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$W^{1,p}$ versus C^1: The nonsmooth case involving critical growth

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In this paper, we study a class of generalized and not necessarily differentiable functionals of the form

$$J(u)=\int_\Omega G(x,\nabla u)dx-\int_\Omega j_1(x,u)dx-\int_{\partial\Omega} j_2(x,u)d\sigma$$

with functions $j_1\colon \Omega\times\mathbb{R}\to\mathbb{R}$, $j_2\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ that are only locally Lipschitz in the second argument and involving critical growth for the elements of their generalized gradients $\partial j_k(x,\cdot), k=1,2$ even on the boundary $\partial\Omega$. We generalize the famous result of Brezis and Nirenberg [H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.* **317**(5) (1993) 465–472] to a more general class of functionals and extend all the other generalizations of this result which has been published in the last decades.

Keywords: Nonhomogeneous partial differential operator; local minimizer; Clarke's generalized gradient; critical growth; Neumann problem.

Mathematics Subject Classification: 35-XX

1. Introduction

Consider the following functional $\Phi\colon H_0^1(\Omega)\to\mathbb{R}$ defined by

$$\Phi(u)=\frac{1}{2}\int_\Omega |\nabla u|^2dx-\int_\Omega F(x,u)dx,$$

where $F(x,s)=\int_0^s f(x,t)dt$ with a Carathéodory function $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ that satisfies the growth condition

$$|f(x,u)|\le C(1+|u|^p)\quad\text{with } p\le\frac{N+2}{N-2}.$$

It is well known that a local $C_0^1(\overline{\Omega})$-minimizer of Φ is also a local $H_0^1(\Omega)$-minimizer of Φ. Such a result is originally due to Brezis and Nirenberg [3] for functionals on H_0^1 and the critical points of Φ are weak solutions of the equation

$$-\Delta u=f(x,u)\quad\text{in }\Omega,$$

$$u=0\quad\text{on }\partial\Omega,$$

where Δ denotes the well-known Laplace differential operator. An extension of the result of Brezis and Nirenberg to functionals related with the p-Laplace differential operator was done by García Azorero *et al.* [6] who considered the functional $J_p\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$J_p(u)=\frac{1}{p}\int_\Omega|\nabla u|^pdx-\int_\Omega F(x,u)dx,$$

where $F(x,s)=\int_0^s f(x,t)dt$ and $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the following growth condition:

$$|f(x,s)|\le C(1+|s|^{r-1})\quad\text{with } r<\begin{cases}\dfrac{Np}{N-p} & \text{if } p<N,\\ \infty & \text{if } p\ge N.\end{cases}$$

A simpler proof than those in [6] but only in case $p>2$ was done by Guo and Zhang [11]. A nonsmooth version for functionals defined on $W_0^{1,p}(\Omega)$ with $p\ge 2$ has been studied by Motreanu and Papageorgiou [17].

The first paper concerning local minimizers of functional corresponding to nonlinear parametric Neumann problems was written by Motreanu *et al.* [16]. Therein, the potential $\Phi_0\colon W_n^{1,p}(\Omega) \to \mathbb{R}$ is defined by

$$\Phi_0(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F_0(z, x(z))dz, \quad 1 < p < \infty$$

with

$$W_n^{1,p}(\Omega) = \left\{ x \in W^{1,p}(\Omega)\colon \frac{\partial x}{\partial n} = 0 \right\},$$

where $\frac{\partial x}{\partial n}$ is the outer normal derivative of x and $F_0(z,x) = \int_0^x f_0(z,s)ds$. The first result dealing with nonsmooth functionals defined on $W_n^{1,p}(\Omega)$ for the case $2 \le p < \infty$ was proved by Barletta and Papageorgiou [2] while the general case $1 < p < \infty$ has been treated by Iannizzotto and Papageorgiou [13]. The first result concerning functionals defined on $W^{1,p}(\Omega)$ involving a boundary term was published by the third author in the smooth [21] and in the nonsmooth [22] case. Moreover, a singular functional $I\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\|u\|_{W_0^{1,p}(\Omega)} - \int_\Omega F(x, u^+)\, dx - \int_\Omega G(u^+)\, dx,$$

with $F(x,t) = \int_0^t f(x,s)ds$ and $G(t) = \int_0^t g(s)ds$ with $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ being a singular term such that $\lim_{t\to 0^+} g(t) = +\infty$ was studied by Giacomoni and Saoudi [10].

All the above-mentioned works are related to the p-Laplace differential operator. A first result concerning local minimizers and nonhomogeneous operators was presented in the work of Motreanu and Papageorgiou [18] who studied functionals of the form

$$\varphi_0(u) = \int_\Omega G(x, \nabla u)dx - \int_\Omega F_0(x,u)dx, \quad u \in W_n^{1,p}(\Omega),$$

where G is the potential of a general nonhomogeneous operator. A prototype of such operator is the (p,q)-Laplace differential operator which is the sum of the p- and q-Laplacian. A nonsmooth version of functionals related to nonhomogeneous operators defined on the space $W^{1,p}(\Omega)$ has been studied by Gasiński and Papageorgiou [8].

Recently, Papageorgiou and Rădulescu [19] studied functionals that are not only related to nonhomogeneous operator but also have a boundary term and the potential term in the domain is related to a Carathéodory function that has critical growth. Namely, they considered the functional $\varphi_0\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega F_0(z,u)dz,$$

where $F_0(z,x) = \int_0^x f_0(z,s)ds$ and $f_0(x,\cdot)$ has critical growth.

In this paper, we are interested in a generalization of all the above-mentioned results. The idea is to study functionals on $W^{1,p}(\Omega)$ which are related to nonhomogeneous operators and involving boundary terms that allow critical growth also at the boundary.

To this end, let $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ be a bounded domain with a $C^{1,\alpha}$-boundary $\partial\Omega$ and consider the following functional $J\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega G(x, \nabla u)dx - \int_\Omega j_1(x,u)dx - \int_{\partial\Omega} j_2(x,u)d\sigma, \tag{1.1}$$

where $G(x,\cdot)$ is the primitive of a function $a(x,\cdot)$ and the nonlinearities $j_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $j_2\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are measurable in the first argument and locally Lipschitz in the second one, that is, for every $s \in \mathbb{R}$ there exist a neighborhood $U_{s,k}$ of s and a constant $L_{s,k} \geq 0$ such that

$$|j_k(x,r) - j_k(x,t)| \leq L_{s,k}|r - t| \quad \text{for all } r,t \in U_{s,k}, \text{ for } k = 1,2,$$

and for all $x \in \Omega$ and for all $x \in \partial\Omega$, respectively. It is easy to see that $J\colon W^{1,p}(\Omega) \to \mathbb{R}$ need not to be differentiable and clearly it corresponds to the following elliptic inclusion:

$$-\operatorname{div} a(x,\nabla u) \in \partial j_1(x,u) \quad \text{in } \Omega,$$

$$a(x,\nabla u)\cdot\nu \in \partial j_2(x,\gamma u) \quad \text{on } \partial\Omega,$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$ and $\partial j_k(x,u), k = 1,2$, stands for Clarke's generalized gradient given by

$$\partial j_k(x,s) = \{\xi \in \mathbb{R} : j_k^\circ(x,s;r) \geq \xi r, \text{for all } r \in \mathbb{R}\},$$

where the term $j_k^\circ(x,s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $s \mapsto j_k(x,s)$ at s in the direction r defined by

$$j_k^\circ(x,s;r) = \limsup_{y\to s, t\downarrow 0} \frac{j_k(x,y+tr) - j_k(x,y)}{t},$$

see [5, Chap. 2]. Based on the Hahn–Banach theorem, the set $\partial j_k(x,s)$ is nonempty. An element $u \in \mathbb{R}$ is said to be a critical point of a locally Lipschitz function $f : X \to \mathbb{R}$ if there holds

$$f^\circ(x;y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$ (see [4]).

2. Preliminaries and Hypotheses

For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ and $L^p(\Omega,\mathbb{R}^N)$ the standard Lebesgue spaces equipped with the norm $\|\cdot\|_p$ and, for $1 < p < \infty$, $W^{1,p}(\Omega)$ denotes the Sobolev spaces endowed with the norm $\|\cdot\|_{1,p}$. Duality pairing between $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)^*$ will be denoted by $\langle\cdot,\cdot\rangle$.

On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Having this measure, we can consider the boundary Lebesgue spaces $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$ with norm $\|\cdot\|_{q,\partial\Omega}$. Furthermore, we know that there exists a unique linear, continuous map $\gamma\colon W^{1,p}(\Omega) \to L^q(\partial\Omega)$ for $1 \le q \le p_*$ called the trace map such that

$$\gamma(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}),$$

where p_* is the critical exponent on the boundary given by

$$p_* = \begin{cases} \dfrac{(N-1)p}{N-p} & \text{if } p < N, \\ \text{any } q \in (1,\infty) & \text{if } p \ge N. \end{cases} \tag{2.1}$$

Having the trace operator, we can talk about the boundary values for an arbitrary Sobolev function. Within the paper, we will omit the usage of the trace operator γ, for the sake of notational simplicity. Whenever considering the values of a Sobolev function on $\partial\Omega$, we understand that the trace operator is applied.

Furthermore, the Sobolev embedding theorem guarantees the existence of a linear, continuous map $i\colon W^{1,p}(\Omega) \to L^{p^*}(\Omega)$ with the critical exponent in the domain given by

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \text{any } q \in (1,\infty) & \text{if } p \ge N. \end{cases} \tag{2.2}$$

For more information on the Sobolev embeddings we refer to Gasiński and Papageorgiou [9] or Adams [1].

For $s \in (1,+\infty)$ we denote by $s' = \frac{s}{s-1}$ its conjugate, the inner product in $\mathbb{R}^N$ is denoted by $\cdot$ and the norm of $\mathbb{R}^N$ is given by $|\cdot|$. Moreover, $\mathbb{R}_+ = [0,+\infty)$ and the Lebesgue measure is denoted by $|\cdot|_N$.

Next, let $\vartheta \in C^1(0,\infty)$ be any function satisfying

$$0 < a_1 \le \frac{t\vartheta'(t)}{\vartheta(t)} \le a_2 \quad \text{and} \quad a_3 t^{p-1} \le \vartheta(t) \le a_4\left(t^{q-1} + t^{p-1}\right) \tag{2.3}$$

for all $t > 0$, with some constants $a_i > 0$, $i \in \{1,2,3,4\}$ and for $1 < q < p < \infty$. The hypotheses on $a\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ are listed as follows:

H(a): $a(x,\xi) = a_0(x,|\xi|)\,\xi$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$ for all $\xi \in \mathbb{R}^N$ and with $a_0(x,t) > 0$ for all $x \in \overline{\Omega}$, for all $t > 0$ and

(i) $a_0 \in C^1(\overline{\Omega} \times (0,\infty))$, $t \mapsto ta_0(x,t)$ is strictly increasing in $(0,\infty)$, $\lim_{t\to 0^+} ta_0(x,t) = 0$ for all $x \in \overline{\Omega}$ and

$$\lim_{t\to 0^+} \frac{ta_0'(x,t)}{a_0(x,t)} = c > -1 \quad \text{for all } x \in \overline{\Omega};$$

(ii) $|\nabla_\xi a(x,\xi)| \le a_5 \frac{\vartheta(|\xi|)}{|\xi|}$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and for some $a_5 > 0$;

(iii) $\nabla_\xi a(x,\xi) y \cdot y \ge \frac{\vartheta(|\xi|)}{|\xi|}|y|^2$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and for all $y \in \mathbb{R}^N$.

Remark 2.1. The idea in the choice of the special structure in H(a) is the usage of the nonlinear regularity theory due to Lieberman [14] coupled with the nonlinear maximum principle of Pucci and Serrin [20] as well as Zhang [23] when considering certain differential equations. If we set

$$G_0(x,t) = \int_0^t a_0(x,s)sds,$$

then $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ and the function $G_0(x,\cdot)$ is increasing and strictly convex for all $x \in \overline{\Omega}$. We set $G(x,\xi) = G_0(x,|\xi|)$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$ and obtain that $G \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and that the function $\xi \to G(x,\xi)$ is convex. Moreover, we easily derive that

$$\nabla_\xi G(x,\xi) = (G_0)'_t(x,|\xi|)\frac{\xi}{|\xi|} = a_0(x,|\xi|)\xi = a(x,\xi)$$

for all $\xi \in \mathbb{R}^N\backslash\{0\}$ and $\nabla_\xi G(x,0) = 0$. In other words, $G(x,\cdot)$ occurs to be the primitive of $a(x,\cdot)$. Combining this with convexity of $G(x,\cdot)$ and the fact that $G(x,0) = 0$ for all $x \in \overline{\Omega}$ we get

$$G(x,\xi) \le a(x,\xi)\cdot\xi \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{2.4}$$

The following lemma summarizes some properties of the function $a\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$.

Lemma 2.2. *If hypotheses H(a) hold, then*:

(i) $a \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N\backslash\{0\}), \mathbb{R}^N)$ *and for all* $x \in \overline{\Omega}$ *the map* $\xi \mapsto a(x,\xi)$ *is continuous, strictly monotone and so maximal monotone as well*;
(ii) *there exists* $a_6 > 0$, *such that* $|a(x,\xi)| \le a_6\left(1 + |\xi|^{p-1}\right)$ *for all* $x \in \overline{\Omega}$ *and* $\xi \in \mathbb{R}^N$;
(iii) $a(x,\xi)\cdot\xi \ge \frac{a_3}{p-1}|\xi|^p$ *for all* $x \in \overline{\Omega}$ *and for all* $\xi \in \mathbb{R}^N$.

Lemma 2.2 together with (2.4) allow to obtain the following growth estimates on $G(x,\cdot)$.

Corollary 2.3. *If hypotheses H(a) hold, then there exists* $a_7 > 0$ *such that*

$$\frac{a_3}{p(p-1)}|\xi|^p \le G(x,\xi) \le a_7\left(1 + |\xi|^p\right)$$

for all $x \in \overline{\Omega}$ *and* $\xi \in \mathbb{R}^N$.

The nonlinear operator $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u),\varphi\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi dx \quad \text{for all } u,\varphi \in W^{1,p}(\Omega), \tag{2.5}$$

possesses the following useful properties (see Gasiński and Papageorgiou [9]).

Proposition 2.4. *If hypotheses* H(a) *hold and the operator* $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *is defined by* (2.5), *then* A *is bounded, monotone, continuous, hence maximal monotone and of type* (S_+).

The following examples expose some operators fitting in our setting.

Example 2.5. In the definitions of the operators a, we drop the dependence on x just for simplicity. All the following maps satisfy hypotheses H(a):

(i) If $a(\xi) = |\xi|^{p-2}\xi$ with $1 < p < \infty$, then the corresponding operator is the classical p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

In this case $G(\xi) = \frac{1}{p}|\xi|^p$ for all $\xi \in \mathbb{R}^N$.

(ii) If $a(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$ with $1 < q < p < \infty$ and $\mu > 0$ then the corresponding operator is the so-called weighted (p,q)-Laplacian defined by $\Delta_p u + \mu\Delta_q u$ for all $u \in W^{1,p}(\Omega)$. In this case $G(\xi) = \frac{1}{p}|\xi|^p + \frac{\mu}{q}|\xi|^q$ for all $\xi \in \mathbb{R}^N$.

(iii) If $a(\xi) = (1+|\xi|^2)^{\frac{p-2}{2}}\xi$ with $1 < p < \infty$, then this map represents the generalized p-mean curvature differential operator defined by

$$\operatorname{div}[(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] \quad \text{for all } u \in W^{1,p}(\Omega).$$

In this case $G(\xi) = \frac{1}{p}(1+|\xi|^2)^{\frac{p}{2}}$ for all $\xi \in \mathbb{R}^N$.

Next, let us give the hypotheses on the nonsmooth potentials $j_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

H(j_1) (i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$;

(ii) $s \mapsto j_1(x,s)$ is locally Lipschitz for almost all $x \in \Omega$;

(iii) for some constants $c_1 > 0$ and $1 < q_1 \le p^*$ (where p^* is the given in (2.2)), we have

$$|\xi_1| \le c_1(1+|s|^{q_1-1})$$

for almost all $x \in \Omega$ and for all $\xi_1 \in \partial j_1(x,s)$.

H(j_2) (i) $x \mapsto j_2(x,s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$;

(ii) $s \mapsto j_2(x,s)$ is locally Lipschitz for almost all $x \in \partial\Omega$;

(iii) for some constants $c_2 > 0$ and $1 < q_2 \le p_*$ (where p_* is given in (2.1)), we have

$$|\xi_2| \le c_2(1+|s|^{q_2-1})$$

for almost all $x \in \partial\Omega$ and all $\xi_2 \in \partial j_2(x,s)$;

(iv) for any $u \in W^{1,p}(\Omega)$ and $\xi_3 \in \partial j_2(x,u)$ we have

$$|\xi_3(x_1) - \xi_3(x_2)| \le L|x_1 - x_2|^\alpha,$$

for all x_1, x_2 in $\partial\Omega$ with $\alpha \in (0,1]$.

3. Main Result

The following main result of this paper gives an answer about the relation between local Sobolev and Hölder minimizers of functionals of type J given in (1.1). We point out again that our functional is more general than the functionals of all the other cited papers above because we have a general, nonhomogeneous operator and we allow critical growth even on the boundary.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ be a bounded domain with a $C^{1,\alpha}$-boundary $\partial\Omega$ and let the assumptions $H(a)$, $H(j_1)$, and $H(j_2)$ be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, that is, there exists $\rho_0 > 0$ such that*

$$J(u_0) \le J(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of J, that is, there exists $\rho_1 > 0$ such that

$$J(u_0) \le J(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \le \rho_1.$$

Proof. First, from hypotheses $H(a)$, $H(j_1)$, $H(j_2)$ and Hu and Papageorgiou [12, p. 313], we know that the functional $J\colon W^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Let $h \in C^1(\overline{\Omega})$ and let $t > 0$ be small. Since u_0 is a local $C^1(\overline{\Omega})$-minimizer of J, we have

$$0 \le \frac{J(u_0 + th) - J(u_0)}{t}.$$

This implies

$$0 \le J^\circ(u_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$

Note that the function $h \mapsto J^\circ(u_0; h)$ is upper semicontinuous and $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, hence

$$0 \le J^\circ(u_0; h) \quad \text{for all } h \in W^{1,p}(\Omega).$$

Obviously, we have

$$0 \in \partial J(u_0).$$

This means that there exist functions $g_1 \in L^{q_1'}(\Omega)$ with $g_1(x) \in \partial j_1(x, u_0(x))$ for almost all $x \in \Omega$ and $g_2 \in L^{q_2'}(\partial\Omega)$ with $g_2(x) \in \partial j_2(x, u_0(x))$ for almost all $x \in \partial\Omega$ such that

$$\int_\Omega a(x, \nabla u_0) \cdot \nabla v dx = \int_\Omega g_1 v dx + \int_{\partial\Omega} g_2 v d\sigma \quad \text{for all } v \in W^{1,p}(\Omega). \tag{3.1}$$

Equation (3.1) stands for the weak formulation of the following nonhomogeneous Neumann boundary value problem:

$$-\operatorname{div} a(x, \nabla u_0) = g_1 \quad \text{in } \Omega, \quad a(x, \nabla u_0) \cdot \nu = g_2 \quad \text{on } \partial\Omega.$$

It follows from Marino and Winkert [15, Theorem 3.1] that $u_0 \in L^\infty(\overline{\Omega})$. This combined with the regularity results due to Lieberman [14] implies the existence of

$\eta \in (0,1)$ and $M > 0$ such that

$$u_0 \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_0\|_{C^{1,\eta}(\overline{\Omega})} \leq M. \tag{3.2}$$

To obtain out thesis, we need to show that u_0 is also a local minimizer of J in the $W^{1,p}(\Omega)$-norm. For this purpose, consider the minimizing problem

$$m_0^\varepsilon = \inf_{h \in \overline{B}_\varepsilon} J(u_0 + h), \tag{3.3}$$

where

$$\overline{B}_\varepsilon = \{h \in W^{1,p}(\Omega) \,|\, \|h\|_{1,p} \leq \varepsilon\}.$$

Arguing by contradiction, assume that u_0 is not a local minimizer of the functional J in the $W^{1,p}(\Omega)$-topology. Then we find $\varepsilon_0 \in (0,1]$ such that

$$m_0^\varepsilon < J(u_0) \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{3.4}$$

Fix $\varepsilon \in (0,\varepsilon_0)$ and let $\{h_n\}_{n\geq 1} \subset \overline{B}_\varepsilon$ be a minimizing sequence for (3.3), that is

$$\lim_{n\to\infty} J(u_0 + h_n) = m_0^\varepsilon. \tag{3.5}$$

From (3.4), we see that $\|\nabla h_n\|_p$ is bounded and since $u \mapsto \|\nabla u\|_p + \|u\|_{p^*}$ is an equivalent norm on $W^{1,p}(\Omega)$ (we can also use the norm $u \mapsto \|\nabla u\|_p + \|u\|_{p_*,\partial\Omega}$), it is clear that the sequence $\{h_n\}_{n\geq 1} \subseteq \overline{B}_\varepsilon$ is bounded in $W^{1,p}(\Omega)$ and so we can assume that

$$\begin{gathered} h_n \rightharpoonup h_\varepsilon \quad \text{in } W^{1,p}(\Omega), \text{ in } L^{p^*}(\Omega) \text{ and in } L^{p_*}(\partial\Omega), \\ h_n(x) \to h_\varepsilon(x) \quad \text{for almost all } x \in \Omega \text{ and for almost all } x \in \partial\Omega, \end{gathered} \tag{3.6}$$

by the Sobolev and the trace embedding theorem, respectively.

Applying the Extended Fatou Lemma (see, [7, Theorem A.2.8]), we can obtain that φ is sequentially weakly semicontinuous. From (3.5) and (3.6) it follows that

$$m_0^\varepsilon = \inf_{h\in\overline{B}_\varepsilon} J(u_0 + h) \leq J(u_0 + h_\varepsilon) \leq \liminf_{n\to\infty} J(u_0 + h_n) \leq \lim_{n\to\infty} J(u_0 + h_n) = m_0^\varepsilon,$$

and hence, due to (3.4), $h_\varepsilon \neq 0$.

We are now in the position to apply the nonsmooth Lagrange multiplier rule, see [5, Theorem 1 and Proposition 13], which guarantees the existence of a multiplier $\lambda_\varepsilon \geq 0$ such that

$$0 \in \partial J(u_0 + h_\varepsilon) + \lambda_\varepsilon K(h_\varepsilon),$$

where the function $K\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is defined by

$$\langle K(h_\varepsilon), v\rangle = \int_\Omega |\nabla h_\varepsilon|^{p-2}\nabla h_\varepsilon \cdot \nabla v dx + \int_\Omega |h_\varepsilon|^{p-2} h_\varepsilon v dx \quad \text{for all } v \in W^{1,p}(\Omega).$$

Therefore,

there exist $\hat{g}_1 \in L^{q_1'}(\Omega)$ and $\hat{g}_2 \in L^{q_2'}(\partial\Omega)$ with $\hat{g}_1(x) \in \partial j_1(x,(u_0+h_\varepsilon)(x))$ for almost all $x \in \Omega$ and $\hat{g}_2(x) \in \partial j_2(x,(u_0+h_\varepsilon)(x))$ for almost all $x \in \partial\Omega$ such that

$$\int_\Omega a(x,\nabla(u_0+h_\varepsilon))\cdot\nabla v dx-\int_\Omega \hat{g}_1 v dx-\int_{\partial\Omega}\hat{g}_2 v d\sigma$$
$$+\lambda_\varepsilon\int_\Omega |h_\varepsilon|^{p-2}h_\varepsilon v dx+\lambda_\varepsilon\int_\Omega |\nabla h_\varepsilon|^{p-2}\nabla h_\varepsilon\cdot\nabla v dx=0 \tag{3.7}$$

for all $v\in W^{1,p}(\Omega)$. We need to prove that h_ε belongs to $L^\infty(\Omega)$ and hence to $C^{1,\eta}(\overline{\Omega})$ for some $\eta\in(0,1)$ due to the regularity results due to Lieberman [14]. To end this, let us consider three cases for the multiplier λ_ε.

Case 1. $\lambda_\varepsilon=0$ with $\varepsilon\in(0,1]$
In this case, Eq. (3.7) becomes

$$\int_\Omega a(x,\nabla(u_0+h_\varepsilon))\cdot\nabla v dx=\int_\Omega \hat{g}_1 v dx+\int_{\partial\Omega}\hat{g}_2 v d\sigma\quad\text{for all } v\in W^{1,p}(\Omega).$$

As before, by applying the *a priori* results of Marino and Winkert [15, Theorem 3.1], the regularity results due to Lieberman [14, Theorem 2] and the fact that $u_0\in C^{1,\eta}(\overline{\Omega})$ for some $\eta\in(0,1)$ gives

$$h_\varepsilon\in C^{1,\hat{\eta}}(\overline{\Omega})\quad\text{and}\quad \|h_\varepsilon\|_{C^{1,\hat{\eta}}(\overline{\Omega})}\le M \tag{3.8}$$

for some $\hat{\eta}\in(0,1)$ and $M>0$.

Case 2. $0<\lambda_\varepsilon\le 1$ with $\varepsilon\in(0,1]$
Multiplying (3.1) by $\lambda_\varepsilon>0$ and adding this to (3.7) results in

$$\int_\Omega a(x,\nabla(u_0+h_\varepsilon))\cdot\nabla v dx+\lambda_\varepsilon\int_\Omega a(x,\nabla u_0)\cdot\nabla v dx$$
$$+\lambda_\varepsilon\int_\Omega |\nabla h_\varepsilon|^{p-2}\nabla h_\varepsilon\cdot\nabla v dx$$
$$=\int_\Omega(-\lambda_\varepsilon|h_\varepsilon|^{p-2}h_\varepsilon+\hat{g}_1+\lambda_\varepsilon g_1)v dx+\int_{\partial\Omega}(\hat{g}_2+\lambda_\varepsilon g_2)v d\sigma. \tag{3.9}$$

Now we introduce the map $T_\varepsilon\colon\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ defined by

$$T_\varepsilon(x,\xi)=a(x,\xi)+\lambda_\varepsilon a(x,H(x))+\lambda_\varepsilon|\xi-H(x)|^{p-2}(\xi-H(x))$$

for all $\xi\in\mathbb{R}^N$ and for almost all $x\in\Omega$, where $H(x)=\nabla u_0(x)$ and $H\in C^\eta(\overline{\Omega};\mathbb{R}^N)$ for some $\eta\in(0,1)$, thanks to (3.2). Since $a\colon\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ is continuous (see Lemma 2.2(i)), let $m_H=\max_{x\in\overline{\Omega}}|a(x,H(x))|=\max_{x\in\overline{\Omega}}|a(x,\nabla u_0(x))|$. It is easy to see that $T_\varepsilon\in C(\overline{\Omega}\times\mathbb{R}^N;\mathbb{R}^N)$. On the other side, we can apply Lemma 2.2(iii) and Young's inequality to obtain

$$T_\varepsilon(x,\xi)\cdot\xi=a(x,\xi)\cdot\xi+\lambda_\varepsilon a(x,H(x))\cdot\xi+\lambda_\varepsilon|\xi-H(x)|^{p-2}(\xi-H(x))\cdot\xi$$
$$\ge\frac{a_3}{p-1}|\xi|^p-\lambda_\varepsilon|a(x,H(x))|\cdot|\xi|+\lambda_\varepsilon|\xi-H(x)|^p$$
$$-\lambda_\varepsilon|\xi-H(x)|^{p-2}(\xi-H(x))\cdot H(x)$$

$$
\begin{aligned}
&\geq \frac{a_3}{p-1}|\xi|^p - \lambda_\varepsilon m_H|\xi| - \lambda_\varepsilon|\xi - H(x)|^{p-1}|H(x)| \\
&\geq \frac{a_3}{p-1}|\xi|^p - \lambda_\varepsilon m_H|\xi| - \lambda_\varepsilon m_H|\xi - H(x)|^{p-1} \\
&\geq \frac{a_3}{p-1}|\xi|^p - \delta|\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta),
\end{aligned}
$$

where $\delta = \frac{a_3}{2(p-1)}$ and $d_1(\lambda_\varepsilon, m_H, \delta) > 0$ is a constant, which is independent of ξ. Hence, we have

$$
T_\varepsilon(x,\xi)\cdot\xi \geq \frac{a_3}{2(p-1)}|\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta)
$$

for all $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$. This means that T_ε satisfies a strong ellipticity condition. Note that Eq. (3.9) can be written in the form

$$
\begin{aligned}
-\operatorname{div}(T_\varepsilon(x, \nabla(u_0 + h_\varepsilon))) &= -\lambda_\varepsilon|h_\varepsilon|^{p-2}h_\varepsilon + \hat{g}_1 + \lambda_\varepsilon g_1 \quad \text{in } \Omega, \\
T_\varepsilon(x, \nabla(u_0 + h_\varepsilon))\cdot\nu &= \hat{g}_2 + \lambda_\varepsilon g_2 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{3.10}
$$

Now are able to apply the again the results of Marino and Winkert [15, Theorem 3.1] which gives $u_0 + h_\varepsilon \in L^\infty(\overline{\Omega})$. However, $u_0 \in C^{1,\eta}(\overline{\Omega})$ leads to $h_\varepsilon \in L^\infty(\overline{\Omega})$. Moreover, by using (2.3) and hypothesis H(a)(ii), we obtain

$$
\begin{aligned}
|\nabla_\xi T_\varepsilon(x,\xi)| &\leq |\nabla_\xi a(x,\xi)| + \lambda_\varepsilon|\nabla_\xi[|\xi - H(x)|^{p-2}(\xi - H(x))]| \\
&\leq a_5\frac{\vartheta(|\xi|)}{|\xi|} + b_1 + b_2|\xi|^{p-2} \\
&\leq a_5 a_4(1 + 2|\xi|^{p-2}) + b_1 + b_2|\xi|^{p-2} \\
&= (2a_4a_5 + b_2)|\xi|^{p-2} + b_1 + a_4a_5
\end{aligned}
\tag{3.11}
$$

for all $\xi \in \mathbb{R}^N\backslash\{0\}$, for almost all $x \in \Omega$ and for some $b_1, b_2 > 0$ which are independent of ξ. In the same way, applying (2.3) and hypothesis H(a)(iii) leads to

$$
\begin{aligned}
\nabla_\xi T_\varepsilon(x,\xi)y\cdot y &= \nabla_\xi a(x,\xi)y\cdot y + \lambda_\varepsilon\nabla_\xi[|\xi - H(x)|^{p-2}(\xi - H(x))]y\cdot y \\
&\geq \frac{\vartheta(|\xi|)}{|\xi|}|y|^2 + \lambda_\varepsilon|\xi - H(x)|^{p-2}|y|^2 \\
&\quad + \lambda_\varepsilon(p-2)|\xi - H(x)|^{p-4}(\xi - H(x))\cdot y \\
&\geq c_1|\xi|^{p-2}|y|^2 + \lambda_\varepsilon\min\{1, p-1\}|\xi - H(x)|^{p-2}|y|^2 \\
&\geq c_1|\xi|^{p-2}|y|^2.
\end{aligned}
\tag{3.12}
$$

Finally, since $h_\varepsilon \in L^\infty(\Omega)$ satisfies (3.10) and because of H(a), (3.11), (3.12) along with hypotheses H(j_1) and H(j_2) we are able to apply the regularity results due to Lieberman [14] which gives (3.8) in Case 2 as well.

Case 3. $\lambda_\varepsilon > 1$ with $\varepsilon \in (0,1]$

Multiplying (3.1) by -1 and adding this to (3.7) results in

$$\int_\Omega a(x,\nabla(u_0+h_\varepsilon))\cdot\nabla v dx - \int_\Omega a(x,\nabla u_0)\cdot\nabla v dx + \lambda_\varepsilon \int_\Omega |\nabla h_\varepsilon|^{p-2}\nabla h_\varepsilon \cdot \nabla v dx$$
$$= \int_\Omega (\hat{g}_1 - g_1 - \lambda_\varepsilon |h_\varepsilon|^{p-2}h_\varepsilon) v(x) dx + \int_{\partial\Omega} (\hat{g}_2 - g_2) d\sigma. \tag{3.13}$$

As before, we define a map $T_\varepsilon\colon \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ by

$$T_\varepsilon(x,\xi) = \frac{1}{\lambda_\varepsilon}(a(x,H(x)+\xi) - a(x,H(x))) + |\xi|^{p-2}\xi$$

for all $\xi \in \mathbb{R}^N$ and for almost all $x\in\Omega$, where $H(x) = \nabla u_0(x)$ with $H \in C^\eta(\overline{\Omega};\mathbb{R}^N)$ for some $\eta \in (0,1)$ because of (3.2). Applying the notation for T_ε we can rewrite (3.13) in the following sense:

$$-\mathrm{div}(T_\varepsilon(x,\nabla h_\varepsilon)) = \frac{1}{\lambda_\varepsilon}(\hat{g}_1 - g_1) - |h_\varepsilon|^{p-2}h_\varepsilon \quad \text{in } \Omega,$$
$$T_\varepsilon(x,\nabla h_\varepsilon)\cdot\nu = \frac{1}{\lambda_\varepsilon}(\hat{g}_2 - g_2) \quad \text{on } \partial\Omega.$$

As before we can easily show that

$$\nabla_\xi T_\varepsilon(x,\xi) y\cdot y \ge b_3|\xi|^{p-2}|y|^2,$$
$$T_\varepsilon(x,\xi)\cdot\xi \ge b_4|\xi|^p + b_5,$$
$$|\nabla_\xi T_\varepsilon(x,\xi)| \le b_6|\xi|^{p-2} + b_7,$$

for some positive constants b_3, b_4, b_5, b_6, b_7. Finally, applying Marino and Winkert [15, Theorem 3.1] and Lieberman [14, Theorem 2] we reach again (3.8) in Case 3.

Let $\varepsilon \downarrow 0$. By the compactness of the embedding $C^{1,\hat{\eta}}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ (see [1, p.11]), there exists a subsequence $\{h_{\varepsilon_n}\}_{n\ge1}$ of $\{h_\varepsilon\}$ and a function $h^* \in C^1(\overline{\Omega})$ such that

$$h_{\varepsilon_n} \to h^* \quad \text{in } C^1(\overline{\Omega}).$$

Note that $h_{\varepsilon_n} \in \overline{B}_{\varepsilon_n}$ which gives $h^* = 0$. Therefore, we are able to find $N_0 \in \mathbb{N}$ large enough such that

$$\|h_{\varepsilon_n}\|_{C^1(\overline{\Omega})} \le r_1 \quad \text{for all } n \ge N_0.$$

Because u_0 is a minimizer of J in the $C^1(\overline{\Omega})$-topology, we have

$$J(u_0) \le J(u_0 + h_{\varepsilon_n}).$$

However, by the choice of $\{h_{\varepsilon_n}\}_{n\ge1}$, it holds

$$J(u_0+h_{\varepsilon_n}) = m^0_{\varepsilon_n} < J(u_0).$$

which is a contradiction. Therefore, we conclude that u_0 is a local minimizer of J in the $W^{1,p}(\Omega)$-topology. □

Let us comment on the case where the functional is smooth. Let $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $h\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ be Carathéodory functions, that means, we assume measurability in the first argument and continuity in the second one. We define $F(x,s)=\int_0^s f(x,t)dt$, $H(x,s)=\int_t^s h(x,t)dt$ and consider the functional $I\colon W^{1,p}(\Omega)\to\mathbb{R}$ given by

$$I(u)=\int_\Omega G(x,\nabla u)dx-\int_\Omega F(x,u)dx-\int_{\partial\Omega} H(x,u)d\sigma. \tag{3.14}$$

Of course, $I\in C^1(W^{1,p}(\Omega))$. For the functions $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $h\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ we suppose the existence of constants $c_1,c_2>0$ such that

$$\begin{aligned}|f(x,s)|&\le c_1(1+|s|^{q_1-1})\quad\text{for almost all } x\in\Omega,\\ |h(x,s)|&\le c_2(1+|s|^{q_2-1})\quad\text{for almost all } x\in\partial\Omega,\end{aligned} \tag{3.15}$$

for all $s\in\mathbb{R}$ and for $1<q_1\le p^*$ as well as $1<q_2\le p_*$. Moreover, $h\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the condition

$$|h(x,s)-h(y,t)|\le L[|x-y|^\alpha+|s-t|^\alpha],\quad |g(x,s)|\le L \tag{3.16}$$

for all $(x,s),(y,t)\in\partial\Omega\times[-M_0,M_0]$ with $\alpha\in(0,1]$ and constants $M_0>0$ and $L\ge 0$.

Then, Theorem 3.1 states the following for the functional $I\colon W^{1,p}(\Omega)\to\mathbb{R}$ defined in (3.14).

Theorem 3.2. *Let $\Omega\subseteq\mathbb{R}^N$ with $N>1$ be a bounded domain with a $C^{1,\alpha}$-boundary $\partial\Omega$ and let the assumptions $H(a)$, (3.15) and (3.16) be satisfied. If $u_0\in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of I, that is, there exists $\rho_0>0$ such that*

$$I(u_0)\le I(u_0+h)\quad\text{for all } h\in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})}\le\rho_0,$$

then $u_0\in C^{1,\eta}(\overline{\Omega})$ for some $\eta\in(0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of I, that is, there exists $\rho_1>0$ such that

$$I(u_0)\le I(u_0+h)\quad\text{for all } h\in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p}\le\rho_1.$$

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