On the Existence of Eigenvalues of a Boundary Value Problem with Transmitting Condition of the Integral Form for a Parabolic-Hyperbolic Equation

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Abstract: In the paper, we investigate a local boundary value problem with transmitting condition of the integral form for mixed parabolic-hyperbolic equation with non-characteristic line of type changing. Theorem on strong solvability of the considered problem has been proved and integral representation of the solution is obtained in a functional space. Using Lidskii Theorem on coincidences of matrix and spectral traces of nuclear operator and Gaal's formula for evaluating traces of nuclear operator, which is represented as a product of two Hilbert-Schmidt operators, we prove the existence of eigenvalues of the considered problem.

Keywords: transmitting condition; parabolic-hyperbolic equation; Green's function

MSC: 35M10

1. Introduction

Main problems of the spectral theory of boundary value problems (BVPs) for mixed type equations can be divided as follows:

(1) characterization of the spectrum of boundary problems;

(2) construction of root (eigenfunctions and associated functions) functions;

(3) investigation of the completeness and basis property of root functions in various functional spaces.

Investigation of BVPs for mixed type equations becomes one of the main problems of the general theory of partial differential equations due to several applications of it in both in practice and theory. Nevertheless, despite the great attention to this problem by mathematicians, questions of the spectral theory of BVPs, in particular, for equations of mixed parabolic-hyperbolic type equations with integral transmitting conditions, remained hitherto unexplored.

In the work [1], an analog problem to the generalized Tricomi problem, with integral gluing conditions for mixed parabolic-hyperbolic equation, was studied. Theorems on strong solvability and on the absence of eigenvalues were proved. In [1] one can find historical information and notation on main scientific results on the related field.

Omitting huge amount of work, we just note some of them, which are closely related to the present problem. One of the first investigations of BVPs with non-continuous transmitting conditions for parabolic-hyperbolic equations was work [2]. In [3] authors investigated initial-boundary value

problems for mixed type equations in multi-dimensional domains, which appear at studying problems on motion of conducting fluid in an electromagnetic field.

In the work [4] the propagation of electrical oscillations in composite lines, when on the interval $0 < x < l$ of the semi-infinite line losses are neglected, and the rest of the line is considered as a cable without leakage was reduced to solving the system of equations

$$\left. \begin{array}{l} L\dfrac{\partial I_1}{\partial t} + \dfrac{\partial U_1}{\partial x} = 0, \quad C_1\dfrac{\partial U_1}{\partial t} + \dfrac{\partial I_1}{\partial x} = 0, \quad 0 < x < l \\[2ex] RI_2 + \dfrac{\partial U_2}{\partial x} = 0, \quad C_2\dfrac{\partial U_2}{\partial t} + \dfrac{\partial I_2}{\partial x} = 0, \quad l < x < \infty \end{array} \right\}$$

with initial

$$U_1|_{t=0} = 0, \quad I_1|_{t=0} = 0, \quad U_2|_{t=0} = 0$$

and boundary conditions

$$U_1|_{x=0} = E(t), \quad \lim_{x \to \infty} U_2 = 0,$$

together with requirement of the continuity of the voltage and current

$$U_1|_{x=l} = U_2|_{x=l}, \quad I_1|_{x=l} = I_2|_{x=l}.$$

Here L, C_1 are inductance and capacitance (per unit length) of the first part of the line; R, C_2 are resistance and capacitance of the second part.

It is not difficult to verify that if one excludes current from the system, the following parabolic-hyperbolic equation

$$0 = \begin{cases} a_1^2 U_{xx} - U_{yy}, & 0 < x < l, \\[2ex] a_2^2 U_{xx} - U_y, & l < x < +\infty \end{cases}$$

can be deduced together with boundary conditions

$$U(x,0) = 0, \; U_y(x,0) = 0, \; 0 < x < l, \; U(x,0) = 0, \; l < x < \infty,$$

$$U(0,y) = E(y), \quad \lim_{x \to +\infty} U(x,y) = 0.$$

In this case transmitting condition will have form of

$$U(l-0,y) = U(l+0,y), \; U_x(l+0,y) = \frac{R}{L}\int_0^x U_x(l-0,\eta)d\eta,$$

where

$$a_1^2 = \frac{1}{LC_1}, \quad a_2^2 = \frac{1}{RC_2}.$$

Another example of an application can be found in the work [5].

Investigation of the unique solvability and spectral questions of some BVPs with integral transmitting conditions for parabolic-hyperbolic equations were done in works [6–10]. Regarding the investigation of semilinear parabolic-equations we refer the readers to the works [11,12].

We note that in [6,8,10], classical and generalized solvability questions of local and nonlocal problems for mixed parabolic-hyperbolic type equations were discussed. Spectral properties of boundary problems with a shift for wave equation was studied in [13]. The spectrum and basis properties of the operator eigenfunction systems, comparable to the boundary value problem for some

linear systems of PDEs have been studied in [14]. Properties of the spectrum of an elliptic boundary value problem were the subject of study in [15].

In general, spectral theory for elliptic type equations is well-developed, while similar theory for the wave type and mixed type equations is still under development. Strong solvability and the Volterra property (the absence of eigenvalues) for the analog of the Tricomi problem for mixed parabolic-hyperbolic type equation with non-characteristic line of type-changing (in the domain with the deviation from characteristics) was studied in [1]. The existence of eigenvalues for a class of local boundary problems for the second and third order parabolic-hyperbolic type equations were investigated in [7,9].

In this regard, there is a question on the possibility of formulation and investigation of boundary problems with special transmitting condition for such equations, whose eigenvalues does exist. Therefore, the main aim of the present work is to answer this question. We formulate the correct problem with integral transmitting condition for a mixed parabolic-hyperbolic type equation and prove the existence of eigenvalues for this problem.

It is well-known that general spectral theory has many applications in different branches of mathematics. Especially, spectral theory of differential operators becomes main tool of many investigations related to real-life problems [16,17]. We would like also note that spectral properties of BVPs for mixed type equations will be used at studying qualitative properties of higher order mixed type PDEs.

2. Formulation of the Problem

Let $\Omega \subset \mathbb{R}^2$ be a finite simple-connected domain (See Figure 1), bounded at $y > 0$ by segments AA_0, A_0B_0, B_0B $(A = (0,0), A_0 = (0,1), B_0 = (1,1), B(1,0))$, and at $y < 0$ by characteristics $AC : x + y = 0$, $BC : x - y = 1$ of the equation

$$Lu = f(x,y), \tag{1}$$

where $C(1/2, -1/2)$,

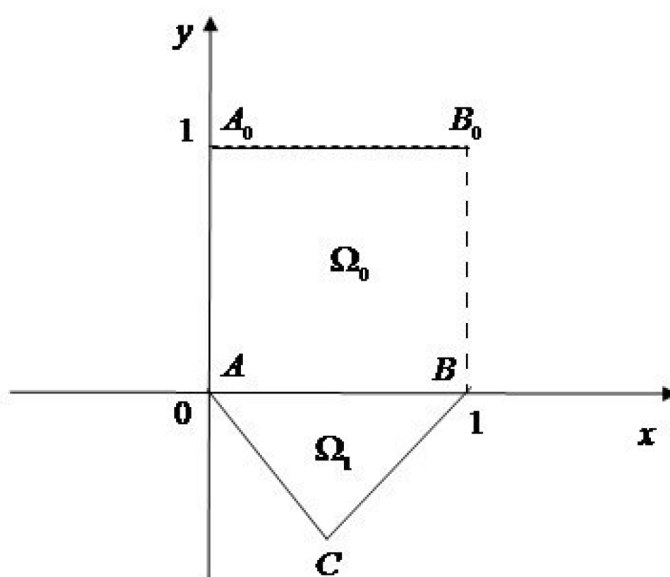$$Lu \equiv \begin{cases} u_x - u_{yy}, & y > 0, \\ u_{xx} - u_{yy}, & y < 0. \end{cases}$$

Figure 1. Domain Ω.

We consider the following variant of the Tricomi problem for a parabolic-hyperbolic equation.

Problem A. Find a solution of the Equation (1), satisfying boundary conditions

$$u\,|_{AA_0 \cup A_0 B_0} = 0, \tag{2}$$

$$u_x + u_y\,|_{BC} = 0 \tag{3}$$

and transmitting condition on the type-changing line

$$u_x(x, +0) = u_x(x, -0), \quad u_y(x, +0) = \alpha u_y(x, -0) - \beta \int_0^x u_y(t, -0)dt,\ 0 < x < 1, \tag{4}$$

where α, $\beta \in \mathbb{R}$, such that $\alpha^2 + \beta^2 > 0$.

Denote $\Omega_0 = \Omega \cap \{y > 0\}$, $\Omega_1 = \Omega \cap \{y < 0\}$. We introduce the following set of functions:

$$\mathbb{W} := \{u : \Omega \to \mathbb{R},\ \text{such that } u \in C\,(\bar{\Omega}) \cap C^{1,2}\,(\bar{\Omega}_0) \cap C^{2,2}\,(\bar{\Omega}_1)\ \text{satisfying (2)–(4)}\}.$$

Function $u \in W_2^1\,(\Omega)$ we call as *strong solution* of the problem, if there exists a sequence of functions $\{u_n\}$, $u_n \in \mathbb{W}$, such that $\|u_n - u\|_{W_2^1(\Omega_0)} + \|u_n - u\|_{W_2^1(\Omega_1)} \to 0$, $\|Lu_n - f\|_0 \to 0$ for $n \to \infty$. Here and further, by $\|\cdot\|_l$ we denote norm of the Sobolev space $W_2^l\,(\Omega)$, where $W_2^0\,(\Omega) = L_2\,(\Omega)$.

Remark 1. *We note that, generally speaking, the strong solution (in our understanding) does not satisfy neither equation nor boundary conditions (in classical sense) and it is associated with the closure of differential operator L in $L_2(\Omega)$ of the corresponding problem.*

If by $\mathcal{L}$ we denote the closure of differential operator L in $L_2(\Omega)$ on the set W, then, $D(\mathcal{L})$, the set of definition of operator $\mathcal{L}$, consists of the strong solutions to the Problem A in the sense of our definition

3. Main Result

Theorem 1. *For any function $f \in L_2\,(\Omega)$ there exists a unique strong solution of Problem A. This solution belongs to the class of functions $W_2^1\,(\Omega) \cap W_2^{1,2}\,(\Omega_0) \cap C\,(\bar{\Omega})$, satisfies the following inequality*

$$\|u\|_{W_2^1(\Omega_0)} + \|u\|_{W_2^1(\Omega_1)} \le c\,\|f\|_0 \tag{5}$$

and represented as

$$u\,(x, y) = \iint_\Omega K\,(x, y; x_1, y_1)\,f\,(x_1, y_1)\,dx_1 dy_1, \tag{6}$$

where $K \in L_2\,(\Omega \times \Omega)$.

Remark 2. *It should be noted that the kernel $K(x, y; x_1, y_1)$ is expressed in the explicit form through the Green's function and the solutions of the second kind Volterra type integral equation. The kernel K is presented in the Proof of Theorem 1 (see the Formula (20)).*

Proof of Theorem 1. According to the unique solvability of the first boundary problem for the heat equation with conditions (2) and $u\,(x, 0) = \tau\,(x)$, $\tau\,(0) = 0$ [18], and the Darboux problem for the

wave equation with conditions (3), $u(x,0) = \tau(x)$, $\tau(0) = 0$ [19], solution of the Equation (1) can be represented as follows

$$
u(x,y) = \begin{cases} \int\limits_0^x dx_1 \int\limits_0^1 G(x - x_1, y, y_1) f(x_1, y_1) \, dy_1 + \\ + \int\limits_0^x G_{y_1}(x - x_1, y, 0) \tau(x_1) \, dx_1, \quad y > 0, \\ \\ \int\limits_\xi^\eta d\xi_1 \int\limits_\eta^1 f_1(\xi_1, \eta_1) \, d\eta_1 + \tau(\eta), \quad y < 0, \end{cases} \tag{7}
$$

where $f_1(\xi, \eta) = \frac{1}{4} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$, $\xi = x + y$, $\eta = x - y$, and $G(x - x_1, y, y_1)$ is Green's function of the first boundary problem for heat equation in a rectangle AA_0B_0B, which has a form [20]:

$$
G(x, y, y_1) = \frac{1}{2\sqrt{\pi x}} \sum_{n=-\infty}^{+\infty} \left[e^{-\frac{(y-y_1+2n)^2}{4x}} - e^{-\frac{(y+y_1+2n)^2}{4x}} \right]. \tag{8}
$$

Considering (8), calculating derivative $\frac{\partial u}{\partial y}$, and passing to the limit as y tends to zero in Ω_0, we obtain first functional relation between functions $\tau(x)$ and $\nu_1(x) = \frac{\partial u}{\partial y}(x, +0)$ given as

$$
\nu_1(x) = -\int\limits_0^x k(x - t) \tau'(t) \, dt + \Phi_0(x), \tag{9}
$$

where

$$
k(x) = \frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{+\infty} e^{-\frac{n^2}{x}}, \tag{10}
$$

$$
\Phi_0(x) = \int\limits_0^x dx_1 \int\limits_0^1 G_0(x - x_1, y_1) f(x_1, y_1) \, dy_1, \tag{11}
$$

$$
G_0(x, y_1) \equiv G_y(x, 0, y_1) = \frac{1}{4\sqrt{\pi}x^{3/2}} \sum_{n=-\infty}^{+\infty} \left[-(2n - y_1) e^{-\frac{(2n-y_1)^2}{4x}} + (2n + y_1) e^{-\frac{(2n+y_1)^2}{4x}} \right]
$$

$$
= \frac{1}{4\sqrt{\pi}x^{3/2}} \left(\sum_{n=-\infty}^{+\infty} -(-2n - y_1) e^{-\frac{(-2n-y_1)^2}{4x}} + \sum_{n=-\infty}^{+\infty} (2n + y_1) e^{-\frac{(2n+y_1)^2}{4x}} \right)
$$

or, which is the same,

$$
G_0(x, y_1) \equiv \frac{1}{2\sqrt{\pi}x^{3/2}} \left(\sum_{n=-\infty}^{+\infty} (2n + y_1) e^{-\frac{(2n+y_1)^2}{4x}} \right). \tag{12}
$$

Similarly, we find another integral-differential relation between functions $\tau(x)$ and $\nu_2(x) = \frac{\partial u}{\partial y}(x, -0)$ on AB, reduced from the domain Ω_1. It has the form

$$
\nu_2(x) = -\tau'(x) - 2\int\limits_x^1 f_1(x, t) \, dt, \quad 0 < x < 1. \tag{13}
$$

Let $\alpha \neq 0$. From (9) and (13) based on transmitting conditions (4), we deduce integral equation with respect to $\tau'(x)$:

$$\tau'(x) + \int_0^x k_1(x-t)\tau'(t)dt = F(x). \tag{14}$$

Here

$$k_1(x-t) = \frac{1}{\alpha}\left[k(x-t) + \beta\right], \tag{15}$$

$$F(x) = -\frac{1}{\alpha}\Phi_0(x) - 2\int_x^1 f_1(x,\eta_1)d\eta_1 + \frac{2\beta}{\alpha}\int_0^x dt \int_t^1 f_1(x,\eta_1)d\eta_1. \tag{16}$$

Thus, the problem is equivalently reduced to the second kind Volterra integral Equation (14). Since by (10), the kernel $k(x)$ can be represented as

$$k(x) = \frac{1}{\sqrt{\pi x}} + \tilde{k}(x),$$

where $\tilde{k}(x) \in C^\infty[0;1]$, then from (15) it follows that $k_1(x)$ has a weak singularity. Therefore, Problem (14) has a unique solution and it has a form

$$\tau'(x) = F(x) + \int_0^x \hat{\Gamma}(x-t) F(t) dt, \tag{17}$$

where $\hat{\Gamma}(x)$ is the resolvent kernel of (14), which is defined as

$$\hat{\Gamma}(x) = \sum_{j=1}^{\infty} k_j(x),$$

$$k_{j+1}(x) = \int_0^x k_1(x-t) k_j(t) dt, \ j \in N.$$

Considering $\tau(0) = 0$ after some evaluations, from (17) we get

$$\tau(x) = -\frac{1}{\alpha}\int_0^x dx_1 \int_0^1 G_1(x-x_1, y_1) f(x_1, y_1) dy_1 -$$

$$-2\int_0^x d\xi_1 \int_{\xi_1}^1 \Gamma_1(x, \xi_1) f_1(\xi_1, \eta_1) d\eta_1 + \frac{2\beta}{\alpha}\int_0^x d\xi_1 \int_{\xi_1}^1 f_1(\xi_1, \eta_1) d\eta_1 \int_{\xi_1}^x \Gamma_1(x, t) dt \tag{18}$$

where

$$\Gamma_1(x,t) = 1 + \int_t^x \hat{\Gamma}(z-t) dz, \tag{19}$$

$$G_1(x-x_1, y_1) = \int_{x_1}^x G_0(t-x_1, y_1) \Gamma_1(x,t) dt.$$

Substituting (18) into (7), we deduce the Formula (6), where

$$
K(x, y; x_1, y_1) = \theta(y)\{\theta(y_1)\theta(x - x_1)G_2(x - x_1, y, y_1) -
$$

$$
-\theta(-y_1)\left[\theta(x - \xi_1)G_3(x - \xi_1, y) + \tfrac{\beta}{\alpha}\theta(x - \xi_1)G_4(x - \xi_1, y)\right]\} +
$$

$$
+\theta(-y)\left\{-\tfrac{1}{\alpha}\theta(y_1)\theta(\eta - x_1)G_1(\eta - x_1, y_1) + \theta(-y_1)\left[\tfrac{1}{2}\theta(\eta - \xi_1)\times\right.\right. \tag{20}
$$

$$
\times\theta(\xi_1 - \xi)\theta(\eta_1 - \eta) - \theta(\eta - \xi_1)\left[\Gamma_1(\eta, \xi_1) + \tfrac{\beta}{\alpha}\int_{\xi_1}^{\eta}\hat{\Gamma}(\eta - t)\,dt\right]\right\},
$$

$$
G_2(x - x_1, y) = G(x - x_1, y, y_1) - \frac{1}{\alpha}\int_{x_1}^{x} G_{y_1}(x - t, y, 0)G_1(t - x_1, y_1)\,dt,
$$

$$
G_3(x - \xi_1, y) = \int_{\xi_1}^{x} G_{y_1}(x - t, y, 0)\Gamma_1(t, \xi_1)\,dt,
$$

$$
G_4(x - \xi_1, y) = \int_{\xi_1}^{x} dt \int_{\xi_1}^{x} G_{y_1}(x - t, y, 0)\Gamma_1(t, z)\,dz,
$$

$$
\theta(y) = \begin{cases} 1, & if \ \ y > 0, \\ 0, & if \ \ y < 0. \end{cases}
$$

Similarly as in [1] (see proof of the Lemma 1), one can prove that

$$
K(x, y; x_1, y_1) \in L_2(\Omega \times \Omega).
$$

Here one can face bulky expressions, which should be carefully considered.

Considering (11) by virtue of direct calculations from (16) we can state that the estimate

$$
\|F(x)\|_{L_2(0,1)} \le C\|f\|_0
$$

is valid. Therefore, from (17) we have

$$
\|\tau'(x)\|_{L_2(0,1)} \le C\|F(x)\|_{L_2(0,1)} \le C\|f\|_0.
$$

Based on this and properties of solution for the first boundary problem for heat equation [9], it follows that solution of the Problem A belongs to the class of functions $W_2^1(\Omega) \cap W_2^{1,2}(\Omega_0) \cap C(\bar{\Omega})$ and satisfies inequality (5).

Now we show that the obtained solution will be strong. Since $C_0^1(\bar{\Omega})$ is dense in $L_2(\Omega)$, then for any function $f \in L_2(\Omega)$ there exists a sequence of functions $f_n \in C_0^1(\bar{\Omega})$ such that $\|f_n - f\| \to 0$, $n \to 0$. Here $C_0^1(\bar{\Omega})$ is a set of differentiable functions in Ω, which are equal to zero in neighborhood of $\partial\Omega$ ($\partial\Omega$ is a boundary of the domain Ω). Denote that $u_n = \mathcal{L}^{-1}f_n$, where $\mathcal{L}^{-1}$ is inverse of the operator $\mathcal{L}$ of Problem A defined as

$$
\mathcal{L}^{-1}f(x, y) := \iint_{\Omega} K(x, y; x_1, y_1)f(x_1, y_1)\,dx_1dy_1.
$$

It is not difficult to verify that $F_n(x) \in C^1[0; 1]$ for any $f_n \in C_0^1(\bar{\Omega})$. Here by $F_n(x)$ we denote representation similar to (16), where $f(x, y)$ should be replaced with $f_n(x, y)$. Hence, Equation (14) we can consider as the second kind Volterra integral equation in the space $C^1[0; 1]$. Consequently,

$\tau'_n(x) = u_{nx}(x, 0) \in C^1[0; 1]$. Based on properties of the first boundary problem for heat equation and the Darboux problem for wave equation [9], considering representation (6), we get that $u_n \in \mathbb{W}$ for all $f_n \in C_0^1(\bar{\Omega})$.

By virtue of the inequality (5) we obtain

$$\|u_n - u\|_{W_2^1(\Omega_0)} + \|u_n - u\|_{W_2^1(\Omega_1)} \leq c \|f_n - f\|_0 \to 0.$$

Therefore, $\{u_n\}$ sequence satisfies all requirements of the definition of the strong solution. Now we can state that the Problem A is strongly solvable for any $f \in L_2(\Omega)$, and strong solution belongs to the class of functions $W_2^1(\Omega) \cap W_2^{1,2}(\Omega_0) \cap C(\bar{\Omega})$.

Theorem 1 is proved. $\square$

From Theorem 1 we can conclude that operator $\mathcal{L}$ of the Problem A is invertible, and inverse operator $\mathcal{L}^{-1}$ is Hilbert-Schmidt operator. There is a natural question on the existence of eigenvalues of the operator $\mathcal{L}^{-1}$, consequently, of the Problem A as well.

Before answering this question, we introduce the following result

Lemma 1. [21] If operator T is nuclear in a Hilbert space H, then for any orthonormal basis ϕ_i $(i = 1, 2, ...)$ in H, the equality

$$Sp(T) \equiv \sum_{k=1}^{\infty} (T\phi_k, \phi_k) = \sum_{k=1}^{\infty} \lambda_k(T) \tag{21}$$

holds true. Here λ_k are the eigenvalues of the operator T.

Theorem 2. *Let $\alpha > 0$, $\beta > 0$. There exists $\lambda \in \mathbb{C}$ such that the equation*

$$Lu = \lambda u$$

has a non-trivial solution $u \in \mathbb{W}$.

Proof of Theorem 2. We denote by $\mathcal{L}$ a closure in $L_2(\Omega)$ of the differential operator given in $\mathbb{W}$ by equality (1). From Theorem 1 it follows that $\mathcal{L}$ is invertible and $\mathcal{L}^{-1}$ defined by (6) is the Hilbert-Schmidt operator. Then operator $\mathcal{L}^{-2} \equiv \mathcal{L}^{-1} \circ \mathcal{L}^{-1}$ is nuclear in $L_2(\Omega)$. Therefore, we apply the result of the Lidskii [21] on the coincidences of the matrix and spectral traces to the operator $\mathcal{L}^{-2}$.

It is very well known that if operator T is nuclear in $L_2(\Omega)$, it can be represented as the composition $T = P \circ R$ of two Hilbert-Schmidt operators

$$(Pf)(z) = \int_{\Omega} P(z, z_1) f(z_1) dz_1$$

and

$$(Rf)(z) = \int_{\Omega} R(z, z_1) f(z_1) dz_1.$$

Then, by using Gaal's formula for calculating traces [22], we have that

$$Sp(T) = \int_{\Omega} \left[\int_{\Omega} P(z, z_1) R(z_1, z) dz_1 \right] dz \tag{22}$$

is true.

From (21) and (22) we deduce that

$$Sp\left(\mathcal{L}^{-2}\right) = \iint\limits_{\Omega} dxdy \iint\limits_{\Omega} K\left(x,y;x_1,y_1\right) K\left(x_1,y_1;x,y\right) dx_1dy_1.$$

From (20) it follows that

$$K\left(x,y;x_1,y_1\right) K\left(x_1,y_1;x,y\right) = \theta\left(y\right)\theta\left(-y_1\right)\theta\left(x-\xi_1\right)\theta\left(\eta_1-x\right) \times$$

$$\times\left[\tfrac{1}{\alpha}G_1(\eta_1-x,y)G_3(x-\xi_1,y) + \tfrac{\beta}{\alpha}G_1\left(\eta_1-x,y\right)G_4\left(x-\xi_1,y\right)\right] +$$

$$+\theta\left(-y\right)\theta\left(y_1\right)\theta\left(\eta-x_1\right)\theta\left(x_1-\xi\right)\left[\tfrac{1}{\alpha}G_1(\eta-x_1,y_1)\, G_3(x_1-\xi,y_1)+\right.$$

$$+\left.\tfrac{\beta}{\alpha^2}G_1\left(\eta-x_1,y_1\right)G_4\left(x_1-\xi,y_1\right)\right] + \theta\left(-y\right)\theta\left(-y_1\right)\left\{-\tfrac{1}{2}\theta\left(\eta-\xi_1\right)\theta\left(\xi_1-\xi\right)\times\right.$$

$$\times\theta\left(\eta_1-\eta\right)\theta\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta_1,\xi\right) - \tfrac{\beta}{\alpha}\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt\right]d\eta_1 - \tfrac{1}{2}\theta\left(\eta-\xi_1\right)\times$$

$$\times\theta\left(\eta_1-\xi\right)\theta\left(\xi-\xi_1\right)\theta\left(\eta-\eta_1\right)\left[\Gamma_1\left(\eta_1,\xi\right) - \tfrac{\beta}{\alpha}\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt\right] +$$

$$+\theta\left(\eta-\xi_1\right)\theta\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta_1,\xi\right)\Gamma_1\left(\eta,\xi_1\right) + \tfrac{\beta}{\alpha}\Gamma_1\left(\eta_1,\xi\right)\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt +\right.$$

$$+\left.\tfrac{\beta}{\alpha}\Gamma_1\left(\eta,\xi_1\right)\int\limits_{\xi}^{\eta_1}\hat{\Gamma}\left(\eta_1-t\right)dt + \left(\tfrac{\beta}{\alpha}\right)^2\int\limits_{\xi}^{\eta_1}\hat{\Gamma}\left(\eta_1-t\right)dt\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt\right]\right\}.$$

Therefore,

$$Sp\mathcal{L}^{-2} = \tfrac{2}{\alpha}\iint\limits_{\Omega_1} dxdy \iint\limits_{\Omega_2}\theta\left(x-\xi_1\right)\theta\left(\eta_1-x\right)\left[G_1(\eta_1-x,y)G_4(x-\xi_1,y)+\right.$$

$$+\left.\tfrac{\beta}{\alpha}G_1\left(\eta_1-x,y\right)G_3\left(x-\xi_1,y\right)\right]dx_1dy_1 - \iint\limits_{\Omega_2} dxdy \iint\limits_{\Omega_2}\theta\left(\eta-\xi_1\right)\times$$

$$\times\theta\left(\xi_1-\xi\right)\theta\left(\eta_1-\eta\right)\theta\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta_1,\xi\right) + \tfrac{\beta}{\alpha}\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt\right]dx_1dy_1+$$

$$+ \iint\limits_{\Omega_2} dxdy \iint\limits_{\Omega_2}\theta\left(\eta-\xi_1\right)\theta\left(\eta_1-\xi\right)\left[\Gamma_1\left(\eta_1,\xi\right)\Gamma_1\left(\eta,\xi_1\right) + \tfrac{\beta}{\alpha}\Gamma_1\left(\eta_1,\xi\right)\times\right.$$

$$\times\int\limits_{\xi_1}^{\eta}\hat{\Gamma}\left(\eta-t\right)dt + \tfrac{\beta}{\alpha}\Gamma_1\left(\eta,\xi_1\right)\int\limits_{\xi}^{\eta_1}\hat{\Gamma}\left(\eta_1-t\right)dt+$$

$$+\left.\left(\tfrac{\beta}{\alpha}\right)^2\int\limits_{\xi}^{\eta_1}\hat{\Gamma}\left(\eta_1-t\right)dt\int\limits_{\xi_1}^{\eta}\Gamma\left(\eta-t\right)dt\right]dx_1dy_1 = \sum_{k=1}^{3} I_k,$$

(23)

where

$$I_1 = \tfrac{2}{\alpha}\iint\limits_{\Omega_1} dxdy \iint\limits_{\Omega_2}\theta\left(x-\xi_1\right)\theta\left(\eta_1-x\right)\left[G_1(\eta_1-x,y)G_4(x-\xi_1,y)+\right.$$

$$+\left.\tfrac{\beta}{\alpha}G_1\left(\eta_1-x,y\right)G_3\left(x-\xi_1,y\right)\right]dx_1dy_1,$$

$$I_2 = - \iint\limits_{\Omega_2} dx dy \iint\limits_{\Omega_2} \theta\left(\eta - \xi_1\right) \theta\left(\xi_1 - \xi\right) \theta\left(\eta_1 - \eta\right) \theta\left(\eta_1 - \xi\right) \left[\Gamma_1\left(\eta_1, \xi\right) + \frac{\beta}{\alpha} \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt\right] dx_1 dy_1,$$

$$I_3 = \iint\limits_{\Omega_2} dx dy \iint\limits_{\Omega_2} \theta\left(\eta - \xi_1\right) \theta\left(\eta_1 - \xi\right) \left[\Gamma_1\left(\eta_1, \xi\right) \Gamma_1\left(\eta, \xi_1\right) + \frac{\beta}{\alpha} \Gamma_1\left(\eta_1, \xi\right) \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt + \right.$$

$$\left. + \frac{\beta}{\alpha} \Gamma_1\left(\eta, \xi_1\right) \int\limits_{\xi}^{\eta_1} \hat{\Gamma}\left(\eta_1 - t\right) dt + \left(\frac{\beta}{\alpha}\right)^2 \int\limits_{\xi}^{\eta_1} \hat{\Gamma}\left(\eta_1 - t\right) dt \int\limits_{\xi_1}^{\eta} \Gamma\left(\eta - t\right) dt\right] dx_1 dy_1.$$

Let us show that $\sum\limits_{k=1}^{3} I_k > 0$. In fact

$$I_3 + I_2 = \frac{1}{4} \int\limits_0^1 d\eta \int\limits_0^{\eta} d\xi \int\limits_0^1 d\xi_1 \int\limits_{\xi_1}^1 \theta\left(\eta - \xi_1\right) \theta\left(\eta_1 - \xi\right) \left[\Gamma_1\left(\eta, \xi_1\right) \Gamma_1\left(\eta_1, \xi\right) + \right.$$

$$+ \frac{\beta}{\alpha} \Gamma_1\left(\eta_1, \xi\right) \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt + \frac{\beta}{\alpha} \Gamma_1\left(\eta, \xi_1\right) \int\limits_{\xi}^{\eta_1} \hat{\Gamma}\left(\eta_1 - t\right) dt +$$

$$\left. + \left(\frac{\beta}{\alpha}\right)^2 \int\limits_{\xi}^{\eta_1} \hat{\Gamma}\left(\eta_1 - t\right) dt \int\limits_{\xi_1}^{\eta} \Gamma\left(\eta - t\right) dt\right] - \frac{1}{4} \int\limits_0^1 d\eta \int\limits_0^{\eta} d\xi \int\limits_0^1 d\xi_1 \int\limits_{\xi_1}^1 \theta\left(\eta - \xi_1\right) \times$$

$$\times \theta\left(\xi_1 - \xi\right) \theta\left(\eta_1 - \eta\right) \theta\left(\eta_1 - \xi\right) \left[\Gamma_1\left(\eta_1, \xi\right) + \frac{\beta}{\alpha} \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt\right] d\eta_1 = \tag{24}$$

$$= \frac{1}{4} \int\limits_0^1 d\eta \int\limits_0^{\eta} d\xi \int\limits_0^{\eta} d\xi_1 \int\limits_{\xi_1}^1 \theta\left(\eta_1 - \xi\right) \left[\Gamma_1\left(\eta_1, \xi\right) + \frac{\beta}{\alpha} \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt\right] \times$$

$$\times \left\{\Gamma_1\left(\eta, \xi_1\right) + \frac{\beta}{\alpha} \int\limits_{\xi}^{\eta} \hat{\Gamma}\left(\eta_1 - t\right) dt - \theta\left(\xi_1 - \xi\right) \theta\left(\eta_1 - \eta\right)\right\} d\eta_1 > 0,$$

Since, due to (10), (15), (19) $\Gamma_1\left(\eta, \xi_1\right) \geq 1$ and $\alpha > 0$, $\beta > 0$, then

$$\Gamma_1\left(\eta, \xi_1\right) + \frac{\beta}{\alpha} \int\limits_{\xi_1}^{\eta} \hat{\Gamma}\left(\eta - t\right) dt - \theta\left(\xi_1 - \xi\right) \theta\left(\eta_1 - \eta\right) > 0.$$

Now consider I_1. We have

$$
I_1 = \frac{2}{\alpha} \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 d\xi_1 \int\limits_{\xi_1}^1 \theta\left(x - \xi_1\right) \theta\left(\eta_1 - x\right) G_1(\eta_1 - x, y) \left[G_4(x - \xi_1, y) + \right.
$$

$$
\left. + \frac{\beta}{\alpha} G_3\left(x - \xi_1, y\right) \right] d\eta_1 = \frac{2}{\alpha} \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^x d\xi_1 \int\limits_x^1 G_1\left(\eta_1 - x, y\right) \left[G_4(x - \xi_1, y) + \right.
$$

$$
\left. + \frac{\beta}{\alpha} G_3\left(x - \xi_1, y\right) \right] d\eta_1 = \frac{2}{\alpha} \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^x d\xi_1 \int\limits_0^{1-x} G_1\left(x - \xi_1, y\right) \times \tag{25}
$$

$$
\left[G_4(\eta_2, y) + \frac{\beta}{\alpha} G_3\left(\eta_2, y\right) \right] d\eta_2 = \frac{2}{\alpha} \int\limits_0^1 dx \int\limits_0^1 dy \left(\int\limits_0^x G_1\left(\xi_2, y\right) d\xi_2 \right) \times
$$

$$
\left(\int\limits_0^{1-x} \left[G_4(\eta_2, y) + \frac{\beta}{\alpha} G_3\left(\eta_2, y\right) \right] d\eta_2 \right).
$$

Function G_1 we represent as

$$
G_1\left(\xi, y\right) = \int\limits_0^\xi G_0\left(t, y\right) dt + \int\limits_0^\xi G_0\left(t, y\right) dt \int\limits_0^{\xi - t} \hat{\Gamma}\left(\tau\right) d\tau. \tag{26}
$$

Taking (12) into account, we study the sign of the first term. For this, we use the following transformations:

$$
\int\limits_0^\xi G_0\left(t, y\right) dt = \frac{1}{2\sqrt{\pi}} \sum\limits_{n=-\infty}^\infty \int\limits_0^\xi \frac{y + 2n}{t^{3/2}} e^{-\frac{(y+2n)^2}{4t}} dt
$$

$$
= -\frac{2}{\sqrt{\pi}} \sum\limits_{n=-\infty}^{-1} \int\limits_{-\infty}^{\frac{y+2n}{2\sqrt{\xi}}} e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \sum\limits_{n=0}^\infty \int\limits_{\frac{y+2n}{2\sqrt{\xi}}}^{+\infty} e^{-t^2} dt =
$$

$$
= \frac{2}{\sqrt{\pi}} \left[\sum\limits_{n=0}^\infty \int\limits_{\frac{y+2n}{2\sqrt{\xi}}}^\infty e^{-t^2} dt - \sum\limits_{n=1}^\infty \int\limits_{-\frac{y+2n}{2\sqrt{\xi}}}^\infty e^{-t^2} dt \right] =
$$

$$
= \frac{2}{\sqrt{\pi}} \left[\sum\limits_{n=0}^\infty \int\limits_{\frac{y+2n}{2\sqrt{\xi}}}^\infty e^{-t^2} dt - \sum\limits_{n=0}^\infty \int\limits_{\frac{2n-y+2}{2\sqrt{\xi}}}^\infty e^{-t^2} dt \right] = \frac{2}{\sqrt{\pi}} \sum\limits_{n=0}^\infty \int\limits_{\frac{y+2n}{2\sqrt{\xi}}}^{\frac{2n+2-y}{2\sqrt{\xi}}} e^{-t^2} dt.
$$

From here we get

$$
\int\limits_0^\xi G_0\left(t, y\right) dt \geq 0. \tag{27}
$$

The equality in (27) will be true only when $y = 1$, i.e.,

$$
\int\limits_0^\xi G_0\left(t, y\right) dt \neq 0 \quad \text{on } [0, 1].
$$

Then considering $\hat{\Gamma}(\tau) > 0$, with respect to the second item of (26) we have

$$\int\limits_0^\xi G_0(t,y)\,dt \int\limits_0^{\xi-t} \hat{\Gamma}(\tau)\,d\tau = \int\limits_0^\xi \hat{\Gamma}(\tau)\,d\tau \int\limits_0^{\xi-\tau} G_0(t,y)\,dt \geq 0 \ (\neq 0).$$

Similarly, we can prove that the second item of (25) is as well positive. Hence, from (25) we can state that $I_1 > 0$. From (23)–(25) it follows that $Sp\left(\mathcal{L}^{-2}\right) > 0$.

Then by virtue of (20), we have

$$\sum_{k=1}^\infty \lambda_k\left(\mathcal{L}^{-2}\right) \equiv \sum_{k=1}^\infty \lambda_k^2\left(\mathcal{L}^{-1}\right) > 0,$$

where $\lambda_k\left(\mathcal{L}^{-2}\right)$ are eigenvalues of $\mathcal{L}^{-2}$. It means that $\sum_{k=1}^\infty \frac{1}{\lambda_k^2} > 0$, where λ_k are eigenvalues of the problem (1)–(3). From here, the existence of eigenvalues of the Problem A follows.

Theorem 2 is proved. $\square$

4. Conclusions

In this work, we formulate correct boundary problem with integral transmitting condition for parabolic-hyperbolic type equation and proved strong solvability and the existence of eigenvalues for considered problem. Due to integral transmitting conditions, expressions obtained during evaluations have composite forms and we carefully used necessary actions on them in order to get required results. General idea of investigation tool is similar to previous works, cited in the paper, but one needs some modifications due to specific form of transmitting condition. This work shows that a transmitting condition has a specific role on solvability of considered problem. The same problem with different transmitting conditions will have different spectral properties (See [1]).

Conflicts of Interest: The authors declare no conflict of interest.

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