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Characterizations of a Banach Space through the Strong Lacunary and the Lacunary Statistical Summabilities

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Abstract: In this manuscript we characterize the completeness of a normed space through the strong lacunary (N_θ) and lacunary statistical convergence (S_θ) of series. A new characterization of weakly unconditionally Cauchy series through N_θ and S_θ is obtained. We also relate the summability spaces associated with these summabilities with the strong p-Cesàro convergence summability space.

Keywords: lacunary statistical summability; strong lacunary summability; weak unconditionally Cauchy series

1. Introduction

Let X be a normed space, a sequence $(x_k) \subset X$ is said to be *strongly 1-Cesàro summable (briefly, $|\sigma_1|$-summable) to* $L \in X$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\| = 0.$$

Hardy-Littlewood [1] and Fekete [2] introduced this type of summability, which is related to the convergence of Fourier series (see [3,4]). The $|\sigma_1|$ summability along with the statistical convergence [5] started a very striking theory with important applications [6–8]. Some years later, the *strong lacunary summability* N_θ was presented by Freedman et al. [9] by introducing lacunary sequences and showed that N_θ is a larger class of BK-spaces which had many of the characteristics of $|\sigma_1|$. Later on, Fridy [10,11] showed the concept of statistical lacunary summability and they related it with the statistical convergence and the N_θ summability.

The characterization of a Banach space through different types of convergence has been dealt by authors such as Kolk [12], Connor, Ganichev, and Kadets [13],...

Consider X a normed space and $\sum x_i$ a series in X. In [14] the authors introduced the space of convergence $S(\sum x_i)$ associated with the series $\sum x_i$, which is defined as the space of sequences (a_j) in ℓ_∞ such that $\sum a_i x_i$ converges. They also proved that the necessary and sufficient condition for X to be a complete space is that for every weakly unconditionally Cauchy series $\sum x_i$, the space $S(\sum x_i)$ is complete. Recall that $\sum x_i$ is a weakly unconditionally Cauchy (wuC) series if for every permutation π of the set of natural numbers, the sequence $(\sum_{i=1}^{n} x_{\pi(i)})$ is a weakly Cauchy sequence. We will also rely on a powerful known result that states that a series $\sum x_i$ is wuC if and only if $\sum |f(x_i)| < \infty$ for all $f \in X^*$ (see [15] for Diestel's complete monograph about series in Banach spaces).

In [16,17] a Banach space is characterized by means of the strong p-Cesàro summability (w_p) and ideal-convergence. In this manuscript, the N_θ and S_θ summabilities are used along with the concept of weakly unconditionally series to characterize a Banach space. In Section 2 we introduce these two kinds of summabilities which are regular methods and we recall some properties. In Sections 3 and 4 we introduce the spaces $S_{S_\theta}(\sum_i x_i)$ and $S_{N_\theta}(\sum_i x_i)$ which will be used in Section 5 to characterize the completeness of a space.

2. Preliminaries

In this section, we present the definition of N_θ and S_θ summabilities for Banach spaces and the relations between them. First, we recall the concept of lacunary sequences.

Definition 1. *A* lacunary sequence *is an increasing sequence of natural numbers $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1}$ tends to infinite as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, the ratio $\dfrac{k_r}{k_{r-1}}$ will be denoted by q_r.*

We now give the definition of strong lacunary summability for Banach spaces based on the one given by Freedman for real-valued sequences [9].

Definition 2. *Let X be a Banach space and $\theta = (k_r)$ a lacunary sequence. A sequence $x = (x_k)$ in X is* lacunary strongly convergent *or $N_\theta-$summable to $L \in X$ if $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|x_k - L\| = 0$, and we write $N_\theta\text{-}\lim x_k = L$ or $x_k \underset{N_\theta}{\to} L$.*

Let N_θ be the space of all lacunary strongly convergent sequences,

$$N_\theta = \left\{ (x_k) \subseteq X : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|x_k - L\| = 0 \text{ for some } L \right\}.$$

The space N_θ is a BK$-$space endowed with the norm $\|x_k\|_\theta = \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|x_k\|$.

In 1993, Fridy and Orhan [11] introduced a generalization of the statistical convergence, the lacunary statistical convergence, using lacunary sequences. To accomplish this, they substituted the set $\{k : k \leq n\}$ by the set $\{k : k_{r-1} < k \leq k_r\}$. We recall now the definition of $\theta-$density of a subset $K \subset \mathbb{N}$.

Definition 3. *Let $\theta = (k_r)$ be a lacunary sequence. If $K \subset \mathbb{N}$, the $\theta-$density of K is denoted by $d_\theta(K) = \lim_r \frac{1}{h_r} \text{card}(\{k \in I_r : k \in K\})$, whenever this limit exists.*

It is easy to show that this density is a finitely additive measure and we can define the concept of lacunary statistically convergent sequences for Banach spaces.

Definition 4. *Let X be a Banach space and $\theta = (k_r)$ a lacunary sequence. A sequence $x = (x_k)$ is a* lacunary statistically convergent sequence *to $L \in X$ if given $\varepsilon > 0$,*

$$d_\theta(\{k \in I_r : \|x_k - L\| \geq \varepsilon\}) = 0,$$

or equivalently,

$$d_\theta(\{k \in I_r : \|x_k - L\| < \varepsilon\}) = 1,$$

we say that (x_k) *is* S_θ*-convergent and we write* $x_k \underset{S_\theta}{\to} L$.

Theorem 1. *Let* X *be a Banach space and* (x_k) *a sequence in* X. *Notice that* S_θ *and* N_θ *are regular methods.*

Proof.

1. If $(x_k) \to L$, then $(x_k) \underset{N_\theta}{\to} L$.

   Let $\varepsilon > 0$, then there exists k_0 such that if $k \geq k_0$, then

$$\|x_k - L\| < \varepsilon.$$

   Hence there exists $r_0 \in \mathbb{N}$ with $r_0 \geq k_0$ such that if $r \geq r_0$ we have

$$\frac{1}{h_r}\sum_{k\in I_r}\|x_k - L\| < \frac{1}{h_r}\sum_{k\in I_r}\varepsilon = \frac{h_r}{h_r}\varepsilon = \varepsilon$$

   which implies that $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|x_k - L\| = 0$.

2. If $(x_k) \to L$, then $(x_k) \underset{S_\theta}{\to} L$.

   Simply observe that since $(x_k) \to L$, given $\varepsilon > 0$ there exists k_0 such that for every $k \geq k_0$ we get $\operatorname{card}(\{k \in I_r : \|x_k - L\| \geq \varepsilon\}) = 0$, which implies $d_\theta(\{k \in I_r : \|x_k - L\| \geq \varepsilon\}) = 0$ for every $k \geq k_0$.

□

The reverse is not true, as we will show in Example 1, in which we introduce an unbounded sequence that is N_θ-summable and Example 2 where an unbounded S_θ convergent sequence is presented.

Example 1. *There exist unbounded sequences which are* N_θ*-summable.*

Let $\theta = (k_r)$ *be the lacunary sequence with* $k_0 = 0$ *and* $k_r = 2^r$. *Notice that*

- $h_1 = k_1 - k_0 = 2$ *and* $h_r = 2^{r-1}$ *for every* $r \geq 2$.
- $I_1 = (k_0, k_1] = (0, 2]$ *and* $I_r = (2^{r-1}, 2^r]$ *for every* $r \geq 2$.

Consider the sequence defined by

$$x_k = \begin{cases} 0 & \text{if } k \neq 2^j \text{ for all } j, \\ j-1 & \text{if } k = 2^j \text{ for some } j. \end{cases}$$

Notice that (x_k) *is unbounded and observe that*

$$\frac{\sum_{k\in I_r}|x_k - 0|}{h_r} = \left\{ \begin{array}{ll} 0 & \text{if } r = 1, \\ \frac{r-1}{2^{r-1}} & \text{if } r \geq 2, \end{array} \right\} \underset{r\to\infty}{\longrightarrow} 0,$$

which implies that $x_k \underset{N_\theta}{\to} 0$.

Fridy and Orhan [10] showed that N_θ and S_θ are equivalent for real-valued bounded sequences. This fact also holds for Banach spaces and we include the proof for the sake of completeness.

Theorem 2. *Let* X *be a Banach space,* (x_k) *a sequence in* X *and* $\theta = (k_r)$ *a lacunary sequence. Then:*

1. *$(x_k) \underset{N_\theta}{\rightarrow} L$ implies $(x_k) \underset{S_\theta}{\rightarrow} L$.*
2. *(x_k) bounded and $(x_k) \underset{S_\theta}{\rightarrow} L$ imply $(x_k) \underset{N_\theta}{\rightarrow} L$.*

Proof. 1. If $(x_k) \underset{N_\theta}{\rightarrow} L$, then for every $\varepsilon > 0$,

$$\sum_{k \in I_r} \|x_k - L\| \geq \sum_{\substack{k \in I_r \\ \|x_k - L\| \geq \varepsilon}} \|x_k - L\| \geq \varepsilon \, \mathrm{card}(\{k \in I_r : \|x_k - L\| \geq \varepsilon\}),$$

which implies that $(x_k) \underset{S_\theta}{\rightarrow} L$.

2. Let us suppose that (x_k) is bounded and $(x_k) \underset{S_\theta}{\rightarrow} L$. Since (x_k) is bounded, there exists $M > 0$ such that $\|x_k - L\| \leq M$ for every $k \in \mathbb{N}$. Given $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} \|x_k - L\| &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|x_k - L\| \geq \varepsilon}} \|x_k - L\| + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|x_k - L\| < \varepsilon}} \|x_k - L\| \\ &\leq \frac{M}{h_r} \mathrm{card}(\{k \in I_r : \|x_k - L\| \geq \varepsilon\}) + \varepsilon, \end{aligned}$$

so we deduce that $(x_k) \underset{N_\theta}{\rightarrow} L$.

□

Next, we give an example to illustrate that the hypothesis over the sequence to be bounded is necessary and cannot be removed.

Example 2. *There exist unbounded S_θ-convergent sequences to L which are not N_θ-summable to L.*

Let $\theta = (k_r)$ be the lacunary sequence with $k_0 = 0$ and $k_r = 2^r$. Notice that

- *$h_1 = k_1 - k_0 = 2$ and $h_r = 2^{r-1}$ for every $r \geq 2$.*
- *$I_1 = (k_0, k_1] = (0, 2]$ and $I_r = (2^{r-1}, 2^r]$ for every $r \geq 2$.*

Consider the sequence defined by

$$x_k = \begin{cases} 0 & \text{if } k \neq 2^j \text{ for all } j, \\ 2^j & \text{if } k = 2^j \text{ for some } j. \end{cases}$$

Given $\varepsilon > 0$, it is easy to show that $\dfrac{\mathrm{card}(\{k \in I_r : |x_k - 0| \geq \varepsilon\})}{h_r} \to 0$ as $r \to \infty$, which implies that $(x_k) \underset{S_\theta}{\rightarrow} 0$. Also, notice that (x_k) is an unbounded sequence. However,

$$\frac{\sum_{k \in I_r} |x_k - 0|}{h_r} = \left\{ \begin{array}{l} \dfrac{2}{2} = 1 \quad \text{if } r = 1, \\ \dfrac{2^r}{2^{r-1}} = 2 \text{ if } r \geq 2, \end{array} \right\} \underset{r \to \infty}{\longrightarrow} 2,$$

which implies that $x_k \underset{N_\theta}{\nrightarrow} 0$.

We now give the definition of lacunary statistically Cauchy sequences in Banach spaces as a generalization of the definition for real-valued sequences by Fridy and Orhan in [11].

Definition 5. *Let X be a Banach space and $\theta = (k_r)$ a lacunary sequence. A sequence $x = (x_k)$ is a* lacunary statistically Cauchy *sequence if there exists a subsequence $x_{k'(r)}$ of x_k such that $k'(r) \in I_r$ for every $r \in \mathbb{N}$, $\lim_{r\to\infty} x_{k'(r)} = L$ for some $L \in X$ and for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{k \in I_r : \|x_k - x_{k'(r)}\| \geq \varepsilon\}) = 0,$$

or equivalently,

$$\lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{k \in I_r : \|x_k - x_{k'(r)}\| < \varepsilon\}) = 1.$$

In this case, we say that (x_k) is S_θ-Cauchy.

An important result in [11] is the S_θ-Cauchy Criterion and some of the next theorems in this work rely on it. This result can also be obtained for sequences in Banach spaces, and we include the proof for the sake of completeness.

Theorem 3. *Let X be a Banach space. A sequence (x_k) in X is S_θ-convergent if and only if it is S_θ-Cauchy.*

Proof. Let (x_k) be an S_θ-convergent sequence in X and for every $k \in \mathbb{N}$, we define $K_j = \{k \in \mathbb{N} : \|x_k - L\| < 1/j\}$. Observe that $K_j \supseteq K_{j+1}$ and $\dfrac{\text{card}(K_j \cap I_r)}{h_r} \to 1$ as $r \to \infty$.

Set m_1 such that if $r \leq m_1$ then $\text{card}(K_1 \cap I_r)/h_r > 0$, i.e., $K_1 \cap I_r \neq \varnothing$. Next, choose $m_2 > m_1$ such that if $r \geq m_2$, then $K_2 \cap I_r \neq \varnothing$. Now, for each $m_1 \leq r \leq m_2$, we choose $k'_r \in I_r$ such that $k'_r \in I_r \cap K_1$, i.e., $\|x_{k'_r} - L\| < 1$. Inductively, we choose $m_{p+1} > m_p$ such that if $r > m_{p+1}$, then $I_r \cap K_{p+1} \neq \varnothing$. Thus, for all r such that $m_p \leq r < m_{p+1}$, we choose $k'_r \in I_r \cap K_p$, and we have $\|x_{k'_r} - L\| < 1/p$.

Therefore, we have a sequence k'_r such that $k'_r \in I_r$ for every $r \in \mathbb{N}$ and $\lim_{r\to\infty} x_{k'_r} = L$. Finally,

$$\begin{aligned}\frac{1}{h_r}\text{card}(\{k \in I_r : \|x_k - x_{k'_r}\| \geq \varepsilon\}) \leq &\frac{1}{h_r}\text{card}(\{k \in I_r : \|x_k - L\| \geq \varepsilon/2\}) \\ &+ \frac{1}{h_r}\text{card}(\{k \in I_r : \|x_{k'_r} - L\| \geq \varepsilon/2\}).\end{aligned}$$

Since $(x_k) \underset{S_\theta}{\to} L$ and $\lim_{r\to\infty} x_{k'_r} = L$ we deduce that (x_k) is S_θ- Cauchy.

Conversely, if (x_k) is a Cauchy sequence, for every $\varepsilon > 0$,

$$\begin{aligned}\text{card}(\{k \in I_r : \|x_k - L\|\}\| \geq \varepsilon\}) \leq &\text{card}(\{k \in I_r : \|x_k - x_{k'_r}\| \geq \varepsilon/2\}) \\ &+ \text{card}(\{k \in I_r : \|x_{k'_r} - L\| \geq \varepsilon/2\}).\end{aligned}$$

Since (x_k) is S_θ-Cauchy and $\lim_{r\to\infty} x_{k'_r} = L$, we deduce that $(x_k) \underset{S_\theta}{\to} L$.

□

3. The Statistical Lacunary Summability Space

Let us consider X a real Banach space, $\sum_i x_i$ a series in X and $\theta = (k_r)$ a lacunary sequence. We define

$$S_{S_\theta}\left(\sum_i x_i\right) = \left\{(a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ is } S_\theta\text{-summable}\right\}$$

endowed with the supremum norm. This space will be named as the space of S_θ-summability associated with $\sum_i x_i$. We will characterize the completeness of the space $S_{S_\theta}(\sum_i x_i)$ in Theorem 4, but first we need a lemma.

Lemma 1. *Let X be a real Banach space and suppose that the series $\sum x_i$ is not wuC. Then there exist $f \in X^*$ and a null sequence $(a_i)_i \in c_0$ such that*

$$\sum_i a_i f(x_i) = +\infty$$

and

$$a_i f(x_i) \geq 0.$$

Proof. Since $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$, there exists m_1 such that $\sum_{i=1}^{m_1} |f(x_i)| > 2 \cdot 2$.

We define $a_i = \frac{1}{2}$ if $f(x_i) \geq 0$ and $a_i = -\frac{1}{2}$ if $f(x_i) < 0$ for $i \in \{1, 2, \ldots, m_1\}$.

This implies that $\sum_{i=1}^{m_1} a_i f(x_i) > 2$ and $a_i f(x_i) \geq 0$ if $i \in \{1, 2, \ldots, m_1\}$.

Let $m_2 > m_1$ be such that $\sum_{i=m_1+1}^{m_2} |f(x_i)| > 2^2 \cdot 2^2$.

We define $a_i = \frac{1}{2^2}$ if $f(x_i) \geq 0$ and $a_i = -\frac{1}{2^2}$ if $f(x_i) < 0$ for $i \in \{m_1 + 1, \ldots, m_2\}$. Hence $\sum_{i=m_1+1}^{m_2} a_i f(x_i) > 2^2$ and $a_i f(x_i) \geq 0$ if $i \in \{m_1 + 1, \ldots, m_2\}$. So we have obtained a sequence $(a_i)_i \in c_0$ with the above properties. □

Theorem 4. *Let X be a real Banach space and $\theta = (k_r)$ a lacunary sequence. The following are equivalent:*

(1) The series $\sum_i x_i$ is weakly unconditionally Cauchy (wuC).
(2) The space $S_{S_\theta}(\sum_i x_i)$ is complete.
(3) The space of all null sequences c_0 is contained in $S_{S_\theta}(\sum_i x_i)$.

Proof. (1)⇒(2): Since $\sum x_i$ is wuC, the following supremum is finite:

$$H = \sup \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : |a_i| \leq 1, 1 \leq i \leq n, n \in \mathbb{N} \right\} < +\infty.$$

Let $(a^m)_m \subset S_{S_\theta}(\sum_i x_i)$ such that $\lim_m \|a^m - a^0\|_\infty = 0$, with $a^0 \in \ell_\infty$. We will show that $a^0 \in S_{S_\theta}(\sum_i x_i)$. Let us suppose without any loss of generality that $\|a^0\|_\infty \leq 1$. Then, the partial sums $S_k^0 = \sum_{i=1}^{k} a_i^0 x_i$ satisfy $\|S_k^0\| \leq H$ for every $k \in \mathbb{N}$, i.e., the sequence (S_k^0) is bounded. Then, $a^0 \in S_{S_\theta}(\sum_i x_i)$ if and only if (S_k^0) is S_θ-summable to some $L \in X$. According to Theorem 3, (S_k^0) is lacunary statistically convergent to $L \in X$ if and only if (S_k^0) is a lacunary statistically Cauchy sequence.

Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we obtain statement (2) if we show that there exists a sub-sequence $(S_{k'(r)})$ such that $k'(r) \in I_r$ for every r, $\lim_{r \to \infty} S_{k'(r)} = L$ and

$$d_\theta(\{k \in I_r : ||S_k^0 - S_{k'(r)}^0|| < \varepsilon\}) = 1.$$

Since $a^m \to a^0$ in ℓ_∞, there exists $m_0 > n$ such that $\|a^m - a^0\|_\infty < \frac{\varepsilon}{4H}$ for all $m > m_0$, and since $S_k^{m_0}$ is $S_\theta-$Cauchy, there exists $k'(r) \in I_r$ such that $\lim_{r \to \infty} S_{k'(r)}^{m_0} = L$ for some L and

$$d_\theta\left(\left\{k \in I_r : \|S_k^{m_0} - S_{k'(r)}^{m_0}\| < \frac{\varepsilon}{2}\right\}\right) = 1.$$

Consider $r \in \mathbb{N}$ and fix $k \in I_r$ such that

$$\|S_k^{m_0} - S_{k'(r)}^{m_0}\| < \frac{\varepsilon}{2}. \tag{1}$$

We will show that $\|S_k^0 - S_{k'(r)}^0\| < \varepsilon$, and this will prove that

$$\left\{k \in I_r : \|S_k^{m_0} - S_{k'(r)}^{m_0}\| < \frac{\varepsilon}{2}\right\} \subset \{k \in I_r : \|S_k^0 - S_{k'(r)}^0\| < \varepsilon\}.$$

Since the first set has density 1, the second will also have density 1 and we will be done.

Let us observe first that for every $j \in \mathbb{N}$,

$$\left\| \sum_{i=1}^{j} \frac{4H}{\varepsilon}(a_i^m - a_i^{m_0})x_i \right\| \leq H,$$

for every $m > m_0$, therefore

$$\|S_j^0 - S_j^{m_0}\| = \left\| \sum_{i=1}^{j}(a_i^0 - a_i^{m_0})x_i \right\| \leq \frac{\varepsilon}{4}. \tag{2}$$

Then, by applying the triangular inequality,

$$\begin{aligned} \|S_k^0 - S_{k'(r)}^0\| &\leq \|S_k^0 - S_k^{m_0}\| + \|S_k^{m_0} - S_{k'(r)}^{m_0}\| + \|S_{k'(r)}^{m_0} - S_{k'(r)}^0\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

where the last inequality follows by applying (1) and (2), which yields the desired result.

(2) $\Rightarrow$ (3): Let us observe that if $S_{S_\theta}(\sum_i x_i)$ is complete, then it contains the space of eventually zero sequences c_{00} and therefore the thesis comes, since the supremum norm completion of c_{00} is c_0.

(3) $\Rightarrow$ (1): By way of contradiction, suppose that the series $\sum x_i$ is not wuC. Therefore there exists $f \in X^*$ such that $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$. By Lemma 1 we can construct inductively a sequence $(a_i)_i \in c_0$ such that

$$\sum_i a_i f(x_i) = +\infty$$

and

$$a_i f(x_i) \geq 0.$$

Now we will prove that the sequence $S_k = \sum_{i=1}^{k} a_i f(x_i)$ is not S_θ-summable to any $L \in \mathbb{R}$. By way of contradiction, suppose that it is S_θ-summable to $L \in \mathbb{R}$, then we have

$$\frac{1}{h_r}\text{card}(\{k \in I_r : |S_k - L| \geq \varepsilon\}) = \frac{1}{h_r} \sum_{\substack{k=k_{r-1} \\ |S_k - L| \geq \varepsilon}}^{k_r} 1 \underset{r\to\infty}{\longrightarrow} 0.$$

Since S_k is an increasing sequence and $S_k \to \infty$, there exists k_0 such that $|S_k - L| \geq \varepsilon$ for every $k \geq k_0$. Let us suppose that $k_r > k_0$ for every r. Hence,

$$\frac{1}{h_r} \sum_{\substack{k=k_{r-1} \\ |S_k - L| \geq \varepsilon}}^{k_r} 1 = \frac{h_r}{h_r} = 1 \underset{r\to\infty}{\not\to} 0,$$

which is a contradiction. This implies that S_k is not S_θ-convergent and this is a contradiction with (3). □

4. The Strong Lacunary Summability Space

Let X be a real Banach space, $\sum_i x_i$ a series in X and $\theta = (k_r)$ a lacunary sequence. We define

$$S_{N_\theta}\left(\sum_i x_i\right) = \left\{ (a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ is } N_\theta\text{-summable} \right\}$$

endowed with the supremum norm. This will be named as the space of N_θ-summability associated with the series $\sum_i x_i$. We can now present a theorem very similar to that of Theorem 4 but for the case of N_θ-summability. Indeed Theorem 5 characterizes the completeness of the space $S_{N_\theta}(\sum_i x_i)$.

Theorem 5. *Let X be a real Banach space and $\theta = (k_r)$ a lacunary sequence. The following are equivalent:*

(1) The series $\sum_i x_i$ is weakly unconditionally Cauchy (wuC).
(2) The space $S_{N_\theta}(\sum_i x_i)$ is complete.
(3) The space of all null sequences c_0 is contained in $S_{N_\theta}(\sum_i x_i)$.

Proof. (1) $\Rightarrow$ (2): Since $\sum x_i$ is wuC, the following supremum is finite

$$H = \sup\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\| : |a_i| \leq 1, 1 \leq i \leq n, n \in \mathbb{N}\right\} < +\infty.$$

Let $(a^m)_m \subset S_{N_\theta}(\sum_i x_i)$ such that $\lim_m \|a^m - a^0\|_\infty = 0$, with $a^0 \in \ell_\infty$.

We will show that $a^0 \in S_{N_\theta}(\sum_i x_i)$.

Without loss of generality we can suppose that $\|a^0\|_\infty \leq 1$. Therefore the partial sums $S_k^0 = \sum_{i=1}^k a_i^0 x_i$ satisfy $\|S_k^0\| \leq H$ for every $k \in \mathbb{N}$, i.e., the sequence (S_k^0) is bounded. Hence $a^0 \in S_{N_\theta}(\sum_i x_i)$ if and only if (S_k^0) is N_θ-summable to some $L \in X$. Since (S_k^0) is bounded, it is sufficient to show that (S_k) is S_θ-convergent, thanks to to Fridy and Orhan's Theorem ([10], Theorem 2.1) (see Theorem 2). The result follows analogously as in Theorem 4.

(2) $\Rightarrow$ (3): It is sufficient to notice that $S_{S_\theta}(\sum_i x_i)$ is a complete space and it contains the space of eventually zero sequences c_{00}, so it contains the completion of c_{00} with respect to the supremum norm, hence it contains c_0.

(3) $\Rightarrow$ (1): By way of contradiction, suppose that the series $\sum x_i$ is not wuC. Therefore there exists $f \in X^*$ such that $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$. By Lemma 1 we can construct inductively a sequence $(a_i)_i \in c_0$ such that $\sum_i a_i f(x_i) = +\infty$ and $a_i f(x_i) \geq 0$.

The sequence $S_k = \sum_{i=1}^k a_i f(x_i)$ is not N_θ-summable to any $L \in \mathbb{R}$.

As $S_k \to \infty$, for every $A > 0$, there exists k_0 such that $|S_k| > A$ if $k \geq k_0$. Then we have

$$\frac{1}{h_r} \sum_{k \in I_r} |S_k| > \frac{h_r A}{h_r} = A.$$

Hence S_k is not N_θ-summable to any $L \in \mathbb{R}$, otherwise

$$\infty \leftarrow \frac{1}{h_r} \sum_{k \in I_r} |S_k| \leq |L| + \frac{1}{h_r} \sum_{k \in I_r} |S_k - L| \to |L|$$

We can conclude that S_k is not N_θ-convergent, a contradiction with (3).
□

5. Characterizations of the Completeness of a Banach Space

A Banach space X can be characterized by the completeness of the space $S_{N_\theta}(\sum_k x_k)$ for every wuC series $\sum_k x_k$, as we will show next.

Theorem 6. *Let X be a normed real vector space. Then X is complete if and only if $S_{N_\theta}(\sum_k x_k)$ is a complete space for every weakly unconditionally Cauchy series (wuC) $\sum_k x_k$.*

Proof. Thanks to Theorem 4, the condition is necessary.

Now suppose that X is not complete, hence there exists a series $\sum x_k$ in X such that $\|x_k\| \leq \frac{1}{k2^k}$ and $\sum x_k = x^{**} \in X^{**} \setminus X$.

We will provide a wuC series $\sum_k y_k$ such that $S_{N_\theta}(\sum_k y_k)$ is not complete, a contradiction.

Set $S_N = \sum_{k=1}^{N} x_k$. As X^{**} is a Banach space endowed with the dual topology, $\sup_{\|y^*\|\leq 1} |y^*(S_N) - x^{**}(y^*)|$ tends to 0 as $N \to \infty$, i.e.,

$$\lim_{N\to+\infty} y^*(S_N) = \lim_{N\to+\infty} \sum_{k=1}^{N} y^*(x_k) = x^{**}(y^*), \text{ for every } \|y^*\| \leq 1. \tag{3}$$

Put $y_k = kx_k$ and let us observe that $\|y_k\| < \frac{1}{2^k}$. Therefore $\sum y_k$ is absolutely convergent, thus it is unconditionally convergent and weakly unconditionally Cauchy.

We claim that the series $\sum_k \frac{1}{k} y_k$ is not N_θ-summable in X.

By way of contradiction suppose that $S_N = \sum_{k=1}^{N} \frac{1}{k} y_k$ is N_θ-summable in X, i.e., there exists L in X such that $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|S_k - L\| = 0$. This implies that

$$\lim_{r\to+\infty} \frac{1}{h_r} \sum_{k\in I_r} y^*(S_k) = y^*(L), \text{ for every } \|y^*\| \leq 1. \tag{4}$$

From Equations (3) and (4), the uniqueness of the limit and since N_θ is a regular method, we have $x^{**}(y^*) = y^*(L)$ for every $\|y^*\| \leq 1$, so we obtain $x^{**} = L \in X$, a contradiction. Hence $S_N = \sum_{k=1}^{N} \frac{1}{k} y_k$ is not N_θ-summable to any $L \in X$.

Finally, let us observe that since $\sum_k y_k$ is a weakly unconditionally Cauchy series and $S_N = \sum_{k=1}^{N} \frac{1}{k} y_k$ is not N_θ-summable, we have $(\frac{1}{k}) \notin S_{N_\theta}(\sum_k y_k)$ and this means that $c_0 \nsubseteq S_{N_\theta}(\sum_k y_k)$ which is a contradiction with Theorem 5(3), so the proof is complete. □

By a similar argument and taking into account Theorem 2, we have also the characterization for the S_θ-summability:

Theorem 7. *Let X be a normed real vector space. Then X is complete if and only if $S_{S_\theta}(\sum_i x_i)$ is a complete space for every weakly unconditionally Cauchy series (wuC) $\sum_i x_i$.*

Let $0 < p < +\infty$, the sequence (x_n) is said to be strongly p-Cesàro or w_p-summable if there is $L \in X$ such that

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} \|x_i - L\|^p = 0;$$

in this case we will write $(x_k) \underset{\mathrm{w}_p}{\to} L$ and $L = \mathrm{w}_p - \lim_n x_n$. Let $\sum x_i$ be a series in a real Banach space X, let us define

$$S_{\mathrm{w}_p}\left(\sum_i x_i\right) = \left\{(a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ is } \mathrm{w}_p\text{-summable}\right\}$$

endowed with the supremum norm. We refer to [16] for other properties of the space $S_{\mathrm{w}_p}(\sum_i x_i)$.

Finally, from Theorem 6, Theorem 7 and ([16], Theorem 3.5), we derive the following corollary.

Corollary 1. *Let X be a normed real vector space and $p \geq 1$. The following are equivalent:*

1. *X is complete.*
2. *$S_{N_\theta}(\sum_k x_k)$ is complete for every weakly unconditionally Cauchy series (wuC) $\sum_k x_k$.*
3. *$S_{S_\theta}(\sum_k x_k)$ is complete for every weakly unconditionally Cauchy series (wuC) $\sum_k x_k$.*
4. *$S_{\mathrm{w}_p}(\sum_k x_k)$ is complete for every weakly unconditionally Cauchy series (wuC) $\sum_k x_k$.*

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