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Article

Plane and Frieze Symmetry Group Determination for Educational Purposes

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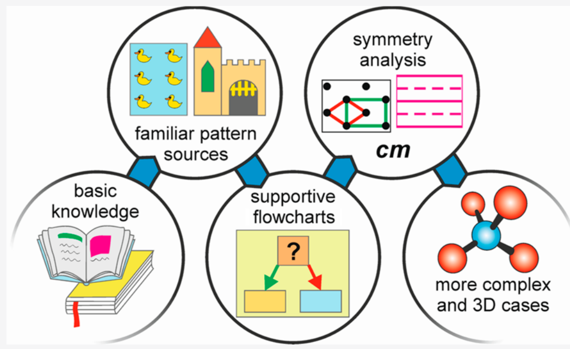
ABSTRACT: Three-dimensional symmetry plays a crucial role in crystallography education. The educational process can be facilitated by introducing the analysis of friezes and plane patterns such as parquets before presenting more sophisticated crystal structures. Analysis of the symmetry of parquets can serve as an opportunity to follow the full routine of finding symmetry group symbols through incorporation of a two-step procedure involving the determination of Bravais lattice type. The aim of this paper is to provide a simple description of the analysis of Bravais lattices and pattern symmetry that can be used by students as well as crystallography educators. The procedure is also summarized in the form of comprehensive tree charts. This form is easy to use and popular with students. It may also aid in the retention of certain facts concerning crystallographic methodology. Examples of patterns in frieze and parquet forms were taken from the interiors of Kórnik Castle (Poland), which exhibit Moorish-style ornamentation including dichroic patterns and constitute an important locus of cultural heritage capable of strengthening students' motivation.



KEYWORDS: *Second-Year Undergraduate, Inorganic Chemistry, Hands-On Learning/Manipulatives, Misconceptions/Discrepant Events, Distance Learning/Self Instruction, Crystals/Crystallography, Group Theory/Symmetry*

INTRODUCTION

Crystallography, which is so important in chemistry, materials, and life sciences, is one of the disciplines increasingly considered a routine technique only[1] and which, as a separate course, is gradually disappearing from higher education curricula. Moreover, students are not eager to choose crystallography courses as the subject of their studies. Given the available hardware and software tools used by crystallographers today, many scientists believe that they are able to solve crystallographic problems without a profound knowledge of crystallography.[1] One of the basic principles of this field is symmetry, which plays a crucial role in crystal structure description.

Although many articles,[2−5] computer programs, websites,[6−13] and even games[14−16] concerning molecular symmetry (usually in Schoenflies notation) have been published, the description and discussion of symmetry of crystal structures is more complex. To describe the symmetry completely, it is necessary to use not point but rather space groups[17,18] (Hermann−Mauguin notation). Since spatial symmetry is difficult, it may be beneficial to begin with simpler examples from frieze and wallpaper groups, if the time frames of classes allow this. The application of frieze and wallpaper patterns was already suggested long ago in the *Journal of Chemical Education*[19−21] and other journals.[22,23]

Some books and papers contain relatively simple and well-organized flowcharts, which enable the user to find wallpaper symmetry by checking whether the appropriate symmetry elements/operations are present.[24−29] However, these flowcharts do not involve terms such as *unit cell* or *Bravais lattice* and do not show that lattice and actual pattern symmetry may differ. Moreover, there is a risk that students will use these flowcharts automatically. We recommend the introduction of more formal and elegant (although more time-consuming) procedures using Bravais lattices that may be more beneficial from the didactic point of view. These procedures are summarized in flowcharts. Students can use them outside the classroom, in museums, castles, and other places decorated with symmetrical patterns.[30]

We have observed that when we introduce, as a teaching model, an item from everyday life[31,32] which is well-known to our students, it may reinforce our teaching efforts and enhance our results. In this paper, we analyze some floor and frieze patterns from a castle in Kórnik (western Poland). A short description of the historical background of the castle is included in the Supporting Information (in the subsection: *Historical background*). Though the patterns in the residence were inspired

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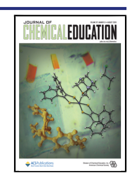

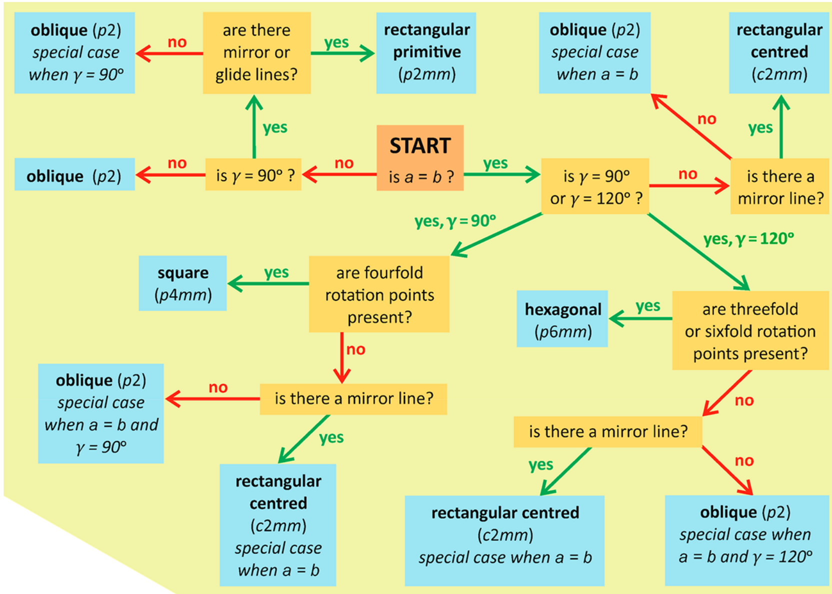

Figure 1. Schema illustrating a procedure for choosing Bravais lattice type starting from a reduced cell.

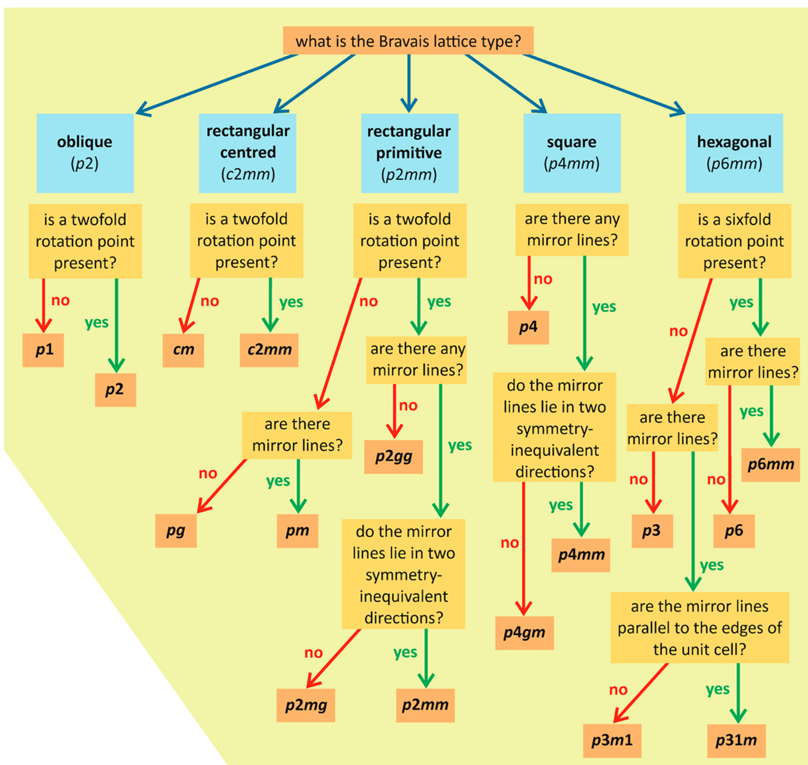

Figure 2. Schema showing how to find the pattern symmetry group, starting with the Bravais lattice type of a pattern.

by those in the Alhambra,[28,33−37] they are significantly simpler and therefore more suitable for educational purposes.

■ SYMMETRY OF FRIEZES AND WALLPAPER-TYPE DECORATIONS

Here we assume that students are familiar with the basics of crystallography: they know the simplest terms, such as *unit cell*, and recognize basic symmetry operations such as reflection, glide, and rotation (if not, books that simply describe these issues can be found in the list of references[27,29]). Hence, the material presented in this paper is suitable for presentation during a regular course in crystallography and structural chemistry.

Wallpaper (or parquet) decorations are two-dimensional patterns in which the repetition of the motif proceeds in two noncollinear directions. The following symmetry elements may be present in wallpapers: one-, two-, three-, four-, and sixfold rotation points, mirror line, and glide line (glide is a combination of reflection by a mirror line with translation by half of the length of the unit cell edge or diagonal in a direction parallel to the mirror line).[38] Since parquets are placed in a plane with no third

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dimension, it is more proper to introduce the terms *rotation point*, *mirror*, or *glide line* instead of speaking of a rotation axis, mirror plane, or glide plane perpendicular to the plane in which the pattern lies, as recommended in the *International Tables for Crystallography Vol. A*.[39] It is worth noting that, in two-dimensional space, each twofold point is identical to the inversion centre. A unit cell with fivefold, sevenfold, or greater than sevenfold points cannot be used to fill a plane so as to completely avoid gaps between cells. Accordingly, such rotation axes are not allowed in planar symmetry groups.[40]

There are 17 wallpaper symmetry groups.[21,41] To find the wallpaper symmetry group of a pattern, we first recommend determination of the Bravais lattice type and selection of the conventional unit cell. An expanded discussion on this topic is provided in the Supporting Information (*Expanded notes on Bravais lattices*).

To explain the rules connected with the procedure of Bravais lattice type selection, we use a schema such as the one presented in Figure 1. The three parameters *a*, *b*, and *γ* describing the reduced (the smallest) cell are crucial in this procedure.

There are five planar types of Bravais lattice for wallpapers: parallelogram (primitive), rectangular primitive, rectangular centred, square (primitive), and hexagonal (primitive).[27] These lattice types exemplify the *p*2, *p*2*mm*, *c*2*mm*, *p*4*mm*, and *p*6*mm* symmetry groups, respectively. Therefore, a conventional elementary cell in the case of wallpaper decorations takes the shape of a parallelogram, a rectangle, a square, or a rhombus with an obtuse angle of 120° (hexagonal cell).

A Bravais lattice type cannot be derived directly from the shape of the unit cell; moreover, the selection of a Bravais lattice type must be correlated with the actual pattern symmetry. To explain this, a suitable example from the Supporting Information can be considered (*An example of a pattern whose symmetry is at a lower level than that of its Bravais lattice*).

The rules governing the choice of pattern symmetry group (once its Bravais lattice type is known) are summarized in the schema in Figure 2. Although some simpler schemata of symmetry group determination are presented in the literature,[24−29] the approach suggested here (Figures 1 and 2) enables us to consider the Bravais lattice type as well, which may be beneficial in terms of crystallography education, even if this method is more time-consuming.

Using the tree charts from Figures 1 and 2, we can work out the symmetry of the parquets from Kórnik Castle and find their symmetry group symbols in accordance with the following instructions:

(i) identify the motif and replace it with points throughout the pattern;
(ii) draw the reduced unit cell;
(iii) follow the schema from Figure 1 to find the Bravais lattice type and the lattice symmetry; when the lattice type has been identified, the standard unit cell is also known (the standard cell may or may not be the same as the reduced cell);
(iv) draw the standard cell;
(v) use the schema in Figure 2 to find the pattern symmetry group symbol; draw the symmetry elements inside the standard cell and in its edges and corners.

Finding symmetry groups of friezes is much simpler. The procedure is described, with examples, in the Supporting Information (*Description and symmetry group determination of friezes with examples from the castle in Kórnik*).

SYMMETRY OF THE PATTERNS FROM KÓRNIK CASTLE

Some wallpaper-type patterns from the castle are shown in Figure 3.

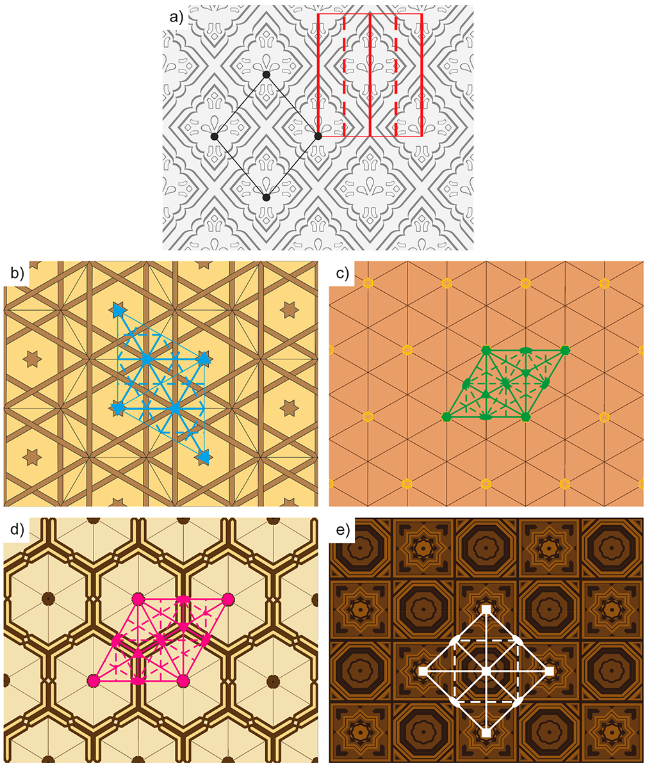

Figure 3. Standard unit cell with symmetry elements drawn within it for (a) a pattern under the arches in the Moorish Room (*cm* symmetry) (reduced cell in black); (b) part of the parquet in Zamoyska's Room (*p*3*m*1); (c) the parquet in the Guest Room (*p*6*mm*); (d) the floor in the Dining Room (*p*6*mm*); (e) the parquet pattern in the Boudoir (*p*4*mm*).

The unit cell of the pattern from the most representative interior of the castle, the Moorish Room[42,43] (Figure 3a), characterized by the smallest possible *a*, *b*, and *γ* parameters, is a rhombus where *a* and *b* are the same and the *γ* angle is not 90°. We can find mirror lines; thus, according to the schema from Figure 1, the Bravais lattice type of this pattern is centred rectangular and exhibits *c*2*mm* symmetry. Accordingly, we should select another unit cell, where the lattice parameters *a* and *b* are not the smallest, the unit cell is a rectangle ($\gamma = 90°$), and $a \neq b$. The lattice is centred and the centre of the cell contains the same motif as the corners (the cell is marked in Figure 3a). As shown in Figure 2, the pattern exhibits *c*2*mm* symmetry if twofold rotation centres are present or *cm* if rotation centres are absent. In this case, there is no twofold rotation point in the actual pattern, so its symmetry group symbol is *cm*. In Figure 3a, symmetry elements included in this group are drawn in the centred unit cell. The mirror lines are shown as thick solid red lines, the glide lines (not included in the group symbol) as thick dotted lines. The thin horizontally oriented lines indicate the upper and lower edges of the unit cell, while the left and right edges overlap the mirror lines. The other examples from Figure 3 are discussed in the Supporting Information (*Discussion on symmetry of parquets from Figure* 3b−e).

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Examples of so-called "dichroic" symmetry[26] can also be found in the castle (Figure 4). The pattern from Figure 4a is constructed of elements shaped like fig (or maple) leaves of the same size but different colors (red or black). This pattern is derived from *Arabian Antiquities*,[44] where it is shown in the black-and-white variant.

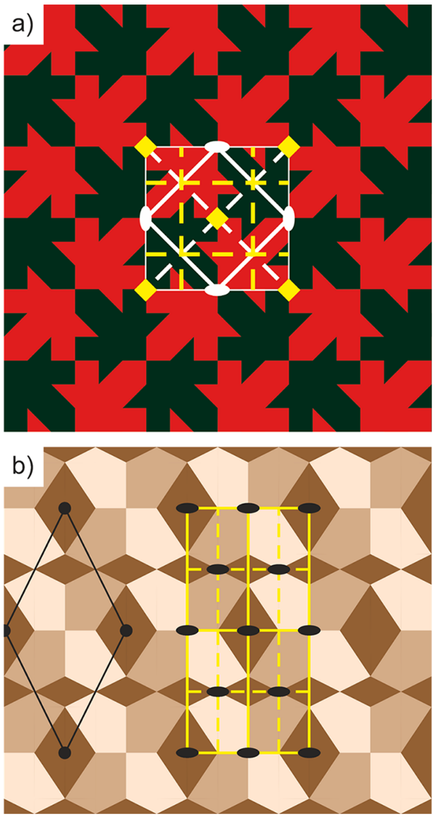

Figure 4. (a) Dichroic pattern ($p4'g'm$) from the staircase; symmetry elements which relate to dichroic symmetry are marked in yellow; (b) parquet exhibiting dichroic symmetry ($c2m'm'$) from the second floor (reduced cell in black).

The pattern in Figure 4b is also composed of the same elements but uses two different colors. As we can see, the color choice is not accidental, and the patterns exhibit a symmetry that also takes colors, even if they vary, into account. The first step to take when we see dichroic symmetry is to imagine components of different colors as if they were the same color. Then we identify the Bravais lattice type and pattern symmetry group in the same way as in "normal" wallpaper-type patterns. The lattice type of the single-color pattern in Figure 4a is square; the pattern exhibits $p4gm$ symmetry. In cases where dichroic symmetry is being investigated, symbols of elements included in the group symbol referring to dichroic symmetry should be marked with the prime symbol ($'$). Symmetry operations associated with these elements also change one color into its opposite. Hence, the symmetry group symbol of this floor pattern is $p4'g'm$. The pattern in Figure 4b possesses $c2m'm'$ symmetry.

■ TEACHING EXPERIMENT

The methodology of planar symmetry teaching presented above was tested on two groups of undergraduate students during classes related to crystallography. The students already knew the basics of symmetry, i.e., definitions of reduced and standard unit cells, Bravais and crystal lattices, symmetry elements in three-dimensional space, and Hermann−Mauguin notation.

Their task was to find the symmetry group symbols of the patterns shown in Figures 3 and 4, and SI-Figure 6 (in the Supporting Information: *Additional patterns used in the teaching experiment*). Additionally, they were asked to draw reduced and standard unit cells and symmetry elements within the standard cell, and to assign a symmetry group symbol to the Bravais lattice. To complete this task, one group of students used traditional schema from ref 30, while the other group was equipped with the diagrams presented here (Figures 1 and 2). Before attempting the test, all the students were also informed about the origin of the patterns from Kórnik and some historical facts about the castle were briefly presented to them by their teacher. The purpose of this test was to assess the effectiveness of the methodology described in this paper when used by students as well as whether it helped them to obtain correct answers.

In the second test, the task was the same but the groups were switched: the first group was asked to use the tree charts elaborated in this paper, and the other group was instructed to follow the schema from ref 30. The switch was done to give both groups the opportunity to work in both modes and then to assess them. Accordingly, all the students were asked the following questions at the end of the second test: (i) Which method is easier to use in practice? (ii) Which method enables you to learn more? (iii) Does the method involving two tree charts (presented in the paper) help you in understanding the difference between Bravais lattice symmetry and the symmetry of the actual pattern? (iv) Did the information about the origin of the patterns and historical facts about the castle influence your motivation to attempt the task? If so, what was the impact? Moreover, after the test, the teacher initiated a discussion during which students were encouraged to express their comments.

In the first test, both groups correctly answered the questions about the symmetry group symbol of the analyzed patterns. However, when asked to find the reduced and standard unit cells and to work out Bravais lattice symmetry, the students who had used the procedures described here obtained better results. This can be explained simply by the fact that the methodology presented in this paper includes instructions for finding Bravais lattice type and symmetry, and indirectly differentiates reduced and standard unit cells, whereas the traditional method concentrates only on the ultimate aim, i.e. finding the symmetry of the pattern.

It was also observed that the students who followed the traditional schema spent more time on their task despite using a method considered faster. These students needed more time than those in the other group to think about Bravais lattice type and symmetry, and this prolonged their work.

In the second test, its participants did not identify either method as easier or more difficult in terms of determination of the symmetry group symbol of a pattern but pointed out that the traditional method seems less time-consuming. However, they unanimously chose the method presented here as enabling them to learn more and to gain a deeper knowledge of symmetry analysis. All of the students agreed as well that the schemata from the present paper are helpful in understanding the difference between Bravais lattice symmetry and the symmetry of the actual pattern. They indicated that the historical context of the given patterns increased their motivation to solve the tasks. During the discussion after the tests, students asked questions about the history of the castle to obtain additional information, and talked about other patterns they had seen. It was observed that

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students' interest in finding repetitive patterns in their surroundings had grown, as they began to analyze designs in the clothes they wore, etc.

A detailed report on the teaching experiment for one of the tested populations of students is available in the Supporting Information.

SUMMARY

The application of crystallographic methodology enabled the analysis of symmetry and the classification of frieze and parquet patterns from a historical residence. The research resulted in the following conclusions: (i) For friezes, the $p211$ group is the most common and, significantly, most of the friezes found in Kórnik Castle do not exhibit mirror symmetry (the friezes are discussed in the Supporting Information). (ii) Of 17 wallpaper symmetry groups, examples of four groups, cm, $p4mm$, $p3m1$, and $p6mm$, were found as well as two examples of dichroic symmetry, $c2m'm'$ and $p4'g'm$. Parquets belonging to the symmetry groups $p6mm$ and $p3m1$ appear most often; their occurrence may indicate oriental influences revealed in the course of art history research. (iii) Hence, symmetry analysis can be used as an additional tool for visual art studies for pattern classification and to support style analysis and investigations of historical objects.

Working on such objects may strengthen students' motivation and may constitute a pleasant introduction to the basics of symmetry and three-dimensional crystallography. Some of the presented patterns can be also used in teaching the symmetry of layer or rod ornaments which can serve as a "bridge" between frieze or plane groups and space groups. Of course, all over the world, educators can find other interesting and unusual objects which are familiar to the students and which will reinforce their motivation and interest in symmetry.[20,21,45]

Although some methods of symmetry group determination are even faster and simpler than the technique presented here (i.e., methods not involving Bravais lattice type), from the didactic point of view it may sometimes prove beneficial to use the procedures described in this paper. As we have observed in our educational practice, students often confuse and fail to differentiate between (i) symmetry of the unit cell parallelepiped, (ii) symmetry of the Bravais lattice, and (iii) symmetry of the pattern[46] in cases where these three aspects are not the same. Following the full process involving Bravais lattice determination may help in understanding differences between the terms listed above. Moreover, the procedure proposed in this paper enables users to break down the matter of symmetry analysis into two steps. If a student makes a mistake, this could make it easier to determine where it was made.

The procedures were tested on groups of undergraduate students. Students who used the procedures described here obtained better results, thanks to the deeper knowledge of symmetry analysis they had gained. Also, knowledge of certain historical details about the patterns increased their motivation.

In this Article, only the simplest and most necessary terms were mentioned; it is left to the teacher to decide whether to introduce terms such as *holohedral* and *merohedral groups* or *Bravais flocks*. Since the discussed procedures are summarized in flowcharts, it is possible for students to print them out and use them outside the classroom as a support in their own symmetry studies.

ASSOCIATED CONTENT

Supporting Information

The Supporting Information is available at https://pubs.acs.org/doi/10.1021/acs.jchemed.0c00093.

Historical background; expanded notes on Bravais lattices; an example of a pattern whose symmetry is at a lower level than that of its Bravais lattice; description and symmetry group determination of friezes with examples from the castle in Kórnik; discussion on symmetry of parquets from Figure 3b–e; additional patterns used in the teaching experiment; a detailed report on the teaching experiment for one of the tested populations of students (PDF)

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Notes

The authors declare no competing financial interest.

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