Advances in Operator Theory (2020) 5:1061–1077
https://doi.org/10.1007/s43036-020-00076-4

Tusi Mathematical Research Group

ORIGINAL PAPER

Check for updates

The Cauchy dual subnormality problem for cyclic 2-isometries

Akash Anand[1] · Sameer Chavan[1] · Zenon Jan Jabłoński[2] · Jan Stochel[2]

Received: 17 February 2020 / Accepted: 23 April 2020 / Published online: 29 June 2020

Abstract

The Cauchy dual subnormality problem asks whether the Cauchy dual operator of a 2-isometry is subnormal. Recently this problem has been solved in the negative. Here we show that it has a negative solution even in the class of cyclic 2-isometries.

Keywords Cauchy dual operator · 2-isometry · Subnormal operator · Weighted composition operator

Mathematics Subject Classification 47B20 · 47B37 · 44A60

1 Introduction

Very recently the Cauchy dual subnormality problem was solved negatively in the class of 2-isometric operators (see [4]). Originally formulated for completely hyperexpansive operators (see [12, Question 2.11]), the problem entails on determining whether the Cauchy dual operator of a member of the underlying class is subnormal. It is worth mentioning that there are relatively broad subclasses

This paper is dedicated to Professor Franciszek Hugon Szafraniec on the occasion of his 80th birthday.

Communicated by Mox S. Moslehian.

✉ Jan Stochel
Jan.Stochel@im.uj.edu.pl

Akash Anand
akasha@iitk.ac.in

Sameer Chavan
chavan@iitk.ac.in

Zenon Jan Jabłoński
Zenon.Jablonski@im.uj.edu.pl

1 Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, India

2 Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, 30348 Kraków, Poland

of 2-isometric or 2-hyperexpansive operators for which this problem has an affirmative solution (see [4] and [7] respectively). In the present paper we show that the Cauchy dual subnormality problem has a negative solution even in the class of cyclic 2-isometric operators. The counterexample is implemented with the help of a weighted composition operator on L^2 space over a directed graph with a circuit (see Theorem 4.4).

Apart from Introduction, the paper consists of three parts. The first one gives the theoretical background on weighted composition operators on L^2 spaces needed in this paper. In the next one, we construct a concrete class of weighted composition operators on an L^2 space and characterize the subnormality of their Cauchy duals. In the last part, by specifying the weights and using Hausdorff's moment problem technique along with subtle classical analysis, we get the required counterexample.

We will now provide the necessary concepts and facts related to the issues discussed, placing more emphasis on the Hausdorff moment problem. Given a complex Hilbert space $\mathcal{H}$, we denote by $\boldsymbol{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Let $T \in \boldsymbol{B}(\mathcal{H})$. We write $|T|$ for $(T^*T)^{1/2}$ and call it the *modulus* of T. We say that T is *cyclic* if there exists a vector e_0, called a *cyclic* vector of T, such that the linear span of the set $\{T^n e_0\}_{n=0}^{\infty}$ is dense in $\mathcal{H}$. We call T *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$ such that $\mathcal{H} \subseteq \mathcal{K}$ (an isometric embedding) and $Th = Nh$ for all $h \in \mathcal{H}$. If T is left-invertible (or equivalently T is bounded from below), then T^*T is an invertible element of $\boldsymbol{B}(\mathcal{H})$ and the operator $T' := T(T^*T)^{-1}$ is called the *Cauchy dual operator* of T (abbreviated to: the *Cauchy dual* of T). Finally, T is said to be a 2-*isometry* (or that T is 2-*isometric*) if

$$I - 2T^*T + T^{*2}T^2 = 0.$$

We refer the reader to [1–3, 13] and [22] for more information on subnormal operators, 2-isometric operators and the Cauchy dual operation, respectively.

Hereafter $\mathbb{C}$ stands for the field of complex numbers. The ring of all polynomials in one complex variable z with complex coefficients is denoted by $\mathbb{C}[z]$. Given a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of complex numbers, we say that γ_n *is a polynomial in* n (of degree d) if there exists $p \in \mathbb{C}[z]$ (of degree d) such that $\gamma_n = p(n)$ for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. The following fact will be used later (cf. [14, Exercise 7.2]).

If γ_n is a polynomial in n of degree d, then

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n} \gamma_n = 0, \quad m \geqslant \max\{d+1, 0\}. \tag{1}$$

A sequence $\{\gamma_n\}_{n=0}^{\infty}$ of real numbers is called a *Stieltjes moment sequence* (resp. *Hausdorff moment sequence*) if there exists a positive Borel measure μ on $[0, \infty)$ (resp. [0, 1]) such that

$$\gamma_n = \int t^n d\mu(t), \quad n \in \mathbb{Z}_+.$$

Clearly, any Hausdorff moment sequence is a Stieltjes moment sequence, but not conversely. Using Lebesgue's monotone convergence theorem, one can show the following.

$$\text{Any bounded Stieltjes moment sequence is a Hausdorff moment sequence.} \tag{2}$$

Recall that Hausdorff moment sequences can be characterized as follows (see [8, Proposition 6.11]).

A sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ of real numbers is a Hausdorff moment sequence if and only if

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n} \gamma_{n+j} \geqslant 0, \quad j, m \in \mathbb{Z}_+. \tag{3}$$

In this paper, we adhere to the conventions:

$$\sum\nolimits_{j=m}^{n} a_j = 0 \text{ and } \prod\nolimits_{l=m}^{n} a_l = 1 \text{ whenever } m > n \text{ and whatever } a_j'\text{s are.} \tag{4}$$

To simplify the notation, we write

$$\{f = 0\} = \{x \in X : f(x) = 0\} \quad \text{and} \quad \{f \neq 0\} = \{x \in X : f(x) \neq 0\},$$

whenever f is a $\mathbb{C}$-valued or a $[0, \infty]$-valued function on a set X.

2 Weighted composition operators on L^2-spaces

Given a measure space $(X, \mathscr{A}, \mu)$, we denote by $L^2(\mu)$ the complex Hilbert space of all square μ-integrable $\mathscr{A}$-measurable complex functions on X endowed with the standard inner product.

Definition 2.1 Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, $\phi : X \to X$ be an $\mathscr{A}$-measurable map and $\mathsf{w} : X \to \mathbb{C}$ be an $\mathscr{A}$-measurable function. By a *weighted composition operator* in $L^2(\mu)$ we mean a mapping $C_{\phi,\mathsf{w}} : L^2(\mu) \supseteq \mathscr{D}(C_{\phi,\mathsf{w}}) \to L^2(\mu)$ defined by

$$\mathscr{D}(C_{\phi,\mathsf{w}}) = \left\{ f \in L^2(\mu) : \mathsf{w} \cdot (f \circ \phi) \in L^2(\mu) \right\},$$
$$C_{\phi,\mathsf{w}} f = \mathsf{w} \cdot (f \circ \phi), \quad f \in \mathscr{D}(C_{\phi,\mathsf{w}}).$$

We call ϕ and w the *symbol* and the *weight* of $C_{\phi,\mathsf{w}}$, respectively.

As a matter of fact, $C_{\phi,\mathsf{w}}$ may not be well defined. The well-definiteness of $C_{\phi,\mathsf{w}}$ means that $\mathsf{w} \cdot (f \circ \phi) = \mathsf{w} \cdot (g \circ \phi)$ a.e. $[\mu]$ whenever $f, g : X \to \mathbb{C}$ are $\mathscr{A}$-measurable functions such that $f = g$ a.e. $[\mu]$, $f \in L^2(\mu)$ and $\mathsf{w} \cdot (f \circ \phi) \in L^2(\mu)$. Below we recall several basic properties of weighted composition operators including well-definiteness (see [11, Proposition 7]).

Proposition 2.2 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, $\phi : X \to X$ be an $\mathscr{A}$-measurable map and $\mathsf{w} : X \to \mathbb{C}$ be an $\mathscr{A}$-measurable function. Then $C_{\phi,\mathsf{w}}$ is well* *defined if and only if* $\mu_{\mathsf{w}} \circ \phi^{-1} \ll \mu$*, where* μ_{w} *and* $\mu_{\mathsf{w}} \circ \phi^{-1}$ *are measures on* $\mathscr{A}$ *defined by*

$$\mu_{\mathsf{w}}(\sigma) = \int_\sigma |\mathsf{w}|^2 d\mu \quad \text{and} \quad \mu_{\mathsf{w}} \circ \phi^{-1}(\sigma) = \mu_{\mathsf{w}}(\phi^{-1}(\sigma)) \quad \text{for } \sigma \in \mathscr{A}. \tag{5}$$

Moreover, if $C_{\phi,\mathsf{w}}$ *is well defined,* $\mathsf{u} : X \to \mathbb{C}$ *is* $\mathscr{A}$*-measurable and* $\mathsf{u} = \mathsf{w}$ *a.e.* $[\mu]$*, then* $C_{\phi,\mathsf{u}}$ *is well defined and* $C_{\phi,\mathsf{u}} = C_{\phi,\mathsf{w}}$*.*

To avoid repetition, we distinguish the following assumption, which we will often refer to in this article.

$$\begin{aligned} &(X, \mathscr{A}, \mu) \text{ is a } \sigma\text{-finite measure space, } \phi : X \to X \text{ is an } \mathscr{A}\text{-measurable} \\ &\text{map and } \mathsf{w} : X \to \mathbb{C} \text{ is an } \mathscr{A}\text{-measurable function such that } \mu_{\mathsf{w}} \circ \phi^{-1} \ll \mu. \end{aligned} \tag{6}$$

If the condition (6) holds, then by the Radon-Nikodym theorem (see [5, Theorem 2.2.1]), there exists a unique (up to a set of μ-measure zero) $\mathscr{A}$-measurable function $h_{\phi,\mathsf{w}} : X \to [0, \infty]$ such that

$$\mu_{\mathsf{w}} \circ \phi^{-1}(\sigma) = \int_\sigma h_{\phi,\mathsf{w}} d\mu, \quad \sigma \in \mathscr{A}. \tag{7}$$

Before going further, we make an important observation.

Lemma 2.3 *Suppose that* (6) *holds. If* $h_{\phi,\mathsf{w}} > 0$ *a.e.* $[\mu]$*, then*

$$\mu(\{\mathsf{w} \neq 0\} \cap \{h_{\phi,\mathsf{w}} \circ \phi = 0\}) = 0. \tag{8}$$

Proof Since $\mu(\{h_{\phi,\mathsf{w}} = 0\}) = 0$ and $\{h_{\phi,\mathsf{w}} \circ \phi = 0\} = \phi^{-1}(\{h_{\phi,\mathsf{w}} = 0\})$, we infer from (6) that $\mu_{\mathsf{w}}(\{h_{\phi,\mathsf{w}} \circ \phi = 0\}) = 0$, which yields (8). □

In view of [11, Proposition 8(v) and Theorem 18], the following is valid.

$$\begin{aligned} &\textit{If (6) holds, then } C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu)) \textit{ if and only if } h_{\phi,\mathsf{w}} \in L^\infty(\mu); \\ &\textit{if this is the case, then } \|C_{\phi,\mathsf{w}}\|^2 = \|h_{\phi,\mathsf{w}}\|_{L^\infty(\mu)} \textit{ and the modulus} \\ &|C_{\phi,\mathsf{w}}| \textit{ of } C_{\phi,\mathsf{w}} \textit{ equals the operator } M_{h_{\phi,\mathsf{w}}^{1/2}} \textit{ of multiplication by } h_{\phi,\mathsf{w}}^{1/2} \textit{ in } L^2(\mu). \end{aligned} \tag{9}$$

From now on, we assume that $h_{\phi,\mathsf{w}}$ takes finite values whenever $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$. This assumption is justified by (9).

Proposition 2.4 *Suppose that (6) holds and* $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$*. Then*

(i) *if* $c \in (0, \infty)$*, then* $\|C_{\phi,\mathsf{w}} f\| \geqslant c\|f\|$ *for all* $f \in L^2(\mu)$ *if and only if* $h_{\phi,\mathsf{w}} \geqslant c^2$ *a.e.* $[\mu]$*,*

(ii) *if* $C_{\phi,\mathsf{w}}$ *is bounded from below, then the Cauchy dual* $C'_{\phi,\mathsf{w}}$ *of* $C_{\phi,\mathsf{w}}$ *equals* $C_{\phi,\mathsf{w}'}$*, where* $\mathsf{w}' : X \to \mathbb{C}$ *is any* $\mathscr{A}$*-measurable function such that*

$$\mathsf{w}' = \begin{cases} \dfrac{\mathsf{w}}{h_{\phi,\mathsf{w}} \circ \phi} & \text{on } \{\mathsf{w} \neq 0\} \cap \{h_{\phi,\mathsf{w}} \circ \phi > 0\}, \\ 0 & \text{on } \{\mathsf{w} = 0\}; \end{cases} \tag{10}$$

in this case, $h_{\phi,\mathsf{w}'} = \frac{1}{h_{\phi,\mathsf{w}}}$ *a.e.* $[\mu]$.

Proof

(i) Adapting the proof of [9, Proposition 4], we get the statement (i).

(ii) Note that by (9) and (i), there exists $c_1, c_2 \in (0, \infty)$ such that $c_1 \leqslant h_{\phi,\mathsf{w}} \leqslant c_2$ a.e. $[\mu]$. By virtue of (9), we have

$$C'_{\phi,\mathsf{w}} = C_{\phi,\mathsf{w}}(|C_{\phi,\mathsf{w}}|^2)^{-1} = C_{\phi,\mathsf{w}} M^{-1}_{h_{\phi,\mathsf{w}}} = C_{\phi,\mathsf{w}} M_g, \tag{11}$$

where $g : X \to [0, \infty)$ is an $\mathscr{A}$-measurable function such that $g = \frac{1}{h_{\phi,\mathsf{w}}}$ on the set $\{h_{\phi,\mathsf{w}} > 0\}$. It is easily seen that

$$\mathsf{w}' \overset{(10)}{=} \mathsf{w} \cdot (g \circ \phi) \quad \text{on} \quad (\{\mathsf{w} \neq 0\} \cap \{h_{\phi,\mathsf{w}} \circ \phi > 0\}) \cup \{\mathsf{w} = 0\}.$$

Hence, by Lemma 2.3, $\mathsf{w}' = \mathsf{w} \cdot (g \circ \phi)$ a.e. $[\mu]$. Proposition 2.2 yields

$$C'_{\phi,\mathsf{w}} \overset{(11)}{=} C_{\phi,\mathsf{w}\cdot(g \circ \phi)} = C_{\phi,\mathsf{w}'}.$$

Now applying [5, Theorem 1.6.12] and (7), we obtain

$$\begin{aligned} \mu_{\mathsf{w}'} \circ \phi^{-1}(\varDelta) = \mu_{\mathsf{w}\cdot(g\circ\phi)} \circ \phi^{-1}(\varDelta) &\overset{(5)}{=} \int_X (\chi_\varDelta \circ \phi)(g^2 \circ \phi) d\mu_{\mathsf{w}} \\ &= \int_\varDelta g^2 h_{\phi,\mathsf{w}} d\mu \\ &= \int_\varDelta \frac{1}{h_{\phi,\mathsf{w}}} d\mu, \quad \varDelta \in \mathscr{A}, \end{aligned}$$

which shows that $h_{\phi,\mathsf{w}'} = \frac{1}{h_{\phi,\mathsf{w}}}$ a.e. $[\mu]$. This completes the proof. □

Recall that (see [11, Lemma 26]) if (6) holds and $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$, then for any integer $n \geqslant 1$, the nth power $C^n_{\phi,\mathsf{w}}$ of $C_{\phi,\mathsf{w}}$ equals $C_{\phi^n,\mathsf{w}_{[n]}}$, where ϕ^n denotes the n-fold composition of ϕ with itself (ϕ^0 is the identity map on X) and $\mathsf{w}_{[n]} : X \to \mathbb{C}$ is the function given by

$$\mathsf{w}_{[0]} = 1 \text{ and } \mathsf{w}_{[n]} = \prod_{j=0}^{n-1} \mathsf{w} \circ \phi^j \text{ for } n \geqslant 1. \tag{12}$$

Moreover, $h_{\phi^0,\mathsf{w}_{[0]}} = 1$ a.e. $[\mu]$ and the following recurrence formula holds:

$$h_{\phi^{n+1},\mathsf{w}_{[n+1]}} = h_{\phi,\mathsf{w}} \cdot \mathsf{E}_{\phi,\mathsf{w}}\big(h_{\phi^n,\mathsf{w}_{[n]}}\big) \circ \phi^{-1} \text{ a.e. } [\mu], \quad n \in \mathbb{Z}_+, \tag{13}$$

where $\mathsf{E}_{\phi,\mathsf{w}}(f)$ stands for the conditional expectation of an $\mathscr{A}$-measurable function $f : X \to [0,\infty)$ with respect to the σ-algebra $\phi^{-1}(\mathscr{A})$ and the measure μ_{w}; we refer the reader to [11, Sect. 2.4] for the precise definitions of $\mathsf{E}_{\phi,\mathsf{w}}(f)$ and $\mathsf{E}_{\phi,\mathsf{w}}(f) \circ \phi^{-1}$. The above discussion and Proposition 2.4 yield the following.

Proposition 2.5 *Suppose that (6) holds, $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is bounded from below. Then for every $n \in \mathbb{Z}_+$, the n th power $C'^n_{\phi,\mathsf{w}}$ of the Cauchy dual $C'_{\phi,\mathsf{w}}$ of $C_{\phi,\mathsf{w}}$ equals $C_{\phi^n,\mathsf{w}'_{[n]}}$, where $\mathsf{w}'_{[n]} : X \to \mathbb{C}$ is the function given by*

$$\mathsf{w}'_{[0]} = 1 \text{ and } \mathsf{w}'_{[n]} = \prod_{j=0}^{n-1} \mathsf{w}' \circ \phi^j \text{ for } n \geqslant 1, \tag{14}$$

with w' as in (10). Moreover, $h_{\phi^0,\mathsf{w}'_{[0]}} = 1$ a.e. $[\mu]$ and

$$h_{\phi^{n+1},\mathsf{w}'_{[n+1]}} = \frac{1}{h_{\phi,\mathsf{w}}} \mathsf{E}_{\phi,\mathsf{w}'}(h_{\phi^n,\mathsf{w}'_{[n]}}) \circ \phi^{-1} \text{ a.e. } [\mu], \quad n \in \mathbb{Z}_+.$$

The subnormality of the Cauchy dual $C'_{\phi,\mathsf{w}}$ of a left-invertible bounded weighted composition operator $C_{\phi,\mathsf{w}}$ can be characterized as follows (cf. [18, Theorem 4.5]).

Proposition 2.6 *Suppose that (6) holds, $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is bounded from below. Then*

(i) *$C'_{\phi,\mathsf{w}}$ is subnormal if and only if $\{h_{\phi^n,\mathsf{w}'_{[n]}}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for μ-a.e. $x \in X$,*
(ii) *if $\|C_{\phi,\mathsf{w}}(f)\| \geqslant \|f\|$ for all $f \in L^2(\mu)$ and $C'_{\phi,\mathsf{w}}$ is subnormal, then the sequence $\{h_{\phi^n,\mathsf{w}'_{[n]}}(x)\}_{n=0}^\infty$ is a Hausdorff moment sequence for μ-a.e. $x \in X$.*

Proof

(i) Combine [11, Theorem 49] with Proposition 2.5.
(ii) Suppose now that $\|C_{\phi,\mathsf{w}}(f)\| \geqslant \|f\|$ for all $f \in L^2(\mu)$ and $C'_{\phi,\mathsf{w}}$ is subnormal. Then $C'_{\phi,\mathsf{w}}$ is a contraction and consequently so is its nth power $C'^n_{\phi,\mathsf{w}}$ for every $n \in \mathbb{Z}_+$. In view of (9) and Proposition 2.5, the sequence $\{h_{\phi^n,\mathsf{w}'_{[n]}}(x)\}_{n=0}^\infty$ is bounded by 1 for μ-a.e. $x \in X$. Applying (i) and (2) completes the proof. □

We conclude this section by characterizing bounded 2-isometric weighted composition operators (see [17, Lemma 2.3] for the case of composition operators).

Proposition 2.7 *Suppose that (6) holds and* $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$. *Then the following conditions are equivalent:*

(i) $C_{\phi,\mathsf{w}}$ *is a* 2*-isometry,*
(ii) $1 - 2h_{\phi,\mathsf{w}} + h_{\phi^2,\mathsf{w}_{[2]}} = 0$ *a.e.* $[\mu]$,
(iii) $1 - 2h_{\phi,\mathsf{w}} + h_{\phi,\mathsf{w}} \cdot \mathsf{E}_{\phi,\mathsf{w}}(h_{\phi,\mathsf{w}}) \circ \phi^{-1} = 0$ *a.e.* $[\mu]$.

Proof Since $C^n_{\phi,\mathsf{w}} = C_{\phi^n,\mathsf{w}_{[n]}}$, we infer from (9) that $C^{*n}_{\phi,\mathsf{w}} C^n_{\phi,\mathsf{w}} = M_{h_{\phi^n,\mathsf{w}_{[n]}}}$ for any $n \in \mathbb{Z}_+$. Applying the definition of 2-isometricity, one can show that (i) and (ii) are equivalent. That (ii) and (iii) are equivalent follows from (13). □

3 A family of weighted composition operators on $\ell^2(\mathbb{Z}_+)$

In this section we concentrate on a family of weighted composition operators coming from [9, Example 42] (see also [10, Section 3.2]).

Example 3.1 Denote by $\mathscr{A}$ the power set $2^{\mathbb{Z}_+}$ of $\mathbb{Z}_+$ and by μ the counting measure on $2^{\mathbb{Z}_+}$. Clearly, all selfmaps of $\mathbb{Z}_+$ and complex functions on $\mathbb{Z}_+$ are $\mathscr{A}$-measurable. Note that $(\mathbb{Z}_+, \mathscr{A}, \mu)$ is a σ-finite measure space. For $n \in \mathbb{Z}_+$, we denote by e_n the element of $L^2(\mu)$ given by

$$e_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L^2(\mu)$. Define the map $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ by

$$\phi(n) = \begin{cases} 0 & n = 0, \\ n-1 & \text{otherwise.} \end{cases}$$

Let $\mathsf{w} : \mathbb{Z}_+ \to \mathbb{C}$ be any function. It is obvious that $\mu_{\mathsf{w}} \circ \phi^{-1} \ll \mu$ and so $C_{\phi,\mathsf{w}}$ is well defined (cf. Proposition 2.2). It follows from (7) that

$$h_{\phi,\mathsf{w}}(n) = \sum_{j \in \phi^{-1}(\{n\})} |\mathsf{w}(j)|^2, \quad n \in \mathbb{Z}_+, \tag{15}$$

which yields

$$h_{\phi,\mathsf{w}}(0) = \alpha_{\mathsf{w}} \text{ and } h_{\phi,\mathsf{w}}(n) = |\mathsf{w}(n+1)|^2 \text{ for } n \geqslant 1, \tag{16}$$

where

$$\alpha_{\mathsf{w}} := |\mathsf{w}(0)|^2 + |\mathsf{w}(1)|^2.$$

Combined with (9), this implies that

$C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ *if and only if* $\sup_{n\geqslant 0} |\mathsf{w}(n)| < \infty$; *if this is the case, then*

$$\|C_{\phi,\mathsf{w}}\|^2 = \max\left\{\alpha_{\mathsf{w}}, \sup_{n\geqslant 2} |\mathsf{w}(n)|^2\right\}. \tag{17}$$

Moreover, by Proposition 2.4(i), $C_{\phi,\mathsf{w}}$ is bounded from below if and only if

$$\min\left\{\alpha_{\mathsf{w}}, \inf_{n\geqslant 2} |\mathsf{w}(n)|^2\right\} > 0. \tag{18}$$

Concerning the cyclicity of $C_{\phi,\mathsf{w}}$, one can make the following observation.

If $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ *and* $\mathsf{w}(n) \neq 0$ *for every* $n \geqslant 1$, *then* $C_{\phi,\mathsf{w}}$ *is cyclic with the cyclic vector* e_0. (19)

This can be deduced from the equality (20) below which is a direct consequence of the definition.

$$C_{\phi,\mathsf{w}} e_n = \begin{cases} \mathsf{w}(0)e_0 + \mathsf{w}(1)e_1 & \text{if } n = 0, \\ \mathsf{w}(n+1)e_{n+1} & \text{if } n \geqslant 1. \end{cases} \tag{20}$$

Using Proposition 2.4, we can describe the Cauchy dual $C'_{\phi,\mathsf{w}}$ of a left-invertible $C_{\phi,\mathsf{w}}$ as follows:

$$\begin{gathered} \textit{If}\, C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu)) \textit{ and } (18) \textit{ holds, then } C'_{\phi,\mathsf{w}} = C_{\phi,\mathsf{w}'}, \textit{where} \\ \mathsf{w}'(n) = \begin{cases} \dfrac{\mathsf{w}(n)}{\alpha_{\mathsf{w}}} & \text{if } n = 0, 1, \\ \dfrac{1}{\overline{\mathsf{w}(n)}} & \text{if } n \geqslant 2. \end{cases} \end{gathered} \tag{21}$$

As a consequence, we get

$$C'_{\phi,\mathsf{w}} e_n = \begin{cases} \dfrac{\mathsf{w}(0)}{\alpha_{\mathsf{w}}} e_0 + \dfrac{\mathsf{w}(1)}{\alpha_{\mathsf{w}}} e_1 & \text{if } n = 0, \\ \dfrac{1}{\overline{\mathsf{w}(n+1)}} e_{n+1} & \text{if } n \geqslant 1. \end{cases}$$

$\diamondsuit$

Below, we give necessary and sufficient conditions for the weighted composition operator $C_{\phi,\mathsf{w}}$ from Example 3.1 to be 2-isometric. Before we do this, we define the functions $\xi_n : [1,\infty) \to [1,\infty)$, where $n \in \mathbb{Z}_+$, by

$$\xi_n(x) = \sqrt{\frac{1 + (n+1)(x^2-1)}{1 + n(x^2-1)}}, \quad x \in [1,\infty), \quad n \in \mathbb{Z}_+. \tag{22}$$

Proposition 3.2 *Let* $\mathscr{A}$, μ, ϕ *and* w *be as in Example 3.1. Assume that* $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$. *Then* $C_{\phi,\mathsf{w}}$ *is a 2-isometry if and only if*

$$|\mathsf{w}(2)| \geq 1, \quad |\mathsf{w}(n+2)| = \xi_n(|\mathsf{w}(2)|), \quad n \in \mathbb{Z}_+, \tag{23}$$

and

$$\begin{cases} |\mathsf{w}(0)| = 1 & \text{if } \mathsf{w}(1) = 0, \\ |\mathsf{w}(2)| = \dfrac{\sqrt{\alpha_{\mathsf{w}}(2 - |\mathsf{w}(0)|^2) - 1}}{|\mathsf{w}(1)|} & \text{if } \mathsf{w}(1) \neq 0. \end{cases} \tag{24}$$

Moreover, if $C_{\phi,\mathsf{w}}$ is a 2-isometry, then

(i) *if $\mathsf{w}(1) \neq 0$ and either $|\mathsf{w}(0)| = 1$ or $\alpha_{\mathsf{w}} = 1$, then $|\mathsf{w}(n)| = 1$ for all $n \geqslant 2$,*
(ii) $(\alpha_{\mathsf{w}} - 1)(1 - |\mathsf{w}(0)|^2) \geqslant 0.$

Proof Arguing as in (15) with $(\phi^2, \mathsf{w}_{[2]})$ in place of (ϕ, w) and using (12), we obtain

$$h_{\phi^2,\mathsf{w}_{[2]}}(n) = \begin{cases} |\mathsf{w}(n+1)|^2|\mathsf{w}(n+2)|^2 & \text{if } n \geqslant 1, \\ |\mathsf{w}(0)|^4 + |\mathsf{w}(0)|^2|\mathsf{w}(1)|^2 + |\mathsf{w}(1)|^2|\mathsf{w}(2)|^2 & \text{if } n = 0. \end{cases}$$

Combined with Proposition 2.7(ii), we deduce that $C_{\phi,\mathsf{w}}$ is a 2-isometry if and only if

$$|\mathsf{w}(1)|^2|\mathsf{w}(2)|^2 = \alpha_{\mathsf{w}}(2 - |\mathsf{w}(0)|^2) - 1 \tag{25}$$

and

$$1 - 2|\mathsf{w}(n+2)|^2 + |\mathsf{w}(n+2)|^2|\mathsf{w}(n+3)|^2 = 0, \quad n \in \mathbb{Z}_+. \tag{26}$$

It is easily seen that (24) is equivalent to (25). Observe that the condition (26) holds if and only if the unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$ with weights $\{|w(n+2)|\}_{n=0}^{\infty}$ is 2-isometric. Hence, by [19, Lemma 6.1], the condition (26) is equivalent to (23). Using $|\mathsf{w}(2)| \geq 1$ and the fact that the expression under the sign of the square root in (24) is nonnegative, we get the "moreover" part. □

It is well known that any 2-isometry is expansive (see [20, Lemma 1]), so we can consider its Cauchy dual operator. Below, we follow the conventions (4).

Theorem 3.3 *Let $\mathscr{A}$, μ, ϕ and w be as in Example 3.1. Assume that $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is a 2-isometry. Then the following assertions hold :*

(i) *the Cauchy dual $C'_{\phi,\mathsf{w}}$ of $C_{\phi,\mathsf{w}}$ is subnormal if and only if $\{h_{\phi^n,\mathsf{w}'_{[n]}}(0)\}_{n=0}^{\infty}$ is a Hausdorff moment sequence,*
(ii) *the value of $h_{\phi^n,\mathsf{w}'_{[n]}}$ at 0 is given by the following explicit formula*

$$h_{\phi^n,\mathsf{w}'_{[n]}}(0) = \frac{|\mathsf{w}(0)|^{2n}}{\alpha_{\mathsf{w}}^{2n}} + \sum_{j=0}^{n-1} \frac{|\mathsf{w}(0)|^{2(n-j-1)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2(n-j)}(1 + j(|\mathsf{w}(2)|^2 - 1))}, \quad n \in \mathbb{Z}_+. \tag{27}$$

Proof

(i) The "only if" part is a direct consequence Proposition 2.6(ii). In view of Proposition 2.6(i) to prove the "if" part, it suffices to show that the sequence $\{h_{\phi^n,\mathsf{w}'_{[n]}}(k)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $k \geqslant 1$. Fix $k \geqslant 1$. Arguing as in (15) with $(\phi^n, \mathsf{w}'_{[n]})$ in place of (ϕ, w), we verify that

$$h_{\phi^n,\mathsf{w}'_{[n]}}(k) = |\mathsf{w}'_{[n]}(k+n)|^2 \overset{(12)}{=} \prod_{j=1}^{n} |\mathsf{w}'(k+j)|^2 \overset{(21)}{=} \frac{1}{\prod_{j=1}^{n} |\mathsf{w}(k+j)|^2}, \quad n \geqslant 1.$$

Since the unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$ with weights $\{|\mathsf{w}(k+1+n)|\}_{n=0}^{\infty}$ is 2-isometric (see the proof of Proposition 3.2), we deduce from[1] [6, Remark 4] that the unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$ with weights $\left\{\frac{1}{|\mathsf{w}(k+1+n)|}\right\}_{n=0}^{\infty}$ is subnormal, which by Berger-Gellar-Wallen theorem (see [15, 16]) is equivalent to the fact that the sequence $\{h_{\phi^n,\mathsf{w}'_{[n]}}(k)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence.

(ii) Arguing as in (i), we get

$$\begin{aligned}
h_{\phi^n,\mathsf{w}'_{[n]}}(0) &= \sum_{j=0}^{n} |\mathsf{w}'_{[n]}(j)|^2 \\
&\overset{(14)}{=} \sum_{j=0}^{n} \prod_{l=0}^{j-1} |\mathsf{w}'(\phi^l(j))|^2 \prod_{l=j}^{n-1} |\mathsf{w}'(\phi^l(j))|^2 \\
&= |\mathsf{w}'(0)|^{2n} + \sum_{j=1}^{n} |\mathsf{w}'(0)|^{2(n-j)} \prod_{l=1}^{j} |\mathsf{w}'(l)|^2 \\
&\overset{(21)}{=} \frac{|\mathsf{w}(0)|^{2n}}{\alpha_{\mathsf{w}}^{2n}} + \frac{|\mathsf{w}(0)|^{2(n-1)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2n}} + \sum_{j=2}^{n} \frac{|\mathsf{w}(0)|^{2(n-j)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2(n-j+1)} \prod_{l=2}^{j} |\mathsf{w}(l)|^2} \\
&\overset{(23)}{=} \frac{|\mathsf{w}(0)|^{2n}}{\alpha_{\mathsf{w}}^{2n}} + \frac{|\mathsf{w}(0)|^{2(n-1)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2n}} + \sum_{j=1}^{n-1} \frac{|\mathsf{w}(0)|^{2(n-j-1)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2(n-j)}(1+j(|\mathsf{w}(2)|^2-1))} \\
&= \frac{|\mathsf{w}(0)|^{2n}}{\alpha_{\mathsf{w}}^{2n}} + \sum_{j=0}^{n-1} \frac{|\mathsf{w}(0)|^{2(n-j-1)}|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2(n-j)}(1+j(|\mathsf{w}(2)|^2-1))}, \quad n \geqslant 2.
\end{aligned}$$

It is a matter of simple verification that (27) holds for $n = 0, 1$ as well. □

[1] Recall that a 2-isometry is m-isometric for every integer $m \geqslant 2$ (see [1, §1]), and thus by [20, Lemma 1(a)] it is completely hyperexpansive.

Corollary 3.4 *Let $\mathscr{A}$, μ, ϕ and* w *be as in Example 3.1. Assume that* $|\mathsf{w}(0)| = |\mathsf{w}(n)| = 1$ *for every integer* $n \geq 2$. *Then* $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$, $C_{\phi,\mathsf{w}}$ *is a 2-isometry and* $C'_{\phi,\mathsf{w}}$ *is subnormal. Moreover,* $C'_{\phi,\mathsf{w}}$ *is an isometry if and only if* $C_{\phi,\mathsf{w}}$ *is an isometry or, equivalently, if and only if* $\mathsf{w}(1) = 0$.

Proof By (17) and Proposition 3.2, $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is a 2-isometry. Using Theorem 3.3(ii), we see that if $\mathsf{w}(1) \neq 0$, then

$$\begin{aligned} h_{\phi^n,\mathsf{w}'_{[n]}}(0) &= \frac{1}{\alpha_{\mathsf{w}}^{2n}} + \sum_{j=0}^{n-1} \frac{|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^{2(n-j)}} \\ &= \frac{1}{\alpha_{\mathsf{w}}^{2n}} + \frac{|\mathsf{w}(1)|^2(\alpha_{\mathsf{w}}^{2n} - 1)}{\alpha_{\mathsf{w}}^{2n}(\alpha_{\mathsf{w}}^2 - 1)} \\ &= \frac{|\mathsf{w}(1)|^2}{\alpha_{\mathsf{w}}^2 - 1} + \frac{\alpha_{\mathsf{w}}}{\alpha_{\mathsf{w}} + 1}(\alpha_{\mathsf{w}}^{-2})^n \\ &= \frac{1}{2 + |\mathsf{w}(1)|^2} + \frac{1 + |\mathsf{w}(1)|^2}{2 + |\mathsf{w}(1)|^2}(\alpha_{\mathsf{w}}^{-2})^n, \quad n \in \mathbb{Z}_+, \end{aligned}$$

which implies that the sequence $\{h_{\phi^n,\mathsf{w}'_{[n]}}(0)\}_{n=0}^{\infty}$ is a Hausdorff moment sequence. The same is true if $\mathsf{w}(1) = 0$ because then by (27), $h_{\phi^n,\mathsf{w}'_{[n]}}(0) = 1$ for all $n \in \mathbb{Z}_+$. Applying Theorem 3.3(i), we see that $C'_{\phi,\mathsf{w}}$ is subnormal. The first equivalence in the "moreover" part is true for arbitrary left-invertible operators, while the second is a direct consequence of [11, (2.22)] and (16). This completes the proof. $\square$

4 Main example

The following example shows that the Cauchy dual subnormality problem has a negative solution even in the class of cyclic operators.

Example 4.1 (Example 3.1 continued). Let $\mathscr{A}$, μ, ϕ and w be as in Example 3.1. To achieve the main purpose of this paper, we will begin by specifying the weight w. Let x is any positive real number and let w be the weight function constructed as follows (for notational convenience the dependence of w on x will not be expressed explicitly). Set

$$\mathsf{w}(0) = \frac{1}{\sqrt{2}}, \quad \mathsf{w}(1) = \sqrt{\frac{1}{2} + x}. \tag{28}$$

Then clearly $\alpha_{\mathsf{w}}(2 - \mathsf{w}(0)^2) - 1 > 0$ and

$$\mathsf{w}(2) := \frac{\sqrt{\alpha_{\mathsf{w}}(2 - \mathsf{w}(0)^2) - 1}}{\mathsf{w}(1)} = \sqrt{\frac{1 + 3x}{1 + 2x}} > 1. \tag{29}$$

Set

$$\mathsf{w}(n+2) := \xi_n(\mathsf{w}(2)), \quad n \geqslant 1, \tag{30}$$

where the functions ξ_n are as in (22). It follows from the definition of w and (22) that $\sup_{n\geqslant 0} \mathsf{w}(n) < \infty$, so by (17), $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$. Using Proposition 3.2, we deduce that $C_{\phi,\mathsf{w}}$ is a 2-isometry. It follows from (27) that

$$h_{\phi^n,\mathsf{w}'_{[n]}}(0) = \frac{1}{2^n(1+x)^{2n}} + \sum_{j=0}^{n-1} \frac{1+2x}{2^{n-j}(1+x)^{2(n-j)}(1+j(\mathsf{w}(2)^2-1))}, \quad n \in \mathbb{Z}_+.$$

By (29), we have

$$1 + j(\mathsf{w}(2)^2 - 1) = \frac{1+(j+2)x}{1+2x},$$

which yields

$$h_{\phi^n,\mathsf{w}'_{[n]}}(0) = \frac{1}{2^n(1+x)^{2n}}\left(1 + (1+2x)^2 \sum_{j=0}^{n-1} \frac{2^j(1+x)^{2j}}{1+(j+2)x}\right), \quad n \in \mathbb{Z}_+. \tag{31}$$

$\diamondsuit$

Below we will continue the necessary preparations to achieve the main goal of this paper. For every $n \in \mathbb{Z}_+$, we define the real valued function ω_n on $\Omega_n := (-\frac{1}{n+1}, \infty)$ by

$$\omega_n(x) = \frac{1}{2^n(1+x)^{2n}}\left(1 + (1+2x)^2 \sum_{j=0}^{n-1} \frac{2^j(1+x)^{2j}}{1+(j+2)x}\right), \quad x \in \Omega_n,\ n \in \mathbb{Z}_+.$$

Clearly, the following holds

$$\omega_n(x) = \frac{1}{2^n(1+x)^{2n}}\Big(1 + (1+2x)^2 S_n(x)\Big), \qquad x \in \Omega_n,\ n \in \mathbb{Z}_+, \tag{32}$$

where

$$S_n(x) := \sum_{j=0}^{n-1} \frac{2^j(1+x)^{2j}}{1+(j+2)x}, \quad x \in \Omega_n,\ n \in \mathbb{Z}_+.$$

Set

$$D_m(x) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \omega_n(x), \quad x \in \Omega_m,\ m \in \mathbb{Z}_+. \tag{33}$$

Then

$$D_m^{(l)}(x) = \sum_{n=0}^{m}(-1)^n\binom{m}{n}\omega_n^{(l)}(x), \quad x \in \Omega_m,\ m \in \mathbb{Z}_+,\ l \in \mathbb{Z}_+, \tag{34}$$

where $D_m^{(l)}$ (resp. $\omega_n^{(l)}$) stands for the l-th derivative of D_m (resp. ω_n). Applying the general Leibniz rule, we see that the l-th derivative $S_n^{(l)}$ of S_n is given by

$$S_n^{(l)}(x) = \sum_{j=0}^{n-1} 2^j \sum_{k=0}^{l}\binom{l}{k}\Big(\frac{1}{1+(j+2)x}\Big)^{(k)}((1+x)^{2j})^{(l-k)}, \quad x \in \Omega_n,\ l,n \in \mathbb{Z}_+.$$

In particular, for every $n \in \mathbb{Z}_+$ we have

$$\begin{aligned} S_n(0) &= 2^n - 1,\\ S_n^{(1)}(0) &= n2^n - 4(2^n-1),\\ S_n^{(2)}(0) &= 2n^2 2^n - 10n2^n + 24(2^n-1),\\ S_n^{(3)}(0) &= 2n^3 2^n - 30n^2 2^n + 100n2^n - 192(2^n-1),\\ S_n^{(4)}(0) &= 8n^4 2^n - 56n^3 2^n + 460n^2 2^n - 1324n2^n + 2208(2^n-1). \end{aligned} \tag{35}$$

Now we compute the lth derivative $\omega_n^{(l)}$ of ω_n.

Lemma 4.2 *The following assertions are valid*:

(i) *if $n \in \mathbb{Z}_+$, l is a positive integer and x varies over Ω_n, then* (see (4))

$$\begin{aligned} \omega_n^{(l)}(x) = {} & \left(\frac{1}{(1+x)^{2n}}\right)^{(l)}\left[\frac{1+(1+2x)^2 S_n(x)}{2^n}\right]\\ & + l\left(\frac{1}{(1+x)^{2n}}\right)^{(l-1)}\left[\frac{2^2(1+2x)S_n(x)+(1+2x)^2 S_n^{(1)}(x)}{2^n}\right]\\ & + \sum_{i=2}^{l}\binom{l}{i}\left(\frac{1}{(1+x)^{2n}}\right)^{(l-i)}\\ & \times\left[\frac{2^3\binom{i}{2}S_n^{(i-2)}(x) + 2^2\binom{i}{1}(1+2x)S_n^{(i-1)}(x) + (1+2x)^2 S_n^{(i)}(x)}{2^n}\right], \end{aligned} \tag{36}$$

(ii) *if $l \in \{0,1,2,3\}$ and $m \in \{4,5,6,\ldots\}$, then $D_m^{(l)}(0) = 0$.*

Proof

(i) Applying the general Leibniz rule twice, we get

$$
\begin{aligned}
2^n\omega_n^{(l)}(x) &\overset{(32)}{=} \sum_{i=0}^{l}\binom{l}{i}\left(\frac{1}{(1+x)^{2n}}\right)^{(l-i)}\left(1+(1+2x)^2S_n(x)\right)^{(i)}\\
&= \left(\frac{1}{(1+x)^{2n}}\right)^{(l)}\left(1+(1+2x)^2S_n(x)\right)\\
&\quad + \sum_{i=1}^{l}\binom{l}{i}\left(\frac{1}{(1+x)^{2n}}\right)^{(l-i)}\left((1+2x)^2S_n(x)\right)^{(i)}\\
&= \left(\frac{1}{(1+x)^{2n}}\right)^{(l)}\left(1+(1+2x)^2S_n(x)\right)\\
&\quad + l\left(\frac{1}{(1+x)^{2n}}\right)^{(l-1)}\left(2^2(1+2x)S_n(x)+(1+2x)^2S_n^{(1)}(x)\right)\\
&\quad + \sum_{i=2}^{l}\binom{l}{i}\left(\frac{1}{(1+x)^{2n}}\right)^{(l-i)}\sum_{m=0}^{i}\binom{i}{m}((1+2x)^2)^{(m)}S_n^{(i-m)}(x),
\end{aligned}
$$

which implies that (36) holds for $l \geqslant 2$, $n \in \mathbb{Z}_+$ and $x \in \Omega_n$. It is a matter of routine to verify that (36) holds for $l = 1$, $n \in \mathbb{Z}_+$ and $x \in \Omega_n$ as well. This yields (i).

(ii) Using (35) we verify that the factors in the square brackets appearing in (36) (the third one for $i = 2, 3$) when calculated at 0 are of the form:

$$
\begin{aligned}
&\frac{S_n(0)+1}{2^n} = p_0(n), \text{ where } p_0 \in \mathbb{C}[z] \text{ is of degree } 0,\\
&\frac{4S_n(0)+S_n^{(1)}(0)}{2^n} = p_1(n), \text{ where } p_1 \in \mathbb{C}[z] \text{ is of degree } 1,\\
&\frac{8S_n(0)+8S_n^{(1)}(0)+S_n^{(2)}(0)}{2^n} = p_2(n), \text{ where } p_2 \in \mathbb{C}[z] \text{ is of degree } 2,\\
&\frac{24S_n^{(1)}(0)+12S_n^{(2)}(0)+S_n^{(3)}(0)}{2^n} = p_3(n), \text{ where } p_3 \in \mathbb{C}[z] \text{ is of degree } 3.
\end{aligned}
\tag{37}
$$

This together with (36) implies that $\omega_n^{(l)}(0)$ is a polynomial in n of degree at most l whenever $l \in \{0, 1, 2, 3\}$. Therefore, (ii) is a direct consequence of (1) and (34). □

Lemma 4.3 *For every integer $m \geqslant 5$, there exists $\varepsilon_m \in (0, \infty)$ such that $D_m(x) < 0$ for every $x \in (0, \varepsilon_m)$.*

Proof It follows from (35) that that the third factor in the square brackets appearing in (36) for $i = 4$ when calculated at 0 is of the form

$$\frac{48S_n^{(2)}(0)+16S_n^{(3)}(0)+S_n^{(4)}(0)}{2^n}=p_4(n)-\frac{288}{2^n},\quad x\in\Omega_n,\ n\in\mathbb{Z}_+,$$

where $p_4\in\mathbb{C}[z]$ is of degree 4. Combined with (37) and (36), this yields

$$\begin{aligned}\omega_n^{(4)}(0)=&\,p_0(n)\left(\frac{1}{(1+x)^{2n}}\right)^{(4)}\Big|_{x=0}+4p_1(n)\left(\frac{1}{(1+x)^{2n}}\right)^{(3)}\Big|_{x=0}\\&+6p_2(n)\left(\frac{1}{(1+x)^{2n}}\right)^{(2)}\Big|_{x=0}+4p_3(n)\left(\frac{1}{(1+x)^{2n}}\right)^{(1)}\Big|_{x=0}\\&+\left(p_4(n)-\frac{288}{2^n}\right)\left(\frac{1}{(1+x)^{2n}}\right)^{(0)}\Big|_{x=0},\quad n\in\mathbb{Z}_+.\end{aligned}$$

Hence, $\omega_n^{(4)}(0)+\frac{288}{2^n}$ is a polynomial in n of degree at most 4. This fact, together with (1) and (34) implies that

$$D_m^{(4)}(0)=-288\sum_{n=0}^{m}(-1)^n\binom{m}{n}2^{-n}=-\frac{288}{2^m}<0,\quad m\geq 5.$$

Now applying Lemma 4.2(ii) and Taylor's theorem to D_m (see [21, Theorem 5.15] with $n=4$ and $\alpha=0$) completes the proof. □

Concerning Lemma 4.3, the reader is referred to Fig. 1. Now we are ready to state the main result of the paper.

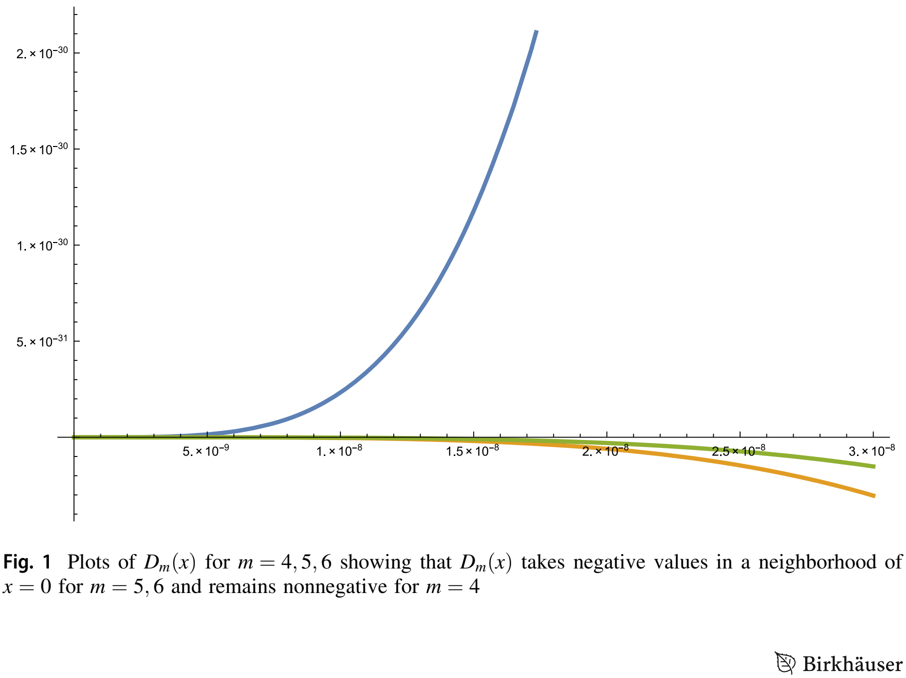

Fig. 1 Plots of $D_m(x)$ for $m=4,5,6$ showing that $D_m(x)$ takes negative values in a neighborhood of $x=0$ for $m=5,6$ and remains nonnegative for $m=4$

Theorem 4.4 *Let $\mathscr{A}$, μ, ϕ and* w *be as in Example 4.1 (with* w *given by (28)–(30)). Then there exists $\varepsilon \in (0, \infty)$ such that for every $x \in (0, \varepsilon)$, $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is a cyclic 2-isometry such that $C'_{\phi,\mathsf{w}}$ is not subnormal.*

Proof It follows from (19) and Example 4.1 that if $x \in (0, \infty)$, then $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and $C_{\phi,\mathsf{w}}$ is a cyclic 2-isometry. In turn, by Lemma 4.3, there exists $\varepsilon \in (0, \infty)$ such that $D_5(x) < 0$ for every $x \in (0, \varepsilon)$. Applying (3), (33) and (31), we deduce that whenever $x \in (0, \varepsilon)$, $\{h_{\phi^n, \mathsf{w}'_{[n]}}(0)\}_{n=0}^{\infty}$ is not a Hausdorff moment sequence and consequently, by Theorem 3.3(i), $C'_{\phi,\mathsf{w}}$ is not subnormal. $\square$

We conclude the paper with the following observation.

Remark 4.5 It is worth noting that if the parameter x in Example 4.1 equals 0, then $C_{\phi,\mathsf{w}} \in \boldsymbol{B}(L^2(\mu))$ and, by (16), (28), (29) and (30) with $x = 0$, $h_{\phi,\mathsf{w}}(n) = 1$ for all $n \in \mathbb{Z}_+$, which implies that $C_{\phi,\mathsf{w}}$ is an isometry[2] (see [11, (2.22)]). It follows that $C'_{\phi,\mathsf{w}}$ is an isometry and, as such, is subnormal (because each isometry has a unitary extension possibly in a larger Hilbert space, see e.g., [23, Proposition I.2.3]). Therefore, Theorem 4.4 breaks down at the point $x = 0$. $\diamondsuit$

References

1. Agler, J., Stankus, M.: *m*-isometric transformations of Hilbert spaces, I. Integr. Equ. Oper. Theory **21**, 383–429 (1995)
2. Agler, J., Stankus, M.: *m*-isometric transformations of Hilbert spaces, II. Integr. Equ. Oper. Theory **23**, 1–48 (1995)
3. Agler, J., Stankus, M.: *m*-isometric transformations of Hilbert spaces, III. Integr. Equ. Oper. Theory **24**, 379–421 (1996)
4. Anand, A., Chavan, S., Jabłoński, Z.J., Stochel, J.: A solution to the Cauchy dual subnormality problem for 2-isometries. J. Funct. Anal. **277**, 108292 (2019). **(p 51)**
5. Ash, R.B.: Probability and Measure Theory. Harcourt/Academic Press, Burlington (2000)
6. Athavale, A.: On completely hyperexpansive operators. Proc. Am. Math. Soc. **124**, 3745–3752 (1996)
7. Badea, C., Suciu, L.: The Cauchy dual and 2-isometric liftings of concave operators. J. Math. Anal. Appl. **472**, 1458–1474 (2019)
8. Berg, C., Christensen, J.P.R., Ressel, P.: Harmonic Analysis on Semigroups. Springer, Berlin (1984)
9. Budzyński, P., Jabłoński, Z.J., Jung, I.B., Stochel, J.: Unbounded subnormal composition operators in L^2-spaces. J. Funct. Anal. **269**, 2110–2164 (2015)
10. Budzyński, P., Jabłoński, Z.J., Jung, I.B., Stochel, J.: Subnormality of unbounded composition operators over one-circuit directed graphs: exotic examples. Adv. Math. **310**, 484–556 (2017)

[2] In fact, in view of (20), $\left(C_{\phi,\mathsf{w}}(L^2(\mu))\right)^{\perp} = \mathbb{C} \cdot (e_0 - e_1)$, so $C_{\phi,\mathsf{w}}$ is not unitary.

11. Budzyński, P., Jabłoński, Z.J., Jung, I.B., Stochel, J.: Unbounded weighted composition operators in L^2-spaces, Lect. Notes Math., vol. 2209, pp. xii+180. Springer (2018)
12. Chavan, S.: On operators Cauchy dual to 2-hyperexpansive operators. Proc. Edinb. Math. Soc. **50**, 637–652 (2007)
13. Conway, J.B.: The Theory of Subnormal Operators, Math. Surveys Monographs, vol. 36. Amer. Math. Soc. Providence (1991)
14. Dickinson, D.R.: Operators; An Algebraic Synthesis. Macmillan, London (1967)
15. Gellar, R., Wallen, L.J.: Subnormal weighted shifts and the Halmos–Bram criterion. Proc. Jpn. Acad. **46**, 375–378 (1970)
16. Halmos, P.R.: Ten problems in Hilbert space. Bull. Am. Math. Soc. **76**, 887–933 (1970)
17. Jabłoński, Z.J.: Hyperexpansive composition operators. Math. Proc. Camb. Philos. Soc. **135**, 513–526 (2003)
18. Jabłoński, Z.J., Kośmider, J.: *m*-isometric composition operators on a directed graph with one circuit (2019) **(preprint)**
19. Jabłoński, Z.J., Stochel, J.: Unbounded 2-hyperexpansive operators. Proc. Edinb. Math. Soc. **44**, 613–629 (2001)
20. Richter, S.: Invariant subspaces of the Dirichlet shift. J. Reine Angew. Math. **386**, 205–220 (1988)
21. Rudin, W.: Principles of Mathematical Analysis, International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York (1976)
22. Shimorin, S.: Wold-type decompositions and wandering subspaces for operators close to isometries. J. Reine Angew. Math. **531**, 147–189 (2001)
23. Sz.-Nagy, B., Foiaş, C.: Harmonic Analysis of Operators on Hilbert Space. Translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest (1970)