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EXISTENCE OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS WITH FRACTIONAL ORDER AND IMPULSES

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Abstract. We establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential inclusions involving the Caputo fractional derivative. We consider the cases when the multivalued nonlinear term takes convex values as well as nonconvex values. The topological structure of the set of solutions is also considered.

1. Introduction

This article studies the existence and uniqueness of solutions for the initial value problems (IVP for short), for fractional order differential inclusions,

$$^cD^\alpha y(t) \in F(t, y(t)), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m,\ 1 < \alpha \leq 2, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.3}$$

$$y(0) = y_0, \quad y'(0) = y_1, \tag{1.4}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), I_k and $\overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, and $y_0, y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \dots, m$.

Differential equations of fractional order have recently proved to be valuable tools in modeling many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4, 19, 26, 32, 35]). There has been a significant development in fractional differential and inclusions in recent years; see the monographs of Kilbas *et al* [29], Podlubny [34], Samko *et al* [36] and the papers of Agarwal *et al* [1], Belarbi *et al* [6, 7], Benchohra *et al* [8, 9, 11, 12], Chang and Nieto [15], Diethelm *et al* [19, 20], Furati and Tatar [23], Henderson and Ouahab [25], Kilbas and Marzan [28], Mainardi [32], Ouahab [33], and Zhang [40] and the references therein.

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Impulsive integer order differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra *et al* [10], Lakshmikantham *et al* [31], and Samoilenko and Perestyuk [37] and the papers [2, 38, 39]. To the best knowledge of the authors, no papers exist in the literature devoted to differential inclusions with fractional order and impulses. In [3, 13] some classes of of fractional differential equations with impulses have been considered. The aim of this paper is to continue this study. Thus the results of the present paper initiate this subject.

This paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present an existence result for the problem (1.1)-(1.4), when the right hand side is convex valued using the nonlinear alternative of Leray-Schauder type. In Section 4 two results are given for nonconvex valued right hand side. The first one is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler, and the second on the nonlinear alternative of Leray-Schauder type [24] for single-valued maps, combined with a selection theorem due to Bressan-Colombo [14] for lower semicontinuous multivalued maps with decomposable values. The topological structure of the solutions set is considered in Section 5. An example is presented in the last section. These results extend to the multivalued case the paper by Benchohra and Slimani [13] and those considered in the above cited literature in the absence of impulsive effect. The present results constitute a contribution to this emerging field of research.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of continuous functions from J to $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

The space $AC^1(J,\mathbb{R})$ consists of functions $y : J \to \mathbb{R}$, which are absolutely continuous, whose first derivative, y' is absolutely continuous. Let $(X, \|\cdot\|)$ be a Banach space. let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e.

$x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [5], Deimling [18] and Hu and Papageorgiou [27].

Definition 2.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in \mathbb{R}$;
(ii) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in PC(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space [30].

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x, y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.3 ([17]). *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Definition 2.4 ([29, 34]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.5 ([29, 34]). For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

Sufficient conditions for the fractional differential and fractional integrals to exist are given in [29].

3. The Convex Case

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.4) when the right hand side has convex values. Initially, we assume that F is a compact and convex valued multivalued map. Consider the Banach space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : y \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,\dots,m+1 \text{ and there exist } y(t_k^-), y(t_k^+),\ k = 1,\dots,m \text{ with } y(t_k^-) = y(t_k)\}.$$

with the norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Set $J' := [0,T]\backslash\{t_1,\dots,t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R}) \bigcap \cup_{k=0}^m AC^1((t_k,t_{k+1}),\mathbb{R})$ with its α-derivative exists on J' is said to be a solution of (1.1)–(1.4) if there exists a function $v \in L^1([0,T],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. $t \in J$ satisfies the differential equation ${}^cD^\alpha y(t) = v(t)$ on J', and conditions

$$\begin{gathered}\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m,\\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m,\\ y(0) = y_0, \quad y'0) = y_1\end{gathered}$$

are satisfied.

Let $h : [a,b] \to \mathbb{R}$ be a continuous function. For the existence of solutions for the problem (1.1)–(1.4), we need the following auxiliary lemmas.

Lemma 3.2 ([40])**.** *Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \dots + c_{n-1}(t-a)^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

Lemma 3.3 ([40])**.** *Let $\alpha > 0$, then*

$$I^{\alpha\,c}D^\alpha h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \dots + c_{n-1}(t-a)^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

As a consequence of Lemma 3.2 and Lemma 3.3 we have the following result which is useful in what follows.

Lemma 3.4. *Let $1 < \alpha \le 2$ and let $\rho \in PC(J,\mathbb{R})$. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + y_1 t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)ds & \text{if } t\in[0,t_1],\\ y_0 + y_1 t + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\rho(s)ds & \\ +\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^k (t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\rho(s)ds & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\rho(s)ds & \\ +\sum_{i=1}^k I_i(y(t_i^-)) + \sum_{i=1}^k (t-t_i)\overline{I}_i(y(t_i^-)), & \text{if } t\in(t_k,t_{k+1}]\\ & k=1,\dots,m \end{cases} \tag{3.1}$$

if and only if y is a solution of the fractional initial-value problem

$${}^cD^\alpha y(t) = \rho(t), \quad \textit{for each } t \in J', \tag{3.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.3}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.4}$$

$$y(0) = y_0, \quad y'(0) = y_1. \tag{3.5}$$

Proof. Assume y satisfies (3.2)-(3.5). If $t \in [0, t_1]$ then ${}^cD^\alpha y(t) = \rho(t)$. Lemma 3.3 implies

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds.$$

Hence $c_0 = y_0$ and $c_1 = y_1$. Thus

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds.$$

If $t \in (t_1, t_2]$ then Lemma 3.3 implies

$$y(t) = c_0 + c_1(t - t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \rho(s) ds. \tag{3.6}$$

$$\begin{aligned} \Delta y|_{t=t_1} &= y(t_1^+) - y(t_1^-) \\ &= c_0 - \Big(y_0 + y_1 t_1 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \rho(s) ds \Big) \\ &= I_1(y(t_1^-)). \end{aligned}$$

Hence

$$c_0 = y_0 + y_1 t_1 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \rho(s) ds + I_1(y(t_1^-)). \tag{3.7}$$

$$\begin{aligned} \Delta y'|_{t=t_1} &= y'(t_1^+) - y'(t_1^-) \\ &= c_1 - \Big(y_1 + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} \rho(s) ds \Big) \\ &= \overline{I}_1(y(t_1^-)), \end{aligned}$$

and

$$c_1 = y_1 + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} \rho(s) ds + \overline{I}_1(y(t_1^-)). \tag{3.8}$$

Then by (3.6)-(3.8), we have

$$\begin{aligned} y(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \rho(s) ds \\ &\quad + \frac{(t - t_1)}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} \rho(s) ds \\ &\quad + I_1(y(t_1^-)) + (t - t_1)\overline{I}_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \rho(s) ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 3.3 we obtain (3.1).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). If $t \in [0, t_1]$ then $y(0) = y_0$, $y'(0) = y_1$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$${}^cD^\alpha y(t) = \rho(t), \quad \text{for each } t \in [0, t_1].$$

If $t \in [t_k, t_{k+1})$, $k = 1, \dots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = \rho(t), \quad \text{for each } t \in [t_k, t_{k+1}). $$

Also, we can easily show that

$$ \begin{gathered} \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m. \end{gathered} $$

Our first result is based on the nonlinear alternative of Leray-Schauder type for multivalued maps [24]. We assume the following hypotheses:

(H1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multi-valued map;

(H2) there exist $p \in C(J, \mathbb{R}^+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$ \|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, u)\} \le p(t)\psi(|u|) $$

for $t \in J$ and $u \in \mathbb{R}$;

(H3) There exist $\psi^*, \overline{\psi}^* : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$ \begin{gathered} |I_k(u)| \le \psi^*(|u|) \quad \text{for } u \in \mathbb{R}, \\ |\overline{I}_k(u)| \le \overline{\psi}^*(|u|) \quad \text{for } u \in \mathbb{R}; \end{gathered} $$

(H4) There exists a number $\overline{M} > 0$ such that

$$ \frac{M}{|y_0| + T|y_1| + a\psi(M) + m\psi^*(M) + mT\overline{\psi}^*(M)} > 1, \tag{3.9} $$

where $p^0 = \sup\{p(t) : t \in J\}$ and

$$ a = \frac{mT^\alpha p^0}{\Gamma(\alpha + 1)} + \frac{mT^\alpha p^0}{\Gamma(\alpha)} + \frac{T^\alpha p^0}{\Gamma(\alpha + 1)}. $$

(H5) there exists $l \in L^1(J, \mathbb{R}^+)$ such that

$$ \begin{gathered} H_d(F(t, u), F(t, \overline{u})) \le l(t)|u - \overline{u}| \quad \text{for a.e. } t \in J.\ u, \overline{u} \in \mathbb{R}, \\ d(0, F(t, 0)) \le l(t), \quad \text{a.e. } t \in J. \end{gathered} $$

Theorem 3.5. *Under Assumptions* (H1)-(H5)*, the initial-value problem* (1.1)-(1.4) *has at least one solution on J.*

Proof. We transform (1.1)–(1.3) into a fixed point problem. Consider the multivalued operator

$$ \begin{aligned} N(y) = \Big\{ h \in PC(J, \mathbb{R}) : h(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s) ds \\ &+ \frac{1}{\Gamma(\alpha - 1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} v(s) ds \\ &+ \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t - t_k)\overline{I}_k(y(t_k^-)),\ v \in S_{F,y}. \Big\} \end{aligned} $$

Clearly, from Lemma 3.4, fixed points of N are solutions to (1.1)–(1.4). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [24]. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in PC(J,\mathbb{R})$. Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned}
h_i(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_i(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v_i(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_i(s)ds \\
&\quad + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)), \quad i = 1,2.
\end{aligned}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
&(dh_1 + (1-d)h_2)(t) \\
&= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)]ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} [dv_1(s) + (1-d)v_2(s)]ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)]ds \\
&\quad + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)).
\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: N maps bounded sets into bounded sets in $PC(J,\mathbb{R})$. Let $B_{\eta^*} = \{y \in PC(J,\mathbb{R}) : \|y\|_\infty \le \eta^*\}$ be bounded set in $PC(J,\mathbb{R})$ and $y \in B_{\eta^*}$. Then for each $h \in N(y)$ and $t \in J$, we have by (H2)-(H3),

$$\begin{aligned}
|h(t)| &\le |y_0| + |y_1|T + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |v(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} |v(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |v(s)|ds \\
&\quad + \sum_{0<t_k<t} |I_k(y(t_k^-))| + \sum_{0<t_k<t} (t-t_k)|\overline{I}_k(y(t_k^-))| \\
&\le |y_0| + |y_1|T + \frac{mT^\alpha p^0}{\Gamma(\alpha+1)}\psi(\eta^*) + \frac{T^\alpha p^0}{\Gamma(\alpha)}\psi(\eta^*)
\end{aligned}$$

$$+\frac{T^\alpha p^0}{\Gamma(\alpha+1)}\psi(\eta^*)+m\psi^*(\eta^*)+m\overline{\psi}^*(\eta^*).$$

Thus

$$\begin{aligned}\|h\|_\infty &\le |y_0|+|y_1|T+\frac{mT^\alpha p^0}{\Gamma(\alpha+1)}\psi(\eta^*)+\frac{T^\alpha p^0}{\Gamma(\alpha)}\psi(\eta^*)\\ &\quad+\frac{T^\alpha p^0}{\Gamma(\alpha+1)}\psi(\eta^*)+m\psi^*(\eta^*)+m\overline{\psi}^*(\eta^*):=\ell.\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, let $y\in B_{\eta^*}$ and $h\in N(y)$, then

$$\begin{aligned}|h(\tau_2)-h(\tau_1)| &\le |y_1|(\tau_2-\tau_1)+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\tau_2-\tau_1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}||v(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}|(\tau_2-s)^{\alpha-1}||v(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}|v(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<\tau_1}(\tau_2-\tau_1)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}|v(s)|ds\\ &\quad+\sum_{0<t_k<\tau_2-\tau_1}|I_k(y(t_k^-))|+\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)|\overline{I}_k(y(t_k^-))|\\ &\quad+(\tau_2-\tau_1)\sum_{0<t_k<\tau_1}|\overline{I}_k(y(t_k^-))|.\end{aligned}$$

Using (H2), (H3) we can easily prove that the right-hand side of the above inequality tends to zero independently of y as $\tau_1\to\tau_2$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N: PC(J,\mathbb{R})\to\mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Step 4: N has a closed graph. Let $y_n\to y_*$, $h_n\in N(y_n)$ and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$\begin{aligned}h_n(t) &= y_0+y_1t+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}v_n(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds\\ &\quad+\sum_{0<t_k<t}I_k(y_n(t_k^-))+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(y_n(t_k^-)).\end{aligned}$$

We must show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_*(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v_*(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_*(s) ds \\ &\quad + \sum_{0<t_k<t} I_k(y_*(t_k^-)) + \sum_{0<t_k<t} (t-t_k) \overline{I}_k(y_*(t_k^-)). \end{aligned} \tag{3.10}$$

Since $F(t,\cdot)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$\begin{gathered} v_{n_m}(\cdot) \to v_*(\cdot) \quad \text{as } m \to \infty, \\ v_*(t) \in F(t, y_*(t)), \quad \text{a.e. } t \in J. \end{gathered}$$

Using the fact that the functions I_k and $\overline{I}_k$, $k = 1, \dots, m$ are continuous, it can be easily shown that h_* and v_* satisfy (3.10).

Step 5: A priori bounds on solutions. Let $y \in PC(J,\mathbb{R})$ be such that $y \in \lambda N(y)$ for $\lambda \in (0,1)$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} |y(t)| &\leq |y_0| + |y_1| T + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} p(s) \psi(|y(s)|) ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} p(s) \psi(|y(s)|) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} p(s) \psi(|y(s)|) ds \\ &\quad + \sum_{0<t_k<t} \psi^*(|y(s)|) + \sum_{0<t_k<t} \overline{\psi}^*(|y(s)|) \\ &\leq |y_0| + |y_1| T + \psi(\|y\|_\infty) \frac{mT^\alpha p^0}{\Gamma(\alpha+1)} + \psi(\|y\|_\infty) \frac{mT^\alpha p^0}{\Gamma(\alpha)} \\ &\quad + \frac{mT^\alpha p^0}{\Gamma(\alpha+1)} \psi(\|y\|_\infty) + m\psi^*(\|y\|_\infty) + m\overline{\psi}^*(\|y\|_\infty). \end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{|y_0| + |y_1| T + a\psi(\|y\|_\infty) + m\psi^*(\|y\|_\infty) + m\overline{\psi}^*(\|y\|_\infty)} \leq 1.$$

Then by (H4), there exists M such that $\|y\|_\infty \neq M$. Let

$$U = \{y \in PC(J,\mathbb{R}) : \|y\|_\infty < M\}.$$

The operator $N : \overline{U} \to \mathcal{P}(PC(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type

[24], we deduce that N has a fixed point y in $\overline{U}$ which is a solution of the problem (1.1)–(1.4). This completes the proof. $\square$

4. The nonconvex case

This section is devoted to the existence of solutions for the problem (1.1)-(1.4) with a nonconvex valued right hand side. Our first result is based on the fixed point theorem for contraction multivalued map given by Covitz and Nadler [17], and the second one on a selection theorem due to Bressan and Colombo [14] for lower semicontinuous operators with decomposable values combined with the nonlinear Leray-Schauder alternative. Some existence results for nonconvex valued differential inclusions can be found in [5, 27].

For the next theorem, we use the following assumptions:

- (H6) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ has the property that $F(\cdot, u) : J \to P_{cp}(\mathbb{R})$ is measurable, convex valued and integrable bounded for each $u \in \mathbb{R}$;
- (H7) There exist constants $l^*, \bar{l}^* > 0$ such that

$$|I_k(u) - I_k(\overline{u})| \le l^*|u - \overline{u}|, \quad \text{for each } u, \overline{u} \in \mathbb{R}, \text{ and } k = 1, \dots, m,$$
$$|\overline{I}_k(u) - \overline{I}_k(\overline{u})| \le \bar{l}^*|u - \overline{u}|, \quad \text{for each } u, \overline{u} \in \mathbb{R}, \text{ and } k = 1, \dots, m.$$

Theorem 4.1. *Assume* (H5)–(H7). *If*

$$\Big[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\bar{l}^*)\Big] < 1, \tag{4.1}$$

where $l = \sup\{l(t) : t \in J\}$, *then* (1.1)-(1.4) *has one solution on* J.

Proof. For each $y \in PC(J, \mathbb{R})$, the set $S_{F,y}$ is nonempty since by (H6), F has a measurable selection (see [16, Theorem III.6]). We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(PC(J, \mathbb{R}))$ for each $y \in PC(J, \mathbb{R})$. Indeed, let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $PC(J, \mathbb{R})$. Then, $\tilde{y} \in PC(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) = y_0 + y_1 t &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_n(s) ds \\ &+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v_n(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} v_n(s) ds \\ &+ \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t - t_k)\overline{I}_k(y(t_k^-)). \end{aligned}$$

Using the fact that F has compact values and from (H5), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J, \mathbb{R})$ (the space endowed with the weak topology). A standard argument shows that v_n converges strongly to v and hence $v \in S_{F,y}$. Then, for each $t \in J$,

$$y_n(t) \to \tilde{y}(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s) ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}v(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds\\
&+\sum_{0<t_k<t}I_k(y(t_k^-))+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(y(t_k^-)).
\end{aligned}
$$

So, $\tilde{y}\in N(y)$.

Step 2: There exists $\gamma<1$ such that

$$
H_d(N(y),N(\overline{y}))\leq\gamma\|y-\overline{y}\|_\infty \text{ for each } y,\overline{y}\in PC(J,\mathbb{R}).
$$

Let $y,\overline{y}\in PC(J,\mathbb{R})$ and $h_1\in N(y)$. Then there exists $v_1(t)\in F(t,y(t))$ such that for each $t\in J$,

$$
\begin{aligned}
h_1(t)&=y_0+y_1t+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_1(s)ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_i}(t_k-s)^{\alpha-2}v_1(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_1(s)ds\\
&+\sum_{0<t_k<t}I_k(y(t_k^-))+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(y(t_k^-)).
\end{aligned}
$$

From (H5) it follows that

$$
H_d(F(t,y(t)),F(t,\overline{y}(t)))\leq l(t)|y(t)-\overline{y}(t)|.
$$

Hence, there exists $w\in F(t,\overline{y}(t))$ such that

$$
|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|,\quad t\in J.
$$

Consider $U:J\to\mathcal{P}(\mathbb{R})$ given by

$$
U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|\}.
$$

Since the multivalued operator $V(t)=U(t)\cap F(t,\overline{y}(t))$ is measurable (see [16, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t)\in F(t,\overline{y}(t))$, and for each $t\in J$,

$$
|v_1(t)-v_2(t)|\leq l(t)|y(t)-\overline{y}(t)|.
$$

Let us define for each $t\in J$,

$$
\begin{aligned}
h_2(t)&=y_0+y_1t+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_2(s)ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}v_2(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_2(s)ds\\
&+\sum_{0<t_k<t}I_k(y(t_k^-))+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(y(t_k^-)).
\end{aligned}
$$

Then for $t \in J$,

$$
\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |v_1(s)) - v_2(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} |v_1(s)) - v_2(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
&\quad + \sum_{0<t_k<t} |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| + \sum_{0<t_k<t} |\overline{I}_k(y(t_k^-)) - \overline{I}_k(\overline{y}(t_k^-))| \\
&\leq \frac{l}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |y(s) - \overline{y}(s)| ds \\
&\quad + \frac{l}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} |y(s) - \overline{y}(s)| ds \\
&\quad + \frac{l}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |y(s) - \overline{y}(s)| ds \\
&\quad + \sum_{k=1}^{m} l^* |y(t_k^-) - \overline{y}(t_k^-)| + \sum_{k=1}^{m} \overline{l}^* |y(t_k^-) - \overline{y}(t_k^-)| \\
&\leq \frac{mlT^\alpha}{\Gamma(\alpha+1)} \|y - \overline{y}\|_\infty + \frac{mlT^\alpha}{\Gamma(\alpha)} \|y - \overline{y}\|_\infty \\
&\quad + \frac{T^\alpha l}{\Gamma(\alpha+1)} \|y - \overline{y}\|_\infty + ml^* \|y - \overline{y}\|_\infty + mT\overline{l}^* \|y - \overline{y}\|_\infty.
\end{aligned}
$$

Thus

$$
\|h_1 - h_2\|_\infty \leq \big[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*)\big] \|y - \overline{y}\|_\infty.
$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$
H_d(N(y), N(\overline{y})) \leq \big[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*)\big] \|y - \overline{y}\|_\infty.
$$

So by (4.1), N is a contraction and thus, by Lemma 2.3, N has a fixed point y which is solution to (1.1)–(1.4). The proof is complete. $\square$

Now we present a result for problem (1.1)-(1.4) in the spirit of the nonlinear alternative of Leray-Schauder type [24] for single-valued maps, combined with a selection theorem due to Bressan-Colombo for lower semicontinuous multivalued maps with decomposable values. Details on multivalued maps with decomposable values and their properties can be found in the recent book by Fryszkowski [22].

Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, $u\chi_{\mathcal{J}} + v\chi_{[0,T]-\mathcal{J}} \in A$, where χ stands for the characteristic function.

Let $G : X \to \mathcal{P}(X)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in X.

Definition 4.2. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has property (BC) if

(1) N is lower semi-continuous (l.s.c.);
(2) N has nonempty closed and decomposable values.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator $\mathcal{F} : PC([0,T],\mathbb{R}) \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ by letting

$$\mathcal{F}(y) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 4.3. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo [14].

Theorem 4.4 ([14]). *Let Y be a separable metric space and let the operator $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued satisfying property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \to L^1([0,1],\mathbb{R})$ such that $\tilde{g}(y) \in N(y)$ for every $y \in Y$.*

Let us introduce the following hypotheses:

(H8) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact valued multivalued map such that:
  (a) $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
  (b) $y \mapsto F(t,y)$ is lower semi-continuous for a.e. $t \in [0,T]$;

(H9) for each $q > 0$, there exists a function $h_q \in L^1([0,T],\mathbb{R}^+)$ such that $\|F(t,y)\|_{\mathcal{P}} \leq h_q(t)$ for a.e. $t \in [0,T]$ and for $y \in \mathbb{R}$ with $|y| \leq q$.

The following lemma is crucial in the proof of our main theorem.

Lemma 4.5. [21]. *Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume that* (H8), (H9) *hold. Then F is of l.s.c.*

Theorem 4.6. *Suppose that hypotheses* (H2)-(H4), (H8), (H9) *are satisfied. Then the problem* (1.1)–(1.4) *has at least one solution.*

Proof. (H8) and (H9) imply by Lemma 4.5 that F is of lower semi-continuous type. Then from Theorem 4.4 there exists a continuous function $f : PC([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in PC([0,T],\mathbb{R})$. Consider the problem

$$^cD^\alpha y(t) \in f(y)(t), \quad \text{for a.e. } t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m,\ 1 < \alpha \leq 2, \tag{4.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{4.3}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{4.4}$$

$$y(0) = y_0, \quad y'(0) = y_1. \tag{4.5}$$

Clearly, if y is a solution of (4.2)–(4.5), then y is a solution of (1.1)-(1.4). Problem (4.2)-(4.5) can be reformulated as a fixed point problem for the operator $N_1 : PC([0,T,\mathbb{R}) \to PC([0,T],\mathbb{R})$ defined by

$$\begin{aligned} N_1(y)(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(y)(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} f(y)(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(y)(s)ds \\ &\quad + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)). \end{aligned}$$

Using (H2)-(H4) we can easily show (using similar argument as in Theorem 3.5) that the operator N_1 satisfies all conditions in the Leray-Schauder alternative. □

5. Topological structure of the solution set

In this section, we present a result on the topological structure of the set of solutions to (1.1)–(1.4).

Theorem 5.1. *Assume that* (H1), (H5) *and the following hypotheses hold:*

- (H10) *there exists* $p_1 \in C(J,\mathbb{R}^+)$ *such that* $\|F(t,u)\|_{\mathcal{P}} \le p_1(t)$ *for* $t \in J$ *and* $u \in \mathbb{R}$;
- (H11) *There exist* $d_1, d_2 > 0$ *such that*

$$\begin{aligned} |I_k(u)| &\le d_1 \quad \text{for } u \in \mathbb{R}, \\ |\overline{I}_k(u)| &\le d_2 \quad \text{for } u \in \mathbb{R}. \end{aligned}$$

Then the solution set of (1.1)-(1.4) *in not empty and is compact in* $PC(J,\mathbb{R})$.

Proof. Let

$$S = \{y \in PC(J,\mathbb{R}) : y \text{ is solution of } (1.1)-(1.4)\}.$$

From Theorem 3.5, $S \neq \emptyset$. Now, we prove that S is compact. Let $(y_n)_{n\in\mathbb{N}} \in S$, then there exists $v_n \in S_{F,y_n}$ and $t \in J$ such that

$$\begin{aligned} y_n(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_n(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v_n(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_n(s)ds \\ &\quad + \sum_{0<t_k<t} I_k(y_n(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y_n(t_k^-)). \end{aligned}$$

From (H1), (H10) and (H11) we can prove that there exists an $M_1 > 0$ such that $\|y_n\|_\infty \le M_1$ for every $n \ge 1$. As in Step 3 in Theorem 3.5, we can easily show that the set $\{y_n : n \ge 1\}$ is equicontinuous in $PC(J,\mathbb{R})$, hence by Arzéla-Ascoli Theorem we can conclude that, there exists a subsequence (denoted again by $\{y_n\}$)

of $\{y_n\}$ such that y_n converges to y in $PC(J,\mathbb{R})$. We shall show that there exist $v(.) \in F(.,y(.))$ and $t \in J$ such that

$$\begin{aligned} y(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)). \end{aligned}$$

Since $F(t,.)$ is upper semicontinuous, for every $\varepsilon > 0$, there exists $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(.,.)$ has compact values, there exists subsequence $v_{n_m}(.)$ such that

$$\begin{gathered} v_{n_m}(.) \to v(.) \quad \text{as } m \to \infty, \\ v(t) \in F(t, y(t)), \quad \text{a.e. } t \in J. \end{gathered}$$

It is clear that

$$|v_{n_m}(t)| \leq p_1(t), \quad \text{a.e. } t \in J.$$

By Lebesgue's dominated convergence theorem, we conclude that $v \in L^1(J,\mathbb{R})$ which implies that $v \in S_{F,y}$. Thus, for $t \in J$, we have

$$\begin{aligned} y(t) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} v(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)). \end{aligned}$$

Then $S \in \mathcal{P}_{cp}(PC(J,\mathbb{R}))$. $\square$

6. An Example

As an application of the main results, we consider the fractional differential inclusion

$$^cD^\alpha y(t) \in F(t,y), \quad \text{a.e. } t \in J = [0,1],\ t \neq \frac{1}{2},\ 1 < \alpha \leq 2, \tag{6.1}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{1}{3 + |y(\frac{1}{2}^-)|}, \tag{6.2}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{1}{5 + |y(\frac{1}{2}^-)|}, \tag{6.3}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{6.4}$$

We have $T = 1$, $m = 1$, $t_1 = 1/2$ and $y_0 = y_1 = 0$. Set

$$F(t,y) = \{v \in \mathbb{R} : f_1(t,y) \leq v \leq f_2(t,y)\},$$

where $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ are given functions,

$$I_1(y(t_1)) = \frac{1}{3 + |y(\frac{1}{2}^-)|}, \quad \overline{I}_1(y(t_1)) = \frac{1}{5 + |y(\frac{1}{2}^-)|}.$$

Then (6.1)-(6.4) takes the form (1.1)-(1.4). We assume that for each $t \in J$, the function $f_1(t, \cdot)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J, f_2(t, \cdot)$ is upper semi-continuous (i.e the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C(J, \mathbb{R}^+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, y)|, |f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in J, \text{ and } y \in \mathbb{R}.$$

Assume there exists a constant $M > 0$ such that

$$\frac{M}{\big(\frac{2p^0}{\Gamma(\alpha+1)} + \frac{p^0}{\Gamma(\alpha)}\big)\psi(M) + \frac{8}{15}} > 1.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [18]). Since all the conditions of Theorem 3.5 are satisfied, the problem (6.1)-(6.4) has at least one solution y on J.

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