EXISTENCE OF EXTREMAL SOLUTIONS FOR QUADRATIC FUZZY EQUATIONS

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Some results on the existence of solution for certain fuzzy equations are revised and extended. In this paper, we establish the existence of a solution for the fuzzy equation $Ex^2+Fx+G=x$, where E, F, G, and x are positive fuzzy numbers satisfying certain conditions. To this purpose, we use fixed point theory, applying results such as the well-known fixed point theorem of Tarski, presenting some results regarding the existence of extremal solutions to the above equation.

1. Preliminaries

In [1], it is studied the existence of extremal fixed points for a map defined in a subset of the set E^1 of fuzzy real numbers, that is, the family of elements $x:\mathbb{R}\to[0,1]$ with the properties:

(i) x is normal: there exists $t_0\in\mathbb{R}$ with $x(t_0)=1$.
(ii) x is upper semicontinuous.
(iii) x is fuzzy convex,

$$x(\lambda t_1+(1-\lambda)t_2)\geq \min\{x(t_1),x(t_2)\},\quad \forall t_1,t_2\in\mathbb{R},\lambda\in[0,1]. \tag{1.1}$$

(iv) The support of x, $\operatorname{supp}(x)=\operatorname{cl}(\{t\in\mathbb{R}:x(t)>0\})$ is a bounded subset of $\mathbb{R}$.

In the following, for a fuzzy number $x\in E^1$, we denote the α-level set

$$[x]^\alpha=\{t\in\mathbb{R}:x(t)\geq\alpha\} \tag{1.2}$$

by the interval $[x_{\alpha l},x_{\alpha r}]$, for each $\alpha\in(0,1]$, and

$$[x]^0=\operatorname{cl}\left(\cup_{\alpha\in(0,1]}[x]^\alpha\right)=[x_{0l},x_{0r}]. \tag{1.3}$$

Note that this notation is possible, since the properties of the fuzzy number x guarantee that $[x]^\alpha$ is a nonempty compact convex subset of $\mathbb{R}$, for each $\alpha\in[0,1]$.

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We consider the partial ordering $\leq$ in E^1 given by

$$x, y \in E^1, \quad x \leq y \Longleftrightarrow x_{\alpha l} \leq y_{\alpha l}, \quad x_{\alpha r} \leq y_{\alpha r}, \quad \forall \alpha \in (0,1], \tag{1.4}$$

and the distance that provides E^1 the structure of complete metric space is given by

$$d_\infty(x, y) = \sup_{\alpha \in [0,1]} d_H([x]^\alpha, [y]^\alpha), \quad \text{for } x, y \in E^1, \tag{1.5}$$

being d_H the Hausdorff distance between nonempty compact convex subsets of $\mathbb{R}$ (that is, compact intervals).

For each fuzzy number $x \in E^1$, we define the functions $x_L : [0,1] \to \mathbb{R}$, $x_R : [0,1] \to \mathbb{R}$ given by $x_L(\alpha) = x_{\alpha l}$ and $x_R(\alpha) = x_{\alpha r}$, for each $\alpha \in [0,1]$.

Theorem 1.1 [1, Theorem 2.3]. *Let u_0, $v_0 \in E^1$, $u_0 < v_0$. Let*

$$B \subset [u_0, v_0] = \{x \in E^1 : u_0 \leq x \leq v_0\} \tag{1.6}$$

be a closed set of E^1 such that $u_0, v_0 \in B$. Suppose that $A : B \to B$ is an increasing operator such that

$$u_0 \leq Au_0, \qquad Av_0 \leq v_0, \tag{1.7}$$

and A is condensing, that is, A is continuous, bounded and $r(A(S)) < r(S)$ for any bounded set $S \subset B$ with $r(S) > 0$, where $r(S)$ denotes the measure of noncompactness of S. Then A has a maximal fixed point x^* and a minimal fixed point x_* in B, moreover

$$x^* = \lim_{n \longrightarrow +\infty} v_n, \qquad x_* = \lim_{n \longrightarrow +\infty} u_n, \tag{1.8}$$

where $v_n = Av_{n-1}$ and $u_n = Au_{n-1}$, $n = 1, 2, \ldots$ and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{1.9}$$

Corollary 1.2 [1, Corollary 2.4]. *In the hypotheses of Theorem 1.1, if A has a unique fixed point $\bar{x}$ in B, then, for any $x_0 \in B$, the successive iterates*

$$x_n = Ax_{n-1}, \quad n = 1, 2, \ldots \tag{1.10}$$

converge to $\bar{x}$, that is, $d_\infty(x_n, \bar{x}) \to 0$ as $n \to +\infty$.

Theorem 1.1 is used in [1] to solve the fuzzy equation

$$Ex^2 + Fx + G = x, \tag{1.11}$$

where E, F, G and x are positive fuzzy numbers satisfying some additional conditions. In this direction, consider the class of fuzzy numbers $x \in E^1$ satisfying

(i) $x > 0$, $x_L(\alpha)$, $x_R(\alpha) \leq 1/6$, for each $\alpha \in [0,1]$.

(ii) $|x_L(\alpha) - x_L(\beta)| < (M/6)|\alpha - \beta|$ and $|x_R(\alpha) - x_R(\beta)| < (M/6)|\alpha - \beta|$, for every $\alpha, \beta \in [0,1]$.

Denote this class by $\mathcal{F}$.

Theorem 1.3 [1, Theorem 2.9]. *Let $M > 0$ be a real number. Suppose that $E,F,G \in \mathcal{F}$. Then (1.11) has a solution in*

$$B_M = \{x \in E^1 : 0 \le x \le 1,\ |x_L(\alpha) - x_L(\beta)| \le M|\alpha - \beta|,\ |x_R(\alpha) - x_R(\beta)| \le M|\alpha - \beta|,\ \forall \alpha,\beta \in [0,1]\}. \tag{1.12}$$

Here, 0,1 referred to fuzzy numbers represent, respectively, the characteristic functions of 0 and 1, that is, $\chi_{\{0\}}$ and $\chi_{\{1\}}$.

In the proof of Theorem 1.3, in addition to Theorem 1.1, the following results are used.

Theorem 1.4 [1, Theorem 2.6]. *For each fuzzy number x, functions*

$$x_L : [0,1] \longrightarrow \mathbb{R}, \qquad x_R : [0,1] \longrightarrow \mathbb{R} \tag{1.13}$$

are continuous.

Theorem 1.5 [1, Theorem 2.7]. *Suppose that x and y are fuzzy numbers, then*

$$d_\infty(x,y) = \max\{\|x_L - y_L\|_\infty, \|x_R - y_R\|_\infty\}. \tag{1.14}$$

Theorem 1.6 [1, Theorem 2.8]. *B_M is a closed subset of E^1.*

Lemma 1.7 [1, Lemma 2.10]. *Suppose that $B \subset E^1$. If*

$$B_L = \{x_L : x \in B\}, \quad B_R = \{x_R : x \in B\} \tag{1.15}$$

are compact in $(C[0,1], \|\cdot\|_\infty)$, then B is a compact set in E^1.

In Section 2, we point out some considerations about the previous results and justify the validity of the proof of Theorem 1.3 given in [1], presenting a more general existence result. Then, in Section 3, we study the existence of solution to (1.11) by using some fixed point theorems such as Tarski's fixed point theorem, proving the existence of extremal solutions to (1.11) under less restrictive hypotheses.

2. Revision and extension of results in [1]

First of all, Theorem 1.4 [1, Theorem 2.6] is not valid. Indeed, take for example, $x : \mathbb{R} \to [0,1]$ defined as

$$t \in \mathbb{R} \longrightarrow x(t) = \begin{cases} \dfrac{1}{2}, & t \in [-1,0) \cup (0,1], \\ 1, & t = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

which represents [2, Proposition 6.1.7] and [3, Theorem 1.5.1] a fuzzy real number since the level sets of x are the nonempty compact convex sets

$$[x]^\alpha = \begin{cases} [-1,1], & \text{if } 0 \le \alpha \le \dfrac{1}{2}, \\ \{0\}, & \text{if } \dfrac{1}{2} < \alpha \le 1. \end{cases} \tag{2.2}$$

Then, $x_L : [0,1] \to \mathbb{R}$ is given by

$$x_L(\alpha) = \begin{cases} -1, & \text{if } 0 \le \alpha \le \dfrac{1}{2}, \\ 0, & \text{if } \dfrac{1}{2} < \alpha \le 1, \end{cases} \tag{2.3}$$

and $x_R : [0,1] \to \mathbb{R}$ is

$$x_R(\alpha) = \begin{cases} 1, & \text{if } 0 \le \alpha \le \dfrac{1}{2}, \\ 0, & \text{if } \dfrac{1}{2} < \alpha \le 1, \end{cases} \tag{2.4}$$

which are clearly discontinuous. Note that x_L and x_R are left-continuous see [3, Theorem 1.5.1] and [2, Propositions 6.1.6 and 6.1.7]. In the proof of Theorem 1.4 [1, Theorem 2.6], it is considered a sequence $\alpha_n > \alpha$ with $\alpha_n \to \alpha$ as $n \to +\infty$. Then $x_L(\alpha_n)$ is a nonincreasing and bounded sequence, hence, $x_L(\alpha_n)$ converges to a number L. At this point, one cannot affirm that $x(L) \le \alpha_n$. For example, in the previous case, taking $\alpha = 1/2$ and $\alpha_n = 1/2 + 1/n$, with $n > 2$, then $x_L(\alpha_n) = 0$. Hence $x_L(\alpha_n)$ converges to $L = 0$, but $x(L) = x(0) = 1 > \alpha_n = 1/2 + 1/n$ for all $n > 2$.

A fuzzy number is not necessarily a continuous function, just upper semicontinuous, thus Theorem 1.4 [1, Theorem 2.6] is not valid in the general context of fuzzy real numbers. However, it is valid for continuous fuzzy numbers, that is, fuzzy numbers continuous in its membership grade, as we state below. Here $\mathcal{K}_C^1$ denotes the space of nonempty compact convex subsets of $\mathbb{R}$ furnished with the Hausdorff metric d_H.

Definition 2.1. We say that a fuzzy number $x : \mathbb{R} \to [0,1]$ is continuous if the function

$$[x]^{\cdot} : [0,1] \longrightarrow \mathcal{K}_C^1 \tag{2.5}$$

given by $\alpha \to [x]^\alpha$ is continuous on $(0,1]$, that is, for every $\alpha \in (0,1]$, and $\epsilon > 0$, there exists a number $\delta(\epsilon,\alpha) > 0$ such that $d_H([x]^\alpha, [x]^\beta) < \epsilon$, for every $\beta \in (\alpha - \delta, \alpha + \delta) \cap [0,1]$.

THEOREM 2.2. *Let x be a fuzzy number, then x is continuous if and only if functions*

$$x_L : [0,1] \longrightarrow \mathbb{R}, \qquad x_R : [0,1] \longrightarrow \mathbb{R} \tag{2.6}$$

are continuous.

Proof. Suppose that $x \in E^1$ is continuous and let $\alpha \in (0,1]$ and $\epsilon > 0$. Since x is continuous at α, then there exists $\delta(\epsilon,\alpha) > 0$ such that for every $\beta \in (\alpha - \delta, \alpha + \delta) \cap [0,1]$,

$$\begin{aligned} d_H([x]^\alpha, [x]^\beta) &= \max\{|x_{\alpha l} - x_{\beta l}|, |x_{\alpha r} - x_{\beta r}|\} \\ &= \max\{|x_L(\alpha) - x_L(\beta)|, |x_R(\alpha) - x_R(\beta)|\} < \epsilon, \end{aligned} \tag{2.7}$$

which implies that

$$|x_L(\alpha) - x_L(\beta)| < \epsilon, \qquad |x_R(\alpha) - x_R(\beta)| < \epsilon, \tag{2.8}$$

for every $\beta \in (\alpha-\delta,\alpha+\delta)\cap[0,1]$, proving the continuity of x_L and x_R at α. Reciprocally, continuity of x_L and x_R trivially implies the continuity of x. □

Remark 2.3. For a given $x \in E^1$, x, $[x]^{\cdot}$, x_L and x_R are trivially continuous at $\alpha = 0$. Indeed, let $\epsilon > 0$. The 0-level set of x (support of x) is the closure of the union of all of the level sets, that is,

$$[x]^0 = \mathrm{cl}\left(\cup_{\beta\in(0,1]}\left[x_L(\beta),x_R(\beta)\right]\right). \tag{2.9}$$

Since $x_L(\beta)$ is nondecreasing in β and $x_R(\beta)$ is nonincreasing in β and those values are bounded, then

$$x_L(0) = \inf_{\beta\in(0,1]} x_L(\beta), \qquad x_R(0) = \sup_{\beta\in(0,1]} x_R(\beta). \tag{2.10}$$

For $\epsilon > 0$, there exist $\beta_{1,\epsilon},\beta_{2,\epsilon} \in (0,1]$, such that

$$\begin{gathered} x_L(0) \le x_L\left(\beta_{1,\epsilon}\right) < x_L(0)+\epsilon, \\ x_R(0)-\epsilon < x_R\left(\beta_{2,\epsilon}\right) \le x_R(0). \end{gathered} \tag{2.11}$$

By monotonicity,

$$\begin{aligned} x_L(0) \le x_L(\beta) \le x_L\left(\beta_{1,\epsilon}\right) < x_L(0)+\epsilon, &\quad \text{for } 0<\beta\le\beta_{1,\epsilon}, \\ x_R(0)-\epsilon < x_R\left(\beta_{2,\epsilon}\right) \le x_R(\beta) \le x_R(0), &\quad \text{for } 0<\beta\le\beta_{2,\epsilon}. \end{aligned} \tag{2.12}$$

Hence, taking $\delta = \min\{\beta_{1,\epsilon},\beta_{2,\epsilon}\} > 0$, we obtain

$$x_L(0) \le x_L(\beta) < x_L(0)+\epsilon, \qquad x_R(0)-\epsilon < x_R(\beta) \le x_R(0), \tag{2.13}$$

for every $0<\beta\le\delta$, and

$$d_H\left([x]^0,[x]^\beta\right) = \max\left\{\left|x_L(0)-x_L(\beta)\right|,\left|x_R(0)-x_R(\beta)\right|\right\} < \epsilon, \quad \forall\beta\in[0,\delta]. \tag{2.14}$$

As a particular case of continuous fuzzy numbers, we present Lipschitzian fuzzy numbers.

Definition 2.4. We say that $x \in E^1$ is a Lipschitzian fuzzy number if it is a Lipschitz function of its membership grade, in the sense that

$$d_H\left([x]^\alpha,[x]^\beta\right) \le K|\alpha-\beta|, \tag{2.15}$$

for every $\alpha,\beta \in [0,1]$ and some fixed, finite constant $K \ge 0$.

This property of fuzzy numbers is equivalent (see [2, page 43]) to the Lipschitzian character of the support function $s_x(\cdot,p)$ uniformly in $p \in S^0$, where

$$s_x(\alpha,p) = s(p,[x]^\alpha) = \sup\{\langle p,a\rangle : a \in [x]^\alpha\}, \quad (\alpha,p) \in [0,1] \times S^0, \tag{2.16}$$

and S^0 is the unit sphere in $\mathbb{R}$, that is, the set $\{-1,+1\}$.

If we consider a Lipschitzian fuzzy number x, then x is continuous and, in consequence, x_L and x_R are continuous functions. Moreover, we prove that these are Lipschitzian functions.

Theorem 2.5. *Let $x \in E^1$. Then x is a Lipschitzian fuzzy number, with Lipschitz constant $K \geq 0$, if and only if $x_L : [0,1] \to \mathbb{R}$ and $x_R : [0,1] \to \mathbb{R}$ are K-Lipschitzian functions.*

Proof. It is deduced from the identity

$$\begin{aligned} d_H([x]^\alpha,[x]^\beta) &= \max\{|x_{\alpha l} - x_{\beta l}|, |x_{\alpha r} - x_{\beta r}|\} \\ &= \max\{|x_L(\alpha) - x_L(\beta)|, |x_R(\alpha) - x_R(\beta)|\}, \quad \text{for every } \alpha,\beta \in [0,1]. \end{aligned} \tag{2.17}$$

□

Note that Theorem 1.5 [1, Theorem 2.7] is valid for $\|\cdot\|_\infty$ considered in the space $L^\infty[0,1]$, but not in $C[0,1]$, since for an arbitrary fuzzy number x, x_L and x_R are not necessarily continuous. Nevertheless, from Theorem 2.2, we deduce that the distance d_∞ can be characterized for continuous fuzzy numbers in terms of the sup norm in $C[0,1]$, and also for Lipschitzian fuzzy numbers.

Theorem 2.6. *Suppose that x and y are continuous fuzzy numbers (in the sense of Definition 2.1), then*

$$d_\infty(x,y) = \max\{\|x_L - y_L\|_\infty, \|x_R - y_R\|_\infty\}. \tag{2.18}$$

Proof. Indeed,

$$\begin{aligned} d_\infty(x,y) &= \sup_{\alpha\in[0,1]} d_H([x]^\alpha,[y]^\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\{|x_L(\alpha) - y_L(\alpha)|, |x_R(\alpha) - y_R(\alpha)|\} \\ &= \max\left\{\sup_{\alpha\in[0,1]} |x_L(\alpha) - y_L(\alpha)|, \sup_{\alpha\in[0,1]} |x_R(\alpha) - y_R(\alpha)|\right\} \\ &= \max\{\|x_L - y_L\|_\infty, \|x_R - y_R\|_\infty\}. \end{aligned} \tag{2.19}$$

□

For $M > 0$ fixed, consider the set

$$B_M = \{x \in E^1 : \chi_{\{0\}} \leq x \leq \chi_{\{1\}},\ x \text{ is } M\text{-Lipschitzian}\}. \tag{2.20}$$

Note that B_M coincides with the set with the same name defined in Theorem 1.3 [1, Theorem 2.9] and that B_M is a closed set in E^1. For the sake of completeness, we give here another proof. Let x_n a sequence in B_M such that $\lim_{n\to+\infty} x_n = x \in E^1$ in E^1. We prove that $x \in B_M$. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d_\infty(x_n, x) = \sup_{\alpha\in[0,1]} d_H([x_n]^\alpha, [x]^\alpha) < \epsilon, \quad \text{for } n \geq n_0. \tag{2.21}$$

Then, for $n \geq n_0$,

$$\begin{aligned} d_H([x]^\alpha, [x]^\beta) &\leq d_H([x]^\alpha, [x_n]^\alpha) + d_H([x_n]^\alpha, [x_n]^\beta) + d_H([x_n]^\beta, [x]^\beta) \\ &< 2\epsilon + M|\alpha - \beta|, \quad \text{for every } \alpha, \beta \in [0,1]. \end{aligned} \tag{2.22}$$

Since $\epsilon > 0$ is arbitrary, this means that

$$d_H([x]^\alpha, [x]^\beta) \leq M|\alpha - \beta|, \quad \text{for every } \alpha, \beta \in [0,1], \tag{2.23}$$

and x is M-Lipschitzian. We can easily prove that $\chi_{\{0\}} \leq x_n \leq \chi_{\{1\}}$, for all n implies that $\chi_{\{0\}} \leq x \leq \chi_{\{1\}}$. Therefore, $x \in B_M$.

Concerning Lemma 1.7 [1, Lemma 2.10] we have to restrict our attention to relatively compact sets, since we are not considering closed sets. On the other hand, if B contains noncontinuous fuzzy numbers, B_L and B_R are not subsets of $C[0,1]$. We prove the corresponding result.

Lemma 2.7. *Suppose that $B \subset E^1$ consists of continuous fuzzy numbers, hence*

$$B_L = \{x_L : x \in B\}, \qquad B_R = \{x_R : x \in B\} \tag{2.24}$$

are subsets of $C[0,1]$. If B_L and B_R are relatively compact in $(C[0,1], \|\cdot\|_\infty)$, then B is a relatively compact set in E^1.

Proof. Let $\{x_n\}_n \subseteq B$ a sequence in B and $I = [0,1]$. Since B_L is relatively compact in $(C(I), \|\cdot\|_\infty)$, then $\{(x_n)_L\}_n$ has a subsequence $\{(x_{n_k})_L\}_k$ converging in $C(I)$ to $f_1 \in C(I)$. Using that B_R is relatively compact in $(C(I), \|\cdot\|_\infty)$, then $\{(x_{n_k})_R\}_k$ has a subsequence $\{(x_{n_l})_R\}_l$ converging in $C(I)$ to $f_2 \in C(I)$. We have to prove that $\{[f_1(\alpha), f_2(\alpha)] : \alpha \in [0,1]\}$ is the family of level sets of some fuzzy number $x \in E^1$ and, hence, $x_L = f_1$, $x_R = f_2$. Indeed, intervals $[f_1(\alpha), f_2(\alpha)]$ are nonempty compact convex subsets of $\mathbb{R}$, since

$$(x_{n_l})_L(\alpha) \leq (x_{n_l})_R(\alpha), \quad \forall \alpha \in [0,1], l \in \mathbb{N}, \tag{2.25}$$

and, thus, passing to the limit as $l \to +\infty$,

$$f_1(\alpha) \leq f_2(\alpha), \quad \forall \alpha \in [0,1]. \tag{2.26}$$

Moreover, if $0 \le \alpha_1 \le \alpha_2 \le 1$,

$$(x_{n_l})_L(\alpha_1) \le (x_{n_l})_L(\alpha_2), \quad (x_{n_l})_R(\alpha_1) \ge (x_{n_l})_R(\alpha_2), \quad \forall l, \tag{2.27}$$

so that

$$f_1(\alpha_1) \le f_1(\alpha_2), \qquad f_2(\alpha_1) \ge f_2(\alpha_2), \tag{2.28}$$

then

$$[f_1(\alpha_2), f_2(\alpha_2)] \subseteq [f_1(\alpha_1), f_2(\alpha_1)]. \tag{2.29}$$

Finally, let $\alpha > 0$ and $\{\alpha_i\} \uparrow \alpha$, then $\{[f_1(\alpha_i), f_2(\alpha_i)]\}$ is a contractive sequence of compact intervals, and, by continuity of f_1 and f_2,

$$\cap_{i\ge 1}[f_1(\alpha_i), f_2(\alpha_i)] = \Big[\lim_{\alpha_i \longrightarrow \alpha^-} f_1(\alpha_i), \lim_{\alpha_i \longrightarrow \alpha^-} f_2(\alpha_i)\Big] = [f_1(\alpha), f_2(\alpha)]. \tag{2.30}$$

Applying [2, Proposition 6.1.7] or also [3, Theorem 1.5.1], there exists $x \in E^1$ such that

$$[x]^\alpha = [f_1(\alpha), f_2(\alpha)], \quad \forall \alpha \in (0,1], \tag{2.31}$$

and

$$[x]^0 = \mathrm{cl}\,(\cup_{0<\alpha\le 1}[f_1(\alpha), f_2(\alpha)]) = \Big[\lim_{\alpha \longrightarrow 0^+} f_1(\alpha), \lim_{\alpha \longrightarrow 0^+} f_2(\alpha)\Big] = [f_1(0), f_2(0)] \tag{2.32}$$

again by continuity of f_1, f_2. Note that $x_L = f_1$ and $x_R = f_2$ are continuous, thus x is a continuous fuzzy number and also x_{n_l} is, for every l. Then, by Theorem 2.6,

$$d_\infty(x_{n_l}, x) = \max\{\|(x_{n_l})_L - f_1\|_\infty, \|(x_{n_l})_R - f_2\|_\infty\} \xrightarrow{l \to +\infty} 0, \tag{2.33}$$

and $\{x_{n_l}\}_l \to x$ in E^1, completing the proof. □

Recall equation (1.11)

$$Ex^2 + Fx + G = x. \tag{2.34}$$

Here, the product $x \cdot y$ of two fuzzy numbers x and y is given by the Zadeh's extension principle:

$$\begin{gathered} x \cdot y : \mathbb{R} \longrightarrow [0,1] \\ (x \cdot y)(t) = \sup_{s \cdot s' = t} \min\{x(s), y(s')\}. \end{gathered} \tag{2.35}$$

Note that $[x \cdot y]^\alpha = [x]^\alpha \cdot [y]^\alpha$, for every $\alpha \in [0,1]$. See [2, page 4] and [3, page 3].

In the following, we make reference to the canonical partial ordering $\leq$ on E^1 as well as the order $\preceq$ defined by

$$x, y \in E^1,\ x \preceq y \Longleftrightarrow [x]^\alpha \subseteq [y]^\alpha, \quad \forall \alpha \in (0,1], \tag{2.36}$$

that is,

$$x_{\alpha l} \geq y_{\alpha l}, \quad x_{\alpha r} \leq y_{\alpha r}, \quad \forall \alpha \in (0,1]. \tag{2.37}$$

Remark 2.8. Note that, for a given $x \in E^1$, it is not true in general that

$$x^2 \geq \chi_{\{0\}}, \qquad x^2 \succeq \chi_{\{0\}}. \tag{2.38}$$

Indeed, for $x = \chi_{[-3,3]}$,

$$\left(\chi_{[-3,3]}\right)^2 = \chi_{[-9,9]} \ngeq \chi_{\{0\}}, \tag{2.39}$$

and, for $y = \chi_{[1,2]}$, we obtain

$$\left(\chi_{[1,2]}\right)^2 = \chi_{[1,4]} \nsucceq \chi_{\{0\}}. \tag{2.40}$$

The proof of Theorem 1.3 [1, Theorem 2.9] can be completed using the revised results. In fact, the same proof is valid for a more general situation. Note that, if $G = \chi_{\{0\}}$, then $x = \chi_{\{0\}}$ is a solution to (1.11).

Theorem 2.9. *Let $M > 0$ be a real number, and E, F, G fuzzy numbers such that*
(i) $E, F, G \geq \chi_{\{0\}}$, $d_\infty(E, \chi_{\{0\}}) \leq 1/6$, $d_\infty(F, \chi_{\{0\}}) \leq 1/6$, $d_\infty(G, \chi_{\{0\}}) \leq 4/6$.
(ii) *E, F, G are $(M/6)$-Lipschitzian.*
Then (1.11) has a solution in B_M.

Proof. We define the mapping

$$A : B_M \longrightarrow B_M, \tag{2.41}$$

by $Ax = Ex^2 + Fx + G$. To check that A is well-defined, let $x \in B_M$ and then

$$\begin{aligned}
&\left|(Ax)_L(\alpha) - (Ax)_L(\beta)\right| \\
&\quad = \left|E_L(\alpha)x_L^2(\alpha) + F_L(\alpha)x_L(\alpha) + G_L(\alpha) - E_L(\beta)x_L^2(\beta) - F_L(\beta)x_L(\beta) - G_L(\beta)\right| \\
&\quad \leq \left|E_L(\alpha) - E_L(\beta)\right| x_L^2(\alpha) + E_L(\beta)\left|x_L(\alpha) + x_L(\beta)\right| \cdot \left|x_L(\alpha) - x_L(\beta)\right| \\
&\qquad + \left|F_L(\alpha) - F_L(\beta)\right| x_L(\alpha) + F_L(\beta)\left|x_L(\alpha) - x_L(\beta)\right| + \left|G_L(\alpha) - G_L(\beta)\right| \\
&\quad \leq \frac{M}{6}|\alpha - \beta| + \frac{2M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| \\
&\quad = M|\alpha - \beta|, \quad \forall \alpha, \beta \in [0,1],
\end{aligned} \tag{2.42}$$

and, analogously,

$$\left|(Ax)_R(\alpha)-(Ax)_R(\beta)\right| \leq M|\alpha-\beta|, \quad \text{for every } \alpha,\beta\in[0,1], \tag{2.43}$$

therefore, by Theorem 2.5, $Ax \in E^1$ is M-Lipschitzian and, using the hypotheses and $\chi_{\{0\}} \leq x \leq \chi_{\{1\}}$, we obtain

$$\begin{aligned} 0 &\leq E_L(\alpha)x_L^2(\alpha)+F_L(\alpha)x_L(\alpha)+G_L(\alpha)=(Ax)_L(\alpha) \\ &\leq (Ax)_R(\alpha)=E_R(\alpha)x_R^2(\alpha)+F_R(\alpha)x_R(\alpha)+G_R(\alpha) \\ &\leq \frac{1}{6}+\frac{1}{6}+\frac{4}{6}=1, \end{aligned} \tag{2.44}$$

for $\alpha\in[0,1]$, achieving $Ax\in B_M$. Moreover, A is a nondecreasing and continuous mapping (use Theorem 2.6). A is bounded, since

$$d_\infty\left(Ax,\chi_{\{0\}}\right)=d_\infty\left(Ex^2+Fx+G,\chi_{\{0\}}\right)\leq 1, \quad \text{for } x\in B_M. \tag{2.45}$$

Let $S\subset B_M$ a bounded set (consisting of continuous fuzzy numbers) with $r(S)>0$, and prove that $A(S)$ is relatively compact. In that case,

$$r(A(S))=0<r(S) \tag{2.46}$$

and the proof is complete by application of Theorem 1.1 [1, Theorem 2.3]. Let $A(S)\subset E^1$ and prove that $A(S)_L$ and $A(S)_R$ are relatively compact in $C[0,1]$. Indeed, using that for $y\in A(S)$, $\chi_{\{0\}}\leq y\leq\chi_{\{1\}}$, we obtain that $A(S)_L$ is a bounded set in $C[0,1]$,

$$\|y_L\|_\infty \leq d_\infty\left(y,\chi_{\{0\}}\right)\leq 1, \quad y\in A(S). \tag{2.47}$$

Let $f\in A(S)_L$, then f is M-Lipschitzian, and $A(S)_L$ is equicontinuous. This proves that $A(S)_L$ is relatively compact by Arzelà-Ascoli theorem, and the same for $A(S)_R$. Lemma 2.7 guarantees that $A(S)$ is relatively compact and, therefore, A is condensing. Besides, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are elements in B_M and $\chi_{\{0\}}\leq A\chi_{\{0\}}$, $A\chi_{\{1\}}\leq\chi_{\{1\}}$. This completes the proof. In fact, there exist extremal solutions between $\chi_{\{0\}}$ and $\chi_{\{1\}}$. □

Remark 2.10. Note that our Theorem 2.9 do not impose $G_R(\alpha)\leq 1/6$ for all $\alpha\in[0,1]$ and, therefore, improves the results of [1].

THEOREM 2.11. *Let E,F,G be Lipschitzian fuzzy numbers with $E,F,G\geq\chi_{\{0\}}$. Moreover, suppose that there exist $k>0$, $S\geq 0$ such that*

$$E_R(0)k^2+F_R(0)k+G_R(0)\leq k, \tag{2.48}$$

$$M_Ek^2+E_R(0)2kS+M_Fk+F_R(0)S+M_G\leq S, \tag{2.49}$$

where M_E, M_F, M_G are, respectively, the Lipschitz constants of E, F and G. Then (1.11) has a solution in

$$B_{k,S} := \{x \in E^1 : \chi_{\{0\}} \le x \le \chi_{\{k\}}, x \text{ is } S\text{-Lipschitzian}\}. \tag{2.50}$$

Proof. Define

$$A : B_{k,S} \longrightarrow E^1, \tag{2.51}$$

by $Ax = Ex^2 + Fx + G$. We show that $A(B_{k,S}) \subseteq B_{k,S}$. Indeed, for $x \in B_{k,S}$, and every $\alpha \in [0,1]$,

$$\begin{aligned} 0 &\le E_L(\alpha)x_L^2(\alpha) + F_L(\alpha)x_L(\alpha) + G_L(\alpha) = (Ax)_L(\alpha) \\ &\le (Ax)_R(\alpha) = E_R(\alpha)x_R^2(\alpha) + F_R(\alpha)x_R(\alpha) + G_R(\alpha) \\ &\le E_R(\alpha)k^2 + F_R(\alpha)k + G_R(\alpha) \le E_R(0)k^2 + F_R(0)k + G_R(0) \\ &\le k, \end{aligned} \tag{2.52}$$

which proves that $\chi_{\{0\}} \le Ax \le \chi_{\{k\}}$. Besides, for $x \in B_{k,S}$, and $\alpha, \beta \in [0,1]$,

$$\begin{aligned} &|(Ax)_L(\alpha) - (Ax)_L(\beta)| \\ &\quad \le |E_L(\alpha) - E_L(\beta)|x_L^2(\alpha) + E_L(\beta)|x_L(\alpha) + x_L(\beta)| \cdot |x_L(\alpha) - x_L(\beta)| \\ &\qquad + |F_L(\alpha) - F_L(\beta)|x_L(\alpha) + F_L(\beta)|x_L(\alpha) - x_L(\beta)| + |G_L(\alpha) - G_L(\beta)| \\ &\quad \le (M_E k^2 + E_L(\beta)2kS + M_F k + F_L(\beta)S + M_G)|\alpha - \beta| \\ &\quad \le (M_E k^2 + E_R(0)2kS + M_F k + F_R(0)S + M_G)|\alpha - \beta| \le S|\alpha - \beta|, \end{aligned} \tag{2.53}$$

and, similarly,

$$|(Ax)_R(\alpha) - (Ax)_R(\beta)| \le S|\alpha - \beta|, \tag{2.54}$$

proving $Ax \in B_{k,S}$. The proof is completed in the same way of Theorem 2.9. □

Remark 2.12. Inequalities (2.48) and (2.49) in Theorem 2.11 are equivalent to

$$d_\infty(E, \chi_{\{0\}})k^2 + d_\infty(F, \chi_{\{0\}})k + d_\infty(G, \chi_{\{0\}}) \le k, \tag{2.55}$$

$$M_E k^2 + d_\infty(E, \chi_{\{0\}})2kS + M_F k + d_\infty(F, \chi_{\{0\}})S + M_G \le S, \tag{2.56}$$

since, for $x \in E^1$, $x \ge \chi_{\{0\}}$,

$$d_\infty(x, \chi_{\{0\}}) = \sup_{\alpha \in [0,1]} \max\{|x_L(\alpha)|, |x_R(\alpha)|\} = x_R(0). \tag{2.57}$$

Corollary 2.13. *In Theorem 2.11, take* $E_R(0) \leq 1/6$, $F_R(0) \leq 1/6$, $G_R(0) \leq 4/6$, *and* $M_E = M_F = M_G = M/6$, *with* $M > 0$, *to obtain Theorem 2.9.*

Proof. Conditions in Theorem 2.11 are valid for $k = 1$ and $S = M$. Indeed,

$$\begin{gathered} E_R(0)k^2 + F_R(0)k + G_R(0) \leq 1 = k, \\ M_E k^2 + E_R(0)2kS + M_F k + F_R(0)S + M_G \\ = \frac{M}{6} + E_R(0)2M + \frac{M}{6} + F_R(0)M + \frac{M}{6} \leq M. \end{gathered} \tag{2.58}$$

□

3. Other existence results

Now, we present some results on the existence of extremal solutions to (1.11), based on Tarski's fixed point Theorem [6]. For the sake of completeness, we present it here, and note that the proof is not constructive.

Theorem 3.1. *Let* X *be a complete lattice and*

$$F : X \longrightarrow X \tag{3.1}$$

a nondecreasing function, that is, $F(x) \leq F(y)$ *whenever* $x \leq y$. *Suppose that there exists* $x_0 \in X$ *such that* $F(x_0) \geq x_0$. *Then* F *has at least one fixed point in* X.

Proof. Consider the set $Y = \{x \in X : F(x) \geq x\}$, which is a nonempty set since $x_0 \in Y$. Let $z = \sup Y$ $(x_0 \leq z)$. Note that, for every $x \in Y$, $F(x) \geq x$, so that $F(F(x)) \geq F(x) \geq x$ and $F(x) \in Y$. Let $x \in Y$, then $x \leq z$, and $x \leq F(x) \leq F(z)$, which implies that $z \leq F(z)$. On the other hand, $z \in Y$, so that $F(z) \in Y$, then $F(z) \leq z$ and z is a fixed point for F in X. Note that z is thus the maximal fixed point in X. □

Remark 3.2. In the hypotheses of the previous result, if there exists $x_1 \in X$ such that $F(x_1) \leq x_1$, we obtain the minimal fixed point as the infimum of the set $Z = \{x \in X : F(x) \leq x\}$. If, at the same time, there exist x_0 and x_1 such that $F(x_0) \geq x_0$ and $F(x_1) \leq x_1$, then

$$\begin{gathered} z = \sup Y = \sup\{x \in X : F(x) \geq x\}, \\ \hat{z} = \inf Z = \inf\{x \in X : F(x) \leq x\} \end{gathered} \tag{3.2}$$

are, respectively, the maximal and minimal fixed points of F in X. Indeed, since there exists at least one fixed point for F, then $\hat{z} \leq z$, and any fixed point for F is between $\hat{z}$ and z.

Lemma 3.3. *If* $E, x, y \in E^1$ *are such that* $E \geq \chi_{\{0\}}$ *and* $\chi_{\{0\}} \leq x \leq y$, *then* $\chi_{\{0\}} \leq Ex \leq Ey$.

Proof. By hypotheses,

$$\begin{gathered} 0 \leq x_L(a) \leq y_L(a), \quad 0 \leq x_R(a) \leq y_R(a), \quad \forall a \in [0,1], \\ 0 \leq E_L(a), \quad 0 \leq E_R(a), \quad \forall a \in [0,1], \end{gathered} \tag{3.3}$$

so that, for $a \in [0,1]$,

$$[Ex]^a = [E_L(a)x_L(a), E_R(a)x_R(a)], \qquad [Ey]^a = [E_L(a)y_L(a), E_R(a)y_R(a)], \tag{3.4}$$

where

$$0 \le E_L(a)x_L(a) \le E_L(a)y_L(a), \quad 0 \le E_R(a)x_R(a) \le E_R(a)y_R(a), \quad \forall a \in [0,1], \tag{3.5}$$

hence

$$\chi_{\{0\}} \le Ex \le Ey. \tag{3.6}$$

□

Theorem 3.4. *Let E, F, G be fuzzy numbers such that*

$$E, F, G \ge \chi_{\{0\}}, \tag{3.7}$$

and suppose that there exists $p > 0$ such that

$$E_R(0)p^2 + F_R(0)p + G_R(0) \le p. \tag{3.8}$$

Then (1.11) has extremal solutions in the interval

$$[\chi_{\{0\}}, \chi_{\{p\}}] := \{x \in E^1 : \chi_{\{0\}} \le x \le \chi_{\{p\}}\}. \tag{3.9}$$

Proof. Since $p > 0$, $\chi_{\{0\}} < \chi_{\{p\}}$. Define

$$A : [\chi_{\{0\}}, \chi_{\{p\}}] \longrightarrow E^1, \tag{3.10}$$

by $Ax = Ex^2 + Fx + G$. We show that $A([\chi_{\{0\}}, \chi_{\{p\}}]) \subseteq [\chi_{\{0\}}, \chi_{\{p\}}]$. Indeed,

$$\begin{aligned} A\chi_{\{0\}} &= E(\chi_{\{0\}})^2 + F\chi_{\{0\}} + G = \chi_{\{0\}} + \chi_{\{0\}} + G = G \ge \chi_{\{0\}}, \\ A\chi_{\{p\}} &= E(\chi_{\{p\}})^2 + F\chi_{\{p\}} + G, \end{aligned} \tag{3.11}$$

so that, using the conditions, for every $a \in [0,1]$, we have

$$\begin{aligned} [A\chi_{\{p\}}]^a &= [E_L(a), E_R(a)]\{p^2\} + [F_L(a), F_R(a)]\{p\} + [G_L(a), G_R(a)] \\ &= [E_L(a)p^2 + F_L(a)p + G_L(a), E_R(a)p^2 + F_R(a)p + G_R(a)]. \end{aligned} \tag{3.12}$$

By hypotheses and using the properties of $E_L, E_R, F_L, F_R, G_L, G_R$, we obtain, for all $a \in [0,1]$,

$$\begin{aligned} E_L(a)p^2 + F_L(a)p + G_L(a) &\le E_R(a)p^2 + F_R(a)p + G_R(a) \\ &\le E_R(0)p^2 + F_R(0)p + G_R(0) \le p. \end{aligned} \tag{3.13}$$

This proves that $A\chi_{\{p\}} \le \chi_{\{p\}}$. Moreover, A is a nondecreasing operator. Indeed, for $\chi_{\{0\}} \le x \le y$, we have

$$0 \le x_L(a) \le y_L(a), \quad 0 \le x_R(a) \le y_R(a), \quad \forall a \in [0,1], \tag{3.14}$$

and thus

$$0 \le \left(x_L(a)\right)^2 \le \left(y_L(a)\right)^2, \quad 0 \le \left(x_R(a)\right)^2 \le \left(y_R(a)\right)^2, \quad \forall a \in [0,1]. \tag{3.15}$$

Hence

$$\chi_{\{0\}} \le x^2 \le y^2. \tag{3.16}$$

This fact could have also been deduced from application of Lemma 3.3. Using that $E,F \ge \chi_{\{0\}}$ and applying Lemma 3.3, we obtain

$$Ax = Ex^2 + Fx + G \le Ey^2 + Fy + G = Ay. \tag{3.17}$$

Therefore, $A : [\chi_{\{0\}},\chi_{\{p\}}] \to [\chi_{\{0\}},\chi_{\{p\}}]$ is nondecreasing and $[\chi_{\{0\}},\chi_{\{p\}}]$ is a complete lattice. Tarski's fixed point theorem provides the existence of extremal fixed points for A in $[\chi_{\{0\}},\chi_{\{p\}}]$, that is, extremal solutions to (1.11) in the same interval. □

Remark 3.5. Suppose that $E_R(0) > 0$. To find an appropriate $p > 0$, we can solve the inequality

$$E_R(0)p^2 + \left(F_R(0) - 1\right)p + G_R(0) \le 0, \tag{3.18}$$

and, of course, study the discriminant

$$\left(F_R(0) - 1\right)^2 - 4E_R(0)G_R(0). \tag{3.19}$$

For instance, if it is equal to zero, the function

$$\varphi(p) = E_R(0)p^2 + \left(F_R(0) - 1\right)p + G_R(0) \tag{3.20}$$

is nonnegative and has a unique zero $(1 - F_R(0))/(2E_R(0))$. Then, if $F_R(0) < 1$, we can take $p = (1 - F_R(0))/(2E_R(0)) > 0$. If the discriminant is negative, then $G_R(0) > 0$ and φ is positive (φ has no zeros). Hence hypothesis (3.8) is not verified. If the discriminant is positive, there exist two zeros for φ and, if $F_R(0) \le 1$, we can take

$$p = \frac{(1 - F_R(0)) + \sqrt{\left(F_R(0) - 1\right)^2 - 4E_R(0)G_R(0)}}{2E_R(0)} > 0. \tag{3.21}$$

In the case $E_R(0) = 0$ ($E = \chi_{\{0\}}$), we have to calculate $p > 0$ satisfying

$$(F_R(0) - 1)p + G_R(0) \le 0. \tag{3.22}$$

If $F_R(0) > 1$, there is no such value of p; if $F_R(0) = 1$, the unique possibility is that $G = \chi_{\{0\}}$, and any $p > 0$ is valid; and for $F_R(0) < 1$, we can take $p > 0$, $p \ge G_R(0)/(1 - F_R(0))$.

Remark 3.6. If $0 \le E_R(0) + F_R(0) < 1$, $E_R(0) > 0$ and

$$\frac{G_R(0)}{1 - E_R(0) - F_R(0)} \le 1, \tag{3.23}$$

then we can take $0 < p \le 1$ such that

$$p \ge \frac{G_R(0)}{1 - E_R(0) - F_R(0)}. \tag{3.24}$$

In this case,

$$\begin{gathered} E_R(0)p^2 \le E_R(0)p, \\ p(1 - E_R(0) - F_R(0)) \ge G_R(0), \end{gathered} \tag{3.25}$$

hence

$$E_R(0)p^2 + F_R(0)p + G_R(0) \le E_R(0)p + F_R(0)p + G_R(0) \le p. \tag{3.26}$$

Remark 3.7. Note that condition (3.8) in Theorem 3.4 coincides with estimate (2.48) for $k = p$. Hence, similarly to the statement in Remark 2.12, condition (3.8) can be written equivalently, using the hypotheses on E, F, G, as

$$d_\infty(E, \chi_{\{0\}})p^2 + d_\infty(F, \chi_{\{0\}})p + d_\infty(G, \chi_{\{0\}}) \le p. \tag{3.27}$$

In particular, for $p = 1$, we obtain

$$d_\infty(E, \chi_{\{0\}}) + d_\infty(F, \chi_{\{0\}}) + d_\infty(G, \chi_{\{0\}}) \le 1, \tag{3.28}$$

and we can take, for instance,

$$d_\infty(E, \chi_{\{0\}}) \le \frac{1}{6}, \qquad d_\infty(F, \chi_{\{0\}}) \le \frac{1}{6}, \qquad d_\infty(G, \chi_{\{0\}}) \le \frac{4}{6}, \tag{3.29}$$

to generalize Theorem 2.9.

Theorem 3.8. *Let E, F, G be fuzzy numbers such that*

$$E, F, G \geq \chi_{\{0\}}, \tag{3.30}$$

and suppose that there exists $u_0 \in E^1$ such that $u_0 > \chi_{\{0\}}$ and

$$Eu_0^2 + Fu_0 + G \leq u_0, \tag{3.31}$$

that is, for all $a \in [0,1]$,

$$\begin{aligned} E_L(a)\big((u_0)_L(a)\big)^2 + F_L(a)(u_0)_L(a) + G_L(a) &\leq (u_0)_L(a), \\ E_R(a)\big((u_0)_R(a)\big)^2 + F_R(a)(u_0)_R(a) + G_R(a) &\leq (u_0)_R(a). \end{aligned} \tag{3.32}$$

Then (1.11) has extremal solutions in

$$[\chi_{\{0\}}, u_0] := \{x \in E^1 : \chi_{\{0\}} \leq x \leq u_0\}. \tag{3.33}$$

Proof. Define

$$A : [\chi_{\{0\}}, u_0] \longrightarrow E^1, \tag{3.34}$$

by $Ax = Ex^2 + Fx + G$. Again $A\chi_{\{0\}} \geq \chi_{\{0\}}$, and, by hypothesis, $Au_0 \leq u_0$. Moreover, A is nondecreasing and $A : [\chi_{\{0\}}, u_0] \to [\chi_{\{0\}}, u_0]$. Using that $[\chi_{\{0\}}, u_0]$ is a complete lattice, we obtain the existence of extremal fixed points for A in $[\chi_{\{0\}}, u_0]$, using again Tarski's fixed point theorem. □

Remark 3.9. Taking $p > 0$ and $u_0 = \chi_{\{p\}} > \chi_{\{0\}}$ in Theorem 3.8, we obtain Theorem 3.4.

Now, we present analogous results for the partial ordering $\preceq$ in E^1. In this case, the intervals of the type $[\chi_{\{0\}}, \chi_{[-p,p]}]$, with $p > 0$, or $[\chi_{\{0\}}, u_0]$, with $u_0 \succ \chi_{\{0\}}$, are complete lattices.

Lemma 3.10. *If $E, x, y \in E^1$ are such that $E \succeq \chi_{\{0\}}$ and $\chi_{\{0\}} \preceq x \preceq y$, then $\chi_{\{0\}} \preceq Ex \preceq Ey$.*

Proof. By hypotheses,

$$\begin{gathered} E_L(a) \leq 0, \quad E_R(a) \geq 0, \quad \forall a \in [0,1], \\ y_L(a) \leq x_L(a) \leq 0 \leq x_R(a) \leq y_R(a), \quad \forall a \in [0,1], \end{gathered} \tag{3.35}$$

so that

$$\begin{aligned} E_R(a)y_L(a) \leq E_R(a)x_L(a) \leq 0, \quad 0 \geq E_L(a)x_R(a) \geq E_L(a)y_R(a), \quad \forall a, \\ E_L(a)y_L(a) \geq E_L(a)x_L(a) \geq 0, \quad 0 \leq E_R(a)x_R(a) \leq E_R(a)y_R(a), \quad \forall a, \end{aligned} \tag{3.36}$$

which imply, for all $a \in [0,1]$, that

$$\begin{aligned} &\min\{E_L(a)y_R(a),E_R(a)y_L(a)\} \le \min\{E_L(a)x_R(a),E_R(a)x_L(a)\} \le 0,\\ &0 \le \max\{E_L(a)x_L(a),E_R(a)x_R(a)\} \le \max\{E_L(a)y_L(a),E_R(a)y_R(a)\}. \end{aligned} \tag{3.37}$$

In consequence, for every $a \in [0,1]$,

$$\begin{aligned} \{0\} &\subseteq [Ex]^a\\ &= [\min\{E_L(a)x_R(a),E_R(a)x_L(a)\},\max\{E_L(a)x_L(a),E_R(a)x_R(a)\}]\\ &\subseteq [\min\{E_L(a)y_R(a),E_R(a)y_L(a)\},\max\{E_L(a)y_L(a),E_R(a)y_R(a)\}]\\ &= [Ey]^a, \end{aligned} \tag{3.38}$$

hence

$$\chi_{\{0\}} \preceq Ex \preceq Ey. \tag{3.39}$$

□

Theorem 3.11. *Let E,F,G be fuzzy numbers such that*

$$E,F,G \succeq \chi_{\{0\}}, \tag{3.40}$$

and suppose that there exists $p>0$ satisfying

$$-p \le \min\{E_L(0),-E_R(0)\}p^2 + \min\{F_L(0),-F_R(0)\}p + G_L(0), \tag{3.41}$$

$$\max\{-E_L(0),E_R(0)\}p^2 + \max\{-F_L(0),F_R(0)\}p + G_R(0) \le p. \tag{3.42}$$

Then (1.11) has extremal solutions in

$$[\chi_{\{0\}},\chi_{[-p,p]}] := \{x \in E^1 : \chi_{\{0\}} \preceq x \preceq \chi_{[-p,p]}\}. \tag{3.43}$$

Proof. Since $p>0$, $\chi_{\{0\}} \prec \chi_{[-p,p]}$. Define

$$A : [\chi_{\{0\}},\chi_{[-p,p]}] \longrightarrow E^1, \tag{3.44}$$

by $Ax = Ex^2 + Fx + G$. We show that $A([\chi_{\{0\}},\chi_{[-p,p]}]) \subseteq [\chi_{\{0\}},\chi_{[-p,p]}]$. It is easy to prove that

$$A\chi_{\{0\}} = E(\chi_{\{0\}})^2 + F\chi_{\{0\}} + G = \chi_{\{0\}} + \chi_{\{0\}} + G = G \succeq \chi_{\{0\}}, \tag{3.45}$$

and, for every $a \in [0,1]$,

$$\begin{aligned} [A\chi_{[-p,p]}]^a &= [E_L(a),E_R(a)][-p^2,p^2] + [F_L(a),F_R(a)][-p,p] + [G_L(a),G_R(a)]\\ &= [\min\{E_L(a)p^2,-E_R(a)p^2\},\max\{-E_L(a)p^2,E_R(a)p^2\}]\\ &\quad + [\min\{F_L(a)p,-F_R(a)p\},\max\{-F_L(a)p,F_R(a)p\}]\\ &\quad + [G_L(a),G_R(a)], \end{aligned} \tag{3.46}$$

so that, for $a \in [0,1]$,

$$\begin{aligned}
[A\chi_{[-p,p]}]_L(a) &= \min\{E_L(a), -E_R(a)\}p^2 + \min\{F_L(a), -F_R(a)\}p + G_L(a),\\
[A\chi_{[-p,p]}]_R(a) &= \max\{-E_L(a), E_R(a)\}p^2 + \max\{-F_L(a), F_R(a)\}p + G_R(a).
\end{aligned} \tag{3.47}$$

By hypotheses and using the monotonicity properties of E_L, E_R, F_L, F_R, G_L, G_R, we obtain, for all $a \in [0,1]$,

$$\begin{aligned}
-p &\le \min\{E_L(0), -E_R(0)\}p^2 + \min\{F_L(0), -F_R(0)\}p + G_L(0)\\
&\le \min\{E_L(a), -E_R(a)\}p^2 + \min\{F_L(a), -F_R(a)\}p + G_L(a)\\
&= [A\chi_{[-p,p]}]_L(a),
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
[A\chi_{[-p,p]}]_R(a) &= \max\{-E_L(a), E_R(a)\}p^2 + \max\{-F_L(a), F_R(a)\}p + G_R(a)\\
&\le \max\{-E_L(0), E_R(0)\}p^2 + \max\{-F_L(0), F_R(0)\}p + G_R(0)\\
&\le p.
\end{aligned} \tag{3.49}$$

This proves that $A\chi_{[-p,p]} \preceq \chi_{[-p,p]}$. Besides, A is a nondecreasing operator. Take $\chi_{\{0\}} \preceq x \preceq y$, then

$$\begin{gathered}
y_L(a) \le x_L(a) \le 0 \le x_R(a) \le y_R(a), \quad \forall a \in [0,1],\\
\{0\} \subseteq [x^2]^a = [x_L(a), x_R(a)]^2 = [x_L(a)x_R(a), \max\{(x_L(a))^2, (x_R(a))^2\}], \quad \forall a.
\end{gathered} \tag{3.50}$$

Analogously for y. Hence, since $y_R(a) \ge 0$ and $x_L(a) \le 0$, then

$$y_L(a)y_R(a) \le x_L(a)y_R(a) \le x_L(a)x_R(a), \quad a \in [0,1], \tag{3.51}$$

and, using that $(x_L(a))^2 \le (y_L(a))^2$, $(x_R(a))^2 \le (y_R(a))^2$, we obtain

$$\max\{(x_L(a))^2, (x_R(a))^2\} \le \max\{(y_L(a))^2, (y_R(a))^2\}, \quad a \in [0,1], \tag{3.52}$$

which proves that

$$\begin{gathered}
\{0\} \subseteq [x^2]^a \subseteq [y^2]^a, \quad \forall a,\\
\chi_{\{0\}} \preceq x^2 \preceq y^2.
\end{gathered} \tag{3.53}$$

Using that $E, F \succeq \chi_{\{0\}}$ and Lemma 3.10, we obtain the nondecreasing character of A,

$$Ax = Ex^2 + Fx + G \preceq Ey^2 + Fy + G = Ay, \quad \text{for } \chi_{\{0\}} \preceq x \preceq y. \tag{3.54}$$

Tarski's fixed point theorem gives the existence of extremal fixed points for

$$A : [\chi_{\{0\}}, \chi_{[-p,p]}] \longrightarrow [\chi_{\{0\}}, \chi_{[-p,p]}] \tag{3.55}$$

in the complete lattice $[\chi_{\{0\}}, \chi_{[-p,p]}]$. □

Remark 3.12. In the hypotheses of Theorem 3.11, conditions (3.41) and (3.42) can be written, equivalently, as

$$d_\infty(E,\chi_{\{0\}})p^2 + d_\infty(F,\chi_{\{0\}})p + d_\infty(G,\chi_{\{0\}}) \leq p. \tag{3.56}$$

Compare with condition obtained in Remark 3.7 for the ordering $\leq$. Indeed, for $x \in E^1$, $x \succeq \chi_{\{0\}}$, we have $x_L(0) \leq x_L(a) \leq 0 \leq x_R(a) \leq x_R(0)$, for all $a \in [0,1]$, hence

$$\begin{aligned} d_\infty(x,\chi_{\{0\}}) &= \sup_{a\in[0,1]} \max\{|x_L(a)|, |x_R(a)|\} \\ &= \max\{|x_L(0)|, |x_R(0)|\} = \max\{-x_L(0), x_R(0)\}, \\ -d_\infty(x,\chi_{\{0\}}) &= -\max\{-x_L(0), x_R(0)\} = \min\{x_L(0), -x_R(0)\}. \end{aligned} \tag{3.57}$$

Now, since $E,F \succeq \chi_{\{0\}}$, conditions (3.41) and (3.42) are equivalent to

$$\begin{aligned} -p &\leq -d_\infty(E,\chi_{\{0\}})p^2 - d_\infty(F,\chi_{\{0\}})p + G_L(0), \\ d_\infty(E,\chi_{\{0\}})p^2 &+ d_\infty(F,\chi_{\{0\}})p + G_R(0) \leq p, \end{aligned} \tag{3.58}$$

or also

$$\begin{aligned} d_\infty(E,\chi_{\{0\}})p^2 + d_\infty(F,\chi_{\{0\}})p &\leq p + G_L(0), \\ d_\infty(E,\chi_{\{0\}})p^2 + d_\infty(F,\chi_{\{0\}})p &\leq p - G_R(0), \end{aligned} \tag{3.59}$$

that is,

$$\begin{aligned} d_\infty(E,\chi_{\{0\}})p^2 + d_\infty(F,\chi_{\{0\}})p &\leq \min\{p + G_L(0), p - G_R(0)\} \\ &= p + \min\{G_L(0), -G_R(0)\} \\ &= p - d_\infty(G,\chi_{\{0\}}). \end{aligned} \tag{3.60}$$

Hence, we have obtained the equivalent condition

$$d_\infty(E,\chi_{\{0\}})p^2 + d_\infty(F,\chi_{\{0\}})p + d_\infty(G,\chi_{\{0\}}) \leq p. \tag{3.61}$$

If $E,F,G \in E^1$, $E,F,G \succeq \chi_{\{0\}}$, and

$$d_\infty(E,\chi_{\{0\}}) + d_\infty(F,\chi_{\{0\}}) + d_\infty(G,\chi_{\{0\}}) \leq 1, \tag{3.62}$$

conditions in Theorem 3.11 are verified for $p = 1$, and (1.11) has extremal solutions in $[\chi_{\{0\}}, \chi_{[-1,1]}]$. We can choose, for instance,

$$d_\infty(E, \chi_{\{0\}}) \le \frac{1}{6}, \qquad d_\infty(F, \chi_{\{0\}}) \le \frac{1}{6}, \qquad d_\infty(G, \chi_{\{0\}}) \le \frac{4}{6}, \tag{3.63}$$

to obtain a result similar to Theorem 2.9.

THEOREM 3.13. *Let E, F, G be fuzzy numbers such that*

$$E, F, G \succeq \chi_{\{0\}}, \tag{3.64}$$

and suppose that there exists $u_0 \in E^1$ with $u_0 \succ \chi_{\{0\}}$ and

$$Eu_0^2 + Fu_0 + G \preceq u_0, \tag{3.65}$$

that is, for all $a \in [0,1]$,

$$\begin{gathered}
\min\{E_L(a) \cdot \max\{((u_0)_L(a))^2, ((u_0)_R(a))^2\}, E_R(a) \cdot (u_0)_L(a) \cdot (u_0)_R(a)\} \\
+ \min\{F_L(a) \cdot (u_0)_R(a), F_R(a) \cdot (u_0)_L(a)\} + G_L(a) \ge (u_0)_L(a), \\
\max\left\{E_L(a) \cdot (u_0)_L(a) \cdot (u_0)_R(a), E_R(a) \cdot \max\left\{((u_0)_L(a))^2, ((u_0)_R(a))^2\right\}\right\} \\
+ \max\{F_L(a) \cdot (u_0)_L(a), F_R(a) \cdot (u_0)_R(a)\} + G_R(a) \le (u_0)_R(a).
\end{gathered} \tag{3.66}$$

Then (1.11) has extremal solutions in

$$[\chi_{\{0\}}, u_0] := \{x \in E^1 : \chi_{\{0\}} \preceq x \preceq u_0\}. \tag{3.67}$$

Proof. Define

$$A : [\chi_{\{0\}}, u_0] \longrightarrow E^1, \tag{3.68}$$

by $Ax = Ex^2 + Fx + G$. Again $A\chi_{\{0\}} \succeq \chi_{\{0\}}$, and, by hypothesis, $Au_0 \preceq u_0$. Moreover, A is nondecreasing and $A : [\chi_{\{0\}}, u_0] \to [\chi_{\{0\}}, u_0]$. Using that $[\chi_{\{0\}}, u_0]$ is a complete lattice, the existence of extremal fixed points for A in $[\chi_{\{0\}}, u_0]$ follows from application of Tarski's fixed point theorem. □

Remark 3.14. If we take $p > 0$ and $u_0 = \chi_{[-p,p]} \succ \chi_{\{0\}}$ in Theorem 3.13, we get estimates in Theorem 3.11.

The following results (Theorems 3.15–3.18) are valid for the order $\le$ as well as for the order $\preceq$ with the obvious changes. We give them only for the order $\le$.

THEOREM 3.15. *Let E, F, G be fuzzy numbers such that*

$$E, F \ge \chi_{\{0\}}, \tag{3.69}$$

and suppose that there exist $\alpha, \beta \in E^1$ *with* $\beta > \alpha \geq \chi_{\{0\}}$ *and*

$$E\alpha^2 + F\alpha + G \geq \alpha, \\ E\beta^2 + F\beta + G \leq \beta. \tag{3.70}$$

Then (1.11) has extremal solutions in $[\alpha, \beta] := \{x \in E^1 : \alpha \leq x \leq \beta\}$*. Moreover, if* $\alpha = \beta$*,* α *is a solution to (1.11).*

Theorem 3.16. *Let* $F : E^1 \to E^1$ *be nondecreasing and suppose that there exist* $\alpha, \beta \in E^1$ *with* $\alpha \leq \beta$ *and*

$$F(\alpha) \geq \alpha, \\ F(\beta) \leq \beta. \tag{3.71}$$

Then equation

$$F(x) = x \tag{3.72}$$

has extremal solutions in $[\alpha, \beta]$*. Note that, if* $\alpha = \beta$*, this is a fixed point for* F*.*

Theorem 3.17. *Let* S *be a closed bounded interval in* E^1*, and* $F : S \to S$ *nondecreasing. Suppose that there exists* $\alpha \in S$ *with*

$$F(\alpha) \geq \alpha. \tag{3.73}$$

Then equation

$$F(x) = x \tag{3.74}$$

has a solution in S*. A solution is obtained as the supremum of the set*

$$X = \{x \in S : F(x) \geq x\}, \tag{3.75}$$

taking into account that S *is a complete lattice.*

Theorem 3.18. *Let* $F : E^1 \to E^1$ *be monotone nondecreasing (or nonincreasing and continuous). Suppose that there exists* $K \in [0, 1)$ *such that*

$$d_\infty(F(x), F(y)) \leq K d_\infty(x, y), \quad \forall x \geq y, \tag{3.76}$$

and there exists $x_0 \in E^1$ *with*

$$F(x_0) \geq x_0 \qquad \text{or} \qquad F(x_0) \leq x_0. \tag{3.77}$$

Then there exists exactly one solution for equation

$$F(x) = x. \tag{3.78}$$

Proof. Following the ideas in [4], if $F(x_0) \neq x_0$ and F is nondecreasing, we consider the sequence $\{F^n(x_0)\}_{n\in\mathbb{N}}$, which is a Cauchy sequence in E^1 and monotone. Since E^1 is a complete metric space, then there exists $y \in E^1$ such that

$$\lim_{n\longrightarrow+\infty} F^n(x_0) = y, \tag{3.79}$$

and y is a fixed point of F. For more details, see [4, Theorems 2.1, 2.2, and 2.4] and [5, Theorem 2.1]. □

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