3-1-2019

The Odd Log-Logistic Gompertz Lifetime Distribution: Properties and Applications

Morad Alizadeh

Saeid Tahmasebi

Mohammad Reza Kazemi

Hamideh Siyamar Arabi Nejad

Gholamhossein G. Hamedani

Studia Scientiarum Mathematicarum Hungarica, Vol. 56, No. 1 (March 1, 2019): 55-80. DOI.

The Odd Log-Logistic Gompertz Lifetime Distribution: Properties and Applications

Morad Alizadeh
Department of Statistics, Persian Gulf University, Bushehr, Iran
Saeid Tahmasebi
Department of Statistics, Persian Gulf University, Bushehr, Iran
Mohammad Reza Kazemi
Department of Statistics, Fasa University, Fasa, Iran
Hamideh Siyamar Arabi Nejad
Department of Statistics, Persian Gulf University, Bushehr, Iran
G. Hossein G. Hamedani
Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI

Abstract

In this paper, we introduce a new three-parameter generalized version of the Gompertz model called the odd log-logistic Gompertz (OLLGo) distribution. It includes some well-known lifetime

distributions such as Gompertz (Go) and odd log-logistic exponential (OLLE) as special sub-models. This new distribution is quite flexible and can be used effectively in modeling survival data and reliability problems. It can have a decreasing, increasing and bathtub-shaped failure rate function depending on its parameters. Some mathematical properties of the new distribution, such as closed-form expressions for the density, cumulative distribution, hazard rate function, the *k*th order moment, moment generating function and the quantile measure are provided. We discuss maximum likeli- hood estimation of the OLLGo parameters as well as three other estimation methods from one observed sample. The flexibility and usefulness of the new distribution is illustrated by means of application to a real data set.

1. Introduction

The Gompertz distribution is a flexible distribution that can be skewed to the right or to the left. This distribution is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis [23, page 25]. The Gompertz distribution is considered for the analysis of survival, in some areas such as gerontology [5], computer [28], biology [10] and marketing science [3]. The hazard rate function of this distribution is an increasing function and often employed to describe the distribution of adult human life spans by actuaries and demographers [32]. The Gompertz distribution with parameters $a > 0$ and $b > 0$ denoted by $Go(a, b)$, has the cumulative distribution function (cdf)

(1.1) $G(x, a, b) = 1 - e^{-\frac{b}{a}}(e^{ax} - 1), x > 0, a > 0, b > 0,$

and the probability density function (pdf)

(1.2) $g(x, a, b) = be^{ax}e^{-\frac{b}{a}}(e^{ax} - 1), x > 0:$

In this paper, we introduce a new generalization of Gompertz distribution which by product of the application of the Gompertz distribution to the odd log-logistic generator proposed by [15], called the OLLGo distribution. Several generalized distributions have been proposed via this methodology. The idea is similar to introducing some generalization of the well-known distributions such as: odd-log-logistic Weibull distribution [7] and odd log logistic exponential distribution (OLLE), [15]. The OLLGo distribution includes some well-known distributions and offers a more flexible distribution for modeling lifetime data in terms of its hazard rate functions that are decreasing, increasing, upside-down bathtub and bathtub shaped. Several mathematical properties of this new model are provided in order to attract wider applications in reliability, engineering and other areas of research.

The paper is organized as follows: In Section 2, we define cumulative distribution, probability density, and failure rate functions, and outline some special cases of the OLLGo distribution. In Section 3 we provide some extensions and properties of the cdf, pdf, rth moment and moment generating function (mgf) of the OLLGo distribution in the form of power series. Furthermore, in this section, we derive the corresponding expressions for the cdf and pdf of the order statistics and quantile measure from the proposed distribution. Section 4 deals with certain characterizations of OLLGo distribution. In Section 5, we discuss maximum likelihood estimation (MLE) of the OLLGo parameters from one observed sample. Section 6 contains Monte Carlo simulation results on the finite sample behavior of

MLEs as well as three other estimation methods. Finally, application of the OLLGo model using a real data set is considered in Section 7.

2. The OLLGo model

In this section, we introduce the three-parameter OLLGo distribution. The idea of this distribution rises from the following general class: If $G(\cdot)$ denotes the cdf of a random variable then a generalized class of distributions, called the Generalized log-logistic family, introduced by Gleaton and Lynch (2006) is

(2.1) $F(x) = \frac{G(x;\xi)^{\alpha}}{G(x;\xi)^{\alpha} + \bar{G}(x;\xi)^{\alpha}}, x \in \mathbb{R}.$

Let $g(x;\ \xi) = \frac{dG(x;\xi)}{dx}$ be the density of the baseline distribution. Then, the probability density function corresponding to (2.1) is

(2.2) $f(x) = \frac{\alpha g(x;\xi) G(x;\xi)^{\alpha-1}\, \bar{G}(x;\xi)^{\alpha-1}}{[G(x;\xi)^{\alpha} + G(x;\xi)^{\alpha}]^2}, x \in \mathbb{R}:$

As stated in the paper by Gleaton and Lynch, if two distributions are generated from a common baseline distribution by generalized log-logistic transformations, then the log-odds function for one of the distributions is proportional to the log-odds function for the other distribution, with the constant of proportionality being the ratio of the transformation parameters.

Inserting $G(x)$ given in (1.1) into (2.1), we have

(2.3) $F(x) = \frac{\left[1 - e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha}}{\left[1 - e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha} + e^{-\frac{b\alpha}{a}(e^{ax}-1)}}, x > 0.$

The model of (2.3) is called the OLLGo distribution with parameters α, b and a (we use the notation $X \sim OLLGo(a, b, \alpha)$. Hence, the pdf of OLLGo distribution is

(2.4) $f(x) = \frac{\alpha b e^{ax} e^{-\frac{b\alpha}{a}(e^{ax}-1)}\left[1 - e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha-1}}{\left(\left[1 - e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha} + e^{-\frac{b\alpha}{a}(e^{ax}-1)}\right)^2}, x > 0.$

The hazard rate function of OLLGo distribution is given by

(2.5) $h(x) = \frac{\alpha b \mathrm{e}^{\mathrm{ax}}\left[1 - \mathrm{e}^{\frac{-\mathrm{b}}{a}(\mathrm{e}^{\mathrm{ax}}-1)}\right]^{\alpha-1}}{\left[1 - \mathrm{e}^{\frac{-b}{a}(e^{\mathrm{ax}}-1)}\right]^{\alpha} + \mathrm{e}^{\frac{-b\alpha}{a}(e^{\mathrm{ax}}-1)}}, x > 0.$

The OLLGo distribution includes some well-known distributions such as: $Go(a;\ b)$ if $\alpha = 1$, which is the generalization of exponential distribution (E) and $OLLE(b;\ \alpha)$ if a tends to 0^+. Also, we can state the following propositions related to a OLLGo distribution.

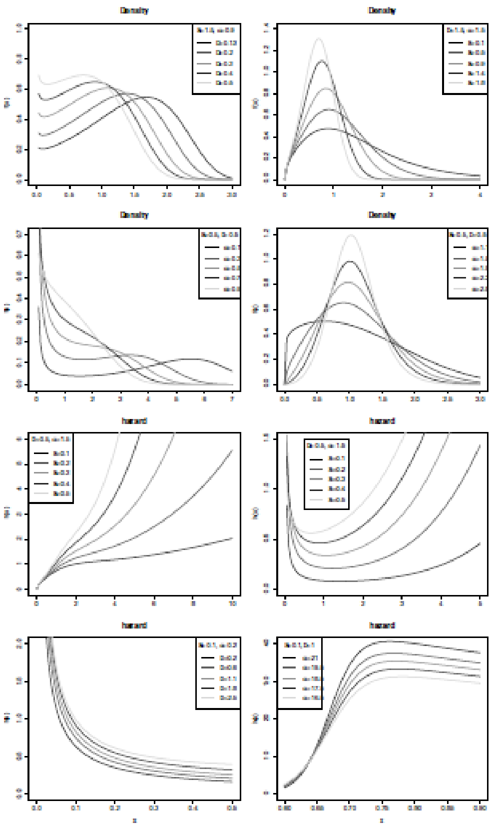

Fig. 1. pdf and hrf of OLL-Go model for some parameter values

PROPOSITION 2.1. *Let X have an $OLLGo(a;\ b;\ \alpha)$ distribution, then the random variable* $Y = 1 - e^{-\frac{b}{a}(e^{aX}-1)}$ *satisfies the odd log-logistic unit uniform distribution with parameter α and therefore, the random variable* $T = \frac{b}{a}\,(e^{aX} - 1)$ *satisfies the odd log-logistic exponential distribution with parameters α and 1, i.e. $(OLLE(\alpha, 1))$ which is a sub-model of the OLLGo distribution.*

PROPOSITION 2.2. If $U \sim U(0, 1)$, then

$$X_u = \frac{1}{a}\log\left\{1 - \frac{\mathrm{a}}{\mathrm{b}}\log\left[\frac{(1 - \mathrm{U})^{\frac{1}{\alpha}}}{\mathrm{U}^{\frac{1}{\alpha}} + (1 - \mathrm{U})^{\frac{1}{\alpha}}}\right]\right\},$$

follows $OLLGo(a, b, \alpha)$ distribution.

The result of Proposition 2.2 helps in simulating data from the OLLGo distribution. Figure 1 illustrates some of the possible shapes of density and hazard functions for selected values of the parameters. For instance, these plots show the hazard function of the new model is much more exible than the beta Gompertz (BG) introduced by [22] and Go (Gompertz) distributions. The hazard rate function can be bathtub shaped, monotonically increasing or decreasing and upside-down bathtub shaped depending on the parameter values.

3. General properties

In this section using some existing rules for the power series such as division and multiplication of power series and a power series raised to a positive integer n and so on, some properties of OLLGo distribution are mentioned.

3.1. A useful expansion

First, using generalized binomial expansion we have

$$\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha}{i} e^{-\frac{ib}{a}(e^{ax}-1)}$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k}$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k}$$

$$=\sum_{k=0}^{\infty}a_k\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k}.$$

where $a_k = \sum_{i=0}^{\infty} (-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}$. Similarly, we obtain

$$\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha} + e^{-\frac{b\alpha}{a}(e^{ax}-1)} = \sum_{k=0}^{\infty}b_k\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k},$$

where $b_k = a_k + (-1)^k\binom{\alpha}{k}$. Now using the rule for division of two power series, we have

(3.1) $F(x) = \dfrac{\sum_{k=0}^{\infty} a_k\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k}}{\sum_{k=0}^{\infty} b_k\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k}} = \sum_{k=0}^{\infty} c_k\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{k},$

where $c_0 = a_0 b_0$ and for $k \geq 1$ we have $c_k = b_0^{-1}\ [a_k - b_0^{-1}\sum_{r=0}^{k-1} b_r c_{k-r}]$.

The following Propositions show that the cdf and pdf of the OLLGo distribution can be rewritten as a mixture of cdf and pdf of the generalized Gompertz (GG) distributions. The GG distribution was introduced by [11] which has the following cdf and pdf as

$$F(x) = \left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{\alpha}, a \geq 0, b, \alpha > 0, x \geq 0,$$

and

$$f(x) = \alpha b e^{ax} e^{-\frac{b}{a}(e^{ax}-1)}\left[1- e^{-\frac{b}{a}(e^{ax}-1)}\right]^{(\alpha-1)}.$$

PROPOSITION 3.1. *The equation (3.1) shows that we can write cdf of OLLGo distribution as a mixture of cdfs of GG distributions as follows:*

$$F(x) = \sum_{k=0}^{\infty} c_k H_k(x),$$

where $H_k(x)$ denotes the cdf of GG distribution with power parameter k.

PROPOSITION 3.2. *The pdf of OLLGo distribution can be expressed as a mixture of pdfs of GG distributions as follows:*

(3.2) $f(x) = \sum_{k=0}^{\infty} c_{k+1} h_{k+1}(x),$

where $h_{k+1}(x)$ denotes the pdf of GG distribution with power parameter $k + 1$.

From (3.2), the rth ordinary moment of OLLGo distribution is given by

$$\acute{\mu}_r = E(X^r) = \sum_{k=0}^{\infty} c_{k+1} E(Y_{k+1}^r),$$

where, using Theorem 2 of [11], the rth moment of Y_{k+1} (a GG random variable with power parameter $k + 1$) can be written as

$$E(Y_{k+1}^r) = (k + 1) b \Gamma(r + 1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \binom{k}{j} \frac{(-1)^{j+i}}{\Gamma(i + 1)} \frac{(e)^{\frac{b}{a}(j+1)}}{\left[\frac{b}{a}(j + 1)\right]^{-i}} \left[\frac{-1}{a(i + 1)}\right]^{r+1}.$$

Therefore, the rth moment of OLLGo distribution is

$$\acute{\mu}_r = E(X^r)$$

(3.3) $= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \binom{k}{j} c_{k+1} (k + 1) b \Gamma(r + 1) \frac{(-1)^{j+i}}{\Gamma(i + 1)} \frac{(e)^{\frac{b}{a}(j+1)}}{\left[\frac{b}{a}(j + 1)\right]^{-i}} \left[\frac{-1}{a(i + 1)}\right]^{r+1}.$

The skewness and kurtosis measures can be calculated recursively using the rth moment of OLLGo distribution in (3.3). Furthermore, the cumulants (κ_n) of X can be written recursively as

$$\kappa_n = \acute{\mu}_n - \sum_{r=1}^{n-1} \binom{n - 1}{r - 1} \kappa_r \, \acute{\mu}_{n-r},$$

where $\kappa_1 = \acute{\mu}_1, \kappa_2 = \acute{\mu}_2 - \acute{\mu}_1^2, \kappa_3 = \acute{\mu}_3 - 3\acute{\mu}_1\acute{\mu}_2 + \acute{\mu}_1^3$. Figure 2 shows the behaviour of the skewness and kurtosis of $\mathrm{OLLGo}(a, 1, \alpha)$ distribution.

3.2. Moment generating function

We give two representations for moment generating function (mgf) $M(s)$ of the OLLGo distribution. For the first representation, we use the Maclaurin series expansion of an exponential function, as follows

$$M(s) = \sum_{r=0}^{\infty} \frac{s^r}{r!} E(X^r).$$

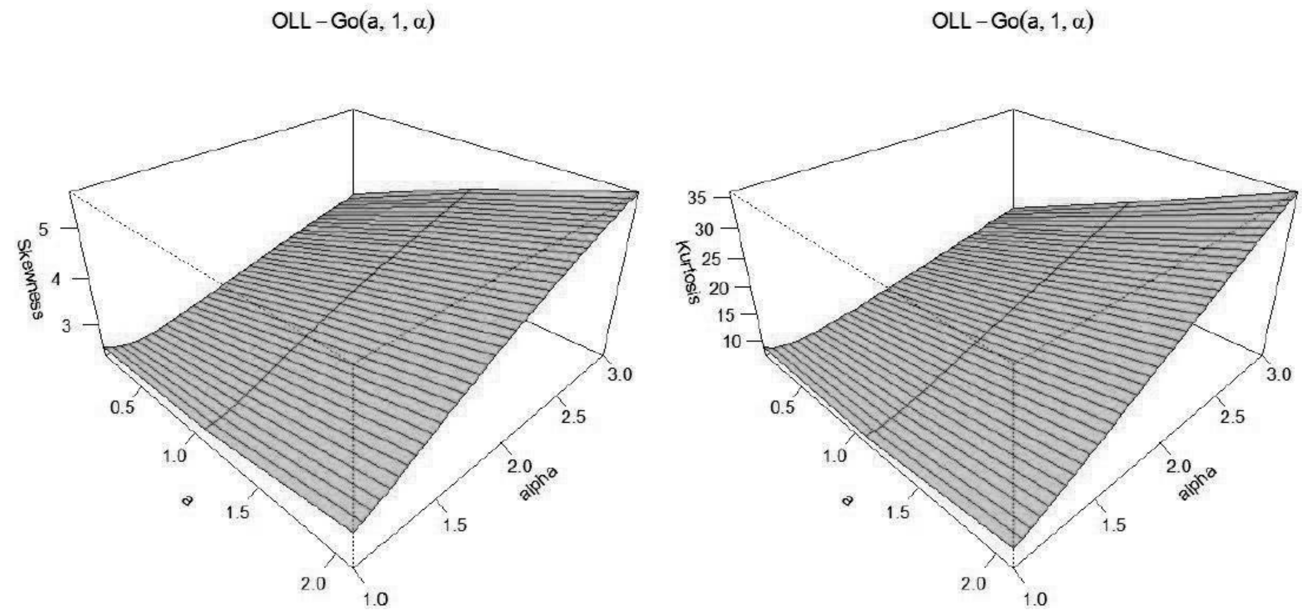

Fig. 2. Skewness and Kurtosis for OLL-Go$(a, 1, \alpha)$

Thus the mgf of the $OLLGo(a, b, \alpha)$ distribution can be obtain as

$$M(s) = \sum_{r;k;j;i=0}^{\infty} \binom{k}{j} \frac{s^r}{r!} c_{k+1}(k+1)b\Gamma(r+1) \frac{(-1)j+i}{\Gamma(i+1)} \frac{(e)^{\frac{b}{a}(j+1)}}{\left[\frac{b}{a}(j+1)\right]^{-i}} \left[\frac{-1}{a(i+1)}\right]^{r+1}.$$

For the second representation, using equation (3.2), we can obtain another formula for $M(s)$ as follows:

$$M(s) = \sum_{k=0}^{\infty} c_{k+1} M_{k+1}(s),$$

where

$$M_{k+1}(s) = (k+1)be^{\frac{b}{a}} \sum_{0}^{\infty} e^{sx} e^{ax} e^{\frac{-b}{a}e^{ax}} \left(1 - e^{\frac{-b}{a}(e^{ax}-1)}\right)^k dx.$$

Using the binomial series expansion for $\left(1 - e^{\frac{-b}{a}(e^{ax}-1)}\right)^k$, we arrive at

$$M_{k+1}(s) = (k+1)b \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{\frac{b}{a}(j+1)} \int_{0}^{\infty} e^{sx} e^{ax} e^{\frac{-b}{a}e^{ax}} e^{\frac{-b}{a}(j+1)e^{ax}} dx.$$

The change of variable $z = e^{ax}$ yields

$$M_{k+1(s)} = \frac{(k+1)b}{a}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{\frac{b}{a}(j+1)}\int_{1}^{\infty} z^{\frac{s}{a}}\, e^{\frac{b}{a}(j+1)z}dz.$$

Then, setting $y = \frac{b}{a}(j+1)z$, we obtain

$$M_{k+1(s)} = (k+1)\left(\frac{a}{b}\right)^{\frac{s}{a}-1}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j e^{\frac{b}{a}(j+1)}}{(j+1)^{\frac{s}{a}}}\Gamma\left(\frac{s}{a}+1, \frac{b}{a}(j+1)\right),$$

where $\Gamma(u,v)$ is the incomplete gamma function defined by

$$\Gamma(u,v) = \int_{v}^{\infty} x^{u-1}e^{-x}dx.$$

Therefore,

$$M(s) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{a}{b}\right)^{\frac{s}{a}-1}\frac{c_{k+1}(k+1)(-1)^j e^{\frac{b}{a}(j+1)}}{(j+1)^{\frac{s}{a}}}\Gamma\left(\frac{s}{a}+1, \frac{b}{a}(j+1)\right).$$

3.3. Mean deviations

The mean deviations can be used as measures of spread in a population. They are defined by

$$\delta_1 = \int_0^{\infty}|x-\mu|f(x)dx \text{ and } \delta_2 = \int_0^{\infty}|x-M|f(x)dx,$$

where μ and M are the first moment and median of the corresponding distribution. These measures can be expressed as

$$\delta_1 = 2\mu F(\mu) - 2\int_{\mu}^{\infty} xf(x)dx \text{ and } \delta_2 = \mu - 2\int_{M}^{\infty} xf(x)dx.$$

Using the above equations we obtain closed form formulas for δ_1 and δ_2. From (3.2), we have

$$h_{GG}(x;\, a,b,k+1) = (k+1)b\mathrm{e}^{\mathrm{ax}}\mathrm{e}^{\frac{-\mathrm{b}}{\mathrm{a}}(\mathrm{e}^{\mathrm{ax}}-1)}\left(1-\mathrm{e}^{\frac{-\mathrm{b}}{\mathrm{a}}(\mathrm{e}^{\mathrm{ax}}-1)}\right)^k,$$

and then for a given η we have

$$\int_{\eta}^{\infty} xf(x)dx = \frac{b}{a^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(k+1)c_{k+1}e^{\frac{(j+1)b}{a}}\int_{e^{a\eta}}^{\infty} e^{-\frac{b}{a}(j+1)z}\ln(z)dz.$$

Upon changing variables and integrating by parts we have

$$\int_{\eta}^{\infty} xf(x)dx = \frac{b}{a^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(k+1)c_{k+1}e^{\frac{(j+1)b}{a}}.\left[\frac{a\eta e^{-\frac{b}{a}(j+1)e^{a\eta}}}{\frac{b}{a}(j+1)}+\frac{\Gamma\left(0,\frac{b}{a}(j+1)e^{a\eta}\right)}{\frac{b}{a}(j+1)}\right]$$

$$=\frac{1}{a}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j(k+1)c_{k+1}e^{\frac{(j+1)b}{a}}}{(j+1)}\left[a\eta e^{-\frac{b}{a}(j+1)e^{a\eta}}+\Gamma\left(0,\frac{b}{a}(j+1)e^{a\eta}\right)\right].$$

Therefore,

$$\delta_1 = 2\mu F(\mu)-\frac{2}{a}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j(k+1)c_{k+1}e^{\frac{(j+1)b}{a}}}{(j+1)}.\left[\mu a e^{-\frac{b}{a}(j+1)e^{a\mu}}+\Gamma\left(0,\frac{b}{a}(j+1)e^{a\mu}\right)\right],$$

and

$$\delta 2 = \mu-\frac{2}{a}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j(k+1)c_{k+1}e^{\frac{(j+1)b}{a}}}{(j+1)}.\left\{Mae^{-\frac{b}{a}(j+1)e^{aM}}+\Gamma\left(0,\frac{b}{a}(j+1)e^{aM}\right)\right\}.$$

3.4. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \dots, X_n$ is a random sample from any OLLGo distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = cf(x)F^{i-1}(x)\{1-F(x)\}^{n-i} = c\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f(x)F(x)^{j+i-1},$$

where $c = \frac{1}{B(I,n-i+1)}$.

Upon integration, the cdf of ith order statistic is

$$F_{i:n}(x) = c\sum_{j=0}^{n-i}\frac{(-1)^j}{j+i}\binom{n-i}{j}F(x)^{j+i}.$$

From Gradshteyn and Ryzhik (2000), we can use the following expansion for a power series raise to a positive integer as

$$\left(\sum_{i=0}^{\infty}a_iu^i\right)^n = \sum_{i=0}^{\infty}c_{n,i}u^i;\ n \geq 1,$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \dots$) are determined from the recurrence equation (with $c_{n,0} = a_0^n$)

(3.4) $c_{n,i} = (ia_0)^{-1}\sum_{m=1}^{i}\ [m(n+1)-i]a_m c_{n,i-m}$.

Applying the above equations to the power series form of the $F(x)^{j+i}$ with $F(x)$ given by (3.1), the cdf of ith order statistic from OLLGo distribution can be rewritten as

$$F_{i:n}(x) = c\sum_{j=0}^{n-i}\sum_{k=0}^{\infty} e_{i,j}c_{i+j,k}H_k(x),$$

where $e_{i,j} = \frac{(-1)^j}{j+i}\binom{n-i}{j}$, $c_{i+j,0} = c_0^{i+j}$ and $c_{i+j,k}$ satisfy in the following recursive equation

$$c_{i+j,k} = (kc_0)^{-1}\sum_{m=1}^{k}[m(i + j + 1) - k]c_m c_{i+j,k-m}.$$

The pdf of the ith order statistic can be used to obtain the moments, mgf and mean deviations of the OLLGo order statistics from above equations and some properties of the GG model.

3.5. Asymptotics and shapes

In the following Proposition we give some asymptotes for cdf, pdf and hazard rate functions of the OLLGo model.

PROPOSITION 3.3. Let the random variable X have $OLLGo(a;\ b;\ \alpha)$ distribution with cdf $F(x)$, pdf $f(x)$ and failure rate function $h(x)$, then,

$$F(x) \sim (bx)^{\alpha} \text{ as } x \to 0,$$

$$f(x) \sim \alpha b^{\alpha}x^{\alpha-1} \text{ as } x \to 0,$$

$$h(x) \sim \alpha b^{\alpha}x^{\alpha-1} \text{ as } x \to 0,$$

$$1 - F(x) \sim \mathrm{e}^{\frac{-\mathrm{b}}{\mathrm{a}}}\,\mathrm{e}^{\mathrm{ax}} \text{ as } x \to 1,$$

$$f(x) \sim b\alpha \mathrm{e}^{\mathrm{ax}}\mathrm{e}^{\frac{-\mathrm{b}\alpha}{\mathrm{a}}\mathrm{e}^{\mathrm{ax}}} \text{ as } x \to 1;$$

$$h(x) \sim b\alpha\mathrm{e}^{\mathrm{ax}} \text{ as } x \to 1:$$

These equations show the effect of parameters on the tail of OLLGo distribution.

3.6. Quantile measure

The quantile function of OLLGo distribution is given by

$$Q(u) = \frac{1}{a}\log\left\{1 - \frac{a}{b}\log\frac{(1 - u)^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}} + (1 - u)^{\frac{1}{\alpha}}}\right\},$$

where $u \in (0,1)$.

4. Characterization Results

This section deals with the characterizations of the OLLGo distribution in different directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) in terms of the reverse hazard function and (iv) based on the conditional expectation of a function of the random variable. Note that (i) can be employed also when the cdf does not have a closed form. We would also like to mention that due to the nature of OLLGo distribution, our characterizations may be the only possible ones. We present our characterizations (i)-(iv) in four subsections.

4.1. Characterizations based on two truncated moments

This subsection is devoted to the characterizations of OLLGo distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glänzel (1987), see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

PROPOSITION 4.1. *Let* $X\colon\ \Omega\ (0,\infty)$ *be a continuous random variable and let*

$$q_1(x) = \left(\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right] +\ e^{-\frac{b}{a}(e^{ax-1})}\right)^2$$

and

$$q_2(x) =\ q_1(x)\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}$$

for $x\ >\ 0$. *The random variable* X *has pdf (2.4) if and only if the function* η *defined in Theorem 1 is of the form*

$$\eta(x) = \frac{1}{2}\left\{1\ +\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}\right\},\qquad x\ >\ 0.$$

PROOF. Suppose the random variable X has pdf (2.4), then

$$(1-\ F(x))E[q_1(X)\,|\,X\ \geq\ x]\ =\ 1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha},\qquad x\ >\ 0,$$

and

$$(1-\ F(x))E[q_2(X)\,|\,X\ \geq\ x]\ = \frac{1}{2}\left\{1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{2\alpha}\right\},\qquad x\ >\ 0.$$

Further,

$$\eta(x)q_1(x) -\ q_2(x) = \frac{q_1(x)}{2}\left\{1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}\right\} >\ 0,\qquad \text{for } x\ >\ 0:$$

Conversely, if η is of the above form, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) -\ q_2(x)} = \frac{\alpha b e^{ax} e^{-\frac{b}{a}(e^{ax-1})}\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha-1}}{1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}},\qquad x\ >\ 0,$$

and consequently

$$s(x) =\ -\log\left\{\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}\right\},\qquad x\ >\ 0.$$

Now, according to Theorem 1, X has density (2.4).

COROLLARY 4.1. Let X: Ω $(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. The random variable X has pdf (2.4) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the following differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-\ q_2(x)}=\frac{\alpha b e^{ax}e^{-\frac{b}{a}(e^{ax-1})}\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha-1}}{1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}},\qquad x\ >\ 0.$$

COROLLARY 4.2. *The general solution of the differential equation in Corollary 4.1 is*

$$\eta(x)=\left\{1-\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha}\right\}^{-1}$$
$$\times\left[-\int\alpha b e^{ax}e^{-\frac{b}{a}(e^{ax-1})}\left[1-\ e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha-1}\left(q_1(x)\right)^{-1}q_2(x)dx\ +\ D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 4.1 with $D\ =\ \frac{1}{2}$. Clearly, there are other triplets (q_1,q_2,η) which satisfy conditions of Theorem 1.

4.2. Characterization in terms of hazard function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'\ (x)}{h_F\ (x)}-\ h_F\ (x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of OLLGo distribution, for $\alpha\ =\ 1$, in terms of the hazard function.

PROPOSITION 4.2. Let X: Ω $(0,\infty)$ be a continuous random variable. The random variable X has pdf (2.4), for $\alpha=\ 1$, if and only if its hazard function $h_F\ (x)$ satisfies the following differential equation

$$h_F'\ (x)-\ ah_F\ (x)=\ 0,\qquad x\ >\ 0.$$

PROOF. The proof is straightforward and hence omitted. _

4.3. Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F\ (x)=\frac{f(x)}{F(x)},x\ \in\ \text{support of}\,F.$$

In this subsection we present a characterization of OLLGo distribution in terms of the reverse hazard function.

PROPOSITION 4.3. *Let X: $\Omega\ (0,\infty)$ be a continuous random variable. The random variable X has pdf (2.4) if and only if its reverse hazard function $r_F\ (x)$ satisfies the following differential equation*

$$r_F'\ (x) + be^{ax} r_F\ (x) = \alpha b e^{-\frac{b}{a}(e^{ax-1})}\ \frac{d}{dx}\left\{\frac{e^{ax}\left[1 - e^{-\frac{b}{a}(e^{ax-1})}\right]^{-1}}{\left[1 - e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha} + e^{-\frac{b\alpha}{a}(e^{ax-1})}}\right\}, \qquad x > 0.$$

PROOF. If X has pdf (2.4), then clearly the above differential equation holds. Now, if this equation holds, then

$$\frac{d}{dx}\left\{e^{\frac{b}{a}(e^{ax-1})} r_F\ (x)\right\} = ab\frac{d}{dx}\left\{\frac{e^{ax}\left[1 - e^{-\frac{b}{a}(e^{ax-1})}\right]^{-1}}{\left[1 - e^{-\frac{b}{a}(e^{ax-1})}\right]^{\alpha} + e^{-\frac{b\alpha}{a}(e^{ax-1})}}\right\},$$

from which we obtain the reverse hazard function corresponding to the pdf (2.4).

REMARK 4.1. For $\alpha = 1$, the above differential equation has the following much simpler form.

$$r_F'\ (x) + be^{ax} r_F\ (x) = be^{ax-\frac{b}{a}(e^{ax}-1)}\left\{\frac{a - e^{-\frac{b}{a}(e^{ax-1})}\left[a - be^{ax}\right]}{\left[1 - e^{-\frac{b}{a}(e^{ax-1})}\right]^{2}}\right\}, \qquad x > 0.$$

4.4. Characterization based on the conditional expectation of certain function of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here. It can be used to characterize an OLLGo distribution for $\alpha = 1$.

PROPOSITION 4.4. *Let X: $\Omega\ (e;\ f)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on $(e;\ f)$ with $\lim_{x\to e+} \psi(x) = 1$. Then for $\delta \neq 1$,*

$$E[\,\psi(X)\,|\,X \geq x] = \delta\psi(x), x \in (e,f)$$

if and only if

$$\psi(x) = \left(1 - F(x)\right)^{\frac{1}{\delta}-1}, \qquad x \in (e;\ f).$$

REMARK 4.2. For $(e,f) = (0,\infty), \alpha = 1,\ \psi(x) = e^{-(e^{ax-1})}$ and $\delta = \frac{b}{a+b}$, Proposition 4.4 provides a characterization of OLLGo for $\alpha = 1$.

5. Estimation

Let $X_1, \dots, X_n$ be a random sample of size n from the $OLLGo(a;\ b;\ \alpha)$ distribution and $\mathbf{\Theta} = (a;\ b;\ \alpha)$ be the unknown parameter vector. The loglikelihood function is given by

$$l(\mathbf{\Theta}) = n\log(\alpha b) + a\sum_{i=1}^{n} x_i + \alpha\sum_{i=1}^{n}\log(1-t_i) + (\alpha-1)\sum_{i=1}^{n}\log(t_i) - 2\sum_{i=1}^{n}\log(t_i^{\alpha} + (1-t_i)^{\alpha}),$$

where $t_i = 1 - \exp(-\frac{b}{a}(e^{ax_i} - 1))$. The maximum likelihood estimation (MLE) of Θ is obtained by solving the nonlinear equations, $U(\mathbf{\Theta}) = (U_a(\mathbf{\Theta}); U_b(\mathbf{\Theta}); U_\alpha(\mathbf{\Theta}))T = \mathbf{0}$, where

$$U_\alpha(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\big(t_i(1-t_i)\big) - 2\sum_{i=1}^{n}\frac{t_i^{\alpha}\log t_i + (1-t_i)^{\alpha}\log(1-t_i)}{t_i^{\alpha} + (1-t_i)^{\alpha}},$$

$$U_a(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial a} = n\bar{x} - \alpha\sum_{i=1}^{n}\frac{t_i^{(a)}}{1-t_i} + (\alpha-1)\sum_{i=1}^{n}\frac{t_i^{(a)}}{t_i} - 2\alpha\sum_{i=1}^{n} t_i^{(a)} \times \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}},$$

$$U_b(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial b} = \frac{n}{b} - \alpha\sum_{i=1}^{n}\frac{t_i^{(b)}}{1-t_i} + (\alpha-1)\sum_{i=1}^{n}\frac{t_i^{(b)}}{t_i} - 2\alpha\sum_{i=1}^{n} t_i^{(b)} \times \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}.$$

Note that

$$t_i^{(a)} = \frac{\partial t_i}{\partial a} = \left(-\frac{bx_i}{a}e^{ax_i}\ \frac{b}{a^2}\ (e^{ax_i} - 1))(1-t_i\right)$$

and

$$t_i^{(b)} = \frac{\partial t_i}{\partial b} = \frac{1}{a}(e^{ax_i} - 1)(1-t_i).$$

By Referring to Gleaton and Rahman (2010, 2014), there are situations in which some of the regularity conditions are not necessarily satisfied for certain subsets of the interior of the parameter space for certain GLL-families of distributions. We will check these regularity conditions for this proposed distribution OLLGo in our future work. So, by assuming that the regularity conditions hold then asymptotically

$$n(\widehat{\mathbf{\Theta}} - \mathbf{\Theta}) \sim N_3(0, I(\mathbf{\Theta})^{-1}),$$

where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. This asymptotic behavior is valid if $I(\mathbf{\Theta})$ replaced by $J_n(\widehat{\mathbf{\Theta}})$, i.e., the observed Fisher information matrix evaluated at $\widehat{\mathbf{\Theta}}$, see [9].

5.1. The other estimation methods

There are several approaches to estimating the parameters of distributions, and each of them has its characteristic features and benefits. In this subsection, three of these methods are briefly mentioned and numerically investigated in the simulation study. A useful summary of these methods can be found in Dey *et al*. (2017). Here $\{t_{i:n};\ i = 1, 2, \ldots, n\}$ is the associated order statistics and F is the distribution function of OLLGo distribution.

5.1.1. Weighted least square estimators

The Weighted least square estimators (WLSE) was introduced by Swain *et al*. (1988). The WLSE's is obtained by minimizing

$$S_{\text{WLSE}}(a,b,\alpha)=\sum_{i=1}^{n}\frac{(n\ +\ 1)^2(n\ +\ 2)}{i(n-\ i\ +\ 1)}\left(F(t_{i:n};\ a,b,\alpha)-\frac{i}{n\ +\ 1}\right)^2$$

with respect to a, b and α.

5.1.2. Cramr-von-Mises estimator

Cramér-von-Mises Estimator (CVM) was introduced by Choi and Bulgren (1968). The CVMs is obtained by minimizing the following function

$$S_{\text{CVM}}(a,b,\alpha)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(t_{i:n};\ a,b,\alpha)-\frac{2i\ -\ 1}{2n}\right)^2.$$

5.1.3. Anderson-Darling and right-tailed Anderson-Darling estimators

The Anderson Darling (ADE) and Right-tailed Anderson Darling estimators (RTADE) were introduced by Anderson and Darling (1952). The ADE's and RTADE's are obtained by minimizing with respect to a, b and α, the function

$$S_{\text{ADE}}(a,b,\alpha)=\ -n-\frac{1}{n}\sum_{i=1}^{n}(2i-\ 1)\{\log F(t_{i:n};\ a,b,\alpha)\ +\ \log\bar{F}(t_{i:n};\ a,b,\alpha)\},$$

where $\bar{F}(\cdot)=\ 1-\ F(\cdot)$.

6. Simulation study

In this subsection, the behaviour of the MLEs of the parameters of the OLLGo distribution has been assessed via simulation. To verify the validity of the MLEs, the bias and the mean square error (MSE) of MLEs have been checked. Samples of size $n\ =\ 20, 25, \ldots\ , 200$, from $OLLGo\ (a,b,\alpha)\ =\ (2, 0.05, 1.1)$, with $(a,b,\alpha)\ =\ (2, , 0.05, 1.1)$ have been generated. To estimate the MLEs, the *optim* function (in the *stat* package) and BFGS method in R software have been used. We replicate these steps for $r\ =\ 200$ times. If $\boldsymbol{\xi}\ =\ (a,b,\alpha)$, for the ith replication $i\ =\ 1, 2, \ldots\ , 200$ the MLEs are obtained as $\hat{\boldsymbol{\xi}}_{\boldsymbol{i}}\ =\ (\hat{a}_i, \hat{b}_i, \hat{\alpha}_i)$. We Compute the biases and MSEs as

(6.1)

$$Bias_r(\widehat{\boldsymbol{\xi}})\ =\frac{1}{r}\sum_{i=1}^{r}(\hat{\boldsymbol{\xi}}_i-\ \boldsymbol{\xi}_i)\,,\qquad \text{and}\quad MSE_r(\widehat{\boldsymbol{\xi}})\ =\frac{1}{r}\sum_{i=1}^{r}(\hat{\boldsymbol{\xi}}_i-\ \boldsymbol{\xi}_i)^2\,,\qquad \text{for}\ \boldsymbol{\xi}\ =\ (a,b,\alpha).$$

Figures 3, 4 respectively reveal how the three biases, MSEs vary with respect to n. As expected, the Biases and MSEs of the estimated parameters converge to zero while n growing.

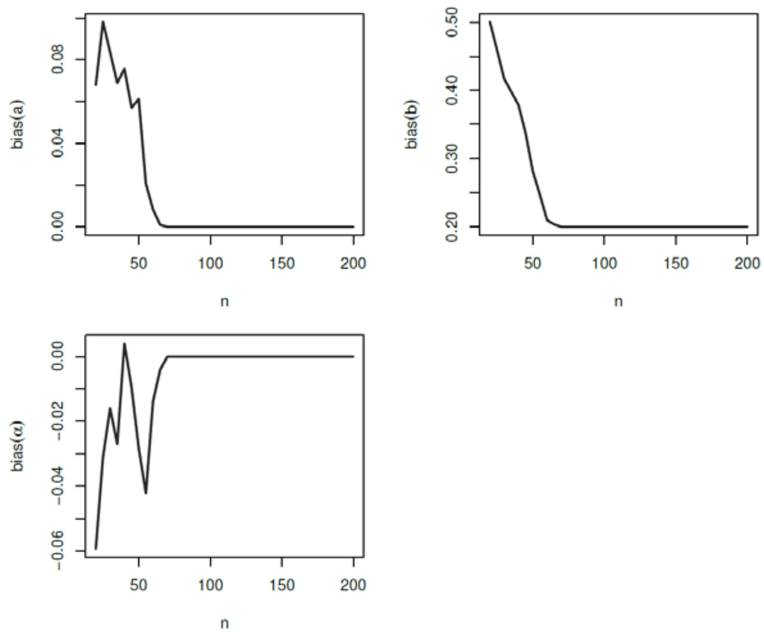

Fig. 3. Bias of $\hat{a}, \hat{b}, \hat{\alpha}$ versus n for OLLGo when $(a, b, \alpha) = (2, 0.05, 1.1)$

In order to explore the estimators introduced above, we consider one model that has been used in this section and investigate MSE of estimators for different samples. We consider the sample sizes of the $n = 30, 35, \cdots 200$ with $r = 200$ replications and we take $(a, b, \alpha) = (2, 0.05, 1.1)$. Then the biases and MSEs formulas that are mentioned in equation (6.1) are calculated.

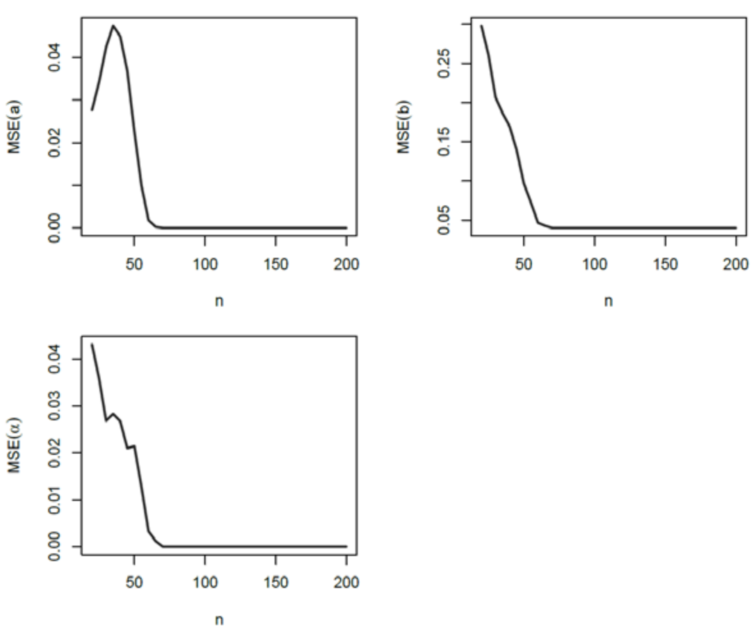

Fig. 4. MSE of $\hat{a}, \hat{b}, \hat{\alpha}$ versus n for OLLGo when $(a, b, \alpha) = (2, 0.05, 1.1)$

To obtain the value of the estimators, we have used the optima function and BFGS method in R, again. The results of the simulations of this subsection are shown in Figures (5) and (6).

A general result about above figures is that MSE plots for three parameters (a, b, α) will approach zero as the sample size increases for all methods WLSE, CVM, and ADE and this verifies the validity of the these estimation methods and numerical calculations for the parameters of OLLGo distribution. Further results are mentioned below

- For estimating a,ADE method has the minimum amount of bias.
- For estimating b, ADE method has the minimum amount of bias.
- For estimating $alpha$, ADE method has the minimum amount of bias.
- For estimating a, ADE method has the minimum amount of MSE.
- For estimating b, ADE method has the minimum amount of MSE.
- For estimating $alpha$, ADE method has the minimum amount of MSE.

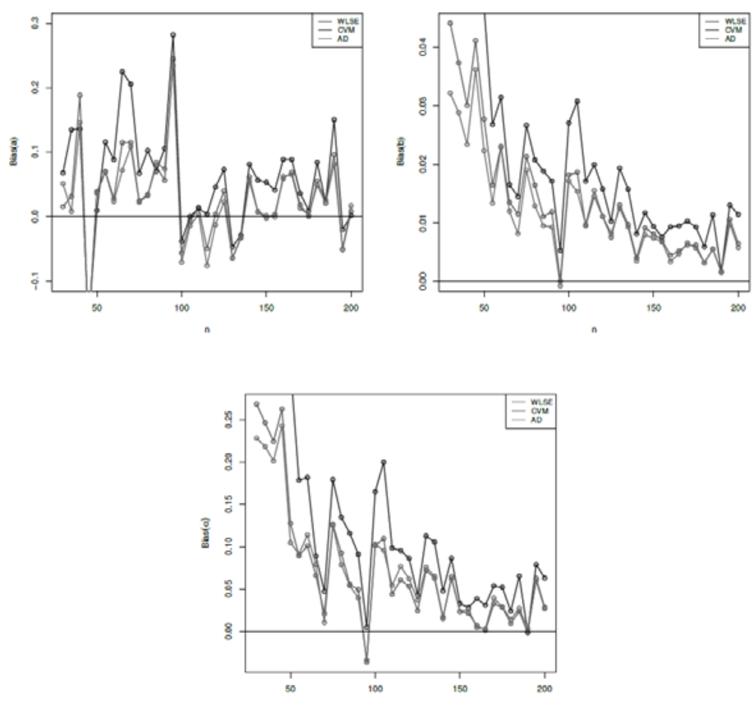

Fig. 5. Biases of $\hat{a}, \hat{b}, \hat{\alpha}$ versus n for OLLGo when $(a, b, \alpha) = (2, 0.05, 1.1)$ for WLSE, CVM, and ADE methods

7. Application of OLLG to real data set

The following data set have been obtained from [1] and represents the lifetimes of 50 devices. The data are:

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.

To analyze above data, we obtained the MLE's of the parameters (with standard deviations) for the distributions OLLGo (proposed distribution in this paper), OLLE ([27]), Go (Gompertz distribution), MO-EGG (Marshal-Olkin extended generalized Gompertz) introduced by [4]), Beta-Go (Beta-Gompertz introduced by [22]), Mc-Go (Mc Donald-Gompertz introduced by [29]), Kw-Go (Kumaraswamy

Gompertz introduced by [31]) and GGo (generalized Gompertz introduced by [11]). Then, we calculated the Akaike information

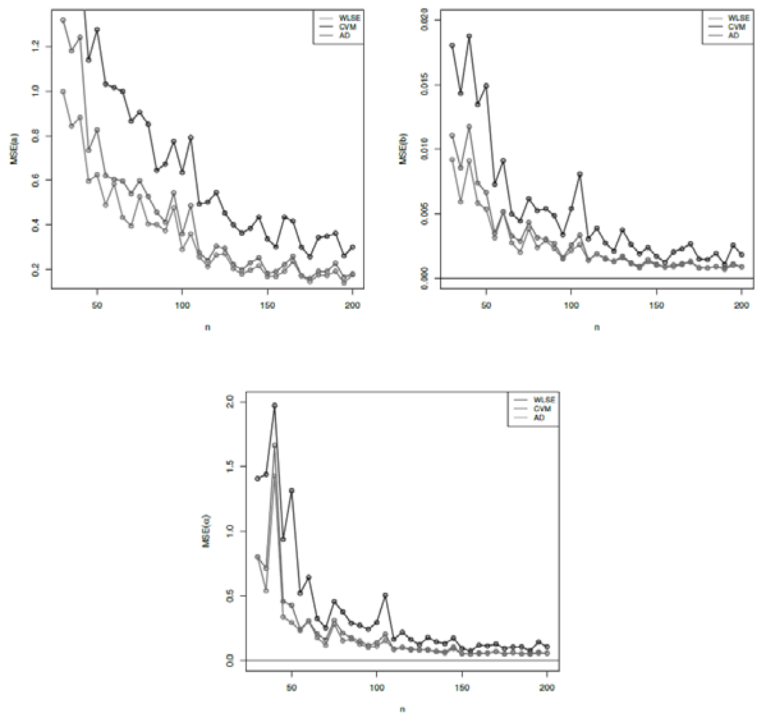

Fig. 6. MSE of $\hat{a}, \hat{b}, \hat{\alpha}$ versus n for OLLGo when $(a, b, \alpha) = (2, 0.05, 1.1)$ for WLSE, CVM, and ADE methods

criterion (AIC), Bayesian information criterion (BIC), Anderson-Darling (AC) and Cramér-von Mises (W^*), Kolmogorov Smirnov (K-S) statistic and the P-Value of K. S test for each model. The results show that the OLLGo distribution yields the best fit. The plots of the densities (together with the data histogram) and cumulative distribution functions in Figure 7 confirm this conclusion.

8. Conclusion

Here we propose a new model, called the OLLGo distribution. Some mathematical properties of this model including explicit expansions for the ordinary and incomplete moments, quantile and mgf, mean deviations, and

	Distribution							
	OLL-Go	OLL-E	Go	MO-EGG	Beta-G	Mc-Go	Kw-Go	GGo
$\hat{a}$(s.e.)	0.0592 (0.0058)	0.0277 (0.0062)	0.0202 (0.0057)	0.0429 (0.0042)	0.0558 (0.0094)	0.0693 (0.0135)	0.0520 (0.0077)	0.0390 (0.0044)
$\hat{b}$ (s.e.)	0.0025 (0.007)	0.7615 (0.1255)	0.0097 (0.0028)	0.0022 (0.0004)	0.0044 (0.0024)	0.0037 (0.0018)	0.0035 (0.0016)	0.0022 (0.0004)
$\hat{\alpha}$ (s.e.)	0.3661 (0.0655)	— (—)	— (—)	0.3178 (0.1406)	0.3285 (0.0862)	0.1242 (0.1226)	0.2661 (0.0922)	0.4540 (0.0701)
$\hat{\beta}$ (s.e.)	— (—)	— (—)	— (—)	2.3613 (1.8029)	0.2108 (0.0924)	0.0957 (0.0847)	0.3374 (0.1353)	— (—)
$\hat{c}$ (s.e.)	— (—)	— (—)	— (—)	— (—)	— (—)	2.3719 (2.3473)	— (—)	— (—)

AIC	443.6125	482.8373	474.6533	454.8368	451.1068	449.3803	453.3934	455.5266
CAIC	444.1343	483.0926	474.9086	455.7257	451.9957	450.7439	454.2823	456.0483
BIC	449.3486	486.6913	478.4773	462.4849	458.7549	458.9404	461.0415	461.2627
HQIC	445.7969	484.2935	476.1095	457.7493	454.0192	453.0208	456.3059	457.7109
W	0.1280	0.4718	0.2891	0.1807	0.1624	0.1441	0.1786	0.2008
A	1.0262	2.8788	1.8916	1.3233	1.2307	1.141	1.3262	1.4111
K-S	0.1385	0.2262	0.1694	0.1346	0.1383	0.1252	0.1337	0.1426
p-value	0.292	0.0119	0.1132	0.3246	0.2938	0.4126	0.333	0.2606

Table 1. MLEs of the model parameters for the lifetimes of 50 devices data, the corresponding SEs and the AIC, CAIC,BIC, HQIC and K-S statistics

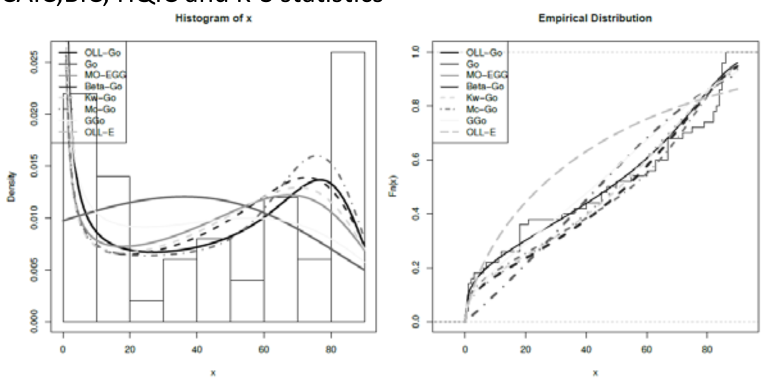

Fig. 7. Plots of the estimated pdfs and cdfs OLLGo, OLLE, Go, Beta-Go, Kw-Go, Mc-Go, MO-EGGo and GGo models using the strengths of 1.5 cm glass fibers data

order statistics are provided. The model parameters are estimated by the maximum likelihood estimation method. We prove empirically, by means of an application to a real data set, that the proposed family can give better fits than other models generated from well-known families.

REFERENCES

[1] Aarset, M. V., How to identify a bathtub hazard rate, IEEE Transactions on Reliability, R-36(1) (1987), 106-108.

[2] Akinsete, A., Famoye, F. and Lee, C., The beta-Pareto distribution, Statistics, 42(6) (2008), 547{563.

[3] Bemmaor, A. C. and Glady, N., Modeling purchasing behavior with sudden \death": A exible customer lifetime model, Management Science58(5) (2012), 1012-1021.

[4] Benkhelifa, L., The Marshall-Olkin extended generalized Gompertz distribution, arXiv preprint arXiv:1603.08242, 2016.

[5] Brown, K. and Forbes, W., A mathematical model of aging processes, Journal of Gerontology, 29(1) (1974), 46-51.

[6] Cintra, R. J., Rêgo, L. C., Cordeiro, G. M. and Nascimento, A. D. C., Beta generalized normal distribution with an application for SAR image processing, Statistics, 48(2) (2012), 279-294.

[7] Cooray, K., Generalization of the Weibull distribution: the odd Weibull family, Statistical Modelling, 6(3) (2006), 265-277.

[8] Cordeiro, G. M. and Nadarajah, S., Closed-form expressions for moments of a class of beta generalized distributions, Brazilian Journal of Probability and Statistics, 25(1) (2011), 14-33.

[9] Cox, D. R. and Hinkley, D. V., Theoretical Statistics, Chapman and Hall, London, 1974.

[10] Economos, A. C., Rate of aging, rate of dying and the mechanism of mortality, Archives of Gerontology and Geriatrics, 1(1) (1982), 46-51.

[11] El-Gohary, A., Alshamrani, A. and Al-Otaibi, A. N., The generalized Gompertz distribution, Applied Mathematical Modelling, 37(1-2) (2013), 13-24.

[12] Eugene, N., Lee, C. and Famoye, F., Beta-normal distribution and its applications, Communications in Statistics - Theory and Methods, 31(4) (2002), 497-512.

[13] Glänzel, W., A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 1987, 75-84.

[14] Glänzel, W., Some consequences of a characterization theorem based on truncated moments, Statistics: A Journal of Theoretical and Applied Statistics, 21(4) (1990), 613-618.

[15] Gleaton, J. U. and Lynch, J. D., Properties of generalized log-logistic families of lifetime distributions, Journal of Probability and Statistical Science, 4(1) (2006), 51-64.

[16] Gleaton, J. U. and Rahman, M. M., Asymptotic properties of MLE's for distributions generated from a 2-parameter Weibull distribution by a generalized log-logistic transformation, Journal of Probability and Statistical Science, 8 (2010), 199-214.

[17] Gleaton, J. U. and Rahman, M. M., Asymptotic properties of MLE's for distributions generated from a 2-parameter inverse Gaussian distribution by a generalized log-logistic transformation, Journal of Probability and Statistical Science, 12 (2014), 85-99.

[18] Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series, and Products, Edited by Alan Je_rey and Daniel Zwillinger, Academic Press, 7th edition, 2007.

[19] Gupta, R. C. and Gupta, R. D., Proportional reversed hazard rate model and its applications, Journal of Statistical Planning and Inference, 137(11) (2007), 3525-3536.

[20] Gupta, R. D. and Kundu, D., Generalized exponential distributions, Australian & New Zealand Journal of Statistics, 41(2) (1999), 173-188.

[21] Hamedani, G. G., On certain generalized gamma convolution distributions II, Technical Report No. 484, 2013, MSCS, Marquette University.

[22] Jafari, A. A., Tahmasebi, S. and Alizadeh, M., The Beta-Gompertz Distribution, Revista Colombiana de Estadística, 37(1) (2014), 139-156.

[23] Johnson, N. L., Kotz, S. and Balakrishnan, N., Continuous Univariate Distributions, volume 2, John Wiley & Sons, New York, second edition, 1995.

[24] Kenney, J. F. and Keeping, E., Mathematics of Statistics, D. Van Nostrand Company, 1962.

[25] Moors, J. J. A., A quantile alternative for kurtosis, Journal of the Royal Statistical Society. Series D (The Statistician), 37(1) (1988), 25-32.

[26] Nadarajah, S. and Kotz, S., The beta Gumbel distribution, Mathematical Problems in Engineering, 2004(4) (2004), 323-332.

[27] Nadarajah, S. and Kotz, S., The beta exponential distribution, Reliability Engineering & System Safety, 91(6) (2006), 689-697.

[28] Ohishi, K., Okamura, H. and Dohi, T., Gompertz software reliability model: estimation algorithm and empirical validation, Journal of Systems and software, 82(3) (2009), 535-543.

[29] Roozegar, R., Tahmasebi, S. and Jafari, A. A., The MCDonald Gompertz distribution: properties and applications, Communication in Statistics: Simulation and Computation, 46(5) (2017), 3341-3355.

[30] Silva, G. O., Ortega, E. M. and Cordeiro, G. M., The beta modified Weibull distribution, Lifetime Data Analysis, 16(3) (2010), 409-430.
[31] Silva, R. C., Sanchez, J. J. D., Lima, _F. P. and Cordeiro, G. M., The Kumaraswamy Gompertz distribution, Journal of Data Science, 13 (2015), 241-260.
[32] Willemse, W. and Koppelaar, H., Knowledge elicitation of gompertz'law of mortality, Scandinavian Actuarial Journal, 2000(2) (2000), 168-179.

9. Appendix A.

THEOREM 1. Let $(\Omega, \mathcal{F}; \boldsymbol{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ ($a = -\infty, b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathrm{E}[q_2(X) \mid X \geq x] = \mathrm{E}[q_1(X) \mid X \geq x]\eta(x); \; x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H), \xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{(u)q_1(u) - q_2(u)} \right| \exp(-s(u))du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel [14]), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and η_n ($n \in \mathbb{N}$) satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1, q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E[q_2(X) \mid X \geq x]}{E[q_1(X) \mid X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and η, respectively. It guarantees, for instance, the `convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions.

For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.