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Python Code of Solving Quaternionic Quadratic Equations

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Abstract

In this project, we introduce Python code for solving Quaternionic Quadratic Equations(QQE). Liping Huang and Wasin So [2] derive explicit formulas for computing the roots of quaternionic quadratic equations. We study and motivated by their mathematical work and able to give convenient Python code to get solutions for any QQE. In section one, we give a brief introduction about quaternions, its history, algebra and geometry[4]. Later we explain Huang and Wasin [2] work of how to derive explicit formula for solving QQE and include the Python code to solve any QQE of the form $x^2 + bx + c = 0$ where $a, b \in \mathbb{H}$ in general. All necessary details about how to install and code using Python can be found from Python official website [5] and wide range of practical examples can be learned from the book, Computational Physics with Python by Mark Newman[3].

Keywords: Quaternions, Quaternionic quadratic equation, Python code

1 Quaternions

1.1 History, Algebra and Geometry of Quaternions

The quaternions were discovered on October 16, 1843 by Sir William Rowan Hamilton [For more information about history [4]]. They form a noncommutative associative algebra over $\mathbb{R}$:

$$\mathbb{H} = \{q_1 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} | q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

For $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$, let $\bar{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$ be the conjugate of q, norm of q is $|q| = \sqrt{q\bar{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$. The real part of q is q_1 and denote as $\mathrm{Re}(q) = q_1$. Imaginary part of q is, $\mathrm{Im}(q) = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$.
quaternionic multiplication table is cyclic and can be written as follows,

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},$$

$$\mathbf{jk} = \mathbf{i} = -\mathbf{kj},$$

$$\mathbf{ki} = \mathbf{j} = -\mathbf{ik},$$

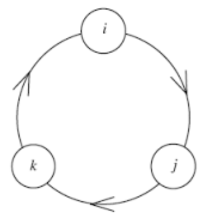

Figure 1.1: The Quaternionic multiplication table

We refer to $\mathbf{i}$,$\mathbf{j}$, and $\mathbf{k}$ as imaginary quaternionic units. Notice that these units anticommute. This multiplication table is shown schematically in Figure 1.
Multiplying two of these quaternionic units together in the direction of the arrow yields the third; going against the arrow contributes an additional minus sign. The quaternions are denoted by $\mathbb{H}$; the H is for "Hamilton", they are spanned by the identity element 1 and three imaginary units, that is , a quaternion q can be represented as four real numbers (q_1, q_2, q_3, q_4), usually written

$$q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \tag{1.1}$$

which can be thought of as a point or vector in $\mathbb{R}^4$. Since (1.1) can be written in the form

$$q = (q_1 + q_2\mathbf{i}) + (q_3 + q_4\mathbf{i})\mathbf{j}$$

we see that a quaternion can be viewed as a pair of complex numbers, we can write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$ in direct analogy to the construction of $\mathbb{C}$ from $\mathbb{R}$.

The only quaternion with norm zero is zero, and every nonzero quaternion has a unique inverse, namely

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

Quaternionic conjugation satisfies the identity $\overline{pq} = \bar{q}\bar{p}$ from which it follows that the norm satisfies $|pq| = |p||q|$.

Squaring both sides and expanding the result in terms of components yields the $4-$ sqaures rule,

$$\begin{aligned}
&(p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4)^2 + (p_2q_1 + p_1q_2 - p_4q_3 + p_3q_4)^2 \\
&+ (p_3q_1 + p_4q_2 + p_1q_3 - p_2q_4)^2 + (p_4q_1 - p_3q_2 + p_2q_3 + p_1q_4)^2 \\
&= (p_1^2 + p_2^2 + p_3^2 + p_4^4)(q_1^2 + q_2^2 + q_3^2 + q_4^2)
\end{aligned}$$

which is not quite as obvious as the $2-$ squars rule. This identity implies that the quaternions form a division algebra, that is, not only are there inverses, but there are no zero divisors-if a product is zero, one of the factors must be zero.
It is important to realize that $\pm\mathbf{i}, \pm\mathbf{j}$, and $\pm\mathbf{k}$ are not the only quaternionic square roots of -1, any imaginary quaternion squares to a negative number, so it is only necessary to choose its norm to be one in order to get a square root of -1. The imaginary quaternions of norm one form a $2-$dimensional sphere$(q_1 = 0)$; in the above notation, this is the set of points

$$q_2^2 + q_3^2 + q_4^2 = 1$$

any such unit imaginary quaternion u can be used to construct a complex subalgebra of $\mathbb{H}$, which we will also denote by $\mathbb{C}$, namely

$$\mathbb{C} = a + bu$$

with $a, b \in \mathbb{R}$. Furthermore, we can use the identity to write

$$e^{u\theta} = \cos\theta + u\sin\theta$$

This means that any quaternion can be written in the form
$q = re^{u\theta}$ where $r = |q|$ and u denotes the direction of the imaginary part of q.

1.2 Further properties of quaternions

Some elementary properties of the algebra of quaternions are listed below as a proposition without proof.

1.2.1 Proposition 1

Let $x, y \in \mathbb{H}$. Then:

1. $\bar{x}x = x\bar{x}$.
2. $|x| = |\bar{x}|$.
3. $|.|$ is indeed a norm on $\mathbb{H}$, For all $x, y \in \mathbb{H}$ we have, $|x| \geq 0$ with equality if and only if $x = 0$; $|x + y| \leq |x| + |y|$; $|xy| = |yx| = |x|.|y|$;
4. $jc\bar{j} = kc\bar{k} = \bar{c}$ for every $c \in \mathbb{C}$.
5. $\overline{(xy)} = \bar{y}\bar{x}$.
6. $x = \bar{x}$ if and only if $x \in \mathbb{R}$.
7. if $a \in \mathbb{H}$, then $ax = xa$ for every $x \in \mathbb{H}$ if and only if $a \in \mathbb{R}$.
8. Every $x \in \mathbb{H} - \{0\}$ has an inverse $x^{-1} = \frac{\bar{x}}{|x|^2} \in \mathbb{H}$;
   in more detail, $x\frac{\bar{x}}{|x|^2} = \frac{\bar{x}}{|x|^2}x = 1$;
9. $|x^{-1}| = |x|^{-1}$ for every $x \in \mathbb{H} - \{0\}$.
10. $x \in \mathbb{H}$ and $\bar{x}$ are solutions of the following quadratic equation with real coefficients $t^2 - 2\mathrm{Re}(x)t + |x|^2 = 0$
11. Cauchy-Schwarz type inequality is $\max\{|\mathrm{Re}(xy)|, |\mathrm{Im}(xy)|\} \leq |x||y|$.
12. $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ for all $x, y \in \mathbb{H}$
13. If $\mathrm{Re}(x) = 0$, then $x^2 = -|x|^2$, we indicate a proof of $|xy| = |x||y|$:
    For, $|xy|^2 = xy\overline{xy} = xy\bar{y}\bar{x} = y\bar{y}x\bar{x} = |y|^2|x|^2$, for all $x, y \in \mathbb{H}$.

Thus, $\mathbb{H}$ is a division ring, i.e., a unital ring in which every nonzero element has a multiplicative inverse, and also a 4-dimensional algebra over the field of real numbers $\mathbb{R}$.

2 Quaternionic quadratic formulas

Consider the monic standard quadratic equation

$$x^2 + bx + c = 0,$$

where $b, c \in \mathbb{H}$ in general.

Lemma 2.1. *Let B, E, and D be real numbers such that*

1. *$D \neq 0$, and*
2. *$B < 0$ implies $B^2 < 4E$*

Then the cubic equation

$$y^3 + 2By^2 + (B^2 - 4E)y - D^2 = 0$$

has exactly one positive solution y

Proof. Let

$$f(y) = y^3 + 2By^2 + (B^2 - 4E)y - D^2 = 0$$

Note that $f(0) = -D^2 < 0$ and $\lim_{y \to +\infty} f(y) = +\infty$

According to the intermediate value Theorem: if f is a continuous function over an interval [a,b], then f takes all values between $f(a)$ and $f(b)$.
Since above cubic polynomial is a continuous function, its graph must intersect the x-axis at some finite point greater than zero. So the equation has at least one positive root.

Now let's prove that f has only one positive root.

Suppose that f has three real roots, r_1, r_2 and r_3. Take $r_1 > 0$ be the positive root we found above. We must show that $r_2, r_3 < 0$. Then we have the result. we know,

$$r_1.r_2.r_3 = D^2 > 0$$

This implies the product of r_2 and r_3 is positive. Therefore, r_2 and r_3 should be in same sign(both positive or negative).

Let's assume $r_2, r_3 > 0$
We know that,

If $B < 0$, then $r_1 + r_2 + r_3 = -2B > 0$ But

$$r_1.r_2 + r_1.r_3 + r_2.r_3 = B^2 - 4E < 0$$

This is a contradiction.Thus, we have only one positive solution to $f(y) = 0$ $\square$

Lemma 2.2. *Let $B, E,$ and D be real numbers such that*

1. *$E \geq 0$, and*
2. *$B < 0$ implies $B^2 < 4E$*

Then the real system

$$N^2 - (B + T^2)N + E = 0, \tag{2.1}$$
$$T^3 + (B - 2N)T + D = 0, \tag{2.2}$$

has at most two solutions (T, N) satisfying $T \in \mathbb{R}$ and $N \geq 0$ as follows.

(a) $T = 0,\ N = (B \pm \sqrt{B^2 - 4E})/2$ provided that $D = 0, B^2 \geq 4E$.
(b) $T = \pm\sqrt{2\sqrt{E} - B},\ N = \sqrt{E}$ provided that $D = 0, B^2 < 4E$.
(c) $T = \pm\sqrt{z},\ N = (T^3 + BT + D)/2T$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$.

Proof. (a) If $D = 0$ by (2.2) we have $T = 0$ and (2.1) gives $N = (B \pm \sqrt{B^2 - 4E})/2$

(b) If $D = 0$ and $B^2 < 4E$, by (2.2) we have

$$T(T^2 - (2N - B)) = 0$$

Therefore,

$$T = \pm\sqrt{2N - B}$$

By (8) we have,

$$N^2 - (B + 2N - B)N + E = 0$$
$$N^2 = E \quad \text{and we have} \quad N = \sqrt{E}$$

Hence,

$$T = \pm\sqrt{2\sqrt{E} - B} \quad \text{and} \quad N = \sqrt{E}$$

(c) If $D \neq 0$ from (2.2) $N = (T^3 + BT + D)/2T$ plug this N value to (2.1) we have

$$\Big(\frac{T^3 + BT + D}{2T}\Big)^2 - (B + T^2)\Big(\frac{T^3 + BT + D}{2T}\Big) + E = 0,$$
$$\Big(T^3 + BT + D\Big)^2 - 2T(B + T^2)\Big(T^3 + BT + D\Big) + 4T^2E = 0,$$
$$\Big(T^3 + BT + D\Big)^2 - 2T(B + T^2)\Big(T^3 + BT + D\Big) + 4T^2E = 0,$$
$$T^6 + 2BT^4 + (B^2 - 4E)T^2 - D^2 = 0,$$

Let $T = \pm\sqrt{z}$ Then we have z as unique positive root satisfies cubic equation $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$ $\square$

Theorem 2.3. . *The solutions of quaternionic quadratic equation* $x^2+bx+c=0$ *can be obtained by formulas according to the following cases.*

case 1. *If* $b, c \in \mathbb{R}$ *and* $b^2 < 4c$, *then*

$$x = \frac{1}{2}(-b + \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k})$$

$$\text{where } \ \alpha^2 + \beta^2 + \gamma^2 = 4c - b^2 \ \text{ and } \ \alpha, \beta, \gamma \in \mathbb{R}$$

case 2. *If* $b, c \in \mathbb{R}$ *and* $b^2 \geq 4c$, *then*

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

case 3. *If* $b \in \mathbb{R}$ *and* $c \notin \mathbb{R}$, *then*

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\boldsymbol{i} \mp \frac{c_2}{\rho}\boldsymbol{j} \mp \frac{c_3}{\rho}\boldsymbol{k},$$

$$\text{where } c = c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k} \ \text{ and}$$

$$\rho = \sqrt{\left(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)}\right)/2}$$

case 4. *If* $b \notin \mathbb{R}$, *then*

$$x = \frac{-Reb}{2} - (b' + T)^{-1}(c' - N),$$

where

$b' = b - Reb = Imb, c' = c - \big((Reb)/2)(b - (Reb)/2\big)$, *and* (T, N) *is chosen as follows.*

1. $T = 0, \ N = (B \pm \sqrt{B^2 - 4E})/2$ *provided that* $D = 0, \ B^2 \geq 4E$.
2. $T = \pm\sqrt{2\sqrt{E} - B}, \ N = \sqrt{E}$ *provided that* $D = 0, B^2 < 4E$.
3. $T = \pm\sqrt{z}, \ N = (T^3 + BT + D)/2T$ *provided that* $D \neq 0$ *and* z *is the unique positive root of the real polynomial* $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$.

where $B = b'\bar{b'} + c' + \bar{c'} = |b'|^2 + 2Rec', \ E = c'\bar{c'} = |c'|^2, D = \bar{b'}c' + \bar{c'}b' = 2Re\,\bar{b'}c'$ *are real numbers.*

Proof. **Case 1.** $b, c \in \mathbb{R}$ and $b^2 < 4c$. Note that x is a solution if and only if $q^{-1}xq$ is also a solution for $q \neq 0$, and there are at least two complex solutions

$$\frac{-b \pm \sqrt{4c - b^2}\mathbf{i}}{2}$$

Hence, the solution set is

Let $R^2 = 4c - b^2 > 0$ and $q \in \mathbb{H}$

$$\left\{ q^{-1}\Big(\frac{-b + \sqrt{4c - b^2}\mathbf{i}}{2}\Big)q : q \neq 0 \right\}$$

$$\left\{ q^{-1}\Big(\frac{-b + R\mathbf{i}}{2}\Big)q : q \neq 0 \right\}$$

$$\left\{ \frac{-b + q^{-1}R\mathbf{i}q}{2} : q \neq 0 \right\}$$

$$\left\{ \frac{-b + Rq^{-1}\mathbf{i}q}{2} : q \neq 0 \right\}$$

Now we have to prove that $Rq^{-1}\mathbf{i}q \in \mathbf{Im}\ \mathbb{H}$

For, $(Rq^{-1}\mathbf{i}q)(Rq^{-1}\mathbf{i}q) = R^2(q^{-1}\mathbf{i})(qq^{-1})(\mathbf{i}q) = -R^2$

This implies,

$$Rq^{-1}\mathbf{i}q = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \in \mathbb{H}$$

and

$$\overline{Rq^{-1}\mathbf{i}q} = -Rq^{-1}\mathbf{i}q$$

Therefore,

$$|Rq^{-1}\mathbf{i}q|^2 = (Rq^{-1}\mathbf{i}q)\overline{(Rq^{-1}\mathbf{i}q)} = -(Rq^{-1}\mathbf{i}q)(Rq^{-1}\mathbf{i}q) = R^2 = 4c - b^2$$

implies,

$$\alpha^2 + \beta^2 + \gamma^2 = 4c - b^2$$

Hence solution set is,

$$= \left\{ \frac{1}{2}(-b + \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}) : \alpha^2 + \beta^2 + \gamma^2 = 4c - b^2 \right\}$$

Case 2. $b, c \in \mathbb{R}$ and $b^2 \geq 4c$. Note that x is a solution if and only if $q^{-1}xq$ is also a solution for $q \neq 0$, and hence, there are at most two solutions, both are real

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Case 3. $b \in \mathbb{R}$ and $c \notin \mathbb{R}$. Let $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $x^2 + bx + c = 0$ becomes the real system

$$(x_0^2 - x_1^2 - x_2^2 - x_3^2) + bx_0 + c_0 = 0$$

$$\left(x_0^2 + \frac{b}{2}\right)^2 - x_1^2 - x_2^2 - x_3^2\Big) = \frac{b^2}{4} - c_0 \tag{2.3}$$

$$(2x_0 + b)x_1 = -c_1 \tag{2.4}$$

$$(2x_0 + b)x_2 = -c_2 \tag{2.5}$$

$$(2x_0 + b)x_3 = -c_3 \tag{2.6}$$

Since c is non-real from (2.4)-(2.6), $(2x_0 + b)$ is non-zero. Then by (2.3)

$$(2x_0 + b)^2 - 4x_1^2 - 4x_2^2 - 4x_3^2 = (b^2 - 4c_0)$$

$$(2x_0 + b)^4 - 4c_1^2 - 4c_2^2 - 4c_3^2 = (2x_0 + b)^2(b^2 - 4c_0)$$

$$(2x_0 + b)^4 - (b^2 - 4c_0)(2x_0 + b)^2 + 4(c_1^2 + c_2^2 + c_3^2) = 0$$

$$(2x_0 + b)^2 = \frac{(b^2 - 4c_0) \pm \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)}}{2}$$

$$(2x_0 + b) = \pm\sqrt{\frac{\left(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)}\right)}{2}}$$

$$x_0 = \frac{(-b \pm \rho)}{2} \qquad \text{where} \qquad \rho \neq 0$$

$$x = x_0 - \frac{c_1}{2x_0 + b}\mathbf{i} - \frac{c_2}{2x_0 + b}\mathbf{j} - \frac{c_3}{2x_0 + b}\mathbf{k}$$

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k}$$

Case 4. $b \notin \mathbb{R}$. Rewrite the equation $x^2 + bx + c = 0$ as

$$y^2 + b'y + c' = 0,$$

where $y = x + (\text{Re } b)/2$, $b' = b - \text{Re } b \notin \mathbb{R}$ and $c' = c - ((\text{Re } b)/2)\big(b - (\text{Re } b)/2\big)$. Following the idea of Niven [1], we observe that the solution of the quadratic equation $y^2 + b'y + c' = 0$ also satisfies

$$y^2 - Ty + N = 0,$$

where $N = \bar{y}y \geq 0$ and $T = y + \bar{y} \in \mathbb{R}$. Hence, $(b' + T)y + (c' - N) = 0$, and so

$$y = -(b' + T)^{-1}(c' - N)$$

because $T \in \mathbb{R}$ and $b' \notin \mathbb{R}$ implies that $b' + T \neq 0$ To solve for T and N, we substitute y back into the definitions $T = y + \bar{y}$ and $N = \bar{y}y$ and simplify to obtain the real system

$$N^2 - (B + T^2)N + E = 0,$$
$$T^3 + (B - 2N)T + D = 0,$$

where $B = b'\bar{b'} + c' + \bar{c'} = |b'|^2 + 2\text{Re}c'$, $E = c'\bar{c'} = |c'|^2, D = \bar{b'}c' + \bar{c'}b' = 2\text{Re } \bar{b'}c'$ are real numbers. Note that $E = |c'|^2 \geq 0$. if $B < 0$, then $c' + \bar{c'} < 0$ and $B^2 - 4E = |b'|^2B + |b'|^2(c' + \bar{c'}) + (c' - \bar{c'})^2 \leq 0$ because of the face that $(c' - \bar{c'})^2 \leq 0$. It follows that $B^2 - 4E < 0$, otherwise $B^2 - 4E = 0$ and so $|b'|^2B = |b'|^2(c' + \bar{c'}) = (c' - \bar{c'})^2 = 0$, i.e., $b' = 0 \in \mathbb{R}$, a contradiction. Hence, by lemma 2 such system can be solved explicitly as claimed.

Consequently, $x = (-\text{Re } b/2) - (b' + T)^{-1}(c' - T)$ $\square$

Example 1 Consider the equation $x^2 + 1 = 0$. Find solutions $x \in \mathbb{H}$

Solution

$b = 0$ and $c = 1$, therefore, $b^2 < 4c$. From theorem 1, case 1.

$$x = \frac{1}{2}(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$$

$$\text{where } \ \alpha^2 + \beta^2 + \gamma^2 = 4 \ \text{ and } \ \alpha, \beta, \gamma \in \mathbb{R}$$

so there are infinitely many solutions to the equation $x^2 + 1 = 0$ in $\mathbb{H}$

Example 2 Consider the equation $x^2 + 5x + 6 = 0$. Find solutions $x \in \mathbb{H}$

Solution

$b = 5$ and $c = 6$, therefore, $b^2 > 4c$. From theorem 1, case 2.

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 6}}{2} = \frac{-5 \pm 1}{2} = -3 \ \text{ or } -2$$

Example 3 Consider the equation $x^2 + x + (2 + 3\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}) = 0$. Find solutions $x \in \mathbb{H}$

Solution

$b = -1$ and $c = 2 + 3\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$, From theorem 1, case 3.

$$x = -\frac{1}{2} \pm \frac{\rho}{2} \mp \frac{3\mathbf{i}}{\rho} \mp \frac{6\mathbf{j}}{\rho} \mp \frac{5\mathbf{k}}{\rho}$$

where

$$\rho = \sqrt{\frac{-7 + \sqrt{1169}}{2}}$$

Example 4 Consider the equation $x^2 + \mathbf{k}x + \mathbf{j} = 0$. Find solutions $x \in \mathbb{H}$

Solution

$b = \mathbf{l} \notin \mathbb{R}$ and $c = \mathbf{j}$, From theorem 2, case 4

$$x = -\frac{\text{Re } b}{2} - (b' + T)^{-1}(c' - N)$$

since Re $b = 0$ we have $x = -(b' + T)^{-1}(c' - N)$

where

$b' = b - \text{Re } b = \mathbf{k}$

$c' = c - ((\text{Re } b)/2)\big(b - (\text{Re } b)/2\big) = \mathrm{j}$

$B = |b'|^2 + 2\text{Re } c' = 1$, $E = |c'|^2 = 1$ and $D = 2\text{Re }(\bar{b}'c') = 0$

note that $B^2 < 4E$

Therefore, $T = \pm\sqrt{2\sqrt{E} - B} = \pm 1$ and $N = \sqrt{E} = 1$

Thus,

$x = -(\mathbf{k}+1)^{-1}(\mathbf{j}-1)$ or $x = -(\mathbf{k}-1)^{-1}(\mathbf{j}-1)$

finally,

$$x = \frac{(\mathbf{k}-1)(1-\mathbf{j})}{-2} = \frac{\mathbf{k}+\mathbf{i}-1+\mathbf{j}}{-2} = \frac{1-\mathbf{i}-\mathbf{j}-\mathbf{k}}{2}.$$

or

$$x = \frac{(\mathbf{k}+1)(1-\mathbf{j})}{-2} = \frac{\mathbf{k}+\mathbf{i}+1-\mathbf{j}}{-2} = \frac{-1-\mathbf{i}+\mathbf{j}-\mathbf{k}}{2}.$$

3 Python and code implementation

Python is an interpreted, interactive, object-oriented programming language. It incorporates modules, exceptions, dynamic typing, very high level dynamic data types, and classes. It supports multiple programming paradigms beyond object-oriented programming, such as procedural and functional programming. Python combines remarkable power with very clear syntax. It has interfaces to many system calls and libraries, as well as to various window systems, and is extensible in C or C++. It is also usable as an extension language for applications that need a programmable interface. Moreover, Python is portable: it runs on many Unix variants including Linux and macOS, and on Windows.

The Python interpreter and the extensive standard library are freely available in source or binary form for all major platforms from the Python Web site, https://www.python.org[5] and may be freely distributed. The same site also contains distributions of and pointers to many free third party Python modules, programs and tools, and additional documentation. In this project, we use Jupyter Notebook which is an open source web application that you can use to create and share documents that contain live code, equations, visualizations, and text. Jupyter Notebooks are a spin-off project from the IPython project, which used to have an IPython Notebook project itself.

Consider the Quaternionic Quadratic Equation(QQE) of the form $x^2 + bx + c = 0$ where b and c are quaternions in general. We define function called "Quatquad" which take 8 values corresponding to real coefficients of quaternions b and c respectively. For example, if you want to find roots of $x^2 + bx + c = 0$ where $b = 1 + 2i + 3j + 4k$ and $c = 5 + 6i + 7j + 8k$. send values $1, 2, 3, 4, 5, 6, 7, 8$ to the function as "quatquad(1,2,3,4,5,6,7,8)". This will return roots of QQE $x^2 + (1 + 2i + 3j + 4k)x + (5 + 6i + 7j + 8k) = 0$. You can use this function to solve any QQE you wish. We include the python code and more examples below. For download python notebook, please refer my blogspot: https://kalpamadhawa.blogspot.com/

Python code for solving quaternionic quadratic equations

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[2]:
```
from math import sqrt
import numpy as np
import sympy as sym
sym.init_printing()
theta,alpha,rho,beta,u,i,j,k,I=sym.symbols('theta alpha rho beta u i j k I')
from pyquaternion import Quaternion


def quat_quad(p_0,p_1,p_2,p_3,p_00,p_11,p_22,p_33):

    b=Quaternion(p_0,p_1,p_2,p_3)                #define Quaternion
    c=Quaternion(p_00,p_11,p_22,p_33)

    if b[1]==0 and b[2]==0 and b[3]==0:            # when b is real

        if  c[1]==0 and c[2]==0 and c[3]==0:       # when c is real

            d=4*c[0]-b[0]**2

            if d>0:
                d_1=abs(sqrt(d/4))
                print("Roots of the Quadratic equation are:",-b[0]/2,"+I")
                print("where I is imaginary Quaternion with norm equal to",d_1)

            else:
                e=-d
                print("Roots of the Quadratic equation are:")
                print((-b[0]+sqrt(e))/2, "or",(-b[0]-sqrt(e))/2)

        else:                                        # when c is not real

            r_1=((b[0]**2-4*c[0])**2)+16*(c[1]**2 + c[2]**2 + c[3]**2)
            r_2=b[0]**2-4*c[0]


            print("Roots are of the form:")
            display((-b[0]/2)+(rho/2)-c[1]*(i/rho)-c[2]*(j/rho)-c[3]*(k/rho))
```

```
        print("OR")
        display((-b[0]/2)-(rho/2)+c[1]*(i/rho)+c[2]*(j/rho)+c[3]*(k/rho))


        h=sqrt(r_1)
        m=r_2+h
        p=sqrt(m/2)
        print("where")
        display(rho)
        print("=",p)

        x_1=(-b[0]/2)-(p/2)
        x_2= c[1]/p
        x_3= c[2]/p
        x_4=c[3]/p
        q_1=Quaternion(x_1,x_2,x_3,x_4)
        q_2=Quaternion(x_1,-x_2,-x_3,-x_4)
        print("Thus, Roots of the Quaternionic Quadratic equation can be␣
↪written as:")
        print(q_1,"and",q_2)

    else:                                       # when b is not real
        b_1=b-b[0]
        c_11=(b[0]/2)*(b-(b[0]/2))
        c_1=c-c_11
        B=(b_1.norm)**2 +2*c_1[0]
        E=(c_1.norm)**2
        D_1=(b_1.conjugate)*(c_1)
        D=2*D_1[0]

        if D==0:
            if B>=2*sqrt(E) or B<= -2*sqrt(E):
                T=0
                N_1=(B+sqrt(B**2 -4*E))/2
                N_2=(B-sqrt(B**2 -4*E))/2

                Q=b_1.inverse
                xx_1=(-b[0]/2)-Q*(c_1-N_1)
                xx_2=(-b[0]/2)-Q*(c_1-N_2)
                print("Roots of the Quaternionic Quadratic equation are")
                print(xx_1,"and",xx_2)

            else:
                T_1=sqrt(2*sqrt(E)-B)
                T_2=-sqrt(2*sqrt(E)-B)
                N=sqrt(E)
```

```
            Q_1=(b_1+T_1).inverse
            Q_2=(b_1+T_2).inverse
            xx_1=(-b[0]/2)-Q_1*(c_1-N)
            xx_2=(-b[0]/2)-Q_2*(c_1-N)
            print("Roots of the Quaternionic Quadratic equation are")
            print((xx_1),"and",(xx_2))
    else:

        p=np.poly1d([1,2*B,B**2 -4*E,-(D**2)])
        rootsp=p.r


        var=[0,1,2]

        for n in var:

            if rootsp[n]>0 :
                root=rootsp[n]

        T_1=sqrt(root)
        T_2=-sqrt(root)
        N_1=(T_1**3 +B*T_1+D)/(2*T_1)
        N_2=(T_2**3 +B*T_2+D)/(2*T_2)


        Q_1=(b_1+T_1).inverse
        Q_2=(b_1+T_2).inverse
        xx_1=(-b[0]/2)-Q_1*(c_1-N_1)
        xx_2=(-b[0]/2)-Q_2*(c_1-N_2)
        print("Roots of the Quaternionic Quadratic equation are :")
        print(xx_1,"and",xx_2)
```

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1 Example 1

$x^2 + (1 + 2i + 3j + 4k)x + (5 + 6i + 7j + 8k) = 0$

[6]:
```
quat_quad(1,2,3,4,5,6,7,8)
```

```
Roots of the Quaternionic Quadratic equation are :
0.775 -2.588i -3.344j -4.776k and -1.775 +0.362i +0.794j +0.550k
```

2 Example 2

$x^2 + 2x + 3 = 0$

[8]:
```
quat_quad(2,0,0,0,3,0,0,0)
```

```
Roots of the Quadratic equation are: -1.0 +I
where I is imaginary Quaternion with norm equal to 1.4142135623730951
```

3 Example 3

$x^2 + x + (2 + 3i + 6j + 5k) = 0$

[9]:
```
quat_quad(1,0,0,0,2,3,6,5)
```

Roots are of the form:

$$-\frac{3.0i}{\rho} - \frac{6.0j}{\rho} - \frac{5.0k}{\rho} + \frac{\rho}{2} - 0.5$$

OR

$$\frac{3.0i}{\rho} + \frac{6.0j}{\rho} + \frac{5.0k}{\rho} - \frac{\rho}{2} - 0.5$$

where

ρ

```
= 3.687183341966976
Thus, Roots of the Quaternionic Quadratic equation can be written as:
-2.344 +0.814i +1.627j +1.356k and -2.344 -0.814i -1.627j -1.356k
```

4 Example 4

$x^2 + kx + j = 0$

[10]:
```
quat_quad(0,0,0,1,0,0,1,0)
```

```
Roots of the Quaternionic Quadratic equation are
0.500 -0.500i -0.500j -0.500k and -0.500 -0.500i +0.500j -0.500k
```

References

[1] I. Niven. *Equations in quaternions*, American Math. Monthly 48, 654-661, (1941)

[2] L. Huang, W. So. *Quadratic formulas for quaternions*, preprint 2000.

[3] Mark Newman, *Computational Physics with python*, Department of Physics, University of Michigan ISBN 9781480145511.

[4] Tevian Dray and Corinne, A. Manogue. *The Geometry of the Octonions*, World Scientific, 2015.

[5] www.python.org, *official python website.*