Vol. 42 (2011) ACTA PHYSICA POLONICA B No 1

EIGENVALUES AND EIGENVECTORS OF THE LAPLACE OPERATOR IN d-DIMENSIONAL CUT FOCK BASIS

Piotr Korcyl

The M. Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received November 10, 2010)

We present exact expressions for the eigenvalues and eigenvectors of the d-dimensional Laplace operator in a cut Fock basis. We also discuss the physical interpretation of the cutoff as well as the validity of the scaling law of the eigenenergies for $d > 1$.

DOI:10.5506/APhysPolB.42.85
PACS numbers: 11.15.Tk

1. Introduction

The ultimate goal of studies involving Supersymmetric Yang–Mills Quantum Mechanics (SYMQM) is the solution of the system defined in $D = 9+1$ space-time dimensions with SU(N) gauge group for any N. The latter appears in such problems as the quantization of a supermembrane [1, 2] or the description of dynamics of $D0$ branes, one of the kinds of objects present in M-theory [3, 4]. It should not be surprising that finding such solution is a difficult task. Therefore, one tries to first investigate the lower dimensional versions of SYMQM. Among them, the $D = 1+1$ SYMQM is the simplest one [5]. It is described by position ϕ^m and momentum π^m bosonic operators and, λ^m and $\bar{\lambda}^m$ fermionic operators. The index m labels the color degrees of freedom, $m = 1, \ldots, \mathcal{D}$, where $\mathcal{D}$ is the dimension of the adjoint representation of the gauge group. Although the $D = 2$ SYMQM Hamiltonian is free, its solutions are not trivial due to the singlet constraint which represents the dimensionally reduced Gauss law. In other words, the physical Hilbert space of $D = 2$ SYMQM is composed of states invariant under the gauge transformations.

A convenient method for analyzing such systems was proposed few years ago by Wosiek [6] (for a recent introductory article see [7]). It was designed mainly as a fully nonperturbative, numerical method for deriving the spectra of SYMQM. One of its advantages is that it treats fermions and bosons on an equal footing which made the calculations in all fermionic sectors possible. Moreover, it can handle any gauge group in any dimension. The main idea is to calculate the matrix elements of the Hamiltonian operator in a Fock basis composed of states containing less than N_B quanta. N_B is called the cutoff. Such finite matrix is then diagonalized numerically in order to obtain the approximated eigenenergies. A suitable choice of the cutoff, preserving the gauge symmetry as well as the rotational symmetry, enabled one to obtain many numerical results [8, 9, 10, 11]. The cut Fock space approach turned out also to be a suitable tool for analytic investigations.

In this short paper, we analyze the $D = 2$ SYMQM models with the SO(d) gauge group. In the language of SYMQM they can be interpreted as an extremely simplified d-dimensional model with only one color or, equivalently, as a model with d colors transforming in the adjoint representation of the SO(d) group. Due to the simplicity of the SO(d) symmetry group we are able to derive analytic expressions for the cut spectrum and eigenstates. Consequently, we check that the infinite cutoff limit of the cut solutions is given by the well-know Bessel functions [12, 13], which are solutions of the free d-dimensional Schrödinger equation carrying zero angular momentum. We expect that such analytic solutions can be used as a starting point for the study of systems with several colors transforming with the SU(N) group. Moreover, once known for any d or N they could be instructive in the further study of physically interesting 'large-d' or 'large-N' limits.

We also discuss the physical interpretation of the cutoff imposed on the Fock basis. We show that the cutoff corresponds to a specific discretization of position and momentum spaces. Afterwards, we briefly analyze the scaling law, *i.e.* the dependence of the eigenenergies on the labeling index. Such law was first investigated by Wosiek and Trzetrzelewski [14,15] in one spatial dimension and was used as a handle to correctly differentiate localized states from the nonlocalized ones. We discuss the validity of such approach in higher dimensional spaces.

We summarize our results in the last section.

2. The system

Traditionally, in the $D = 2$ SYMQM models, ϕ^m and π^m, as well as, λ^m and $\bar{\lambda}^m$ transform in the adjoint representation of the SU(N) gauge symmetry. In this paper, we concentrate on the SO(d) gauge symmetry, and therefore, we assume ϕ^m and λ^m to be in the adjoint representation of SO(d). The SYMQM Hamiltonian is supplemented by the singlet constraint, hence we search for solutions carrying zero angular momentum. The Hamiltonian reads

$$H = \frac{1}{2}\sum_{m=1}^{d} \pi^m \pi^m \,. \tag{1}$$

One introduces standard creation and annihilation operators

$$a^{\dagger m} = \frac{1}{\sqrt{2}}\left(\phi^m - i\pi^m\right) , \qquad a^m = \frac{1}{\sqrt{2}}\left(\phi^m + i\pi^m\right) \tag{2}$$

fulfilling the commutation relation

$$\left[a^n, a^{\dagger m}\right] = \delta^{mn} , \tag{3}$$

and rewrite H as

$$H = a^{\dagger m} a^m + \frac{d}{2} - \frac{1}{2} a^m a^m - \frac{1}{2} a^{\dagger m} a^{\dagger m} , \tag{4}$$

where the summation from 1 to d over repeated indices is understood from now on.

The SO(d) group possesses only one invariant symmetric tensor, namely the Kronecker delta. Hence, the algebra of invariant under rotations creation operators reads

$$\begin{aligned} \left[a^m a^m, a^{\dagger n} a^{\dagger n}\right] &= 2d + 4a^{\dagger n} a^n , \\ \left[a^{\dagger m} a^m, a^{\dagger n} a^{\dagger n}\right] &= 2a^{\dagger n} a^{\dagger n} , \\ \left[a^p, a^{\dagger m} a^{\dagger m}\right] &= 2a^{\dagger p} . \end{aligned} \tag{5}$$

The basis vectors in the singlet sector can be labeled by a single index n, and take the form

$$|n\rangle = \frac{1}{\mathcal{N}_n^0}\left(a^{\dagger m} a^{\dagger m}\right)^n |0\rangle \,. \tag{6}$$

Notice that n is related to the half of the total number of quanta of each type contained in $|n\rangle$, $N_B = 2n$. For a given cutoff N_B only states $|n \leq \lfloor \frac{N_B}{2} \rfloor\rangle$ are present. There are $\lfloor \frac{N_B}{2} \rfloor + 1$ such states, a new one appearing for N_B even.

We calculate the normalization factor in Eq. (6) by commuting consecutively all operators $a^q a^q$ through $(a^{\dagger p} a^{\dagger p})^i$ using Eqs. (5), so that they annihilate the Fock vacuum. This gives a recursive relation for $\mathcal{N}_n^0$

$$\left(\mathcal{N}_n^0\right)^2 = 2n(2n + d - 2)\left(\mathcal{N}_{n-1}^0\right)^2 , \qquad \left(\mathcal{N}_0^0\right)^2 = 1 \tag{7}$$

which is solved by

$$\left(\mathcal{N}_n^0\right)^2 = 4^n n! \frac{\Gamma\left(\frac{d}{2}+n\right)}{\Gamma\left(\frac{d}{2}\right)} \,. \tag{8}$$

Although we are mainly interested in the scalar sector, we can obtain results for higher representations without additional effort. As an example, we present some calculations in the vector sector. The basis states in this sector can be obtained out of the states given by Eq. (6) by the action of an additional single creation operator

$$|n,i\rangle = \frac{1}{\mathcal{N}_n^1} a^{\dagger i} \left(a^{\dagger m} a^{\dagger m}\right)^n |0\rangle \,, \qquad i = 1,\ldots,d \,, \tag{9}$$

where

$$\left(\mathcal{N}_n^1\right)^2 = 4^n n! \frac{\Gamma\left(\frac{d}{2}+n+1\right)}{\Gamma\left(\frac{d}{2}+1\right)} \,. \tag{10}$$

States given by Eq. (9) are orthogonal in n and i: $\langle m,j|n,i\rangle = \delta_{mn}\delta_{ji}\left(\mathcal{N}_n^1\right)^2$.

It is now straightforward to evaluate all elements of the Hamiltonian matrix in the singlet sector

$$\langle n|H|n\rangle = 2n + \frac{d}{2} \,, \quad \langle n+1|H|n\rangle = \langle n|H|n+1\rangle = -\sqrt{(n+1)\left(\frac{d}{2}+n\right)} \,, \tag{11}$$

and in the vector sector

$$\begin{aligned} \langle n,j|H|n,i\rangle &= \delta_{ji}\left(\frac{d}{2}+2n+1\right) \,, \\ \langle n,j|H|n+1,i\rangle = \langle n+1,j|H|n,i\rangle &= -\delta_{ji}\sqrt{(n+1)\left(\frac{d}{2}+n+1\right)} \,. \end{aligned} \tag{12}$$

Having this done, we can calculate numerically the spectra. In order to obtain the physical results, corresponding to the infinite cutoff limit, we investigate the behavior of the eigenenergies with increasing cutoff for some finite values of N_B. Figure 1 contains such results for the 12 lowest eigenenergies in the singlet sector. Each curve falls slowly to zero as expected from the nonlocalized character of the eigenstates of the free Hamiltonian [14, 15, 16]. The dependence on the cutoff will be also discussed in the last section of this note.

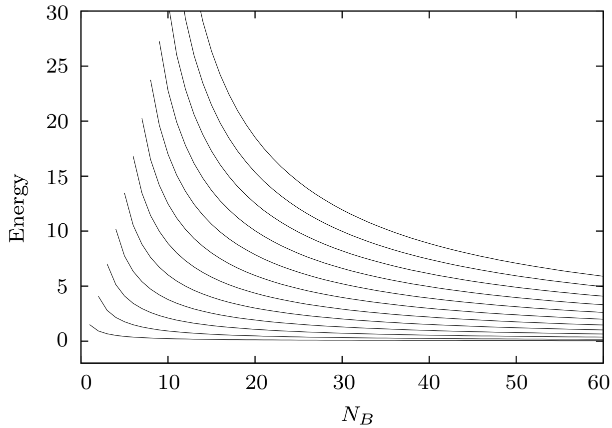

Fig. 1. Spectrum of the Hamiltonian operator in $d = 3$ dimensions in the singlet sector.

3. Character method

There exists an independent way to calculate the number of basis states transforming according to a given representation of the SO(d) group, which exploits the orthogonality of characters of irreducible representations[1]. We start with a formula for the number of singlets appearing in a product of N_B adjoint representations [17]

$$d^d_{N_B} = \int d_{\mu_{\mathrm{SO}(d)}}(\alpha_i)\ 1 \cdot \chi^d_{N_B}(\alpha_i)\,, \tag{13}$$

where $d_{\mu_{\mathrm{SO}(d)}}(\alpha_i)$ is the invariant Haar measure for the SO(d) group parameterized by the set of angles $\{\alpha_i\}_{i=1,\ldots,\lfloor d/2\rfloor}$ running from 0 to 2π and $\chi^d_{N_B}(\alpha_i)$ represents the symmetrized product of N_B characters of the adjoint representation of the SO(d) group and is equal to [18]

$$\chi^d_{N_B}(\alpha_i) = \sum_{\sum_k k i_k = N_B} \prod_{k=1}^{N_B} \frac{1}{i_k!}\frac{1}{k^{i_k}} \left(\chi^d(k\alpha_i)\right)^{i_k} . \tag{14}$$

1 in Eq. (13) stands for the trivial character of the invariant representation of SO(d). It can be replaced by the character of any other representation R, in which case, $d^d_{n_B}$ will be equal to the number of times R appears in the product of N_B adjoint representations. For d even, $d = 2M$ with M integer,

[1] This method was suggested by R. Janik.

the Haar measure for SO(d) is given by [19]

$$d_{\mu_{\mathrm{SO}(d)}} = \frac{2^{(M-1)^2}}{\pi^M M!} \prod_{1\leq j<k\leq M} \left(\cos\left(\alpha_k\right) - \cos\left(\alpha_j\right)\right)^2 , \tag{15}$$

and for d odd, $d = 2M+1$,

$$d_{\mu_{\mathrm{SO}(d)}} = \frac{2^{M^2}}{\pi^M M!} \prod_{1\leq j<k\leq M} \left(\cos(\alpha_k) - \cos\left(\alpha_j\right)\right)^2 \prod_{m=1}^{M} \sin^2\left(\frac{\alpha_m}{2}\right) . \tag{16}$$

The characters of the adjoint representation of SO(d) are given by, for $d = 2M$ [19],

$$\chi^d(\alpha_i) = 2\sum_{m=1}^{M} \cos(\alpha_m) , \tag{17}$$

and for $d = 2M+1$,

$$\chi^d(\alpha_i) = 1 + 2\sum_{m=1}^{M} \cos(\alpha_m) . \tag{18}$$

Explicit evaluation of the integrals confirms that Eqs. (6) and (9) form indeed a complete basis of the Hilbert space in the corresponding sectors.

4. Analytic results

In this section, we derive the set of eigenvalues and eigenvectors of the Hamiltonian operator in the singlet and vector sectors for any finite cutoff N_B.

At cutoff equal to N_B, the basis size in the singlet sector is $n_{\max} = \lfloor \frac{N_B}{2} \rfloor + 1$. Let us denote by $I_{n_{\max}}(\lambda)$ the characteristic polynomial of the Hamiltonian matrix at this cutoff. Then, $I_1 = \langle 0|H-\lambda|0\rangle = \frac{d}{2} - \lambda$. Due to the tridiagonal nature of the Hamiltonian matrix we can write a recursive relation for $I_n(\lambda)$

$$I_n(\lambda) = \left(2n - 2 + \frac{d}{2} - \lambda\right) I_{n-1}(\lambda) - \left((n-1)\left(n - 2 + \frac{d}{2}\right)\right) I_{n-2}(\lambda) . \tag{19}$$

By changing the variables as $I_n(\lambda) \to n! I_n(\lambda)$ we get a recursive equation which is solved by generalized Laguerre polynomials, *i.e.*

$$I_n(\lambda) = L_n^{\frac{d}{2}-1}(\lambda) . \tag{20}$$

These well-known polynomials are defined as the solutions of the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0\,,$$

and the orthogonality relation

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) x^\alpha e^{-x} dx = \delta_{mn}\,.$$

Therefore, the eigenvalues of the Hamiltonian operator in the singlet sector in d-dimensions at cutoff N_B are given by the zeros of the Laguerre polynomial $L^{\frac{d}{2}-1}_{\lfloor \frac{N_B}{2} \rfloor + 1}(\lambda)\,.$ By taking $d = 1$ we recover the results of [15].

An alternative method to derive this result is to consider a general state $|E\rangle$ from the singlet sector and expand it as

$$|E\rangle = \sum_{n=0}^{\lfloor \frac{N_B}{2} \rfloor} a_n(E)|n\rangle\,. \tag{21}$$

The eigenequation of H translates into a recursive relation for the $a_n(E)$ coefficients. After extracting an irrelevant constant factor, $a_n(E) \to 2^{-n} a_n(E)$, the recursion reads

$$a_{n-1}(E) - \left(2n + \frac{d}{2} - E\right) a_n(E) + (n+1)\left(n + \frac{d}{2}\right) a_{n+1}(E) = 0\,. \tag{22}$$

Eq. (22) is solved by

$$a_n(E) = a_0 \varGamma\left(\frac{d}{2}\right) \frac{L_n^{\frac{d}{2}-1}(E)}{\varGamma\left(n + \frac{d}{2}\right)}\,, \qquad n = 1, \ldots, \lfloor \frac{N_B}{2} \rfloor\,, \tag{23}$$

together with the quantization condition $L^{\frac{d}{2}-1}_{\lfloor \frac{N_B}{2} \rfloor + 1}(E) = 0$, which is equivalent to Eq. (20).

Similarly, in the vector sector the recursive relation for the determinant of the Hamiltonian matrix can be solved by

$$I_n(\lambda) = L_n^{\frac{d}{2}}(\lambda)\,. \tag{24}$$

Thus, the spectrum in the vector sector for the cutoff N_B is given by the zeros of the generalized Laguerre polynomials with the index shifted by 1 comparing to the singlet case:

$$H\left|E_n^i\right\rangle = E_n\left|E_n^i\right\rangle$$

with E_n such that

$$L^{\frac{d}{2}}_{\lceil\frac{N_B}{2}\rceil+1}(E_i) = 0\,, \qquad n = 1,\ldots,\lceil\frac{N_B}{2}\rceil + 1\,,$$

where $|E^i_n\rangle$ are the eigenstates of H with the vector index i. As a straightforward consequence of the vector nature of the states $|E^i_n\rangle$ each eigenvalue is d times degenerate. The corresponding eigenvectors are given as an expansion in the basis

$$\left|E^i\right\rangle = \sum_{n=0}^{\lceil\frac{N_B}{2}\rceil} a_{n,i}(E)|n,i\rangle \tag{25}$$

with

$$a_{n,i}(E) = a_0\Gamma\left(\frac{d}{2}\right)\frac{L^{\frac{d}{2}}_n(E)}{\Gamma\left(n+\frac{d}{2}+1\right)}\,, \qquad n = 1,\ldots,\lceil\frac{N_B}{2}\rceil\,. \tag{26}$$

Hence, we obtained simple formulae giving the spectrum of the cut Hamiltonian operator as well as its eigenstates. The dependence on the cutoff is explicit and it is straightforward to perform the infinite cutoff limit.

One can easily generalize the above expressions for the eigenvalues and eigenvectors transforming under any irreducible representation of the SO(d) group.

5. Reconstruction of radial waves

In this section, we confirm the results obtained above by comparing their infinite cutoff limit with the well-known continuous results. Before we present the detailed calculations, we make two observations.

The rescaled Hamiltonian of the quantum harmonic oscillator is given by

$$H_{\mathrm{osc}} = p^2 + x^2\,. \tag{27}$$

It is evident that H_{osc} is invariant under the transformation $x \leftrightarrow p$. Therefore, the functional form of the wave functions in the momentum and position representations is the same up to a constant multiplicative factor. In fact, both are related by the d-dimensional Fourier transform.

Next, the amplitude $a_n(E)$ can be interpreted as the nth wave function of the d-dimensional harmonic oscillator in the momentum representation. To see this, let as denote the plane wave with a wave-vector κ, $\kappa = \sqrt{E}$, which is the eigenstate of the free Hamiltonian H, by $|\kappa\rangle$, and the nth state of the d-dimensional harmonic oscillator carrying zero angular momentum by $|n\rangle$. Then, $a_n(E)$ is simply the scalar product of the free wave $|\kappa\rangle$ and the harmonic oscillator eigenfunction $|n\rangle$

$$a_n(E) = a_n\left(\kappa^2\right) = \langle\kappa|n\rangle\,. \tag{28}$$

Thus,

$$\psi_n(\kappa) \equiv \langle \kappa | n \rangle = f\left(\kappa^2\right) L_n^{\frac{d}{2}-1}\left(\kappa^2\right) , \tag{29}$$

$$\phi_n(r) \equiv \langle r | n \rangle = g\left(r^2\right) L_n^{\frac{d}{2}-1}\left(r^2\right) , \tag{30}$$

where $f(\kappa^2)$ and $g(r^2)$ are functions which do not depend on n and could not be determined from the recursion relations Eq. (22). We can check the above conclusions by explicitly solving the radial d-dimensional harmonic oscillator eigenequation in the position representation

$$\Phi''(r) + \frac{d-1}{r}\Phi'(r) + 2E\Phi(r) - r^2\Phi(r) = 0 \,, \tag{31}$$

where the prime denotes the differentiation with respect to r. Solving the quantization condition we get $E = \frac{1}{2}(d+4n)$ and hence the solutions are

$$\phi_n(r) = c_1 2^{\frac{d}{4}} e^{-\frac{r^2}{2}} U\left(-n, \frac{d}{2}, r^2\right) + c_2 2^{\frac{d}{4}} e^{-\frac{r^2}{2}} L_n^{\frac{d}{2}-1}\left(r^2\right) , \tag{32}$$

where $U(a,b,z)$ is the confluent hypergeometric function. The first term of the solution is singular at $r=0$, so we have to set $c_1 = 0$. Thus,

$$\psi_n(\kappa) = c_2(-1)^n e^{-\frac{\kappa^2}{2}} L_n^{\frac{d}{2}-1}\left(\kappa^2\right) , \tag{33}$$

$$\phi_n(r) = c_2 e^{-\frac{r^2}{2}} L_n^{\frac{d}{2}-1}\left(r^2\right) , \tag{34}$$

where the constant c_2 can be fixed by the normalization condition

$$c_2 = \sqrt{\frac{n!\,\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(n+\frac{d}{2}\right)}} \,. \tag{35}$$

Therefore, indeed, $a_n(E)$ is the nth eigenstate of the d-dimensional harmonic oscillator carrying zero angular momentum.

At this point we can calculate the wave functions in the position representation of the infinite cutoff limit of solutions Eq. (23). We write

$$\begin{aligned} \langle r | \kappa \rangle &= \sum_{n=0}^{\infty} \langle r | n \rangle \langle n | \kappa \rangle = \sum_{n=0}^{\infty} \phi_n(r) \psi_n(\kappa) \\ &= \mathcal{N} e^{-\frac{\kappa^2 + r^2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma\left(n+\frac{d}{2}\right)} L_n^{\frac{d}{2}-1}\left(r^2\right) L_n^{\frac{d}{2}-1}\left(\kappa^2\right) \end{aligned} \tag{36}$$

with all unimportant constant factors gathered in $\mathcal{N}$ and the $(-1)^n$ factor coming from the definition of Fourier transform. Using the known relation for associated Laguerre polynomials [13] we eventually get

$$\langle r|\kappa\rangle = \frac{1}{2}\mathcal{N}\frac{1}{(-\kappa r)^{\frac{d}{2}-1}}J_{\frac{d}{2}-1}(-\kappa r)\,. \tag{37}$$

This result indeed coincides with the solutions of the radial part of the d-dimensional Laplace equation

$$-\Delta\phi(r) - E\phi(r) = \phi''(r) + \frac{d-1}{r}\phi'(r) + E\phi(r) = 0 \tag{38}$$

given by the well-known Bessel functions

$$\phi(r) = c_1 r^{1-\frac{d}{2}} J_{\frac{d}{2}-1}\left(-\sqrt{E}r\right) + c_2 r^{1-\frac{d}{2}} Y_{\frac{d}{2}-1}\left(-\sqrt{E}r\right). \tag{39}$$

Requiring a regular behavior of the solution at $r=0$ fixes $c_2 = 0$, and after setting $c_1 = \frac{1}{2}\mathcal{N}\kappa^{1-\frac{d}{2}}$ and $E = \kappa^2$, we recover the solutions of Eq. (37). Therefore, our solutions obtained for a finite N_B become, in the infinite cutoff limit, the known Bessel functions.

6. Physical interpretation of the cutoff

Physically, the introduction of a cutoff limiting the maximal number of quanta contained in the basis states is equivalent to a specific discretization of the position as well as momentum variables. This can be demonstrated by considering the eigenvalues of the cut matrix representations of those operators. They correspond to possible outcomes of a measurement process. Obviously, in the cut basis the matrices of the $\hat{x}$ and $\hat{p}$ operators are finite and have as many eigenvalues as there are states in the cut basis. Hence, position as well as momentum spaces are discrete.

In order to be more precise, let us consider gauge invariant position and momentum operators of the form $X^2 = x^m x^m$ and $P^2 = p^m p^m$. We have shown in the preceding sections that their spectra consist of zeros of Laguerre polynomials. We denote their eigenvalues by X_i^2 and P_i^2, respectively

$$X_i^2:\ L_n^{\frac{d}{2}-1}\left(X_i^2\right) = 0\,, \qquad P_i^2:\ L_n^{\frac{d}{2}-1}\left(P_i^2\right) = 0\,, \qquad i = 1,\ldots,n\,.$$

A well-known upper bound for the largest zero of $L_n^{\alpha}(x)$ is, provided $|\alpha| \geq \frac{1}{4}$, $\alpha > -1$,

$$x_n < \left(\sqrt{4n+2\alpha+2} - \gamma(4n+2\alpha+2)^{-\frac{1}{6}}\right)^2,$$

$\gamma = 6^{-\frac{1}{3}} i_1$ and i_1 is the smallest zero of the Airy function [20]. Hence, for a cutoff N_B the eigenvalues X_i and P_i lie in the interval $\mathcal{I}$

$$\mathcal{I} = \left(-\sqrt{4\lceil \frac{N_B}{2} \rceil + 3},\ \sqrt{4\lceil \frac{N_B}{2} \rceil + 3} \right) .$$

The interval $\mathcal{I}$ is of the length of $2\sqrt{2N_B}$ and contains roughly $\frac{1}{2}N_B$ such eigenvalues. For the increasing cutoff, $\mathcal{I}$ becomes bigger and bigger whereas the distribution of eigenvalues inside $\mathcal{I}$ becomes denser and denser. One concludes that X_i and P_i tend to cover the real axis in the limit of infinite cutoff. We call this limit also a continuum limit.

7. Continuum limit

Up to now, our results suggest that the continuum limit of the nth eigenenergy vanishes

$$\lim_{N_B \to \infty} E_n(N_B) = 0 . \tag{40}$$

However, this is in contradiction with the general expectation that the spectrum of the kinetic energy operator is continuous, *i.e.*

$$\lim_{N_B \to \infty} E_n(N_B) = \tfrac{1}{2}p^2 , \qquad p \in \mathbb{R} . \tag{41}$$

In order to avoid the situation of Eq. (40), one must assume that the limit in Eq. (41) is taken with

$$n = n(N_B) \tag{42}$$

which we call *scaling law*, referring to a similar procedure in lattice field theory. This phenomenon was investigated in the case of one-dimensional momentum operators by Trzetrzelewski [15]. He was able to derive the scaling properties of the eigenvalues of the momentum operator as well as of the kinetic energy operator and obtained

$$E_n(N_B) = \frac{\pi^2}{2} \frac{\left(n - \frac{1}{2}\right)^2}{2N_B + 5} . \tag{43}$$

It is now straightforward to check the validity of the scaling law in $d > 1$ dimensions. The eigenenergies of the kinetic energy operator being the zeros of appropriate Laguerre polynomials can be approximated at large N_B by the formula [12, 13]

$$E_n(N_B) = \left(L^{\frac{d}{2}-1}_{\lfloor \frac{1}{2} N_B \rfloor + 1}(E) \right)_n \approx \frac{j^2_{\frac{d}{2}-1,n}}{4\lfloor \frac{1}{2} N_B \rfloor + d} + \ldots . \tag{44}$$

Setting $d = 1$, for which $j_{-\frac{1}{2},n} = \pi(n - \frac{1}{2})$, we obtain exactly the expression Eq. (43).

Hence, we see that the simple scaling $E_n \sim n^2$ is correct as long as the following approximation is true

$$j_{\frac{d}{2}-1,n} \approx \gamma_1(d)n + \gamma_2(d)\,. \tag{45}$$

Obviously, (45) will not hold for high d. Indeed, Fig. 2 shows the dependence of the eigenenergies $E_n(N_B)$ on the labeling number n for a given cutoff $N_B = 200$. Distinct curves correspond to different d, namely $d = 1$ for the lower one and $d = 150$ for the upper one. Definitely, in the latter case the dependence is no longer linear. Some particular values of the coefficients γ_1 and γ_2 are given in Table I.

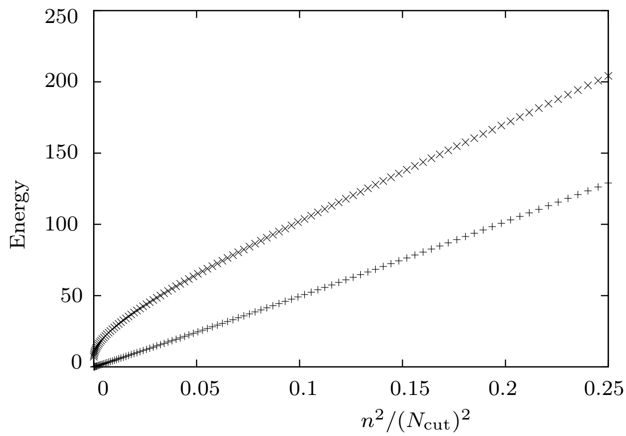

Fig. 2. Scaling laws for $d = 1$ and $d = 150$.

TABLE I

Fitted values of parameters $\gamma_1(d)$ and $\gamma_2(d)$ for $d = 1, 3, 9$.

d	$\gamma_1(d)$	Std. error	$\gamma_2(d)$	Std. error
1	3.14159	$< 10^{-10}$	-1.5708	$< 10^{-10}$
3	3.14159	$< 10^{-10}$	0.0	$< 10^{-10}$
9	3.142	4.4×10^{-5}	4.6192	0.0077

8. Conclusions

In this note we used the cut Fock space approach to investigate the system which can be interpreted as a free quantum particle in d-dimensions. Although solutions to such problem are well known in the continuum ($N_B=\infty$), they were unknown for finite N_B.

Our approach was guided by the numerical algorithm proposed for solving supersymmetric Yang–Mills quantum mechanics in the Hamiltonian formulation. The study of this particular system was motivated by a similar problem with the $\mathrm{SO}(d)$ gauge group replaced by the $\mathrm{SU}(N)$ group [21, 22].

We started by describing the construction of basis states with the use of gauge invariant creation operators. Then, we explicitly derived the Hamiltonian matrix. Due to its simple form it was possible to obtain exact expressions for the eigenvalues and eigenvectors. The formulae are parameterized by the cutoff and hence, are valid for any finite cutoff. Namely, the spectrum at finite cutoff is given by the zeros of an appropriate Laguerre polynomial, whereas in the continuum it corresponds to the positive real axis.

In order to confirm the correctness of our solutions, we calculated their wave functions in the position representation in the continuum limit. Indeed, it was possible to show that such wave functions are equivalent to the well known solutions given in terms of Bessel functions.

Next, we discussed the physical interpretation of the cutoff. We argued that a finite cutoff corresponds to a discretization of both position and momentum space. In the infinite cutoff limit continuum results are recovered.

Finally, we observed that the scaling law proved for the one-dimensional kinetic energy operator holds only approximately in spaces with higher number of spatial dimensions.

The Author would like to acknowledge many useful discussions with J. Wosiek on the subject of this paper.

REFERENCES

[1] B. de Wit, J. Hoppe, H. Nicolai, *Nucl. Phys.* **B305**, 545 (1988).

[2] B. de Wit, M. Lüscher, H. Nicolai, *Nucl. Phys.* **B320**, 135 (1989).

[3] T. Banks, W. Fischler, S. Shenker, L. Susskind, *Phys. Rev.* **D55**, 6189 (1997).

[4] W. Taylor, *Rev. Mod. Phys.* **73**, 419 (2001).

[5] M. Claudson, M.B. Halpern, *Nucl. Phys.* **B250**, 689 (1985).

[6] J. Wosiek, *Nucl. Phys.* **B644**, 85 (2002).

[7] P. Korcyl, *Acta Phys. Pol. B* **41**, 795 (2010) [arXiv:0912.5265 [hep-th]].

[8] J. Wosiek, *Int. J. Mod. Phys.* **A20**, 4484 (2005).

[9] J. Wosiek, *Phys. Lett.* **B619**, 171 (2005).

[10] M. Campostrini, J. Wosiek, *Phys. Lett.* **B550**, 121 (2002).

[11] M. Campostrini, J. Wosiek, *Nucl. Phys.* **B703**, 454 (2004).

[12] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York 1968.

[13] H. Bateman, *Higher Transcendental Functions, vol. II*, Bateman Manuscript Project, McGraw-Hill, 1953.

[14] M. Trzetrzelewski, J. Wosiek, *Acta Phys. Pol. B* **35**, 1615 (2004) [arXiv:hep-th/0308007].

[15] M. Trzetrzelewski, *Acta Phys. Pol. B* **35**, 2393 (2004) [arXiv:hep-th/0407059].

[16] P. Korcyl, *Phys. Rev.* **D74**, 115012 (2006).

[17] M. Trzetrzelewski, Supersymmetric Yang–Mills Quantum Mechanics with Arbitrary Number of Colors, PhD thesis Jagiellonian University, 2006.

[18] M. Hamermesh, *Group Theory and Its Application to Physical Problems*, Addison-Wesely, Reading Mass., 1962.

[19] J.-Q. Chen, J. Ping, F. Wang, *Group Representation Theory for Physicists*, World Scientific, New Jersey 2002.

[20] G. Szegö, *Orthogonal Polynomials*, Vol. 23, 4th ed., Amer. Math. Soc. Colloq. Publ., Providence, RI, 1975.

[21] P. Korcyl, *Acta Phys. Pol. B Proc. Suppl.* **2**, 623 (2009) [arXiv:0911.2152 [hep-th]].

[22] P. Korcyl, arXiv:1008.2975 [math-ph].