Bounding the degrees of a minimal μ-basis for a rational surface parametrization.

Yairon Cid-Ruiz

Department de Matemàtiques i Informàtica, Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585; 08007 Barcelona, Spain.

Abstract

In this paper, we study how the degrees of the elements in a minimal μ-basis of a parametrized surface behave. For an arbitrary rational surface parametrization $P(s, t) = (a_1(s, t), a_2(s, t), a_3(s, t), a_4(s, t)) \in \mathbb{F}[s, t]^4$ over an infinite field $\mathbb{F}$, we show the existence of a μ-basis with polynomials bounded in degree by $O(d^{33})$, where $d = \max(\deg(a_1), \deg(a_2), \deg(a_3), \deg(a_4))$. Under additional assumptions we can obtain tighter bounds.

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1. Introduction

The concept of a μ-basis is an important notion in Computer Aided Geometric Design and Geometric Modeling, that was introduced in Cox et al. (1998) to study the implicitization problem in the case of parametrized curves. The μ-basis of a parametric curve is a well-understood object that provides the implicit equation by computing its resultant (Cox et al., 1998, Section 4) and has a number of applications in the study of rational curves (see e.g. Chen and Sederberg (2002); Chen et al. (2008); Jia and Goldman (2009)). On the other hand, the μ-basis of a parametric surface is a more complicated object and its development took several years of research. In a first attempt, for the particular case of rational ruled surfaces the concept of a μ-basis was defined in Chen et al. (2001) and Chen and Wang (2003) (also, see Dohm (2009)). Later, in Chen et al. (2005), the existence of a μ-basis was proved for an arbitrary rational surface.

The existence of μ-bases for rational surfaces is a strong result whose geometrical meaning is that any rational surface is the intersection of three moving planes without extraneous factors. Additionally, a μ-basis of a rational surface parametrization coincides with a basis of the syzygy module (Chen et al., 2005, Corollary 3.1) and can be used to obtain the implicit equation (Chen et al., 2005, Section 4). Contrary to the case of rational curves, there is no known upper bound for the degrees of the elements in a minimal μ-basis of a rational surface parametrization. The main purpose of this article is to obtain such an upper bound.

In order to describe the main results of this paper, we briefly recall the notion of a μ-basis for a rational parametric surface.

Let $\mathbb{F}$ be an infinite field and R be the polynomial ring $R = \mathbb{F}[s, t]$.

Definition 1. *A rational surface parametrization in homogeneous form is defined by*

$$P(s, t) = (a_1(s, t), a_2(s, t), a_3(s, t), a_4(s, t)) \tag{1}$$

where $a_1, a_2, a_3, a_4 \in R$ and $\gcd(a_1, a_2, a_3, a_4) = 1$.

Definition 2. *A moving plane following the rational parametrization (1) is a quadruple*

$$(A(s, t), B(s, t), C(s, t), D(s, t)) \in R^4$$

such that

$$A(s, t)a_1(s, t) + B(s, t)a_2(s, t) + C(s, t)a_3(s, t) + D(s, t)a_4(s, t) = 0. \tag{2}$$

Definition 3. *Let $\mathbf{p} = (p_1, p_2, p_3, p_4), \mathbf{q} = (q_1, q_2, q_3, q_4), \mathbf{r} = (r_1, r_2, r_3, r_4)$ be three moving planes such that*

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \alpha P(s, t) \tag{3}$$

*for some nonzero constant $\alpha \in \mathbb{F}$. Then $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are said to be a μ − **basis** of the rational surface parametrization (1). Here $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ is defined as the outer product*

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \left(\begin{vmatrix} p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \\ r_2 & r_3 & r_4 \end{vmatrix}, - \begin{vmatrix} p_1 & p_3 & p_4 \\ q_1 & q_3 & q_4 \\ r_1 & r_3 & r_4 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 & p_4 \\ q_1 & q_2 & q_4 \\ r_1 & r_2 & r_4 \end{vmatrix}, - \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \right).$$

As noted before, there is a $1 − 1$ relation between a μ-basis and a basis for the syzygy module $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ given in the following form:

- (Chen et al., 2005, Corollary 3.1) $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ form a μ-basis if and only if $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are a basis of $\mathrm{Syz}(a_1, a_2, a_3, a_4)$.

For any vector $\mathbf{v} \in R^m$ we denote its degree by $\deg(\mathbf{v}) = \max_j\{\deg(\mathbf{v}_j)\}$, where $\deg(\mathbf{v}_j)$ is equal to the total degree of the polynomial $\mathbf{v}_j$ in the variables s and t.

Definition 4. $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *are said to form a **minimal μ-basis** of the rational surface (1) if among all the triples satisfying (3), $\deg(\mathbf{p}) + \deg(\mathbf{q}) + \deg(\mathbf{r})$ is the smallest.*

In Chen et al. (2005) there are left several questions of interest for further research and better understanding. Here we try to address the question:

- **What can be said about the degrees of the polynomials in a minimal μ-basis?**

that was asked in (Chen et al., 2005, Section 5, second question).

Due to the equivalence between being a μ-basis and being a basis for the syzygy module, this question is the same as finding an upper bound for the latter one. We remark that the problem of studying the degrees of the syzygies of an ideal or a module has attracted the attention of several researchers (see e.g. Lazard (1977, 1992); Bayer and Stillman (1988); Yap (1991); Peeva and Sturmfels (1998); Avramov et al. (2015)).

Another interesting feature of the μ-bases is that they form the linear part of the moving curve/surface ideals in the case of curves/surfaces. In Cox (2008), it was noticed that computing the moving curve/surface ideal is the same as determining the defining equations of the Rees algebra of the ideal generated by the parametrization of the curve/surface. The problem of finding

the presentation of the Rees algebra is a long standing problem in commutative algebra and algebraic geometry that is a very active research topic (see e.g. Vasconcelos (1991); Cox et al. (2008); Hong et al. (2008); Busé (2009); Cortadellas Benítez and D'Andrea (2010, 2014, 2015); Kustin et al. (2011, 2017); Cid-Ruiz (2017); Busé et al. (2018)).

In the following construction we homogenize the ideal $I = (a_1, a_2, a_3, a_4) \subset R$ defined by the parametrization (1).

Construction 5. *Given the data* $\{a_1, a_2, a_3, a_4\}$ *that determines* (1)*, then we define the homogeneous ideal* $\hat{I} = (b_1, b_2, b_3, b_4)$ *with generators*

$$b_i(s, t, u) = u^d a_i(\frac{s}{u}, \frac{t}{u}) \in \mathbb{F}[s, t, u], \tag{4}$$

where $d = \max(\deg(a_1), \deg(a_2), \deg(a_3), \deg(a_4))$.

The main result of this paper is the following theorem where we find upper bounds for the degrees of the elements in a minimal μ-basis.

Theorem A (Theorem 26). *Let* $P(s, t)$ *be the parametrization in* (1) *and* d *be the number*

$$d = \max\{\deg(a_1), \deg(a_2), \deg(a_3), \deg(a_4)\}.$$

Then, the following statements hold:

(i) *There exists a* μ-*basis with polynomials bounded in the order of* $O(d^{33})$.

(ii) *If the homogenized ideal* $\hat{I}$ *(obtained in* (4)*) has height* $\mathrm{ht}(\hat{I}) = 3$*, then there exists a* μ-*basis with degree bounded by* $O(d^{22})$.

(iii) *If the homogenized ideal* $\hat{I}$ *is a "general" Artinian almost complete intersection (i.e, like in Remark 20), then there exists a* μ-*basis with degree bounded by* $O(d^{12})$.

(iv) *If the homogenized ideal* $\hat{I}$ *has projective dimension* $\mathrm{pd}(\hat{I}) = 1$*, then there exists a* μ-*basis with degree bounded by* d.

The proof of Theorem A is based on two fundamental ingredients. By using techniques coming from homological and commutative algebra we bound numerical invariants of the minimal free resolution (e.g. regularity and Betti numbers) of the ideal $\hat{I}$ obtained by homogenizing the ideal $I = (a_1, a_2, a_3, a_4)$, and then a process of dehomogenization gives us a presentation of $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ where everything can be bounded in terms of d. Under the assumptions of working over an infinite field $\mathbb{F}$ and having a presentation of $\mathrm{Syz}(a_1, a_2, a_3, a_4)$, then we apply the remarkable results of Caniglia et al. (1993) where an effective version of the Quillen-Suslin Theorem is given.

In the part (*iii*) we use an explicit description of the minimal free resolution of a general Artinian almost complete intersection, that was obtained in Migliore and Miró-Roig (2003). The part (*iv*) follows from Cox (2001) where the case $\mathrm{pd}(\hat{I}) = 1$ was studied and called *strong μ-basis*.

In contrast to our results, the elements of a μ-basis of a parametric rational curve of degree d are bounded in degree by exactly d. This big difference between the case of curves and surfaces comes from the fact that the syzygy module of the homogenized ideal may not be free in the case of surfaces but in the case of curves is always free. Actually, the condition of (*iv*) accounts to say

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that the syzygy module of $\hat{I}$ is free, and the case of a parametric rational surface having a strong μ-basis is treated similarly to the case of rational curves. In the general case where the syzygy module of $\hat{I}$ is not free, then the dehomogenization process that we use does not give us a basis of $\mathrm{Syz}(a_1, a_2, a_3, a_4)$. To overcome this difficulty, we use the effective version of Quillen-Suslin Theorem in Caniglia et al. (1993), and it is in this last step where the complexity of our upper bounds becomes large. As a general opinion, we think that our upper bounds are not sharp.

In our proof of Theorem A we needed to find some upper bounds for the regularity and Betti numbers of the homogeneous ideal $\hat{I}$. Since we think that these auxiliary upper bounds may be of interest on their own, we worked with more general ideals and obtained the following results:

(i) Let $\mathbb{K}$ be an arbitrary field. For a homogeneous ideal $J = (f_1, f_2, \ldots, f_m) \subset \mathbb{K}[s, t, u]$ generated by $m \geq 2$ relatively prime polynomials in $\mathbb{K}[s, t, u]$, in Theorem 11 we give upper bounds for the regularity and the Betti numbers of J.

(ii) For a homogeneous ideal $J = (g_1, g_2, g_3, g_4) \subset \mathbb{F}[s, t, u]$ with $\deg(g_1) = \cdots = \deg(g_4)$ and $\mathrm{ht}(I) = 3$, in Theorem 17 we improve the upper bounds for the Betti numbers of J.

The basic outline of this paper is as follows. In Section 2, we study the syzygies of ideals in a polynomial ring, and in particular we show that $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ is a free module of rank 3. In Section 3, we compute upper bounds for the regularity and the Betti numbers of ideals generated by relatively prime polynomials in three variables. In Section 4, by applying the effective version of Quillen-Suslin Theorem in Caniglia et al. (1993), we prove Theorem A. In Section 5, we briefly discuss the sharpness of our upper bounds. In Section 6, we give a simple example to show the process of computing μ-bases with our method.

Finally, for the sake of completeness we recall some basic definitions that will be used. For notational purposes, let M be an R-module and $J \subset R$ be an ideal. The *projective dimension* of M, denoted by $\mathrm{pd}(M)$, is the smallest possible length of a projective resolution of M (see (Rotman, 1979, page 233)). The *height* of J, denoted by $\mathrm{ht}(J)$, is equal to $\mathrm{ht}(J) = \inf\{\mathrm{ht}(\mathfrak{p}) \mid I \subset \mathfrak{p} \in \mathrm{Spec}(R)\}$, where the height of a prime ideal $\mathfrak{p}$ is the maximum of the lengths of increasing chains of prime ideals contained in $\mathfrak{p}$ (see (Matsumura, 1989, Section 5)). The *grade* of J, denoted by $\mathrm{grade}(J)$, is the maximum of the lengths of the regular sequences contained in J (see (Bruns and Herzog, 1993, Definition 1.2.6)).

Assume in addition that M is a finitely generated graded R-module. The k-th graded component of M is denoted by M_k. The *Hilbert function* of M, denoted by $H_M(k)$, is equal to $\dim_{\mathbb{F}}(M_k)$. The *minimal free resolution* of M is unique up to isomorphism (see (Peeva, 2011, Theorem 7.5)), then as a consequence, we can define the *Betti numbers* of M (see (Peeva, 2011, Section 11)) and the *regularity* of M (see (Peeva, 2011, Section 18)).

2. Dealing with syzygies

The fact that $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ is a free module of rank 3 is an important step in Chen et al. (2005) to show the existence of a μ-basis. In this section we give a different proof for that statement, which we also generalize because we will need the case of three variables after homogenizing the ideal $I = (a_1, a_2, a_3, a_4)$.

In this section we use the following notation.

Notation. *Let $\mathbb{K}$ be an arbitrary field and R be the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ where $n \geq 2$.*

Theorem 6. *Let I be an ideal in R. Then, for any projective resolution*

$$\cdots \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \xrightarrow{d_{n-2}} P_{n-3} \xrightarrow{d_{n-3}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} I \to 0$$

where the P_i's are finitely generated, the corresponding (n-2)-th syzygy $K_{n-2} = \mathrm{Ker}(d_{n-2})$ is free.

Proof. By the Hilbert Syzygy Theorem (Rotman, 1979, Corollary 9.36), there exists a finite free resolution of length at most n for the quotient ring R/I. We assume that it has length n and we denote it by

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0,$$

because if it has length smaller than n then we can simply fill it with zero modules.

With the given projective resolution of I we get the exact sequence

$$0 \to K_{n-2} \to P_{n-2} \xrightarrow{d_{n-2}} P_{n-3} \xrightarrow{d_{n-3}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} R \to R/I \to 0,$$

where $K_{n-2} = \mathrm{Ker}(d_{n-2})$ is the *(n-2)-th* syzygy. Then, from the generalized Schanuel Lemma (see (Kaplansky, 1974, Theorem 189)) we have the isomorphism

$$K_{n-2} \bigoplus \left(\bigoplus_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{n-1-2j} \right) \bigoplus \left(\bigoplus_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} P_{n-3-2j} \right) \cong \left(\bigoplus_{j=0}^{\lfloor \frac{n}{2} \rfloor} F_{n-2j} \right) \bigoplus \left(\bigoplus_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} P_{n-2-2j} \right),$$

which implies that K_{n-2} is a projective module. Since R is Noetherian and P_{n-2} is finitely generated, then P_{n-2} is Noetherian and $K_{n-2} \subset P_{n-2}$ is finitely generated. Finally, the Quillen-Suslin Theorem (Rotman, 1979, Theorem 4.59) implies that the module K_{n-2} is free. $\qquad\square$

Corollary 7. *For any ideal $I \subset R$ we have $\mathrm{pd}(I) \leq n - 1$.*

Proof. Using that R is Noetherian, for the ideal I we can always find a free resolution composed of finitely generated modules. So the corollary follows from Theorem 6. $\qquad\square$

We finish this section by proving that the free module $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ has rank 3.

Lemma 8. *Let A be a Noetherian ring, and $I \subset A$ be a nonzero ideal with a finite free resolution. Then $\mathrm{rank}(I) = 1$.*

Proof. From $0 \to I \to A \to A/I \to 0$ and the additivity of the rank function (Bruns and Herzog, 1993, Proposition 1.4.5), we obtain $\mathrm{rank}(A) = \mathrm{rank}(I) + \mathrm{rank}(A/I)$. We always have $\mathrm{rank}(A) = 1$ and from (Kaplansky, 1974, Theorem 195) we get $\mathrm{rank}(A/I) = 0$. Therefore $\mathrm{rank}(I) = 1$. $\qquad\square$

From the Hilbert Syzygy Theorem we know that any finitely generated module over R has a finite free resolution, so we are free to apply the previous Lemma 8 in this case.

Corollary 9. *Let $I = (f_1, f_2, \ldots, f_m)$ be an ideal in R with $\mathrm{pd}(I) = 1$. Then $\mathrm{Syz}(f_1, f_2, \ldots, f_m)$ is a free module of rank $m - 1$.*

Proof. The hypothesis $\text{pd}(I) = 1$ implies that the ideal I has a projective resolution of the form

$$0 \to P_1 \to P_0 \to I \to 0.$$

To study the module $\text{Syz}(f_1, f_2, \ldots, f_m)$, we consider the short exact sequence

$$0 \to \text{Syz}(f_1, f_2, \ldots, f_m) \to R^m \to I \to 0. \tag{5}$$

Then, the Schanuel lemma (see e.g. (Lang, 2002, page 841, Lemma 2.4)) yields the following isomorphism

$$\text{Syz}(f_1, f_2, \ldots, f_m) \oplus P_0 \;\cong\; P_1 \oplus R^m,$$

which implies that $\text{Syz}(f_1, f_2, \ldots, f_m)$ is a projective module. Since $\text{Syz}(f_1, f_2, \ldots, f_m)$ is also finitely generated, then the Quillen-Suslin Theorem (Rotman, 1979, Theorem 4.59) implies that $\text{Syz}(f_1, f_2, \ldots, f_m)$ is actually free. Finally, the short exact sequence (5), the additivity of the rank function and Lemma 8 give us

$$\text{rank}\left(\text{Syz}(f_1, f_2, \ldots, f_m)\right) = \text{rank}(R^m) - \text{rank}(I) = m - 1,$$

and so the claim of the corollary follows. $\qquad\square$

3. Upper bounds for the regularity and the Betti numbers

This section is devoted to finding an upper bound for the regularity and the Betti numbers of $\hat{I}$. Since the results of this section could be of general interest, we will deal with the case of homogeneous ideals generated by relatively prime polynomials in three variables. Our main reference will be the chapter *"Graded Free Resolutions"* of Peeva (2011).

First we note that the homogenized ideal $\hat{I}$ satisfies the condition of being generated by relatively prime polynomials.

Lemma 10. *For the ideal $\hat{I} = (b_1, b_2, b_3, b_4)$ constructed in (4) we have that b_1, b_2, b_3, b_4 are relatively prime (i.e. $\gcd(b_1, b_2, b_3, b_4) = 1$).*

Proof. Suppose that $g = \gcd(b_1, b_2, b_3, b_4) \neq 1 \in \mathbb{F}[s, t, u]$. Since $g(s, t, 1) \mid a_i(s, t)$ and $\gcd(a_1, a_2, a_3, a_4) = 1$, then we necessarily have that $g \in \mathbb{K}[u]$. By construction one of the b_i's has a term that is free of u, without loss of generality we assume that $b_1(s, t, u) = \lambda s^\alpha t^{d-\alpha} + u p(s, t, u)$ with $p \in \mathbb{F}[s, t, u]$ and $\lambda \neq 0$. So, since b_1 is homogeneous of degree d, we have that $g \mid b_1$ is a contradiction. $\qquad\square$

During the present section we use the following notation.

Notation. *Let $\mathbb{K}$ be an arbitrary field and $\mathbb{F}$ be an infinite field. Let T and S be the polynomial rings $T = \mathbb{K}[s, t, u]$ and $S = \mathbb{F}[s, t, u]$.*

We divide the section into two different parts. In the first part, we consider a homogeneous ideal $J = (f_1, f_2, \ldots, f_m) \subset T$ generated by $m \geq 2$ relatively prime polynomials. In the second part, we deal with the special case of an ideal $J = (g_1, g_2, g_3, g_4) \in S$ with $\deg(g_1) = \cdots = \deg(g_4)$ and $\text{ht}(J) = 3$.

Theorem 11. *Let $m \geq 2$, $J = (f_1, f_2, \ldots, f_m) \subset T$ be a homogeneous ideal, $\gcd(f_1, \ldots, f_m) = 1$ and $\deg(f_1), \ldots, \deg(f_m) \leq d$. Then, the following statements hold:*

(i) $\operatorname{reg}(J) \leq 3d - 2$.

(ii) $\beta_1(J) \leq \beta_2(J) + m - 1$.

(iii) $\beta_2(J) \leq H_J(\operatorname{reg}(J)) \leq H_J(3d - 2) \leq \binom{3d}{2}$.

In addition if $\deg(f_1) = \deg(f_2) = \ldots = \deg(f_m) = d$ then $\beta_2(J) \leq m\binom{2d}{2}$.

We break the proof of Theorem 11 in some steps that now follow. First, we prove that any ideal as J above has two relatively prime elements, but in order to prove it we have to make a more complicated reformulation.

Lemma 12. *Let $m \geq 2$ and $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ be relatively prime polynomials (i.e. $\gcd(f_1, \ldots, f_m) = 1$). Then there exists an infinite sequence of polynomials $\{h_i\}_{i=1}^{\infty} \subset (f_1, \ldots, f_m)$, with $\gcd(h_i, h_j) = 1$ for $i \neq j$.*

Proof. We proceed by an induction argument on m. Fix $m \geq 2$. We compute $g = \gcd(f_1, \ldots, f_{m-1})$ and the new polynomials $f_1' = f_1/g, \ldots, f_{m-1}' = f_{m-1}/g$. In the case $m = 2$ we have $f_1' = 1$, and when $m > 2$ we get $\gcd(f_1', \ldots, f_{m-1}') = 1$. Hence, in both cases, we can obtain an infinite sequence $\{h_i'\}_{i=1}^{\infty} \subset (f_1', \ldots, f_{m-1}')$ with $\gcd(h_i', h_j') = 1$ for $i \neq j$, because when $m = 2$ we have $(f_1') = \mathbb{K}[x_1, \ldots, x_n]$ and when $m > 2$ we can use the induction hypothesis.

For each h_i' we have that $\gcd(f_m, f_m + gh_i') = \gcd(f_m, gh_i')$. From $\gcd(f_1, \ldots, f_m) = 1$ we conclude that $\gcd(f_m, g) = 1$, and for some $j \in \mathbb{N}$ we should have $\gcd(f_m, gh_j') = 1$, because all the h_i''s have different prime factors but f_m can have only a finite amount of prime factors.

Suppose we have computed a sequence of polynomials $h_1, \ldots, h_k$ and a polynomial g_k, with the properties $\gcd(h_i, h_j) = 1$ for $1 \leq i < j \leq k$ and $\gcd(h_i, g_k) = 1$ for $1 \leq i \leq k$. Again, for each h_i' we have

$$\gcd(h_1 \cdots h_k, h_1 \cdots h_k + g_k h_i') = \gcd(h_1 \cdots h_k, g_k h_i') = \gcd(h_1 \cdots h_k + g_k h_i', g_k h_i'), \quad (6)$$

and there must exist some $j \in \mathbb{N}$ with $\gcd(h_1 h_2 \cdots h_k, g_k h_j') = 1$. Thus we define the next elements in the inductive step as $h_{k+1} = h_1 h_2 \cdots h_k + g_k h_j'$ and $g_{k+1} = g_k h_j'$.

From (6) we have that $\gcd(h_1 h_2 \cdots h_k, h_{k+1}) = \gcd(h_1 h_2 \cdots h_k, g_{k+1}) = \gcd(h_{k+1}, g_{k+1}) = 1$, which implies $\gcd(h_i, h_j) = 1$ for $1 \leq i < j \leq k+1$ and $\gcd(h_i, g_{k+1}) = 1$ for $1 \leq i \leq k+1$. Starting with $h_1 = f_m$, $g_1 = g$ and following this iterative process we can construct the required sequence $\{h_i\}_{i=1}^{\infty} \subset (f_1, \ldots, f_m)$ with $\gcd(h_i, h_j) = 1$ for $i \neq j$. $\square$

Corollary 13. *Let $m \geq 2$ and $J = (f_1, f_2, \ldots, f_m) \subset T$ be a homogeneous ideal, where $\gcd(f_1, \ldots, f_m) = 1$. Then $\operatorname{grade}(J) \geq 2$.*

Proof. We choose two relatively prime elements p and q from the previous Lemma 12. Then p is regular on T, and q is regular on T/p because $\gcd(p, q) = 1$. Therefore $\{p, q\}$ is a regular sequence and $\operatorname{grade}(J) \geq 2$. $\square$

As consequence of our translation of the condition $\gcd(f_1, \ldots, f_m) = 1$ in terms of $\operatorname{grade}(J) \geq 2$, we obtain the following upper bound for the regularity of J.

Proposition 14. *Let $m \geq 2$ and $J = (f_1, f_2, \ldots, f_m) \subset T$ be a homogeneous ideal, where $\gcd(f_1, \ldots, f_m) = 1$ and $\deg(f_1), \ldots, \deg(f_m) \leq d$. Then the regularity is bounded by $\operatorname{reg}(J) \leq 3d - 2$.*

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Proof. If we prove $\dim(T/J) \leq 1$, then from (Peeva, 2007, Theorem 1.9.4) we get $\mathrm{reg}(J) = \mathrm{reg}(T/J) + 1 \leq 3d - 2$. Since T is a Cohen-Macaulay ring we obtain $\mathrm{ht}(J) = \mathrm{grade}(J) \geq 2$. Finally, $\mathrm{ht}(J) + \dim(T/J) = \dim(T) = 3$ implies that $\dim(T/J) \leq 1$. $\quad\square$

The following proposition uses the Koszul complex in order to relate the Betti numbers of J with the Hilbert function of J.

Proposition 15. *Let* $J = (f_1, f_2, \ldots, f_m) \subset T$ *be a homogeneous ideal. Then*

$$\beta_{2,p}(J) = \dim_{\mathbb{K}}(\mathrm{Tor}_2^T(J, \mathbb{K})_p) \leq H_J(p-2) - H_J(p-3).$$

Proof. Let $\mathbf{x} = \{s, t, u\}$, we consider the Koszul complex $K(\mathbf{x}; J) = K(\mathbf{x}) \otimes_T J$:

$$0 \to J \otimes_T \bigwedge^3 T(-3)^3 \xrightarrow{id \otimes_T d_3} J \otimes_T \bigwedge^2 T(-2)^3 \xrightarrow{id \otimes_T d_2} J \otimes_T \bigwedge^1 T(-1)^3 \xrightarrow{id \otimes_T d_1} J \otimes_T \bigwedge^0 T^3 \to 0.$$

We need to compute in the graded part $(J \otimes_T \bigwedge^2 T(-2)^3)_p = J_{p-2} \otimes_{\mathbb{K}} \bigwedge^3 \mathbb{K}^3$, so we only take the complex

$$0 \to J_{p-3} \otimes_{\mathbb{K}} \bigwedge^3 \mathbb{K}^3 \xrightarrow{(id \otimes_T d_3)_{p-3}} J_{p-2} \otimes_{\mathbb{K}} \bigwedge^2 \mathbb{K}^3 \xrightarrow{(id \otimes_T d_2)_{p-2}}$$
$$J_{p-1} \otimes_{\mathbb{K}} \bigwedge^1 \mathbb{K}^3 \xrightarrow{(id \otimes_T d_1)_{p-1}} J_p \otimes_{\mathbb{K}} \bigwedge^0 \mathbb{K}^3 \to 0,$$

and we get the formula

$$\mathrm{Tor}_2^T(J, \mathbb{K})_p \cong H_2 K(\mathbf{x}; J)_p = \frac{\mathrm{Ker}(id \otimes_T d_2)_p}{\mathrm{Im}(id \otimes_T d_3)_p} = \frac{\mathrm{Ker}((id \otimes_T d_2)_{p-2})}{\mathrm{Im}((id \otimes_T d_3)_{p-3})}.$$

Then using the fact that $\mathrm{Ker}((id \otimes_T d_2)_{p-2})$ and $\mathrm{Im}((id \otimes_T d_3)_{p-3})$ are $\mathbb{K}$-vector spaces, we can compute $\beta_{2,p}(J) = \dim_{\mathbb{K}}(\mathrm{Ker}((id \otimes_T d_2)_{p-2})) - \dim_{\mathbb{K}}(\mathrm{Im}((id \otimes_T d_3)_{p-3}))$.

From Corollary 7 we know that $\mathrm{pd}_T(J) \leq 2$, then we have that $H_3 K(\mathbf{x}; J) \cong \mathrm{Tor}_3^T(J, \mathbb{K}) = 0$ and so $\mathrm{Ker}(id \otimes_T d_3) = 0$. From this we conclude that $(id \otimes_T d_3)_{p-3}$ is an injective map and $\dim_{\mathbb{K}}(\mathrm{Im}((id \otimes_T d_3)_{p-3})) = \dim_{\mathbb{K}}(J_{p-3} \otimes_{\mathbb{K}} \bigwedge^3 \mathbb{K}^3) = H_J(p-3)$.

Let $h_{12} \otimes_{\mathbb{K}} e_1 \wedge e_2 + h_{13} \otimes_{\mathbb{K}} e_1 \wedge e_3 + h_{23} \otimes_{\mathbb{K}} e_2 \wedge e_3 \in \mathrm{Ker}((id \otimes_T d_2)_{p-2})$. By applying the differential map of the Koszul complex we have

$$sh_{12} \otimes_{\mathbb{K}} e_2 - th_{12} \otimes_{\mathbb{K}} e_1 + sh_{13} \otimes_{\mathbb{K}} e_3 - uh_{13} \otimes_{\mathbb{K}} e_1 + th_{23} \otimes_{\mathbb{K}} e_3 - uh_{23} \otimes_{\mathbb{K}} e_2 = 0.$$

From here we deduce the equations $th_{12} = -uh_{13}$, $sh_{12} = uh_{23}$, $sh_{13} = -th_{23}$. Therefore one of the terms can completely determine the other two. This simple fact implies the inequality $\dim_{\mathbb{K}}(\mathrm{Ker}((id \otimes_T d_2)_{p-2})) \leq H_J(p-2)$, and concludes the proof of the proposition. $\quad\square$

Corollary 16. *Let* $m \geq 2$ *and* $J = (f_1, f_2, \ldots, f_m) \subset T$ *be a homogeneous ideal, with* $\gcd(f_1, \ldots, f_m) = 1$ *and* $\deg(f_1), \ldots, \deg(f_m) \leq d$. *Then* $\beta_2(J) \leq H_J(\mathrm{reg}(J)) \leq H_J(3d - 2)$.

Proof. We have that $\beta_{2,p} = 0$ for $p > \mathrm{reg}(J) + 2$. Then we compute

$$\beta_2(J) = \sum_{p=1}^{\mathrm{reg}(J)+2} \beta_{2,p}(J) \leq \sum_{p=1}^{\mathrm{reg}(J)+2} (H_J(p-2) - H_J(p-3)) = H_J(\mathrm{reg}(J)) \leq H_J(3d-2).$$

The last inequality is obtained from Proposition 14. $\quad\square$

Proof of Theorem 11. *(i)* The upper bound for the regularity has already been proved in Proposition 14.

(ii) Follows from the additivity of the rank function.

(iii) We know that the number of monomials of degree d in $\mathbb{K}[s, t, u]$ is $\binom{d+2}{2}$, hence from Corollary 16 we get the upper bound $\beta_2(J) \leq H_J(\text{reg}(J)) \leq H_J(3d - 2) \leq \binom{3d}{2}$.

Now we add the extra condition that $J = (f_1, f_2, \ldots, f_m)$ is generated by m polynomials of the same degree d. Hence for any $p \geq d$ we have that the $\mathbb{K}$-vector space J_p is generated by elements of the form $g f_i$ $(1 \leq i \leq m)$ where g is a monomial of degree $p - d$. So we have that the Hilbert function of J is bounded by $H_J(p) \leq m\binom{p-d+2}{2}$. $\qquad\square$

Now for the second part of this section we work with an ideal $J = (g_1, g_2, g_3, g_4) \in S$, such that $d = \deg(g_1) = \cdots = \deg(g_4)$ and $\text{ht}(J) = 3$.

Theorem 17. *Let $J = (g_1, g_2, g_3, g_4) \subset S$ be a homogeneous ideal with $d = \deg(g_1) = \cdots = \deg(g_4)$ and $\text{ht}(J) = 3$. Then $\beta_1(J) \leq 2d + 2$ and $\beta_2(J) \leq 2d - 1$.*

The proof of Theorem 17 is divided in some steps that are given below.

Remark 18. *From the Unmixedness Theorem and the fact that $\mathbb{F}$ is an infinite field, we can find a complete intersection inside J (see (Szanto, 2008, Lemma A.10), (Kaplansky, 1974, Theorem 125)). Explicitly, there exist scalars $\alpha_{ij} \in \mathbb{F}$ that give us the following sort of triangular transformation*

$$
\begin{aligned}
h_1 &= g_1 + \alpha_{12}g_2 + \alpha_{13}g_3 + \alpha_{14}g_4, \\
h_2 &= g_2 + \alpha_{23}g_3 + \alpha_{24}g_4, \\
h_3 &= g_3 + \alpha_{34}g_4, \\
h_4 &= g_4,
\end{aligned}
$$

where $\{h_1, h_2, h_3\}$ is a complete intersection. Therefore, we can assume that $J = (h_1, h_2, h_3, h_4)$, where $\{h_1, h_2, h_3\}$ is a complete intersection and $d = \deg(h_1) = \cdots = \deg(h_4)$. Also, we can suppose that $h_4 \notin (h_1, h_2, h_3)$, because in case $J = (h_1, h_2, h_3)$ then the minimal free resolution of S/J can be obtained with the Koszul complex, that trivially satisfies the result of Theorem 17.

We shall take a similar approach to Migliore and Miró-Roig (2003) using a process of linkage or liaison. We make the observation that $J = (h_1, h_2, h_3, h_4)$ can be linked to a Gorenstein ideal G (see (Migliore and Nagel, 2002, Corollary 5.19), (Buchsbaum and Eisenbud, 1977, Proposition 5.2)) via the complete intersection $K = (h_1, h_2, h_3)$, i.e., $G = (K : J)$.

The minimal free resolution of S/K is given by the Koszul complex. Using Buchsbaum and Eisenbud's structure theorem for height 3 Gorenstein ideals (Buchsbaum and Eisenbud, 1977, Theorem 2.1), the minimal free resolution of S/G has the form

$$
0 \to S(-s-3) \xrightarrow{g^*} \bigoplus_{i=1}^{m} S(-p_i) \xrightarrow{f} \bigoplus_{i=1}^{m} S(-q_i) \xrightarrow{g} S \to S/G \to 0,
$$

where s is the socle degree of G (the largest k such that $(S/G)_k \neq 0$), m is odd, f is alternating, and $G = \text{Pf}_{m-1}(f)$ (the ideal generated by the $(m-1)$-th Pfaffians of f).

Lemma 19. *The socle degree of S/G is $s = 2d - 3$.*

Proof. Since K is a complete intersection we know that the socle degree of S/K is $3d-3$. We have that the Hilbert function of an Artinian Gorenstein algebra is symmetric, also we can relate the Hilbert functions of S/K, S/G and S/J (see (Elias et al., 2010, Theorem 2.10, page 308), Migliore and Miró-Roig (2003)) in the following way

$$H_{S/G}(t) = H_{S/K}(3d-3-t) - H_{S/J}(3d-3-t).$$

Then for any $t > 2d-3$ we have $3d-3-t < d$ and $H_{S/J}(3d-3-t) = H_{S/K}(3d-3-t) = \binom{3d-1-t}{2}$, also we can easily check that $H_{S/G}(2d-3) = 1$. $\qquad\square$

Proof of Theorem 17. By using Lemma 19 we substitute the socle degree of S/G in its minimal free resolution, and from the canonical map $S/K \to S/G$ we can lift a comparison map

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & S(-3d) & \xrightarrow{d_3} & S(-2d)^3 & \xrightarrow{d_2} & S(-d)^3 & \xrightarrow{d_1} & S & \longrightarrow & S/K \longrightarrow 0 \\
 & & \downarrow{\phi_3} & & \downarrow{\phi_2} & & \downarrow{\phi_1} & & \downarrow{\phi_0 = id} & & \downarrow{can} \\
0 & \longrightarrow & S(-2d) & \xrightarrow{g^*} & \bigoplus_{i=1}^{m} S(-p_i) & \xrightarrow{f} & \bigoplus_{i=1}^{m} S(-q_i) & \xrightarrow{g} & S & \longrightarrow & S/G \longrightarrow 0.
\end{array}
$$

With a dual mapping cone construction (Buchsbaum and Eisenbud (1977), Migliore and Miró-Roig (2003)) we can obtain the following free resolution for S/J (not necessarily minimal)

$$0 \to \bigoplus_{i=1}^{m} S(-3d+q_i) \to \begin{array}{c} S(-2d)^3 \\ \oplus \\ \bigoplus_{i=1}^{m} S(-3d+p_i) \end{array} \to S(-d)^4 \to S \to S/J \to 0.$$

Thus we have $\beta_2(J) \le m$ and $\beta_1(J) \le m+3$. In (Diesel, 1996, Theorem 3.3) it is proved that given the smallest degree k of the generators of G (i.e., k is the first position in which $H_{S/G}(k) < \binom{k+2}{2}$) then $m \le 2k+1$.

Since S/G has socle degree $2d-3$ and its Hilbert function is symmetric, we have $H_{S/G}(d-2) = H_{S/G}(d-1)$. Hence $k \le d-1$ because otherwise we get the contradiction $\binom{d}{2} = \binom{d+1}{2}$. Therefore, we have obtained $\beta_2(J) \le 2d-1$ and $\beta_1(J) \le 2d+2$. $\qquad\square$

Remark 20. *An interesting fact proved in (Migliore and Miró-Roig, 2003, Corollary 4.4), is that when the ideal $J = (g_1, g_2, g_3, g_4)$ is a general Artinian almost complete intersection of type (d,d,d,d), then the minimal free resolution can be given explicitly. This means that J is generated by "generically chosen" polynomials g_1, g_2, g_3, g_4, where (g_1, g_2, g_3) are a complete intersection, $g_4 \notin (g_1, g_2, g_3)$, and $d = \deg(g_1) = \cdots = \deg(g_4)$. The minimal free resolution of S/J in this case is*

$$0 \to S(-2d-1)^d \to \begin{array}{c} S(-2d)^3 \\ \oplus \\ S(-2d+1)^d \end{array} \to S(-d)^4 \to S \to S/J \to 0.$$

The term "generically chosen" means that g_1, g_2, g_3, g_4 belong to a suitable dense open subset of $S_d \times S_d \times S_d \times S_d$ in the Zariski topology.

4. Projective dimension two

From Corollary 7 we know that $\mathrm{pd}(\hat{I}) \leq 2$. Here we deal with the remaining case $\mathrm{pd}(\hat{I}) = 2$, because $\mathrm{pd}(\hat{I}) = 1$ was studied in Cox (2001). In the rest of this paper we use the following notation.

Notation. *Let $\mathbb{F}$ be an infinite field, R be the polynomial ring $R = \mathbb{F}[s, t]$ and S be the polynomial ring $S = \mathbb{F}[s, t, u]$.*

From Lemma 8 we get a free resolution

$$0 \to S^a \xrightarrow{\hat{d}_2} S^{a+3} \xrightarrow{\hat{d}_1} S^4 \xrightarrow{[b_1, b_2, b_3, b_4]} \hat{I} \to 0, \tag{7}$$

and now we want to compute the value of a. Here we are using an abuse of notation, because we should write

$$0 \to \bigoplus_{i=1}^{a} S(-p_i) \xrightarrow{\hat{d}_2} \bigoplus_{i=1}^{a+3} S(-q_i) \xrightarrow{\hat{d}_1} S(-d)^4 \xrightarrow{[b_1, b_2, b_3, b_4]} \hat{I} \to 0, \tag{8}$$

if we want to take care of the grading.

In (7) we do not know if b_1, b_2, b_3, b_4 is a minimal system of generators. But in the next step of finding the resolution (7) of $\hat{I}$, we can choose a minimal system of generators for $\mathrm{Syz}(b_1, b_2, b_3, b_4)$ because it is a graded module. Therefore, from (Peeva, 2011, Theorem 7.3) we can assure that $\mathrm{Im}(\hat{d}_2) \subseteq \mathfrak{m}S^{a+3}$, where $\mathfrak{m} = (s, t, u)$ is the irrelevant ideal. By exploiting the condition $\mathrm{Im}(\hat{d}_2) \subset \mathfrak{m}S^{a+3}$ we will "*adapt*" the upper bounds obtained in the previous section to (7).

Lemma 21. *For the resolution (7) (more specifically (8)) we have that*

(i) $\max_{1 \leq i \leq a+3}(q_i) \leq 3d - 1$,

(ii) $\max_{1 \leq i \leq a}(p_i) \leq 3d$,

(iii) $a = \beta_2(\hat{I})$.

Proof. Here we use the key fact that for a graded free S-module $F = \bigoplus_{i=1}^{r} S(-\alpha_i)$ we have $(F \otimes_S \mathbb{F})_p = 0$ if and only if $p \neq \alpha_i$ for all $1 \leq i \leq r$.

(i) Let $p > 3d - 2 + 1 = 3d - 1$. The upper bound $\mathrm{reg}(\hat{I}) \leq 3d - 2$ (Theorem 11(*i*)) yields that $\beta_{1,p} = 0$, and this implies that $\mathrm{Tor}_1^S(\hat{I}, \mathbb{F})_p = 0$. The condition $\mathrm{Im}(\hat{d}_2) \subseteq \mathfrak{m}S^{a+3}$ gives us that $\mathrm{Im}(\hat{d}_2 \otimes_S \mathbb{F}) = 0$ and so we get $\mathrm{Ker}((\hat{d}_1 \otimes_S \mathbb{F})_p) = \mathrm{Tor}_1^S(\hat{I}, \mathbb{F})_p = 0$. Since $(\hat{d}_1 \otimes_S \mathbb{F})_p$ is an injective map and $(S(-d)^4 \otimes_S \mathbb{F})_q = 0$ for $q > d$, then we conclude

$$\left(\left(\bigoplus_{i=1}^{a+3} S(-q_i) \right) \bigotimes_S \mathbb{F} \right)_p = 0.$$

Therefore we have the inequality $\max_{1 \leq i \leq a+3}(q_i) \leq 3d - 1$.

(ii), (iii) Deleting $\hat{I}$ from (7) and applying the tensor product $\otimes_S \mathbb{F}$ we get the complex

$$0 \to \mathbb{F}^a \xrightarrow{0} \mathbb{F}^{a+3} \xrightarrow{\hat{d}_1 \otimes_S F} \mathbb{F}^4 \to 0.$$

So $a = \dim_{\mathbb{F}}(\mathrm{Tor}_2^S(\hat{I}, \mathbb{F})) = \beta_2(\hat{I})$, and the grading of the module S^a is just like the one for a minimal free resolution, i.e. $\leq (3d - 2) + 2 = 3d$. $\square$

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So in this case the resolution of $\hat{I}$ is of the form

$$0 \to S^{\beta_2(\hat{I})} \xrightarrow{\hat{d}_2} S^{\beta_2(\hat{I})+3} \xrightarrow{\hat{d}_1} S^4 \xrightarrow{[b_1,b_2,b_3,b_4]} \hat{I} \to 0,$$

where the polynomials in the entries of the matrices $\hat{d}_1$ and $\hat{d}_2$ have degree bounded by $3d-1-d = 2d - 1$, and $3d - d = 2d$ respectively.

Remark 22. *We apply the tensor product with $\otimes_S S/(u - 1)$ to obtain the exact sequence (see (Eisenbud, 1995, Corollary 19.8) or (Bruns and Herzog, 1993, Proposition 1.1.5))*

$$0 \to R^{\beta_2(\hat{I})} \xrightarrow{d_2} R^{\beta_2(\hat{I})+3} \xrightarrow{d_1} R^4 \xrightarrow{[a_1,a_2,a_3,a_4]} I \to 0,$$

where $d_1 = \hat{d}_1 \otimes_S S/(u - 1)$ and $d_2 = \hat{d}_2 \otimes_S S/(u - 1)$ are matrices with entries in R bounded in degree by $2d - 1$ and $2d$ respectively.

For the rest of this section we shall work with the exact sequence

$$0 \to R^{\beta_2(\hat{I})} \xrightarrow{d_2} R^{\beta_2(\hat{I})+3} \xrightarrow{d_1} \mathrm{Syz}(a_1, a_2, a_3, a_4) \, (\subset R^4) \to 0, \tag{9}$$

which is a split exact sequence because we know that $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ is a free module.

An $m \times n$ ($m > n$) polynomial matrix $A \in R^{m \times n}$ is said to be *unimodular* if it satisfies one of the following equivalent conditions (see Lam (2006))

 (i) *A* can be completed into an invertible $m \times m$ square matrix.

 (ii) there exists an $n \times m$ polynomial matrix $B \in R^{n \times m}$ such that $AB = I_m$.

 (iii) there exists an $n \times m$ polynomial matrix $B \in R^{n \times m}$ such that $BA = I_n$.

 (iv) the ideal generated by the $n \times n$ minors of A is equal to R.

We define the degree of a matrix $M = (a_{ij}) \in R^{m \times n}$ as the maximum degree of the polynomial entries of M, i.e., $\deg(M) = \max(\deg(a_{ij}))$. For an *"effective"* solution of completing a unimodular matrix we are going to use the following result from Caniglia et al. (1993).

Theorem 23. *Let $F \in R^{m \times n}$ ($m < n$) be a unimodular matrix. Then there exists a square matrix $M \in R^{n \times n}$ such that*

 (i) *M is unimodular,*

 (ii) *$FM = [I_m, 0] \in R^{m \times n}$,*

 (iii) *$\deg(M) \le 2D(1 + 2D)(1 + D^4)(1 + D)^4$, where $D = m(1 + \deg(F))$.*

Proof. See the Appendix for a discussion. □

This previous result is given for completing rows (i.e., $m < n$), but we want to complete columns (i.e., $m > n$). By simply taking transpose in (*ii*) of the previous theorem we get the following corollary.

Corollary 24. *Let $F \in R^{m \times n}$ ($m > n$) be a unimodular matrix. Then there exists a square matrix $M \in R^{m \times m}$ such that*

(i) M *is unimodular,*

(ii) $MF = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in R^{m \times n}$,

(iii) $\deg(M) \le 2D(1 + 2D)(1 + D^4)(1 + D)^4$, *where* $D = n(1 + \deg(F))$.

For notational purposes we make the following conventions
- we use just β_2 instead of $\beta_2(\hat{I})$,
- $m = \beta_2 + 3$ and $n = \beta_2$,
- $F \in R^{m \times n}$ denotes the $m \times n$ matrix corresponding with the map d_2,
- $\gamma_2 = \deg(F)$,
- $G \in R^{4 \times m}$ denotes the $4 \times m$ matrix corresponding with the map d_1,
- $\gamma_1 = \deg(G)$,
- $D = n(1 + \deg(F)) = \beta_2(1 + \gamma_2)$,

thus we end up with the following short exact sequence

$$0 \to R^n \xrightarrow{F} R^m \xrightarrow{G} \text{Syz}(a_1, a_2, a_3, a_4) \, (\subset R^4) \to 0. \tag{10}$$

Since this sequence splits, there exists a matrix $H \in R^{n \times m}$ with $HF = I_n$ and so the matrix F is unimodular.

Proposition 25. *From the exact sequence (10) we can get a basis for* $\text{Syz}(a_1, a_2, a_3, a_4)$ *made of three vectors* $\mathbf{p}, \mathbf{q}, \mathbf{r} \in R^4$, *with*

$$\max(\deg(\mathbf{p}), \deg(\mathbf{q}), \deg(\mathbf{r})) \le \gamma_1(\beta_2 + 2)2D(1 + 2D)(1 + D^4)(1 + D)^4$$

$$\le 2\gamma_1(\beta_2 + 2)\beta_2(1 + \gamma_2)(1 + 2\beta_2(1 + \gamma_2))(1 + (\beta_2(1 + \gamma_2))^4)(1 + \beta_2(1 + \gamma_2))^4.$$

Proof. We can get a matrix $M \in R^{m \times m}$ that satisfies *(i), (ii), (iii)* from Corollary 24. Let $N \in R^{m \times m}$ be the inverse matrix of M, then we have that $\deg(N) \le (m - 1)\deg(M)$, because the determinant of every $(m - 1) \times (m - 1)$-minor is a polynomial of degree at most $m - 1$ in terms of the entries of M. Also, from the item *(ii)* of Corollary 24 we have that

$$F = N \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

We know that N is an automorphism for $R^m = N(\text{span}(e_1, \ldots, e_n, e_{n+1}, e_{n+2}, e_{n+3}))$ and that $\text{Ker}(G) = \text{Im}(F) = N(\text{span}(e_1, \ldots, e_n))$, where e_i is the i-th column vector of R^m. Hence we have

$$\text{Syz}(a_1, a_2, a_3, a_4) = \text{Im}(G) = G(R^m) = GN(\text{span}(e_1, \ldots, e_m)) = GN(\text{span}(e_{n+1}, e_{n+2}, e_{n+3})),$$

and we define the basis for $\text{Syz}(a_1, a_2, a_3, a_4)$ as

$$\mathbf{p} = GNe_{n+1},$$
$$\mathbf{q} = GNe_{n+2},$$
$$\mathbf{r} = GNe_{n+3}.$$

Finally, we obtain the result

$$\max(\deg(\mathbf{p}), \deg(\mathbf{q}), \deg(\mathbf{r})) \le \deg(G)\deg(N) \le \gamma_1(\beta_2 + 2)2D(1 + 2D)(1 + D^4)(1 + D)^4$$

$$\le 2\gamma_1(\beta_2 + 2)\beta_2(1 + \gamma_2)(1 + 2\beta_2(1 + \gamma_2))(1 + (\beta_2(1 + \gamma_2))^4)(1 + \beta_2(1 + \gamma_2))^4. \quad \square$$

The following theorem contains the main result of this paper, and gives different degree bounds for the generators of the basis depending on the type of exact sequence (presentation) (10) that we can obtain.

Theorem 26. *Given the data* $\{a_1, a_2, a_3, a_4\}$ *defining (1) where*

$$d = \max_{1 \leq i \leq 4}(\deg(a_i)) \quad and \quad \gcd(a_1, a_2, a_3, a_4) = 1.$$

Then, the following statements hold:

(i) *There exists a basis for* $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ *with polynomials bounded in the order of* $O(d^{33})$.

(ii) *If the homogenized ideal* $\hat{I}$ *(obtained in (4)) has height* $\mathrm{ht}(\hat{I}) = 3$, *then there exists a basis with degree bounded by* $O(d^{22})$.

(iii) *If the homogenized ideal* $\hat{I}$ *is a "general" almost complete intersection (i.e, like in Remark 20), then there exists a basis with degree bounded by* $O(d^{12})$.

(iv) *If the homogenized ideal* $\hat{I}$ *has projective dimension* $\mathrm{pd}(\hat{I}) = 1$, *then there exists a basis with degree bounded by* d.

Proof. (*i*) By Lemma 21 we know that $\gamma_1 = \deg(G) \leq 2d - 1$ and $\gamma_2 = \deg(F) \leq 2d$, from Theorem 11 we have $\beta_2 \leq 4\binom{2d}{2} \in O(d^2)$. Therefore substituting in the formula obtained in Proposition 25 we get a basis bounded by $O(d^{33})$.

(*ii*) In this case from Theorem 17 we have $\beta_2 \leq 2d - 1$. So we can reduce the upper bound to $O(d^{22})$.

(*iii*) In Remark 20 we saw that when $\hat{I}$ is a general Artinian almost complete intersection then $S/\hat{I}$ has a very special minimal free resolution, from which we can obtain $\gamma_1 = \deg(G) = d$, $\gamma_2 = \deg(F) = 2$ and $\beta_2 = d$. Therefore we obtain an upper bound in the order of $O(d^{12})$.

(*iv*) From (Cox, 2001, §5.1), the resolution of $\hat{I}$ is given by

$$0 \to S(-d - \mu_1) \oplus S(-d - \mu_2) \oplus S(-d - \mu_3) \to S(-d)^4 \to \hat{I} \to 0,$$

where $\mu_1 + \mu_2 + \mu_3 = d$. Then the same dehomogenization of Remark 22 gives us the result. $\square$

5. Discussions of the upper bounds and a related problem

In this short section we discuss the sharpness of the upper bounds obtained in Theorem 26.

First, from Remark 22 we obtain the interesting fact that the syzygy module $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ can be generated by elements of degree bounded by $2d - 1$. In our particular case, this is almost identical to a remarkable result of Lazard (see Lazard (1992, 1977)). Let $S(n, d)$ be the least integer such that the module of syzygies $\mathrm{Syz}(h_1, h_2, \ldots, h_k)$ can be generated by elements of degree at most $S(n, d)$, where $h_1, h_2, \ldots, h_k$ are arbitrary polynomials in n variables and degree at most d (this definition is independent of the particular polynomials h_i's). In Lazard (1992) there is an important general upper bound for $S(n, d)$, and also the following

$$S(2, d) \leq 2d - \min(2, d)$$

sharp upper bound (see (Lazard, 1977, Proposition 5, Proposition 10)). So, the upper bound that we obtained for the degree of the generators of $\mathrm{Syz}(a_1, a_2, a_3, a_4)$ is "almost sharp".

The main obstacle is that in the general case of non-graded modules we do not have a well-defined concept of "minimal set of generators" (see e.g. (Cox et al., 2005, pages 236 and 237, Excercise 4)). Due to this fact, it is unclear to the author how to obtain a basis of $\text{Syz}(a_1, a_2, a_3, a_4)$ if we are given a set of generators. After obtaining the results of Lemma 21 and Remark 22, we "only needed" to solve a particular case of the following problem.

Problem 27. *Find a function $f : \mathbb{N}^2 \to \mathbb{N}$, such that for any free module $F \subset \mathbb{K}[x_1, \ldots, x_n]^m$ of rank $r < m$ and generated by a set of elements $\{v_1, v_2, \ldots, v_k\} \subset \mathbb{K}[x_1, \ldots, x_n]^m$ with $\deg(v_i) \leq d$, then there exists a basis $\{w_1, w_2, \ldots, w_r\} \subset \mathbb{K}[x_1, \ldots, x_n]^m$ of F satisfying the condition*

$$\deg(w_i) \leq f(d, k).$$

In this paper we use an effective version of the Quillen-Suslin Theorem to solve the problem above. We remark that the best known upper bounds for the effective Quillen-Suslin Theorem are given in Caniglia et al. (1993) (see (Lombardi and Yengui, 2005, page 715, Remark (1))).

In conclusions, the sharpness of the upper bounds in Theorem 26 depends mostly in our ability to solve Problem 27. As a general opinion, we think that they can be improved.

6. Example

The aim of this example is to show some computational aspects of the case studied in this paper (i.e., $\text{pd}(\hat{I}) = 2$). With a simple example, we make all the steps of our method for computing a μ-basis.

Example 28. *Find a μ-basis for the rational surface parametrization*

$$P(s, t) = (s^2, t^2, s^2 - 1, s^2 + 1).$$

Proof. Using a computer algebra system like *Singular* (Decker et al. (2018)), we get the following free resolution

$$0 \to S \xrightarrow{\begin{pmatrix} 0 \\ s^2 \\ -u^2 \\ -t^2 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} -2 & -t^2 & -t^2 & u^2 - s^2 \\ 0 & u^2 & s^2 & 0 \\ 1 & t^2 & 0 & s^2 \\ 1 & 0 & 0 & 0 \end{pmatrix}} S^4 \xrightarrow{\left(\begin{array}{cccc} s^2 & t^2 & s^2 - u^2 & s^2 + u^2 \end{array} \right)} \hat{I} \to 0.$$

Substituting $u = 1$ and cutting the resolution, we obtain

$$0 \to R \xrightarrow{\begin{pmatrix} 0 \\ s^2 \\ -1 \\ -t^2 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} -2 & -t^2 & -t^2 & 1 - s^2 \\ 0 & 1 & s^2 & 0 \\ 1 & t^2 & 0 & s^2 \\ 1 & 0 & 0 & 0 \end{pmatrix}} \text{Syz}(s^2, t^2, s^2 - 1, s^2 + 1) \ (\subset R^4) \to 0.$$

Proceeding as in Proposition 25, we have to complete the unimodular column $(0, s^2, -1, -t^2)^t$ into an invertible matrix $N \in R^{4 \times 4}$. For this we can check that the following matrix

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ s^2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ t^2 & 0 & 0 & 1 \end{pmatrix}$$

has determinant 1. Therefore, a μ-basis for $P(s, t) = (s^2, t^2, s^2 - 1, s^2 + 1)$ is given by the vectors

$$\mathbf{p} = \begin{pmatrix} -2 & -t^2 & -t^2 & 1 - s^2 \\ 0 & 1 & s^2 & 0 \\ 1 & t^2 & 0 & s^2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ s^2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ t^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -t^2 \\ 1 \\ t^2 \\ 0 \end{pmatrix},$$

$$\mathbf{q} = \begin{pmatrix} -2 & -t^2 & -t^2 & 1 - s^2 \\ 0 & 1 & s^2 & 0 \\ 1 & t^2 & 0 & s^2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ s^2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ t^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{r} = \begin{pmatrix} -2 & -t^2 & -t^2 & 1 - s^2 \\ 0 & 1 & s^2 & 0 \\ 1 & t^2 & 0 & s^2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ s^2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ t^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - s^2 \\ 0 \\ s^2 \\ 0 \end{pmatrix}.$$

$\square$

Acknowledgments

Appendix

In this appendix we will discuss the "effective" completion of unimodular matrices that we used in Proposition 25. The main result we shall follow from Caniglia et al. (1993) is the following theorem.

Theorem 29. *(Caniglia et al., 1993, Theorem 3.1) Let $R = \mathbb{F}[x_1, \ldots, x_n]$ and assume that $F \in R^{r \times s}$ $(r < s)$ is unimodular. Then there exists a square matrix $M \in R^{s \times s}$ such that*

(i) M is unimodular,

(ii) $FM = [I_r, 0] \in R^{r \times s}$,

(iii) $\deg(M) = (r(1 + \deg(F)))^{O(n)}$.

In our case $n = 2$ and we will have to make some small *"adjustments"* to find an actual constant. We will follow exactly the same proof as in Caniglia et al. (1993), and in certain steps we will substitute phrases like $n + 3n \in O(n)$ by the exact computation $n + 3n = 4n$. Inside this appendix section by the variable d we denote $d = 1 + \deg(F)$.

Proposition 30. *(Caniglia et al., 1993, Proposition 4.1) Assume that $F \in R^{r \times s}$ ($R = \mathbb{F}[x_1, \ldots, x_n]$, $r < s$) is unimodular. Then there exists a square matrix $M \in R^{s \times s}$ such that:*

(i) *M is unimodular,*

(ii) *$FM = [f_{ij}(x_1, \ldots, x_{n-1}, 0)]$ (i.e., FM is equal to the $r \times s$ matrix obtained by specializing the indeterminate x_n to zero in the matrix F),*

(iii) *$\deg(M) \le D(1 + 2D)(1 + D^{2n})(1 + D)^{2n}$, with $D = r(1 + \deg(F)) = rd$.*

Proof. We denote $F(t)$ as the matrix $F(t) = [f_{ij}(x_1, \ldots, x_{n-1}, t)]$.

    <u>*Claim 1.*</u> *(Caniglia et al., 1993, Procedure 4.6, Step 1 and Step 2)* There exists elements $c_1, \ldots, c_N \in \mathbb{F}[x_1, \ldots, x_{n-1}]$ with $N \le (1 + rd)^{2n}$ such that $1 \in (c_1, \ldots, c_N)$. Also we can find elements $a_1, \ldots, a_N \in x_n \mathbb{F}[x_1, \ldots, x_{n-1}]$, such that $x_n = a_1 c_1 + \ldots + a_N c_N$ and with $\max_{1 \le k \le N}\{\deg(a_k c_k)\} \le 1 + (rd)^{2n}$.

    <u>*Claim 2.*</u> *(Caniglia et al., 1993, Procedure 4.6, Step 3 and Step 4)* For $1 \le k \le N$, let $b_k = \sum_{h=1}^{k} a_h c_h$, then there exist unimodular matrices E_k with the properties

- $F(b_k)E_k = F(b_{k-1})$,
- $\deg(E_k) \le rd(1 + 2rd)\max\{\deg(b_k), \deg(b_{k-1})\} \le rd(1 + 2rd)(1 + (rd)^{2n})$.

Therefore we define $M = E_N E_{N-1} \ldots E_1$ and we have $F(x_n)M = F(0)$, with the upper bound

$$\deg(M) \le Nrd(1 + 2rd)(1 + (rd)^{2n}) \le rd(1 + 2rd)(1 + (rd)^{2n})(1 + rd)^{2n}$$

$$\deg(M) \le D(1 + 2D)(1 + D^{2n})(1 + D)^{2n}.$$

$\square$

Proof of Theorem 29. For a matrix $F = [f_{ij}(x_1, \ldots, x_n)]$ the substitution of a variable x_i for 0 does not increase the degree of the matrix, and keeps the unimodularity. Therefore, applying the previous proposition n times and some elementary transformations, we can find an invertible matrix $M \in R^{s \times s}$, with $FM = [I_r, 0]$ and $\deg(M) \le nD(1 + 2D)(1 + D^{2n})(1 + D)^{2n}$. $\square$

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