Semigroups of Partial Transformations with Kernel and Image Restricted by an Equivalence

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Abstract

For an arbitrary set X and an equivalence relation μ on X, denote by $P_\mu(X)$ the semigroup of partial transformations α on X such that $x\mu \subseteq x(\ker(\alpha))$ for every $x \in \mathrm{dom}(\alpha)$, and the image of α is a partial transversal of μ. Every transversal K of μ defines a subgroup $G = G_K$ of $P_\mu(X)$. We study subsemigroups $\langle G, U\rangle$ of $P_\mu(X)$ generated by $G \cup U$, where U is any set of elements of $P_\mu(X)$ of rank less than $|X/\mu|$. We show that each $\langle G, U\rangle$ is a regular semigroup, describe Green's relations and ideals in $\langle G, U\rangle$, and determine when $\langle G, U\rangle$ is an inverse semigroup and when it is a completely regular semigroup. For a finite set X, the top $\mathcal{J}$-class J of $P_\mu(X)$ is a right group. We find formulas for the ranks of the semigroups J, $G \cup I$, $J \cup I$, and I, where I is any proper ideal of $P_\mu(X)$.

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1 Introduction

In semigroup theory, transformation semigroups play a role analogous to the role of permutation groups in group theory. For a given set X, denote by $T(X)$ the semigroup of full transformations on X, that is, the set of all functions from X to X with function composition as the semigroup operation. (Throughout this paper, we will write functions on the right and compose from left to right; that is, for $f : A \to B$ and $g : B \to C$, we will write xf, rather than $f(x)$, and $x(fg)$, rather than $(gf)(x)$.) The semigroup $T(X)$ is fundamental in semigroup theory since every semigroup can be embedded in some $T(X)$ [7, Theorem 1.1.2]. This result is analogous to Cayley's Theorem for groups, which states that every group can be embedded in some symmetric group $\mathrm{Sym}(X)$ of all permutations on X. A natural generalization of $T(X)$ is the semigroup $P(X)$ of partial transformations on X (that is, functions whose domain and image are included in X). The semigroup $P(X)$ contains as its subsemigroups both $T(X)$ and the symmetric inverse semigroup $\mathcal{I}(X)$ of partial injective transformations on X. The semigroup $\mathcal{I}(X)$ is fundamental for the important class of inverse semigroups (see [13] and [7, Chapter 5]) since every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [7, Theorem 5.1.7], which is another analogue of Cayley's Theorem for groups.

These transformation semigroups can be generalized by introducing an equivalence relation μ on X. In 1976, S. Madhavan [9] generalized the symmetric inverse semigroup $\mathcal{I}(X)$ this way by defining the semigroup $\mathcal{I}_\mu(X)$ consisting of all partial transformations α on X such that $x\mu = x(\ker(\alpha))$ for every $x \in \operatorname{dom}(\alpha)$, and $\operatorname{im}(\alpha)$ is a partial transversal of μ. Madhavan proved that every right normal right inverse semigroup (regular semigroup satisfying $efg = feg$ for all idempotents e, f, g) can be embedded into some $\mathcal{I}_\mu(X)$. In 2005, H. Pei [11] introduced the semigroup $T(X, \mu)$ of full transformations α on X that preserve μ (for all $x, y \in X$, if $(x, y) \in \mu$ then $(x\alpha, y\alpha) \in \mu$). The analogous semigroup $P(X, \mu)$ of partial transformations on X preserving μ was studied in [12]. In 2004, J. Araújo and the second author [2] studied the semigroup $T(X, \mu, K)$ of full transformations α on X that preserve both μ and a fixed transversal K of μ ($K\alpha \subseteq K$). One can consider the analogous semigroup $P(X, \mu, K)$ of partial transformations on X. We also note that some subgroups of $\operatorname{Sym}(X)$, where X is finite, defined by two partitions of X of the same type were studied in [1].

In 2010, S. Mendes-Gonçalves and R.P. Sullivan [10] introduced the semigroup $E(X, \mu)$ of full transformations α on X such that $\mu \subseteq \ker(\alpha)$, and its subsemigroup $T_\mu(X)$ consisting of all $\alpha \in E(X, \mu)$ such that $\operatorname{im}(\alpha)$ is a partial transversal of μ. They proved that $T_\mu(X) = \operatorname{Reg}(E(X, \mu))$ [10, Theorem 2.3], where for a semigroup S, $\operatorname{Reg}(S)$ denotes the set of regular elements of S. Again, one can consider the analogous semigroup $E^p(X, \mu)$ of partial transformations α on X such that $x\mu \subseteq x(\ker(\alpha))$ for every $x \in \operatorname{dom}(\alpha)$, and its subsemigroup $P_\mu(X)$ consisting of all $\alpha \in E^p(X, \mu)$ such that $\operatorname{im}(\alpha)$ is a partial transversal of μ.

The semigroups described in the preceding paragraph, and others, are special cases of the sandwich semigroups studied in [4]. For non-empty sets X and Y, denote by $\mathcal{PT}^a_{XY}$ the semigroup of all partial maps from X to Y, with product $\star$ defined by $f \star g = fag$, where a is a fixed partial map from Y to X [4, 3.2]. The set $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ of regular elements of $\mathcal{PT}^a_{XY}$ is a subsemigroup of $\mathcal{PT}^a_{XY}$ [4, 3.3]. Let μ be an equivalence relation on X, let K be a transversal of μ, and let $a : X \to K$ be the function such that $ka^{-1} = k\mu$ (where $k\mu$ is the μ-class of k). Then $\mathcal{PT}^a_{KX}$ is isomorphic to $E^p(X, \mu)$, and $\operatorname{Reg}(\mathcal{PT}^a_{KX})$ is isomorphic to $P_\mu(X)$. Similarly, the semigroups $E(X, \mu)$ and $T_\mu(X)$ are special cases of the sandwich semigroups studied in [5].

For the semigroups of partial transformations mentioned above, we have

$$\mathcal{I}_\mu(X) \subseteq P_\mu(X) \subseteq E^p(X, \mu) \subseteq P(X, \mu) \quad \text{and} \quad P(X, \mu, K) \subseteq P(X, \mu).$$

For the corresponding full transformation semigroups, we have

$$(\mathcal{I}_\mu(X) \cap T(X)) \subseteq T_\mu(X) \subseteq E(X, \mu) \subseteq T(X, \mu) \quad \text{and} \quad T(X, \mu, K) \subseteq T(X, \mu).$$

Let $\alpha \in P(X)$. We denote the domain of α by $\operatorname{dom}(\alpha)$, the image of α by $\operatorname{im}(\alpha)$, and the rank of α (the cardinality of $\operatorname{im}(\alpha)$) by $\operatorname{rank}(\alpha)$. The kernel of α is an equivalence $\ker(\alpha)$ on $\operatorname{dom}(\alpha)$ defined by

$$\ker(\alpha) = \{(x, y) \in \operatorname{dom}(\alpha) \times \operatorname{dom}(\alpha) : x\alpha = y\alpha\}.$$

Let μ be an equivalence on X. For $x \in X$, we denote by $x\mu$ the μ-equivalence class. Then $X/\mu = \{x\mu : x \in X\}$ is the partition of X induced by μ. A *transversal* of μ is any subset K of X such that K intersects each element of X/μ at exactly one point. Any subset of a transversal of μ is called a *partial transversal* of μ.

Definition 1.1. Let μ be an equivalence on X. An $\alpha \in P(X)$ is called a *μ-transformation* if $x\mu \subseteq x(\ker(\alpha))$ for every $x \in \operatorname{dom}(\alpha)$, and $\operatorname{im}(\alpha)$ is a partial transversal of μ. We denote by $P_\mu(X)$ the set of μ-transformations on X. It is clear that $P_\mu(X)$ is a subsemigroup of $P(X)$.

As we have already pointed out, $P_\mu(X)$ is a special case of the semigroup $\mathrm{Reg}(\mathcal{PT}^a_{XY})$ studied in [4].

Definition 1.2. Let μ be an equivalence on X. Fix a transversal K of μ. An element $\sigma \in P_\mu(X)$ is called a *K-permutation* if $\ker(\sigma) = \mu$ and $\mathrm{im}(\sigma) = K$. We denote by G_K the set of K-permutations. Then G_K is a subgroup of $P_\mu(X)$ isomorphic to the symmetric group $\mathrm{Sym}(K)$ (see Proposition 2.1).

Notation 1.3. For the remainder of the paper, we fix a nonempty set X, an equivalence μ on X, a transversal K of μ, the group $G = G_K$, the cardinal $v = |X/\mu| = |K|$, and a nonempty set U of μ-transformations of rank $< v$. We denote by $\langle G, U\rangle$ the subsemigroup of $P_\mu(X)$ generated by the set $G \cup U$. Note that v may be finite or infinite and that for every $\alpha \in P_\mu(X)$, $\mathrm{rank}(\alpha) \leq v$.

The purpose of this paper is to study the semigroups $\langle G, U\rangle$. These semigroups generalize several well-known semigroups. Let $\mu = \mathrm{id}_X$. Then, $K = X$ is the only transversal of μ and $G = G_K = \mathrm{Sym}(X)$. Let $X = \{1, \ldots, n\}$. Define $\alpha, \beta \in P_\mu(X)$ by $1\alpha = 2$ and $x\alpha = x$ for all $x \neq 1$, $\mathrm{dom}(\beta) = \{2, \ldots, n\}$ and $x\beta = x$ for all $x \in \mathrm{dom}(\beta)$. Then $\langle G, \{\alpha, \beta\}\rangle = P(X)$ [7, Exercise 1.9.13], $\langle G, \alpha\rangle = T(X)$ [7, Exercise 1.9.7], and $\langle G, \beta\rangle = \mathcal{I}(X)$ [7, Exercise 5.11.6]. For an arbitrary set X and any proper ideal I of $P(X)$, $T(X)$, or $\mathcal{I}(X)$, we can select a suitable U such that $\langle G, U\rangle = \mathrm{Sym}(X) \cup I$.

In Section 2, we prove that every semigroup $\langle G, U\rangle$ is regular (Theorem 2.4) and characterize the sets U for which $\langle G, U\rangle$ is an inverse semigroup (Theorem 2.7) and those U for which $\langle G, U\rangle$ is a completely regular semigroup (Theorem 2.10). In Section 3, we describe Green's relations (Theorems 3.1 and 3.2) and the ideals (Theorem 3.4) of $\langle G, U\rangle$, and determine the partial order of ideals of $\langle G, U\rangle$ (Theorem 3.6).

In Section 4, we assume that X is a finite set. Then, every ideal of $P_\mu(X)$ is of the form $I_r = \{\alpha \in P_\mu(X) : \mathrm{rank}(\alpha) \leq r\}$ and every $\mathcal{J}$-class of $P_\mu(X)$ is of the form $J_r = \{\alpha \in P_\mu(X) : \mathrm{rank}(\alpha) = r\}$, where $0 \leq r \leq v$. Moreover, J_v is a right group (Proposition 4.9), and $P_\mu(X) = J_v \cup I_{v-1}$. We find formulas for the ranks of the semigroups $G \cup I_r$ (Theorem 4.7), J_v (Proposition 4.9), $J_v \cup I_r$ (Theorem 4.10), and I_r (Corollary 4.11), where $0 \leq r < v$.

2 Regularity

In this section, we prove that each $\langle G, U\rangle$ is a regular semigroup and determine when $\langle G, U\rangle$ is an inverse semigroup and when it is a completely regular semigroup.

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$. If all elements of S are regular, we say that S is a *regular semigroup*. An element a' in S is called an *inverse* of a in S if $a = aa'a$ and $a' = a'aa'$. Since regular elements are precisely those that have inverses (if $a = axa$ then $a' = xax$ is an inverse of a), we may define a regular semigroup as a semigroup in which each element has an inverse [7, p. 51]. If every element of S has exactly one inverse then S is called an *inverse semigroup*. An alternative definition is that S is an inverse semigroup if it is regular and its idempotents commute [7, Theorem 5.1.1]. If every element of S is in some subgroup of S then S is called a *completely regular semigroup* [7, 4.1].

It is well known that $P(X)$, $T(X)$, and $\mathcal{I}(X)$ are regular semigroups. The semigroup $\mathcal{I}_\mu(X)$ is also regular, but $E^p(X, \mu)$ and $P(X, \mu)$ are not regular semigroups [9, 10, 12].

We first prove that $G = G_K$ is a group.

Proposition 2.1. *The set $G = G_K$ from* Definition 1.2 *is a subgroup of $P_\mu(X)$ isomorphic to the symmetric group* $\mathrm{Sym}(K)$.

Proof. Define $f : \mathrm{Sym}(K) \to P_\mu(X)$ by $\delta f = \sigma$, where for every $a \in K$, $(a\mu)\sigma = \{a\delta\}$. Then $\ker(\sigma) = \mu$ and $\mathrm{im}(\sigma) = K$, so $\sigma \in G$. It is straightforward to check that f is an injective semigroup homomorphism with $\mathrm{im}(f) = G$. The result follows. □

The following lemma will be crucial in our arguments. We will use it often without mentioning it explicitly.

Lemma 2.2. *Let L be a partial transversal of μ with $|L| < |K|$. Suppose $\sigma : L \to K$ is injective. Then σ can be extended to $\overline{\sigma} \in G$.*

Proof. Recall that $v = |K|$ and let $r = |L|$, so $r < v$. Since σ is injective, $|L\sigma| = |L| = r < v$. Let $K' = \{a \in K : a \in b\mu \text{ for some } b \in L\}$, $K_1 = K \setminus K'$, and $K_2 = K \setminus L\sigma$. Since L is a partial transversal of μ, we have $|K'| = |L| = r$. It follows that $|K_1| = |K_2| = v - r$ (if v is finite) and $|K_1| = |K_2| = v$ (if v is infinite). Fix a bijection $g : K_1 \to K_2$.

Define $\overline{\sigma} : X \to X$ as follows. Let $x \in X$. If $x\mu \cap L = \{b\}$, then define $(x\mu)\overline{\sigma} = \{b\sigma\}$. If $x\mu \cap L = \emptyset$, then define $(x\mu)\overline{\sigma} = \{ag\}$, where $\{a\} = K_1 \cap x\mu$. It is then clear that $\overline{\sigma} \in G$ and $\overline{\sigma}$ is an extension of σ. □

Notation 2.3. Let $\alpha \in P_\mu(X)$ with $\operatorname{rank}(\alpha) = r$. Then $0 \le r \le v$. Write $\operatorname{im}(\alpha) = \{x_i\}_{1 \le i \le r}$ and let $A_i = x_i\alpha^{-1} \cap K$. We will write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix},$$

where it will be understood that i is a cardinal ranging from 1 to $r = \operatorname{rank}(\alpha)$. This notation is justified by the fact that $\alpha \in P_\mu(X)$ is determined by its values on $\operatorname{dom}(\alpha) \cap K$.

For example, let $X = \{1, \dots, 8\}$, μ correspond to the partition $\{\{1,2,3\}, \{4,5\}, \{6,7,8\}\}$, and $K = \{1,4,6\}$. Then $\alpha, \beta \in P_\mu(X)$ defined by $\{1,2,3,4,5\}\alpha = \{1\}$, $\{6,7,8\}\alpha = \{4\}$, and $\{4,5,6,7,8\}\beta = \{6\}$ will be written

$$\alpha = \begin{pmatrix} \{1,4\} & \{6\} \\ 1 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \{4,6\} \\ 6 \end{pmatrix}.$$

Theorem 2.4. *Each semigroup $\langle G, U\rangle$ is regular.*

Proof. Let $\alpha \in \langle G, U\rangle$. If $\operatorname{rank}(\alpha) = v$, then $\alpha \in G$, and so α is a regular element of $\langle G, U\rangle$ since G is a group. Suppose $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$ with $\operatorname{rank}(\alpha) < v$. Fix $a_i \in A_i$ (note that $a_i \in K$) and define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$. By Lemma 2.2, σ can be extended to $\overline{\sigma} \in G$. It is clear that $\alpha\overline{\sigma}\alpha = \alpha$, so α is regular. □

Remark 2.5. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in \langle G, U\rangle$ with $\operatorname{rank}(\alpha) < v$ and let $\overline{\sigma} \in G$ be as in the proof of Theorem 2.4. Since $\alpha\overline{\sigma}\alpha = \alpha$, the transformation $\alpha' = \overline{\sigma}\alpha\overline{\sigma}$ is an inverse of α. For this particular inverse, we have

$$x_i\alpha' = x_i(\overline{\sigma}\alpha\overline{\sigma}) = a_i(\alpha\overline{\sigma}) = x_i\overline{\sigma} = a_i.$$

Theorem 2.4 is not true if we allow U to contain transformations of rank v. For example, suppose $|X/\mu| = \aleph_0$ with $K = \{a_1, a_2, \dots\}$. Consider $\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$ of rank $\aleph_0$ with $A_i = \{a_{2i}\}$. Then $\langle G, \alpha\rangle$ is not a regular semigroup. Indeed, suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in \langle G, \alpha\rangle$. Then $a_i = a_{2i}\alpha = a_{2i}(\alpha\beta\alpha) = a_i(\beta\alpha)$, which implies $a_i\beta \in a_{2i}\mu$. Thus $\operatorname{im}(\beta) \ne K$, which is impossible since $\operatorname{im}(\alpha) = K$, and so the image of every element of $\langle G, \alpha\rangle$ is K.

However, $P_\mu(X)$ is a regular semigroup. We fix an idempotent $\varepsilon \in P_\mu(X)$ such that $\ker(\varepsilon) = \mu$. Since then the image of ε is a transversal of μ, we may assume that $\operatorname{im}(\varepsilon) = K$. Note that ε is the identity of the group $G = G_K$ and $(a\mu)\varepsilon = \{a\}$ for every $a \in K$.

For a semigroup S, we denote by $\operatorname{Reg}(S)$ the set of all regular elements of S.

Theorem 2.6. $P_\mu(X) = \mathrm{Reg}(\varepsilon P(X))$. *Consequently, $P_\mu(X)$ is a regular semigroup.*

Proof. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in P_\mu(X)$. Fix $a_i \in A_i$ and consider $\beta = \begin{pmatrix} x_i\mu \\ a_i \end{pmatrix} \in P(X)$. Then $\alpha = \varepsilon\alpha$ and $\beta = \varepsilon\beta$, so $\alpha, \beta \in \varepsilon P(X)$, and $\alpha = \alpha\beta\alpha$. Thus $\alpha \in \mathrm{Reg}(\varepsilon P(X))$.

Conversely, let $\alpha \in \mathrm{Reg}(\varepsilon P(X))$. Then, $\alpha = \alpha\beta\alpha$ for some $\beta \in \varepsilon P(X)$. By the definition of ε, we have $x\mu \subseteq x(\ker(\alpha))$ for every $x \in \mathrm{dom}(\alpha)$. Let $x, y \in \mathrm{dom}(\alpha)$ with $(x\alpha, y\alpha) \in \mu$. Then $x\alpha, y\alpha \in \mathrm{dom}(\beta)$, and so $(x\alpha)\beta = (y\alpha)\beta$ (since $z\mu \subseteq z(\ker(\beta))$ for every $z \in \mathrm{dom}(\beta)$). Thus, $x\alpha = x(\alpha\beta\alpha) = ((x\alpha)\beta)\alpha = ((y\alpha)\beta)\alpha = y(\alpha\beta\alpha) = y\alpha$, which implies that $\mathrm{im}(\alpha)$ is a partial transversal of μ. Hence $\alpha \in P_\mu(X)$ □

We note that $\varepsilon P(X) = E^p(X, \mu)$ (see Section 1).

Theorem 2.7. *A semigroup $\langle G, U\rangle$ is an inverse semigroup if and only if the following conditions are satisfied:*

(a) *for all $\alpha \in U$ and $x \in \mathrm{dom}(\alpha)$, $x\mu = x(\ker(\alpha))$;*

(b) *for all $\alpha, \beta \in U$, $x \in \mathrm{dom}(\alpha)$, and $y \in \mathrm{dom}(\beta)$, if $(x\alpha, y\beta) \in \mu$, then $x\alpha = y\beta$.*

Proof. ($\Rightarrow$) Suppose that (a) does not hold. Then there is $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in U$ such that $|A_j| \geq 2$ for some j. Select $a_i \in A_i$ (for each i) and $a \in A_j$ such that $a \neq a_j$. Define $\sigma_1, \sigma_2 : \{x_i\} \to K$ by $x_i\sigma_1 = a_i$ for all i, $x_i\sigma_2 = a_i$ if $i \neq j$, and $x_j\sigma_2 = a$, and extend these mappings to $\overline{\sigma_1}, \overline{\sigma_2} \in G$ as in the proof of Lemma 2.2. Both $\overline{\sigma_1}\alpha\overline{\sigma_1}$ and $\overline{\sigma_2}\alpha\overline{\sigma_2}$ are inverses of α (see Remark 2.5). Moreover, they are distinct since $x_j(\overline{\sigma_1}\alpha\overline{\sigma_1}) = a_j$ and $x_j(\overline{\sigma_2}\alpha\overline{\sigma_2}) = a$. Thus $\langle G, U\rangle$ is not an inverse semigroup.

Suppose that (b) does not hold. Then there are $\alpha, \beta \in U$ such that, for some $x \in \mathrm{dom}(\alpha)$ and $y \in \mathrm{dom}(\beta)$, $(x\alpha, y\beta) \in \mu$ and $x\alpha \neq y\beta$. Let α' and β' be inverses of α and β in $\langle G, U\rangle$, respectively. Then $\alpha'\alpha$ and $\beta'\beta$ are idempotents. Since $(x\alpha, y\beta) \in \mu$, we have $(x\alpha)\beta' = (y\beta)\beta'$ and $(y\beta)\alpha' = (x\alpha)\alpha'$. Thus

$$(x\alpha)(\alpha'\alpha\beta'\beta) = (x\alpha)(\beta'\beta) = ((x\alpha)\beta')\beta = ((y\beta)\beta')\beta = y\beta.$$

On the other hand,

$$(x\alpha)(\beta'\beta\alpha'\alpha) = (y\beta)(\beta'\beta\alpha'\alpha) = (y\beta)(\alpha'\alpha) = ((y\beta)\alpha')\alpha = ((x\alpha)\alpha')\alpha = x\alpha.$$

Thus the idempotents $\alpha'\alpha$ and $\beta'\beta$ do not commute, and so $\langle G, U\rangle$ is not an inverse semigroup.

($\Leftarrow$) Conversely, suppose that (a) and (b) are satisfied. Note that these conditions are also satisfied by the elements of the group G and they are preserved by the composition of transformations. It follows that (a) and (b) hold for all elements of $\langle G, U\rangle$. We already know that $\langle G, U\rangle$ is a regular semigroup. Let $\varepsilon, \xi \in \langle G, U\rangle$ be idempotents. We will show that ε and ξ commute. Note that $\alpha \in P_\mu(X)$ is an idempotent if and only if for all $x \in \mathrm{dom}(\alpha)$, $(x, x\alpha) \in \ker(\alpha)$. Let $x \in \mathrm{dom}(\varepsilon\xi)$, that is, $x \in \mathrm{dom}(\varepsilon)$ and $x\varepsilon \in \mathrm{dom}(\xi)$. Since ε satisfies (a), we have $(x, x\varepsilon) \in \mu$. Since $x \in \mathrm{dom}(\varepsilon)$ and $x\varepsilon \in \mathrm{dom}(\xi)$, it follows that $x\mu \subseteq \mathrm{dom}(\varepsilon)$ and $x\mu = (x\varepsilon)\mu \subseteq \mathrm{dom}(\xi)$. Further, by (a) applied to ξ, $(x, x\xi) \in \mu$. Thus $x \in \mathrm{dom}(\xi\varepsilon)$, so $\mathrm{dom}(\varepsilon\xi) \subseteq \mathrm{dom}(\xi\varepsilon)$. The reverse inclusion follows by symmetry, so $\mathrm{dom}(\varepsilon\xi) = \mathrm{dom}(\xi\varepsilon)$. Now, both $x(\varepsilon\xi)$ and $x(\xi\varepsilon)$ are in $x\mu$. Thus, by (b), $x(\varepsilon\xi) = x(\xi\varepsilon)$. Hence the idempotents in $\langle G, U\rangle$ commute, and so $\langle G, U\rangle$ is an inverse semigroup. □

Using arguments from the proof of Theorem 2.7, we can obtain the following result.

Theorem 2.8. *If $|X| \geq 2$, then $P_\mu(X)$ and $T_\mu(X)$ are not inverse semigroups.*

Regarding a criterion for $\langle G,U\rangle$ to be a completely regular semigroup, we will use the following result about $P(X)$. This result has been proved for $T(X)$ [3, Theorem 2.10] and extends easily to $P(X)$.

Lemma 2.9. *For every $\alpha \in P(X)$, α is in a subgroup of $P(X)$ if and only if $\mathrm{im}(\alpha)$ is a transversal of $\ker(\alpha)$.*

Theorem 2.10. *A semigroup $\langle G,U\rangle$ is completely regular if and only if for every nonzero $\alpha \in U$, $\ker(\alpha) = X \times X$.*

Proof. ($\Rightarrow$) Suppose $\ker(\alpha) \neq X \times X$ for some nonzero $\alpha \in U$. Note that $\ker(\alpha) \neq \mu$ since $\mathrm{rank}(\alpha) < v$. Thus there are two possible cases.

Case 1. $\ker(\alpha) = \mu \cap (\mathrm{dom}(\alpha) \times \mathrm{dom}(\alpha))$ and $\mathrm{dom}(\alpha) \neq X$.

Then there is $a \in K$ such that $a\mu \cap \mathrm{dom}(\alpha) = \emptyset$. Let $x \in \mathrm{dom}(\alpha)$ (such an x exists since $\alpha \neq 0$) and $y = x\alpha$. Define $\sigma : \{y\} \to K$ by $y\sigma = a$ and extend it to $\overline{\sigma} \in G$. Let $\beta = \alpha\overline{\sigma} \in \langle G,U\rangle$ and note that $x\beta = a$ and $a \notin \mathrm{dom}(\beta)$. Thus $\mathrm{im}(\beta)$ is not a transversal of $\ker(\beta)$, and so $\langle G,U\rangle$ is not completely regular.

Case 2. $\ker(\alpha) \neq \mu \cap (\mathrm{dom}(\alpha) \times \mathrm{dom}(\alpha))$.

Then, since $\alpha \neq 0$, there are $x, y \in \mathrm{dom}(\alpha)$ such that $(x,y) \in \ker(\alpha)$ and $(x,y) \notin \mu$. If $\mathrm{im}(\alpha)$ is not a transversal of $\ker(\alpha)$, then $\langle G,U\rangle$ is not completely regular. Suppose $\mathrm{im}(\alpha)$ is a transversal of $\ker(\alpha)$. We can then assume that $x \in \mathrm{im}(\alpha)$. Let $z \in \mathrm{dom}(\alpha)$ with $z\alpha = x$ and let $w = x\alpha$.

Suppose $(x,w) \in \ker(\alpha)$. Then $w = x$ since $\mathrm{im}(\alpha)$ is a transversal of $\ker(\alpha)$. Thus $x\alpha = x = z\alpha$, and so $(x,z) \in \ker(\alpha)$. Since $\ker(\alpha) \neq X \times X$, there is $u \in X$ such that $(x,u) \notin \ker(\alpha)$. Suppose $u \in \mathrm{dom}(\alpha)$ and let $t = u\alpha$. Note that $t \neq x$ (since $(x,u) \notin \ker(\alpha)$), and so $(x,t) \notin \ker(\alpha)$ (since x is the only element of $\mathrm{im}(\alpha)$ in the $\ker(\alpha)$-class of x). Let $\{a\} = K \cap x\mu$ and $\{b\} = K \cap y\mu$. Then $a \neq b$ (since $(x,y) \notin \mu$). Define $\sigma_1 : \{x,t\} \to K$ by $x\sigma_1 = a$ and $t\sigma_1 = b$ and extend it to $\overline{\sigma_1} \in G$. Let $\beta_1 = \alpha\overline{\sigma_1} \in \langle G,U\rangle$ and note that $\ker(\beta_1) = \ker(\alpha)$, $x\beta_1 = a$ and $u\beta_1 = b$. Thus $a, b \in \mathrm{im}(\beta_1)$ and $(x,y) \in \ker(\beta_1)$, so $\mathrm{im}(\beta_1)$ is not a transversal of $\ker(\beta_1)$. Suppose $u \notin \mathrm{dom}(\alpha)$ and let $\{c\} = K \cap u\mu$. Define $\sigma_2 : \{x\} \to K$ by $x\sigma_2 = c$ and extend it to $\overline{\sigma_2} \in G$. Let $\beta_2 = \alpha\overline{\sigma_2} \in \langle G,U\rangle$ and note that $\ker(\beta_2) = \ker(\alpha)$, $x\beta_2 = c$, and $c \notin \mathrm{dom}(\beta_2)$. Thus $\mathrm{im}(\beta_2)$ is not a transversal of $\ker(\beta_2)$. Hence, when $(x,w) \in \ker(\alpha)$, $\langle G,U\rangle$ is not completely regular.

Finally, suppose $(x,w) \notin \ker(\alpha)$. Let $\{a\} = K \cap x\mu$ and $\{b\} = K \cap y\mu$. Then $a \neq b$ (since $(x,y) \notin \mu$). Define $\sigma : \{x,w\} \to K$ by $x\sigma = a$ and $w\sigma = b$ and extend it to $\overline{\sigma} \in G$. Let $\beta = \alpha\overline{\sigma} \in \langle G,U\rangle$ and note that $\ker(\beta) = \ker(\alpha)$, $z\beta = a$ and $x\beta = b$. Thus $a, b \in \mathrm{im}(\beta)$ and $(x,y) \in \ker(\beta)$, so $\mathrm{im}(\beta)$ is not a transversal of $\ker(\beta)$. Hence, when $(x,w) \notin \ker(\alpha)$, $\langle G,U\rangle$ is not completely regular.

($\Leftarrow$) Conversely, suppose $\ker(\alpha) = X \times X$ for every nonzero $\alpha \in U$. Then, every element of $\langle G,U\rangle$ is either an element of the group G or a constant idempotent with domain X. Thus $\langle G,U\rangle$ is completely regular. □

Using arguments from the proof of Theorem 2.10, we can obtain the following result.

Theorem 2.11. *If $\mu \neq X \times X$, then $P_\mu(X)$ and $T_\mu(X)$ are not completely regular semigroups.*

3 Green's relations and ideals

In this section, we determine Green's relations and ideals in $\langle G,U\rangle$.

Let S be a semigroup and denote by S^1 the semigroup S with an identity adjoined (if necessary). Then, for every $a \in S$, S^1a, aS^1, and S^1aS^1 are, respectively, the principal left ideal, principal right ideal, and principal ideal generated by a. The principal ideals of S have been used to define five equivalence relations on S that are among the most important tools in studying semigroups. For $a, b \in S$, we say that $a \mathcal{L} b$ if $S^1 a = S^1 b$, $a \mathcal{R} b$ if $aS^1 = bS^1$, and $a \mathcal{J} b$ if $S^1 a S^1 = S^1 b S^1$. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$. These equivalences are called *Green's relations*. The relations $\mathcal{L}$ and $\mathcal{R}$ commute [7, Proposition 2.1.3], and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. For a Green relation $\mathcal{F}$ in S and $a \in S$, we denote by F_a the $\mathcal{F}$-equivalence class of a.

Green's relations in the semigroup $P(X)$ are well known: $\alpha \mathcal{L} \beta \Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta)$; $\alpha \mathcal{R} \beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$; $\alpha \mathcal{J} \beta \Leftrightarrow \operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$, and $\mathcal{D} = \mathcal{J}$.

If T is a regular subsemigroup of S and $\mathcal{F} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$, then $\mathcal{F}$ in T is the restriction of $\mathcal{F}$ in S to $T \times T$ [7, Proposition 2.4.2]. Thus, by Theorem 2.4, we have the following result.

Theorem 3.1. *For all* $\alpha, \beta \in \langle G, U \rangle$, $\alpha \mathcal{L} \beta \Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta)$ *and* $\alpha \mathcal{R} \beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$.

The corresponding statements about relations $\mathcal{D}$ and $\mathcal{J}$ in a regular subsemigroup T of a semigroup S are not true. Therefore, the next result requires a proof. First we note that in any semigroup S, the inclusion relation on the set of principal ideals induces the partial order relation $\leq$ on the set of $\mathcal{J}$-classes:

$$J_a \leq J_b \Leftrightarrow S^1 a S^1 \subseteq S^1 b S^1.$$

Theorem 3.2. *In every* $\langle G, U \rangle$, $\mathcal{D} = \mathcal{J}$. *Moreover, if* $\alpha, \beta \in \langle G, U \rangle$, *then* $J_\alpha \leq J_\beta \Leftrightarrow \operatorname{rank}(\alpha) \leq \operatorname{rank}(\beta)$. *Consequently,* $\alpha \mathcal{J} \beta \Leftrightarrow \operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$.

Proof. Let $\alpha, \beta \in \langle G, U \rangle$. If $J_\alpha \leq J_\beta$, then $\alpha = \gamma_1 \beta \gamma_2$ for some $\gamma_1, \gamma_2 \in \langle G, U \rangle^1$, which implies $\operatorname{rank}(\alpha) \leq \operatorname{rank}(\beta)$. Conversely, let $r = \operatorname{rank}(\alpha)$, $t = \operatorname{rank}(\beta)$, and suppose $r \leq t$. If $r = t = v$, then $\alpha, \beta \in G$, and so, since G is a group, $\alpha \mathcal{J} \beta$, which implies $J_\alpha \leq J_\beta$. Suppose $r < v$. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$ and $\beta = \begin{pmatrix} B_j \\ y_j \end{pmatrix}$, with $1 \leq i \leq r$ and $1 \leq j \leq t$. Fix $b_j \in B_j$ and define $\sigma_1 : \{x_i\} \to \{b_i\}$ by $x_i \sigma_1 = b_i$ $(1 \leq i \leq r)$. Then σ_1 is well defined (since $r \leq t$) and injective. Thus, since $r < v$, σ_1 can be extended to $\overline{\sigma_1} \in G$. For every $1 \leq i \leq r$, let $\{c_i\} = x_i \mu \cap K$, and define $\sigma_2 : \{y_i\}_{1 \leq i \leq r} \to \{c_i\}$ by $y_i \sigma_2 = c_i$, and extend σ_2 to $\overline{\sigma_2} \in G$. Let $\alpha' \in \langle G, U \rangle$ be an inverse of α. Then, for all $1 \leq i \leq r$ and all $a_i \in A_i$,

$$\begin{aligned} a_i(\alpha \overline{\sigma_1} \beta \overline{\sigma_2} \alpha' \alpha) &= x_i(\overline{\sigma_1} \beta \overline{\sigma_2} \alpha' \alpha) = b_i(\beta \overline{\sigma_2} \alpha' \alpha) = y_i(\overline{\sigma_2} \alpha' \alpha) \\ &= c_i(\alpha' \alpha) = x_i(\alpha' \alpha) = a_i(\alpha \alpha' \alpha) = a_i \alpha. \end{aligned}$$

It follows that $\alpha = (\alpha \overline{\sigma_1}) \beta (\overline{\sigma_2} \alpha' \alpha)$, and so $J_\alpha \leq J_\beta$.

In every semigroup, $\mathcal{D} \subseteq \mathcal{J}$. Let $\alpha, \beta \in \langle G, U \rangle$ with $\alpha \mathcal{J} \beta$. Then, by the first part of the proof, $\operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$. Let $r = \operatorname{rank}(\alpha)$. If $r = v$, then $\alpha, \beta \in G$, and so $\alpha \mathcal{D} \beta$. Suppose $r < v$. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$, and $\beta = \begin{pmatrix} B_i \\ y_i \end{pmatrix}$, with $1 \leq i \leq r$. Fix $a_i \in A_i$ and $b_i \in B_i$, define $\sigma_1 : \{x_i\} \to \{a_i\}$ by $x_i \sigma_1 = a_i$, $\sigma_2 : \{a_i\} \to \{b_i\}$ by $a_i \sigma_2 = b_i$, and extend σ_1 to $\overline{\sigma_1} \in G$ and σ_2 to $\overline{\sigma_2} \in G$. Let $\gamma = \alpha \overline{\sigma_1}\, \overline{\sigma_2} \beta \in \langle G, U \rangle$. Then, for all $1 \leq i \leq r$ and all $c_i \in A_i$,

$$c_i \gamma = c_i(\alpha \overline{\sigma_1}\, \overline{\sigma_2} \beta) = a_i(\alpha \overline{\sigma_1}\, \overline{\sigma_2} \beta) = x_i(\overline{\sigma_1}\, \overline{\sigma_2} \beta) = a_i(\overline{\sigma_2} \beta) = b_i \beta = y_i.$$

It follows that $\ker(\gamma) = \ker(\alpha)$ and $\operatorname{im}(\gamma) = \operatorname{im}(\beta)$. Thus, by Theorem 3.1, $\alpha \mathcal{R} \gamma$ and $\gamma \mathcal{L} \beta$, and so $\alpha \mathcal{D} \beta$ since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$. Hence $\mathcal{J} \subseteq \mathcal{D}$, and so $\mathcal{D} = \mathcal{J}$. $\square$

Notation 3.3. Let $\beta \in \langle G, U \rangle$ with $r = \operatorname{rank}(\beta)$. By Theorem 3.2, the principal ideal of $\langle G, U \rangle$ generated by β consists of all $\alpha \in \langle G, U \rangle$ such that $\operatorname{rank}(\alpha) \leq r$. We denote this principal ideal by I_r, that is,

$$I_r = \{\alpha \in \langle G, U \rangle : \operatorname{rank}(\alpha) \leq r\}.$$

For a cardinal k, we denote by k^+ the successor cardinal of k [8, p. 162]. For $1 \leq r \leq v^+$, let

$$E_r = \{\alpha \in \langle G, U \rangle : \operatorname{rank}(\alpha) < r\}.$$

It is clear that E_r is an ideal of $\langle G, U \rangle$. Note that $E_{v^+} = \langle G, U \rangle$ and that for every $0 \leq r \leq v$, $I_r = E_{r^+}$.

We will now prove that every ideal of $\langle G, U \rangle$ is equal to some ideal E_r, where $1 \leq r \leq v^+$, and determine the partial order of ideals of $\langle G, U \rangle$.

Theorem 3.4. *Let I be an ideal of $\langle G, U \rangle$. Then $I = E_r$ for some r with $1 \leq r \leq v^+$.*

Proof. Let r be the minimum cardinal such that $r \leq v^+$ and $\operatorname{rank}(\beta) < r$ for every $\beta \in I$. (Such an r exists because $\operatorname{rank}(\beta) < v^+$ for every $\beta \in \langle G, U \rangle$.) Clearly, $I \subseteq E_r$. Let $\alpha \in E_r$. By the minimality of r, there is $\beta \in I$ such that $\operatorname{rank}(\alpha) \leq \operatorname{rank}(\beta)$. By Theorem 3.2, $\alpha = \gamma_1 \beta \gamma_2$ for some $\gamma_1, \gamma_2 \in \langle G, U \rangle^1$. Thus $\alpha \in I$, and so $E_r \subseteq I$. $\square$

It follows from Theorem 3.4 that the ideals of every semigroup $\langle G, U \rangle$ form a chain. To describe this chain, we need the following lemma.

Lemma 3.5. *Let $t < v$ and suppose U contains some α with $\operatorname{rank}(\alpha) = t$. Then:*

(1) *for every cardinal r with $1 \leq r < t$, there is $\beta \in \langle G, U \rangle$ such that $\operatorname{rank}(\beta) = r$;*

(2) *if U contains some γ with $\operatorname{dom}(\gamma) \neq X$, then $0 \in \langle G, U \rangle$.*

Proof. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$, where $1 \leq i \leq t$, and fix $a_i \in A_i$. To prove (1), let $1 \leq r < t$. Suppose $t \geq \aleph_0$ and consider two possible cases.

Case 1. There is a cardinal l with $1 \leq l \leq t$ such that $|A_l| > t$.

Then, there is a subset $\{b_j\}_{r<j\leq t}$ of A_l such that each $b_j \neq a_l$ and $b_{j_1} \neq b_{j_2}$ if $j_1 \neq j_2$. If $l \leq r$, then define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$ if $1 \leq i \leq r$, and $x_i\sigma = b_i$ if $r < i \leq t$; if $l > r$, then define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$ if $1 \leq i < r$, $x_r\sigma = a_l$, and $x_i\sigma = b_i$ if $r < i \leq t$. In either case, we can extend σ to $\overline{\sigma} \in G$. Let $\beta = \alpha\overline{\sigma}\alpha \in \langle G, U \rangle$. Then, $\operatorname{im}(\beta) = \{x_i\}_{1\leq i\leq r}$ if $l \leq r$, and $\operatorname{im}(\beta) = \{x_i\}_{1\leq i<r} \cup \{x_l\}$ if $l > r$. In either case, $\operatorname{rank}(\beta) = r$.

Case 2. For every cardinal i with $1 \leq i \leq t$, $|A_i| \leq t$.

Then, $|\bigcup A_i| \leq t \cdot t = t < v$, and so there is a subset $\{b_j\}_{r<j\leq t}$ of $K \setminus \bigcup A_i$ such that $b_{j_1} \neq b_{j_2}$ if $j_1 \neq j_2$. Define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$ if $1 \leq i \leq r$, and $x_i\sigma = b_i$ if $r < i \leq t$. Then, for $\beta = \alpha\overline{\sigma}\alpha \in \langle G, U \rangle$, we have $\operatorname{im}(\beta) = \{x_i\}_{1\leq i\leq r}$, and so $\operatorname{rank}(\beta) = r$.

Suppose $t < \aleph_0$. We may assume that $r = t - 1$. (The result will then follow by an inductive argument.) Suppose there is $l \in \{1, \ldots, t\}$ such that $|A_l| \geq 2$. We then have some $b \in A_l$ with $b \neq a_l$. Select $j \in \{1, \ldots, t\}$ with $j \neq l$ (possible since $1 \leq r < t$, so $t \geq 2$). Define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$ if $i \neq j$, and $x_j\sigma = b$, and extend σ to $\overline{\sigma} \in G$. Then, for $\beta = \alpha\overline{\sigma}\alpha \in \langle G, U \rangle$, we have $\operatorname{im}(\beta) = \{x_i\}_{i\neq j}$, and so $\operatorname{rank}(\beta) = t - 1$. Suppose $|A_i| = 1$ for all i. Since $t < v$, we then have some $b \in K \setminus \bigcup_{1\leq i\leq t} A_i$. Define $\sigma : \{x_i\} \to K$ by $x_i\sigma = a_i$ if $i < t$ and $x_t\sigma = b$, and extend σ to $\overline{\sigma} \in G$. Then, for $\beta = \alpha\overline{\sigma}\alpha \in \langle G, U \rangle$, we have $\operatorname{im}(\beta) = \{x_i\}_{i\neq t}$, and so $\operatorname{rank}(\beta) = t - 1$.

To prove (2), suppose U contains some γ with $\operatorname{dom}(\gamma) \neq X$. Select $a \in K$ such that $a \notin \operatorname{dom}(\gamma)$. By (1), there exists $\beta \in \langle G, U \rangle$ with $\operatorname{rank}(\beta) = 1$. Let $\operatorname{im}(\beta) = \{y\}$. Define $\sigma : \{y\} \to K$ by $y\sigma = a$ and extend σ to $\overline{\sigma} \in G$. Then $\beta\overline{\sigma}\gamma = 0$. $\square$

For sets A and B, we will write $A \subset B$ to mean $A \subseteq B$ and $A \neq B$.

Theorem 3.6. *Let $m = \min\{r : \operatorname{rank}(\alpha) < r$ for every $\alpha \in U\}$. Then:*

(1) *if U consists of full transformations on X, then the chain of ideals of $\langle G, U\rangle$ is isomorphic to the chain of cardinals $\{r : 2 \le r \le m\} \cup \{v^+\}$;*

(2) *if U contains a strictly partial transformation on X, then the chain of ideals of $\langle G, U\rangle$ is isomorphic to the chain of cardinals $\{r : 1 \le r \le m\} \cup \{v^+\}$.*

Proof. Note that $m \le v$. Suppose U consists of full transformations on X. Let $2 \le r < m$. By the minimality of m, there is $\alpha \in U$ such that $r \le \operatorname{rank}(\alpha)$. By Lemma 3.5, $\langle G, U\rangle$ contains a transformation of rank $< r$. Thus, for every cardinal r with $2 \le r < m$, $E_r \neq \emptyset$. Moreover, $E_m \neq \emptyset$ (since $E_2 \subseteq E_m$) and $E_1 = \emptyset$ (since 0 cannot be a product of full transformations on X). Hence, by Theorem 3.4, $\{E_r : 2 \le r \le m\} \cup \{E_{v^+}\}$ is the set of ideals of $\langle G, U\rangle$. Define

$$f : \{r : 2 \le r \le m\} \cup \{v^+\} \to \{E_r : 2 \le r \le m\} \cup \{E_{v^+}\}$$

by $rf = E_r$ $(2 \le r \le m)$ and $v^+ f = E_{v^+}$. It is then clear that f is surjective and that it preserves the order (for all $s, t \in \{r : 2 \le r \le m\} \cup \{v^+\}$, if $s \le t$, then $E_s \subseteq E_t$). Let $s, t \in \{r : 2 \le r \le m\} \cup \{v^+\}$ with $s < t$. If $t = v^+$, then $E_s \subset E_t$ since $G \subseteq E_{v^+}$ and $G \cap E_s = \emptyset$. Suppose $t \le m$. By the minimality of m, U contains α such that $s \le \operatorname{rank}(\alpha)$. If $\operatorname{rank}(\alpha) < t$, then $\alpha \in E_t \setminus E_s$, so $E_s \subset E_t$. Suppose $\operatorname{rank}(\alpha) \ge t$. Then, by Lemma 3.5, there is $\beta \in \langle G, U\rangle$ with $\operatorname{rank}(\beta) = s$. Thus $\beta \in E_t \setminus E_s$, so $E_s \subset E_t$. Hence f is injective, and so it is a poset isomorphism.

We have proved (1). The proof of (2) is almost identical. The difference is that, if U contains a strictly partial transformation, then $E_1 \neq \emptyset$ by Lemma 3.5. □

For example, denote by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, the sets of integers, rational numbers, and real numbers, respectively. Let $X = \mathbb{R}$, let μ be defined by the partition

$$\{\{-n, n\} : n \in \mathbb{Z}\} \cup \{\mathbb{Q} \setminus \mathbb{Z}\} \cup \{\{x\} : x \in \mathbb{R} \setminus \mathbb{Q}\},$$

and $K = \{0, 1, 2, \ldots\} \cup \{\frac{1}{2}\} \cup \{x : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Then $v = 2^{\aleph_0}$. Consider the idempotent

$$\varepsilon = \begin{pmatrix} \{0\} & \{1\} & \{2\} & \ldots & \{\frac{1}{2}\} & K \setminus \mathbb{Q} \\ 0 & 1 & 2 & \ldots & \frac{1}{2} & \sqrt{2} \end{pmatrix} \in P_\mu(\mathbb{R})$$

(see Notation 2.3) and the semigroup $\langle G, \varepsilon\rangle$. Note that ε is a full transformation on $\mathbb{R}$ and that $\operatorname{rank}(\varepsilon) = \aleph_0$. Thus, the cardinal m from Theorem 3.6 is $\aleph_1$ (see [8, p. 131]) and the chain of ideals of $\langle G, \varepsilon\rangle$ is isomorphic to the chain of cardinals

$$2 < 3 < 4 < \ldots < \aleph_0 < \aleph_1 < (2^{\aleph_0})^+,$$

which, in turn, is isomorphic to the ordinal $\omega_0 + 2$ (see [8, p. 131]).

4 Ranks

Throughout this section, X will be a finite set. By the more general results obtained in [4, 3.4], we can conclude the following. Every ideal of $P_\mu(X)$ is of the form $E_s = \{\alpha \in P_\mu(X) : \operatorname{rank}(\alpha) < s\}$, where $1 \le s \le v^+$. For a finite set X, $E_s = I_r = \{\alpha \in P_\mu(X) : \operatorname{rank}(\alpha) \le r\}$, where $r = s - 1$. Thus

$\{I_r : 0 \leq r \leq v\}$ is the set of ideals of $P_\mu(X)$. Each ideal I_r is principal and is generated by any $\alpha \in P_\mu(X)$ of rank r. Moreover, $I_v = P_\mu(X)$ and if $r < v$, then I_r is a proper ideal of $P_\mu(X)$. Let J_r be the set of elements of $P_\mu(X)$ of rank r, where $0 \leq r \leq v$. Then $\{J_r : 0 \leq r \leq v\}$ is the set of $\mathcal{J}$-classes of $P_\mu(X)$, with $J_0 < J_1 < \ldots < J_v$.

Since v is finite, J_v is the union of groups G_M, where M ranges over all transversals of μ (see Definition 1.2). We will show that J_v is a right group (Proposition 4.9).

In this section, we find formulas for the ranks of the semigroups $G \cup I_r$, J_v, $J_v \cup I_r$, and I_r, where $0 \leq r < v$. (For $r = v - 1$, we have $J_v \cup I_r = P_\mu(X)$.) We also record the corresponding formulas for $T_\mu(X) = P_\mu(X) \cap T(X)$.

Definition 4.1. Let S be a semigroup. The *rank* of S, denoted $\operatorname{rank} S$, is the minimum cardinality of a generating set of S.

The ranks of various transformation semigroups have been found. For example, for a finite set X, $\operatorname{rank} P(X) = 4$, $\operatorname{rank} T(X) = 3$, and $\operatorname{rank} \mathcal{I}(X) = 3$. The following general result for the ranks of finite semigroups proved in [6] is useful when working with transformation semigroups.

Lemma 4.2. ([6, Theorem 10]) *Let S be a finite nontrivial semigroup with a maximal regular class $\mathcal{J}$-class J such that $\langle J \rangle = S$. Suppose that each group $\mathcal{H}$-class of J has rank ≤ 2, and it is not the case that J has exactly one idempotent in every $\mathcal{R}$-class and in every $\mathcal{L}$-class. Then* $\operatorname{rank} S = \max\{m_l, m_r\}$, *where m_l and m_r are the numbers of $\mathcal{L}$- and $\mathcal{R}$-classes in J, respectively.*

Definition 4.3. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in P_\mu(X)$ with $\operatorname{rank}(\alpha) = r$, $0 \leq r \leq v$, as in Notation 2.3. In this section, we will always assume that $|A_1| \geq |A_2| \geq \ldots \geq |A_r|$. Let $n_i = |A_i|$ and $m = |K \setminus \bigcup_{1 \leq i \leq r} A_i|$. Then the sequence

$$(n_1, n_2, \ldots, n_r; m)$$

will be called the *μ-type* of α and denoted $\operatorname{type}_\mu(\alpha)$. We will call the number m the *deficit* of α. Note that $n_1 + n_2 + \cdots + n_r + m = v$ and that if $\alpha \neq 0$, then the sequence $(n_1, n_2, \ldots, n_r)$ is a partition of $v - m$ with r parts [14, p. 235].

By a *μ-type* we will mean any sequence $(n_1, n_2, \ldots, n_r; m)$ with $0 \leq r \leq v$, each $n_i \geq 1$, $m \geq 0$, and $n_1 + n_2 + \cdots + n_r + m = v$.

For example, every $\sigma \in G$ has μ-type $(1, 1, \ldots, 1; 0)$. Let $X = \{1, \ldots, 9\}$, μ be defined by the partition $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9\}\}$, and $K = \{1, 4, 6, 9\}$. Then

$$\alpha = \begin{pmatrix} \{1, 4\} & \{6\} \\ 1 & 2 \end{pmatrix} \in P_\mu(X)$$

has μ-type $(2, 1; 1)$.

Lemma 4.4. *Let $\alpha, \beta \in P_\mu(X)$ with* $\operatorname{rank}(\alpha) = \operatorname{rank}(\beta) = r < v$. *Then:*

(1) $\operatorname{type}_\mu(\sigma\beta) = \operatorname{type}_\mu(\beta)$ *for all $\sigma \in P_\mu(X)$ with* $\operatorname{rank}(\sigma) = v$*;*

(2) *if $\alpha = \beta\gamma$, for some $\gamma \in P_\mu(X)$, then* $\operatorname{type}_\mu(\alpha) = \operatorname{type}_\mu(\beta)$.

Proof. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$ and $\beta = \begin{pmatrix} B_i \\ y_i \end{pmatrix}$, $1 \leq i \leq r$. Then for every $\sigma \in P_\mu(X)$ with $\operatorname{rank}(\sigma) = v$, $\sigma\beta = \begin{pmatrix} B_i\sigma^{-1} \\ y_i \end{pmatrix}$, where $B_i\sigma^{-1} = \{k \in K : k\sigma \in b\mu \text{ for some } b \in B_i\}$. Since σ maps different elements of K to elements in different μ-classes, $|B_i\sigma^{-1}| = |B_i|$ for all i, so $\sigma\beta$ and β have the same μ-type.

Suppose $\alpha = \beta\gamma$ for some $\gamma \in P_\mu(X)$. Let $b \in B_i$. Then either $b \in A_j$, for some j, or $b \in K \setminus \bigcup_{1\le i\le r} A_i$. Suppose $b \in A_j$ and let $c \in B_i$. Then $b\beta = c\beta$, and so $x_j = b\alpha = b(\beta\gamma) = c(\beta\gamma) = c\alpha$. Thus $c \in A_j$.

We have proved that for every i, either $B_i \subseteq A_j$, for some j, or $B_i \subseteq K \setminus \bigcup_{1\le i\le r} A_i$. Let $j \in \{1, \ldots, r\}$ and $a \in A_j$. Then $a \in \mathrm{dom}(\alpha)$, and so $a \in \mathrm{dom}(\beta)$ (since $\alpha = \beta\gamma$). Thus $a \in B_i$ for some i, and so, by the foregoing argument, $a \in B_i \subseteq A_j$.

It then follows that every A_j is a union of some distinct $B_{i_1}, \ldots, B_{i_{k_j}}$. But the number of $A_j s$ is r and the number of $B_i s$ is also r. Hence each k_j must equal to 1, that is, for every j, there is i_j such that $A_j = B_{i_j}$, and $i_{j_1} \ne i_{j_2}$ if $j_1 \ne j_2$. It follows that $\mathrm{type}_\mu(\alpha) = \mathrm{type}_\mu(\beta)$. □

The following proposition will be crucial for the rank results.

Proposition 4.5. *Let $0 \le r < v$ and suppose $U \subseteq J_r$, where J_r is the $\mathcal{J}$-class of $P_\mu(X)$ of rank r. Then $J_r \subseteq \langle G, U\rangle$ if and only if for every μ-type $(n_1, n_2, \ldots, n_r; m)$ and every partial transversal L of μ with $|L| = r$ and $L \not\subseteq K$, there are $\alpha, \beta \in U$ such that $\mathrm{type}_\mu(\alpha) = (n_1, n_2, \ldots, n_r; m)$ and $\mathrm{im}(\beta) = L$.*

Proof. Suppose $J_r \subseteq \langle G, U\rangle$. Let $(n_1, n_2, \ldots, n_r; m)$ be a μ-type and L be a partial transversal of μ with $|L| = r$ and $L \not\subseteq K$. Since $J_r \subseteq \langle G, U\rangle$, there is $\eta \in \langle G, U\rangle$ such that $\mathrm{type}_\mu(\eta) = (n_1, n_2, \ldots, n_r; m)$ and $\mathrm{im}(\eta) = L$. Since $\langle G, U\rangle$ is generated by $G \cup U$, $\eta = \sigma\alpha\gamma$ or $\eta = \alpha\gamma$, where $\sigma \in G$, $\alpha \in U$, and $\gamma \in \langle G, U\rangle$. Since $\varepsilon\alpha = \alpha$, where ε is the identity in $G = G_K$, we may assume that $\eta = \sigma\alpha\gamma$. Thus, by Lemma 4.4, $\mathrm{type}_\mu(\alpha) = \mathrm{type}_\mu(\sigma\alpha) = \mathrm{type}_\mu(\eta) = (n_1, n_2, \ldots, n_r; m)$. Also, $\eta = \theta\beta$, where $\theta \in \langle G, U\rangle$ and $\beta \in U$. (Note that β cannot be followed by any element $\delta \in G$ since $\mathrm{im}(\eta) = L$ and $L \not\subseteq K$.) Then $L \subseteq \mathrm{im}(\beta)$, and so $\mathrm{im}(\beta) = L$ since $|L| = \mathrm{rank}(\beta) = r$.

Conversely, suppose that the set U satisfies the given condition. Let $\gamma = \begin{pmatrix} C_i \\ y_i \end{pmatrix} \in J_r$, and let $(n_1, n_2, \ldots, n_r; m)$ be the μ-type of γ (so $n_i = |C_i|$ for every i) and $L = \mathrm{im}(\gamma) = \{y_i\}$. By the hypothesis, there is $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in U$ with $\mathrm{type}_\mu(\alpha) = (n_1, n_2, \ldots, n_r; m)$. Then, for every i, $|A_i| = |C_i|$, so there is a bijection $f_i : A_i \to C_i$. Define $\sigma : \bigcup_{1\le i\le r} A_i \to K$ by $a\sigma = af_i$ if $a \in A_i$, and extend σ to $\overline{\sigma} \in G$. (Since X is finite, Lemma 2.2 is also true if $|L| = |K|$.) Then

$$(\overline{\sigma})^{-1}\alpha\overline{\sigma} = \begin{pmatrix} A_i\overline{\sigma} \\ x_i\overline{\sigma} \end{pmatrix} = \begin{pmatrix} A_i f_i \\ x_i\overline{\sigma} \end{pmatrix} = \begin{pmatrix} C_i \\ x_i\overline{\sigma} \end{pmatrix}.$$

Suppose $\{y_i\}_{1\le i\le r} \subseteq K$. Define $\delta : \{x_i\overline{\sigma}\} \to K$ by $(x_i\overline{\sigma})\delta = y_i$, and extend δ to $\overline{\delta} \in G$. Then $(\overline{\sigma})^{-1}\alpha\overline{\sigma}\overline{\delta} = \gamma$. Suppose $\{y_i\} \not\subseteq K$. Then, by the hypothesis, there is $\beta = \begin{pmatrix} B_i \\ y_i \end{pmatrix} \in U$. Fix $b_i \in B_i$, define $\delta : \{x_i\overline{\sigma}\} \to K$ by $(x_i\overline{\sigma})\delta = b_i$, and extend δ to $\overline{\delta} \in G$. Then $(\overline{\sigma})^{-1}\alpha\overline{\sigma}\overline{\delta}\beta = \gamma$. Hence $\gamma \in \langle G, U\rangle$, and so $J_r \subseteq \langle G, U\rangle$. □

For positive integers n and $r \le n$, denote by $p_r(n)$ the number of partitions of n with r parts. For example, $(3, 1, 1)$ and $(2, 2, 1)$ are the only partitions of 5 with 3 parts, so $p_3(5) = 2$. There is no known closed formula for calculating $p_r(n)$. For recursive formulas, see [14, Theorem 2.4.4].

Lemma 4.6. *Let $\alpha \in P_\mu(X)$ with $\mathrm{rank}(\alpha) = r < v - 1$. Then, there are $\varepsilon, \gamma \in P_\mu(X)$, both of rank $r + 1$, such that $\alpha = \varepsilon\gamma$.*

Proof. Let $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$ and fix $a_i \in A_i$. Choose $y \in X$ such that $y \notin x_i\mu$, for every i, $1 \leq i \leq r$ (possible since $r < v-1$ and $|X/\mu| = v$), and note that $y \notin \{x_1, \dots, x_r\}$.

Suppose $|A_t| \geq 3$, for some t, and let $b, c \in A_t$ with $b \neq c$ and $b, c \neq a_t$. Define $\varepsilon \in P_\mu(X)$, with $\mathrm{dom}(\varepsilon) = \mathrm{dom}(\alpha)$, by $k\varepsilon = a_i$ if $k \in A_i$ and $i \neq t$, $k\varepsilon = a_t$ if $k \in A_t$ and $k \neq b$, and $b\varepsilon = b$. Then $\mathrm{im}(\varepsilon) = \{a_i\}_{1 \leq i \leq r} \cup \{b\}$. Define $\gamma \in P_\mu(X)$, with $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha)$, by $k\gamma = x_i$ if $k \in A_i$ and $i \neq t$, $k\gamma = x_t$ if $k \in A_t$ and $k \neq c$, and $c\gamma = y$. Then $\mathrm{im}(\varepsilon) = \{x_i\}_{1 \leq i \leq r} \cup \{y\}$.

Suppose $|A_s| = |A_t| = 2$, for some distinct s, t, say $A_s = \{a_s, b\}$ and $A_t = \{a_t, c\}$. Define $\varepsilon \in P_\mu(X)$, with $\mathrm{dom}(\varepsilon) = \mathrm{dom}(\alpha)$, by $k\varepsilon = a_i$ if $k \in A_i$ and $i \neq s$, $a_s\varepsilon = a_s$, and $b\varepsilon = b$. Define $\gamma \in P_\mu(X)$, with $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha)$, by $k\gamma = x_i$ if $k \in A_i$ and $i \neq t$, $a_t\gamma = x_t$, and $c\gamma = y$.

Suppose $|A_s| = 2$, for some s, say $A_s = \{a_s, b\}$, and $|A_i| = 1$ for every $i \neq s$. Since $r < v-1$, there is $c \in K$ such that $c \notin A_i$ for every i. Define ε exactly as in the previous case. Define $\gamma \in P_\mu(X)$, with $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha) \cup c\mu$, by $k\gamma = x_i$ for all $k \in A_i$, and $c\gamma = y$.

Finally, suppose $|A_i| = 1$ for every i. Since $r < v-1$, there are distinct $b, c \in K$ such that $b, c \notin A_i$ for every i. Define $\varepsilon \in P_\mu(X)$, with $\mathrm{dom}(\varepsilon) = \mathrm{dom}(\alpha) \cup b\mu$, by $k\varepsilon = a_i$ for every $k \in A_i$, and $b\varepsilon = b$. Define γ exactly as in the previous case.

In all cases, $\mathrm{im}(\varepsilon) = \{a_i\}_{1 \leq i \leq r} \cup \{b\}$ (so $\mathrm{rank}(\varepsilon) = r+1$), $\mathrm{im}(\gamma) = \{x_i\}_{1 \leq i \leq r} \cup \{y\}$ (so $\mathrm{rank}(\gamma) = r+1$), and $\alpha = \varepsilon\gamma$. □

Let $\{Q_1, Q_2, \dots, Q_v\}$ denote the set of μ-classes.

Theorem 4.7. *Let $v \geq 3$ and let I_r be the ideal of $P_\mu(X)$ consisting of all transformations of rank $\leq r$, where $1 \leq r < v$. Then*

$$\mathrm{rank}\,(G \cup I_r) = \max\left\{ \sum_{1 \leq i_1 < \dots < i_r \leq v} |Q_{i_1}| \cdots |Q_{i_r}| - \binom{v}{r},\ \sum_{m=0}^{v-r} p_r(v-m) \right\} + 2.$$

Proof. Let $s_1 = \sum_{1 \leq i_1 < \dots < i_r \leq v} |Q_{i_1}| \cdots |Q_{i_r}| - \binom{v}{r}$ and $s_2 = \sum_{m=0}^{v-r} p_r(v-m)$. Then s_1 is the number of partial transversals L of μ such that $|L| = r$ and $L \nsubseteq K$, and s_2 is the number of μ-types $(n_1, n_2, \dots, n_r; m)$ with $0 \leq m \leq v-r$.

Suppose $s_1 \geq s_2$. Construct a set U of transformations of rank r as follows. For every μ-type $\tau = (n_1, n_2, \dots, n_r; m)$, select α with $\mathrm{type}_\mu(\alpha) = \tau$ and add it to U. Since $s_1 \geq s_2$, we can make these selections in such a way that every $\alpha \in U$ has image $L \nsubseteq K$ and any two distinct $\alpha_1, \alpha_2 \in U$ have distinct images. At this point, there are $s_1 - s_2$ partial transversals L of μ such that $|L| = r$, $L \nsubseteq K$, and L is not the image of any $\alpha \in U$. For each such an L, select an idempotent (or any element) $\varepsilon \in I_r$ with $\mathrm{im}(\varepsilon) = L$ and add it to U. Then U consists of s_1 elements of rank r.

Suppose $s_1 < s_2$. Construct a set U of transformations of rank r as follows. For every μ-type $\tau = (n_1, n_2, \dots, n_r; m)$, select α with $\mathrm{type}_\mu(\alpha) = \tau$ and add it to U. Since $s_1 < s_2$, we can make these selections in such a way that for every partial transversal L of μ with $|L| = r$ and $L \nsubseteq K$, there is $\alpha \in U$ such that $\mathrm{im}(\alpha) = L$. Then U consists of s_2 elements of rank r.

In either case, the μ-types of elements of U cover all μ-types $(n_1, n_2, \dots, n_r; m)$ and the images of elements of U cover all partial transversals L of μ such that $|L| = r$ and $L \nsubseteq K$. Thus, by Proposition 4.5, $\langle G, U \rangle$ contains all elements of I_r of rank r. Further, by Lemma 4.6, it also contains all elements of I_r of rank $< r$. Hence $G \cup U$ generates $G \cup I_r$. Moreover, by Proposition 4.5, U is a set of the smallest cardinality such that $G \cup U$ generates $G \cup I_r$.

The result now follows since $|U| = \max\{s_1, s_2\}$ and $G \cong \mathrm{Sym}(K)$, so it is generated by 2 elements. □

If $v \geq 1$ and $r = 0$, then $I_0 = \{0\}$, so $G \cup I_0$ has rank 3 if $v \geq 3$ (since then G has rank 2), and it has rank 2 if $v \in \{1, 2\}$ (since then G has rank 1). If $v = 2$ and $r = 1$, then $G \cup I_1$ has rank $\max\{s_1, s_2\} + 1$ (since then G has rank 1).

The following result is a special case of [4, Theorem 4.4].

Lemma 4.8. *Let J_r be the $\mathcal{J}$-class in $P_\mu(X)$ consisting of all transformations of rank r, where $0 \leq r \leq v$. Then:*

(1) *J_r has $S(v+1, r+1)$ $\mathcal{R}$-classes and $\sum_{1 \leq i_1 < \ldots < i_r \leq v} |Q_{i_1}| \cdots |Q_{i_r}|$ $\mathcal{L}$-classes;*

(2) *J_r has $r!S(v+1, r+1) \sum_{1 \leq i_1 < \ldots < i_r \leq v} |Q_{i_1}| \cdots |Q_{i_r}|$ elements.*

Recall that J_v is the top $\mathcal{J}$-class of $P_\mu(X)$. A semigroup S is called a *right group* if $S \cong G \times E$, where G is a group and E is a right zero semigroup [7, Exercise 6].

Proposition 4.9. *Let $v \geq 1$. Then:*

(1) *J_v is a right group;*

(2) *if $\mu \neq \mathrm{id}_X$, then* $\mathrm{rank}\ J_v = |Q_1| \cdots |Q_v|$.

Proof. Recall that J_v is the union of groups G_M, where M ranges over all transversals of μ. Fix one of these groups, say $G = G_K$, and let E be the set of idempotents in J_v. Note each element of E is the identity of some group G_M, and that for all $\varepsilon \in E$ and $\beta \in J_v$, $\varepsilon\beta = \beta$. Thus E is a right zero semigroup. Define $f : G \times E \to J_v$ by $(\alpha, \varepsilon)f = \alpha\varepsilon$. The function f is a homomorphism, since for all $(\alpha, \varepsilon), (\beta, \xi) \in G \times E$,

$$((\alpha, \varepsilon)(\beta, \xi))f = (\alpha\beta, \varepsilon\xi)f = (\alpha\beta, \xi)f = \alpha\beta\xi = \alpha(\varepsilon\beta)\xi = ((\alpha, \varepsilon)f)((\beta, \xi)f).$$

Let $(\alpha, \varepsilon), (\beta, \xi) \in G \times E$ with $\alpha\varepsilon = \beta\xi$. Then $\mathrm{im}(\varepsilon) = \mathrm{im}(\xi)$, which implies $\varepsilon = \xi$ since an idempotent in J_v is completely determined by its image. Let $x \in X$. Then $(x\alpha)\varepsilon = (x\beta)\xi = (x\beta)\varepsilon$, and so $x\alpha$ and $x\beta$ are in the same μ-class (since $\ker(\varepsilon) = \mu$). Thus, since $\mathrm{im}(\alpha) = \mathrm{im}(\beta) = K$ and K is a transversal of μ, it follows that $x\alpha = x\beta$. Hence $(\alpha, \varepsilon) = (\beta, \xi)$, so f is injective. Thus, it is also surjective since $G \times E$ and J_v are finite semigroups of the same size. (Indeed, $|G| = r!$, $|E| = |Q_1| \cdots |Q_v|$, and $|J_v| = r!|Q_1| \cdots |Q_v|$ by Lemma 4.8.) Hence f is an isomorphism, which proves (1).

If $\mu \neq \mathrm{id}_X$, then J_v satisfies the hypotheses of Lemma 4.2. By Lemma 4.8, J_v has one $\mathcal{R}$-class and $|Q_1| \cdots |Q_v|$ $\mathcal{L}$-classes, so (2) follows. □

If $\mu = \mathrm{id}_X$, then $J_v = \mathrm{Sym}(X)$, and so $\mathrm{rank}\ J_v = 2$ if $|X| \geq 3$, and $\mathrm{rank}\ J_v = 1$ if $|X| \leq 2$.

Theorem 4.10. *Let $v \geq 2$ and let I_r be the ideal of $P_\mu(X)$ consisting of all elements of rank $\leq r$, where $1 \leq r < v$. Then*

$$\mathrm{rank}\,(J_v \cup I_r) = \sum_{m=0}^{v-r} p_r(v - m) + \mathrm{rank}\ J_v,$$

where $\mathrm{rank}\ J_v = |Q_1| \cdot |Q_2| \cdots |Q_v|$ *if $\mu \neq \mathrm{id}_X$,* $\mathrm{rank}\ J_v = 2$ *if $\mu = \mathrm{id}_X$ and $|X| \geq 3$, and* $\mathrm{rank}\ J_v = 1$ *if $|X| = 2$.*

Proof. Let A be any set of generators of $J_v \cup I_r$. Let $(n_1, n_2, \ldots, n_r; m)$ be a μ-type. Since A generates $J_v \cup I_r$, there is $\eta \in \langle A \rangle$ such that $\mathrm{type}_\mu(\eta) = (n_1, n_2, \ldots, n_r; m)$. Since $\mathrm{rank}(\eta) = r$, we have $\eta = \sigma\alpha\gamma$ or $\eta = \alpha\gamma$, where $\sigma \in J_v$, $\alpha \in A$ with $\mathrm{rank}(\alpha) = r$, and $\gamma \in J_v \cup J_r$. Since $\varepsilon\alpha = \alpha$ for any idempotent $\varepsilon \in J_v$, we may assume that $\eta = \sigma\alpha\gamma$. Thus, by Lemma 4.4, $\mathrm{type}_\mu(\alpha) =$

$\text{type}_\mu(\sigma\alpha) = \text{type}_\mu(\eta) = (n_1, n_2, \ldots, n_r; m)$. Hence for every μ-type $\tau = (n_1, n_2, \ldots, n_r; m)$, A contains an element α with $\text{type}_\mu(\alpha) = \tau$. Since A must also contain a generating set of J_v and $\sum_{m=0}^{v-r} p_r(v-m)$ is the number of μ-types $(n_1, n_2, \ldots, n_r; m)$, we have $|A| \geq \sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$, and so $\text{rank}\,(J_v \cup I_r) \geq \sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$.

We will now construct a set A of generators of $J_v \cup I_r$ with exactly $\sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$ elements. Begin with A being a set of generators of J_v of the smallest cardinality. Then for every μ-type $\tau = (n_1, n_2, \ldots, n_r; m)$, select α with $\text{type}_\mu(\alpha) = \tau$ and add it to A. Let $L = \{m_1, m_2, \ldots, m_r\}$ be any partial transversal of μ of size r. Then $L \subseteq M$ for some transversal M of μ. Select any $\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix} \in A$ of rank r. Define $\sigma : \{x_i\} \to M$ by $x_i\sigma = m_i$ and extend σ to $\overline{\sigma} \in G_M$ (possible by Lemma 2.2 applied to G_M). Then

$$(\overline{\sigma})^{-1}\alpha\overline{\sigma} = \begin{pmatrix} A_i\overline{\sigma} \\ x_i\overline{\sigma} \end{pmatrix} = \begin{pmatrix} A_i\overline{\sigma} \\ m_i \end{pmatrix}.$$

Thus $(\overline{\sigma})^{-1}\alpha\overline{\sigma} \in \langle A \rangle$ and $\text{im}((\overline{\sigma})^{-1}\alpha\overline{\sigma}) = L$. Hence the μ-types of elements of $\langle A \rangle$ cover all μ-types $(n_1, n_2, \ldots, n_r; m)$ and the images of elements of $\langle A \rangle$ cover all partial transversals L of μ such that $|L| = r$. Thus, by Proposition 4.5, $\langle G \cup U \rangle$, where $U = \{\alpha \in \langle A \rangle : \text{rank}(\alpha) = r\}$, contains all elements of I_r of rank r. Further, by Lemma 4.6, it also contains all elements of I_r of rank $< r$. Hence $\langle A \rangle = J_v \cup I_r$ since $J_v \subseteq \langle A \rangle$ and $G \cup U \subseteq \langle A \rangle$. The cardinality of A is $\sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$ by the construction, so it follows that $\text{rank}\,(J_v \cup I_r) \leq \sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$.

Hence $\text{rank}\,(J_v \cup I_r) = \sum_{m=0}^{v-r} p_r(v-m) + \text{rank } J_v$. Finally, the statements about the rank of J_v are true by Proposition 4.9. □

If $r = 0$, then $I_0 = \{0\}$, so $\text{rank}\,(J_v \cup I_0) = \text{rank } J_v + 1$.

Since $P_\mu(X) = J_v \cup I_{v-1}$ and $\sum_{m=0}^{v-(v-1)} p_{v-1}(v-m) = p_{v-1}(v) + p_{v-1}(v-1) = 2$, we have $\text{rank } P_\mu(X) = \text{rank } J_v + 2$ if $v \geq 2$. If $v = 1$, then $\text{rank } P_\mu(X) = \text{rank } J_v + 1 = |Q_1| + 1 = n + 2$, where $n = |X|$. These facts can also be deduced from the more general [4, Theorem 4.5].

The result for each proper ideal I_r of $P_\mu(X)$ follows from Lemma 4.2.

Corollary 4.11. *Let $v \geq 2$ and let I_r be the ideal of $P_\mu(X)$ consisting of all transformations of rank $\leq r$, where $1 \leq r < v$. Then*

$$\text{rank}\, I_r = \max\left\{ \sum_{1 \leq i_1 < \ldots < i_r \leq v} |Q_{i_1}| \cdots |Q_{i_r}|,\ S(v+1, r+1) \right\}.$$

Proof. The top $\mathcal{J}$-class J_r of I_r satisfies the hypotheses of Lemma 4.2 by Theorem 2.6, Lemma 4.6, and the fact that $v \geq 2$ and $1 \leq r < v$. Thus the result follows by Lemma 4.8. □

If $v \geq 1$ and $r = 0$, Then $I_r = \{0\}$ has rank 1.

The results and proofs of this section carry over to the semigroup $T_\mu(X) = P_\mu(X) \cap T(X)$, where $T(X)$ is the semigroup of full transformations on X. The only differences are that $T_\mu(X)$ has no ideal I_0 and each element of $T_\mu(X)$ has deficit 0, so the sum $\sum_{m=0}^{v-r} p_r(v-m)$ reduces to $p_r(v)$. Note that the $\mathcal{J}$-classes J_v in $P_\mu(X)$ and $T_\mu(X)$ are the same.

Theorem 4.12. *Let $v \geq 3$ and let I_r be the ideal of $T_\mu(X)$ consisting of all elements of rank $\leq r$, where $1 \leq r < v$. Then*

$$\operatorname{rank}(G \cup I_r) = \max\left\{ \sum_{1\leq i_1<\ldots<i_r\leq v} |Q_{i_1}|\cdots|Q_{i_r}| - \binom{v}{r},\, p_r(v) \right\} + 2,$$
$$\operatorname{rank}(J_v \cup I_r) = p_r(v) + \operatorname{rank}\, J_v,$$
$$\operatorname{rank} I_r = \max\left\{ \sum_{1\leq i_1<\ldots<i_r\leq v} |Q_{i_1}|\cdots|Q_{i_r}|,\, S(v,r) \right\}.$$

Consequently, $\operatorname{rank} T_\mu(X) = \operatorname{rank}\, J_v + 1$

The result for $\operatorname{rank} T_\mu(X)$ can also be deduced from the more general [5, Theorem 5.18].

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