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Research Article

The Monotone Iterative Technique for Three-Point Second-Order Integrodifferential Boundary Value Problems with p-Laplacian

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A monotone iterative technique is applied to prove the existence of the extremal positive pseudosymmetric solutions for a three-point second-order p-Laplacian integrodifferential boundary value problem.

1. Introduction

Investigation of positive solutions of multipoint second-order ordinary boundary value problems, initiated by Il'in and Moiseev [1, 2], has been extensively addressed by many authors, for instance, see [3–6]. Multipoint problems refer to a different family of boundary conditions in the study of disconjugacy theory [7]. Recently, Eloe and Ahmad [8] addressed a nonlinear nth-order BVP with nonlocal conditions. Also, there has been a considerable attention on p-Laplacian BVPs [9–18] as p-Laplacian appears in the study of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$), glaciology ($1 \leq p \leq 4/3$), and so forth.

In this paper, we develop a monotone iterative technique to prove the existence of extremal positive pseudosymmetric solutions for the following three-point second-order p-Laplacian integrodifferential boundary value problem (BVP):

$$\begin{gathered}\left(\psi_p\left(x'(t)\right)\right)' + a(t)\left\{f\left(t,x(t)\right) + \int_t^{(1+\eta)/2} K\left(t,\zeta,x(\zeta)\right)d\zeta\right\} = 0, \quad t \in (0,1), \\ x(0) = 0, \quad x(\eta) = x(1), \quad 0 < \eta < 1,\end{gathered} \tag{1.1}$$

where $p > 1$, $\psi_p(s) = s|s|^{p-2}$. Let ψ_q be the inverse of ψ_p.

In passing, we note that the monotone iterative technique developed in this paper is an application of Amann's method [19] and the first term of the iterative scheme may be taken to be a constant function or a simple function. The details of the monotone iterative method can be found in [20–27] and for the abstract monotone iterative method, see [28, 29]. To the best of the authors' knowledge, this is the first paper dealing with the integrodifferential equations in the present configuration. In fact, this work is motivated by [11, 17, 18]. The importance of the work lies in the fact that integrodifferential equations are encountered in many areas of science where it is necessary to take into account aftereffect or delay. Especially, models possessing hereditary properties are described by integrodifferential equations in practice. Also, the governing equations in the problems of biological sciences such as spreading of disease by the dispersal of infectious individuals, the reaction-diffusion models in ecology to estimate the speed of invasion, and so forth are integrodifferential equations.

2. Terminology and preliminaries

Let $E = C[0,1]$ be the Banach space equipped with norm $\|x\| = \max_{0\le t\le 1}|x(t)|$ and let P be a cone in E defined by $P = \{x \in E : x$ is nonnegative, concave on $[0,1]$, and pseudosymmetric about $(1+\eta)/2$ on $[0,1]\}$.

Definition 2.1. A functional $\gamma \in E$ is said to be concave on $[0,1]$ if $\gamma(tu+(1-t)v) \ge t\gamma(u)+(1-t)\gamma(v)$, for all $u,v \in [0,1]$ and $t \in [0,1]$.

Definition 2.2. A function $x \in E$ is said to be pseudosymmetric about $(1+\eta)/2$ on $[0,1]$ if x is symmetric over the interval $[\eta,1]$, that is, $x(t) = x(1-(t-\eta))$ for $t \in [\eta,1]$.

Throughout the paper, it is assumed that

(A_1) $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous nondecreasing in x, and for any fixed $x \in [0,\infty)$, $f(t,x)$ is pseudosymmetric in t about $(1+\eta)/2$ on $(0,1)$;

(A_2) $K : [0,1]\times[0,1]\times[0,\infty) \to [0,\infty)$ is continuous nondecreasing in x, and for any fixed $(\zeta,x) \in [0,1]\times[0,\infty)$, $K(t,\zeta,x)$ is pseudosymmetric in t about $(1+\eta)/2$ on $(0,1)$;

(A_3) $a(t) \in L(0,1)$ is nonnegative on $(0,1)$ and pseudosymmetric in t about $(1+\eta)/2$ on $(0,1)$. Further, $a(t)$ is not identically zero on any nontrivial compact subinterval of $(0,1)$.

Lemma 2.3. *Any $x \in P$ satisfies the following properties:*

(i) $x(t) \ge 2(1+\eta)^{-1}\|x\|\min\{t,(1-(t-\eta))\}$, $t \in [0,1]$;

(ii) $x(t) \ge 2\eta(1+\eta)^{-1}\|x\|$, $t \in [\eta,(1+\eta)/2]$;

(iii) $\|x\| = x((1+\eta)/2)$.

Proof. (i) For any $x \in P$, we define

$$x_\eta = \begin{cases} x(t), & t \in [0,1], \\ x(1-(t-\eta)), & t \in [1,1+\eta], \end{cases} \tag{2.1}$$

and note that x_η is nonnegative, concave, and symmetric on $[0,1+\eta]$ with $\|x_\eta\| = \|x\|$. From the concavity and symmetry of x_η, it follows that

$$x_\eta \geq \begin{cases} 2(1+\eta)^{-1}\|x_\eta\|t, & t \in \left[0, \dfrac{1+\eta}{2}\right], \\ 2(1+\eta)^{-1}\|x_\eta\|(1-(t-\eta)), & t \in \left[\dfrac{1+\eta}{2}, 1+\eta\right], \end{cases} \tag{2.2}$$

which, in view of $x_\eta(t) = x(t)$ on $[0,1]$, yields

$$x(t) \geq 2(1+\eta)^{-1}\|x\| \min\{t,(1-(t-\eta))\}, \quad t \in [0,1]. \tag{2.3}$$

The proof of (ii) is similar to that of (i) while (iii) can be proved using the properties of the cone P. □

Let us define an operator $\Omega : P \to E$ by

$$(\Omega x)(t) = \begin{cases} \displaystyle\int_0^t \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\right\}d\nu\right]dw, \\ \qquad\qquad t \in \left[0, \dfrac{1+\eta}{2}\right], \\ \displaystyle\int_0^\eta \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\right\}d\nu\right]dw \\ \quad + \displaystyle\int_t^1 \psi_q\left[\int_{(1+\eta)/2}^w a(\nu)\left\{f(\nu,x(\nu)) + \int_{(1+\eta)/2}^\nu K(\nu,\zeta,x(\zeta))d\zeta\right\}d\nu\right]dw, \\ \qquad\qquad t \in \left[\dfrac{1+\eta}{2}, 1\right]. \end{cases} \tag{2.4}$$

Obviously, $(\Omega x) \in E$ is well defined and x is a solution of problem (1.1) if and only if $\Omega x = x$. Now, we prove the following lemma which plays a pivotal role to prove the main result.

LEMMA 2.4. *Assume that* (A_1)*,* (A_2)*, and* (A_3) *hold. Then* $\Omega : P \to P$ *is continuous, compact, and nondecreasing.*

Proof. The nondecreasing nature of Ω follows from the fact that f and K are nondecreasing in x and that a is nonnegative. Now, for any $x \in P$, let $y = \Omega x$. Then

$$y'(t) = \psi_q\left[\int_t^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\right\}d\nu\right], \tag{2.5}$$

$$\left(\psi_p((y'(t)))\right)' = -a(t)\left\{f(t,x(t)) + \int_t^{(1+\eta)/2} K(t,\zeta,x(\zeta))d\zeta\right\} \leq 0, \tag{2.6}$$

that is, $y = \Omega x$ is concave. To show that Ω is compact, we take a set $A \subset P$. For $x \in A$, let $y = \Omega x$, which is bounded in E as the nonlinear functions f and K are continuous.

The expression for $(\Omega x)'$ is given by (2.5). If A is bounded, then the set $\{(\Omega x)' : x \in A\}$ is bounded, and hence ΩA is equicontinuous. By the Arzela-Ascoli theorem, ΩA is relatively compact. Now, we show that (Ωx) is pseudosymmetric about $(1+\eta)/2$ on $[0,1]$. For that, we note that $(1-(t-\eta)) \in [(1+\eta)/2, 1]$ for all $t \in [\eta, (1+\eta)/2]$. Thus,

$$\begin{aligned}
&(\Omega x)(1-(t-\eta)) \\
&\quad = \int_0^\eta \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\qquad + \int_{1-(t-\eta)}^1 \psi_q\left[\int_{(1+\eta)/2}^{w} a(\nu)\left\{f(\nu,x(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\quad = \int_0^\eta \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\qquad - \int_t^\eta \psi_q\left[\int_{(1+\eta)/2}^{1-(w-\eta)} a(\nu)\left\{f(\nu,x(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\quad = \int_0^\eta \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\qquad + \int_\eta^t \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_{(1+\eta)/2}^{1-(\nu-\eta)} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\quad = \int_0^\eta \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\qquad + \int_\eta^t \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw \\
&\quad = \int_0^t \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left\{f(\nu,x(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))\,d\zeta\right\}d\nu\right]dw = (\Omega x)(t).
\end{aligned} \tag{2.7}$$

Next, we show that (Ωx) is nonnegative. By the symmetry of (Ωx) on $[(1+\eta)/2, 1]$, it follows that $(\Omega x)'((1+\eta)/2) = 0$. The concavity of (Ωx) implies that $(\Omega x)'(t) \ge 0$, $t \in [0,(1+\eta)/2]$. Therefore, $(\Omega x)(1) = (\Omega x)(\eta) \ge (\Omega x)(0) = 0$. Consequently, we have $(\Omega x)(t) \ge 0$ as (Ωx) is concave. Hence we conclude that $\Omega P \subseteq P$. □

3. Main result

Theorem 3.1. *Assume that (A_1), (A_2), and (A_3) hold. Further, there exist positive numbers θ_1 and θ_2 such that $\theta_2 < \theta_1$ and*

$$\begin{aligned}
\sup_{0\le t\le 1}\left\{f(t,\theta_1) + \int_t^{(1+\eta)/2} K(t,\zeta,\theta_1)\,d\zeta\right\} &\le \psi_p(\theta_1\Theta_1), \\
\inf_{\eta\le t\le (1+\eta)/2}\left\{f(t,2\eta(1+\eta)^{-1}\theta_2) + \int_t^{(1+\eta)/2} K(t,\zeta,2\eta(1+\eta)^{-1}\theta_2)\,d\zeta\right\} &\ge \psi_p(\theta_2\Theta_2),
\end{aligned} \tag{3.1}$$

where

$$\Theta_1 = \frac{1}{\int_0^{(1+\eta)/2} \psi_q\big[\int_w^{(1+\eta)/2} a(\nu)d\nu\big]dw}, \qquad \Theta_2 = \frac{1}{\int_\eta^{(1+\eta)/2} \psi_q\big[\int_w^{(1+\eta)/2} a(\nu)d\nu\big]dw}. \tag{3.2}$$

Then there exist extremal positive, concave, and pseudosymmetric solutions α^*, β^* of (1.1) with $\theta_2 \le \|\alpha^*\| \le \theta_1$, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\Omega^n\alpha_0 = \alpha^*$, where $\alpha_0(t) = \theta_1$, $t \in [0,1]$, and $\theta_2 \le \|\beta^*\| \le \theta_1$, $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\Omega^n\beta_0 = \beta^*$, where $\beta_0(t) = 2\theta_2(1+\eta)^{-1}\min\{t,(1-(\eta-t))\}$, $t\in[0,1]$.

Proof. We define

$$P[\theta_2,\theta_1] = \{\alpha \in P : \theta_2 \le \|\alpha\| \le \theta_1\}, \tag{3.3}$$

and show that $\Omega P[\theta_2,\theta_1] \subseteq P[\theta_2,\theta_1]$. Let $\alpha \in P[\theta_2,\theta_1]$, then

$$0 \le \alpha(t) \le \max_{0\le s\le 1}\alpha(s) = \|\alpha\| \le \theta_1. \tag{3.4}$$

By Lemma 2.3(ii), we have

$$\min_{\eta\le t\le(1+\eta)/2}\alpha(t) \ge 2\eta(1+\eta)^{-1}\|\alpha\| \ge 2\eta(1+\eta)^{-1}\theta_2. \tag{3.5}$$

Now, by assumptions (A_1) and (A_2), and (3.1), for $t \in [\eta,(1+\eta)/2]$, we obtain

$$\begin{aligned}
0 &\le f(t,\alpha(t)) + \int_t^{(1+\eta)/2} K(t,\zeta,\alpha(\zeta))d\zeta \le f(t,\theta_1) + \int_t^{(1+\eta)/2} K(t,\zeta,\theta_1)d\zeta \\
&\le \sup_{0\le t\le 1}\left\{ f(t,\theta_1) + \int_t^{(1+\eta)/2} K(t,\zeta,\theta_1)d\zeta\right\} \le \psi_p(\theta_1\Theta_1),
\end{aligned}$$
$$\begin{aligned}
&f(t,\alpha(t)) + \int_t^{(1+\eta)/2} K(t,\zeta,\alpha(\zeta))d\zeta \\
&\quad \ge f(t,2\eta(1+\eta)^{-1}\theta_2) + \int_t^{(1+\eta)/2} K(t,\zeta,2\eta(1+\eta)^{-1}\theta_2)d\zeta \\
&\quad \ge \inf_{\eta\le t\le(1+\eta)/2}\left\{ f(t,2\eta(1+\eta)^{-1}\theta_2) + \int_t^{(1+\eta)/2} K(t,\zeta,2\eta(1+\eta)^{-1}\theta_2)d\zeta\right\} \ge \psi_p(\theta_2\Theta_2).
\end{aligned} \tag{3.6}$$

By Lemma 2.4, $(\Omega\alpha) \in P$. Therefore, by Lemma 2.3(iii), $\|(\Omega\alpha)\| = (\Omega\alpha)((1+\eta)/2)$. Note that θ_j and Θ_j are constants and $\psi_q(\psi_p(\theta_j\Theta_j)) = \theta_j\Theta_j$, $j = 1,2$. Now, we use (3.2)–(3.6) to obtain

$$\begin{aligned}
||(\Omega\alpha)|| &= (\Omega\alpha)\left(\frac{1+\eta}{2}\right)\\
&= \int_0^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\left\{f(\nu,\alpha(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,\alpha(\zeta))\,d\zeta\right\}d\nu\right]dw\\
&\geq \int_\eta^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\left\{f(\nu,\alpha(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,\alpha(\zeta))\,d\zeta\right\}d\nu\right]dw\\
&\geq \int_\eta^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\psi_p(\theta_2\Theta_2)\,d\nu\right]dw\\
&= \int_\eta^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\,d\nu\right]dw\,\psi_q[\psi_p(\theta_2\Theta_2)]\\
&= \int_\eta^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\,d\nu\right]dw\,(\theta_2\Theta_2)=\theta_2,
\end{aligned} \tag{3.7}$$

where we have used the fact that $\psi_q(s_1s_2)=\psi_q(s_1)\psi_q(s_2)$ as $\psi_q(s)=s^{1/(p-1)}$ for $s>0$. Similarly, we have

$$\begin{aligned}
||(\Omega\alpha)|| &= (\Omega\alpha)\left(\frac{1+\eta}{2}\right)\\
&= \int_0^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\left\{f(\nu,\alpha(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,\alpha(\zeta))\,d\zeta\right\}d\nu\right]dw\\
&\leq \int_0^{(1+\eta)/2}\psi_q\left[\int_w^{(1+\eta)/2}a(\nu)\psi_p(\theta_1\Theta_1)\,d\nu\right]dw=\theta_1.
\end{aligned} \tag{3.8}$$

Thus, it follows that $\theta_2 \leq \|(\Omega\alpha)\| \leq \theta_1$ for $\alpha \in P[\theta_2,\theta_1]$. Hence, $\Omega P[\theta_2,\theta_1] \subseteq P[\theta_2,\theta_1]$.

Now, we set $\alpha_0(t)=\theta_1$ ($\in P[\theta_2,\theta_1]$), $t\in[0,1]$, and $\alpha_1=\Omega\alpha_0$ ($\in P[\theta_2,\theta_1]$). We denote

$$\alpha_{n+1}=\Omega\alpha_n=\Omega^{n+1}\alpha_0,\quad n=1,2,\ldots. \tag{3.9}$$

In view of the fact that $\Omega P[\theta_2,\theta_1]\subseteq P[\theta_2,\theta_1]$, it follows that $\alpha_n\in P[\theta_2,\theta_1]$ for $n=0,1,2,\ldots$. Since Ω is compact by Lemma 2.4, therefore, we assert that the sequence $\{\alpha_n\}_{n=1}^\infty$ has a convergent subsequence $\{\alpha_{n_k}\}_{k=1}^\infty$ such that $\alpha_{n_k}\to\alpha^*$.

Since $\alpha_1\in P[\theta_2,\theta_1]$, therefore, $0\leq\alpha_1(t)\leq\|\alpha_1\|\leq\theta_1=\alpha_0(t)$, $t\in[0,1]$. Applying the nondecreasing property of Ω, we have $\Omega\alpha_1\leq\Omega\alpha_0$, which implies that $\alpha_2\leq\alpha_1$. Hence by induction, we obtain $\alpha_{n+1}\leq\alpha_n$, $n=0,1,2,\ldots$. Thus, $\alpha_n\to\alpha^*$. Taking the limit $n\to\infty$ in (3.9) yields $\Omega\alpha^*=\alpha^*$. Since $\|\alpha^*\|\geq\theta_2>0$ and α^* is a nonnegative concave function on $[0,1]$, we conclude that $\alpha^*(t)>0$, $t\in(0,1)$.

Now, we set $\beta_0(t) = 2\theta_2(1+\eta)^{-1}\min\{t,(1-(\eta-t))\}$, $t \in [0,1]$, and note that $\|\beta_0\| = \theta_2$, $\beta_0 \in P[\theta_2,\theta_1]$. Letting $\beta_1 = \Omega\beta_0$ $(\in P[\theta_2,\theta_1])$, we define

$$\beta_{n+1} = \Omega\beta_n = \Omega^{n+1}\beta_0, \quad n = 1,2,\dots. \tag{3.10}$$

By Lemma 2.3(i), we have

$$\begin{aligned}\beta_1(t) &\geq \|\beta_1\| 2(1+\eta)^{-1}\min\{t,(1-(\eta-t))\}\\ &\geq 2\theta_2(1+\eta)^{-1}\min\{t,(1-(\eta-t))\} = \beta_0(t), \quad t \in [0,1].\end{aligned} \tag{3.11}$$

Again, using the nondecreasing property of Ω, we get $\Omega\beta_1 \geq \Omega\beta_0$, that is, $\beta_2 \geq \beta_1$. Employing the arguments similar to $\{\alpha_n\}_{n=1}^{\infty}$, it is straightforward to show that $\beta_{n_k} \to \beta^*$ and $\beta^*(t) > 0$, $t \in (0,1)$.

Now, utilizing the well-known fact that a fixed point of the operator Ω in P must be a solution of (1.1) in P, it follows from the monotone iterative technique [20] that α^* and β^* are the extremal positive, concave, and pseudosymmetric solutions of (1.1). This completes the proof. □

Remark 3.2. In case the Lipschitz condition is satisfied by the functions involved, the extremal solutions α^* and β^* obtained in Theorem 3.1 coincide, and then (1.1) would have a unique solution in $P[\theta_2,\theta_1]$.

Example 3.3. Let us consider the boundary value problem

$$\begin{gathered}\left(|x'|^3x'\right)'(t) + a(t)\left\{f(t,x(t)) + \int_t^{2/3} K(t,\zeta,x(\zeta))\,d\zeta\right\} = 0, \quad t \in (0,1),\\ x(0) = 0, \qquad x\left(\frac{1}{3}\right) = x(1),\end{gathered} \tag{3.12}$$

where $a(t) = t^{-1/2}(4/3-t)^{-1/2}$, $f(t,x(t)) = (x(t))^3 + \ln[1+(x(t))^2]$, $K(t,\zeta,x(\zeta)) = x(\zeta) + \ln[1+(x(\zeta))^3]$. It can easily be verified that $a(t)$ is nonnegative and pseudo-symmetric about 2/3 on $(0,1)$, $f(t,x(t))$ and $K(t,\zeta,x(\zeta))$ are continuous and nondecreasing in x. Moreover, we observe that

$$\begin{aligned}&\overline{\lim}_{u\to 0} \inf_{t\in[1/3,2/3]} \frac{f(t,u(t)) + \int_t^{2/3} K(t,\zeta,u(\zeta))\,d\zeta}{\psi_5(u)}\\ &\quad = \overline{\lim}_{u\to 0} \inf_{t\in[1/3,2/3]} \frac{u^3 + \ln[1+u^2] + \int_t^{2/3}[u+\ln(1+u^3)]\,d\zeta}{u^4} = +\infty,\\ &\overline{\lim}_{u\to +\infty} \inf_{t\in[0,1]} \frac{f(t,u(t)) + \int_t^{2/3} K(t,\zeta,u(\zeta))\,d\zeta}{\psi_5(u)}\\ &\quad = \overline{\lim}_{u\to +\infty} \inf_{t\in[0,1]} \frac{u^3 + \ln[1+u^2] + \int_t^{2/3}[u+\ln(1+u^3)]\,d\zeta}{u^4} = 0.\end{aligned} \tag{3.13}$$

Thus, by Theorem 3.1, there exist extremal positive, concave, and pseudosymmetric solutions for the boundary value problem (3.12).

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