

## Delay-Dependent Criterion for Exponential Stability Analysis of Neural Networks with Time-Varying Delays

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**Abstract:** This note investigates the problem of exponential stability of neural networks with time-varying delays. To derive a less conservative stability condition, a novel augmented Lyapunov-Krasovskii functional (LKF) which includes triple and quadruple-integral terms is employed. In order to reduce the complexity of the stability test, the convex combination method is utilized to derive an improved delay-dependent stability criterion in the form of linear matrix inequalities (LMIs). The superiority of the proposed approach is demonstrated by two comparative examples.

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### 1. INTRODUCTION

Neural networks have been studied by numerous researchers during the recent years due to their wide range of applications, Du et al. (2014), Yuhás et al. (2012), Cochocki et al. (1993). Dynamical stability is an important issue in the performance analysis of neural networks. Among the notions that were used to define the stability of neural networks, the exponential stability is frequent, because of including the exponential convergence in the evaluation of stability rate.

Appearing of time-delay in the dynamical equations of neural networks makes their stability analysis more challenging. Several criteria were proposed in the literature for instance, Song et al. (2013), Gao et al. (2013), Shen et al. (2008), He et al. (2007), to check the stability of delayed neural networks by including the information of their structures in the construction of LKF and using innovative computational techniques to derive stability condition in terms of linear matrix inequalities. In Zhang et al. (2014), the research on stability of continuous-time recurrent neural networks was surveyed and the recent results in the case of constant and variable delay in recurrent neural networks were discussed and compared. In Zeng et al. (2006), delay-independent stability criteria was developed for neural networks with time-varying delay. However, the mentioned approach leads to conservative result compared to the delay-dependent methods in which the value of delay is incorporated directly in the stability conditions.

In order to decrease the conservativeness of the delay-dependent stability results, two main directions were followed recently. First, in the method of free weighting matrices, some free matrix variables are added to the stability measures to improve their effectiveness by adjustable variables. Second, in delay partitioning approach, the delay interval is divided into some subintervals in order that more information of the varying delay and more free variables can

be used. By combining the mentioned ideas with the innovative augmented LKFs, new analysis methods for delayed neural networks have been proposed. In both of the above mentioned schemes, conservativeness of the stability test is decreased at the expense of more unknown parameters involved in the final sufficient condition.

A new augmented LK functional was proposed in Kwon et al. (2013), to establish a less conservative stability criterion in terms of LMIs. In Xie et al. (2014), by using the delay-partitioning method and the reciprocally convex technique, less conservative stability criteria were obtained for neural networks with time-varying delays in terms of LMIs. In Zhou et al. (2014), for recurrent neural networks with time-varying delays, a novel LKF was introduced; furthermore, reciprocally convex approach was used to improve stability criteria which are derived in terms of LMIs. By construction of an augmented LKF based on delay partitioning idea, the problem of exponential stability analysis of neural networks with time varying-delay was investigated in Hua et al. (2011). Second order convex combination approach was employed in Huaguang et al. (2013), for stability analysis of neural networks with time-varying delay. By using the LKF method, novel stability criteria were derived in Liu G. et al. (2013), for robust stability analysis of uncertain stochastic neural networks of neutral-type with interval time-varying delays. In Liu C. et al (2013), the stability of Hopfield neural networks with time delay and variable-time impulses was addressed. In Zhang et al. (2013), new sufficient conditions were extracted in terms of LMIs to guarantee that the neutral-type delayed projection neural network is globally exponentially convergent to the optimal solution. The problem of stochastic stability was investigated in Ma et al (2015), for perturbed chaotic neural networks with mixed time-delays and Markovian jumping parameters by employing suitable LKF.

In this paper, a less conservative stability criterion is introduced for neural networks with time-varying delays. Inspired by Sun et al. (2010), triple-integral term is utilized in

LKF. Moreover, quadruple-integral terms together with other new augmented terms are added to the energy functional to bring more degree of freedom in the final stability condition. Motivated by Park et al. (2011), the convex combination approach is employed to reduce the parameters of the stability test. The new exponential stability condition is derived in the form of LMIs. The main advantage of the proposed stability condition is to reduce simultaneously the conservativeness and complexity of the stability test; i.e. by the proposed method, much larger allowable delay bound is achieved while lighter computational burden is needed, compared to some of recent methods in Xu et al. (2006), Shen et al. (2008) and Wu et al. (2008), Hua et al. (2011), Huanguanget al. (2013), which analyze the stability of neural networks with variable delay. Two comparative numerical examples are presented to demonstrate that the proposed method can lead to less conservative results compared to some of existing approaches in the literature.

Notations: In this paper,  $\mathfrak{R}$  denotes the real numbers set. The symbol  $*$  stands for the symmetric block in the symmetric matrices.  $I$  is the identity matrix of the appropriate dimensions. The notation  $P > 0$  (respectively,  $P \geq 0$ ) means that  $P$  is real symmetric and positive definite (respectively, positive semidefinite). The superscript  $T$  stands for the matrix transposition.  $col\{\}$  shows the column vector composed of the elements in the bracket.  $diag\{\}$  symbolizes a diagonal matrix of elements in the bracket.  $e_i$  represents block entry matrices, for instance  $e_8^T = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$ .

## 2. PROBLEM FORMULATION

Consider the dynamical model of neural network with time-varying delay as follows:

$$\dot{x}(t) = -Ax(t) + Bf(x(t)) + Cf(x(t - \eta(t))) \quad (1)$$

where,  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathfrak{R}^n$  and  $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t))]^T \in \mathfrak{R}^n$  are the neuron state and neuron activation vectors, respectively.  $A = diag\{a_i\}$  is diagonal matrix with  $a_i > 0$ . The matrices  $B$  and  $C$  are the connection weight matrix and the delayed weight matrix, respectively. The time-varying delay  $\eta(t)$  satisfies the following conditions:

$$0 \leq \eta(t) \leq \eta, \quad 0 \leq \dot{\eta}(t) \leq \mu \quad (2)$$

wherein,  $\eta$  and  $\mu$  are constant known values that denote the upper bounds of delay and delay rate, respectively. Activation function  $f_i(\cdot)$ , for  $i \in \{1, 2, \dots, n\}$  is supposed to be bounded, satisfying the following inequality:

$$0 \leq \frac{f_i(y_1) - f_i(y_2)}{y_1 - y_2} \leq L_i \quad (3)$$

where,  $L_i$ , for  $i \in \{1, 2, \dots, n\}$  is positive scalar. Assume that there exists a vector  $x^* = [x_1^*, x_2^*, \dots, x_n^*] \in \mathfrak{R}^n$  which satisfies:

$$A x^*(t) = B f(x^*(t)) + C f(x^*(t - \eta(t))) \quad (4)$$

So,  $x^*$  is called the equilibrium point of neural network (1).

By employing the transformation  $z = x - x^*$ , the dynamical equation of neural network in (1) is changed into the following:

$$\dot{z}(t) = -Az(t) + Bg(z(t)) + Cg(z(t - \eta(t))) \quad (5)$$

In which,

$$z(t) = [z_1(t), z_2(t), \dots, z_n(t)]^T \in \mathfrak{R}^n$$

$$g(z(t)) = [g_1(z_1(t)), g_2(z_2(t)), \dots, g_n(z_n(t))]^T \in \mathfrak{R}^n$$

$$g_i(z_i(t)) = f_i(z_i(t) + z_i^*) - f_i(z_i^*), \quad i = 1, 2, \dots, n$$

where,  $z_i^*$  is the  $i$  th element of the equilibrium point vector in the new coordinate. Regarding (3) and the transformation  $z = x - x^*$ , the function  $g_i(z_i)$ , for  $i \in \{1, 2, \dots, n\}$  satisfy the following:

$$0 \leq \frac{g_i(z_i)}{z_i} \leq L_i, \quad g_i(0) = 0, \quad \forall z_i \neq 0, \quad (6)$$

Before proceeding further, the following definition and Lemmas are introduced:

*Definition 1:* The neural network in (1) is exponentially stable with the exponential convergence rate  $k$ , if there exist positive constant values  $\gamma$  and  $k$ , stasifying:

$$\|z(t)\| \leq \gamma \Psi e^{-kt}, \quad \Psi = \max_{-\eta \leq \theta \leq 0} \|z(t)\| \quad \forall t > 0 \quad (7)$$

*Lemma 1* (Jensen's Inequality): Suppose  $\eta \in \mathfrak{R}$  and  $x(t) \in \mathfrak{R}^n$ , for any positive definite matrix  $P$  the following inequality holds:

$$-\eta \int_{t-\eta}^t \dot{x}^T(s) P \dot{x}(s) ds \leq \begin{bmatrix} x(t) \\ x(t - \eta) \end{bmatrix}^T \begin{bmatrix} -P & P \\ P & -P \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \eta) \end{bmatrix}.$$

*Lemma 2* Park et al. (2011): Suppose  $\eta \in \mathfrak{R}$  and  $x(t) \in \mathfrak{R}^n$ , for any matrices  $Q = Q^T > 0$  and  $S$  the following inequality holds:

$$-\eta \int_{t-\eta}^t \dot{x}^T(s) Q \dot{x}(s) ds \leq - \begin{bmatrix} (x(t) - x(t - \eta(t)))^T \\ (x(t - \eta(t)) - x(t - \eta))^T \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} (x(t) - x(t - \eta(t)))^T \\ (x(t - \eta(t)) - x(t - \eta))^T \end{bmatrix}$$

where,

$$\begin{bmatrix} Q & S \\ S^T & Q \end{bmatrix} \geq 0.$$

## 3. MAIN RESULTS

Note By presenting an appropriate LKF, a new delay-dependent sufficient condition is derived in Theorem 1 to check the exponential stability of the neural network (1).

*Theorem 1:* For the given  $\eta$ ,  $\mu$ ,  $A$ ,  $B$  and  $C$ , the system (1) with the time-varying delay satisfying (2) is exponentially stable with the exponential convergence rate  $k$ , if there exist arbitrary matrices  $M$ ,  $S_1$ ,  $S_2$ ,  $S_3$ , symmetric matrices  $P > 0$ ,  $T > 0$ ,  $Q > 0$ ,  $X_1 > 0$ ,  $X_2 > 0$ ,  $R_1 > 0$ ,  $R_2 > 0$ ,  $U > 0$

and diagonal matrices  $D = \text{diag}\{d_1, d_2, \dots, d_n\} \geq 0$ ,  $R = \text{diag}\{r_1, r_2, \dots, r_n\} \geq 0$  and  $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$  with appropriate dimensions such that the LMIs (8)-(9) hold:

$$\begin{bmatrix} X_1 & S_1 \\ * & X_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} X_2 & S_2 \\ * & X_2 \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} \geq 0 \quad (8)$$

$$\Omega < 0 \quad (9)$$

wherein,

$$\begin{aligned} \Omega = & \Lambda + \eta^2(e_1 X_1 e_1^T + e_4 X_2 e_4^T) + \frac{\eta^4}{4}(e_1 R_1 e_1^T + e_4 R_2 e_4^T) + \\ & + \frac{\eta^6}{36} e_4 U e_4^T - e^{-2k\eta} \left( \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix} \begin{bmatrix} X_1 & S_1 \\ * & X_1 \end{bmatrix} \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix} \right. \\ & \left. - \begin{bmatrix} (e_1 - e_2)^T \\ (e_2 - e_3)^T \end{bmatrix} \begin{bmatrix} X_2 & S_2 \\ * & X_2 \end{bmatrix} \begin{bmatrix} (e_1 - e_2)^T \\ (e_2 - e_3)^T \end{bmatrix} \right) \\ & - e^{-2k\eta} \left( \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix} \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix} \right) \\ & + (\eta e_1 - e_6 - e_7) R_2 \quad (\eta e_1 - e_6 - e_7)^T \\ & - e^{-2k\eta} \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right) U \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right)^T \end{aligned} \quad (10)$$

with,

$$\Lambda = \pi + \alpha + \alpha^T \quad (11)$$

in which,

$$\alpha = [MA \ 0 \ 0 \ M \ 0 \ 0 \ 0 \ 0 \ 0 \ -MB \ -MC]$$

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ * & \pi_5 & \pi_6 & \pi_7 \\ * & * & \pi_8 & \pi_9 \\ * & * & * & \pi_{10} \end{bmatrix}$$

$$\pi_1 = \begin{bmatrix} \pi_{111} & 0 & P_{23}^T + \eta P_{24}^T + P_{13} + 2kP_{12} \\ * & -(1-\mu)T_{11} & 0 \\ * & * & -P_{23}^T - P_{23} - Q_{11} + 2kP_{22} \end{bmatrix}$$

$$\pi_2 = \begin{bmatrix} P_{11} + e^{2k\eta} Q_{12} & P_{12} & \pi_{213} \\ 0 & 0 & 0 \\ P_{12}^T & P_{22} - Q_{12} & -P_{33} - P_{24} + 2kP_{23} \end{bmatrix}$$

$$\pi_3 = \begin{bmatrix} P_{33} + \eta P_{34}^T - P_{14} + 2kP_{13} & \pi_{312} & \pi_{313} \\ 0 & 0 & 0 \\ -P_{33} - P_{24} + 2kP_{23} & -P_{34} + 2kP_{24} & \pi_{333} \end{bmatrix}$$

$$\pi_4 = \begin{bmatrix} 2kD + e^{2k\eta} T_{12} + LR & 0 \\ 0 & -(1-\mu)T_{12} + LS \\ 0 & 0 \end{bmatrix}$$

$$\pi_5 = \begin{bmatrix} e^{2k\eta} & 0 & P_{13} \\ * & -Q_{22} & P_{23} \\ * & * & -P_{34} - P_{34}^T + 2kP_{33} \end{bmatrix}$$

$$\pi_6 = \begin{bmatrix} P_{13} & P_{14} & P_{14} \\ P_{23} & P_{24} & P_{24} \\ -P_{34} - P_{34}^T + 2kP_{33} & -P_{44} + 2kP_{34} & -P_{44} + 2kP_{34} \end{bmatrix}$$

$$\pi_7 = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\pi_8 = \begin{bmatrix} -P_{34} - P_{34}^T + 2kP_{33} & -P_{44} + 2kP_{34} & -P_{44} + 2kP_{34} \\ * & 2kP_{44} & 2kP_{44} \\ * & * & 2kP_{44} \end{bmatrix}$$

$$\pi_9 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\pi_{10} = \begin{bmatrix} e^{2k\eta} T_{22} - 2R & 0 \\ * & -(1-\mu)T_{22} - 2S \end{bmatrix}$$

$$\pi_{111} = P_{13} + P_{13}^T + \eta(P_{14} + P_{14}^T) + 2kP_{11} + e^{2k\eta} T_{11} + e^{2k\eta} Q_{11}$$

$$\pi_{213} = P_{33} + \eta P_{34}^T - P_{14} + 2kP_{13}$$

$$\pi_{312} = \pi_{313} = P_{34} + \eta P_{44} + 2kP_{14}$$

$$\pi_{333} = -P_{34} + 2kP_{24}$$

where,

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ * & P_{22} & P_{23} & P_{24} \\ * & * & P_{33} & P_{34} \\ * & * & * & P_{44} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ * & T_{22} \end{bmatrix}, \quad (12)$$

$$Q = [Q_{11} \ Q_{12}]$$

*Proof:* Construct the LKF candidate as follows:

$$V(z_t) = V_1(z_t) + V_2(z_t) + V_3(z_t) + V_4(z_t) + V_5(z_t) \quad (13)$$

with,

$$V_1(z_t) = e^{2kt} \xi^T(t) P \xi(t) + 2 \sum_{i=1}^n d_i e^{2kt} \int_0^{z_i(t)} g_i(s) ds$$

$$\begin{aligned} V_2(z_t) = & e^{2k\eta} \int_{t-\eta}^t e^{2ks} \begin{bmatrix} z(s) \\ g(z(s)) \end{bmatrix}^T T \begin{bmatrix} z(s) \\ g(z(s)) \end{bmatrix} ds \\ & + e^{2k\eta} \int_{t-\eta}^t e^{2ks} \begin{bmatrix} z(s) \\ \dot{z}(s) \end{bmatrix}^T Q \begin{bmatrix} z(s) \\ \dot{z}(s) \end{bmatrix} ds \end{aligned}$$

$$\begin{aligned} V_3(z_t) = & \eta \left( \int_{-\eta}^0 \int_{t+\beta}^t e^{2ks} z^T(s) X_1 z(s) ds d\beta \right. \\ & \left. + \int_{-\eta}^0 \int_{t+\beta}^t e^{2ks} \dot{z}^T(s) X_2 \dot{z}(s) ds d\beta \right) \end{aligned}$$

$$\begin{aligned} V_4(z_t) = & \frac{\eta^2}{2} \left( \int_{-\eta}^0 \int_{\beta}^0 \int_{t+\lambda}^t e^{2ks} z^T(s) R_1 z(s) ds d\lambda d\beta \right. \\ & \left. + \int_{-\eta}^0 \int_{\beta}^0 \int_{t+\lambda}^t e^{2ks} \dot{z}^T(s) R_2 \dot{z}(s) ds d\lambda d\beta \right) \end{aligned}$$

$$V_5(z_t) = \frac{\eta^3}{6} \int_{-\eta}^0 \int_{\beta}^0 \int_{\lambda}^0 \int_{t+\varphi}^t e^{2ks} \dot{z}^T(t) U \dot{z}(t) ds d\varphi d\lambda d\beta$$

in which,  $\xi(t)$  is defined as follows:

$$\xi(t) = \text{col} \left\{ z(t), z(t-\eta), \int_{t-\eta}^t z(s) ds, \int_{-\eta}^0 \int_{t+\beta}^t z(s) ds d\beta \right\}$$

Note that regarding (6),  $V_1$  is positive as required. Before proceeding, the following notation is introduced:

$$\zeta(t) = \text{col} \left\{ z(t), z(t - \eta(t)), z(t - \eta), \dot{z}(t), \dot{z}(t - \eta), \int_{t-\eta(t)}^t z(s) ds, \int_{t-\eta}^{t-\eta(t)} z(s) ds, \int_{-\eta(t)}^0 \int_{t+\beta}^t z(s) ds d\beta, \int_{-\eta(t)}^{-\eta} \int_{t+\beta}^t z(s) ds d\beta, g(z(t)), g(z(t - \eta(t))) \right\}.$$

Taking the time derivative of the  $V(z_t)$  along the trajectories of (1) yields:

$$\dot{V}(z_t) = \dot{V}_1(z_t) + \dot{V}_2(z_t) + \dot{V}_3(z_t) + \dot{V}_4(z_t) + \dot{V}_5(z_t) \quad (14)$$

Each term of (14) is upper bounded as follows:

$$\dot{V}_1(z_t) \leq 2ke^{2kt}\zeta^T(t)P\xi(t) + 2e^{2kt}\xi^T(t)P\xi(t) + 4ke^{2kt}g^T(z(t))Dz(t) + 2e^{2kt}g^T(z(t))D\dot{z}(t) \quad (15)$$

$$\begin{aligned} \dot{V}_2(z_t) &\leq e^{2k\eta}e^{2kt} \left( \begin{bmatrix} z(t) \\ g(z(t)) \end{bmatrix}^T T \begin{bmatrix} z(t) \\ g(z(t)) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}^T Q \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \right) \\ &\quad - e^{2kt}(1 - \mu) \begin{bmatrix} z(t - \eta(t)) \\ g(z(t - \eta(t))) \end{bmatrix}^T T \begin{bmatrix} z(t - \eta(t)) \\ g(z(t - \eta(t))) \end{bmatrix} \\ &\quad - e^{2kt} \begin{bmatrix} z(t - \eta) \\ \dot{z}(t - \eta) \end{bmatrix}^T Q \begin{bmatrix} z(t - \eta) \\ \dot{z}(t - \eta) \end{bmatrix} \quad (16) \end{aligned}$$

$$\begin{aligned} \dot{V}_3(z_t) &= \eta^2 e^{2kt} (z^T(t) X_1 z(t) + \dot{z}^T(t) X_2 \dot{z}(t)) \\ &\quad - \eta \left( \int_{t-\eta}^t e^{2ks} z^T(s) X_1 z(s) ds + \int_{t-\eta}^t e^{2ks} \dot{z}^T(s) X_2 \dot{z}(s) ds \right) \\ &\leq \eta^2 e^{2kt} (\zeta^T(t) e_1 X_1 e_1^T \zeta(t) + \zeta^T(t) e_4 X_2 e_4^T \zeta(t)) \\ &\quad - e^{2k(t-\eta)} \zeta^T(t) \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix}^T \begin{bmatrix} X_1 & S_1 \\ * & X_1 \end{bmatrix} \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix} \\ &\quad + \left[ \begin{bmatrix} e_1 - e_2 \\ e_2 - e_3 \end{bmatrix}^T \begin{bmatrix} X_2 & S_2 \\ * & X_2 \end{bmatrix} \begin{bmatrix} e_1 - e_2 \\ e_2 - e_3 \end{bmatrix} \right] \zeta(t) \quad (17) \end{aligned}$$

provided that:

$$\begin{bmatrix} X_1 & S_1 \\ * & X_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} X_2 & S_2 \\ * & X_2 \end{bmatrix} \geq 0$$

$$\begin{aligned} \dot{V}_4(z_t) &= \frac{\eta^4}{4} e^{2kt} \left( z^T(t) R_1 z(t) + \dot{z}^T(t) R_2 \dot{z}(t) \right) \\ &\quad - \frac{\eta^2}{2} \left( \int_{-\eta}^0 \int_{t+\beta}^t e^{2ks} z^T(s) R_1 z(s) ds d\beta \right. \\ &\quad \left. + \int_{-\eta}^0 \int_{t+\beta}^t e^{2ks} \dot{z}^T(s) R_2 \dot{z}(s) ds d\beta \right) \\ &\leq \frac{\eta^4}{4} e^{2kt} (\zeta^T(t) e_1 R_1 e_1^T \zeta(t) + \zeta^T(t) e_4 R_2 e_4^T \zeta(t)) \\ &\quad - e^{2k(t-\eta)} \zeta^T(t) \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix}^T \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix} \\ &\quad + (\eta e_1 - e_6 - e_7) R_2 (\eta e_1 - e_6 - e_7)^T \zeta(t) \quad (18) \end{aligned}$$

on the condition that:

$$\begin{aligned} \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} &\geq 0 \\ \dot{V}_5(z_t) &= \frac{\eta^6}{36} e^{2kt} \dot{z}^T(t) U \dot{z}(t) \\ &\quad - \frac{\eta^3}{6} \int_{-\eta}^0 \int_{\beta}^0 \int_{t+\lambda}^t e^{2ks} \dot{z}^T(t) U \dot{z}(s) ds d\lambda d\beta \\ &\leq \frac{\eta^6}{36} e^{2kt} (\zeta^T(t) e_4 U e_4^T \zeta(t)) \\ &\quad - e^{2k(t-\eta)} \zeta^T(t) \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right) U \\ &\quad \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right)^T \zeta(t) \quad (19) \end{aligned}$$

It should be noted that the details of manipulation to obtain inequality (15) can be found in [30]; also, inequalities in (16), (17) and (18) come from using Lemmas 1 and 2.

The terms  $\theta_i$ s, for  $i = 1, 2$  which are equal to zero are defined as follows:

$$\begin{aligned} \theta_1(t) &= 2e^{2kt} \zeta^T(t) M [\dot{z}(t) + Az(t) \\ &\quad - Bg(z(t)) - cg(z(t - \eta(t)))] = 0 \quad (20) \end{aligned}$$

$$\begin{aligned} \theta_2(t) &= 2e^{2kt} [z^T(t) LR g(z(t)) - g^T(z(t)) R g(z(t)) \\ &\quad + z^T(t - \eta(t)) LS g(z(t - \eta(t))) \\ &\quad - g^T(z(t - \eta(t))) S g(z(t - \eta(t)))] = 0 \quad (21) \end{aligned}$$

Regarding (15)-(21), the following is obtained:

$$\begin{aligned} \dot{V}(z_t) + \sum_{i=1}^2 \theta_i(t) &\leq e^{2kt} \zeta^T(t) \{ \Lambda + \eta^2 (e_1 X_1 e_1^T + e_4 X_2 e_4^T) \\ &\quad + \frac{\eta^4}{4} (e_1 R_1 e_1^T + e_4 R_2 e_4^T) + \frac{\eta^6}{36} e_4 U e_4^T \\ &\quad - e^{-2k\eta} \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix}^T \begin{bmatrix} X_1 & S_1 \\ * & X_1 \end{bmatrix} \begin{bmatrix} e_6^T \\ e_7^T \end{bmatrix} \\ &\quad + \left[ \begin{bmatrix} e_1 - e_2 \\ e_2 - e_3 \end{bmatrix}^T \begin{bmatrix} X_2 & S_2 \\ * & X_2 \end{bmatrix} \begin{bmatrix} e_1 - e_2 \\ e_2 - e_3 \end{bmatrix} \right] \\ &\quad - e^{-2k\eta} \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix}^T \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} \begin{bmatrix} e_8^T \\ e_9^T \end{bmatrix} \\ &\quad + (\eta e_1 - e_6 - e_7) R_2 (\eta e_1 - e_6 - e_7)^T \\ &\quad - e^{-2k\eta} \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right) U \left( \frac{\eta^2}{2} e_1 - e_8 - e_9 \right)^T \} \zeta(t) \\ &= e^{2kt} \zeta^T(t) \Omega \zeta(t) \quad (22) \end{aligned}$$

If  $\Omega < 0$ , then  $\dot{V}(z_t) < 0$ ; so, by the Lyapunov-Krasovskii argument the exponential stability of the system (1) is guaranteed. ■

*Remark 1:* The novelty of the introduced LKF in (13) is threefold. First, the quadruple-integral term is employed in the energy functional. Second, by including the integral term

$\int_{-\eta}^0 \int_{t+\beta}^t z(s) ds d\beta$  in  $\xi(t)$ , quadratic terms with respect to it are created in  $V(z_t)$ . Third, differently from Wu et al. (2008), Hua et al. (2011), Huaguanget al. (2013), the state  $z$  is imported in the integrand of the double-integral term and also triple-integral term.

The combination of the quadruple-integral with quadratic terms containing  $\int_{-\eta}^0 \int_{t+\beta}^t z(s) ds d\beta$  leads to the noticeable reduction of the conservativeness in the obtained stability measure. Furthermore, the existence of the quadruple-integral and the quadratic term with  $\int_{-\eta}^0 \int_{t+\beta}^t z(s) ds d\beta$  permits to handle triple-integral of the state  $z$ .

*Remark 2:* Unlike Wu et al. (2008), Hua et al. (2011) that employ free weighting matrices to handle the integral terms coming from the derivative of LKF, the inequality presented in Lemma 2 was employed in extracting the proposed criterion. It is worth mentioning that alternative approaches exist in the literature to reduce the complexity of stability condition while preserve its conservativeness; for instance in Li et al. (2011), improved stability test was proposed with lighter computational burden for linear discrete-time systems with time-varying delay.

*Remark 3:* For the more general case that the lower bound of the time-varying delay is non-zero, the lower bound of delay interval can be included in the lower limits of the integral terms in the augmented LKF in (13) as done in Liu Y., Wang Z., Liang J. et al. (2009) and Liu Y., Wang Z. et al. (2009) for discrete-time case to obtain delay-range-dependent stability criterion.

### 3. NUMERICAL EXAMPLES

Two numerical examples are represented to compare the results of the proposed method with some of existing ones. The LMI Toolbox of Matlab<sup>®</sup> is utilized to solve the LMI feasibility problems, Gahinet et al (1995). Maximum allowable delay bound (MADB), which is defined as the maximum delay value that retains the stability of the system, is the common measure to evaluate the performance of stability tests of delay systems in the literature. MADBs are reported to compare the conservativeness of the proposed stability test to rival ones.

*Example 1:* Consider the delayed neural network (1) with the following parameters, Hua et al. (2011):

$$A = \text{diag}\{1.2769, 0.6231, 0.9230, 0.4480\}$$

$$B = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & -0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$

$$L_1 = 0.1137, L_2 = 0.1279, L_3 = 0.7994, L_4 = 0.2368$$

It's assumed that the exponential convergence,  $k$  is zero to fairly compare the results of Theorem 1 with the schemes in Xu et al. (2006), Shen et al. (2008), Wu et al. (2008), Hua et al. (2011), Huaguanget al. (2013). The computed MADBs obtained from different methods are shown in Table 1.

**Table 1. MADBs Computed from different methods with various values  $\mu$  for Example 1**

$\mu$	0.1	0.5	0.9
Xu (2006)	3.3039	2.5376	2.0853
Wu (2008)	3.7525	2.7353	2.2760
Shen (2008)	4.4288	4.0089	3.2900
Hua (2011)	5.7803	4.6949	3.6639
Huaguang (2013)	6.4371	4.9210	3.9103
Theorem1	7.4104	5.4626	4.2031

*Example 2:* Consider the delayed neural network (1) with the following parameters [21]:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix}$$

$$L_1 = 3, L_4 = 4$$

It's supposed that the exponential convergence,  $k$  equals 0.8 to properly compare the results of Theorem 1 with Xu et al. (2006), Zheng et al. (2009), Wu et al. (2008), and Hua et al. (2011). The computed MADBs obtained from different methods for are shown in Table 2. The symbol ‘-’ in Table 2 means that the corresponding method is not feasible to determine the stability of the system.

**Table 2. MADBs Computed from different methods with various values  $\mu$  for Example 2**

$\mu$	0.8	0.9	0.95
Zheng (2009)	-	-	-
Xu (2006)	1.2977	0.9880	0.8519
Wu (2008)	1.3521	1.1032	0.9913
Hua (2011)	1.8654	1.6104	1.4030
Theorem 1	2.4743	2.1607	1.8911

Tables 1 and 2 clearly verify that the proposed method leads to less conservative results compared to the approaches listed in them.

### 5. CONCLUSION

In this paper, a new approach has been proposed to analyze the exponential stability of the neural networks with time-varying delay, by constructing an appropriate augmented LKF including quadruple-integral term. In order to reduce the parameters needed for stability analysis, convex combination approach has been employed. A new delay-dependent stability condition has been derived in terms of linear matrix inequalities. Two numerical examples have been given to demonstrate that the proposed criterion is less conservative compared to some of the existing approaches in the literature.

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