

Disclaimer: I have discovered since the completion of this work that there is a 2007 article by K.M. Naralenkov in which he proves Theorem 1.0.2 for the case $p = 1$ and credits A. Pelczyński and G.C. da Rocha Filho with an unpublished generalization of Theorem 4.0.2. The main results of this work are then clearly **not** original to me, although I did derive them independently. I have therefore decided to make this work freely available through the KU ScholarWorks repository in order not only to highlight the still open problem of determining a general characterization for the Property of Lebesgue, but also to bring a satisfying end to a long personal journey.

Asymptotic- ℓ_p Banach Spaces and the Property of Lebesgue

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Abstract

The primary contribution of this work is to nearly characterize the Property of Lebesgue for Banach spaces that behave in a global asymptotic sense like ℓ_p . This generalizes a number of individual results that are collected by Russell Gordon in his 1991 survey article among other notable consequences and also raises the possibility of characterizing the Property of Lebesgue for more general Banach spaces in terms of their local asymptotic structures.

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1 Introduction

A Banach space X with the norm $\|\cdot\|$ is said to have a (Schauder) basis $(e_j)_{j=1}^\infty$ if, for all $x \in X$, there exists a unique sequence of scalars $(\lambda_j)_{j=1}^\infty$ such that

$$\limsup_{N \rightarrow \infty} \left\| x - \sum_{j=1}^N \lambda_j e_j \right\| = 0.$$

The set $\text{supp}(x) = \{j \in \mathbb{N} \mid \lambda_j = e_j^*(x) \neq 0\}$ is said to be the support of $x \in X$ where $e_j^* \in X^*$ denotes the j^{th} biorthogonal functional, the notation $x < y$ means that $\max \text{supp}(x) < \min \text{supp}(y)$, and $(x_i)_{i=1}^\infty$ is called a block sequence if $x_{i-1} < x_i$ holds. The basis $(e_j)_{j=1}^\infty$ is said to be semi-normalized if there exist constants $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \leq \|e_j\| \leq \alpha_2$ for all $j \in \mathbb{N}$ and it is

moreover said to be equivalent to another basis $(e'_j)_{j=1}^\infty$ (possibly in a different Banach space \tilde{X}) if there exist constants $\beta_1, \beta_2 > 0$ such that for all $N \in \mathbb{N}$

$$\beta_1 \left\| \sum_{j=1}^N \lambda_j e'_j \right\|_{\tilde{X}} \leq \left\| \sum_{j=1}^N \lambda_j e_j \right\| \leq \beta_2 \left\| \sum_{j=1}^N \lambda_j e'_j \right\|_{\tilde{X}} \quad (1.0.1)$$

for all scalar sequences $(\lambda_j)_{j=1}^N$. A relatively weak condition that still implies that $(e_j)_{j=1}^\infty$ is semi-normalized is the existence of a constant $\gamma > 0$ such that

$$\left\| \sum_{j \in A} e_j \right\| \leq \gamma \left\| \sum_{j \in B} e_j \right\| \quad (1.0.2)$$

for all finite subsets $A, B \subset \mathbb{N}$ with $|A| = |B|$ and the basis is said to be democratic in this case. Finally, there exists the following global asymptotic concept [3, Definition 2.7] of the proximity of X to ℓ_p .

Definition 1.0.1. Let $(e_j)_{j=1}^\infty$ be a basis for X and fix $p \in [1, \infty)$. If there exist constants $\zeta_1, \zeta_2 > 0$ such that for all $N \in \mathbb{N}$

$$\zeta_1 \left(\sum_{i=1}^N \|x_i\|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^N x_i \right\| \leq \zeta_2 \left(\sum_{i=1}^N \|x_i\|^p \right)^{\frac{1}{p}} \quad (1.0.3)$$

for all block sequences $(x_i)_{i=1}^N$ that satisfy $M = M_N \leq \min \text{supp}(x_1)$, then X is said to be asymptotic- ℓ_p with respect to $(e_j)_{j=1}^\infty$.

The specification of the particular basis with respect to which X is asymptotic- ℓ_p may be omitted if the basis is understood. In addition, there is an analogous definition if $p = \infty$ and X is said to be asymptotic- c_0 in this case because ℓ_∞ has no basis. This terminology, while perhaps more correct, is replaced by “asymptotic- ℓ_∞ ” in this work for notational ease.

The Banach space ℓ_p is, for each $p \in [1, \infty)$, trivially asymptotic- ℓ_p and c_0 is trivially asymptotic- ℓ_∞ . More interesting are those asymptotic- ℓ_p spaces that contain no isomorphic copy of ℓ_p . The prototypical example of such a Banach space is the well-known Figiel-Johnson Tsirelson space, \mathcal{T} , that is formulated as the completion of the set c_{00} of finitely-supported scalar sequences with respect to the implicitly-defined norm

$$\|x\|_{\mathcal{T}} = \max \left\{ \|x\|_\infty, \frac{1}{2} \sup \left\{ \sum_{i=1}^N \|E_i x\|_{\mathcal{T}} \mid N \in \mathbb{N} \right. \right. \\ \left. \left. \text{and } \{N\} \leq E_1 < E_2 < \dots < E_N \right\} \right\} \quad x = (x_j)_{j=1}^\infty \in c_{00}$$

where $E_i \subset \mathbb{N}$ is finite, $E_i x = \sum_{j \in E_i} x_j e_j$, and the notation $E < F$ means that $\max(E) < \min(F)$ for subsets $E, F \subset \mathbb{N}$. This Banach space is, despite its reflexivity, known to be asymptotic- ℓ_1 and to share the following somewhat obscure property with ℓ_1 as an apparent consequence.

Let μ be the Lebesgue measure. A time-honored analysis exercise is to prove, for real-valued functions on $[0, 1]$, that boundedness and μ -almost everywhere (μ -a.e.) continuity is equivalent to

both Darboux and Riemann integrability. It might then be a surprise that this need not be the case for X -valued functions on $[0, 1]$. More precisely, boundedness and μ -a.e. continuity is equivalent to Darboux integrability and implies Riemann integrability in this context, but a Riemann-integrable X -valued function on $[0, 1]$ can be everywhere discontinuous in general. If every Riemann-integrable X -valued function on $[0, 1]$ is μ -a.e. continuous, then X is said to be a PL-space where PL abbreviates ‘‘Property of Lebesgue.’’ It is known from [6, Theorems 26 and 27] that ℓ_1 and \mathcal{T} are PL-spaces and both proofs evidently rely on a 1-norm lower bound of the form (1.0.3). On the other hand, [6, Example 11] shows that ℓ_p is, for each $p \in (1, \infty)$, not a PL-space by means of a p -norm upper bound that can be (but is not necessarily) of the form (1.0.3). These results together motivate the primary contribution of this work, stated below.

Theorem 1.0.2. *Let X be asymptotic- ℓ_p with respect to $(e_j)_{j=1}^\infty$. Then, X is a PL-space if $p = 1$ and X is not a PL-space if $p > 1$ and if $(e_j)_{j=1}^\infty$ is democratic.*

This theorem is noteworthy for three reasons. First, it generalizes a number of individual results that are collected in [6]. Second, it leads to a non-basis-theoretic proof that c_0 and ℓ_p for each $p \in (1, \infty)$ do not embed isomorphically into an asymptotic- ℓ_1 Banach space. This fact is normally proved as an independent and basis-theoretic assertion (e.g. [5, Theorem 3.4.5]) but is here a corollary of Theorem 1.0.2 and the trivial observation that a PL-space contains no isomorphic copy of a non-PL-space. Third, the proof of Theorem 1.0.2 suggests that it can be generalized in terms of the local asymptotic structures of X such as its spreading models. This is because the derived Riemann sum estimates of the form (1.0.3) are uniform with respect to the constants ζ_1 and ζ_2 but would still be valid if these constants were allowed to depend on specific functions. A preliminary result along these lines concludes this work and asserts that a spreading model of a PL-space is equivalent in the sense of (1.0.1) to the canonical basis for ℓ_1 if it is non-trivial, unconditional, and generated by a democratic basic sequence.

The contents of Section 2 are uncelebrated but not novel and can be found, for instance, in [6]. They are included here in order to make this work more or less self-contained. On the other hand, the results and proofs of Section 3 and 4 are original to the best of my knowledge. It should, however, be reiterated that the proof of Theorem 1.0.2 does draw heavily on the methods of [6, Example 11 and Theorem 26 and 27]. The notation used throughout this work is common to many Banach space theory sources. Namely, X denotes an infinite-dimensional (real or complex) Banach space with the norm $\|\cdot\|$ and, if prescribed, the basis $(e_j)_{j=1}^\infty$. The letters Y and Z are subspaces of X and, if appropriate, general subsets thereof. Lowercase x, y , and z are vectors, $s, t \in [0, 1]$ are variables, and a variety of other letters (both English and Greek) are fixed constants. Finally, μ is reserved exclusively to be the Lebesgue measure and \mathbb{N} is the set of positive integers.

2 Preliminary Information

Readers who are familiar with the Darboux and Riemann integrals of X -valued functions on $[0, 1]$ and in particular with details that involve integrability may freely skip ahead to Section 3. Otherwise, recall the following definition.

Definition 2.0.1. A finite and strictly increasing sequence of real numbers $P = (p_i)_{i=0}^d$ is said to be a partition of $[0, 1]$ if $p_0 = 0$ and if $p_d = 1$.

A partition $P = (p_i)_{i=0}^d$ of $[0, 1]$ specifies, for each $i \in \{1, \dots, d\}$, the non-negative real numbers $\Delta_P(i) = p_i - p_{i-1} = \mu([p_{i-1}, p_i]) = \mu((p_{i-1}, p_i))$ and the maximum $\pi(P)$ of these numbers is said to

be its mesh size. If $\Delta_P(i) = \Delta_P(j) = \Delta_P$ for all $i \neq j$, then P is said to be regular. Any partition of $[0, 1]$ whose range contains $\text{ran}(P) = \{p_0, \dots, p_d\}$ as a subset is said to refine P and to every finite collection of partitions of $[0, 1]$, there corresponds a unique coarsest partition of $[0, 1]$ called the common refinement that refines each of them simultaneously.

2.1 Darboux Integrability

The Darboux integrability of a real-valued function on $[0, 1]$ is characterized by the convergence of its upper and lower Darboux sums to the same value (e.g. [11, Theorem 6.6]). These sums can only be defined if it makes sense to discuss infima and suprema, and this is not necessarily the case for subsets of X . If, however, $f : [0, 1] \rightarrow \mathbb{R}$ is bounded, then

$$\sup_{s \in I} f(s) - \inf_{s \in I} f(s) = \sup_{s, s' \in I} |f(s) - f(s')|$$

for all non-empty and compact sub-intervals $I \subset [0, 1]$. This motivates a slightly more nuanced approach to Darboux integrability. For convenience, let

$$\mathcal{B}([0, 1], X) = \left\{ f : [0, 1] \rightarrow X \mid \sup_{s \in [0, 1]} \|f(s)\| < \infty \right\}$$

be the collection of bounded X -valued functions on $[0, 1]$.

Definition 2.1.1. Let $f \in \mathcal{B}([0, 1], X)$, $s_0 \in [0, 1]$, and $\delta > 0$. The non-negative real number defined by

$$\omega_f[\mathcal{N}_\delta(s_0)] = \sup \{ \|f(s) - f(s')\| \mid s, s' \in \mathcal{N}_\delta(s_0) \}$$

is said to be the oscillation of f with respect to the sub-interval $\mathcal{N}_\delta(s_0) = [s_0 - \delta, s_0 + \delta] \cap [0, 1]$.

A natural generalization of [11, Theorem 6.6] is now possible by means of Definition 2.1.1.

Definition 2.1.2. Let $f \in \mathcal{B}([0, 1], X)$. If, for all $\varepsilon > 0$, there exists a partition $P_\varepsilon = P = (p_i)_{i=0}^d$ of $[0, 1]$ such that

$$\sum_{i=1}^d \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] \leq \varepsilon \tag{2.1.1}$$

where $\delta_i = \frac{\Delta_P(i)}{2}$ and $s_i = p_{i-1} + \delta_i$ for each $i \in \{1, \dots, d\}$, then f is said to be Darboux-integrable.

This definition would reduce exactly to [11, Theorem 6.6] if X were equal to \mathbb{R} and would in that case be equivalent to boundedness in μ -a.e. continuity. If $f \in \mathcal{B}([0, 1], X)$, then the facts:

- $\inf_{\delta > 0} \omega_f[\mathcal{N}_\delta(s)] = 0$ if and only if f is continuous at s
- $\Omega_f(\lambda) = \{s \in [0, 1] \mid \inf_{\delta > 0} \omega_f[\mathcal{N}_\delta(s)] < \frac{1}{\lambda}\}$ is a relatively open (and therefore μ -measurable) subset of $[0, 1]$ for all $\lambda > 0$
- If $H \subset \mathbb{R}$, then $\mu(H) = 0$ if and only if, for all $\varepsilon > 0$, there exist open intervals U_1, U_2, \dots that both cover H and satisfy $\sum_{j=1}^{\infty} \mu(U_j) < \varepsilon$

permit a characterization of the Darboux-integrable X -valued functions on $[0, 1]$, $\mathcal{D}([0, 1], X)$, that mirrors the real-valued situation.

Theorem 2.1.3. *Let $f \in \mathcal{B}([0, 1], X)$. Then, $f \in \mathcal{D}([0, 1], X)$ if and only if it is μ -a.e. continuous.*

Proof. The set $H_\lambda = [0, 1] \setminus \Omega_f(\lambda)$ is well-defined, μ -measurable, and compact in \mathbb{R} for all $\lambda > 0$. Moreover,

$$H = \{s \in [0, 1] \mid f \text{ is discontinuous at } s\} = \left\{s \in [0, 1] \mid \inf_{\delta > 0} \omega_f[\mathcal{N}_\delta(s)] > 0\right\} = \bigcup_{n=1}^{\infty} H_n$$

implies that H is μ -measurable and, if $\mu(H_n) = 0$ for all $n \in \mathbb{N}$, that $\mu(H) = 0$.

Let $f \in \mathcal{D}([0, 1], X)$ and suppose for a contradiction that $\mu(H_{n_0}) > 0$ for some $n_0 \in \mathbb{N}$. It follows that $\sum_{j=1}^{\infty} \mu(U_j) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ and for all open intervals U_1, U_2, \dots that cover H_{n_0} . Note that

$$H_{n_0} \subset \left(\bigcup_{i \in A} (p_{i-1}, p_i) \right) \cup (p_0 - \varepsilon, p_0 + \varepsilon) \cup \left(p_1 - \frac{\varepsilon}{2}, p_1 + \frac{\varepsilon}{2} \right) \cup \dots \cup \left(p_d - \frac{\varepsilon}{2^d}, p_d + \frac{\varepsilon}{2^d} \right)$$

where $\varepsilon \in \left(0, \frac{\varepsilon_0}{5n_0}\right)$ is chosen freely, $P_\varepsilon = P = (p_i)_{i=0}^d$ is a partition of $[0, 1]$ such that (2.1.1) holds, and $A = \{i \mid (p_{i-1}, p_i) \cap H_{n_0} \neq \emptyset\}$. This obtains

$$\frac{\varepsilon_0}{n_0} \leq \frac{1}{n_0} \left(\sum_{i \in A} \mu((p_{i-1}, p_i)) + \sum_{i=0}^d \frac{\varepsilon}{2^{i-1}} \right) \leq 4\varepsilon + \sum_{i \in A} \Delta_P(i) \frac{1}{n_0} \leq 4\varepsilon + \sum_{i \in A} \Delta_P(i) \omega_f[\mathcal{N}_{\rho_i}(t_i)] \quad (2.1.2)$$

where $t_i \in (p_{i-1}, p_i) \cap H_{n_0}$ and $\rho_i > 0$ is such that $\mathcal{N}_{\rho_i}(t_i) \subset (p_{i-1}, p_i)$. Now, (2.1.2) can be bounded above by

$$4\varepsilon + \sum_{i=1}^d \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] \leq 4\varepsilon + \varepsilon = 5\varepsilon$$

where $\delta_i = \frac{\Delta_P(i)}{2}$ and $s_i = p_{i-1} + \delta_i$ for each $i \in \{1, \dots, d\}$. This is evidently a contradiction so $\mu(H_n) = 0$ for all $n \in \mathbb{N}$ and $\mu(H) = 0$ as required.

Conversely, let $\mu(H) = 0$ and let $\varepsilon \in (0, 1)$ be given. Fix $\lambda = \frac{2+\theta}{\varepsilon}$ where $\theta = 1 + 2 \sup_{s \in [0, 1]} \|f(s)\|$ and note that $H_\lambda \subset H$ is both μ -measurable and compact in \mathbb{R} . There are then finitely-many open intervals U_1, \dots, U_N that cover H_λ and satisfy $\sum_{j=1}^N \mu(U_j) < \frac{1}{\lambda}$.

It follows that $\Omega_f(\lambda) \supset [0, 1] \setminus \left(\bigcup_{j=1}^N U_j \right) = G$ is both non-empty and, in its own right, compact in \mathbb{R} . Therefore,

$$[0, 1] \subset \left(\bigcup_{j=1}^N U_j \right) \cup \left(\bigcup_{l=1}^M [t_l - \rho_l, t_l + \rho_l] \right) \quad (2.1.3)$$

for some finitely-many $t_l \in G$ and $\rho_l > 0$ such that $\omega_f[\mathcal{N}_{\rho_l}(t_l)] < \frac{1}{\lambda}$. Intersecting (2.1.3) with $[0, 1]$ now yields

$$[0, 1] = \left(\bigcup_{j=1}^N \mathcal{U}_j \right) \cup \left(\bigcup_{l=1}^M \mathcal{N}_{\rho_l}(t_l) \right) \quad \mathcal{U}_j = U_j \cap [0, 1]$$

and define $E = \{s \in [0, 1] \mid s \text{ is an endpoint of either some } \mathcal{U}_j \text{ or of some } \mathcal{N}_{\rho_l}(t_l)\}$. Note that $|E| \leq 2(N + M)$ and let $\sigma = \frac{1}{4\lambda\theta(N+M)}$.

Let $P = (p_i)_{i=0}^d$ be a partition of $[0, 1]$ with $\pi(P) < \sigma$ and deposit each non-zero index into one of three pairwise disjoint sets:

$$\begin{aligned} A_1 &= \{i \mid [p_{i-1}, p_i] \cap E \neq \emptyset\} \\ A_2 &= \{i \mid i \notin A_1 \text{ and } [p_{i-1}, p_i] \subsetneq \mathcal{U}_j \text{ for some } j\} \\ A_3 &= \{i \mid i \notin A_1 \cup A_2\} \end{aligned}$$

It is evident that $|A_1| \leq 4(N + M)$ and that each $i \in A_3$ corresponds to a sub-interval of the form $[p_{i-1}, p_i] \subsetneq \mathcal{N}_{\rho_l}(t_l)$ for some l . Consequently,

$$\begin{aligned} \sum_{i=1}^d \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] &= \sum_{i \in A_1} \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] + \sum_{i \in A_2} \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] \\ &\quad + \sum_{i \in A_3} \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] \leq 4\sigma\theta(N + M) + \frac{\theta}{\lambda} + \frac{1}{\lambda} = \frac{2 + \theta}{\lambda} = \varepsilon \end{aligned}$$

where $\delta_i = \frac{\Delta_P(i)}{2}$ and $s_i = p_{i-1} + \delta_i$ for each $i \in \{1, \dots, d\}$. This completes the proof of Theorem 2.1.3 because $P = P_\varepsilon$ as in Definition 2.1.2. \square

Theorem 2.1.3 shows that $\mathcal{D}([0, 1], X) \subset \mathcal{B}([0, 1], X)$ as a subspace and its proof does not require the completeness of X . On the other hand, the assumption that X is a Banach space is essential for defining the actual Darboux integral of a given $f \in \mathcal{D}([0, 1], X)$. This can be done with the results of the next subsection.

2.2 Riemann Integrability

The Riemann integrability of a real-valued function on $[0, 1]$ is characterized by the convergence of its Riemann sums to the same value (e.g. [7, Definition 11.56]). Recall that (P, T) is said to be a tagged partition of $[0, 1]$ if $P = (p_i)_{i=0}^d$ is a partition of $[0, 1]$ and if $T = (t_i)_{i=1}^d$ is such that $t_i \in [p_{i-1}, p_i]$ for each $i \in \{1, \dots, d\}$. In the case that $f : [0, 1] \rightarrow X$, the vector

$$S_f(P, T) = \sum_{i=1}^d \Delta_P(i) f(t_i)$$

is said to be the Riemann sum of f with respect to (P, T) and the Riemann integrability of f may be assessed in the following familiar manner.

Definition 2.2.1. A function $f : [0, 1] \rightarrow X$ is said to be Riemann-integrable if there exists a vector $x_f \in X$ such that for all $\varepsilon > 0$, there is a $\delta = \delta_\varepsilon > 0$ so that

$$\|x - S_f(P, T)\| \leq \varepsilon$$

for all tagged partitions (P, T) of $[0, 1]$ that satisfy $\pi(P) < \delta$.

It is immediate that $\mathcal{R}([0, 1], X) \subset \mathcal{B}([0, 1], X)$ as a subspace where $\mathcal{R}([0, 1], X)$ denotes the set of Riemann-integrable X -valued functions on $[0, 1]$. Moreover, the vector $x_f \in X$ is unique so that $f \mapsto x_f$ is a well-defined linear function. This function is said to be the Riemann integral of

X -valued functions on $[0, 1]$. Noting now that the convex hull of a general subset $Y \subset X$ is given by

$$\text{co}(Y) = \left\{ \sum_{k=1}^N \lambda_k y_k \mid N \in \mathbb{N}, y_k \in Y, \text{ and } \lambda_k \in (0, 1) \text{ with } \sum_{k=1}^N \lambda_k = 1 \right\}$$

and has the property that $\sum_{i=1}^d \text{co}(Y_i) = \text{co}\left(\sum_{i=1}^d Y_i\right)$ for all finite Minkowski sums of subsets of X , Definition 2.2.1 can be reformulated in several ways.

Theorem 2.2.2. *Let $f : [0, 1] \rightarrow X$. Then, the following assertions are equivalent.*

- (i) *The function f is Riemann-integrable.*
- (ii) *There exists a vector $x_f \in X$ such that for all $\varepsilon > 0$, there is a partition P_ε of $[0, 1]$ so that $\|x_f - S_f(P, T)\| \leq \varepsilon$ for all tagged partitions (P, T) of $[0, 1]$ where P refines P_ε .*
- (iii) *For all $\varepsilon > 0$, there exists a partition P_ε of $[0, 1]$ such that $\|S_f(P_1, T_1) - S_f(P_2, T_2)\| \leq \varepsilon$ for all tagged partitions (P_1, T_1) and (P_2, T_2) where P_1 and P_2 refine P_ε .*
- (iv) *For all $\varepsilon > 0$, there exists a partition P_ε of $[0, 1]$ such that $\|S_f(P_1, T_1) - S_f(P_2, T_2)\| \leq \varepsilon$ for all tagged partitions (P_1, T_1) and (P_2, T_2) where $P_1 = P_2 = P_\varepsilon$.*

Proof. It is clear that (i) \implies (ii) \implies (iii) \implies (iv) so it suffices to prove the converse implications. To that end, let $\varepsilon > 0$ be given and let $P_\varepsilon = (p_i)_{i=0}^d$ be a partition of $[0, 1]$ as in (iv). Define, for each $i \in \{1, \dots, d\}$, the subsets of X given by

$$Y_i = \{\Delta_P(i)(f(s) - f(s')) \mid s, s' \in [p_{i-1}, p_i]\}$$

and note that if $z \in \text{co}\left(\sum_{i=1}^d Y_i\right)$, then

$$\|z\| = \left\| \sum_{k=1}^N \lambda_k y_k \right\| \leq \sum_{k=1}^N \lambda_k \|y_k\| = \sum_{k=1}^N \lambda_k \left\| \sum_{i=1}^d \Delta_P(i)(f(s_{k,i}) - f(s'_{k,i})) \right\| \leq \varepsilon \sum_{k=1}^N \lambda_k = \varepsilon \quad (2.2.1)$$

for some vectors $y_k \in \sum_{i=1}^d Y_i$ and some scalars $\lambda_k \in (0, 1)$ with $\sum_{k=1}^N \lambda_k = 1$. Now, fix $T_\varepsilon = (t_i)_{i=1}^d$ so that $(P_\varepsilon, T_\varepsilon)$ is a tagged partition of $[0, 1]$ and let (P', T') be a tagged partition of $[0, 1]$ such that $P' = (p'_l)_{l=0}^{d'}$ refines P_ε . It follows that

$$\begin{aligned} \|S_f(P_\varepsilon, T_\varepsilon) - S_f(P', T')\| &= \left\| \sum_{i=1}^d \Delta_P(i) f(t_i) - \sum_{l=1}^{d'} \Delta_{P'}(l) f(t'_l) \right\| \\ &= \left\| \sum_{i=1}^d \left(\sum_{l \in A_i} \Delta_{P'}(l) \right) f(t_i) - \sum_{i=1}^d \sum_{l \in A_i} \Delta_{P'}(l) f(t'_l) \right\| \end{aligned} \quad (2.2.2)$$

where $A_i = \{l \mid [p'_{l-1}, p'_l] \subset [p_{i-1}, p_i]\}$ for each $i \in \{1, \dots, d\}$. The quantity (2.2.2) is equivalent to

$$\left\| \sum_{i=1}^d \sum_{l \in A_i} \Delta_{P'}(l) (f(t_i) - f(t'_l)) \right\| = \left\| \sum_{i=1}^d \sum_{l \in A_i} \frac{\Delta_{P'}(l)}{\Delta_P(i)} \Delta_P(i) (f(t_i) - f(t'_l)) \right\| = \left\| \sum_{i=1}^d z_i \right\| \quad (2.2.3)$$

where $z_i = \sum_{l \in A_i} \frac{\Delta_{P'}(l)}{\Delta_P(i)} \Delta_P(i)(f(t_i) - f(t'_l)) \in \text{co}(Y_i)$. Consequently, $z = \sum_{i=1}^d z_i \in \sum_{i=1}^d \text{co}(Y_i) = \text{co}\left(\sum_{i=1}^d Y_i\right)$ so (2.2.3) is bounded above by ε according to (2.2.1). The implication (iv) \implies (iii) now follows by an application of the triangle inequality.

Next, define $\varepsilon_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ and let P_{ε_n} be a partition of $[0, 1]$ as in (iii). Consider any sequence $((P_n, T_n))_{n=1}^{\infty}$ of tagged partitions of $[0, 1]$ where P_n is the common refinement of the set of partitions $P_{\varepsilon_1}, \dots, P_{\varepsilon_n}$. It follows that the sequence of Riemann sums $(S_f(P_n, T_n))_{n=1}^{\infty}$ with respect to these common refinements is Cauchy in X and that $\lim_{n \rightarrow \infty} S_f(P_n, T_n) = x_f$ satisfies (ii) by an application of the triangle inequality.

Finally, let $\varepsilon > 0$ be given and let P_ε be a partition of $[0, 1]$ as in (ii). Define $\sigma = |\text{ran}(P_\varepsilon)|$ and note that $f \in \mathcal{B}([0, 1], X)$ by means of (ii). Let $P = (p_i)_{i=0}^d$ be a partition of $[0, 1]$ with $\pi(P) < \frac{\varepsilon}{\sigma\theta}$ where $\theta = 1 + 2 \sup_{s \in [0, 1]} \|f(s)\|$ and consider the set of indices

$$A = \{i \mid (p_{i-1}, p_i) \cap \text{ran}(P_\varepsilon) \neq \emptyset\}.$$

It follows that $|A| \leq \sigma - 2$ and that p_{i-1} and p_i are consecutive terms of the common refinement $P' = (p'_l)_{l=0}^d$ of P and P_ε if $i \notin A$. Let $T = (t_i)_{i=1}^d$ be given so that (P, T) is an arbitrary tagged partition of $[0, 1]$ that satisfies $\pi(P) < \frac{\varepsilon}{\sigma\theta}$ and choose $T' = (t'_l)_{l=1}^d$ so that $t'_l = t_i$ if $i \notin A$. Then,

$$\begin{aligned} \|S_f(P, T) - S_f(P', T')\| &= \left\| \sum_{i=1}^d \Delta_P(i) f(t_i) - \sum_{l=1}^d \Delta_{P'}(l) f(t'_l) \right\| \\ &= \left\| \sum_{i=1}^d \left(\sum_{l \in A_i} \Delta_{P'}(l) \right) f(t_i) - \sum_{i=1}^d \sum_{l \in A_i} \Delta_{P'}(l) f(t'_l) \right\| \end{aligned} \quad (2.2.4)$$

where $A_i = \{l \mid [p'_{l-1}, p'_l] \subset [p_{i-1}, p_i]\}$ for each $i \in \{1, \dots, d\}$ as in (2.2.2) and (2.2.4) is therefore equivalent to

$$\begin{aligned} \left\| \sum_{i=1}^d \sum_{l \in A_i} \Delta_{P'}(l) (f(t_i) - f(t'_l)) \right\| &= \left\| \sum_{i \in A} \sum_{l \in A_i} \Delta_{P'}(l) (f(t_i) - f(t'_l)) \right\| \\ &\leq \sum_{i \in A} \sum_{l \in A_i} \Delta_{P'}(l) \|f(t_i) - f(t'_l)\| \leq \sum_{i \in A} \theta \Delta_P(i) \leq \sum_{i \in A} \theta \frac{\varepsilon}{\sigma\theta} \leq \varepsilon \end{aligned} \quad (2.2.5)$$

by the definition of T' . The estimate (2.2.5) and the fact that P' refines P_ε now obtain that

$$\|x_f - S_f(P, T)\| \leq \|x_f - S_f(P', T')\| + \|S_f(P', T') - S_f(P, T)\| \leq \varepsilon + \varepsilon = 2\varepsilon$$

where $x_f \in X$ is the vector provided by (ii). This completes the proof of Theorem 2.2.2. \square

The general relationship between the subspaces of Darboux and Riemann-integrable X -valued functions on $[0, 1]$ can be clarified as a result of Theorem 2.2.2.

Corollary 2.2.3. *The inclusion $\mathcal{D}([0, 1], X) \subset \mathcal{R}([0, 1], X)$ is valid.*

Proof. Let $f \in \mathcal{D}([0, 1], X)$, $\varepsilon > 0$, and $P_\varepsilon = P$ be a partition of $[0, 1]$ with $d + 1$ terms such that (2.1.1) holds. If (P_1, T_1) and (P_2, T_2) are tagged partitions of $[0, 1]$ with $P_1 = P_2 = P$, then

$$\|S_f(P_1, T_1) - S_f(P_2, T_2)\| \leq \sum_{i=1}^d \Delta_P(i) \|f(t_{1,i}) - f(t_{2,i})\| \leq \sum_{i=1}^d \Delta_P(i) \omega_f[\mathcal{N}_{\delta_i}(s_i)] \leq \varepsilon$$

where $\delta_i = \frac{\Delta_P(i)}{2}$ and $s_i = p_{i-1} + \delta_i$ for each $i \in \{1, \dots, d\}$. The fourth assertion of Theorem 2.2.2 now implies that $f \in \mathcal{R}([0, 1], X)$. \square

The Darboux integral of X -valued functions on $[0, 1]$ is, in view of Corollary 2.2.3, defined to be the restriction of the Riemann integral $f \mapsto x_f$ to the subspace $\mathcal{D}([0, 1], X)$. In particular, the assumption that X is a Banach space is essential for this definition because it is required for the implication (iii) \implies (ii) in Theorem 2.2.2 and is therefore necessary for the equivalence of the fourth assertion of Theorem 2.2.2 to Definition 2.2.1. Corollary 2.2.3 moreover begs the question of when its asserted inclusion holds with equality.

3 Main Result

To say that $\mathcal{D}([0, 1], X) \subset \mathcal{R}([0, 1], X)$ holds with equality is to say that X has the Property of Lebesgue. This condition can be formulated in terms of Theorem 2.1.3 as is noted in Section 1 and restated below.

Definition 3.0.1. The Banach space X is said to be a PL-space if every $f \in \mathcal{R}([0, 1], X)$ is μ -a.e. continuous.

All finite-dimensional normed vector spaces are PL-spaces while infinite-dimensional Banach spaces may or may not have the Property of Lebesgue. The aim of this section is to prove Theorem 1.0.2 which generalizes not only [6, Example 11 and Theorems 26 and 27], but also [6, Example 10] and indirectly portions of [6, Corollary 24 and Theorem 25]. This theorem applies in addition to certain Orlicz sequence spaces and \mathcal{T} -like spaces that are not mentioned in [6].

Proof of Theorem 1.0.2. Let X be asymptotic- ℓ_p with respect to the basis $(e_j)_{j=1}^\infty$. Suppose first that $p > 1$ and that $(e_j)_{j=1}^\infty$ is democratic. The basis democracy implies that there exist constants $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \leq \|e_j\| \leq \alpha_2$ for all $j \in \mathbb{N}$ (i.e. $(e_j)_{j=1}^\infty$ is semi-normalized). Let r_1, r_2, \dots be a listing of $\mathbb{Q} \cap [0, 1]$ and define $f : [0, 1] \rightarrow X$ by

$$f(s) = \begin{cases} e_j & \text{if } s = r_j \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } s \text{ is irrational} \end{cases}$$

so that f is discontinuous everywhere on $[0, 1]$ and therefore not Darboux-integrable by Theorem 2.1.3. On the other hand, let $\varepsilon > 0$ be given and let P be a regular partition of $[0, 1]$ with $d + 1$ terms such that $\Delta_P < \varepsilon^{\frac{p}{p-1}}$ in the case $p \in (1, \infty)$ and such that $\Delta_P < \varepsilon$ in the case $p = \infty$. Let $T_1 = (t_{1,i})_{i=1}^d$ and $T_2 = (t_{2,i})_{i=1}^d$ be given so that (P, T_1) and (P, T_2) are tagged partitions of $[0, 1]$ and note that

$$\begin{aligned} \|S_f(P, T_1) - S_f(P, T_2)\| &= \left\| \sum_{i=1}^d \Delta_P (f(t_{1,i}) - f(t_{2,i})) \right\| \\ &\leq \left\| \sum_{k=1}^{d'} \Delta_P f(t_{1,i_k}) \right\| + \left\| \sum_{l=1}^{d''} \Delta_P f(t_{2,i_l}) \right\| \end{aligned} \quad (3.0.1)$$

where $(i_k)_{k=1}^{d'}$ and $(i_l)_{l=1}^{d''}$ are the subsequences of indices such that $t_{1,i_k}, t_{2,i_l} \in \mathbb{Q} \cap [0, 1]$. The two summands on the right-hand side of (3.0.1) may be bounded above in an identical manner so it

suffices to derive a suitable upper bound for the first summand. Let $e_k = f(t_{1,i_k})$, $\gamma > 0$ be as in (1.0.2), and $\zeta_2 > 0$ and $M = M_{d'} \in \mathbb{N}$ be specified as in Definition 1.0.1. This obtains

$$\begin{aligned} \left\| \sum_{k=1}^{d'} \Delta_P e_k \right\| &= \Delta_P \left\| \sum_{k=1}^{d'} e_k \right\| \leq \Delta_P \gamma \left\| \sum_{k'=1}^{d'} e_{M+k'} \right\| = \gamma \left\| \sum_{k'=1}^{d'} \Delta_P e_{M+k'} \right\| \\ &\leq \gamma \zeta_2 \left(\sum_{k'=1}^{d'} \Delta_P^p \|e_{M+k'}\|^p \right)^{\frac{1}{p}} \leq \gamma \zeta_2 \alpha_2 \Delta_P^{\frac{p-1}{p}} \left(\sum_{k'=1}^{d'} \Delta_P \right)^{\frac{1}{p}} \leq \gamma \zeta_2 \alpha_2 \varepsilon = C\varepsilon \end{aligned}$$

in the case $p \in (1, \infty)$ where $\{e_{k'}\}_{k'=1}^{d'} = \{e_1, \dots, e_{d'}\}$ and $C = \gamma \zeta_2 \alpha_2 \geq 0$. It follows that (3.0.1) is bounded above by $2C\varepsilon$ so that $f \in \mathcal{R}([0, 1], X)$ and the estimate in the case $p = \infty$ is identical.

Suppose in contrast that $p = 1$. It suffices in this case to prove that for every function $f \in \mathcal{B}([0, 1], X)$ whose set of discontinuities has positive Lebesgue measure, there exists a constant $c_f > 0$ such that for all partitions P of $[0, 1]$

$$\|S_f(P, T_1) - S_f(P, T_2)\| \geq c_f$$

for some sequences T_1 and T_2 so that (P, T_1) and (P, T_2) are tagged partitions of $[0, 1]$. Indeed, let f be a bounded X -valued function on $[0, 1]$ and let $\mu(H) > 0$ where H is its set of discontinuities. Recall that $H = \bigcup_{n=1}^{\infty} H_n$ where $H_n = [0, 1] \setminus \Omega_f(n)$ as in Theorem 2.1.3 so it follows that

$$H_{n_0} = \left\{ s \in [0, 1] \mid \inf_{\delta > 0} \omega_f[\mathcal{N}_\delta(s)] \geq \frac{1}{n_0} \right\}$$

has positive Lebesgue measure for some $n_0 \in \mathbb{N}$ or else $\mu(H) = 0$. Define the sets

$$G_j = \{s \in [0, 1] \mid (e_j^* \circ f)(s) : [0, 1] \rightarrow \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\} \text{ is discontinuous}\}$$

for each $j \in \mathbb{N}$. It follows that $(e_j^* \circ f) \notin \mathcal{R}([0, 1], \mathbb{F})$ if $\mu(G_j) > 0$ because $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and because $\sup_{s \in [0, 1]} |(e_j^* \circ f)(s)| \leq \|e_j^*\|_{\text{op}} \sup_{s \in [0, 1]} \|f(s)\| < \infty$. In particular, $f \notin \mathcal{R}([0, 1], X)$ in this case or else its composition with $e_j^* \in X^*$ would be Riemann-integrable. Assume then that $\mu(G_j) = 0$ for each $j \in \mathbb{N}$ and note that $\mu(G) = 0$ for $G = \bigcup_{j=1}^{\infty} G_j$ as well. Now, let $P = (p_i)_{i=0}^d$ be a partition of $[0, 1]$ and define the non-empty set

$$A_{n_0} = \{i \mid \mu((p_{i-1}, p_i) \cap (H_{n_0} \setminus G)) > 0\}$$

where $(e_j^* \circ f)(s) : [0, 1] \rightarrow \mathbb{F}$ is continuous on $H_{n_0} \setminus G$ for each $j \in \mathbb{N}$. Let i_1, \dots, i_r be the members of A_{n_0} ($r \leq d$) and let $\zeta_1 > 0$ and $M = M_r \in \mathbb{N}$ be as in Definition 1.0.1. Define $m_0 = \max\{1, M - 1\}$ and $\varepsilon = \frac{\zeta_1 \mu(H_{n_0} \setminus G)}{16n_0(1 + \zeta_1)} > 0$.

Choose $s_1 \in (p_{i_1-1}, p_{i_1}) \cap (H_{n_0} \setminus G)$ and, because $\inf_{\delta > 0} \omega_f[\mathcal{N}_\delta(s_1)] \geq \frac{1}{n_0}$, note that for a small enough $\delta_1 > 0$ and for some $u_1, v_1 \in \mathcal{N}_{\delta_1}(s_1) \subset (p_{i_1-1}, p_{i_1})$,

$$\|z_1\| \geq \frac{1}{2n_0} \quad \text{and} \quad \left\| \sum_{j=1}^{m_0} e_j^*(z_1) e_j \right\| \leq \theta_0 \sum_{j=1}^{m_0} |e_j^*(z_1)| < \varepsilon$$

where $z_1 = f(u_1) - f(v_1)$ and $\theta_0 = \max_{1 \leq j \leq m_0} \|e_j\|$ by the continuity of $(e_j^* \circ f)(s) : [0, 1] \rightarrow \mathbb{F}$ for each $j \in \{1, \dots, m_0\}$ and approximating the supremum $\omega_f[\mathcal{N}_{\delta_1}(s_1)]$, respectively. Fix $m_1 > m_0$

so that the series tail satisfies the estimate $\left\| \sum_{j=m_1}^{\infty} e_j^*(z_1)e_j \right\| < \varepsilon$. Proceeding analogously, choose $s_2 \in (p_{i_2-1}, p_{i_2}) \cap (H_{n_0} \setminus G)$ and $u_2, v_2 \in (p_{i_2-1}, p_{i_2})$ such that

$$\|z_2\| \geq \frac{1}{2n_0} \text{ and } \left\| \sum_{j=1}^{m_1} e_j^*(z_2)e_j \right\| \leq \theta_1 \sum_{j=1}^{m_1} |e_j^*(z_2)| < \frac{\varepsilon}{2}$$

where $z_2 = f(u_2) - f(v_2)$ and $\theta_1 = \max_{1 \leq j \leq m_1} \|e_j\|$, and fix $m_2 > m_1$ so that the series tail satisfies the estimate $\left\| \sum_{j=m_2}^{\infty} e_j^*(z_2)e_j \right\| < \frac{\varepsilon}{2}$. It now follows by continuing this selection process that there exist $u_l, v_l \in (p_{i_l-1}, p_{i_l})$ and $m_{l-1}, m_l \in \mathbb{N}$ for each $l \in \{1, \dots, r\}$ so that

$$\|z_l\| \geq \frac{1}{2n_0} \text{ and } \left\| \sum_{j=1}^{m_{l-1}} e_j^*(z_l)e_j \right\| \leq \theta_{l-1} \sum_{j=1}^{m_{l-1}} |e_j^*(z_l)| < \frac{\varepsilon}{2^{l-1}}$$

where $z_l = f(u_l) - f(v_l)$, $\theta_{l-1} = \max_{1 \leq j \leq m_{l-1}} \|e_j\|$, and $\left\| \sum_{j=m_l}^{\infty} e_j^*(z_l)e_j \right\| \leq \frac{\varepsilon}{2^{l-1}}$. Define the sequences $T_1 = (t_{1,i})_{i=1}^d$ and $T_2 = (t_{2,i})_{i=1}^d$ so that $t_{i,1} = u_l$ and $t_{2,i} = v_l$ for each $l \in \{1, \dots, r\}$ and so that $t_{1,i} = t_{2,i} \in [p_{i-1}, p_i]$ for each $i \notin A_{n_0}$. It follows that (P, T_1) and (P, T_2) are tagged partitions of $[0, 1]$ such that

$$\begin{aligned} \|S_f(P, T_1) - S_f(P, T_2)\| &= \left\| \sum_{l=1}^r \Delta_P(i_l) z_l \right\| \\ &= \left\| \sum_{l=1}^r \Delta_P(i_l) \left(\sum_{j=1}^{m_{l-1}} e_j^*(z_l)e_j + \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l)e_j + \sum_{j=m_l}^{\infty} e_j^*(z_l)e_j \right) \right\|. \end{aligned} \quad (3.0.2)$$

The reverse triangle inequality now implies that (3.0.2) is bounded below by

$$\begin{aligned} &\left\| \sum_{l=1}^r \Delta_P(i_l) \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l)e_j \right\| - \left\| \sum_{l=1}^r \Delta_P(i_l) \left(\sum_{j=1}^{m_{l-1}} e_j^*(z_l)e_j + \sum_{j=m_l}^{\infty} e_j^*(z_l)e_j \right) \right\| \\ &\geq \left\| \sum_{l=1}^r \Delta_P(i_l) \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l)e_j \right\| - \sum_{l=1}^r \Delta_P(i_l) \frac{\varepsilon}{2^{l-2}} \end{aligned}$$

and this quantity is, in turn, bounded below by

$$\left\| \sum_{l=1}^r \Delta_P(i_l) \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l)e_j \right\| - 4\varepsilon. \quad (3.0.3)$$

It remains to bound (3.0.3) from below by means of the asymptotic- ℓ_1 condition on X . Namely, the vectors $z'_l = \Delta_P(i_l) \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l)e_j$ define a block sequence $(z'_l)_{l=1}^r$ such that $\min \text{supp}(z'_1) \geq$

$m_0 + 1 \geq (M - 1) + 1 = M$ so an application of (1.0.3) shows that

$$\begin{aligned} \left\| \sum_{l=1}^r \Delta_P(i_l) \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l) e_j \right\| &\geq \zeta_1 \sum_{l=1}^r \Delta_P(i_l) \left\| \sum_{j=m_{l-1}+1}^{m_l-1} e_j^*(z_l) e_j \right\| \\ &\geq \zeta_1 \sum_{l=1}^r \Delta_P(i_l) \left(\|z_l\| - \frac{\varepsilon}{2^{l-2}} \right) \geq \zeta_1 \left(\frac{\mu(H_{n_0} \setminus G)}{2n_0} - 4\varepsilon \right) \end{aligned}$$

and (3.0.3) is therefore bounded below by

$$\zeta_1 \left(\frac{\mu(H_{n_0} \setminus G)}{2n_0} - 4\varepsilon \right) - 4\varepsilon = \frac{\zeta_1 \mu(H_{n_0} \setminus G)}{4n_0} > 0$$

which, in turn, completes the proof of Theorem 1.0.2. \square

It follows immediately from Theorem 1.0.2 that ℓ_1 and \mathcal{T} are PL-spaces and that c_0 and ℓ_p for each $p \in (1, \infty)$ are not PL-spaces. Similar arguments also apply to certain weighted- ℓ_p sequence spaces such as the Lorentz sequence space $d(w, p)$ defined to be the completion of c_{00} with respect to the norm

$$\|x\|_{d(w,p)} = \sup_{\kappa \in \Pi} \left(\sum_{j=1}^{\infty} |x_{\kappa(j)}|^p w_j \right)^{\frac{1}{p}} \quad x = (x_j)_{j=1}^{\infty} \in c_{00}$$

where $p \in (1, \infty)$, Π is the collection of permutations of \mathbb{N} , and $(w_j)_{j=1}^{\infty} \in c_0 \setminus \ell_1$ is a decreasing sequence of real numbers with $w_1 = 1$. This Banach space satisfies a weighted version of (1.0.3) and has a symmetric (and hence, democratic) basis, so it is not a PL-space by the same logic that applies to ℓ_p for each $p \in (1, \infty)$ in Theorem 1.0.2. Note that the case $p = 1$ is not analogous to Theorem 1.0.2 because the sequence of weights is decreasing to zero. It is, of course, well-known that $d(w, p)$ of the above form contains a complemented isomorphic copy of ℓ_p and this exemplifies the easiest way to show that a Banach space does not have the Property of Lebesgue.

Theorem 3.0.2. *A closed and infinite-dimensional subspace of a PL-space is not isomorphic to a non-PL-space.*

Proof. Let Y be a closed and infinite-dimensional subspace of a PL-space X and suppose for a contradiction that Y is isomorphic to a non-PL-space Z . Let $\psi : Z \rightarrow Y$ be an isomorphism and note that there exists $f \in \mathcal{R}([0, 1], Z)$ whose set of discontinuities has positive Lebesgue measure or else Z would be a PL-space. Let $\varepsilon > 0$ be given and let $P_\varepsilon = P$ be a partition of $[0, 1]$ with $d+1$ terms so that

$$\|S_f(P, T_1) - S_f(P, T_2)\| \leq \frac{\varepsilon}{1 + \|\psi\|_{\text{op}}}$$

for all tagged partitions (P, T_1) and (P, T_2) of $[0, 1]$. If $g = (\psi \circ f)(s) : [0, 1] \rightarrow Y \subset X$, then

$$\begin{aligned} \|S_g(P, T_1) - S_g(P, T_2)\| &= \left\| \psi \left(\sum_{i=1}^d \Delta_P(i) (f(t_{1,i}) - f(t_{2,i})) \right) \right\| \\ &\leq \|\psi\|_{\text{op}} \|S_f(P, T_1) - S_f(P, T_2)\| \leq \frac{\|\psi\|_{\text{op}}}{1 + \|\psi\|_{\text{op}}} \varepsilon \leq \varepsilon \end{aligned}$$

by the linearity and continuity of ψ . It follows that $g \in \mathcal{R}([0, 1], Y) \subset \mathcal{R}([0, 1], X)$ and is therefore μ -a.e. continuous because X is a PL-space. Then, $f = (\phi^{-1} \circ g)(s) : [0, 1] \rightarrow Z$ is μ -a.e. continuous by the continuity of ϕ^{-1} and this is a contradiction. \square

This theorem is a slightly modified version of [6, Theorem 21 (a)] and together with Theorem 1.0.2 it leads to four more trivial results. First, a separable Hilbert space is not a PL-space because it is isomorphic to ℓ_2 . Second, the Orlicz space defined by

$$\ell_\Gamma(\mathbb{F}) = \left\{ (x_j)_{j=1}^\infty \in \mathbb{F}^\mathbb{N} \mid \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\} \text{ and there exists } r > 0 \text{ such that } \sum_{j=1}^\infty \Gamma\left(\frac{|x_j|}{r}\right) < \infty \right\}$$

where $\Gamma : [0, \infty) \rightarrow [0, \infty)$ is continuous, increasing, and convex with $\Gamma(0) = 0$ is a Banach space with respect to the (Luxemburg) norm

$$\|x\|_{\ell_\Gamma} = \inf \left\{ r > 0 \mid \sum_{j=1}^\infty \Gamma\left(\frac{|x_j|}{r}\right) \leq 1 \right\}$$

and it is not a PL-space if it is separable and if the estimate

$$1 < \alpha_\Gamma = \sup \left\{ \theta \in \mathbb{R} \mid \sup \left\{ \frac{\Gamma(\lambda t)}{t^\theta \Gamma(\lambda)} \mid \lambda > 0 \text{ and } t \in (0, 1] \right\} < \infty \right\}$$

is valid because the standard result [9, Theorem I.4.a.9] asserts in this case that it contains an isomorphic copy of ℓ_p if $p = \alpha_\Gamma \in (1, \infty)$ and an isomorphic copy of c_0 if $\alpha_\Gamma = \infty$. It is worth noting as well that a separable ℓ_Γ has a symmetric and boundedly complete canonical unit vector basis. Third, Theorems 1.0.2 and 3.0.2 imply that every separable Banach space is isomorphic to a quotient of PL-spaces. This follows because it is well-known that

$$\ell_1/Y \cong X$$

for some closed and infinite-dimensional subspace $Y \subset \ell_1$ if X is separable. Fourth, c_0 and ℓ_p for each $p \in (1, \infty)$ clearly do not embed isomorphically into any PL-space and, in particular, into any asymptotic- ℓ_1 Banach space. This is a non-basis-theoretic fact in the context of this work in the sense that Theorem 1.0.2 does not fundamentally rely on basis-theoretic arguments. If, in addition to c_0 and ℓ_p for each $p \in (1, \infty)$, a PL-space contains no isomorphic copy of ℓ_1 , then it contains a distortable subspace by a classical result attributed in [10] to V. Milman. Recall that X is said to be λ -distortable (or simply, distortable) for some $\lambda > 1$ if there exists an equivalent norm $\|\cdot\|$ that depends on λ such that

$$\lambda \leq \sup \left\{ \frac{\|y\|}{\|y'\|} \mid y, y' \in S(Y) = \{x \in Y \mid \|x\| = 1\} \right\}$$

for all closed and infinite-dimensional subspaces $Y \subset X$. The Banach space X is then said to be arbitrarily distortable if it is λ -distortable for all $\lambda > 1$. One particularly enduring open problem in Banach space theory to ascertain whether or not there exists a distortable Banach space that is not arbitrarily distortable. The Property of Lebesgue is undoubtedly of little assistance in this matter because there are PL-spaces that are not distortable for any $\lambda > 1$ (ℓ_1 by a result due to

R.C. James in [8]) and that are arbitrarily distortable (the asymptotic- ℓ_1 mixed Tsirelson spaces $\mathcal{T}[(F_n, \theta_n)_{n=1}^\infty]$ that satisfy the hypotheses of [1, Theorem 1.5]). A considerably more reasonable direction for additional research does, however, present itself by means of the observation that the upper and lower Riemann sum bounds from Theorem 1.0.2 that are of the form (1.0.3)

$$\gamma\zeta_2\alpha_2\varepsilon = C\varepsilon \geq 0 \text{ and } c_f = \frac{\zeta_1\mu(H_{n_0} \setminus G)}{4n_0} > 0 \quad (3.0.4)$$

are uniform with respect to the constants ζ_1 and ζ_2 (in the sense that ζ_1 and ζ_2 do not depend on specific functions) but need not be.

4 Additional Research and Closing Remarks

The permissible dependence of the constants ζ_1 and ζ_2 from (3.0.4) on specific functions suggests that Theorem 1.0.2 can be generalized in terms of the local asymptotic structures of X and, in particular, its spreading models. It is well-known that the non-trivial spreading models of both ℓ_1 and \mathcal{T} are equivalent in the sense of (1.0.1) to the canonical basis for ℓ_1 and, in fact, a weaker conclusion is valid for general PL-spaces.

Definition 4.0.1. A sequence $(v_k)_{k=1}^\infty$ of vectors from a semi-normed vector space $(V, \|\cdot\|_V)$ is said to be spreading if, for all $N \in \mathbb{N}$,

$$\left\| \sum_{k=1}^N \lambda_k v_k \right\|_V = \left\| \sum_{j=1}^N \lambda_j v_{k_j} \right\|_V$$

for all scalar sequences $(\lambda_k)_{k=1}^N$ and for all positive integers $(k_j)_{j=1}^N$ such that $k_1 < \dots < k_N$.

A spreading sequence $(v_k)_{k=1}^\infty$ is said to be a spreading model of X if it is Schreier almost isometric to a sequence $(x_k)_{k=1}^\infty$ of vectors from X . In other words, there exists a sequence of positive real numbers $(\varepsilon_n)_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that

$$\left| \left\| \sum_{k=1}^N \lambda_k v_k \right\|_V - \left\| \sum_{j=1}^N \lambda_j x_{k_j} \right\| \right| \leq \varepsilon_{k_1}$$

for all positive integers $(k_j)_{j=1}^N$ that satisfy $N \leq k_1 < \dots < k_N$ and for all real sequences $(\lambda_k)_{k=1}^N$ that satisfy $\lambda_k \in [-1, 1]$ for each $k \in \{1, \dots, N\}$. The sequence $(x_k)_{k=1}^\infty$ is in this case said to generate the spreading model $(v_k)_{k=1}^\infty$, which is called trivial if

$$\left\| \sum_{k=1}^N \lambda_k v_k \right\|_V = \left| \sum_{k=1}^N \lambda_k \right| \|v_1\|_V$$

is valid for all $N \in \mathbb{N}$ and for all real sequences $(\lambda_k)_{k=1}^N$. Definition 4.0.1 and the precise notion of a (trivial) spreading model given here can be found in [2]. It is well-known that the vectors $(v_k)_{k=1}^\infty$ are linearly independent and that the semi-norm $\|\cdot\|_V$ restricted to their closed linear span defines a genuine norm if this spreading model is non-trivial. If, in addition, the vectors $(v_k)_{k=1}^\infty$ constitute an unconditional basis for their closed linear span, then [2, Proposition 3.7] is a useful dichotomy: either $(v_k)_{k=1}^\infty$ is equivalent in the sense of (1.0.1) to the canonical basis for ℓ_1 , or it is norm-Cesàro summable to zero.

Theorem 4.0.2. *A spreading model $(v_k)_{k=1}^\infty$ of a PL-space X is equivalent to the canonical basis for ℓ_1 if it is non-trivial, unconditional, and generated by a democratic basic sequence $(x_k)_{k=1}^\infty$.*

Proof. Note that $(x_k)_{k=1}^\infty$ is semi-normalized because it is democratic and define $f : [0, 1] \rightarrow X$ by

$$f(s) = \begin{cases} x_k & \text{if } s = r_k \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } s \text{ is irrational} \end{cases}$$

where r_1, r_2, \dots is a listing of $\mathbb{Q} \cap [0, 1]$. It follows that f is discontinuous everywhere on $[0, 1]$ and therefore that $f \notin \mathcal{D}([0, 1], X)$ by Theorem 2.1.3 as in Theorem 1.0.2. Suppose for a contradiction that $(v_k)_{k=1}^\infty$ is norm-Cesàro summable to zero so that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N v_k = 0.$$

Let $\varepsilon > 0$ be given, choose $N \in \mathbb{N}$ such that $\left\| \frac{1}{N} \sum_{k=1}^N v_k \right\|_V < \varepsilon$, and let P be a regular partition of $[0, 1]$ with $N + 1$ terms so that $\Delta_P = \frac{1}{N}$. If $T_1 = (t_{1,i})_{i=1}^d$ and $T_2 = (t_{2,i})_{i=1}^d$ are given such that (P, T_1) and (P, T_2) are tagged partitions of $[0, 1]$, then

$$\begin{aligned} \|S_f(P, T_1) - S_f(P, T_2)\| &= \left\| \sum_{i=1}^d \frac{1}{N} (f(t_{1,i}) - f(t_{2,i})) \right\| \\ &\leq \left\| \sum_{k=1}^{d'} \frac{1}{N} f(t_{1,i_m}) \right\| + \left\| \sum_{l=1}^{d''} \frac{1}{N} f(t_{2,i_l}) \right\| \end{aligned} \quad (4.0.1)$$

as in (3.0.1) where $(i_m)_{m=1}^{d'}$ and $(i_l)_{l=1}^{d''}$ are the subsequences of indices such that $t_{1,i_m}, t_{2,i_l} \in \mathbb{Q} \cap [0, 1]$. The two summands on the right-hand side of (4.0.1) may be bounded above in an identical manner so it suffices to derive a suitable upper bound for the first summand. Let $x_m = f(t_{1,i_m})$, $\gamma > 0$ be as in (1.0.2), and $\Gamma \geq 0$ be the basis constant of $(x_k)_{k=1}^\infty$ as in [4, Theorem 3.2]. Then,

$$\begin{aligned} \left\| \sum_{m=1}^{d'} \frac{1}{N} x_m \right\| &= \frac{1}{N} \left\| \sum_{m=1}^{d'} x_m \right\| \leq \frac{1}{N} \gamma \left\| \sum_{m'=1}^{d'} x_{m'} \right\| \\ &\leq \frac{1}{N} \gamma \Gamma \left\| \sum_{m'=1}^N x_{m'} \right\| \leq \frac{1}{N} \gamma^2 \Gamma \left\| \sum_{m'=1}^N x_{K+m'} \right\| = \gamma^2 \Gamma \left\| \sum_{m'=1}^N \frac{1}{N} x_{K+m'} \right\| \end{aligned}$$

where $\{x_{m'}\}_{m'=1}^{d'} = \{x_1, \dots, x_{d'}\}$ and $K \in \mathbb{N}$ is such that $K \geq N$ and $\varepsilon_{K+1} \in (0, \varepsilon)$. It now follows that

$$\begin{aligned} \gamma^2 \Gamma \left\| \sum_{m'=1}^N \frac{1}{N} x_{K+m'} \right\| &\leq \gamma^2 \Gamma \left\| \left\| \sum_{m'=1}^N \frac{1}{N} x_{K+m'} \right\| - \left\| \sum_{k=1}^N \frac{1}{N} v_k \right\|_V \right\| \\ &\quad + \gamma^2 \Gamma \left\| \frac{1}{N} \sum_{k=1}^N v_k \right\|_V \leq \gamma^2 \Gamma \varepsilon_{K+1} + \gamma^2 \Gamma \varepsilon \leq 2\gamma^2 \Gamma \varepsilon \end{aligned} \quad (4.0.2)$$

and this implies that $f \in \mathcal{R}([0, 1], X)$. This contradicts the fact that X is a PL-space so the proof of Theorem 4.0.2 is complete. \square

The Riemann sum upper bound (4.0.2) depends implicitly on f in the sense that γ and Γ are specific to the democratic basic sequence from which f is defined. What is more, it is reasonable to expect that the converse of Theorem 4.0.2 is valid, at least for Banach spaces such that every vector in the range of a highly discontinuous function can be expressed in terms of a democratic basic sequence. The proofs of Theorem 4.0.2 and its (possible) converse are clearly local versions of the proof of Theorem 1.0.2 in the cases $p \in (1, \infty]$ and $p = 1$, respectively. However, whether or not this or a more general result that characterizes the Property of Lebesgue in ever larger classes Banach spaces exists is beyond the scope of this work.

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