For each $n \in \mathbb{N}$, take $a'_n \in (a_n, b_n)$. Then, for each $m \in \mathbb{N}$, we have

$$\sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a'_n)) = \mu_x \left(\bigcup_{n=1}^m [a'_n, b_n) \right) \leq \mu_x(V) < \delta,$$

and so $\sum_{n=1}^m |F(b_n) - F(a'_n)| < \varepsilon/2$. By letting a'_n tend to a_n , we obtain

$$\sum_{n=1}^m |F(b_n) - F(a_n^+)| \leq \varepsilon/2, \quad \text{for each fixed } m \in \mathbb{N}.$$

Thus, if $\mu(V) < \delta$, we have that

$$|\mu_F(V)| = \left| \sum_{n=1}^{\infty} \mu_F(a_n, b_n) \right| \leq \sum_{n=1}^{\infty} |F(b_n) - F(a_n^+)| < \varepsilon.$$

Let $E \in \mathcal{B}([a, b])$ be such that $\mu_x(E) = 0$. By outer regularity, there exist open sets $V_n \subset [a, b]$, $n \in \mathbb{N}$ such that $E \subset V_n$, $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \mu_x(V_n) = \mu_x(E), \quad \lim_{n \rightarrow \infty} \mu_i(V_n) = \mu_i(E), \quad i = 1, 2.$$

Now, by the first part of the proof, we know that $\lim_{n \rightarrow \infty} \mu_F(V_n) = \mu_F(E) = 0$ since $\lim_{n \rightarrow \infty} \mu_x(V_n) = \mu_x(E) = 0$, so

$$\mu_F(E) = \mu_1(E) - \mu_2(E) = \lim_{n \rightarrow \infty} \mu_F(V_n) = 0.$$

Hence, $\mu_F \ll \mu_x$, and, since $\mu_x \ll \mu$, the result follows. \square

Lemma 8. Let $f \in \mathcal{L}^1_{\mu}([a, b])$ and consider $F : [a, b] \rightarrow \mathbb{R}$ given by

$$F(x) = \int_{[a, x)} f \, d\mu.$$

Then, F is Δ -absolutely continuous.

Proof. It is enough to consider the case $f \geq 0$, as the general case can be expressed as a difference of two functions of this type.

Fix $\varepsilon > 0$ and $x \in \mathbb{R}$. Hypothesis (H4, iii) implies that there exists $K > 0$ such that $|\gamma(t, x)| < K$ for all $t \in [a, b]$. Since $f \in \mathcal{L}^1_{\mu}([a, b]) = \mathcal{L}^1_{\Delta}([a, b])$, there exists $\delta > 0$ such that, if $E \in \mathcal{M}$, $\mu_x(E) < \delta$, then $\int_E f \, d\mu_x < \varepsilon/K$. Now, noting the inequalities in (12) holds Δ -almost everywhere, it holds μ -almost everywhere and so

$$\int_E f(s) \, d\mu_x(s) = \int_E f(s) h_{x, \alpha(s)}(s) \, d\mu(s) \geq \int_E \frac{f(s)}{\gamma(\alpha(s), x)} \, d\mu(s) > \frac{1}{K} \int_E f(s) \, d\mu(s).$$

Thus, if $E \in \mathcal{M}$, $\mu_x(E) < \delta$, then $\int_E f \, d\mu < \varepsilon$. Consider $\{(a_n, b_n)\}_{n=1}^m$ intervals in the conditions of the definition of Δ_x -absolute continuity. Take $E = \cup [a_n, b_n)$. Then,

$$\mu_x(E) = \sum_{n=1}^m \mu_x([a_n, b_n)) = \sum_{n=1}^m (\Delta_x(b_n^-) - \Delta_x(a_n^-)) = \sum_{n=1}^m (\Delta_x(b_n) - \Delta_x(a_n)) < \delta$$

implies that

$$\sum_{n=1}^m |F(b_n) - F(a_n)| = \sum_{n=1}^m (F(b_n) - F(a_n)) = \sum_{n=1}^m \int_{[a_n, b_n)} f \, d\mu = \int_E f \, d\mu < \varepsilon.$$

□

Theorem 2. A function $F : [a, b] \rightarrow \mathbb{R}$ is Δ -absolutely continuous on $[a, b]$ if and only if the following conditions are fulfilled:

- (i) there exists F^Δ for μ -a.a. $x \in [a, b]$;
- (ii) $F^\Delta \in \mathcal{L}^1_\mu([a, b])$;
- (iii) for each $x \in [a, b]$,

$$F(x) = F(a) + \int_{[a, x)} F^\Delta \, d\mu.$$

Proof. Lemma 8 ensures that the three conditions are sufficient for F to be Δ -absolutely continuous. For the converse, consider μ_F to be the Lebesgue–Stieltjes measure defined by F and let $z \in [a, b]$ be fixed. Lemma 7 and the Radon–Nykodym Theorem guarantee that there exists a measurable function $l : ([a, b), \mathcal{B}) \rightarrow (\mathbb{R}, \mathcal{B})$ such that

$$\mu_F(E) = \int_E l \, d\mu, \quad \text{for any Borel set } E \subset [a, b).$$

In particular,

$$F(x) - F(a) = \mu_F([a, x)) = \int_{[a, x)} l(s) \, d\mu(s).$$

Theorem 1 ensures that $F^\Delta(s) = l(s)$ for μ -a.a. $s \in [a, b)$, and so the result follows. □

5. The Relationship with Stieltjes Derivatives

As commented before, Stieltjes derivatives are, in a first approach, a particular case of Δ -derivatives. However, it can be proven to be equivalent to the displacement derivatives if hypotheses (H1)–(H5) hold. Indeed, we shall prove this equivalence through the following results.

Proposition 15. Let $\Delta : [a, b]^2 \rightarrow \mathbb{R}$ satisfy (H1)–(H5) and $g : [a, b] \rightarrow \mathbb{R}$ be as in expression (16). Then,

$$\lim_{s \rightarrow t} \frac{g(s) - g(t)}{\Delta(t, s)} = 1, \quad \text{for all } t \in [a, b] \setminus C_\Delta.$$

Proof. Fix $t \in [a, b] \setminus C_\Delta$. It follows from the inequalities in (14) that $1/\gamma(t, r) \leq h_{r,t}(r) \leq \gamma(r, t)$ for all $r \in [a, b]$. As a consequence, we have that

$$\frac{1}{\sup_{r \in [t, s)} \gamma(t, r)} \leq h_{r,t}(r) \leq \sup_{r \in [t, s)} \gamma(r, t). \tag{19}$$

Since $t \notin C_\Delta$, we have that

$$\Delta_x(y) > 0 \quad \text{for all } y > x, \quad \text{and/or} \quad \Delta_x(y) < 0 \quad \text{for all } y < x. \tag{20}$$

Assume the first case in condition (20) holds. Hence, for any $s \in [a, b], s > t$, we have that

$$\frac{g(s) - g(t)}{\Delta(t, s)} = \frac{\int_{[a, s]} h_{r,t}(r) \, d\mu_t - \int_{[a, t]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)} = \frac{\int_{[t, s]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)}.$$

Now, for any $s > t$, it follows from the inequalities in (19) that

$$\frac{1}{\sup_{r \in [t, s]} \gamma(t, r)} \Delta(t, s) \leq \int_{[s, t]} h_{r,t}(r) \, d\mu_t \leq \sup_{r \in [t, s]} \gamma(r, t) \Delta(t, s),$$

since $\mu_t([t, s]) = \Delta(t, s)$. Equivalently,

$$\frac{1}{\sup_{r \in [t, s]} \gamma(t, r)} \leq \frac{\int_{[t, s]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)} = \frac{g(s) - g(t)}{\Delta(t, s)} \leq \sup_{r \in [t, s]} \gamma(r, t), \quad s > t,$$

and so, allowing $s \rightarrow t^+$, we obtain

$$1 = \lim_{s \rightarrow t^+} \frac{1}{\sup_{r \in [t, s]} \gamma(t, r)} \leq \lim_{s \rightarrow t^+} \frac{g(s) - g(t)}{\Delta(t, s)} \leq \lim_{s \rightarrow t^+} \sup_{r \in [t, s]} \gamma(r, t) = 1.$$

If $\Delta_x(\cdot) = 0$ on some $[x - \delta, x], \delta > 0$, then the proof is complete. Otherwise, the second condition in (20) holds. In that case, we have that, for any $s \in [a, b], s < t$,

$$\frac{g(s) - g(t)}{\Delta(t, s)} = \frac{\int_{[a, s]} h_{r,t}(r) \, d\mu_t - \int_{[a, t]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)} = -\frac{\int_{[s, t]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)}. \tag{21}$$

Once again, it follows the inequalities in (19) that, for any $s < t$,

$$-\sup_{r \in [t, s]} \gamma(r, t) (-\Delta(t, s)) \leq -\int_{[s, t]} h_{r,t}(r) \, d\mu_t \leq -\frac{1}{\sup_{r \in [t, s]} \gamma(t, r)} (-\Delta(t, s)),$$

since $\mu_t([s, t]) = -\Delta(t, s)$. Equivalently,

$$\sup_{r \in [t, s]} \gamma(r, t) \leq -\frac{\int_{[s, t]} h_{r,t}(r) \, d\mu_t}{\Delta(t, s)} = \frac{g(s) - g(t)}{\Delta(t, s)} \leq \frac{1}{\sup_{r \in [t, s]} \gamma(t, r)}, \quad s < t.$$

Now, the rest of the proof is analogous to the previous case, and we omit it. \square

Bearing in mind the following relation,

$$\lim_{s \rightarrow t} \frac{f(s) - f(t)}{\Delta(t, s)} = \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)} \frac{g(s) - g(t)}{\Delta(t, s)}, \quad \text{for all } t \in [a, b] \setminus C_\Delta,$$

we obtain the next result.

Theorem 3. Let $\Delta : [a, b]^2 \rightarrow \mathbb{R}$ satisfy (H1)–(H5) and $g : [a, b] \rightarrow \mathbb{R}$ be given by (16). Given $f : [a, b] \rightarrow \mathbb{R}$ and $t \in [a, b] \setminus O_\Delta$, $f^\Delta(t)$ exists if and only if $f'_g(t)$ exists.

Therefore, we have that both derivatives are, indeed, equivalent. This shows the interest of studying this type of derivatives. In particular, when looking at differential equations, a wide variety of results exists in several papers such as [4], where the authors showed that Stieltjes differential equations

are good for studying equations on time scales and impulsive differential equations, or [9,20–23] where we can find different types of existence and uniqueness of solution results.

However, obtaining the corresponding function g can be hard. Indeed, although equation (16) gives an explicit expression of the function, it is defined in terms of the Δ -measure, which depends on the functions (9) given by the Radon–Nidokým Theorem. The following result gives a simple way to obtain, under certain hypotheses, the corresponding function g .

Proposition 16. *Let $\Delta : [a, b]^2 \rightarrow \mathbb{R}$ satisfy (H1)–(H5) and $g : [a, b] \rightarrow \mathbb{R}$ be given by expression (16). Suppose that $D_2\Delta$ exists, is continuous, and positive on $[a, b]^2$. Then, $g'(t)$ exists for all $t \in [a, b]$, and*

$$g'(t) = D_2\Delta(t, t).$$

In particular, g is strictly nondecreasing and $g \in C^1([a, b])$.

Proof. Fix $t \in [a, b]$. Since $D_2\Delta(t, t) > 0$, the function Δ_t is strictly nondecreasing in a neighborhood of t . Thus, $t \notin C_\Delta$, and so, if we apply Proposition 15, we have that

$$D_2\Delta(t, t) = \lim_{s \rightarrow t} \frac{\Delta(t, s)}{s - t} = \lim_{s \rightarrow t} \frac{\Delta(t, s)}{s - t} \lim_{s \rightarrow t} \frac{g(s) - g(t)}{\Delta(t, s)} = \lim_{s \rightarrow t} \frac{g(s) - g(t)}{s - t}.$$

Hence, $g'(t)$ exists and equals to $D_2\Delta(t, t)$. The rest of the result now follows. \square

Example 8. *Consider the non-Stieltjes displacement $\Delta : [0, 1]^2 \times [0, 1]^2 \rightarrow \mathbb{R}$ in Example 7, given by*

$$\Delta(x, y) = e^{y^2 - x^2} - e^{x - y}.$$

It follows from expression (8) that the hypotheses of Proposition 16 are satisfied. Hence, we can compute $g : [0, 1] \rightarrow \mathbb{R}$ in expression (16) as

$$g(t) = \int_0^t D_2\Delta(s, s) \, ds = \int_0^t (2s + 1) \, ds = t^2 + t.$$

6. A Model for Smart Surface Textures

In this section, we develop a model for smart surfaces based on a displacement and a derivation similar to that of the diffusion equation.

It is extended more every day to employ biomimetics in order to develop surfaces with extraordinary properties [24]. These meta-materials imitate organic tissues with special microstructures which modify their usual behavior. For instance, a cat’s tongue possesses backwards-facing spines, a disposition which facilitates particles moving towards the interior of the mouth and not in the other direction. This situation is similar to the one on the human respiratory epithelium, with the difference that, in this other tissue, the effect is due to the active motion of the cells’ cilia, which moves mucus and particles upwards, and not a passive result of the microstructure.

With 3D-printing (or other methods), we may obtain these surfaces for which friction depends on direction, position, pressure, etc. We can measure friction in an indirect but simple way using the definition of work: *work is the energy necessary to move and object between two points against a given force field*. Thus, on our surface, which, for convenience in the present discussion, we will consider one-dimensional (the higher dimensional case would be analogous), we can define a function $W(x, y)$ that measures the work necessary to move a point mass from x to y .

If we consider now a distribution of particles on the surface subject to random vibrations, this kind of situation may be described as a Brownian motion, but, in the case that those particles are very small, this model may be approximated by a diffusion process. The derivation of the classical

diffusion process—that is, of Fick’s second law or, equivalently, the heat equation—can be found in many references—see, for instance, [25–27].

By Nernst’s law, in the frictionless setting, the mass flowing through the point x during the time interval $(t, t + dt)$ is equal to

$$Q = -D \frac{\Delta u(x, t)}{dx} dt,$$

where u denotes the concentration of mass, Δu the spatial variation of u , D is the diffusion coefficient and dx, dt are considered to be *infinitesimal quantities*. With friction, this variation of the mass flow is impeded by the work necessary to move the particles; that is, W . We will assume that the variation of the mass flow is inversely proportional to the spatial variation of this work; that is,

$$Q = -D \frac{\Delta u(x, t)}{W(x, x + dx)} dt.$$

On the other hand, the spatial variation of Q can be computed directly as

$$\Delta Q = Q(x, t) - Q(x + dx, t) + h(x) dx dt,$$

where h is a source term in the case we allow for a continuous inflow of particles. At the same time, the variation in the mass flowing through the sectional volume, being proportional to the concentration of mass, can be computed as

$$\Delta Q = \left[-D \frac{\Delta u(x, t)}{W(x, x + dx)} + D \frac{\Delta u(x + dx, t)}{W(x + dx, x + 2 dx)} + h(x) dx \right] dt = c \Delta u(x, t) dx.$$

for some constant c . In the limit $dx \rightarrow 0$,

$$\frac{\Delta u(x, t)}{W(x, x + dx)} \approx \partial_{x,W} \Delta u(x, t), \quad \frac{\Delta u(x + dx, t)}{W(x + dx, x + 2 dx)} \approx \partial_{x,W} \Delta u(x + dx, t),$$

where $\partial_{x,W}$ denotes the displacement derivative for the displacement W with respect to the variable x . Thus,

$$[-D \partial_{x,W} \Delta u(x, t) + D \partial_{x,W} \Delta u(x + dx, t) + h(x) dx] dt = c \Delta u(x, t) dx,$$

that is,

$$D \left[\frac{\partial_{x,W} \Delta u(x + dx, t) - \partial_{x,W} \Delta u(x, t)}{dx} \right] + h(x) = c \frac{\Delta u(x, t)}{dt}.$$

Again, in the limit,

$$D \partial_x \partial_{x,W} u(x, t) + h(x) = c \partial_t u(x, t).$$

Now, we consider, for instance, the stationary problem with mixed two-point boundary conditions

$$u'_W(x) + h(x) = 0, \quad u'_W(0) = 0, \quad u(1) = C. \tag{22}$$

where C is a real constant. Integrating, we obtain the problem

$$u'_W(x) = - \int_0^x h(y) dy, \quad u(1) = C. \tag{23}$$

Under the conditions of Theorem 3—that is, (H1)–(H5)—we can apply theorems such as [20] (Theorem 3.5) to derive the existence of a solution of problem (23).

7. Conclusions

In this work, we have established a theory of Calculus based on the concept of displacement. We have studied the associated topology and measured and proved some general results regarding

their interaction. We have also defined new concepts such as the integral with respect to a path of measures or the displacement derivative, studied their properties, and proved a Fundamental Theorem of Calculus that relates them. Finally, we have set up a framework in order to study displacement equations. We have proved they can be transformed into Stieltjes differential equations, and so the results in [9,20] can be applied to new mathematical models.

We have also left some open problems. First of all, conditions (H1) and (H2) may be weakened further, substituted by a different set of axioms in order to achieve the same results. It will also be interesting to analyze how these relate to other concepts that generalize the notion of metric space, such as those derived from the conditions in [3,10], and to further explore the topological properties of displacement spaces.

To explore how to weaken conditions (H3)–(H5) would be an even more important task. Although very general in nature, they bound displacements to Stieltjes derivatives in a stringent way, such as is shown in Proposition 2. In the same way, it would be interesting to generalize the theory of displacement derivatives to the case where a general displacement is also considered in the numerator. We believe that the natural way to define the derivative in that case is as presented in the following definition.

Definition 11. Let $([a, b], \Delta_1)$ satisfy (H1)–(H5) and (\mathbb{R}, Δ_2) satisfy (H1)–(H3). The derivative with respect to the pair of displacements (Δ_1, Δ_2) (or Δ_1^2 -derivative) of a function $f : [a, b] \rightarrow \mathbb{R}$ at a point $x \in [a, b] \setminus O_{\Delta_1}$ is defined as follows, provided that the corresponding limits exist:

$$f^{\Delta_1^2}(x) = \begin{cases} \lim_{y \rightarrow x} \frac{\Delta_2(f(x), f(y))}{\Delta_1(x, y)}, & x \notin D_{\Delta_1}, \\ \lim_{y \rightarrow x^+} \frac{\Delta_2(f(x), f(y))}{\Delta_1(x, y)}, & x \in D_{\Delta_1}. \end{cases}$$

Again, this definition would establish a more general setting than Stieltjes derivatives, but would also include absolute derivatives and some very well known operators, such as the φ -Laplacian [28]. If we consider two functions $f, \varphi : \mathbb{R} \rightarrow \mathbb{R}$, the φ -Laplacian of y is given by

$$(\varphi \circ f')' = (f')^{\Delta_\varphi^2} = (f^{\Delta_\varphi^2})^{\Delta_\varphi^2},$$

where $\Delta_2(x, y) = \varphi(y) - \varphi(x)$ for $x, y \in \mathbb{R}$. Note, however, that since displacements need not to be linear in any sense, this definition could make it more difficult to prove a Fundamental Theorem of Calculus.

We have also hinted at the possibility of developing a theory of non-symmetric length spaces, based on the concept of displacement that generalizes the results in [12]. Furthermore, fixed point theorems in both displacement spaces and vector displacement spaces should be studied, focusing, in the last case, on the compatibility of the displacement with the underlying order topology (cf. [10,29]).

Finally, the properties of the integral with respect to a path of measures have to be thoroughly studied, as this concept lays many possibilities ahead.

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