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Fractional Langevin Equation Involving Two Fractional Orders: Existence and Uniqueness Revisited

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Abstract: We consider the nonlinear fractional Langevin equation involving two fractional orders with initial conditions. Using some basic properties of Prabhakar integral operator, we find an equivalent Volterra integral equation with two parameter Mittag–Leffler function in the kernel to the mentioned equation. We used the contraction mapping theorem and Weissinger's fixed point theorem to obtain existence and uniqueness of global solution in the spaces of Lebesgue integrable functions. The new representation formula of the general solution helps us to find the fixed point problem associated with the fractional Langevin equation which its contractivity constant is independent of the friction coefficient. Two examples are discussed to illustrate the feasibility of the main theorems.

Keywords: fractional Langevin equation; Mittag–Leffler function; Prabhakar integral operator; existence; uniqueness

1. Introduction

Dynamical behavior of physical processes are usually represented by differential equations. If the model of physical system in some ways possesses a memory and hereditary properties, for instance, viscoelastic deformation [1], anomalous diffusion [2], stock market [3], bacterial chemotaxis [4] and complex networks [5], relaxation in filled polymer networks [6], relaxation and reaction kinetics of polymers [7], description of mechanical systems subject to damping [8], Behavior of Biomedical Materials [9]; the corresponding models can be described by the fractional differential equations.

Langevin equation is a fundamental theory of the Brownian motion to describe the evolution of physical phenomena in fluctuating environments [10,11]. Fractional Langevin equation as a generalization of classical one gives a fractional Gaussian process parametrized by two indices, which is more flexible for modeling fractal processes [12–16].

The virtually simultaneous development of fractional derivatives, various generalizations of the Langevin equation were proposed and studied by various researchers during recent years. Despite the widespread use of many of the applications [17–22], the fractional Langevin equation is extensively studied in literature both from theoretical and numerical points of view. Authors in [23] studied nonlinear fractional Langevin equation involving two fractional orders in different intervals as a generalized form of three point third order nonlocal boundary value problem of nonlinear ordinary differential equations. In [24], the authors have studied fractional Langevin equations with nonlocal integral boundary conditions. Recently, anti-periodic boundary value problem for Langevin equation involving two fractional orders has been studied in [25]. Existence and uniqueness results for coupled and uncoupled systems of fractional Langevin equations of Riemann-Liouville and Hadamard types has been discussed in [26]. Guo et al. [27] gave an efficient numerical method for solving the fractional

Langevin equation with or without an external force. Some more recent work on Langevin equation can be found in [28–37].

In the current paper, we mainly focus on the existence and uniqueness result for the fractional Langevin equation involving two fractional orders:

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\lambda\right)x(t)=f(t,x(t)), & 0<t\leq 1,\\ x^{(i)}(0)=\mu_i, & 0\leq i<l,\\ x^{(i+\alpha)}(0)=\nu_i, & 0\leq i<n, \end{cases} \tag{1}$$

where $m-1<\alpha\leq m, n-1<\beta\leq n, l=\max\{m,n\}, m,n\in\mathbb{N}$, D^{α} is the Caputo fractional derivative, $x(t)$ is the particle displacement, $x^{(i+\alpha)}(0)$ equals $D^{i}D^{\alpha}x(0)$, in the sequential sense, $\lambda\in\mathbb{R}$ is the friction coefficient and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given function which represents a noise term.

Based on the criteria specified in [38], the problem (1) is a general form of anomalous systems governed by a generalized Langevin equation with long-range memory. In contrast to the classical Langevin equation, we use $D^{\beta}x(t)$ and $D^{\beta}D^{\alpha}x(t)$ instead of the ordinary definition of the velocity and acceleration as the first and second derivatives of the displacement to derive a generalized Langevin equation involving friction memory kernel. For example, if $0<\beta\leq 1, \alpha=1$, then according to the standard definition of the Caputo fractional derivatve operator, we have a special case of generalized Langevin equation involving friction memory kernel equal to $\dfrac{\lambda}{\Gamma(1-\beta)}t^{\beta-1}$. Based on the calculations in ([39], Section B), in this case, the resulting motion is in fact subdiffusive. Furthermore, it is worth noting that, if $\alpha+\beta>2$, then we do not have any physical meaning for the main problem. For this case, it is only a valuable problem in the thory of fractional differential equations as a sequential fractional differential equation with initial conditions.

As we have seen in the papers cited above about analysis of fractional Langevin equation, using various classical fixed point theorems is a common and useful technique for obtaining the existence and uniqueness results for fractional Langevin equation involving different initial or boundary conditions. In the mentioned papers, the contractivity constant of the fixed point problem associated with the fractional Langevin equation depended on the friction coefficient λ. For example, in the obtained existence and unique results in [33,34], the contractivity constants R_1, R_2 satisfy the following conditions

$$R_1=\sup_{0\leq t\leq 1}\left(\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}a(s)\,ds+\frac{|\lambda|}{\Gamma(\alpha+1)}\right)<1, \tag{2}$$

and

$$R_2=k\left(\frac{\|a\|_p}{\Gamma(\alpha+\beta)}\,ds+\frac{|\lambda|}{\Gamma(\alpha)}\right)<1, \tag{3}$$

where $k=\left(\frac{1}{1-q(1-\alpha)}\right)^{\frac{1}{q}}$ and $p^{-1}+q^{-1}=1$, respectively. As stated in relations (2) and (3), the contractivity constants R_1, R_2 depend on the friction constant λ. Therefore, from (2), we can not discuss the problems involving the friction constant $|\lambda|\geq\Gamma(\alpha+1)$. Similarly, from (3), we can not study the problems involving the friction constant $|\lambda|\geq\Gamma(\alpha)$. Note that $0<1-q(1-\alpha)<1$. Therefore, we cannot discuss the existence and uniqueness of solutions for the problems involving large friction coefficient λ. In this paper, we strive to overcome this major limitation. First we propose a new construction of the general solution for the Equation (1) using two parameter Mittag–Leffler functions and some of the basic properties of Prabhakar operator. This is done in Section 2. Then we obtain a new existence and uniqueness results under some weak conditions by using contractive mapping theorem and Weissinger's fixed point theorem. This is content of Section 3. Two examples are given in Section 4 to illustrate our results.

2. Preliminaries and Auxiliary Results

In the following section, we apply some technical calculations related to fractional calculus to build a new general solution corresponding to initial value problem (1) which provides an extremely powerful tool for the proof of the main result. Furthermore, we present some preliminaries and notations regarding fractional calculus for the reader's convenience. For details, see [40–46].

Definition 1. *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *for the function* $x : [0,1] \to \mathbb{R}$, $x \in L^1[0,1]$ *is defined as*

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds.$$

Definition 2. *The Caputo fractional derivative of order* $\alpha > 0$ *of a function* $x : [0,1] \to \mathbb{R}$ *is defined as*

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where $n-1 < \alpha \leq n$ *and* $n \in \mathbb{N}$, *provided that the right-hand-side integral exists and is finite.*

Definition 3 ([46]). *Let* $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$ *and* $x \in L^1[0,1]$. *The Prabhakar integral can be written as*

$$\mathbb{E}[\alpha,\beta,\lambda]x(t) = \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}\left(\lambda(t-s)^\alpha\right) x(s) ds,$$

where $E_{\alpha,\beta}(\cdot)$ *is the so-called two parameter Mittag-Leffler function, defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)},$$

and $E_\alpha(\cdot) = E_{\alpha,1}(\cdot)$. *Like the Mittag–Leffler function* $E_\alpha(z)$, $E_{\alpha,\beta}(z)$ *is an entire function of order* $\frac{1}{\alpha}$.

Lemma 1 ([46]). *Let* $\alpha, \beta, \gamma \geq 0$ *and* $x \in L^1[0,1]$. *Then*

$$I^\gamma \mathbb{E}[\alpha,\beta,\lambda]x(t) = \mathbb{E}[\alpha,\beta,\lambda] I^\gamma x(t) = \mathbb{E}[\alpha,\beta+\gamma,\lambda]x(t),$$

holds almost everywhere on $[0,1]$. *Furthermore,* $\mathbb{E}[\alpha,\beta,\lambda]t^\gamma = \Gamma(\gamma+1)t^{\gamma+\beta} E_{\alpha,\beta}(\lambda t^\alpha)$.

Lemma 2. *The general solution of* (1) *is given by*

$$\begin{aligned} x(t) = &\sum_{j=0}^{m-1} \mu_j t^j E_{\alpha+j}\left(-\lambda t^\alpha\right) + \sum_{i=0}^{n-1} \nu_i t^{\alpha+i} E_{\alpha,\alpha}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \mu_i t^i \left(\frac{1}{\Gamma(i+1)} - E_\alpha(-\lambda t^\alpha)\right) \\ &+ \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(t-s)^\alpha\right) f(s,x(s))\, ds. \end{aligned} \tag{4}$$

Proof. Let $x(t)$ be a solution of the problem (1), we have

$$\left(D^\alpha + \lambda\right) x(t) = \sum_{i=0}^{n-1} a_i t^i + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s)) ds.$$

By using the initial conditions for the initial problem (1), we find that $a_i = \frac{\nu_i+\lambda\mu_i}{\Gamma(i+1)}$, $i = 0, 1, \cdots, n-1$. Therefore, we have

$$(D^\alpha + \lambda)\, x(t) = \sum_{i=0}^{n-1} \frac{\nu_i + \lambda\mu_i}{\Gamma(i+1)} t^i + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, x(s)) ds. \tag{5}$$

Now, using the approach of Kilbas et al. ([40], Section 3.1), the solution of the Equation (5) is given by the following expression

$$x(t) = \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha,j+1}\left(-\lambda t^\alpha\right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) \left(\sum_{i=0}^{n-1} \frac{\nu_i + \lambda\mu_i}{\Gamma(i+1)} s^i + I^\beta f(\cdot, x(\cdot))(s) \right) ds. \tag{6}$$

Note $E_{\alpha,\alpha}(z) = \alpha E'_\alpha(z)$ and so $(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) = \frac{d}{ds}\left(\frac{1}{\lambda} E_\alpha\left(-\lambda(t-s)^\alpha\right)\right)$. This yields that $\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) ds = \frac{1}{\lambda}\left(1 - E_\alpha\left(-\lambda t^\alpha\right)\right)$. On the other hand, an integration by parts reveals

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) s^i ds = \frac{1}{\lambda}\left(s^i E_\alpha(-\lambda(t-s)^\alpha)\Big|_0^t - i\int_0^t E_\alpha(-\lambda(t-s)^\alpha) s^{i-1} ds \right), \tag{7}$$

for each $i \in \mathbb{N}$. Applying Lemma 1 to the second term in the right-hand side of (7), we conclude

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) s^i ds = \frac{1}{\lambda}\left(t^i - \Gamma(i+1) t^i E_\alpha(-\lambda t^\alpha)\right),$$

for each $i \in \mathbb{N}$. Therefore

$$\begin{aligned}
x(t) &= \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha,j+1}\left(-\lambda t^\alpha\right) + \sum_{i=0}^{n-1} \frac{\nu_i}{\Gamma(i+1)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) s^i\, ds \\
&\quad + \sum_{i=0}^{n-1} \frac{\lambda\mu_i}{\Gamma(i+1)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) s^i\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(t-s)^\alpha\right) \left(I^\beta f(\cdot, x(\cdot))(s)\right) ds \\
&= \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha,j+1}\left(-\lambda t^\alpha\right) + \sum_{i=0}^{n-1} \nu_i t^{\alpha+i} E_{\alpha,\alpha}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \frac{\lambda\mu_i}{\Gamma(i+1)} \left(\frac{1}{\lambda}\left(t^i - \Gamma(i+1) t^i E_\alpha(-\lambda t^\alpha)\right)\right) \\
&\quad + \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(t-s)^\alpha\right) f(s, x(s))\, ds \\
&= \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha,j+1}\left(-\lambda t^\alpha\right) + \sum_{i=0}^{n-1} \nu_i t^{\alpha+i} E_{\alpha,\alpha}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \mu_i t^i \left(\frac{1}{\Gamma(i+1)} - E_\alpha(-\lambda t^\alpha)\right) \\
&\quad + \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(t-s)^\alpha\right) f(s, x(s))\, ds,
\end{aligned}$$

which is the desired result. □

Now, we state Weissinger's fixed point theorem ([41], Theorem D.7) as a generalization of the so-called contraction mapping theorem which is needed to prove Theorem 3.

Theorem 1. *Let X to be a Banach space and let $\theta_n \geq 0$ for every $n \in \mathbb{N} \cup \{0\}$ such that $\sum_{n=0}^\infty \theta_n$ converges. Furthermore, assume $T : X \to X$ is a nonlinear mapping which satisfies the inequality $\|T^n x - T^n y\| \leq \theta_n \|x - y\|$ for every $n \in \mathbb{N}$ and every $x, y \in X$. Then, T has a unique fixed point x^*. Moreover, the sequence $\{T^n x_0\}_{n=0}^\infty$ converges to this fixed point x^*, for any $x_0 \in X$.*

3. Existence and Uniqueness

Our aim in the following section is to deeply investigate the existence and uniqueness results for the main problem (1) in the Lebesgue space.

Theorem 2. *Let $\max\{1, \frac{1}{\alpha+\beta}\} \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$ and the following hypotheses 1–3 hold:*

Hypothesis 1. $f(t,0) \in L^q[0,1]$.

Hypothesis 2. *There exists nonnegative* $a \in L^p[0,1]$ *such that* $|f(t,x_2) - f(t,x_1)| \le a(t)|x_2 - x_1|$, *for each* $t \in [0,1]$ *and* $x_1, x_2 \in \mathbb{R}$.

Hypothesis 3. $R := \frac{M_1 \|a\|_p}{(1-q+q(\alpha+\beta))^{\frac{1}{q}}} < 1$ *where* $M_1 = \sup_{t\in[0,1]} \left|E_{\alpha,\alpha+\beta}(-\lambda t^\alpha)\right|$.

Then the integral Equation (4) has a unique solution in $L^q[0,1]$.

Proof. We define the operator T as follows:

$$Tx(t) = \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(t-s)^\alpha\right) f(s,x(s))ds + \phi(t), \tag{8}$$

where

$$\phi(t) = \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha,j+1}\left(-\lambda t^\alpha\right) + \sum_{i=0}^{n-1} \nu_i t^{\alpha+i} E_{\alpha,\alpha}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \mu_i t^i \left(\frac{1}{\Gamma(i+1)} - E_\alpha(-\lambda t^\alpha)\right). \tag{9}$$

Let $M(t) = t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda t^\alpha)$, $M_1 = \sup_{t\in[0,1]} \left|E_{\alpha,\alpha+\beta}(-\lambda t^\alpha)\right|$ and $M_2 = \sup_{t\in[0,1]} |\phi(t)|$. Note that the generalized Mittag–Leffler functions are entire functions [43,44]. For each $x \in L^q[0,1]$, we have

$$\begin{aligned}
|Tx(t)| &\le \left|\int_0^t M(t-s)f(s,x(s))ds\right| + M_2 \\
&\le \int_0^t |M(t-s)|\,|f(s,0)| + |M(t-s)|\,|f(s,x(s)) - f(s,0)|\,ds + M_2 \\
&\le \int_0^t |M(t-s)|^{\frac{1}{q}}\,|f(s,0)|\,|M(t-s)|^{\frac{1}{p}}\,ds + \int_0^t |M(t-s)|\,|x(s)|\,|a(s)|\,ds + M_2 \\
&\le \left(\int_0^t |M(t-s)|\,|f(s,0)|^q\,ds\right)^{\frac{1}{q}} \left(\int_0^t |M(t-s)|\,ds\right)^{\frac{1}{p}} \\
&\quad + M_1 \left(\int_0^t \frac{|x(s)|^q}{(t-s)^{q-q(\alpha+\beta)}}ds\right)^{\frac{1}{q}} \left(\int_0^t |a(s)|^p\,ds\right)^{\frac{1}{p}} + M_2 \\
&\le M_1^{\frac{1}{q}} \left(\int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha+\beta}}ds\right)^{\frac{1}{q}} \cdot \frac{M_1^{\frac{1}{p}}}{(\alpha+\beta)^{\frac{1}{p}}} + M_1\|a\|_p \left(\int_0^t \frac{|x(s)|^q}{(t-s)^{q-q(\alpha+\beta)}}ds\right)^{\frac{1}{q}} + M_2 \\
&= \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}} \left(\int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha+\beta}}ds\right)^{\frac{1}{q}} + M_1\|a\|_p \left(\int_0^t \frac{|x(s)|^q}{(t-s)^{q-q(\alpha+\beta)}}ds\right)^{\frac{1}{q}} + M_2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|Tx\|_q &\le \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}} \left(\int_0^1 \int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha+\beta}}dsdt\right)^{\frac{1}{q}} + M_1\|a\|_p \left(\int_0^1 \int_0^t \frac{|x(s)|^q}{(t-s)^{q-q(\alpha+\beta)}}dsdt\right)^{\frac{1}{q}} + M_2 \\
&= \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}} \left(\int_0^1 \int_s^1 \frac{|f(s,0)|^q}{(t-s)^{1-\alpha+\beta}}dtds\right)^{\frac{1}{q}} + M_1\|a\|_p \left(\int_0^1 \int_s^1 \frac{|x(s)|^q}{(t-s)^{q-q(\alpha+\beta)}}dtds\right)^{\frac{1}{q}} + M_2 \\
&= \frac{M_1}{\alpha+\beta}\|f(s,0)\|_q + \frac{M_1}{(1-q+q(\alpha+\beta))^{\frac{1}{q}}}\|a\|_p\|x\|_q + M_2,
\end{aligned}$$

which yields $T : L^q[0,1] \to L^q[0,1]$. Now, for $x, y \in L^q[0,1]$, we obtain

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^t |M(t-s)|\,|f(s,x(s)) - f(s,y(s))|\,ds \\
&\leq \int_0^t |M(t-s)|\,|x(s) - y(s)|\,|a(s)|\,ds \\
&\leq M_1 \left(\int_0^t \frac{|x(s)-y(s)|^q}{(t-s)^{q-q(\alpha+\beta)}} ds \right)^{\frac{1}{q}} \left(\int_0^t |a(s)|^p\,ds \right)^{\frac{1}{p}} \\
&= M_1 \|a\|_p \left(\int_0^t \frac{|x(s)-y(s)|^q}{(t-s)^{q-q(\alpha+\beta)}} ds \right)^{\frac{1}{q}},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|Tx - Ty\|_q &= M_1 \|a\|_p \left(\int_0^1 \int_0^t \frac{|x(s)-y(s)|^q}{(t-s)^{q-q(\alpha+\beta)}} dsdt \right)^{\frac{1}{q}} \\
&= M_1 \|a\|_p \left(\int_0^1 \int_s^1 \frac{|x(s)-y(s)|^q}{(t-s)^{q-q(\alpha+\beta)}} dtds \right)^{\frac{1}{q}} \\
&= M_1 \|a\|_p \left(\int_0^1 \frac{(1-s)^{1-q+q(\alpha+\beta)}}{1-q+q(\alpha+\beta)} |x(s)-y(s)|^q\,ds \right)^{\frac{1}{q}} \\
&\leq \frac{M_1 \|a\|_p}{(1-q+q(\alpha+\beta))^{\frac{1}{q}}} \|x-y\|_q, \\
&= R\|x-y\|_q.
\end{aligned}$$

Note that $1 - q + q(\alpha + \beta) \geq 0$ because of $p \geq \frac{1}{\alpha+\beta}$. Therefore, T is a contraction since $R < 1$. By the Banach contraction principle, T has a unique fixed point, which is the unique solution of the initial problem (1). □

Remark 1. *We recall from* [43,44] *that $E_{\alpha,\beta}(-z)$ is completely monotonic function for $0 < \alpha \leq 1$ and $\beta \geq \alpha$, that is, $E_{\alpha,\beta}(-z)$ possesses derivatives $\frac{d^n}{dz^n}\left(E_{\alpha,\beta}(-z)\right)$ for all $n = 0,1,2,\cdots$ and $(-1)^n \frac{d^n}{dz^n}\left(E_{\alpha,\beta}(-z)\right) \geq 0$ for all $z > 0$. Therefore, $E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) \leq E_{\alpha,\alpha+\beta}(0) = \frac{1}{\Gamma(\alpha+\beta)}$ for $\lambda \geq 0$, $0 < \alpha \leq 1$ and $0 \leq t \leq 1$.*

Theorem 3. *Let $1 \leq q \leq \infty$ and the following hypotheses 4 and 5 hold:*

Hypothesis 4. *$f(t,0) \in L^q[0,1]$.*

Hypothesis 5. *There exists $L > 0$ such that $|f(t,x_2) - f(t,x_1)| \leq L|x_2 - x_1|$, for almost every $t \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$.*

Then the integral Equation (4) has a unique solution in $L^q[0,1]$.

Proof. With notations as in the proof of Theroem 2, and using the same arguments, we obtain

$$|Tx(t)| \leq \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}} \left(\int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha-\beta}} ds \right)^{\frac{1}{q}} + \frac{M_1 L}{(\alpha+\beta)^{\frac{1}{p}}} \left(\int_0^t \frac{|x(s)|^q}{(t-s)^{1-\alpha-\beta}} ds \right)^{\frac{1}{q}} + M_2,$$

and therefore,

$$
\begin{aligned}
\|Tx\|_q &\leq \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}}\left(\int_0^1\int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha-\beta}}dsdt\right)^{\frac{1}{q}} + \frac{M_1L}{(\alpha+\beta)^{\frac{1}{p}}}\left(\int_0^1\int_0^t \frac{|x(s)|^q}{(t-s)^{1-\alpha-\beta}}dsdt\right)^{\frac{1}{q}} + M_2 \\
&= \frac{M_1}{(\alpha+\beta)^{\frac{1}{p}}}\left(\int_0^1\int_s^1 \frac{|f(s,0)|^q}{(t-s)^{1-\alpha-\beta}}dtds\right)^{\frac{1}{q}} + \frac{M_1L}{(\alpha+\beta)^{\frac{1}{p}}}\left(\int_0^1\int_s^1 \frac{|x(s)|^q}{(t-s)^{1-\alpha-\beta}}dtds\right)^{\frac{1}{q}} + M_2 \\
&\leq \frac{M_1}{\alpha+\beta}\left(\|f(0,s)\|_q + L\|x\|_q\right) + M_2,
\end{aligned}
$$

which yields $T : L^q[0,1] \to L^q[0,1]$. On the other hand, for every $n \in \mathbb{N}$ and for each $t \in [0,1]$, we have

$$
\begin{aligned}
|T^n x(t) - T^n y(t)| &\leq \int_0^t |M(t-s_1)| \left|f(s_1, T^{n-1}x(s_1)) - f(s_1, T^{n-1}y(s_1))\right| ds_1 \\
&\leq M_1 L \int_0^t (t-s_1)^{\alpha+\beta-1} \left|T^{n-1}x(s_1) - T^{n-1}y(s_1)\right| ds_1 \\
&\leq (M_1L)^2 \int_0^t (t-s_1)^{\alpha+\beta-1}\left(\int_0^{s_1}(s_1-s_2)^{\alpha+\beta-1}\left|T^{n-2}x(s_2) - T^{n-2}y(s_2)\right| ds_2\right) ds_1 \\
&= (M_1L)^2 \int_0^t \int_{s_2}^t (t-s_1)^{\alpha+\beta-1}(s_1-s_2)^{\alpha+\beta-1}\left|T^{n-2}x(s_2) - T^{n-2}y(s_2)\right| ds_1\, ds_2 \\
&= (M_1L)^2 \int_0^t \left(\int_{s_2}^t (t-s_1)^{\alpha+\beta-1}(s_1-s_2)^{\alpha+\beta-1} ds_1\right)\left|T^{n-2}x(s_2) - T^{n-2}y(s_2)\right| ds_2 \\
&= \frac{(\Gamma(\alpha+\beta)M_1L)^2}{\Gamma(2\alpha+2\beta)} \int_0^t (t-s_2)^{2\alpha+2\beta-1}\left|T^{n-2}x(s_2) - T^{n-2}y(s_2)\right| ds_2 \\
&\ \ \vdots \\
&\leq \frac{(\Gamma(\alpha+\beta)M_1L)^n}{\Gamma(n\alpha+n\beta)} \int_0^t (t-s_n)^{n\alpha+n\beta-1} |x(s_n) - y(s_n)|\, ds_n \\
&= \frac{(\Gamma(\alpha+\beta)M_1L)^n}{\Gamma(n\alpha+n\beta)} \int_0^t (t-s_n)^{\frac{n\alpha+n\beta-1}{q}} |x(s_n) - y(s_n)| (t-s_n)^{\frac{n\alpha+n\beta-1}{p}} ds_n \\
&\leq \frac{(\Gamma(\alpha+\beta)M_1L)^n}{\Gamma(n\alpha+n\beta)} \left(\int_0^t (t-s_n)^{n\alpha+n\beta-1} |x(s_n) - y(s_n)|^q\, ds_n\right)^{\frac{1}{q}} \left(\int_0^t (t-s_n)^{n\alpha+n\beta-1} ds_n\right)^{\frac{1}{p}} \\
&\leq \frac{(\Gamma(\alpha+\beta)M_1L)^n}{(n\alpha+n\beta)^{\frac{1}{p}}\Gamma(n\alpha+n\beta)} \left(\int_0^t (t-s_n)^{n\alpha+n\beta-1} |x(s_n) - y(s_n)|^q\, ds_n\right)^{\frac{1}{q}}.
\end{aligned}
$$

Therefore, we conclude

$$
\|T^n x - T^n y\|_q \leq \frac{(\Gamma(\alpha+\beta)M_1L)^n}{\Gamma(n(\alpha+\beta)+1)}\|x-y\|_q,
$$

for every $n \in \mathbb{N}$ and all $x, y \in L^q[0,1]$. Now let $\theta_n = \frac{(\Gamma(\alpha+\beta)M_1L)^n}{\Gamma(n(\alpha+\beta)+1)}$. From the definition of the generalized Mittag–Leffler functions, we have $\sum_{n=0}^{\infty}\theta_n = E_{\alpha+\beta}\left(\Gamma(\alpha+\beta)M_1L\right)$ and hence the series $\sum_{n=0}^{\infty}\theta_n$ converges. Therefore, the existence of the unique fixed point of T follows from Weissinger's fixed point Theorem. □

4. Illustrative Examples

In this section, some examples are provided to show the applicability of the analytical achievements of the paper.

Example 1. *Consider the initial value problem*

$$
\begin{cases}
D^{\frac{4}{5}}\left(D^{\frac{1}{2}}+\lambda\right)x(t) = 1 + t^2 + \dfrac{\sin t + \arctan x(t)}{2e^t\sqrt[3]{t}} & 0 < t \leq 1, \\
x(0) = 1, \quad D^{\frac{1}{2}}x(0) = 1.
\end{cases}
\tag{10}
$$

Here $f(t,x) = 1 + t^2 + \frac{\sin t + \arctan x}{2e^t \sqrt[3]{t}}$, $\alpha = \frac{1}{2}, \beta = \frac{4}{5}$ *and the friction constant* $\lambda \geq 0$.

Let $p = q = 2$. *Clearly,* $f(t,0) = 1 + t^2 + \frac{\sin t}{2e^t \sqrt[3]{t}}$ *and* $f(t,0) \in L^2[0,1]$. *In fact, it is easily seen that* $\|f(t,0)\|_2 \leq 1 + \left(\frac{1}{5}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{\Gamma(\frac{1}{3})}{\sqrt[3]{2}}\right)^{\frac{1}{2}}$. *On the other hand,* $|f(t,x) - f(t,y)| \leq \frac{1}{2e^t \sqrt[3]{t}}|x-y|$ *with* $a(t) = \frac{1}{2e^t \sqrt[3]{t}}$. *Similarly, we see that* $a \in L^2[0,1]$ *and* $\|a\|_2 \leq \frac{1}{2}\left(\frac{\Gamma(\frac{1}{3})}{\sqrt[3]{2}}\right)^{\frac{1}{2}}$. *Further, from Remark 1 it follows that* $M_1 = \sup_{t\in[0,1]}\left|E_{\frac{1}{2},\frac{13}{10}}(-\lambda t^{\frac{1}{2}})\right| \leq \frac{1}{\Gamma(\frac{13}{10})}$. *Therefore,*

$$R = \frac{M_1\|a\|_p}{(1-q+q(\alpha+\beta))^{\frac{1}{q}}} < \frac{\frac{1}{2}\sqrt{\frac{\Gamma(\frac{1}{3})}{\sqrt[3]{2}}}}{\sqrt{1.6}\Gamma\left(\frac{13}{10}\right)} = 0.64226 < 1.$$

Note that the contraction constant R is independent of friction constant λ. *Thus, by Theorem 2, the initial value problem (10) has a unique solution in* $L^2[0,1]$.

Example 2. *Consider the initial value problem*

$$\begin{cases} D^{\frac{1}{3}}\left(D^{\frac{5}{4}} + \lambda\right)x(t) = g(t)\dfrac{|x(t)|}{1+|x(t)|} & 0 < t \leq 1, \\ x(0) = 1, \quad x'(0) = -1, \quad D^{\frac{5}{4}}x(0) = 1, \end{cases} \tag{11}$$

where $g \in L^\infty[0,1]$ *and the friction constant* $\lambda \in \mathbb{R}$.

Observe that $f(t,0) = 0$ *and* $|f(t,x) - f(t,y)| \leq L|x-y|$ *for almost every* $t \in [0,1]$ *with* $L = \|g\|_\infty$. *Thus, by Theorem 3, the initial value problem (11) has a unique solution in* $L^\infty[0,1]$.

5. Conclusions

In this article, we have considered initial value problem of nonlinear fractional Langevin equation involving two fractional orders. As a first step, by applying the tools of fractional calculus and using some basic properties of Prabhakar integral operator, we build a general structure of solutions associated with our proposed model. Once the fixed point operator equation is available, the existence results are established by means of contraction mapping theorem and Weissinger's fixed point theorem. Finally, two examples were presented to support the result.

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