The characteristic polynomial of some anti-tridiagonal 2-Hankel matrices of even order

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Abstract

In this paper we derive the characteristic polynomial for a family of anti-tridiagonal 2-Hankel matrices of even order in terms of Chebyshev polynomials giving also a representation of its eigenvectors. An orthogonal diagonalization for these type of matrices having null northeast-to-southwest diagonal is also provided using prescribed eigenvalues.

Keywords: Anti-tridiagonal 2-Hankel matrix, eigenvalue, eigenvector, Chebyshev polynomials

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1 Introduction

The concept of an r-Toeplitz matrix was introduced by Gover and Barnett in the eighties (see Gover & Barnett, 1985) which established also many of its properties (see Gover & Barnett, 1985; Gover, 1989). They defined an r-Toeplitz matrix as an $n\times n$ matrix $\mathbf{A}_n$ such that $[\mathbf{A}_n]_{k+r,\ell+r} = [\mathbf{A}_n]_{k,\ell}$ for all $k,\ell = 1,2,\ldots,n-r$. Following this idea, we say that an $n \times n$ matrix $\mathbf{B}_n$ is an *r-Hankel matrix* if $[\mathbf{B}_n]_{k+r,\ell-r} = [\mathbf{B}_n]_{k,\ell}$ for every $k = 1,2,\ldots,n-r$ and $\ell = r+1,\ldots,n$. Note that, when $r = 1$, the matrix $\mathbf{B}_n$ becomes a Hankel matrix.

Let us point out that Hankel matrices appear not only in engineering problems of system and control theory (see Olshevsky & Stewart, 2001 and the references therein) but also in computational mathematics (see Bultheel & Van Barel, 1997, among others).

In this note, we shall consider a particular type of anti-tridiagonal 2-Hankel matrices of even order, concretely, real $2n \times 2n$ matrices of the form

$$\mathbf{H}_{2n} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 & b_1 & c \\ \vdots & & & \iddots & a_2 & d & a_1 \\ \vdots & & \iddots & \iddots & c & b_2 & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a_2 & d & \iddots & \iddots & & \vdots \\ b_1 & c & b_2 & \iddots & & & \vdots \\ d & a_1 & 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix} \tag{1.1}$$

with $cd = 0$. It is our goal to obtain an explicit expression for the characteristic polynomial of $\mathbf{H}_{2n}$ as well as a representation of its eigenvectors for eigenvalues given *a priori*. As a consequence, sufficient conditions are announced to get an orthogonal diagonalization of anti-tridiagonal 2-Hankel matrices of even order having null northeast-to-southwest diagonal. We emphasize that, in general, $\mathbf{H}_{2n}$ is not a persymmetric matrix which makes some recent approaches concerning this issue unfeasible (see Akbulak, da Fonseca & Yilmaz, 2013 or Wu, 2010). Therefore, our results emerge as a complement for these and other papers about spectral properties of anti-tridiagonal matrices.

2 Main results

For any integer $p \geqslant -1$, we shall denote by $U_p(x)$ the pth degree Chebyshev polynomial of second kind

$$U_p(x) = \frac{\sin[(p+1)\arccos x]}{\sin(\arccos x)}, \quad -1 < x < 1,$$

with $U_p(\pm 1) = (\pm 1)^p (p+1)$ (see Mason & Handscomb, 2003). This expression as a sum of powers of x can, of course, be evaluated for any x. The symbols $\lfloor x \rfloor$ and $\otimes$ will be used to indicate the largest integer not greater than x and the Kronecker product, respectively. The Euclidean norm will be denoted by $\| \cdot \|$.

Let ξ, b_1, b_2 be real numbers such that $b_1 b_2 \neq 0$. Throughout, we shall consider the sequence of polynomials $\{Q_k(x,\xi)\}_{k\geqslant 0}$ defined by

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$$
Q_k(x,\xi) := \begin{cases} x(b_1b_2)^{\frac{k-1}{2}} U_{\frac{k-1}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right), & k \text{ odd} \\[2ex] (b_1b_2)^{\frac{k}{2}} U_{\frac{k}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) + \\ \quad \xi^2 (b_1b_2)^{\frac{k}{2}-1} U_{\frac{k}{2}-1}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right), & k \text{ even} \end{cases} \tag{2.1}
$$

as well as the $n \times n$ matrix $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ whose (k,ℓ)-entry is

$$
\begin{cases} -b_{\frac{3-(-1)^k}{2}}^{\left\lfloor\frac{\ell-k}{2}\right\rfloor} b_{\frac{3+(-1)^k}{2}}^{\left\lfloor\frac{\ell-k+1}{2}\right\rfloor} \dfrac{Q_{k-1}(\lambda,b_2) Q_{n-\ell}\left[\lambda, b_{\frac{3+(-1)^n}{2}}\right]}{Q_n(\lambda,b_2)}, & k \leqslant \ell \\[3ex] -b_{\frac{3-(-1)^\ell}{2}}^{\left\lfloor\frac{k-\ell}{2}\right\rfloor} b_{\frac{3+(-1)^\ell}{2}}^{\left\lfloor\frac{k-\ell+1}{2}\right\rfloor} \dfrac{Q_{\ell-1}(\lambda,b_2) Q_{n-k}\left[\lambda, b_{\frac{3+(-1)^n}{2}}\right]}{Q_n(\lambda,b_2)}, & k > \ell \end{cases} \tag{2.2}
$$

and the $n \times n$ matrix $\mathbf{S}_n\left[x, b_{\frac{3+(-1)^n}{2}}, b_2\right]$ given by

$$
\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right] - \frac{b_{\frac{3+(-1)^n}{2}} Q_n(x,b_2)}{Q_n(x,b_2) - b_{\frac{3+(-1)^n}{2}} Q_{n-1}(x,b_2)} \cdot \mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right] \mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]^\top \tag{2.3}
$$

with $\mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ the last column of $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$. Further, we shall suppose the $n \times n$ matrix $\mathbf{T}_n(x,y)$ defined by

$$
\begin{cases} \begin{bmatrix} 0 & x & 0 & \ldots & \ldots & \ldots & 0 \\ x & 0 & y & 0 & & & \vdots \\ 0 & y & 0 & x & \ddots & & \vdots \\ \vdots & 0 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & x \\ 0 & \ldots & \ldots & \ldots & 0 & x & y \end{bmatrix}, & n \text{ even} \\[10ex] \begin{bmatrix} 0 & x & 0 & \ldots & \ldots & \ldots & 0 \\ x & 0 & y & 0 & & & \vdots \\ 0 & y & 0 & x & \ddots & & \vdots \\ \vdots & 0 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & y \\ 0 & \ldots & \ldots & \ldots & 0 & y & x \end{bmatrix}, & n \text{ odd.} \end{cases} \tag{2.4}
$$

Set

$$
\mathbf{J}_n := \left[\delta_{k+\ell,n+1}\right]_{k,\ell},
$$

$$
\mathbf{E}_n := \left[\tfrac{1+(-1)^k}{2}\delta_{k,\ell}\right]_{k,\ell},
$$

$$
\mathbf{K}_n := \left[\tfrac{1-(-1)^k}{2}\delta_{k,\ell}\right]_{k,\ell}
$$

where δ is the Kronecker delta. For $ab \neq 0$, let $\mathbf{u}_n(x,a,b)$ be the n-dimensional vector whose the kth component is

$$
\begin{cases} U_{\frac{k-1}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) + \frac{b}{a} U_{\frac{k-3}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ odd} \\[2ex] \frac{x}{a} U_{\frac{k}{2}-1}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ even} \end{cases} \tag{2.5}
$$

In what follows, we shall assume the anti-tridiagonal 2-Hankel matrix $\mathbf{H}_{2n}$ defined in (1.1) with $d = 0$. Notwithstanding, similar results hold for any real number d and $c = 0$, *mutatis mutandis*.

Theorem 1 *Let n be a positive integer, c a real number, $\{Q_k(x,\xi)\}_{k\geqslant 0}$ the sequence of polynomials* (2.1) *and $\mathbf{T}_n(a_1,a_2)$, $\mathbf{T}_n(b_1,b_2)$ the matrices defined by* (2.4) *for nonzero reals a_1, a_2, b_1, b_2.*

(a) *If n is even, then the eigenvalues of $\mathbf{H}_{2n}$ in* (1.1) *are precisely the zeros of*

$$
f(x) = (a_1a_2b_1b_2)^{\frac{n}{2}} \cdot \left[U_{\frac{n}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right) + \frac{a_2-x}{a_1} U_{\frac{n}{2}-1}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right)\right] \cdot \left[U_{\frac{n}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) + \frac{b_2-x}{b_1} U_{\frac{n}{2}-1}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right)\right] \tag{2.6}
$$

Moreover, if λ is an eigenvalue of $\mathbf{T}_n(a_1,a_2)$, μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$, $Q_n(\lambda,b_2) \neq b_2 Q_{n-1}(\lambda,b_2)$ and $\det\left[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n\right] \neq 0$, *then*

$$
\mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{u}_n(\lambda,a_1,a_2) \\ -c\,\mathbf{S}_n(\lambda,b_2,b_2)\mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix} \tag{2.7}
$$

and

$$
\mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_n(\mu,b_1,b_2) \end{bmatrix} \tag{2.8}
$$

are eigenvectors of $\mathbf{H}_{2n}$ associated to λ and μ, respectively, where $\mathbf{P}_{2n}$ is the $2n \times 2n$ permutation matrix

$$
\mathbf{P}_{2n} := \left[\begin{array}{c|c} \mathbf{E}_n & \mathbf{J}_n\mathbf{E}_n \\ \hline \mathbf{K}_n & \mathbf{J}_n\mathbf{K}_n \end{array}\right] \tag{2.9}
$$

$\mathbf{u}_n(\lambda,a_1,a_2)$, $\mathbf{u}_n(\mu,b_1,b_2)$ are the n-dimensional vectors defined by (2.5) *and $\mathbf{S}_n(\lambda,b_2,b_2)$ is the $n \times n$ matrix given in* (2.3).

(b) *If n is odd, then the eigenvalues of $\mathbf{H}_{2n}$ in* (1.1) *are precisely the zeros of*

$$
f(x) = (a_1a_2b_1b_2)^{\frac{n-1}{2}} \cdot \left[(x-a_1)U_{\frac{n-1}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right) - a_2 U_{\frac{n-3}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right)\right] \cdot \left[(x-b_1)U_{\frac{n-1}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) - b_2 U_{\frac{n-3}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right)\right] \tag{2.10}
$$

Furthermore, if λ is an eigenvalue of $\mathbf{T}_n(a_1,a_2)$, μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$, $Q_n(\lambda,b_2) \neq b_1 Q_{n-1}(\lambda,b_2)$ and $\det\left[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n\right] \neq 0$, *then*

$$
\mathbf{P}_{2n}^\top \begin{bmatrix} -c\,\mathbf{S}_n(\lambda,b_1,b_2)\mathbf{u}_n(\lambda,a_1,a_2) \\ \mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix} \tag{2.11}
$$

and

$$\mathbf{P}_{2n}^{\top}\left[\begin{array}{c}\mathbf{u}_n(\mu,b_1,b_2)\\ \mathbf{0}\end{array}\right] \tag{2.12}$$

are eigenvectors of $\mathbf{H}_{2n}$ *associated to* λ *and* μ*, respectively, where* $\mathbf{P}_{2n}$ *is the* $2n\times 2n$ *permutation matrix*

$$\mathbf{P}_{2n} := \left[\begin{array}{c|c}\mathbf{K}_n & \mathbf{E}_n\mathbf{J}_n\\ \hline \mathbf{E}_n & \mathbf{K}_n\mathbf{J}_n\end{array}\right] \tag{2.13}$$

$\mathbf{u}_n(\lambda,a_1,a_2)$*,* $\mathbf{u}_n(\mu,b_1,b_2)$ *are the* n*-dimensional vectors defined by* (2.5) *and* $\mathbf{S}_n(\lambda,b_1,b_2)$ *is the* $n\times n$ *matrix given in* (2.3).

Remark It is worthy to note that by taking $c=0$ and $a_2=b_1$, $a_1=b_2$ in (2.6) or (2.10), we recover the expressions obtained in section 4 of da Fonseca, 2018 for the matrices of even order analysed therein.

The previous result leads us to an orthogonal diagonalization for anti-tridiagonal 2-Hankel matrices (1.1) with null northeast-to-southwest diagonal, i.e. for matrices of the form

$$\mathbf{H}_{2n}^{*} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & b_1 & 0\\ \vdots & & & \iddots & a_2 & 0 & a_1\\ \vdots & & \iddots & \iddots & 0 & b_2 & 0\\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots\\ 0 & a_2 & 0 & \iddots & \iddots & & \vdots\\ b_1 & 0 & b_2 & \iddots & & & \vdots\\ 0 & a_1 & 0 & \cdots & \cdots & \cdots & 0\end{bmatrix} \tag{2.14}$$

Put

$$\begin{aligned}\mathbf{V}_n &:= \left[\begin{array}{ccc}\frac{\mathbf{u}_n(\lambda_1,a_1,a_2)}{\|\mathbf{u}_n(\lambda_1,a_1,a_2)\|} & \cdots & \frac{\mathbf{u}_n(\lambda_n,a_1,a_2)}{\|\mathbf{u}_n(\lambda_n,a_1,a_2)\|}\end{array}\right]\\ \mathbf{W}_n &:= \left[\begin{array}{ccc}\frac{\mathbf{u}_n(\mu_1,b_1,b_2)}{\|\mathbf{u}_n(\mu_1,b_1,b_2)\|} & \cdots & \frac{\mathbf{u}_n(\mu_n,b_1,b_2)}{\|\mathbf{u}_n(\mu_n,b_1,b_2)\|}\end{array}\right]\end{aligned} \tag{2.15}$$

where $\mathbf{u}_n(\lambda_k,a_1,a_2)$ and $\mathbf{u}_n(\mu_k,b_1,b_2)$ are the n-dimensional vectors whose the kth components are defined by (2.5).

Corollary 1 *Let* n *be a positive integer,* a_1,a_2,b_1,b_2 *nonzero real numbers,* $\mathbf{H}_{2n}^{*}$ *the* $2n\times 2n$ *matrix* (2.14)*,* $\mathbf{T}_n(a_1,a_2)$ *and* $\mathbf{T}_n(b_1,b_2)$ *matrices defined by* (2.4) *having eigenvalues* $\lambda_1,\ldots,\lambda_n$ *and* $\mu_1,\ldots,\mu_n$*, respectively. Suppose that* $\det\left[\mathbf{I}_n\otimes\mathbf{T}_n(a_1,a_2)-\mathbf{T}_n(b_1,b_2)\otimes\mathbf{I}_n\right]\neq 0$ *and the sequence of polynomials* $\{Q_k(x,\xi)\}_{k\geqslant 0}$ *given by* (2.1) *satisfies* $Q_n(\lambda_k,b_2)\neq b_{\frac{3+(-1)^n}{2}}Q_{n-1}(\lambda_k,b_2)$ *for each* $k=1,\ldots,n$.

(a) *If* n *is even, then*

$$\mathbf{H}_{2n}^{*} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)\mathbf{U}_{2n}^{\top}, \tag{2.16}$$

where

$$\mathbf{U}_{2n} = \mathbf{P}_{2n}^{\top}\left[\begin{array}{c|c}\mathbf{V}_n & \mathbf{O}\\ \hline \mathbf{O} & \mathbf{W}_n\end{array}\right], \tag{2.17}$$

$\mathbf{P}_{2n}$ *is the permutation matrix* (2.6) *and* $\mathbf{V}_n,\mathbf{W}_n$ *are the* $n\times n$ *matrices in* (2.15).

(b) *If* n *is odd, then*

$$\mathbf{H}_{2n}^{*} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)\mathbf{U}_{2n}^{\top} \tag{2.18}$$

where

$$\mathbf{U}_{2n} = \mathbf{P}_{2n}^{\top}\left[\begin{array}{c|c}\mathbf{O} & \mathbf{W}_n\\ \hline \mathbf{V}_n & \mathbf{O}\end{array}\right],$$

$\mathbf{P}_{2n}$ *is the permutation matrix* (2.13) *and* $\mathbf{V}_n,\mathbf{W}_n$ *are the* $n\times n$ *matrices in* (2.15).

Remark More generally, Theorem 1 also leads to an eigendecomposition for $\mathbf{H}_{2n}$ in (1.1) with $d=0$, taking eigenvector matrices formed by the column vectors (2.7), (2.8) or (2.11), (2.12) according as n is even or odd, respectively.

3 Lemmata and proofs

In order to prove Theorem 1, we will need some auxiliary results. The first one is well-known in the literature (see Akbulak, da Fonseca & Yilmaz, 2013) and locates the eigenvalues of tridiagonal matrices having the form (2.4). Indeed, the characteristic polynomial of $\mathbf{T}_n(a,b)$ is

$$(ab)^{\frac{n}{2}}\left[U_{\frac{n}{2}}\left(\tfrac{x^2-a^2-b^2}{2ab}\right)+\tfrac{b-x}{a}U_{\frac{n}{2}-1}\left(\tfrac{x^2-a^2-b^2}{2ab}\right)\right]$$

when n is even and

$$(ab)^{\frac{n-1}{2}}\left[(x-a)U_{\frac{n-1}{2}}\left(\tfrac{x^2-a^2-b^2}{2ab}\right)-bU_{\frac{n-3}{2}}\left(\tfrac{x^2-a^2-b^2}{2ab}\right)\right]$$

whenever n is odd. Next, we shall provide a representation of its eigenvectors.

Lemma 1 *Let* n *be a positive integer and* $\mathbf{T}_n(a,b)$ *the* $n\times n$ *matrix* (2.4) *with* a,b *nonzero reals. If* λ *is an eigenvalue of* $\mathbf{T}_n(a,b)$*, then* $\mathbf{u}_n(\lambda,a,b)$ *given in* (2.5) *is an eigenvector of* $\mathbf{T}_n(a,b)$ *associated to* λ.

Proof. Suppose a positive integer n and reals a,b such that $a\neq 0$, $b\neq 0$. Consider the three-term recurrence relation,

$$\begin{cases}P_{-1}(x)\equiv 0,\\ P_0(x)\equiv 1,\\ P_k(x)=\frac{x-\beta_k}{\alpha_k}P_{k-1}(x)-\frac{\gamma_{k-1}}{\alpha_k}P_{k-2}(x), \quad 1\leqslant k\leqslant n\end{cases}$$

with $\gamma_0=\alpha_n=1$,

$$\alpha_k=\gamma_k=\begin{cases}a, & k \text{ odd}\\ b, & k \text{ even}\end{cases}$$

and

$$\beta_k=\begin{cases}0, & k<n\\ b, & k=n \text{ and } n \text{ even}\\ a, & k=n \text{ and } n \text{ odd.}\end{cases}$$

Hence, $P_k(x)$ is expressed by

$$\begin{cases} U_{\frac{k}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right)+\frac{b}{a}U_{\frac{k}{2}-1}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ even} \\ \frac{x}{a}U_{\frac{k-1}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ odd} \end{cases}$$

for each $0 \leqslant k \leqslant n-1$ and $[P_0(\lambda), P_1(\lambda), \ldots, P_{n-1}(\lambda)]^\top$ is an eigenvector of $\mathbf{T}_n(a,b)$ associated to the eigenvalue λ (see da Fonseca, 2005). The thesis is established. □

The following auxiliary statement is an explicit formula for the inverse of a sort of perturbed tridiagonal 2-Toeplitz matrices.

Lemma 2 *Let n be a positive integer, λ a real number, $\{Q_k(x,\xi)\}_{k\geqslant 0}$ the sequence of polynomials defined by (2.1) and $\mathbf{T}_n(b_1,b_2)$ the $n \times n$ matrix defined by (2.4) with nonzero reals b_1, b_2. If $Q_n(\lambda, b_2) \neq b_{\frac{3+(-1)^n}{2}} Q_{n-1}(\lambda, b_2)$, then*

$$[\mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n]^{-1} = \mathbf{S}_n\left[\lambda, b_{\frac{3+(-1)^n}{2}}, b_2\right] \quad (3.1)$$

where $\mathbf{S}_n\left[\lambda, b_{\frac{3+(-1)^n}{2}}, b_2\right]$ is the $n \times n$ matrix given by (2.3).

Proof. Suppose a positive integer n and real numbers λ, b_1, b_2 such that $b_1 \neq 0$, $b_2 \neq 0$. Employing the Second Principle of Mathematical Induction on the variable n we can state that $\det[\mathbf{T}_n(b_1,b_2)] = (-1)^{\lfloor \frac{n}{2} \rfloor} b_1^n$ which ensures the nonsingularity of $\mathbf{T}_n(b_1,b_2)$. Denoting by $\mathbf{e}_n$ the n-dimensional vector $(0,\ldots,0,1)$, the inverse of $\mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n - b_{\frac{3+(-1)^n}{2}}\mathbf{e}_n$ is the matrix $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ in (2.2) (see Theorem 4.1 of da Fonseca & Petronilho, 2001) and the thesis is a direct consequence of the well-known Sherman-Morrison-Woodbury formula. □

Proof of Theorem 1. Since both assertions can be proven in the same way, we only prove (a). Let n be an even positive integer. It is straightforward to see that

$$\mathbf{P}_{2n}\mathbf{H}_{2n}\mathbf{P}_{2n}^\top = \left[\begin{array}{c|c} \mathbf{T}_n(a_1,a_2) & \mathbf{O} \\ \hline c\mathbf{I}_n & \mathbf{T}_n(b_1,b_2) \end{array}\right] \quad (3.2)$$

where $\mathbf{P}_{2n}$ is the permutation matrix (2.9). Thus,

$$\det(t\mathbf{I}_{2n} - \mathbf{H}_{2n}) = \det[t\mathbf{I}_n - \mathbf{T}_n(a_1,a_2)]\det[t\mathbf{I}_n - \mathbf{T}_n(b_1,b_2)]$$

and from Lemma 1 we obtain (2.6). Let λ be an eigenvalue of $\mathbf{T}_n(a_1,a_2)$. According to (3.3) we can rewrite the relation $(\mathbf{H}_{2n} - \lambda\mathbf{I}_{2n})\mathbf{x} = \mathbf{0}$ as

$$\left[\begin{array}{c|c} \mathbf{T}_n(a_1,a_2)-\lambda\mathbf{I}_n & \mathbf{O} \\ \hline c\mathbf{I}_n & \mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n \end{array}\right]\mathbf{P}_{2n}\mathbf{x} = \mathbf{0},$$

that is,

$$\begin{aligned} &[\mathbf{T}_n(a_1,a_2)-\lambda\mathbf{I}_n]\,\mathbf{y}^{(1)} = \mathbf{0}, \\ &c\mathbf{y}^{(1)} + [\mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n]\,\mathbf{y}^{(2)} = \mathbf{0}, \\ &\begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \mathbf{P}_{2n}\mathbf{x}. \end{aligned} \quad (3.3)$$

Since $\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$, the matrices $\mathbf{T}_n(a_1,a_2)$ and $\mathbf{T}_n(b_1,b_2)$ have no eigenvalues in common (see Laub, 2005, page 145) which implies $\det[\mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n] \neq 0$ and Lemma 1 ensures that the solution of (3.4) is

$$\mathbf{x} = \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{u}_n(\lambda,a_1,a_2) \\ -c\,[\mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n]^{-1}\,\mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix}$$

where $\mathbf{u}_n(\lambda,a_1,a_2)$ is given by (2.5). From Lemma 2,

$$[\mathbf{T}_n(b_1,b_2)-\lambda\mathbf{I}_n]^{-1} = \mathbf{S}_n(\lambda,b_2,b_2)$$

and (2.7) is an eigenvector of $\mathbf{H}_{2n}$ associated to the eigenvalue λ. On the other hand, suppose that μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$. Since $\mathbf{H}_{2n}\mathbf{x} = \mu\mathbf{x}$ is equivalent to

$$\begin{aligned} &[\mathbf{T}_n(a_1,a_2)-\mu\mathbf{I}_n]\,\mathbf{y}^{(1)} = \mathbf{0}, \\ &c\mathbf{y}^{(1)} + [\mathbf{T}_n(b_1,b_2)-\mu\mathbf{I}_n]\,\mathbf{y}^{(2)} = \mathbf{0}, \\ &\begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \mathbf{P}_{2n}\mathbf{x}, \end{aligned}$$

and $\det[\mathbf{T}_n(a_1,a_2)-\mu\mathbf{I}_n] \neq 0$, we obtain

$$\mathbf{x} = \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_n(\mu,b_1,b_2) \end{bmatrix},$$

where $\mathbf{u}_n(\mu,b_1,b_2)$ is defined in (2.5). Therefore, (2.8) is an eigenvector of $\mathbf{H}_{2n}$ associated to the eigenvalue μ. □

Proof of Corollary 1. Consider an even positive integer n. From Lemma 1 and

$$\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$$

we can guarantee that all eigenvalues of $\mathbf{H}_{2n}^*$ are distinct. Setting

$$\mathbf{v}_n(\lambda_k) := \mathbf{u}_n(\lambda_k,a_1,a_2),$$

$$\mathbf{w}_n(\mu_k) := \mathbf{u}_n(\mu_k,b_1,b_2)$$

and

$$\widehat{\mathbf{v}}_n(\lambda_k) := \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{v}_n(\lambda_k) \\ \mathbf{0} \end{bmatrix},$$

$$\widehat{\mathbf{w}}_n(\mu_k) := \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_n(\mu_k) \end{bmatrix}$$

it follows that

$$\left\{\frac{\widehat{\mathbf{v}}_n(\lambda_1)}{\|\widehat{\mathbf{v}}_n(\lambda_1)\|},\ldots,\frac{\widehat{\mathbf{v}}_n(\lambda_n)}{\|\widehat{\mathbf{v}}_n(\lambda_n)\|},\frac{\widehat{\mathbf{w}}_n(\mu_1)}{\|\widehat{\mathbf{w}}_n(\mu_1)\|},\ldots,\frac{\widehat{\mathbf{w}}_n(\mu_n)}{\|\widehat{\mathbf{w}}_n(\mu_n)\|}\right\} \quad (3.4)$$

is a complete set of orthogonal eigenvectors according to Theorem 1. Hence,

$$\mathbf{H}_{2n}^{*} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\mathbf{U}_{2n}^{-1}$$

where

$$\mathbf{U}_{2n} = \left[\begin{array}{cccccc} \frac{\widehat{\mathbf{v}}_n(\lambda_1)}{\|\widehat{\mathbf{v}}_n(\lambda_1)\|} & \cdots & \frac{\widehat{\mathbf{v}}_n(\lambda_n)}{\|\widehat{\mathbf{v}}_n(\lambda_n)\|} & \frac{\widehat{\mathbf{w}}_n(\mu_1)}{\|\widehat{\mathbf{w}}_n(\mu_1)\|} & \cdots & \frac{\widehat{\mathbf{w}}_n(\mu_n)}{\|\widehat{\mathbf{w}}_n(\mu_n)\|} \end{array}\right]$$

$$= \mathbf{P}_{2n}^{\top} \left[\begin{array}{cccccc} \frac{\mathbf{v}_n(\lambda_1)}{\|\mathbf{v}_n(\lambda_1)\|} & \cdots & \frac{\mathbf{v}_n(\lambda_n)}{\|\mathbf{v}_n(\lambda_n)\|} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\mathbf{w}_n(\mu_1)}{\|\mathbf{w}_n(\mu_1)\|} & \cdots & \frac{\mathbf{w}_n(\mu_n)}{\|\mathbf{w}_n(\mu_n)\|} \end{array}\right]$$

provided that $\mathbf{P}_{2n}^{\top}$ is an orthogonal matrix. Since (3.4) is an orthonormal set, $\mathbf{U}_{2n}$ is an orthogonal matrix and (2.16) is established. The proof of (b) is analogous and so will be omitted. □

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