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# EXACT SOLUTIONS TO $D = 2$ SUPERSYMMETRIC YANG–MILLS QUANTUM MECHANICS WITH SU(3) GAUGE GROUP*

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In this article we present the cut Fock space approach to the $D=d+1=2$, Supersymmetric Yang–Mills Quantum Mechanics (SYMQM). We start by briefly introducing the main features of the framework. We concentrate on those properties of the method which make it a convenient set up not only for numerical calculations but also for analytic computations. In the main part of the article a sample of results are discussed, namely, analytic and numerical analysis of the $D = 2$, SYMQM systems with SU(2) and SU(3) gauge symmetry.

## 1. Introduction

Supersymmetric Yang–Mills Quantum Mechanics (SYMQM) turned out to be not only a class of supersymmetric systems possessing some interesting physical features, but also to play an important rôle in many areas of theoretical physics. Among the problems where SYMQM are relevant, the two most notable examples are: their relation to a particular limit of M-theory [1], and the description of regularized dynamics of relativistic quantum membranes and supermembranes [2, 3]. Hence, an efficient way of investigating the spectra of SYMQM with various gauge groups and defined in spaces with different dimensionality would be of importance.

In short, $D = 2$, SYMQM are supersymmetric, $D = 2$ dimensional, Yang–Mills quantum field theories reduced to one point in space. Therefore, the original local gauge symmetry is transformed into a global symmetry of the quantum mechanical system. The difficulty of solving these systems even in the simplest cases comes from the singlet constraint which is the remnant of the Gauss law.

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The cut Fock space approach [4] was proposed few years ago as a non-perturbative way of investigating numerically the SYMQM systems. The energy eigenstates are constructed as linear combinations of physical basis states and thus are gauge invariant by construction. The approach allowed the calculation of low lying eigenenergies and their eigenstates through a numerical diagonalization of the Hamiltonian matrix. Its main application was the study of $D = 2$, SYMQM [5–7], but it was also generalized to the $D = 4$ case [8]. In principle, the method can be used in numerical investigations of systems with any gauge group and in any dimension. Nevertheless, we would like to stress in this article that it is also convenient for analytic treatment. The cut Fock space approach provides a systematic way of controlling the Fock basis in terms of which the energy eigenstates can be decomposed. As an example, we will present such decomposition of the solutions for two simple systems.

The paper is composed as follows. We start by briefly describing the cut Fock space approach, concentrating on the construction of the Fock basis. Then, we present the numerical algorithm together with the numerical spectra of the $D = 2$, SYMQM with SU(3) gauge group in the sectors with $n_{\mathrm{F}} = 0$ and $n_{\mathrm{F}} = 2$ fermionic quanta. In the third part of the paper, we derive analytically the bosonic solutions of the $D = 2$, SYMQM Hamiltonians with SU(2) and SU(3) gauge groups. We conclude by indicating possible directions of further studies.

## 2. The framework

For the reasons of simplicity the framework will be presented in the context of $D = 2$, supersymmetric Yang–Mills quantum mechanics with the SU($N$) gauge group. Nevertheless, the method is more flexible and systems with other gauge groups as well as in higher dimensional spaces can be studied. Particularly, creation and annihilation operators introduced in the following subsection can be labeled by additional spatial indices and transform in some representation of the SO($d$) group.

### *2.1. Basic degrees of freedom*

A two-dimensional SYMQM system is described [9] by a bosonic variable $\phi_A$ and a complex fermion $\lambda_A$, where $A$ is a color index. Being remnants of the gauge field in the original field theory, the bosonic and fermionic variables transform in the adjoint representation of the SU($N$) group. Thus, the system contains $N^2-1$ bosonic and $N^2-1$ fermionic degrees of freedom.

The characteristic feature of SYMQM is that its Hilbert space is composed of states invariant under the SU($N$) group. We would like to incorporate this constraint in the approach from the beginning. Hence, we introduce a matrix notation, in which any singlet can be written in terms of traces of

appropriate matrices [10]. We define

$$\phi_{i,j} = \sum_{A=1}^{N^2-1} \phi_A T^A_{i,j}\,, \qquad \pi_{i,j} = \sum_{A=1}^{N^2-1} \pi_A T^A_{i,j}\,,$$
$$f^\dagger_{i,j} = \sum_{A=1}^{N^2-1} f^\dagger_A T^A_{i,j}\,, \qquad f_{i,j} = \sum_{A=1}^{N^2-1} f_A T^A_{i,j}\,,$$

where $T^A_{i,j}$ are the generators of the SU($N$) group in the fundamental representation, $i,j = 1,\ldots,N$. Now, all operators become operator valued matrices. Particularly, the gauge-invariant occupation number operators can be defined as

$$\text{Tr } \left(a^\dagger a\right) = \sum_{A=1}^{N^2-1} a^\dagger_A a_A\,, \qquad \text{Tr } (f^\dagger f) = \sum_{A=1}^{N^2-1} f^\dagger_A f_A\,, \tag{1}$$

where we have introduced standard commuting creation and annihilation operators $a^\dagger_A, a_A$ defined by $\phi_A = \frac{1}{\sqrt{2}}(a_A + a^\dagger_A)$ and $\pi_A = \frac{1}{i\sqrt{2}}(a_A - a^\dagger_A)$. The sum over the adjoint indices ensures the invariance of these operators under the SU($N$) transformations.

In the following we will consider the $D = 2$, SYMQM systems with SU(2) and SU(3) gauge groups. Their Hamiltonians read, respectively, [9],

$$H = \tfrac{1}{2}\,\pi_A\pi_A = \text{Tr } \left(a^\dagger a\right) + \tfrac{3}{4} - \tfrac{1}{2}\left(\text{Tr } (a^\dagger a^\dagger) - \text{Tr } (aa)\right), \tag{2}$$

$$H = \tfrac{1}{2}\,\pi_A\pi_A = \text{Tr } \left(a^\dagger a\right) + 2 - \tfrac{1}{2}\left(\text{Tr } (a^\dagger a^\dagger) - \text{Tr } (aa)\right). \tag{3}$$

### *2.2. Fock basis*

The fundamental part of the approach is a systematic and recursive construction of the Fock basis. Fock states are eigenstates of some occupation number operators, and in the case of SYMQM models, we choose them to be eigenstates of the gauge-invariant occupation number operators Eq. (1).

It is convenient when the Fock basis states can be labeled by as many quantum numbers conserved by the Hamiltonian as possible. Hence, since for most of systems the fermionic occupation number is conserved, one usually constructs the Fock basis independently in each subspace of the Hilbert space with a definite fermionic occupation number. As far as the bosonic occupation number is concerned, it is in general not conserved. Nevertheless, we will further divide the fermionic sectors into subspaces with a given number of bosonic quanta, in order to facilitate the recursive approach.

In the following we introduce the concept of bosonic elementary bricks, which are necessary to obtain the Fock basis in the bosonic sector. Then, we will proceed in full analogy with the fermionic sectors.

#### 2.2.1. Bosonic elementary bricks

We define the set of bosonic elementary bricks as the set of $N$ linearly independent single traces of bosonic creation operators. Traces with more than $N-1$ operators can be reduced by the Cayley–Hamilton theorem. Table I contains examples of such sets for $N=2$, $N=3$ and $N=4$.

TABLE I

Elementary bosonic bricks for SU(2), SU(3) and SU(4).

| SU(2) | SU(3) | SU(4) |
|---|---|---|
| Tr $(a^\dagger a^\dagger)$ | Tr $(a^\dagger a^\dagger)$ | Tr $(a^\dagger a^\dagger)$ |
| — | Tr $(a^\dagger a^\dagger a^\dagger)$ | Tr $(a^\dagger a^\dagger a^\dagger)$ |
| — | — | Tr $(a^\dagger a^\dagger a^\dagger a^\dagger)$ |

Let us consider the set of states[1]

$$\Big\{(a^{\dagger 2})^{k_2}(a^{\dagger 3})^{k_3}\dots(a^{\dagger N})^{k_N}|0\rangle\Big\}_{\sum_{j=2}^{N} jk_j=n_{\mathrm{B}}} \equiv \big|\{n_{\mathrm{B}}\}\big\rangle \tag{4}$$

composed of the products of powers of elementary bosonic bricks acting on the Fock vacuum. One can show [7] that it spans the subspace of the Hilbert space with $n_{\mathrm{B}}$ bosonic quanta. In Eq. (4), we introduced a generalized notation in which $|\{n_{\mathrm{B}}\}\rangle$ is a vector of all states with $n_{\mathrm{B}}$ quanta.

Suppose that we have constructed such basis up to sectors containing less than $n_{\mathrm{B}}$ bosonic quanta. The Fock basis in the sector with $n_{\mathrm{B}}$ bosonic quanta can be build as the sum of all states obtained by the action of appropriate bricks on the already generated Fock basis states. Using the generalized notation this can be written in a compact form as

$$\big|\{n_{\mathrm{B}}\}\big\rangle = \sum_{k=2}^{N}(a^{\dagger k})\big|\{n_{\mathrm{B}}-k\}\big\rangle.$$

Note that the same state may appear in several copies, differing in the order of successive bricks used to build it. The basis is obtained once this redundancy is removed and the remaining states orthonormalized.

[1] We adopt here the notation in which $(X)$ designs Tr $(X)$. We will use this notation only when it is self-evident.

### 2.2.2. Fermionic bricks

In order to obtain the Fock basis in the fermionic sectors we must define bricks which contain fermionic creation operators. The set of *elementary fermionic bricks* can be defined in full analogy to the set of elementary bosonic bricks. We, thus, consider all single traces with $n_{\mathrm{F}}$ fermionic creation operators, which cannot be further reduced by the Cayley–Hamilton theorem. Subsequently, such set of elementary fermionic bricks must be enlarged by operators, which are products of fermionic elementary bricks with smaller number of fermionic quanta and contain $n_{\mathrm{F}}$ fermionic creation operators in total. The inclusion of these composite operators ensures that all possible invariant contractions of $n_{\mathrm{F}}$ fermionic and some bosonic creation operators are taken into account. The enlarged set is called the set of *composite fermionic bricks*. Its elements will be denoted by $C^{\dagger}(n_{\mathrm{B}}, n_{\mathrm{F}}, \alpha)$, where the first argument describes the number of bosonic creation operators contained in $C^{\dagger}$, and the last argument is an additional label needed in the cases where $n_{\mathrm{B}}$ and $n_{\mathrm{F}}$ are not enough to distinguish different operators. Table II presents the set of composite fermionic bricks for the SU(3) gauge group. In analogy to the bosonic case, we can define the set of states,

$$\left\{C^{\dagger}(n, n_{\mathrm{F}}, \alpha)\left(a^{\dagger 2}\right)^{k_2}\left(a^{\dagger 3}\right)^{k_3}\ldots\left(a^{\dagger N}\right)^{k_N}|0\rangle\right\}_{\sum_{j=2}^{N} jk_j+n=n_{\mathrm{B}}} \equiv \left|\left\{n_{\mathrm{B}}, n_{\mathrm{F}}\right\}\right\rangle, \tag{5}$$

which after orthonormalization will give the basis in the subspace of Hilbert space with $n_{\mathrm{B}}$ and $n_{\mathrm{F}}$ bosonic and fermionic quanta, respectively.

TABLE II

Fermionic SU(3) bricks.

| $F=1$ | $F=2$ | $F=3$ | $F=4$ |
|---|---|---|---|
| $(f^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}f^{\dagger})$ | $(f^{\dagger}f^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger})$ |
| $(f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger})$ | $(f^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}f^{\dagger})$ |
| | $(f^{\dagger}a^{\dagger}a^{\dagger}f^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ |
| | $(f^{\dagger}a^{\dagger})(f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger})$ | $(f^{\dagger}a^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}f^{\dagger})$ |
| | | $(f^{\dagger}a^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}a^{\dagger})(a^{\dagger}f^{\dagger}f^{\dagger}f^{\dagger})$ |
| | | $(f^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger})$ |
| | | $(f^{\dagger}a^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger})$ | $(f^{\dagger}a^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}f^{\dagger}a^{\dagger})$ |
| | | $(f^{\dagger}a^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ | $(f^{\dagger}f^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ |
| | | | $(f^{\dagger}a^{\dagger})(f^{\dagger}a^{\dagger}a^{\dagger})(f^{\dagger}f^{\dagger}a^{\dagger})$ |
| | | | $(f^{\dagger}f^{\dagger}a^{\dagger})(f^{\dagger}a^{\dagger}f^{\dagger}a^{\dagger}a^{\dagger})$ |

#### 2.2.3. Correctness of the Fock basis

One can show that the sets of states Eqs. (4) and (5) provide indeed a good basis of the Hilbert space. On one hand, its completeness follows from the fact that the states constructed with powers of bosonic and fermionic bricks represent the most general contractions of invariant tensors with creation operators. On the other hand, after removing the trivially linear dependent states, the linear independence of the remaining ones can be check explicitly by calculating the Gram's matrix. Obviously, the rank of the Gram's matrix corresponds to the number of linearly independent states in a sector with given $n_{\mathrm{B}}$ and $n_{\mathrm{F}}$. Fortunately, there exists also an independent way of calculating this number [11]. It exploits the orthogonality of the characters of the SU($N$) group, and can be used as a crosscheck that the Fock basis obtained through the recursive construction spans correctly the physical Hilbert space of SYMQM.

### *2.3. Extraction of approximate eigenenergies and eigenstates*

Once the Hamiltonian operator is expressed as an operator function of creation and annihilation operators, its action is straightforward in the Fock basis. However, the numerical analysis requires an additional step, namely the introduction of a cut-off $N_{\mathrm{cut}}$ (see Fig. 1) on the countably infinite Fock basis. There are many ways to introduce such a cut-off depending on the symmetries of the system. A practical cut-off is a limit on the total number of quanta contained in the Fock basis states. Once the cut Hamiltonian matrix is obtained, its eigenvalues correspond to an approximation of the

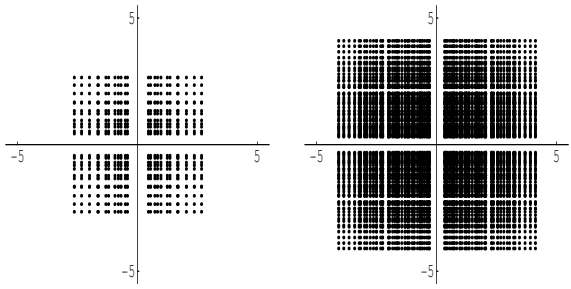


Fig. 1. Physical interpretation of the introduced cut-off. At finite cut-off the position and momentum operators have discrete spectra. As an example eigenvalues of the pair of operators $\phi_1$ and $\phi_2$ as well as $\pi_1$ and $\pi_2$ are shown for different cut-offs ($N_{\mathrm{cut}} = 5, 11$ for the left and right figures). With increasing cut-off the points become more and more dense and tend to cover a bigger area on the plane. In the limit of infinite cut-off the operators have a continuum spectrum and the whole plane is covered.

eigenenergies of the quantum system, and its eigenvectors to the eigenstates. Finally, calculations with several increasing $N_{\rm cut}$ have to be performed and the physical results extracted from the limit of infinite cut-off. The properties of such a procedure were analyzed in [12–15], where a different behavior of the eigenenergies corresponding to localized states and those corresponding to nonlocalized states was observed. Therefore, the method offers a tool to distinguish these two types of states, and we will indeed exploit this possibility when discussing the numerical results in subsection 3.2.

## 3. Numerical results

In this section we briefly describe a recursive algorithm which can be used to efficiently evaluate the matrix elements of the Hamiltonian operator. Then, we present the spectra of $D = 2$, SYMQM system with SU(3) gauge group in the sectors with $n_{\rm F} = 0$ and $n_{\rm F} = 2$.

### *3.1. Recursive numerical approach*

The recursive algorithm is based on relations connecting the desired matrix element of an operator to simpler matrix elements of some other operators, which have been already evaluated at the earlier stage of calculations. We will not describe here the full algorithm in details (for a full presentation of the approach see [16]). Instead, we will concentrate exclusively on the bosonic sector and give one example of such recursive relation.

Let us now assume that we want to evaluate the matrix elements of the normally-ordered operator $O(n_{\rm B}^O)$ between states containing $n_{\rm B}$ and $n'_{\rm B}$ quanta, $\langle\{n'_{\rm B}\}|O(n_{\rm B}^O)|\{n_{\rm B}\}\rangle$. The argument of $O$ means that the difference between the number of creation operators and the number of annihilation operators is $n_{\rm B}^O$. Therefore, the matrix elements $\langle\{n'_{\rm B}\}|O(n_{\rm B}^O)|\{n_{\rm B}\}\rangle$ will be nonzero only when $n'_{\rm B} = n_{\rm B}^O + n_{\rm B}$. The strategy to evaluate a matrix element of $O$ is to drag $O$ over the operators constituting the state $|\{n_{\rm B}\}\rangle$ so that $O$ annihilates the Fock vacuum. We thus have

$$\begin{aligned}\langle\{n'_{\rm B}\}|O(n_{\rm B}^O)|\{n_{\rm B}\}\rangle \;=\; &\Big(\langle\{n'_{\rm B}\}|\big[O(n_{\rm B}^O),(a^{\dagger p})\big]|\{n_{\rm B}-p\}\rangle \\ &+\langle\{n'_{\rm B}\}|(a^{\dagger p})O(n_{\rm B}^O)|\{n_{\rm B}-p\}\rangle\Big)\,R(n_{\rm B})\,, \quad (6)\end{aligned}$$

where the matrix $R(n_{\rm B})$ is obtained from the matrix of scalar products $S(n_{\rm B}) = \langle\{n_{\rm B}\}|\{n_{\rm B}\}\rangle$, and is used to remove redundant basis vectors and orthonormalize the remaining ones. Thus, $R(n_{\rm B})$ satisfies

$$R^{\rm T}(n_{\rm B})S(n_{\rm B})R(n_{\rm B}) = I\,,$$

where $I$ is the unity matrix which dimension is equal to the size of the subspace of the Hilbert space with $n_{\mathrm{B}}$ bosonic quanta. Note that we have expressed the desired matrix element through Eq. (6) in terms of matrix elements of operators between states with lower number of quanta, which should have been already evaluated during some previous calculations. Hence, applying successively such relation one can evaluate $\langle \{n'_{\mathrm{B}}\} | O(n_{\mathrm{B}}^{O}) | \{n_{\mathrm{B}}\} \rangle$. One can similarly organize the calculations in the fermionic sectors.

### *3.2. SU(3) model*

As an application of the above algorithm we present the spectra of the system given by the Hamiltonian Eq. (3). Figs. 2 and 3 contain the results in the sectors with $n_{\mathrm{F}} = 0$ and $n_{\mathrm{F}} = 2$, respectively. We notice that in both cases all eigenenergies fall down to zero. Thus, we can conclude that the corresponding eigenstates are nonlocalized. It is a feature of the approach that the eigenvalues corresponding to such states do not converge with the increasing cut-off. Such behavior results from approximating a plane wave by a finite set of localized harmonic oscillator eigenstates and its dependence on $N_{\mathrm{cut}}$ can be parameterized in a power like manner. A precise procedure for obtaining the correct infinite cut-off limit of the energy was described in [12,13,15]. In particular, it was shown that the hyperbolic fall off contains

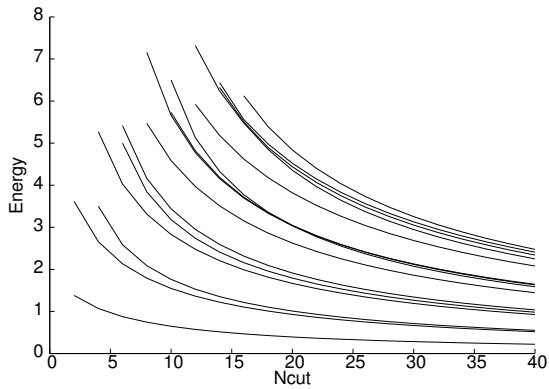


Fig. 2. Dependence of the eigenenergies on the cut-off for the SU(3) model in the $n_{\mathrm{F}} = 0$ sector.

information about the dispersion relation. Note that for a finite $N_{\mathrm{cut}}$ there can be only a finite number of eigenenergies. With increasing $N_{\mathrm{cut}}$ more and more new eigenvalues should appear on plots Figs. 2 and 3, in the limit giving the continuum spectrum of a free system. In the next section we will present analytic calculations which enable one to completely reconstruct the above spectra, and thus provide a full understanding of their features.

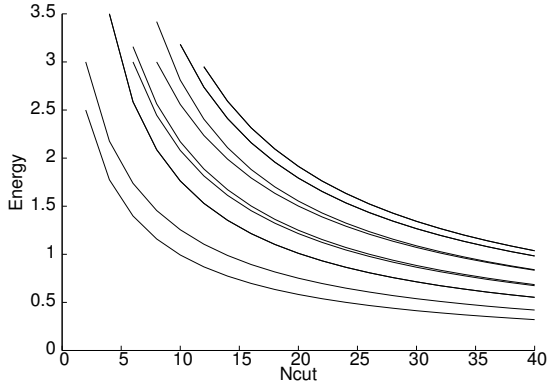


Fig. 3. Dependence of the eigenenergies on the cut-off for the SU(3) model in the $n_{\mathrm{F}} = 2$ sector.

## 4. Analytic solutions

Our procedure to derive exact solutions for the $D = 2$, SYMQM models consists in three steps. We decompose a general state $|E\rangle$ from the cut Hilbert space in the Fock basis. In this step a convenient parametrization of the Fock basis is necessary. Then, we translate the requirement that $|E\rangle$ is an eigenstate of the appropriate Hamiltonian into a recurrence relation on the decomposition coefficients. The last step is the solution of this recurrence relation at finite cut-off and the investigation of the infinite cut-off limit of these solutions.

Before we deal with the Hamiltonian (3), we will analyze a simpler model with the SU(2) symmetry group, (2). The solution of the SU(2) model was first found by Claudson and Halpern [9]. It can be obtained from the generalized solutions of Samuel [17] and was also recently rederived by Trzetrzelewski [7] by algebraic methods. This model is particularly simple because the SU(2) group is isomorphic with the SO(3) group and can be parameterized by spherical coordinates.

### *4.1. SU(2) model*

#### 4.1.1. Recurrence relation

At cut-off $N_{\mathrm{cut}}$ the general state from the bosonic sector can be decomposed as

$$|E\rangle = \sum_{j=0}^{N_{\mathrm{cut}}} a_j(E)\ \mathrm{Tr}\ (a^{\dagger 2})^j\,|0\rangle\,.$$

The eigenequation

$$H|E\rangle = E|E\rangle\,,$$

yields the recurrence relation for the $a_j(E)$ coefficients,

$$a_{j-1}(E) - \left(2j + \tfrac{3}{2} - 2E\right) a_j(E) + (j+1)\left(j + \tfrac{3}{2}\right) a_{j+1}(E) = 0 \,, \qquad (7)$$

which can be solve analytically as will be discussed in the following subsections.

#### 4.1.2. Finite cut-off solutions

One can show [18] that Eq. (7) admits $N_{\rm cut} + 1$ solutions. The possible eigenenergies are given by the zeros of an appropriate associated Laguerre polynomial[2],

$$L^{1/2}_{N_{\rm cut}+1}(2E) = 0 \,.$$

For each $E$ satisfying the above equation there exist an eigenstate with $N_{\rm cut} + 1$ decomposition coefficients are given by

$$a_j(E) = a_0 \Gamma\left(\tfrac{3}{2}\right) L^{1/2}_j(2E) \,, \qquad 0 \le j \le N_{\rm cut} \,,$$

where $a_0$ is some arbitrary constant.

#### 4.1.3. Infinite cut-off solutions

In the infinite cut-off limit, $N_{\rm cut} \to \infty$, the set of possible eigenenergies is given by the whole real positive axis. This reflects the fact that the physical spectrum is continuous. Hence, for any real number $E$, there exists an eigenstate of $H$ which decomposition coefficients are given by

$$a_j(E) = a_0 \Gamma\left(\tfrac{3}{2}\right) L^{1/2}_j(2E) \,, \qquad j \ge 0 \,.$$

Therefore, the exact eigenstate can be written as,

$$|E\rangle = a_0 \Gamma\left(\tfrac{3}{2}\right) \sum_{j=0}^{\infty} L^{1/2}_j(2E) \ {\rm Tr} \ (a^{\dagger 2})^j \, |0\rangle \,.$$

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[2] Laguerre polynomials $\mathcal{L}^{\alpha}_n(x)$ are defined as the solutions of the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \,,$$

and the orthogonality relation

$$\int_0^{\infty} \mathcal{L}^{\alpha}_m(x) \mathcal{L}^{\alpha}_n(x) x^{\alpha} e^{-x} dx = \delta_{mn} \,.$$

$L^{\alpha}_n(x)$ denotes the Sonine polynomials related to Laguerre polynomials by

$$L^{\alpha}_n(x) = \frac{1}{\Gamma(\alpha + n + 1)} \mathcal{L}^{\alpha}_n(x) \,.$$

**4.1.4. Reconstruction of wavefunctions**

In the case of the SU(2) model one can explicitly check the correctness of the above solutions. Let us denote by $\psi_n(r)$ the wavefunction in the position representation of the $n$-th basis state. $r$ is the radial variable which parameterizes the SU(2) group manifold. Since we consider only SU(2) invariant states, $\psi_n(r)$ do not depend on angular variables. The $n$-th basis state being the eigenstates of the gauge-invariant particle number operator Tr $\left(a^\dagger a\right)$, its wavefunction satisfies the following equation

$$-\frac{1}{2}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+3-r^2\right)\psi_n(r)=2n\ \psi_n(r)\,.$$

Such equation can be solved yielding

$$\psi_n(r)=\alpha(n)\,e^{-\frac{r^2}{2}}(-1)^n 2^{2n+1}n!\,\mathcal{L}_n^{\frac{1}{2}}(r^2)+\beta e^{-\frac{r^2}{2}}\frac{1}{r}\,{}_1F_1\left(-\frac{2n+1}{2},\frac{1}{2},r^2\right),$$

where $\alpha(n)$ is some constant depending on $n$ and ${}_1F_1(a,b,z^2)$ is the Kummer's function of the first kind. We are only interested in normalizable solutions, therefore since the function $\frac{1}{z}{}_1F_1(a,b,z^2)$ is singular at $z=0$, we set $\beta=0$. Thus, $\psi_n(r)$ turns out to be the wavefunction of a three dimensional harmonic oscillator carrying zero angular momentum. In order to find explicitly the Claudson–Halpern solutions, we write the eigensolution with energy $E$ as

$$\begin{aligned}\langle r|E\rangle &= \sum_{j=0}^{\infty}\langle r|\mathrm{Tr}\ (a^{\dagger 2})^j|0\rangle\,\langle 0|\mathrm{Tr}\ (a^2)^j|E\rangle\\ &= a_0\Gamma(\tfrac{3}{2})\,e^{-\frac{r^2}{2}}\phi(E)\sum_{j=0}^{\infty}\frac{\alpha(j)2^{2j+1}(-1)^j\ j!}{\Gamma\left(j+\frac{3}{2}\right)}\,\mathcal{L}_j^{\frac{1}{2}}(2E)\,\mathcal{L}_j^{\frac{1}{2}}(r^2)\,,\end{aligned}$$

where $\phi(E)$ is some function of the variable $E$ which was not determined by the recursion relation Eq. (7). Choosing $\alpha(n)=2^{-2j-1}$ and setting $2E=k^2$ we can use a known formula for the sum of products of associated Laguerre polynomials of the same index [19],

$$\sum_{j=0}^{\infty}\frac{(-1)^j j!}{\Gamma(j+\frac{3}{2})}\,\mathcal{L}_j^{\frac{1}{2}}(r^2)\,\mathcal{L}_j^{\frac{1}{2}}(k^2)=\frac{1}{2}\exp\left(\frac{k^2+r^2}{2}\right)\frac{1}{\sqrt{ikr}}I_{\frac{1}{2}}\left(\sqrt{ikr}\right)\,.$$

Next, by exploiting some properties of the Bessel functions [19], and setting $\phi(E)=e^{-E}$, we can transform the above result into

$$\langle r|E\rangle=\frac{a_0}{2}\Gamma(\tfrac{3}{2})\sqrt{\tfrac{2}{\pi}}\,\frac{\sin(kr)}{kr},$$

which is, up to a multiplicative factor, the Claudson–Halpern solution of the SU(2) model.

### *4.2. SU(3) model*

#### 4.2.1. Recursion relation

We now decompose $|E\rangle$ in the Fock basis of the SU(3) model

$$|E\rangle = \sum_{2j+3k\leq N_{\rm cut}} a_{j,k}(E)\ {\rm Tr}\ (a^{\dagger 2})^j\ {\rm Tr}\ \left(a^{\dagger 3}\right)^k |0\rangle.$$

For $|E\rangle$ being an eigenstate, $a_{j,k}$ must obey the following recursion relation [18]

$$a_{j-1,k} - \big(2j+3k+4-2E\big)a_{j,k} + (j+1)(j+3k+4)a_{j+1,k} + \tfrac{3}{8}(k+1)(k+2)a_{j-2,\,k+2} = 0\,. \tag{8}$$

Notice that the first three terms are diagonal in the $k$ index and are similar to the recurrence relation Eq. (7). The last term in Eq. (8) mixes the coefficients with different values of the $k$ index. However, coefficients $a_{j,k}$ with even and odd $k$ remain not related. Therefore, we can solve separately for the amplitudes $a_{j,2k}$ and $a_{j,2k+1}$.

#### 4.2.2. Finite cut-off solutions

For reasons of clarity we consider here only the situation when the cut-off $N_{\rm cut}$ is even and the solutions contain an even number, $2m$, of cubic bricks $(a^{\dagger 3})$. The derivation of solutions with an odd number of cubic bricks is similar.

It can be shown [18] that the solutions to Eq. (8) can be classified into several separate sets. A solution belongs to the set $f_m$ if $a_{j,k} \equiv 0, k > 2m$ and $a_{j,k} \neq 0, k \leq 2m$. In words, this means that the eigenstate can be decomposed into basis states containing at most $2m$ cubic bricks. $f_0$ is the simplest set of solutions, for which only $a_{j,0}$ are nonzero, *i.e.* is only composed of bilinear bricks. Each set of solutions has its separate quantization condition for the possible values of the $E$ parameter.

We will now give the general form of solutions belonging to the set $f_m$. The set $f_m$ contains $d_m \equiv \frac{1}{2}(N_{\rm cut} - 6m) + 1$ solutions with $E$ such that $L_{d_m}^{6m+3}(2E) = 0$. They can be written as[3] [18]

$$|E\rangle = \sum_{j=0}^{d_m} L_j^{6m+3}(2E)\Big(|j, 2m\rangle + \sum_{p=1}^{m} \Gamma^E(m,p)|j+3p, 2m-2p\rangle\Big)$$

with

$$\Gamma^E(m,p) = \prod_{t=p}^{m} -\frac{1}{3}\frac{t(2t-1)}{(2m+1)^2-(2t-1)^2}\,.$$

[3] We adopt a simplified notation in which $\big({\rm Tr}\ (a^{\dagger 2})\big)^j\ \big({\rm Tr}\ (a^{\dagger 3})\big)^k |0\rangle \equiv |j,k\rangle$.

In order to illustrate these formulas, let us present explicitly the solutions from the sets $f_0$ and $f_1$. They read, respectively,

$$|E\rangle = a_0 \sum_{j=0}^{d_0} L_j^3(2E)|j,0\rangle,$$
$$|E\rangle = a_0 \sum_{j=0}^{d_1} L_j^9(E)\Big(|j,2\rangle - \tfrac{1}{24}|j+3,0\rangle\Big).$$

The new feature of the $f_{>0}$ solutions is the degeneracy, which appears because several states can contain the same total number of quanta, *i.e.* the equation $2j+3k=n_{\mathrm{B}}$ can have several solutions. Particularly, the degeneracy of the states containing 6 quanta, namely $|2,0\rangle$ and $|0,3\rangle$ is responsible for the structure of the solutions from the set $f_1$. Fig. 4 demonstrates graphically the structure of these solutions.

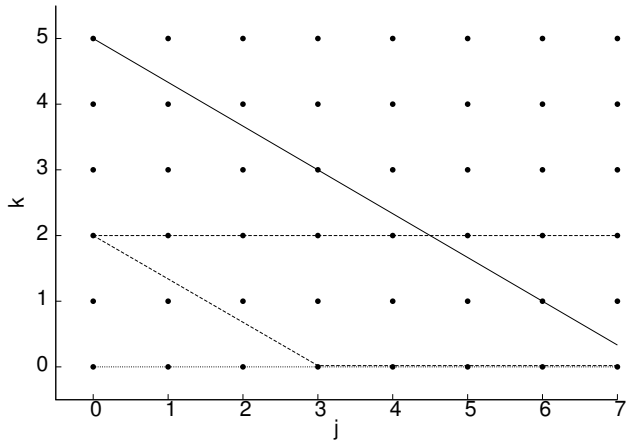


Fig. 4. The structure of the solutions of the recursion relation Eq. (8). Each dot represents a coefficient $a_{j,k}$ with appropriate values of the $j$ and $k$ indices. The oblique, straight line denotes a cut-off with a fixed number of quanta, here $N_{\mathrm{cut}}=15$. The coefficients lying below and on this line are included in the Fock basis, whereas coefficients lying outside are not. By increasing the cut-off we push this line to the right and include more states into the cut Fock basis. The remaining lines represent the sets of amplitudes of particular solutions of the recursion relation. The lowest, dotted line corresponds to the solution involving only quadratic bricks *i.e.* the solution from the set $f_0$. The dashed triple represents a solution from the set $f_1$. The two horizontal parts of the triple denote the amplitudes $a_{j,2}$ and $a_{j,0}$. The mixing of these amplitudes starts at the number of quanta equal to 6, *i.e.* both the amplitudes $a_{0,2}$ and $a_{3,0}$ contain 6 quanta.

The complete solution to the eigenvalue problem, *i.e.* the set of all eigenstates $\{|E\rangle\}$ is given by the union

$$\{|E\rangle\} = \bigcup_{m=0}^{\lfloor \frac{1}{6} N_{\rm cut} \rfloor} f_m \ \cup \bigcup_{m=0}^{\lfloor \frac{1}{6}(N_{\rm cut}-3)\rfloor} g_m \,, \tag{9}$$

where $g_m$ are the corresponding sets of eigenstates with an odd number of cubic bricks, and $d'_m$ their multiplicities. The spectrum, *i.e.* the set of all values of the $E$ parameter, $\{E\}$, for which a nonzero eigenstate exists, can be written as

$$\{E\} = \bigcup_{m=0}^{\lfloor \frac{1}{6} N_{\rm cut} \rfloor} \left\{ L^{6m+3}_{d_m}(2E) = 0 \right\} \ \cup \bigcup_{m=0}^{\lfloor \frac{1}{6}(N_{\rm cut}-3)\rfloor} \left\{ L^{6m+6}_{d'_m}(2E) = 0 \right\}. \tag{10}$$

#### 4.2.3. Infinite cut-off solutions

Eventually, one can show that the solutions retain their structure in the infinite cut-off limit. The proof relies on the observation that the mixing coefficients $\Gamma^E(m,p)$ do not depend on $N_{\rm cut}$. Therefore, in order to obtain the exact solution, the sum over $j$ can be safely extended to infinity,

$$|E\rangle = \sum_{j=0}^{\infty} L^{6m+3}_j(2E) \left( |j, 2m\rangle + \sum_{p=1}^{m} \Gamma^E(m,p) |j+3p, 2m-2p\rangle \right),$$

where $E$ can be now any real number. Notice that in the limit $N_{\rm cut} \to \infty$ the number of separate sets of solutions $f_m$ will become also infinite.

## 5. Conclusions

In this paper we have described the cut Fock space approach to $D = 2$, supersymmetric Yang–Mills quantum mechanics. We have briefly presented the numerical algorithm as well as numerical results, namely the spectra of the SYMQM system with the SU(3) gauge group in the bosonic and $n_{\rm F} = 2$ sectors. Subsequently, we showed that the cut Fock space approach is also a convenient framework for analytic calculations. We have derived exact solutions for the SYMQM system with the SU(2) gauge group and compared them with the original solutions of Halpern and Claudson. Then, we have applied the method to the SU(3) SYMQM system and obtained the spectra and eigenstates in the bosonic sector. Hence, for a given cut-off, one can explain analytically all features of Fig. 2.

Our analytic results can be extended in several directions. First of all, it is possible to obtain recursive relations and solve them in all fermionic sectors of the model with SU(3) gauge group. Obviously, an exact, complete solution of this model enables one to calculate the Witten's index [18]. Second, the knowledge of exact solutions can be helpful for the investigation of systems with interactions. One can use the free solutions at finite cut-off as a starting point of perturbative expansion. In both these problems the generalization to other SU($N$) gauge groups can be achieved. Third, the exact form of the solutions in the bosonic as well as fermionic sectors enables one to study their large-$N$ limit, which is an important point in the investigations of the SYMQM systems. Last but not least, the method can be extended to higher dimensional systems, its application to $D = 4$, SYMQM with SU(2) gauge group is now being investigated.

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