UNIVERSIDADE ABERTA

Conjugation in Abstract Semigroups

Maria de Fátima Lopes Borralho

Doctor's Degree in Computational Algebra

Doctoral dissertation supervised by:
Professor João Jorge Ribeiro Soares Gonçalves de Araújo
Professor Michael K. Kinyon

2019

Resumo

Dado um semigrupo S e um elemento fixo $c \in S$, podemos definir uma nova operação associativa $\cdot_c$ em S por

$$x \cdot_c y = xcy$$

para todo $x, y \in S$, obtendo-se assim um novo semigrupo, o variante de S (em c). Os elementos $a, b \in S$ dizem-se primariamente conjugados ou apenas p-conjugados, se existirem $x, y \in S^1$ tais que $a = xy$ e $b = yx$. Em grupos, a relação $\sim_p$ coincide com a conjugação usual, mas em semigrupos, em geral, não é transitiva. Localizar classes de semigrupos nas quais a conjugação primária é transitiva é um problema em aberto.

Kudryavtseva (2006) provou que a transitividade é válida para semigrupos completamente regulares e, mais recentemente, Araújo et al. (2017) provaram que a transitividade também se aplica aos variantes de semigrupos completamente regulares. Fizeram-no introduzindo uma variedade $\mathcal{W}$ de epigrupos contendo todos os semigrupos completamente regulares e seus variantes, e provaram que a conjugação primária é transitiva em $\mathcal{W}$. Colocaram o seguinte problema: a conjugação primária é transitiva nos variantes dos semigrupos em $\mathcal{W}$? Nesta tese, respondemos a isso afirmativamente como parte de uma abordagem mais geral do estudo de variedades de epigrupos e seus variantes, e mostramos que para semigrupos satisfazendo $xy \in \{yx, (xy)^n\}$ para algum $n > 1$, a conjugação primária também é transitiva.

Numa fase inicial da tese foi feita uma revisão da literatura referente à problemática em estudo com uma breve discussão e declaração dos principais resultados. No Capítulo 2, referimos que seguindo Petrich e Reilly (1999) para semigrupos completamente regulares e Shevrin (1994, 2005) para epigrupos, é habitual ver um epigrupo $(S, \cdot)$ como um semigrupo unário $(S, \cdot, ')$ em que $x \mapsto x'$ é a aplicação que a cada elemento faz corresponder o seu pseudoinverso. Na tese usámos, com frequência, as seguintes igualdades, que

são válidas em todos os epigrupos (Shevrin, 2005):

$$x'xx' = x', \tag{1}$$
$$xx' = x'x, \tag{2}$$
$$x''' = x', \tag{3}$$
$$xx'x = x'', \tag{4}$$
$$(xy)'x = x(yx)', \tag{5}$$
$$(x^p)' = (x')^p\,, \tag{6}$$

para qualquer p primo.

Para cada $n \in \mathbb{N}$, seja $\mathcal{E}_n$ a variedade (classe equacional) de todos os semigrupos unários $(S, \cdot, ')$ que satisfazem (2.2), (2.3) e $x^{n+1}x' = x^n$. Cada $\mathcal{E}_n$ é uma variedade de epigrupos, e as inclusões $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ são válidas para todos os n. Todo o semigrupo finito está contido em algum $\mathcal{E}_n$ e $\mathcal{E}_1$ é a variedade de semigrupos completamente regulares. Demonstrámos um Lema, para usar mais tarde, que nos diz que para cada $n \in \mathbb{N}$, a variedade $\mathcal{E}_n$ é precisamente a variedade de semigrupos unários que satisfazem as identidades referidas.

Variantes de semigrupos completamente regulares não são, em geral, completamente regulares; por exemplo, se um semigrupo completamente regular tiver um zero 0, o variante em 0 será um semigrupo nulo, que nem é regular. Esta dificuldade foi contornada, em Araújo et al. (2017), introduzindo a seguinte classe $\mathcal{W}$ de semigrupos:

$$S \in \mathcal{W} \iff xy \text{ é completamente regular, para todo } x, y \in S\,.$$

Equivalentemente, $\mathcal{W}$ consiste em todos os semigrupos S, de modo que o subsemigrupo $S^2 = \{ab \mid a, b \in S\}$ é completamente regular. A classe $\mathcal{W}$ inclui todos os semigrupos completamente regulares e todos os semigrupos nulos (semigrupos satisfazendo $xy = uv$ para todos os x, y, u, v). Na Proposição 2 resumimos os resultados relevantes, para esta tese, de Araújo et al. (2017): (Proposição 4.14) $\mathcal{W}$ é a variedade de epigrupos em $\mathcal{E}_2$ satisfazendo a igualdade adicional $(xy)'' = xy$; (Teorema 4.15) Se S é um epigrupo em $\mathcal{W}$, então $\sim_p$ é transitiva em S; (Teorema 4.17) Todo o variante de um semigrupo completamente regular está em $\mathcal{W}$ e (Corolário 4.18) Se S é um variante de um semigrupo completamente regular, então $\sim_p$ é transitiva em S. O Teorema 4.15 também comparava $\sim_p$ com outras noções de conjugação. Na forma simplificada aqui escrita, o resultado segue facilmente do teorema de Kudryavtseva (2006): se $a \sim_p b$, $b \sim_p c$, e $a \neq b \neq c \neq a$, então existe $x, y, u, v \in S$ tal que $a = xy$, $b = yx = uv$ e $c = vu$. Assim $a, b, c \in \mathcal{W}$ são completamente regulares, logo $a \sim_p c$.

Melhorámos um pouco a Proposição 4.14 dizendo que a variedade $\mathcal{W}$ é precisamente a variedade de semigrupos unários que satisfazem as identidades (2.2), (2.3), (2.5), (2.7) (para $p = 2$) e $(xy)'' = xy$. A variedade $\mathcal{W}$ tem outra caracterização que não foi mencionada em Araújo et al. (2017). Sendo S um semigrupo, as seguintes afirmações são equivalentes: S é um epigrupo em $\mathcal{W}$, para cada $c \in S$, o ideal principal esquerdo Sc é um subsemigrupo completamente regular e para cada $c \in S$, o ideal principal direito cS é um subsemigrupo completamente regular.

Em vista do Lema 4, também devemos mencionar o estudo semelhante, em Liu, Chen & Han (2016), dos epigrupos S, no qual todo o submonóide local eSe é completamente regular.

A ferramenta chave na demonstração da Proposição 2 (3) foi a seguinte operação unária:

$$x^* = (xc)'x(cx)'\,. \tag{*}$$

De facto, se $(S, \cdot, ')$ é completamente regular, então $(S, \cdot_c, ^*)$ é um epigrupo na variedade $\mathcal{W}$. No entanto, (*) foi introduzida em Araújo et al. (2017) de forma "ad hoc". Para mostrar que é bastante natural, observamos que o ideal de um epigrupo é um subepigrupo (Shevrin1994). Em particular, para cada c num epigrupo S, Sc é um subepigrupo. Assim, para qualquer $x \in S$, o pseudoinverso $(xc)'$ deve ter a forma yc para algum $y \in S$. É exactamente isso que (*) faz. Seja S um epigrupo e fixemos $c \in S$. Para todo $x \in S$,

$$(xc)' = x^*c\,. \tag{7}$$

Se $(S, \cdot, ')$ é um epigrupo vamos referirmo-nos a $(S, \cdot_c, ^*)$ como o variante unário de $(S, \cdot, ')$ em c. A Proposição 2 (3) afirma que, se $(S, \cdot, ') \in \mathcal{E}_1$ é completamente regular, então $(S, \cdot_c, ^*) \in \mathcal{W}$. O nosso primeiro resultado principal irá melhorar e estender isso. Primeiro temos que introduzir uma família de variedades de semigrupos unários. Para cada $n \in \mathbb{N}$, a variedade $\mathcal{V}_n$ é definida pela associatividade e pelas seguintes identidades: (2.2), (2.3),

$$x\underbrace{y\cdots y}_{n-1}y'' = x\underbrace{y\cdots y}_{n} \tag{8}$$

$$x''\underbrace{x\cdots x}_{n-1}y = \underbrace{x\cdots x}_{n}y \tag{9}$$

Definindo $y = x$ em, digamos, (2.9), vemos pelo Lema 1 que $\mathcal{V}_n$ é uma variedade de epigrupos e, em particular,

$$\mathcal{E}_n \subseteq \mathcal{V}_n \subseteq \mathcal{E}_{n+1}\,. \tag{10}$$

É fácil verificar que todo o variante de um epigrupo é um epigrupo, mas o que não é tão óbvio é o que acontece com a operação pseudoinversa. O nosso primeiro resultado principal esclarece isso e também o papel das variedades $\mathcal{V}_n$. Seja $(S, \cdot, ')$ um epigrupo. Para cada $c \in S$, o variante unário $(S, \cdot_c, ^*)$ é um epigrupo. Se $(S, \cdot, ') \in \mathcal{V}_n$ para algum $n > 0$, então $(S, \cdot_c, *) \in \mathcal{V}_n$. Portanto para $n \in \mathbb{N}$, a variedade $\mathcal{V}_n$ é fechada para variantes. Assim, seja $(S, \cdot, ')$ um semigrupo completamente regular. Para cada $c \in S$, o variante unário $(S, \cdot_c, ^*)$ está em $\mathcal{V}_1$.

Observámos que para um elemento c de um semigrupo S, a aplicação $\rho_c : S \to Sc; x \mapsto xc$ é um homomorfismo do variante $(S, \cdot_c)$ em $(Sc, \cdot)$ pois $(x \cdot_c y)c = xc \cdot yc$. Se S também é um epigrupo, como já observámos, o mesmo acontece com Sc. Todo o homomorfismo de semigrupos entre epigrupos é um homomorfismo de epigrupos, mas (2.8) mostra mais explicitamente como ρ_c preserva pseudoinversos.

Comparar a Proposição 2(3) com o Corolário 7 levanta a questão de como as variedades $\mathcal{V}_1$ e $\mathcal{W}$ estão relacionadas, além do facto de que ambas contêm $\mathcal{E}_1$. O nosso segundo resultado principal aborda isso e o seu corolário liga esta discussão à transitividade de $\sim_p$. Assim, $\mathcal{V}_1 \subset \mathcal{W}$ e se $(S, \cdot, ')$ é um epigrupo em $\mathcal{W}$, então, para cada $c \in S$, o variante unário $(S, \cdot_c, ^*)$ está em $\mathcal{V}_1$.

Em particular, todo o variante de um epigrupo em $\mathcal{W}$ na verdade está numa subvariedade adequada de $\mathcal{W}$, uma declaração mais forte do que a afirmação de que $\mathcal{W}$ é fechado para variantes.

Obtivémos assim, uma resposta definitiva ao Problema 6.18 em Araújo et al. (2017). Como $\sim_p$ é transitiva em $\mathcal{W}$, o teorema implica imediatamente que a conjugação primária $\sim_p$ é transitiva em todos os variantes de qualquer epigrupo em $\mathcal{W}$.

O objectivo do terceiro capítulo desta tese foi provar o Teorema 11. Seja $n > 1$ um número inteiro e S um semigrupo satisfazendo o seguinte: para todo $x, y \in S$,

$$xy \in \{yx, (xy)^n\} .$$

Então a conjugação primária $\sim_p$ é transitiva em S.

Existem várias motivações para estudar esta classe específica de semigrupos. Primeiro, ela naturalmente generaliza duas classes de semigrupos em que $\sim_p$ é transitiva: seja S um semigrupo. Se S é comutativo, então $\sim_p$ é transitiva e se S satisfaz $xy = (xy)^2$ para todo $x, y \in S$, então $\sim_p$ é transitiva.

A outra motivação para estudar esta classe de semigrupos é que ela tem sido de interesse recente em outros contextos. Em particular, J. P. Araújo e Kinyon (2015) mostraram que um semigrupo satisfazendo $x^3 = x$ e $xy \in \{yx, (xy)^2\}$ para todo x, y é um semireticulado de bandas rectangulares e grupos de expoente 2.

A demonstração do Teorema 11 foi encontrada demonstrando primeiro os casos especiais $n = 2, 3, 4$.

Como já referimos, no Capítulo 2, com as demonstrações dos Teoremas 6 e 10, resolvemos o Problema 6.18 em Araújo et al. (2017). Estes resultados foram aceites para publicação na revista Communications in Algebra. No Capítulo 3 demonstrámos, com o Teorema 16, que a conjugação primária é transitiva no semigrupo S apresentado. Este resultado foi publicado em maio de 2020, na revista Quasigroups and Related Systems.

Como trabalho futuro, apresentamos na tese seis problemas. Os quatro primeiros, sugeridos pelo Professor Michael Kinyon, são direcionados para o estudo das variedades. Os dois últimos são problemas de Araújo et al. (2017) reformulados pelo Professor João Araújo, no contexto desta tese.

Palavras-chave: semigrupos, conjugação primária, variantes, variedades, epigrupos

Abstract

On a semigroup S with a fixed element c, we can define a new binary operation $x \cdot_c y := xcy$ for all $x, y \in S$. Then $(S, \cdot_c)$ is a semigroup called the *variant* of S at c. Elements $a, b \in S$ are said to be *primarily conjugate* or just *p-conjugate*, if there exist $x, y \in S^1$ such that $a = xy, b = yx$. In groups this coincides with the usual conjugation, but in semigroups, it is not transitive in general. Finding classes of semigroups in which primary conjugacy is transitive is an interesting open problem. Kudryavtseva proved that transitivity holds for completely regular semigroups, and more recently Araújo *et al.* proved that transitivity also holds in the variants of completely regular semigroups. They did this by introducing a variety $\mathcal{W}$ of epigroups containing all completely regular semigroups and their variants, and proved that primary conjugacy is transitive in $\mathcal{W}$. They posed the following problem: is primary conjugacy transitive in the variants of semigroups in $\mathcal{W}$? In this thesis, we answer this affirmatively as part of a more general study of varieties of epigroups and their variants, and we show that for semigroups satisfying $xy \in \{yx, (xy)^n\}$ for some $n > 1$, primary conjugacy is also transitive.

Keywords: semigroups, primary conjugacy, variants, varieties, epigroups

Dedication

To Pedro, Joana and António, for everything.

Acknowledgements

I am very grateful to Professor João Araújo and Professor Michael Kinyon for all the ideas, problems suggested and support throughout my dissertation period.

I am pleased to acknowledge the use of the Prover9 automated theorem prover, developed by McCune [39].

I am also grateful to Professor Michael Kinyon for suggesting the first four problems in Future Work (Section 4.1), and to Professor João Araújo for suggesting the last two.

I am very grateful to Pedro Borralho for all the ideas and support.

I would like to thank my children Pedro, Joana, and António for all their unconditional support, and other friends and family who contributed in some way.

Contents

Chapter 1

Introduction

By a notion of conjugacy for a class of semigroups we mean an equivalence relation defined in the language of that class of semigroups and coinciding with the group theory notion of conjugacy whenever the semigroup is a group.

According to [15], it is unreasonable to consider a generalization of groups to semigroups that is suitable for all purposes. It is therefore necessary to choose the notion of conjugation that best fits the class of semigroups under study, and to understand how it interacts with other notions of conjugation and other mathematical concepts.

Before introducing the notions of conjugacy that will occupy us, we recall some standard definitions and notation (we generally follow [26]). Other needed definitions will be given in context.

For a semigroup S, we denote by $E(S)$ the set of idempotents of S; S^1 is the semigroup S if S is a monoid, or otherwise denotes the monoid obtained from S by adjoining an identity element 1. The relation $\leq$ on $E(S)$ defined by $e \leq f$ if $ef = fe = e$ is a partial order on $E(S)$ [26]. A commutative semigroup of idempotents is said to be a *semilattice*.

An element a of a semigroup S is said to be *regular* if there exists $b \in S$ such that $aba = a$. Setting $c = bab$, we get $aca = a$ and $cac = c$, so c is an *inverse* of a. Since a is also an inverse of c, we often say that a and c are *mutually inverse*. A semigroup S is *regular* if all elements of S are regular, and it is an *inverse semigroup* if every element of S has a *unique* inverse.

If S is a semigroup and $a, b \in S$, we say that $a\mathcal{L}b$ if $S^1a = S^1b$, $a\mathcal{R}b$ if $aS^1 = bS^1$, and $a\mathcal{J}b$ if $S^1aS^1 = S^1bS^1$. We define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, that is, $\mathcal{D}$ is the smallest equivalence relation on $\mathcal{S}$ containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalence relations are known as *Green's relations* [26], and are among the most important tools in studying semigroups.

We now introduce the four notions of conjugacy that we will consider. As noted, we expect any reasonable notion of semigroup conjugacy to coincide in groups with the usual notion. For elements a, b, g of a group G, if $a = g^{-1}bg$, then we say that a and b are *conjugate* and g (or g^{-1}) is a *conjugator* of a and b. Conjugacy in groups has several equivalent formulations that avoid inverses, and hence generalize syntactically to any semigroup. For example, if G is a group, then $a, b \in G$ satisfy $a = g^{-1}bg$ (for some $g \in G$) if and only if $a = uv$ and $b = vu$ for some $u, v \in G$ (namely $u = g^{-1}b$ and $v = g$). This last formulation has been used to define the following relation on a free semigroup S (see [34]):

$$a \sim_p b \Leftrightarrow \exists_{u,v \in S^1} a = uv \text{ and } b = vu. \tag{1.1}$$

If S is a free semigroup, then $\sim_p$ is an equivalence relation on S [34], and so it can be considered as a notion of conjugacy in S. In a general semigroup S, the relation $\sim_p$ is reflexive and symmetric, but not transitive. If $a \sim_p b$ in a semigroup, we say that a and b are *primarily related* [32] (hence the subscript in $\sim_p$). The transitive closure $\sim_p^*$ of $\sim_p$ has been defined as a conjugacy relation in a general semigroup [25, 31, 32]. Lallement credited the idea of the relation $\sim_p$ to Lyndon and Schutzenberger [36].

Again looking to group conjugacy as a model, for a, b in a group G, $a = g^{-1}bg$ for some $g \in G$ if and only if $ga = bg$ for some $g \in G$ if and only if $bh = ha$, for some $h \in G$. A corresponding semigroup conjugacy is defined as follows:

$$a \sim_o b \Leftrightarrow \exists_{g,h \in S^1} ag = gb \text{ and } bh = ha. \tag{1.2}$$

This relation was defined by Otto for monoids presented by finite Thue systems [40], and, unlike $\sim_p$, it is an equivalence relation in any semigroup. However, $\sim_o$ is the universal relation in any semigroup S with zero. Since it is generally believed [19, 23, 43] that $lim_{n \to \infty} \frac{z_n}{s_n} = 1$, where s_n [z_n] is the number of semigroups [with zero] of order n, it would follow that "almost all" finite semigroups have a zero and hence this notion of conjugacy might be of interest only in particular classes of semigroups.

In [10] a new notion of conjugacy was introduced. This notion coincides with Otto's concept for semigroups without zero, but does not reduce to the universal relation when S has a zero. The key idea was to restrict the set from which conjugators can be chosen. For a semigroup S with zero and $a \in S \setminus \{0\}$, let $\mathbb{P}(a)$ be the set of all elements $g \in S$ such that $(ma)g \neq 0$ for all $ma \in S^1 a \setminus \{0\}$. We also define $\mathbb{P}(0) = \{0\}$. If S has no zero, we set $\mathbb{P}(a) = S$ for every $a \in S$. Let $\mathbb{P}^1(a) = \mathbb{P}(a) \cup \{1\}$ where $1 \in S^1$. Define a relation $\sim_c$ on any semigroup S by

$$a \sim_c b \Leftrightarrow \exists_{g \in \mathbb{P}^1(a)} \exists_{h \in \mathbb{P}^1(b)} ag = gb \text{ and } bh = ha. \tag{1.3}$$

(See [10] for the motivation of using the sets $\mathbb{P}^1(a)$). Restricting the choice of conjugators, as happens in the definition of $\sim_c$, is not unprecedented for semigroups. For example, if S is a monoid and G is the group of units of S, we say that a and b in S are G-conjugated and write $a \sim_G b$ if there exists $g \in G$ such that $b = g^{-1}ag$ [31]. The restrictions proposed in the definition of $\sim_c$ are much less stringent. Their choice was motivated by considerations in the context of semigroups of transformations. The translation of these considerations into abstract semigroups resulted in the sets $\mathbb{P}^1(a)$. Roughly speaking, conjugators selected from $\mathbb{P}^1(a)$ satisfy the minimal requirements needed to avoid the pitfalls of $\sim_o$.

The relation $\sim_c$ turns out to be an equivalence relation on an arbitrary semigroup S. Moreover, if S is a semigroup without zero, then $\sim_c = \sim_o$. If S is a free semigroup, then $\sim_c = \sim_o = \sim_p$. In the case where S has a zero, the conjugacy class of 0 with respect to $\sim_c$ is $\{0\}$.

The last notion of conjugacy that we will consider has been inspired by considerations in the representation theory of finite semigroups (for details we refer the reader to Steinberg's book [46]). Let M be a finite monoid and let $a, b \in M$. We say that $a \sim_{tr} b$ if there exist $g, h \in M$ such that $ghg = g, hgh = h, hg = a^\omega, gh = b^\omega$, and $ga^{\omega+1}h = b^{\omega+1}$, where, for $a \in M$, a^ω denotes the unique idempotent in the monogenic semigroup generated by a (see [26]) and $a^{\omega+1} = aa^\omega$. The relation $\sim_{tr}$ is an equivalence relation in any finite monoid.

The same notion can be alternatively introduced (see, for example, Kudryavtseva and Mazorchuk [32]) via characters of finite-dimensional representations. Given a finite-dimensional complex representation $\varphi : S \to End_{\mathbb{C}}(V)$ of a semigroup S, the character of φ is the function $\chi_\varphi : S \to \mathbb{C}$ defined by $\chi_\varphi(s) = trace(\varphi(s))$ for all $s \in S$. In a finite monoid S, $a \sim_{tr} b$ if and only if $\chi_\varphi(a) = \chi_\varphi(b)$ ([38] or [46]) This explains the subscript notation $\sim_{tr}$.

The relation $\sim_{tr}$, in its equational definition, can be naturally extended from the class of finite monoids to the class of epigroups. We need some definitions first. Let S be a semigroup. An element $a \in S$ is an *epigroup element* (or, more classically, a *group-bound element*) if there exists a positive integer n such that a^n belongs to a subgroup of S, that is, the $\mathcal{H}$-class H_{a^n} of a^n is a group. If this positive integer is 1, then a is said to be *completely regular*. If we denote by e the identity element of H_{a^n}, then ae is in H_{a^n} and we define the *pseudo-inverse* a' of a by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of ae in the group H_{a^n} [45]. An *epigroup* is a semigroup consisting entirely of epigroup elements, and a *completely regular semigroup* is a semigroup consisting entirely of completely regular elements. Finite semigroups and completely regular semigroups are examples of epigroups. Following Petrich and Reilly [42] for completely regular semigroups and Shevrin [45] for epigroups, it is now customary to view an epigroup $(S,\cdot)$ as a *unary* semigroup $(S,\cdot,')$ where $x \longmapsto x'$ is the map sending each element to its pseudo-inverse. In addition, the ω notation introduced above for finite semigroups can be extended to an epigroup S [45], where, for $a \in S, a^\omega$ denotes the idempotent of the group to which some power of a belongs. (In the finite case, a^ω itself is a power of a). We can therefore extend the definition of $\sim_{tr}$ from finite monoids to epigroups: for all a, b in a epigroup S,

$$a \sim_{tr} b \Leftrightarrow \exists_{g,h \in S^1} ghg = g, hgh = h, ga^{\omega+1}h = b^{\omega+1}, hg = a^\omega, \text{ and } gh = b^\omega. \tag{1.4}$$

In any epigroup, we have $a^\omega = aa'$ ([45]), and therefore $a^{\omega+1} = aa'a = a''$. Thus in epigroups, as is sometimes convenient, we can express the conjugacy relation $\sim_{tr}$ entirely in terms of pseudo-inverses: for all $a, b \in S$,

$$a \sim_{tr} b \Leftrightarrow \exists_{g,h \in S^1} ghg = g, hgh = h, ga''h = b'', hg = aa', \text{ and } gh = bb'. \tag{1.5}$$

We will refer to $\sim_p$, $\sim_p^*$, $\sim_o$, $\sim_c$, and $\sim_{tr}$ as p-conjugacy, p^*-conjugacy, o-conjugacy, c-conjugacy, and trace conjugacy, respectively. Of course, $\sim_p$ is a valid notion of conjugacy only in the class of semigroups in which it is transitive, and trace conjugacy is only defined for epigroups.

Let S be a semigroup and $a \in S$. A *variant* of S is a semigroup whose universe equals that of S and the multiplication is defined by: for all $x, y \in S, x *_a y = xay$. There are a number of open problems on the relation between the notions of conjugation above in a semigroup S and in its variants. The goal of this thesis is to solve some of those problems.

Chapter 2

Variants of Epigroups and Primary Conjugacy

2.1 Introduction

Let S be a semigroup. Given $c \in S$ we can define a new binary operation $\cdot_c$ on S by

$$a \cdot_c b = acb \tag{2.1}$$

for all $a, b \in S$. The operation $\cdot_c$ is clearly associative, and the semigroup $(S, \cdot_c)$ is called the *variant* of S at c (see [24] and also [21, 20, 28, 30, 27, 37, 41]).

Elements a, b of a semigroup S are said to be *primarily conjugate*, denoted $a \sim_p b$, if there exists $x, y \in S^1$ such that $a = xy$ and $b = yx$. Here as usual, S^1 denotes S if S is a monoid; otherwise $S^1 = S \cup \{1\}$ where 1 is an adjoined identity element. Primary conjugacy is reflexive and symmetric, but it is not transitive in general. The transitive closure $\sim_p^*$ of $\sim_p$ can be considered to be a conjugacy relation in general semigroups [25, 31, 32]. Primary conjugacy is transitive in groups (where it coincides with the usual notion of conjugacy) and free semigroups [34]. To describe additional classes where primary conjugacy is known to be transitive, we must first recall the notion of epigroup.

An element a of a semigroup S is an *epigroup element* (also known as a *group-bound element*) if there exists a positive integer n such that a^n belongs to a subgroup of S, that is, the $\mathcal{H}$-class H_{a^n} of a^n is a group. The smallest such n is the *index* of a. If H_a itself is a group, that is, if a has index 1, then a is said to be *completely regular*. If we let e denote the identity element of H_{a^n}, then ae is in H_{a^n} and we define the *pseudo-inverse* a' of a by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of ae in the group H_{a^n} [44]. If every element of a semigroup is an epigroup element, then the semigroup itself is said to be an *epigroup*, and if every element is completely regular, then the semigroup is said to be *completely regular*. Every finite semigroup, and in fact every periodic semigroup, is an epigroup. Following Petrich and Reilly [42] for completely regular semigroups and Shevrin [44, 45] for epigroups, it is now customary to view an epigroup $(S, \cdot)$ as a *unary* semigroup $(S, \cdot, ')$ where $x \mapsto x'$ is the map sending each element to its pseudoinverse. We will make considerable use of the following identities which hold in all epigroups [45]:

$$x'xx' = x', \tag{2.2}$$

$$xx' = x'x, \tag{2.3}$$

$$x''' = x', \tag{2.4}$$

$$xx'x = x'', \tag{2.5}$$

$$(xy)'x = x(yx)', \tag{2.6}$$

$$(x^p)' = (x')^p, \tag{2.7}$$

for any prime p.

For each $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the variety (equational class) of all unary semigroups $(S, \cdot, ')$ satisfying (2.2), (2.3) and $x^{n+1}x' = x^n$. Each $\mathcal{E}_n$ is a variety of epigroups, and the inclusions $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ hold for all n. Every finite semigroup is contained in some $\mathcal{E}_n$. $\mathcal{E}_1$ is the variety of completely regular semigroups.

The following observation will be useful later.

Lemma 1 *For each $n \in \mathbb{N}$, the variety $\mathcal{E}_n$ is precisely the variety of unary semigroups satisfying* (2.2), (2.3) *and $x^{n-1}x'' = x^n$.*

Proof. If S is an epigroup in $\mathcal{E}_n$, then $x^n = x^{n+1}x' = x^{n-1}xx'x = x^{n-1}x''$ using (2.3) and (2.5). Conversely, suppose S satisfies (2.2), (2.3) and $x^{n-1}x'' = x^n$. Then $x^{n+1}x' = x^n x'' x' = x^{n-1}x''x''x' = x^{n-1}x''x'x'' = x^{n-1}x'' = x^n$, using (2.3) in the third equality and (2.2) in the fourth equality. ■

Kudryavtseva [33] proved that the restriction of $\sim_p$ to the set of all completely regular elements of a semigroup is transitive. More recently, it was shown in [15] that $\sim_p$ is transitive in all variants of completely regular semigroups. Variants of completely regular semigroups are not, in general, completely regular themselves; for example, if a completely regular semigroup has a zero 0, then the variant at 0 is a null semigroup, which is not even regular. This difficulty was circumvented in [15] by introducing the following class $\mathcal{W}$ of semigroups:

$$S \in \mathcal{W} \iff xy \text{ is completely regular for all } x, y \in S\,.$$

Equivalently $\mathcal{W}$ consists of all semigroups S such that the subsemigroup $S^2 = \{ab \mid a, b \in S\}$ is completely regular. The class $\mathcal{W}$ includes all completely regular semigroups and all *null semigroups* (semigroups satisfying $xy = uv$ for all x, y, u, v). The following summarizes the relevant results of [15].

Proposition 2 *1. ([15], Prp. 4.14) $\mathcal{W}$ is the variety of epigroups in $\mathcal{E}_2$ satisfying the additional identity $(xy)'' = xy$.*

2. ([15], Thm. 4.15) If S is a epigroup in $\mathcal{W}$, then $\sim_p$ is transitive in S;

3. ([15], Thm. 4.17) Every variant of a completely regular semigroup is in $\mathcal{W}$;

4. ([15], Cor. 4.18) If S is a variant of a completely regular semigroup, then $\sim_p$ is transitive in S.

Part (2) of this proposition had more to it, comparing $\sim_p$ with other notions of conjugation. In the simplified form stated here, the result follows easily from Kudryavtseva's theorem [33]: if $a \sim_p b$, $b \sim_p c$, and $a \neq b \neq c \neq a$, then there exist $x, y, u, v \in S$ such that $a = xy$, $b = yx = uv$ and $c = vu$. Thus $a, b, c \in \mathcal{W}$ are completely regular, so $a \sim_p c$.

We can slightly improve Proposition 2(1) as follows.

Lemma 3 *The variety $\mathcal{W}$ is precisely the variety of unary semigroups satisfying the identities* (2.2), (2.3), (2.5), (2.7) *(for $p = 2$) and $(xy)'' = xy$.*

Proof. One implication follows from Proposition 2(1), so suppose $(S, \cdot, ')$ is a unary semigroup satisfying the identities listed in the lemma. Then for all $x \in S$,

$$\begin{aligned} x^3 x' &\overset{(2.2)}{=} x^3 \underbrace{x' x}\, x' \overset{(2.3)}{=} x^4 x' x' \\ &\overset{(2.7)}{=} x^2 \underbrace{x^2 (x^2)'} \overset{(2.3)}{=} x^2 (x^2)' x^2 \\ &\overset{(2.5)}{=} (x^2)'' \qquad = x^2 . \end{aligned}$$

Therefore $(S, \cdot, ')$ lies in $\mathcal{E}_2$, hence in $\mathcal{W}$. ■

The variety $\mathcal{W}$ has another characterization that was not mentioned in [15].

Lemma 4 *Let S be a semigroup. The following are equivalent:*

1. S is an epigroup in $\mathcal{W}$.

2. *For each $c \in S$, the principal left ideal Sc is a completely regular subsemigroup.*

3. *For each $c \in S$, the principal right ideal cS is a completely regular subsemigroup.*

Proof. An element of a semigroup is completely regular if and only if it lies in some subgroup, so the desired equivalences follow from the definition of $\mathcal{W}$. ■

In view of Lemma 4, we should also mention the kindred study in [35] of epigroups S in which every local submonoid eSe is completely regular.

The key tool in the proof of Proposition 2(3) was the following unary operation:

$$x^* = (xc)'x(cx)' . \tag{*}$$

Indeed, if $(S, \cdot, ')$ is completely regular, then $(S, \cdot_c, ^*)$ is an epigroup in the variety $\mathcal{W}$. However, (*) was introduced in [15] in an *ad hoc* fashion. To show that it is quite natural, we note that an ideal of an epigroup is a subepigroup [44]. In particular, for each c in an epigroup S, Sc is a subepigroup. Thus for any $x \in S$, the pseudoinverse $(xc)'$ must have the form yc for some $y \in S$. This is exactly what (*) does for us.

Lemma 5 *Let S be an epigroup and fix $c \in S$. For all $x \in S$,*

$$(xc)' = x^*c . \tag{2.8}$$

Proof. We compute

$$(xc)' = (xc)'xc(xc)' = (xc)'(xc)'xc = (xc)x(cx)'c = x^*c ,$$

using (2.2), (2.3) and (2.6). ■

If $(S, \cdot, ')$ is an epigroup, we will refer to $(S, \cdot_c, ^*)$ as the *unary variant* of $(S, \cdot, ')$ at c. Proposition 2(3) states that if $(S, \cdot, ') \in \mathcal{E}_1$ is completely regular, then $(S, \cdot_c, ^*) \in \mathcal{W}$. Our first main result will both improve and extend this. First we must introduce a family of varieties of unary semigroups. For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n$ is defined by associativity and the following identities: (2.2), (2.3),

$$x\underbrace{y\cdots y}_{n-1}y'' = x\underbrace{y\cdots y}_{n} \tag{2.9}$$

$$x''\underbrace{x\cdots x}_{n-1}y = \underbrace{x\cdots x}_{n}y \tag{2.10}$$

Setting $y = x$ in, say, (2.9), we see from Lemma 1 that $\mathcal{V}_n$ is a variety of epigroups and in particular,

$$\mathcal{E}_n \subseteq \mathcal{V}_n \subseteq \mathcal{E}_{n+1} \,. \tag{2.11}$$

That every variant of an epigroup is an epigroup is easy to see, but what is not so obvious is what happens to the pseudoinverse operation. Our first main result clarifies this and also the role of the varieties $\mathcal{V}_n$.

Theorem 6 *Let $(S, \cdot, ')$ be an epigroup. For each $c \in S$, the unary variant $(S, \cdot_c, {}^*)$ is an epigroup. If $(S, \cdot, ') \in \mathcal{V}_n$ for some $n > 0$, then $(S, \cdot_c, *) \in \mathcal{V}_n$. Therefore for $n \in \mathbb{N}$, the variety $\mathcal{V}_n$ is closed under taking variants.*

Corollary 7 *Let $(S, \cdot, ')$ be a completely regular semigroup. For each $c \in S$, the unary variant $(S, \cdot_c, {}^*)$ lies in $\mathcal{V}_1$.*

Example 8 *Not every unary semigroup in $\mathcal{V}_1$ is a variant of a completely regular semigroup. Using* Mace4*, we found that the smallest examples have order 4, and there are three of them up to isomorphism:*

$\cdot$	0	1	2	3
0	1	1	1	1
1	1	1	1	1
2	1	1	2	1
3	1	1	1	3

$\cdot$	0	1	2	3
0	1	1	2	2
1	1	1	2	2
2	2	2	2	2
3	2	2	2	3

$\cdot$	0	1	2	3
0	2	2	1	1
1	2	2	1	1
2	1	1	2	2
3	1	1	2	3

The corresponding pseudoinverse operation is the same for all three epigroups: $0' = 1$ and $x' = x$ for $x = 1, 2, 3$.

Remark 9 *For an element c of a semigroup S, the mapping $\rho_c : S \to Sc; x \mapsto xc$ is a homomorphism from the variant $(S, \cdot_c)$ to $(Sc, \cdot)$ since $(x \cdot_c y)c = xc \cdot yc$. If S is also an epigroup, then as already noted, so is Sc. Every semigroup homomorphism between epigroups is an epigroup homomorphism, but* (2.8) *shows more explicitly how ρ_c preserves pseudoinverses.*

Comparing Proposition 2(3) with Corollary 7 raises the question of how the varieties $\mathcal{V}_1$ and $\mathcal{W}$ are related other than just the fact that both contain $\mathcal{E}_1$. Our second main result addresses this and its corollary connects this discussion to the transitivity of $\sim_p$.

Theorem 10 *1.* $\mathcal{V}_1 \subset \mathcal{W}$*;*

2. *If $(S,\cdot,')$ is an epigroup in $\mathcal{W}$, then for each $c \in S$, the unary variant $(S,\cdot_c,{}^*)$ lies in $\mathcal{V}_1$.*

In particular, every variant of an epigroup in $\mathcal{W}$ actually lies in a proper subvariety of $\mathcal{W}$, a stronger statement than the assertion that $\mathcal{W}$ is closed under taking variants.

We can now give an affirmative answer to Problem 6.18 in [15]. Since $\sim_p$ is transitive in $\mathcal{W}$, the previous theorem immediately implies the following.

Corollary 11 *Primary conjugacy $\sim_p$ is transitive in every variant of any epigroup in $\mathcal{W}$.*

The proofs of Theorems 6 and 10 will be given in §2.2.

Most of the proofs were obtained with the assistance of the automated theorem prover PROVER9 developed by McCune [39].

2.2 Proofs

Let $(S,\cdot,')$ be an epigroup and fix $c \in S$. To verify Theorem 6, we start with a couple of lemmas.

Lemma 12 *$(S,\cdot_c,{}^*)$ satisfies* (2.2) *and* (2.3).

Proof. First we compute

$$\begin{aligned} x^* \cdot_c x \cdot_c x^* &= (xc)'\underbrace{x(cx)'}cxc(xc)'\underbrace{x(cx)'} \overset{(2.6)}{=} (xc)'\underbrace{(xc)'xc}\,\underbrace{xc(xc)'}(xc)'x \\ &\overset{(2.3)}{=} (xc)'xc(xc)' \cdot (xc)'xc(xc)' \cdot x \overset{(2.2)}{=} (xc)'\underbrace{(xc)'x} \\ &\overset{(2.6)}{=} (xc)'x(cx)' \qquad\qquad\qquad\qquad = x^*\,, \end{aligned}$$

which establishes (2.2).

Next we have

$$\begin{aligned} x \cdot_c x^* &= \underbrace{xc(xc)'}x(cx)' \overset{(2.3)}{=} (xc)'xc\,\underbrace{x(cx)'} \\ &\overset{(2.6)}{=} (xc)'x\,\underbrace{c(xc)'}x \overset{(2.6)}{=} (xc)'x(cx)'cx \\ &= x^* \cdot_c x\,, \end{aligned}$$

which establishes (2.3). ■

We will denote powers of elements in $(S,\cdot_c)$ with parentheses in the exponent, that is, $x^{(1)} = x$ and $x^{(n)} = x \cdot_c x^{(n-1)}$ for $n > 1$.

Lemma 13 *If ca has index n in $(S,\cdot,')$, then a is an epigroup element of index n or $n+1$ in $(S,\cdot_c,{}^*)$.*

Proof. For any $k > 0$, we have $a^{(k+1)} \cdot_c a^* = (ac)^{k+1}a^* = a(ca)^k \underbrace{ca^*} = a\underbrace{(ca)^k(ca)'}$ using (2.8), and $a^{(k)} = a(ca)^k$. Thus if ca has index n, then $a^{(n+2)} \cdot_c a^* = a^{(n+1)}$, and so a is an epigroup element of index at most $n+1$ in $(S,\cdot_c,{}^*)$. If a has index $k \le n+1$ in $(S,\cdot_c,{}^*)$, then $a(ca)^k(ca)' = a\,(ca)^k$ and so $(ca)^{k+1}(ca)' = (ca)^k$ and so $k \ge n$. ■

Lemma 14 *For all $x \in S$,*

$$cx^{**} = (cx)''\,. \tag{2.12}$$

Proof. Using (*), we compute

$$\begin{aligned} cx^{**} = c(x^*)^* \qquad\qquad & = \underbrace{c(x^*c)'}x^*(cx^*)' \\ \overset{(2.6)}{=} (cx^*)'cx^*(cx^*)' & \overset{(2.2)}{=} (cx^*)' \\ = (\underbrace{c(xc)'}x(cx)')' \quad & \overset{(2.6)}{=} ((cx)'cx(cx)')' \overset{(2.2)}{=} (cx)''. \end{aligned}$$

■

Proof of Theorem 6. Assume $(S,\cdot,')$ is an epigroup. Then by Lemmas 12 and 13, $(S,\cdot_c,{}^*)$ is also an epigroup.

Suppose now that for some $n \in \mathbb{N}$, $(S,\cdot,') \in \mathcal{V}_n$. Then for all $x,y \in S$,

$$\begin{aligned} x \cdot_c y^{(n-1)} \cdot_c y^{**} &= x(cy)^{n-1}cy^{**} \overset{(2.12)}{=} x(cy)^{n-1}(cy)'' \\ &= x(cy)^n \qquad\quad = x \cdot_c y^{(n)}\,, \end{aligned}$$

using $(S,\cdot,') \in \mathcal{V}_n$ in the third equality. Thus $(S,\cdot_c,{}^*) \in \mathcal{V}_n$. This completes the proof. ■

Now we turn to Theorem 10.

Lemma 15 *$\mathcal{V}_1 \subset \mathcal{W}$.*

Proof. Fix $(S,\cdot,') \in \mathcal{V}_1$. We already know that S is an epigroup in $\mathcal{E}_2$ by (2.11) and so by Proposition 2(1), we just need to verify the identity $(xy)'' = xy$. We compute

$$\begin{aligned} (xy)'' &\overset{(2.5)}{=} \underbrace{xy}(xy)'xy \quad \overset{(2.10)}{=} \underbrace{x''}y(xy)'xy \overset{(2.2)}{=} x''x'\underbrace{x''y}(xy)'xy \\ &\overset{(2.10)}{=} x''x'\underbrace{xy(xy)'xy} \overset{(2.5)}{=} x''\underbrace{x'(xy)''} \quad \overset{(2.9)}{=} \underbrace{x''x'}xy \\ &\overset{(2.10)}{=} \underbrace{xx'x}y \qquad\quad \overset{(2.5)}{=} x''y \qquad\quad \overset{(2.10)}{=} xy\,. \end{aligned}$$

To see that the inclusion is proper, consider the unary semigroup given by the multiplication table

$\cdot$	0	1	2	3
0	2	3	2	2
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

and the unary operation $0' = 2$, $1' = 1$, $2' = 2$, $3' = 3$. This is easily checked to be an epigroup in $\mathcal{W}$ with $'$ as the pseudoinverse operation, but $0'' \cdot 1 = 2 \cdot 1 = 2 \neq 3 = 0 \cdot 1$, so (2.10) does not hold. ∎

Proof of Theorem 10. Lemma 15 takes care of (1), so we need to prove (2).

Let $(S, \cdot, ')$ be an epigroup in $\mathcal{W}$ and fix $c \in S$. Since $\mathcal{W} \subseteq \mathcal{E}_2 \subseteq \mathcal{V}_2$ (by Proposition 2(1) and (2.11)), we know that the unary variant $(S, \cdot_c, ^*)$ is an epigroup in $\mathcal{V}_2$ (Theorem 6). What remains is to prove that $(S, \cdot_c, ^*)$ satisfies (2.9) and (2.10) with $n = 1$. We compute

$$x \cdot_c y^{**} = xcy^{**} \overset{(2.12)}{=} x(cy)'' = xcy = x \cdot_c y\,.$$

This establishes (2.9) in $(S, \cdot_c, ^*)$ and the proof of (2.10) is similar. ∎

Chapter 3

The Transitivity of Primary Conjugacy in a Class of Semigroups

By a notion of conjugacy for a class of semigroups, we mean an equivalence relation defined in the language of that class of semigroups such that when restricted to groups, it coincides with the usual notion of conjugacy.

Before introducing the notion of conjugacy that will occupy us, we recall some standard definitions and notation (we generally follow [26]). For a semigroup S, we denote by S^1 the semigroup S if S is a monoid; otherwise S^1 denotes the monoid obtained from S by adjoining an identity element 1.

Any reasonable notion of semigroup conjugacy should coincide in groups with the usual notion. Elements a, b of a group G are conjugate if there exists $g \in G$ such that $a = g^{-1}bg$. Conjugacy in groups has several equivalent formulations that avoid inverses, and hence generalize syntactically to any semigroup. For many of these notions including the one we focus on here, we refer the reader to [15, 29, 32].

For example, if G is a group, then $a, b \in G$ are conjugate if and only if $a = uv$ and $b = vu$ for some $u, v \in G$. Indeed, if $a = g^{-1}bg$, then setting $u = g^{-1}b$ and $v = g$ gives $uv = a$ and $vu = b$; conversely, if $a = uv$ and $b = vu$ for some $u, v \in G$, then setting $g = v$ gives $g^{-1}bg = v^{-1}vuv = uv = a$.

This last formulation was used to define the following relation on a free semigroup S (see [34]):

$$a \sim_p b \iff \exists_{u,v \in S^1}\, a = uv \text{ and } b = vu.$$

If S is a free semigroup, then $\sim_p$ is an equivalence relation on S, and so it can be considered as a notion of conjugacy in S. In a general semigroup S, the relation $\sim_p$ is reflexive and symmetric, but not transitive. If $a \sim_p b$ in a semigroup, we say that a and b are *primarily conjugate* or just p-conjugate for short (hence the subscript in $\sim_p$); a and b were said to be "primarily related" in [32]. Lallement [34] credited the idea of the relation $\sim_p$ to Lyndon and Schützenberger [36].

In spite of its name, $\sim_p$ is a valid notion of conjugacy only in the class of semigroups in which it is transitive. Otherwise, the transitive closure $\sim_p^*$ of $\sim_p$ has been defined as a conjugacy relation in a general semigroup [25, 31, 32]. Finding classes of semigroups in which $\sim_p$ itself is transitive, that is, $\sim_p = \sim_p^*$, is an open problem. The aim of this note is to prove the following theorem.

Theorem 16 *Let $n > 1$ be an integer and let S be a semigroup satisfying the following: for all $x, y \in S$,*

$$xy \in \{yx, (xy)^n\} .$$

Then primary conjugation $\sim_p$ is transitive in S.

There are various motivations for studying this particular class of semigroups. First, it naturally generalizes two classes of semigroups in which $\sim_p$ is transitive.

Proposition 17 *Let S be a semigroup.*

1. *If S is commutative, then $\sim_p$ is transitive.*

2. *If S satisfies $xy = (xy)^2$ for all $x, y \in S$, then $\sim_p$ is transitive.*

Proof. 1. In a commutative semigroup, $\sim_p$ is the identity relation and hence it is trivially transitive.

2. If $a \sim_p b$, then $a = uv$ and $b = vu$ for some $u, v \in S^1$. Thus $a^2 = (uv)^2 = uv = a$ and $b^2 = (vu)^2 = vu = b$ so that a, b are idempotents. In particular, a, b are completely regular elements of S. The restriction of $\sim_p$ to the set of completely regular elements is a transitive relation [33]. ■

The other motivation for studying this class of semigroups is that it has been of recent interest in other contexts. In particular, J. P. Araújo and Kinyon [16] showed that a semigroup satisfying $x^3 = x$ and $xy \in \{yx, (xy)^2\}$ for all x, y is a semilattice of rectangular bands and groups of exponent 2.

The proof of Theorem 16 was found by first proving the special cases $n = 2, 3, 4$ using the automated theorem prover `Prover9` developed by McCune [39]. After studying these proofs, the pattern became apparent, leading to the proof of the general case. Note that `Prover9` and other automated theorem provers usually cannot handle statements like our theorem directly because there is not a way to specify that n is a fixed positive integer. Thus the approach of examining a few special cases and then extracting a human proof of the general case is the most efficient way to use an automated theorem prover in these circumstances.

Proof of Theorem 16. Suppose $a, b, c \in S$ satisfy $a \sim_p b$ and $b \sim_p c$. Since $a \sim_p b$, there exist $a_1, a_2 \in S^1$ such that $a = a_1 a_2$ and $b = a_2 a_1$. Similarly, since $b \sim_p c$, there exist $b_1, b_2 \in S^1$ such that $b = b_1 b_2$ and $c = b_2 b_1$. We want to prove there exist $x, y \in S^1$ such that $a = xy$ and $c = yx$. If $a = b$ or if $b = c$, then there is nothing to prove. Thus we may assume without loss of generality that $a_1 a_2 \neq a_2 a_1$ and $b_2 b_1 \neq b_1 b_2$.

Assume first that $n = 2$. Then

$$a = a_1 a_2 = (a_1 a_2)(a_1 a_2) = a_1(a_2 a_1)a_2 = a_1 b a_2 = (a_1 b_1)(b_2 a_2)\,,$$

and

$$c = b_2 b_1 = (b_2 b_1)(b_2 b_1) = b_2(b_1 b_2)b_1 = b_2 b b_1 = (b_2 a_2)(a_1 b_1)\,.$$

Thus setting $x = a_1 b_1$ and $y = b_2 a_2$, we have $a \sim_p c$ in this case.

Now assume $n > 2$. We have

$$\begin{aligned} a = a_1 a_2 = (a_1 a_2)^n &= \underbrace{(a_1 a_2) \cdots (a_1 a_2)}_{n} \\ &= a_1 \underbrace{(a_2 a_1) \cdots (a_2 a_1)}_{n-1} a_2 \\ &= a_1 b^{n-1} a_2 \\ &= a_1 b b^{n-2} a_2 \\ &= a_1 (b_1 b_2) b^{n-2} a_2 \\ &= (a_1 b_1)(b_2 b^{n-2} a_2) \end{aligned}$$

and

$$\begin{aligned} c = b_2 b_1 = (b_2 b_1)^n &= \underbrace{(b_2 b_1) \cdots (b_2 b_1)}_{n} \\ &= b_2 \underbrace{(b_1 b_2) \cdots (b_1 b_2)}_{n-1} b_1 \\ &= b_2 b^{n-1} b_1 \\ &= b_2 b^{n-2} b b_1 \\ &= b_2 b^{n-2} (a_2 a_1) b_1 \\ &= (b_2 b^{n-2} a_2)(a_1 b_1)\,. \end{aligned}$$

Thus setting $x = a_1 b_1$ and $y = b_2 b^{n-2} a_2$, we have that $a \sim_p c$. ∎

Chapter 4

Conclusion

With this thesis we tried to solve some open problems about the relation between notions of conjugation in a semigroup S and in its variants.

In Chapter 2, with the proofs of Theorem 6 and Theorem 10, we solved Problem 6.18 in [15]. The results of Chapter 2 have been submitted for publication [17].

In Chapter 3 we prove, with Theorem 16, that primary conjugation is transitive in the semigroup S presented. The result of Chapter 3 has been accepted for publication [18].

In the following Section we suggest some problems for future work.

4.1 Future Work

Completely regular semigroups can be defined conceptually (unions of groups) or as unary semigroups satisfying certain identities. The same is true of the variety $\mathcal{W}$; the conceptual definition given in [15] is that S lies in $\mathcal{W}$ if S^2 is completely regular or $\mathcal{W}$ can be defined as a variety of unary semigroups (Lemma 3).

On the other hand, the epigroup varieties $\mathcal{V}_n$ only have a definition as unary semigroups. Since they are close under taking variants (Theorem 6), they are clearly interesting varieties interlacing the varieties $\mathcal{E}_n$ (see (2.11)). In this context, the following four problems were suggested by Professor Michael Kinyon.

Problem 1 *Is there a conceptual characterization of the varieties $\mathcal{V}_n$, or even just $\mathcal{V}_1$, analogous to the characterizations of $\mathcal{E}_1$ and $\mathcal{W}$?*

From (2.11) and Theorem 10, we have the following chain of varieties:

$$\mathcal{E}_1 \subset \mathcal{V}_1 \subset \mathcal{W} \subset \mathcal{E}_2 \subset \mathcal{V}_2 \subset \mathcal{E}_3 \cdots .$$

Problem 2 *Is there a natural family of varieties $\mathcal{W}_n$ interlacing the varieties in the chain above and such that $\mathcal{W}_1 = \mathcal{W}$? In addition, does the appropriate generalization of Theorem 10(2) hold?*

An interesting direction for the study of the varieties $\mathcal{V}_n$ or $\mathcal{W}$ is to consider the subvarieties in which idempotents commute, that is, so-called E-semigroups (see [1] and the references therein). These are subvarieties because every idempotent in an epigroup has the form $x'x$, and so E-epigroups are characterized by the identity $x'xy'y = y'yx'x$.

Problem 3 *Study the varieties of E-epigroups in $\mathcal{V}_n$ or $\mathcal{W}$.*

Many classes of algebras can be characterized by forbidden subalgebras or forbidden divisors (quotients). For example, distributive lattices can be characterized by two forbidden sublattices; similarly, stable semigroups can be characterized by forbidding the bicyclic monoid as a subsemigroup [34]; see also [9] for another example. The considerations in the paper prompt the following natural problems.

Problem 4 *Can any of the inclusions of varieties considered here, especially $\mathcal{E}_1 \subset \mathcal{V}_1$ and $\mathcal{V}_1 \subset \mathcal{W}$, be characterized by forbidden subepigroups or forbidden epidivisors?*

Finally, returning to primary conjugacy, we rephrase two problems from [15] to the context of this paper, suggested by Professor João Araújo.

Problem 5 *Characterise and enumerate primary conjugacy classes in various types of transformation semigroups and their variants such as, for example, those appearing in the problem list of [8] or those appearing in the list of transformation semigroups included in [22]. Especially interesting would be a characterization of primary conjugacy classes in variants of centralizers of idempotents [2, 4, 5], or in variants of semigroups in which the group of units has a rich structure [3, 13, 14, 12, 7, 6].*

In [11], a problem on independence algebras was solved using their classification; the same technique might perhaps be used to extend the results in [21] and to solve the following.

Problem 6 *Characterize $\sim_p$ in the variants of the endomorphism monoid of a finite dimensional independence algebra.*

Bibliography

[1] J. Almeida, J.-E. Pin and P. Weil, Semigroups whose idempotents form a subsemigroup, *Math. Proc. Camb. Phi. Soc.* **111** (1992), 241–253.

[2] J.M. André, J. Araújo and J. Konieczny, Regular centralizers of idempotent transformations, *Semigroup Forum* **82** (2) (2011), 307–318.

[3] J.M. André, J. Araújo and P.J. Cameron, The classification of partition homogeneous groups with applications to semigroup theory, *J. Algebra* **452** (2016), 288–310.

[4] J. Araújo and J. Konieczny, Automorphisms groups of centralizers of idempotents, *J. Algebra* **269** (2003), 227–239.

[5] J. Araújo and J. Konieczny, Semigroups of transformations preserving an equivalence relation and a cross-section, *Comm. Algebra* **32** (2004), 1917–1935.

[6] J. Araújo and C. Schneider, The rank of the endomorphism monoid of a uniform partition, *Semigroup Forum* **78** 3 (2009), 498–510.

[7] J. Araújo, P.J. Cameron, J.D. Mitchell and M. Neunhöffer, The classification of normalizing groups, *J. Algebra*, **373** (2013), 481–490.

[8] J. Araújo and J. Konieczny, Centralizers in the full transformation semigroup, *Semigroup Forum* **86** (2013), 1–31.

[9] J. Araújo, M. Kinyon and A. Malheiro, A characterization of adequate semigroups by forbidden subsemigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), no. 6, 1115–1122.

[10] J. Araújo and J. Konieczny, and A. Malheiro, Conjugation in semigroups, *J. Algebra* **403** (2014), 93–134.

[11] J. Araújo, W. Bentz and J. Konieczny, The largest subsemilattices of the endomorphism monoid of an independence algebra, *Linear Algebra Appl.* **458** (2014), 60–79.

[12] J. Araújo and P. J. Cameron, Primitive groups synchronize non-uniform maps of extreme ranks, *J. Combin. Theory Ser. B* **106** (2014), 98–114.

[13] J. Araújo, W. Bentz, J.D. Mitchell and C. Schneider, The rank of the semigroup of transformations stabilising a partition of a finite set, *Math. Proc. Cambridge Philos. Soc.* **159** (2015), no. 2, 339–353.

[14] J. Araújo and P. J. Cameron, Two generalizations of homogeneity in groups with applications to regular semigroups, *Trans. Amer. Math. Soc.* **368** (2016), no. 2, 1159–1188.

[15] J. Araújo, M. Kinyon, J. Konieczny and A. Malheiro, Four notions of conjugacy for abstract semigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **147** (2017), no. 6, 1169–1214.

[16] J. P. Araújo and M. Kinyon, A natural characterization of semilattices of rectangular bands and groups of exponent two, *Semigroup Forum* **91** (2015), 295–298.

[17] M. Borralho and M. Kinyon, Variants of epigroups and primary conjugacy, arXiv.org: 1911.05157.

[18] M. Borralho, The transitivity of primary conjugacy in a class of semigroups, *Quasigroups and Related Systems* **28** (2020), arXiv.org: 1911.07296.

[19] A. Distler and J. D. Mitchell, The number of nilpotent semigroups of degree 3, *Electron. J. Combin.* **19** (2012), no. 2, Paper 51, 19 pp.

[20] I. Dolinka and J. East, Variants of finite full transformation semigroups, *Internat. J. Algebra Comput.* **25** (2015), no. 8, 1187–1222.

[21] I. Dolinka and J. East, Semigroups of rectangular matrices under a sandwich operation, *Semigroup Forum* **96** (2018), no. 2, 253–300.

[22] V.H. Fernandes, Presentations for some monoids of partial transformations on a finite chain: a survey, *Semigroups, algorithms, automata and languages* (Coimbra, 2001), 363–378, World Sci. Publ., River Edge, NJ, 2002.

[23] P.A. Grillet, Counting semigroups, *Comm. Algebra* **43** (2015), 574-596.

[24] J. B. Hickey, On variants of a semigroup, *Bull. Austral. Math. Soc.* **34** (1986), no. 3, 447–459.

[25] P.M. Higgins, The semigroup of conjugates of a word, *Internat. J. Algebra Comput.* **16** (2006), 1015–1029.

[26] J.M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Clarendon Press, Oxford, 1995.

[27] R. Jongchotinon, S. Chaopraknoi and Y. Kemprasit, Isomorphism theorems for variants of semigroups of linear transformations, *Int. J. Algebra* **4** (2010), no. 25–28, 1407–1412.

[28] T.A. Khan and M.V. Lawson, Variants of regular semigroups, *Semigroup Forum* **62** (2001), no. 3, 358–374.

[29] J. Konieczny, A new definition of conjugacy for semigroups, *J. Algebra Appl.* **17** (2018), 1850032, 20 pp.

[30] G. Kudryavtseva and V. Maltcev, On representations of variants of semigroups, *Bull. Austral. Math. Soc.* **73** (2006), no. 2, 273–283.

[31] G. Kudryavtseva and V. Mazorchuk, On conjugation in some transformation and Brauer-type semigroups, *Publ. Math. Debrecen* **70** (2007), 19-43.

[32] G. Kudryavtseva and V. Mazorchuk, On three approaches to conjugacy in semigroups, *Semigroup Forum* **78** (2009), 14-20.

[33] G. Kudryavtseva, On conjugation in regular epigroups, arXiv.org:0605698.

[34] G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley & Sons, New York, 1979.

[35] J. Liu, Q. Chen and C. Han, Locally completely regular epigroups, *Comm. Algebra* **44** (2016), 4546–4563.

[36] R. C. Lyndon and M. P. Schutzenberger, The equation $a^m = b^n c^p$ in a free group, *Michigan Math. J.* **9**(4) (1962), 289–298.

[37] V. Mazorchuk and G. Tsyaputa, Isolated subsemigroups in the variants of T_n, *Acta Math. Univ. Comenian.* **77** (2008), no. 1, 63–84.

[38] D. B. McAlister, Characters of finite semigroups, *J. Algebra* **22** (1972), 183-200.

[39] W. McCune, `Prover9` and `Mace4`, https://www.cs.unm.edu/ ~mccune/prover9/.

[40] F. Otto, Conjugacy in monoids with a special Church-Rosser presentation is decidable, *Semigroup Forum* **29** (1984), 223-240.

[41] H. Pei, L. Sun and H. Zhai, Green's relations for the variants of transformation semigroups preserving an equivalence relation, *Comm. Algebra* **35** (2007), no. 6, 1971–1986.

[42] M. Petrich and N.R. Reilly, *Completely Regular Semigroups*, John Wiley & Sons, New York, 1999.

[43] J. Rhodes and B. Steinberg, *The q-theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.

[44] L. N. Shevrin, On the theory of epigroups. I. (Russian) *Mat. Sb.* **185** (1994), 129–160; translation in *Russian Acad. Sci. Sb. Math.* **82** (1995), 485–512

[45] L. N. Shevrin, Epigroups, in V. B. Kudryavtsev and I. G. Rosenberg (eds.), *Structural Theory of Automata, Semigroups, and Universal Algebra,* 331–380, NATO Sci. Ser. II Math. Phys. Chem., 207, Springer, Dordrecht, 2005.

[46] B. Steinberg, *The Representation Theory of Finite Monoids*, Springer Monographs in Mathematics, Springer, New York, 2015.