ABSTRACT

Title of dissertation: NEW APPROACHES FOR ANALYZING SYSTEMS WITH HISTORY-DEPENDENT EFFICIENCY

Michael Lin
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Dissertation directed by: Professor Nuno Martins
Professor Richard La
Department of Electrical and Computer Engineering

In my dissertational work, I propose two novel models for analyzing systems in which the operational efficiency depends on the past history, e.g., systems with human-in-the-loop and energy harvesting sensors.

First, I investigate a queuing system with a single server that serves multiple queues with different types of tasks. The server has a state that is affected by the current and past actions. The task completion probability of each kind of task is a function of the server state. A task scheduling policy is specified by a function that determines the probability of assigning a task to the server. The main results with multiple types of tasks include: (i) necessary and sufficient conditions for the existence of a randomized stationary policy that stabilizes the queues; and (ii) the existence of threshold type policies that can stabilize any stabilizable system. For a single type system, I also identify task scheduling policies under which the utilization rate is arbitrarily close to that of an optimal policy that minimizes the utilization

rate. Here, the utilization rate is defined to be the long-term fraction of time the server is required to work.

Second, I study a remote estimation problem over an activity packet drop link. The link undergoes packet drops and has an (activity) state that is influenced by past transmission requests. The packet-drop probability is governed by a given function of the link's state. A scheduler determines the probability of a transmission request regarding the link's state. The main results include: (i) necessary and sufficient conditions for the existence of a randomized stationary policy that stabilizes the estimation error in the second-moment sense; and (ii) the existence of deterministic policies that can stabilize any stabilizable system. The second result implies that it suffices to search for deterministic strategies for stabilizing the estimation error. The search can be further narrowed to threshold policies when the function for the packet-drop probability is non-decreasing.

NEW APPROACHES FOR ANALYZING SYSTEMS WITH HISTORY-DEPENDENT PERFORMANCE

by

Michael Lin

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Advisory Committee:
Professor Richard J. La, Co-Chair/Advisor
Professor Nuno C. Martins, Co-Chair/Advisor
Professor Sennur Ulukus
Professor Steven I. Marcus
Professor Nikhil Chopra

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List of Abbreviations

CMC	Controlled Markov Chain
LTI	Linear Time Invariant
MC	Markov Chain
MDP	Markov Decision Process
PMF	Probability Mass Function
PRCC	Positive Recurrent Communicating Class
WCDD	Weakly Chained Diagonally Dominant
WDD	Weakly Diagonally Dominant
SDD	Strongly Diagonally Dominant

Chapter 1: Introduction

Recent developments in information and communication networks and sensor techniques have made a wide variety of new applications a reality. These include wireless sensor networks in which devices coupled with communication modules that are powered by renewable resources, such as solar and geothermal energy, as well as evolving technologies, such as unmanned aerial vehicles (UAVs). In many cases, the component system performance, as well as the overall system performance, can be analyzed using appropriate models.

Unfortunately, most existing models are insufficient because server efficiency (e.g., a human supervisor monitoring and controlling various UAVs) is not time-invariant in many of these new applications and typically depends on previous workload history or the types of services performed. As a consequence, little is known about the efficiency and stability of such systems, and new methodologies and theories are therefore required.

In this study, we first propose two new frameworks for studying the stability of systems in which the efficiency of servers is time-varying and is dependent on their past utilization. Using the proposed frameworks, we then explore the problems of designing a task scheduling policy with a simple structure, which is optimal in that it keeps the system stable whenever doing so is possible using some policies or for minimizing the fraction of time that the server is working.

1.1 Motivation

Systems with history-dependent performance appear in a wide number of real-world applications. A few interesting examples are provided in the following.

Human operators:

Though modern technological advances automate many tasks which were formerly performed by the human, human operators are still vital for some critical missions such as military operations. With recent psychology research and development in the biological sensing devices, it is possible to model human performance and study task management policies to increase the productivity of overall human-in-the-loop systems.

The notion of mental workload in human-assisted systems generally relates to the burden placed on a human operator by the complexity and frequency of assignment of duties [1]. Therefore, given that it is known to affect a human operator's efficiency, the effect of arousal is essential for the work assignment policy design. Literature exists which studied the performance of human operators for services such as classification [2], supervision [3], or assembly work within the production system [4].

In the psychology community, the well-known Yerkes–Dodson law [5, 6] describes the relationship between the performance of human and mental arousal. The

Law states that the performance increases with psychological arousal to a point and decreases after that. Moreover, recent studies [7] found that the inverted U shape curve only applied to complex tasks such as decision-making and multitasking. High arousal does not impair the performance of simple tasks such as tasks that only require flashbulb memory. Therefore, it is essential for task management policies for human operators to consider these factors.

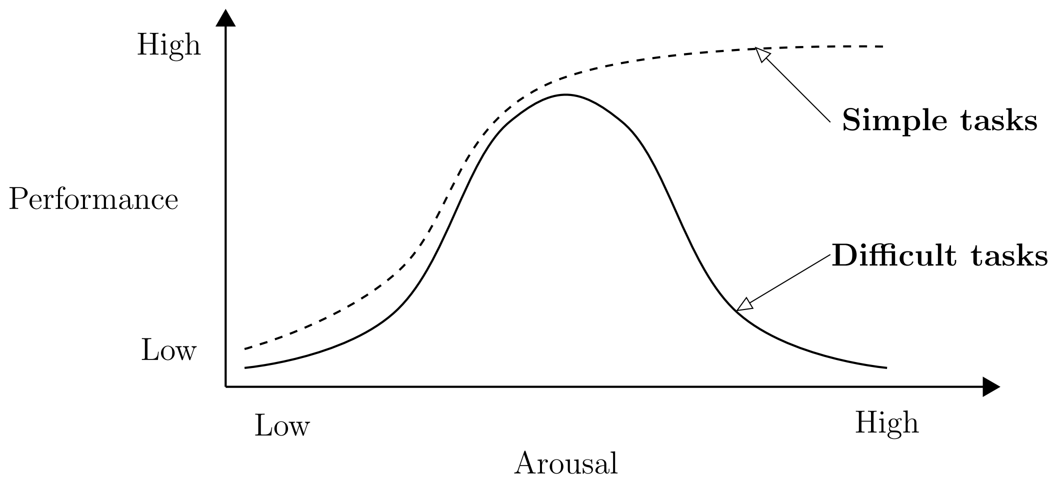

Figure 1.1: Illustration of relationship between performance and arousal

Energy Harvesting devices:

Energy harvesting technologies provide devices with the ability to obtain energy from the surrounding environments and the potential to operate indefinitely without human interactions [8]. The capabilities are particularly useful for the devices within the human body [9] or at remote places since the cost of replacing batteries is significant. In the recent IoT trend of attaching smart sensors to every household item, the self-sustain sensors are also preferable since maintaining the massive amount of sensors might be bothersome for the customers.

There exist several techniques that acquire energy from different sources, such as solar and thermal [10]. However, most of them have one universal challenge: the randomness of available power and the limited volume of rechargeable batteries. Besides the randomness of power, the performance of a device might be affected by the battery state of charge [11,12], which is subject to the past working and energy harvesting history. For example, the typical discharge curve of a battery specifies how the power it can deliver decreases as it loses charge. Therefore, it is essential to manage when and how the devices use the energy to achieve the desired objectives.

Other examples exist, such as systems with thermal dependent performance, and past working history governs the temperature [13–16].

An important observation is that these applications have similar structures. First, an activity state summarizes the historical events such as arousal, battery level, or temperature for human operators, energy harvesting devices, or thermal dependent devices, respectively. Second, the instantaneous performance of the server is affected by the activity state. Inspired by the common structures, we propose two novel mathematic models for analyzing systems in which the operational performance depends on history. A queuing framework is proposed for the systems where the number of completed tasks is critical, while an estimation model focus on estimation error, and the number of transferred packets is not the main focus. We also investigate the formats of optimal task management policies for various objectives, such as stability and minimization of average working time.

1.2 Thesis Outline

In Chapter 2, we survey and present related literature for our two frameworks, the queuing model and the remote estimation model, respectively.

In Chapter 3, we introduce the queuing server model subject to activity performance. The model consists of three components: multiple first-in-first-out queues, a server, and a scheduler. We model the evolution of the whole system state, server states, and queues lengths as a Controlled Markov Chain (CMC). We also introduce an auxiliary CMC with finite state space that describes the evolution of server states with infinite queue lengths. The auxiliary CMC shares some stochastic properties with the original system. These properties simplified our analysis from the complex infinite states system CMC to the simple finite states auxiliary CMC in the following chapters.

In Chapter 4, we present our main results when there is only one type of task and one queue. We address two main challenges for designing policies, stability and utilization rate. We first show that there exists an upper-bound on the arrival rate such that the system can still be stable, and the value can be computed efficiently. Moreover, we prove the existence of a threshold policy that keeps the queue stable whenever doing so is possible using some scheduler. For the utilization rate minimization problem, we find the minimum utilization rate given an arrival rate and propose policies that achieve near-optimal utilization rate while keeping the queue stable.

In Chapter 5, we relax the single type of task constraint and study the stability problem for general m types of tasks. The region for stabilizable arrival rates is identified. We proposed a policy that stabilizes the queues whenever the arrival rates are in the interior of the stability region. Furthermore, the computation of the policy does not rely on the knowledge of arrival rates. It only depends on m threshold values, while each of the threshold values can be computed by considering a single type system and applying the analysis in the previous chapter.

In Chapter 6, we model and study the usage dependent efficiency effect for remote estimation. A single agent setup is presented where past transmission history affects the link quality. Similar to the previous chapter where we study when the server should work, the question that we ask here is, "When should the agent transmits?". This problem leads us to some structural insights for scheduling policies that guarantee the stability for estimation.

Chapter 7 concludes the thesis and outlines future research problems.

Chapter 2: Literature review

In this chapter, we survey and present the related literature for our two frameworks, the queuing model and the remote estimation model, respectively.

2.1 Literature Review for The Queuing Model

Queuing systems with time-varying performance have been of great interest to the decision and control community. Most of the early literature considered systems whose parameters, such as service rates or arrival rates, are dependent on queue lengths [17–19], where [20] is a comprehensive survey. Some of the researches were motivated by the observation that some real-world production systems tend to raise the service rates or slow down the new tasks injection when the number of awaiting works increases. Recent psychological studies also suggested that the queue length influences the performance of a human [21], [22]. If a unimodal function governs the relation of the service rate and queue length for a single server, Bekker and Borst [23] showed that threshold policies on the queue length are optimal in terms of maximizing throughput.

Another line of research considered time-varying behavior as an intrinsic property of the server. For instance, with the fast development of wireless networks, there has been a great interest in understanding and developing effective scheduling strategies with time-varying channel conditions affecting the likelihood of successful transmissions. Various formulations were investigated for designing sound scheduling polices. Some have embraced an optimization structure aimed at maximizing the aggregate utility of flows/users (e.g., [24–28]). This strategy enables designers to assess the trade-off between the aggregate throughput, queue length, and fairness between flows/users. Others focused on designing optimal throughput schedulers that can stabilize the system for any arrival rate (vector) that lies in the stability region (e.g., [29–31]), which is more relevant to our study. However, there is a notable distinction between these studies and our model.

In wireless networks, channel conditions and the likelihood of successful transmission/decoding vary independently, regardless of the scheduling decisions chosen by the scheduler. In other words, the scheduler (along with the physical layer systems) is trying to deal with or take advantage of time-varying channel situations that are beyond resource managers' control and are not influenced by policy choices. On the other hand, in our research, the probability of finishing a job within an epoch relies on the past history of scheduling actions. The present scheduling choice, therefore, impacts the server's future performance.

With the recent development of energy harvesting technologies, many articles studied remote transmitters that are powered by renewable energy. Most of them modeled the energy buffer as a server inner state, and the policies determine the amount of energy to use. The number of packets sent only depended on the amount of energy used. Algorithm are proposed to promote maximize throughput or minimize transmission delay in an information-theoretic context, such as in [32–35]. The continuous-time model [36] is similar to the one in [32]. However, the time of energy and packets arrivals is assumed to be known, and the number of packets was finite. They mainly focused on short-term behavior, such as finding the policy to send all packets with minimal time.

In our model, the server only has the option to work or rest. The transmitter always sends the packet successfully in their model, while our structure takes into account the energy consumption for retransmission due to the packets lost. Our framework might be suitable for the lower-level energy harvesting sensors in which the battery state of charge determines the amount of used energy.

Another area that is closely related to our study is task scheduling for human operators/servers. In the past, many studies have been conducted on the efficiency and management of human operators and servers (e.g., bank tellers, toll collectors, physicians, nurses, emergency dispatchers), e.g., [37–39]. Recently, with rapid advances in information and sensor technologies, human supervisory control, which requires processing a large amount of information in a short period, potentially causing information overload, became an active research area [3].

Various frameworks exist for analyzing human-assisted systems and design policies that reduce error rates [40, 41] or minimize processing time [42]. In closely related studies, Savla and Frazzoli [43, 44] investigated the problem of designing a task release control policy. They assumed periodic task arrivals and modeled the dynamics of server utilization, which determines the service time of the server, using a differential equation; the server utilization increases when the server is busy and decreases when it is idle. They showed that, when all tasks bring identical workload, a policy that allows a new task to be released to the server only when its utilization is below a suitably chosen threshold, is maximally stabilizing [44, Theorems III.1 and III.2].

Researchers who studied production systems consider similar issues, where machines can fail with time-varying rates and (preventive) maintenance planning is crucial. The problems of production scheduling and maintenance scheduling are considered separately in more traditional approaches [45–48], and equipment failures are treated as random events which need to be coped with. However, when the probability or rate of machine failure is time-varying and depends on the age since last (preventive) maintenance, the overall efficiency of production can be enhanced by considering both issues together [49–52]. For example, [53] formulated the problem using an MDP model with the state consisting of the age of the system (since the last preventive maintenance) and the queue length, and examined the structural properties of optimum policies.

While most of the previous models consider deterministic arrivals and services, we propose a discrete-time stochastic queuing framework that couples the server performance with the frequency and recency of the working period. This framework allows us to analyze the structural properties of optimal policies without relying on numerical methods and is general enough to capture the behavior of real-world applications.

2.2 Literature Review for The Remote Estimation Model

Several articles have reported research on a wide range of interesting problems of remote estimation and control. The packet drop event was regarded as multiplicative noise by early works, and the minimum packet arrival rates for stable systems were researched [54, 55]. Different formulations with packet-drop links connecting various components, such as from the sensor to the remote estimator [56] or from the controller to the actuator [57], have been suggested. [58] considered the case where both connections are packet-drop links while [59] studied a similar structure and showed that when there is no feedback from the actuator to the controller, the separation principle does not hold. Settings for two or more links connecting components were studied in [60–62].

The seminal work by Gupta et al. [63] studied not only stabilizability but also encoder and controller schemes that are optimal when information on the state of the plant is relayed over a Markovian packet-drop link. The work in [64], [65], [59] has investigated other interesting formulations and [66] is a comprehensive survey.

Another line of research considered the case in which the estimator or some other component of the system has the authority to decide when to request a transmission [67–69] while the communication does not suffer from erasure. The design of strategies for all the components that are jointly optimal with respect to a figure of merit that also includes communication cost was considered in [70, 71]. Other articles designed optimal policies with constraints such as a limited number of transmissions [72, 73].

The communication link considered in this thesis has an activity state that governs the packet-drop probability. In addition, the state reacts to the history of current and past requests according to a controlled Markov chain. Ward and Martins [74] investigated a similar framework, but focused on finite time horizon problem in which stability was not addressed. They also considered that the activity state was governed by a (deterministic) finite state machine.

Recent work focusing on energy harvesting [75] considered models similar to ours. Policies that promote error stability or minimize certain estimation error metrics using event-triggered strategies or transmission power management are reported in [76], [77] and [78]. Interesting formulations have been proposed, such as multiple energy harvesting sensors measuring the same LTI plant [79] or imperfect feedback from estimators to sensors [80, 81]. Event-triggered policies to ensure mean-squared stability were investigated in Tallapragada et al. [82].

In contrast with existing work, our framework couples transmission performance with the frequency and recency of requests, while stability of the estimation error is assessed in the second-moment sense, which is often relevant for real-time decision and control problems. In addition, we provide tight necessary and sufficient conditions for stabilizability that are also constructive in that they specify the structure of a stabilizing policy when one exists.

Chapter 3: Queuing Servers Subject to Activity Server Performance

In this chapter, we introduce our queuing server model. The model consists of multiple first-in-first-out queues that store incoming tasks. A non-preemptive server with history-dependent performance processes the assigned tasks. A scheduler assigns tasks to the server base on system states. We also design an auxiliary process that shares some stochastic properties with the queuing server model. This supplemental process greatly simplified our analysis from infinite state space to finite state space and is notably helpful for proving our Theorems in the later Chapters.

The chapter is organized as follows. We begin by presenting an overview of each component in the model. Then, we state the main challenges addressed in this thesis. A stochastic discrete-time model is described in Section 3.3. In it, we also introduce notation, key concepts, and propose a Controlled Markov Chain (CMC) framework that is amenable to performance analysis and optimization. In Section 3.4, we design an auxiliary CMC based on the original framework and prove that they have identical marginal stationary Probability Mass Functions (PMFs) under some conditions.

3.1 Introduction

We consider a queuing system comprising the following three components:

- m first-in first-out unbounded **queues** store m types of tasks. Queue i registers a new type i task when it arrives and removes it as soon as work on it is completed. Each queue has an internal state, its queue size, which indicates the number of uncompleted tasks in the queue.
- The **server** performs the work required by each task assigned to it. It has an internal state with two components. The first is the *availability* state, which indicates whether the server is available to start a new task or is busy on a type of task. We assume that the server is non-preemptive, which in our context means that the server gets busy when it starts work on a new task, and it becomes available again only after the task is completed. The second component of the state is termed *activity* and takes values in a finite set of positive integers, which represent parameters that affect the performance of the server. More specifically, we assume that the activity state and the type of task determine the probability that, within a given time-period, the server can complete a task of that type. Hence, a decrease in performance causes an increase in the expected time needed to service a task. Such an activity state could, for instance, represent the battery charge level of an energy harvesting module that powers the server or the status of arousal or fatigue of a human operator that assists the server or supervises the work.
- The **scheduler** has access to the queues sizes and the entire state of the server.

When the server is available and the queues are not empty, the scheduler decides whether to assign a new task from a non-empty queue or to allow for a rest period. Our formulation admits non-work-conserving policies whereby the scheduler may choose to assign rest periods even when the queues are not empty. This allows the server to rest as a way to steer the activity state towards a range that can deliver better long-term performance. A controlled Markov chain, denoted as CMC S, models how the history of actions governs the probabilistic evolution of activity state.

We adopt a stochastic discrete-time framework in which time is uniformly partitioned into epochs, within which new tasks arrive according to a Bernoulli process. The probability of arrival per epoch is termed arrival rate[1]. We constrain our analysis to *stationary schedulers* characterized by policies that are invariant under epoch shifts. We discuss our assumptions and provide a detailed description of our framework in Section 3.3.

3.2 Main problems

The following are the main challenges address in the queuing server model:

3.2.1 Stability

Problem 3.1. *A set of arrival rates for each type of tasks is qualified as stabilizable when there is a scheduler that stabilizes the queues. Given a server, we seek to*

[1]Notice that, unlike the nomenclature we adopt here, *arrival rate* is commonly used in the context of Poisson arrival processes. This distinction is explained in detail in Section 3.3.2.1.

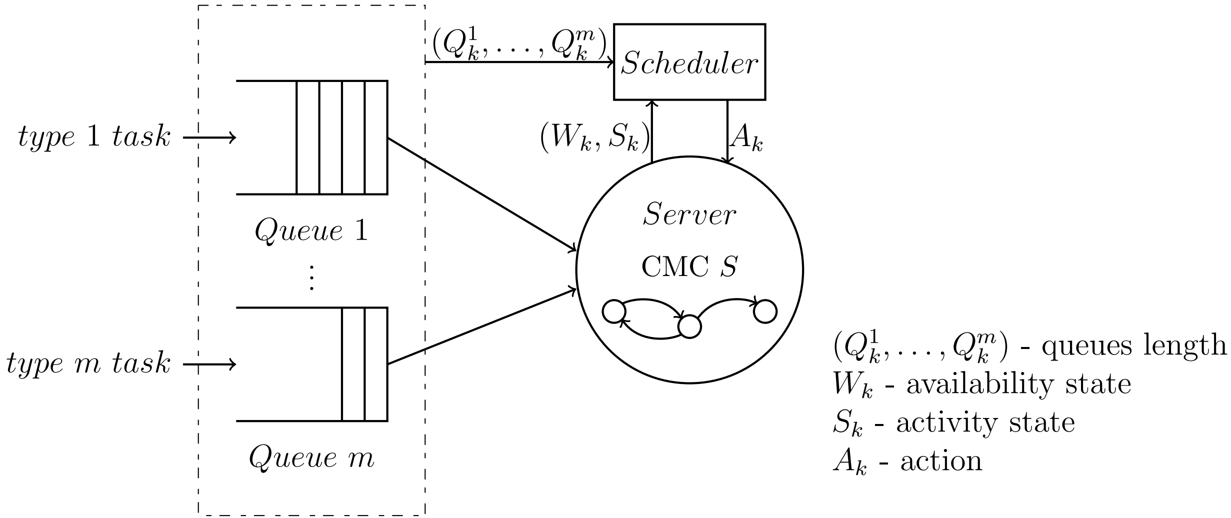

Figure 3.1: Basic queuing server architecture.

identified all stabilizable sets of arrival rates.

Problem 3.2. *We seek to propose schedulers that have a simple structure and are guaranteed to stabilize the queues for any stabilizable set of arrival rates.*

Notice that, as alluded to above in the scheduler description, we allow non-work-conserving policies. This means that, in addressing Problem 3.1, we must allow policies that are a function not only of the queue size, but also of the activity and availability states of the server. The design process for good policies is complicated by the fact that they are a function of these states with intricate dependence, illustrating the importance of addressing Problem 3.2.

3.2.2 Utilization Rate

Problem 3.3. *Given a server and a stabilizable arrival rate, determine a tractable method to compute the infimum of all utilization rates achievable by a stabilizing scheduling policy. Such a fundamental limit is important to determine how effective* *any given policy is in terms of the utilization rate.*

Problem 3.4. *Given a server and a stabilizable arrival rate, determine a tractable method to design policies whose utilization rate is arbitrarily close to the fundamental limit.*

3.3 Stochastic Discrete-Time Framework

In the following section, we describe a discrete-time framework that follows from assumptions on when the states of the queue and the server are updated and how actions are decided. In doing so, we will also introduce the notation used to represent these discrete-time processes. A probabilistic description that leads to a tractable CMC formulation is deferred to Section 3.3.2.

3.3.1 Timing and Notation

We consider an infinite horizon problem in which the *physical* (continuous) time set is $\mathbb{R}_+$, which we partition uniformly into half-open intervals of positive duration Δ as follows:

$$\mathbb{R}_+ = \cup_{k=0}^{\infty} \left[\ k\Delta, (k+1)\Delta\ \right)$$

Each interval is called an *epoch*, and epoch k refers to $[k\Delta, (k+1)\Delta)$. Our formulation and results are valid regardless of the epoch duration Δ. We reserve t to denote continuous time, and k is the discrete-time index we use to represent epochs.

Each epoch is subdivided into three half-open subintervals denoted by stages $S1$, $S2$ and $S3$ (see Fig. 3.2). As we explain below, stages $S1$ and $S2$ are allocated

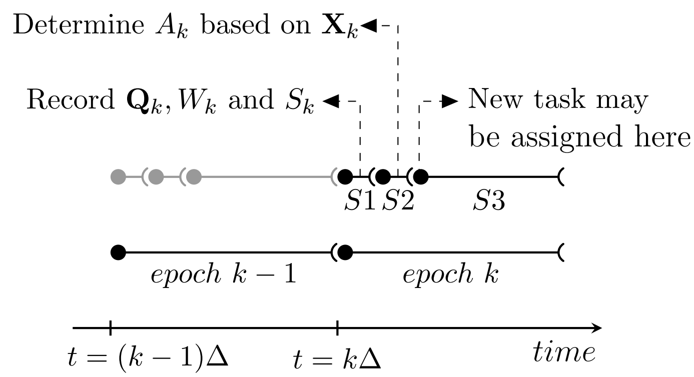

Figure 3.2: Illustration of time uniformly divided into epochs and when updates and actions are taken. (Assuming $k \geq 1$)

for basic operations of record keeping, updates and scheduling decisions. Although, in practice, the duration of these stages is a negligible fraction of Δ, we discuss them here in detail to clarify the causality relationships among states and actions. We also introduce notation used to describe certain key discrete-time processes that are indexed with respect to epoch number.

Our model allows m different types of tasks for any positive m. In addition, even though more general arrival processes can be handled, for simplicity of exposition, we assume that new type i task arrivals occur in accordance with a Bernoulli process, and the all Bernoulli processes are assumed independent. Thus, at most one new task arrives at queue i during each epoch. We denote the number of type i tasks that arrive during epoch k by B_k^i, which takes values in $\{0, 1\}$.

Furthermore, during each epoch, the scheduler assigns at most one task to the server, according to an employed scheduling policy and, hence, the server either works on a single task or remains idle at any given time. We denote the number of type i tasks that the server completes during epoch k by D_k^i, which takes values in $\{0, 1\}$.

Stage $S1$:

The following updates take place during stage $S1$ of epoch $k+1$:

For each $i \in \mathbb{T}$, the number of uncompleted type i tasks at time $t = k\Delta$ is denoted by Q_k^i. We refer to Q_k^i as the length or size of queue i at epoch k, and it is updated according to a Lindley's equation [83]:

$$Q_{k+1}^i = \max\left\{0, Q_k^i + B_k^i - D_k^i\right\}, \ k \in \mathbb{N} \tag{3.1}$$

The *availability state of the server* at time $t = k\Delta$ is denoted by W_k and takes values in

$$\mathbb{W} \stackrel{def}{=} \{\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m\}.$$

We use $W_k = \mathcal{B}_i$ to indicate that the server is busy working on a type i task at time $t = k\Delta$. If it is available at time $t = k\Delta$, then $W_k = \mathcal{A}$. The update mechanism for W_k is as follows:

- If $W_k = \mathcal{A}$, then $W_{k+1} = \mathcal{A}$ when either no new task was assigned during epoch k, or a new task was assigned and completed during epoch k. If $W_k = \mathcal{A}$ and a new type i task is assigned during epoch k which is not completed until $t = (k+1)\Delta$, then $W_{k+1} = \mathcal{B}_i$.

- If $W_k = \mathcal{B}_i$ and the server completes the task by time $t = (k+1)\Delta$, then $W_{k+1} = \mathcal{A}$. Otherwise, $W_{k+1} = \mathcal{B}_i$.

We use S_k to denote the *activity state* at time $t = k\Delta$, and we assume that it takes values in

$$\mathbb{S} \stackrel{def}{=} \{1, \ldots, n_s\}.$$

The activity state is non-decreasing while the server is working and is non-increasing when it is idle. In Section 3.3.2, we describe an CMC that specifies probabilistically how S_k transitions to S_{k+1}, conditioned on whether the server worked or rested during epoch k.

Without loss of generality, we assume that Q_k^i, W_k and S_k are initialized as follows:

$$Q_0^i = 0, \quad W_0 = \mathcal{A}, \quad S_0 = 1, \quad \forall i \in \mathbb{T}.$$

The overall state of the server is represented compactly by $\mathbf{Y}_k$, which takes values in $\mathbb{Y}$, defined as follows:

$$\mathbf{Y}_k \stackrel{\text{def}}{=} (S_k, W_k), \qquad \mathbb{Y} \stackrel{\text{def}}{=} \mathbb{S} \times \mathbb{W}.$$

In a like manner, we define the overall state the overall state of the queues as follow $\mathbf{Q}_k$, which takes values in $\mathbb{N}^m$, defined as follows:

$$\mathbf{Q}_k \stackrel{\text{def}}{=} (Q_k^1, \ldots, Q_k^m).$$

Finally, we define the overall state for the CMC taking values in $\mathbb{X}$ as follows:

$$\mathbf{X}_k \stackrel{\text{def}}{=} (\mathbf{Y}_k, \mathbf{Q}_k), \qquad \mathbb{X} \stackrel{\text{def}}{=} \mathbb{S} \times \Big((\mathbb{W} \times \mathbb{N}^m) \setminus \cup_{i=1}^m \left(\mathcal{B}_i \times \mathbb{N}^{(i-1)} \times 0 \times \mathbb{N}^{(m-i)}\right) \Big).$$

From the definition of $\mathbb{X}$, it follows that when the queue i is empty, there is no type i task for the server to work on and, hence, it cannot be busy on type i task.

Stage $S2$:

It is during stage $S2$ of epoch k that the scheduler issues a decision based on $\mathbf{X}_k$: let $\mathbb{A} \stackrel{\text{def}}{=} \{\mathcal{R}, \mathcal{W}_1, \ldots, \mathcal{W}_m\}$ represent the set of possible actions that the scheduler can request from the server, where $\mathcal{R}$ and $\mathcal{W}_i$ represent 'rest' and 'work on a type i task', respectively. The assumption that the server is non-preemptive and the fact that no new tasks can be assigned when the queue is empty, lead to the following set of available actions for each possible state $\mathbf{x} = (s, w, q_1, \ldots, q_m)$ in $\mathbb{X}$:

$$\mathbb{A}_{\mathbf{x}} = \begin{cases} \{\mathcal{R}, \cup_{i:q_i \neq 0} \mathcal{W}_i\} & \text{if } w = \mathcal{A}, \quad \text{(cannot assign task from empty queues)} \\ \{\mathcal{W}_i\} & \text{if } w = \mathcal{B}_i. \quad \text{(non-preemptive server)} \end{cases} \tag{3.2}$$

We denote the action chosen by the adopted policy at epoch k by A_k, which takes values in $\mathbb{A}_{\mathbf{X}_k}$.As we discuss in Section 3.3.3, we focus on the design of stationary policies that determine A_k as a function of $\mathbf{X}_k$.

Stage $S3$:

A task can arrive at any time during each epoch, but we assume that work on a new task can be assigned to the server only at the beginning of stage $S3$. More specifically, the scheduler acts as follows:

- If $W_k = \mathcal{A}$ and $A_k = \mathcal{W}_i$, then the server starts working on a new type i task at the head of the queue i when stage $S3$ of epoch k begins.

- When $W_k = \mathcal{A}$, the scheduler can also select $A_k = \mathcal{R}$ to signal that no work will be performed by the server during the remainder of epoch. Once this 'rest' decision is made, a new task can be assigned no earlier than the beginning of stage $S3$ of epoch $k + 1$. Since the scheduler is non-work-conserving, it may decide to assign such 'rest' periods as a way to possibly reduce S_{k+1} and to improve future performance.

- If $W_k = \mathcal{B}_i$, the server was still working on a task at time $t = k\Delta$. In this case, because the server is non-preemptive, the scheduler picks $A_k = \mathcal{W}_i$ to indicate that work on the current task is ongoing and must continue until it is completed and no new task can be assigned during epoch k.

3.3.2 Probabilistic Model

Based on the formulation outlined in Section 3.3.1, we proceed to describe a discrete-time CMC that models how the states of the server and queue evolve over time for any given scheduling policy.

3.3.2.1 Arrival Process

We assume that tasks arrive during each epoch according to a set of i.i.d. Bernoulli process $\{B_k^i;\ k \in \mathbb{N}\ i \in \mathbb{T}\}$. The probability of a type i arrival for each epoch ($\Pr(B_k^i = 1)$)[2] is called the *arrival rate* and is denoted by λ_i for all i from 1 to m, which is assumed to belong to $(0, 1)$. Although we assume Bernoulli arrivals to simplify our analysis and discussion, more general arrival distributions (e.g., Poisson distributions) can be handled only with minor changes as it will be clear.

Notice that, as we discuss in Remark 1 below, our nomenclature for λ_i should not be confused with the standard definition of arrival rate for Poisson arrivals. Since our results are valid irrespective of Δ, including when it is arbitrarily small, the remark also gives a sound justification for our adoption of the Bernoulli arrival model by viewing it as a *discrete-time approximation* of the widely used Poisson arrival model.

Remark 1. *It is a well-known fact that, as Δ tends to zero, a Poisson process in continuous time t, with arrival rate $\tilde{\lambda}$, is arbitrarily well approximated by $B_{\lfloor t/\Delta \rfloor}$ with $\lambda = \Delta\tilde{\lambda}$.*

3.3.2.2 Activity Server Performance

In our formulation, the efficiency or performance of the server during an epoch is modeled with the help of a *service rate* function $\mu : \mathbb{S} \times \mathbb{T} \to (0, 1)$. More specifically, if the server works on a type i task during epoch k, the probability that it completes the task by the end of the epoch is $\mu(S_k, i)$. This holds irrespective of whether the task is newly assigned or inherited as ongoing work from a previous epoch.[3] Thus, the service rate function μ quantifies the effect of the activity state on

[2] Note that we use Pr for probability and $\mathbb{E}$ for expected value in this thesis.

[3] This assumption is introduced to simplify the exposition. However, more general scenarios in which the probability of task completion within an epoch depends on the total service received by

the performance of the server. **The results presented throughout this section are valid for *any* choice of μ with codomain** $(0,1)^m$.

3.3.2.3 Dynamics of the Activity State

We assume that (i) S_{k+1} is equal to either S_k or $S_k + 1$ when A_k is in $\{\mathcal{W}_i, \ldots, \mathcal{W}_m\}$ and (ii) S_{k+1} is either S_k or $S_k - 1$ if A_k is $\mathcal{R}$. This is modeled by the following transition probabilities specified for every s and s' in $\mathbb{S}$.

$$\mathcal{P}_{S_{k+1}|S_k,A_k}(s' \mid s, \mathcal{W}_i) = \begin{cases} \rho^i_{s,s+1} & \text{if } s < n_s \text{ and } s' = s+1, \\ 1 - \rho^i_{s,s+1} & \text{if } s < n_s \text{ and } s' = s, \\ 1 & \text{if } s = n_s \text{ and } s' = n_s, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3a}$$

$$\mathcal{P}_{S_{k+1}|S_k,A_k}(s' \mid s, \mathcal{R}) = \begin{cases} \rho^0_{s,s-1} & \text{if } s > 1 \text{ and } s' = s-1, \\ 1 - \rho^0_{s,s-1} & \text{if } s > 1 \text{ and } s' = s, \\ 1 & \text{if } s = 1 \text{ and } s' = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3b}$$

where the parameters $\rho^i_{s,s'}$, which take values in $(0,1)$, model the likelihood that the activity state will transition to a greater or lesser value, depending on the action.

the task prior to epoch k can be handled by extending the state space and explicitly modeling the total service received by the task in service.

3.3.2.4 Transition probabilities for $\mathbf{X}_k$

We assume that S_{k+1} is independent of $(W_{k+1}, \mathbf{Q}_{k+1})$ when conditioned on $(\mathbf{X}_k, A_k)$. Under this assumption, the transition probabilities for $\mathbf{X}_k$ can be written as follows:

$$\begin{aligned}\mathcal{P}_{\mathbf{X}_{k+1}|\mathbf{X}_k,A_k}(\mathbf{x}' \mid \mathbf{x}, a) &= \mathcal{P}_{S_{k+1}|\mathbf{X}_k,A_k}(s' \mid \mathbf{x}, a) \times \mathcal{P}_{W_{k+1},\mathbf{Q}_{k+1}|\mathbf{X}_k,A_k}(w', \mathbf{q}' \mid \mathbf{x}, a) \\ &= \mathcal{P}_{S_{k+1}|S_k,A_k}(s' \mid s, a) \times \mathcal{P}_{W_{k+1},\mathbf{Q}_{k+1}|\mathbf{X}_k,A_k}(w', \mathbf{q}' \mid \mathbf{x}, a) \quad (3.4)\end{aligned}$$

for every $\mathbf{x}$, $\mathbf{x}'$ in $\mathbb{X}$ and a in $\mathbb{A}_{\mathbf{x}}$.

We assume that, within each epoch k, the events that (a) there is a new task arrival during the epoch and (b) a task being serviced during the epoch is completed by the end of the epoch are independent when conditioned on $\mathbf{X}_k$ and $\{A_k = \mathcal{W}_i\}$. Hence, the transition probability $\mathcal{P}_{W_{k+1},\mathbf{Q}_{k+1}|\mathbf{X}_k,A_k}$ in (3.4) is given by the following:

$$\mathcal{P}_{W_{k+1},\mathbf{Q}_{k+1}|\mathbf{X}_k,A_k}(w', \mathbf{q}' \mid \mathbf{x}, \mathcal{W}_i) \quad (3.5a)$$

$$= \begin{cases} \mu(s_k, i) \Pr(\mathbf{B}_k = \mathbf{q}' - \mathbf{q} - \mathbf{e}_i) & \text{if } w' = \mathcal{A}, \\ \big(1 - \mu(s_k, i)\big) \Pr(\mathbf{B}_k = \mathbf{q}' - \mathbf{q}) & \text{if } w' = \mathcal{B}_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{P}_{W_{k+1},Q_{k+1}|\mathbf{X}_k,A_k}(w', q' \mid \mathbf{x}, \mathcal{R}) \quad (3.5b)$$

$$= \begin{cases} \Pr(\mathbf{B}_k = \mathbf{q}' - \mathbf{q}) & \text{if } w' = \mathcal{A}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{B}_k \overset{def}{=} (B_k^1, \ldots, B_K^m)$ is the vector of Bernoulli random variables that determine arrival of new tasks, and $\mathbf{e}_i \in \mathbb{N}^m$ has all elements equal zero except for the ith element which equals one.

Definition 3.1. *(***CMC X***) The CMC with input A_k and state $\mathbf{X}_k$, which at this point is completely defined, is denoted by* $\mathbf{X}$.

Table 3.1 summarizes the notation for CMC $\mathbf{X}$.

$\mathbb{T}$	set of task types $\{1, \ldots, m\}$
$\mathbb{S}$	set of activity states $\{1, \ldots, n_s\}$
$\mathbb{W} \overset{\text{def}}{=} \{\mathcal{A}, \mathcal{B}_i, \ldots, \mathcal{B}_m\}$	server availability ($\mathcal{A}$ = available, $\mathcal{B}_i$ = busy on a type i task)
W_k	server availability at epoch k (takes values in $\mathbb{W}$)
$\mathbb{Y}$	server state components $\mathbb{S} \times \mathbb{W}$
$\mathbf{Y}_k \overset{\text{def}}{=} (S_k, W_k)$	server state at epoch k (takes values in $\mathbb{Y}$)
$\mathbb{N}$	natural number system $\{0, 1, 2, \ldots\}$.
$\mathbf{Q}_k \overset{def}{=} (Q_k^1, \ldots, Q_k^m)$	queue sizes at epoch k (takes values in $\mathbb{N} * m$)
$\mathbb{X}$	state space formed by $\mathbb{S} \times \Big((\mathbb{W} \times \mathbb{N}) \diagdown \cup_{i=1}^m (\mathcal{B}_i \times \mathbb{N}^{(i-1)} \times 0 \times \mathbb{N}^{(m-i)})\Big)$
$\mathbf{X}_k \overset{\text{def}}{=} (\mathbf{Y}_k, \mathbf{Q}_k)$	system state at epoch k (takes values in $\mathbb{X}$)
$\mathbb{A} \overset{\text{def}}{=} \{\mathcal{R}, \mathcal{W}_1, \ldots, \mathcal{W}_m\}$	possible actions ($\mathcal{R}$ = rest, $\mathcal{W}_i$ = work on a type i task)
$\mathbf{X}$	CMC whose state is $\mathbf{X}_k$ at epoch $k \in \mathbb{N}$
$\mathbb{A}_{\mathbf{x}}$	set of actions available at a given state $\mathbf{x}$ in $\mathbb{X}$
A_k	action chosen at epoch k.
PMF	probability mass function

Table 3.1: A summary of notation describing CMC $\mathbf{X}$.

3.3.3 Evolution of the system state under a stationary policy

We start by defining the class of policies that we consider throughout the section.

Definition 3.2. *A stationary randomized policy is specified by a mapping* $\theta : \mathbb{X} \to [0,1]^m$ *that determines the probability that the server is assigned to work on a specific type of task or rest, as a function of the system state, according to*

$$\mathcal{P}_{A_k|\mathbf{X}_k,\ldots,\mathbf{X}_0}(\mathcal{W}_i|x_k,\ldots,x_0) = \theta(x_k)_i, \text{ for all } i \in \mathbb{T} \text{ and,}$$

$$\mathcal{P}_{A_k|\mathbf{X}_k,\ldots,\mathbf{X}_0}(\mathcal{R}|x_k,\ldots,x_0) = 1 - \sum_{i\in\mathbb{T}} \theta(x_k)_i.$$

Definition 3.3. *The set of admissible stationary randomized policies satisfying (3.2) is denoted by* Θ_R.

Convention We adopt the convention that, unless stated otherwise, a set of positive arrival rates $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_m\}$ is pre-selected and fixed. Although the statistical properties of $\mathbf{X}$ and associated quantities subject to a given policy depend on $\boldsymbol{\lambda}$, we simplify our notation by not labeling them with $\boldsymbol{\lambda}$.

From (3.4) - (3.5b), we conclude that $\mathbf{X}$ subject to a policy θ in Θ_R evolves according to a time-homogeneous Markov chain (MC), which we denote by $\mathbf{X}^\theta = \{\mathbf{X}_k^\theta;\ k \in \mathbb{N}\}$. Also, provided that it is clear from the context, we refer to $\mathbf{X}^\theta$ as *the system*.

The following is the notion of system stability we adopt in our study.

Definition 3.4 (System stability)**.** *For a given policy* θ *in* Θ_R*, the system* $\mathbf{X}^\theta$ *is stable if it satisfies the following properties:*

i. There exists at least one recurrent communicating class.

ii. All recurrent communicating classes are positive recurrent.

iii. The number of transient states is finite.

We find it convenient to define $\Theta_S(\boldsymbol{\lambda})$ to be the set of randomized policies in Θ_R, which stabilize the system for the fixed set of arrival rates $\boldsymbol{\lambda}$.

Before we proceed, let us point out a useful fact under any stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$.

Lemma 3.1. *A stable system $\mathbf{X}^\theta$ has a unique positive recurrent communicating class (PRCC), which is aperiodic. Therefore, there is a unique stationary probability mass function (PMF) for $\mathbf{X}^\theta$.*

Proof. We will prove the claim by contradiction. The decomposition theorem of MCs tells us that $\mathbb{X}$ can be partitioned into a set consisting of transient states and a collection of *irreducible*, *closed* recurrent communicating classes $\{\mathbb{C}^1, \mathbb{C}^2, \ldots\}$ [84]. Since $\mathbf{X}^\theta$ is assumed stable, all recurrent communicating classes $\mathbb{C}^\ell$, $\ell = 1, 2, \ldots$, are positive recurrent. Suppose that the claim is false and there is more than one positive recurrent communicating class. We demonstrate that this leads to a contradiction.

First, we show that each set $\mathbb{C}^\ell$, $\ell = 1, 2, \ldots$ contains states with busy working on type i task for $i \in \mathbb{T}$. More specifically, $(s_\ell^i, \mathcal{B}_i, \mathbf{q}_\ell^i) \subset \mathbb{C}^\ell$ for $i \in \mathbb{T}$ and $\ell = 1, 2, \ldots$, where each $s_\ell^i \in \mathbb{S}$, and $\mathbf{q}_\ell^i > (0, \ldots, 0)$ element-wise. If this is not true, the policy θ never choose to work on a type i' task for every state in $\mathbb{C}^\ell$ with the form $(s, \mathcal{A}, \mathbf{q})$ for an $i' \in \mathbb{T}$ because $\mathbb{C}^\ell$ is closed [84]. But, this implies that, starting with any state in $\mathbb{C}^\ell$, the scheduler will never assign a type i' task to the server and, consequently, all states in $\mathbb{C}^\ell$ must be transient, which contradicts that $\mathbb{C}^\ell$ is positive recurrent.

For the same reason, each $\mathbb{C}_\ell$ must include a state $\tilde{\mathbf{x}}_\ell = (\tilde{s}_\ell, \mathcal{A}, \tilde{q}_\ell)$ with $\theta(\tilde{\mathbf{x}}_\ell)_i > 0$ for an $i \in \mathbb{T}$, which implies that $\mathbb{C}_\ell$ is aperiodic.

Second, if some state $(s, \mathcal{B}_i, \mathbf{q})$ is in $\mathbb{C}^\ell$, $\ell = 1, 2, \ldots$, then so are all the states $(s', w, \mathbf{q}')$ for all $s' \geq s$, w in $\{\mathcal{B}_i, \mathcal{A}\}$, and $\mathbf{q}' \geq \mathbf{q}$ element-wise: the fact that $(s, \mathcal{B}_i, \mathbf{q})$ communicates with $(s', w, \mathbf{q})$, $s' \geq s$ and w in $\{\mathcal{B}_i, \mathcal{A}\}$, which means that these states belong to $\mathbb{C}^\ell$ as well, is obvious. In addition, it is evident that $(s', \mathcal{B}_i, \mathbf{q})$ communicates with $(s', \mathcal{B}_i, \mathbf{q}')$ for all $\mathbf{q}' \geq \mathbf{q}$. In order to see why $(s', \mathcal{A}, \mathbf{q}')$, $s' \geq s$ and $\mathbf{q}' \geq \mathbf{q}$, also lie in $\mathbb{C}^\ell$, consider the following two cases: if $\theta(s', \mathcal{A}, \mathbf{q})_i = 0$ for all $i \in \mathbb{T}$, then clearly $(s', \mathcal{A}, \mathbf{q})$ communicates with $(s', \mathcal{A}, \mathbf{q} + \mathbf{e}_i)$ for all $i\mathbb{T}$ where $\mathbf{e}_i \in \mathbb{N}^m$ has all elements equals zero except for the ith element which equals one. On the other hand, if $\theta(s', \mathcal{A}, \mathbf{q})_i > 0$ for an $i \in \mathbb{T}$, $(s', \mathcal{A}, \mathbf{q})$ communicates with $(s', \mathcal{B}_i, \mathbf{q} + \mathbf{e}_{i'})$ for all $i' \in \mathbb{T}$, which in turn communicates with $(s', \mathcal{A}, \mathbf{q} + \mathbf{e}_{i'})$. The claim now follows by induction.

Note that, the above two observations together imply that there exists finite $\mathbf{q}^* \stackrel{def}{=} \max\{\mathbf{q}^1, \mathbf{q}^2\}$ (element-wise maximum) such that all states $(n_s, w, \mathbf{q})$, w in $\mathbb{W}$ and $\mathbf{q} \geq \mathbf{q}^*$ element-wise, belong to both $\mathbb{C}^1$ and $\mathbb{C}^2$. This, however, contradicts the earlier assumption that $\mathbb{C}^1$ and $\mathbb{C}^2$ are *disjoint* recurrent communicating classes. $\square$

Definition 3.5. *Given a set of arrival rates* $\boldsymbol{\lambda} > 0$ *and a stabilizing policy* θ *in* $\Theta_S(\boldsymbol{\lambda})$*, we denote the unique stationary PMF and positive recurrent communicating class of* $\mathbf{X}^\theta$ *by* $\boldsymbol{\pi}^\theta = (\pi^\theta(\mathbf{x});\ \mathbf{x} \in \mathbb{X})$ *and* $\mathbb{C}_\theta$*, respectively.*

3.4 The Server State Process: An Auxiliary CMC $\overline{\mathbf{Y}}$

In this section, we describe an *auxiliary* CMC whose state takes values in $\mathbb{Y}$ and is obtained from $\mathbf{X}$ by artificially removing the queue-length component. We shall show that this auxiliary CMC share some statistical properties with CMC $\mathbf{X}$. Thus, we can analyze the finite state space auxiliary CMC instead of the infinite state space CMC $\mathbf{X}$. We denote this auxiliary CMC by $\overline{\mathbf{Y}}$ and its state at epoch k by $\overline{\mathbf{Y}}_k = (\overline{S}_k, \overline{W}_k)$ in order to emphasize that it takes values in $\mathbb{Y}$. The action chosen at epoch k is denoted by $\overline{A}_k$. We use the overline to denote the auxiliary CMC and any other variables associated with it, in order to distinguish them from those of the server state in $\mathbf{X}$.

As it will be clear, we can view $\overline{\mathbf{Y}}$ as the server state of the original CMC $\mathbf{X}$ for which infinitely many tasks are waiting in the queues at the beginning, i.e., $Q_i = \infty$ for all i from 1 to m. As a result, there is always a task waiting for service when the server becomes available.

The reason for introducing $\overline{\mathbf{Y}}$ is the following: (i) $\mathbb{Y}$ is finite and, hence, $\overline{\mathbf{Y}}$ is easier to analyze than $\mathbf{X}$, and (ii) we can establish a relation between $\mathbf{X}$ and $\overline{\mathbf{Y}}$, which allows us to prove the main results in the previous section by studying $\overline{\mathbf{Y}}$ instead of $\mathbf{X}$. This simplifies the proofs of the theorems in the previous section.

• **Admissible action sets:** As the queue sizes are no longer a component of the state of $\overline{\mathbf{Y}}$, we eliminate the dependence of admissible action sets on $\mathbf{q}$, which was explicitly specified in (3.2) for MDP $\mathbf{X}$, while still ensuring that the server is non-preemptive. More specifically, the set of admissible actions at each element

$\overline{\mathbf{y}} = (\overline{s}, \overline{w})$ of $\mathbb{Y}$ is given by

$$\overline{\mathbb{A}}_{\overline{w}} \stackrel{\text{def}}{=} \begin{cases} \{\mathcal{W}_i\} & \text{if } \overline{w} = \mathcal{B}_i, \quad \text{(non-preemptive server)} \\ \\ \mathbb{A} & \text{if } \overline{w} = \mathcal{A}. \end{cases} \tag{3.6}$$

Consequently, for any given realization of the current state $\overline{\mathbf{y}}_k = (\overline{s}_k, \overline{w}_k)$, $\overline{A}_k$ is required to take values in $\overline{\mathbb{A}}_{\overline{w}_k}$.

• **Transition probabilities:** We define the transition probabilities that specify $\overline{\mathbf{Y}}$, as follows:

$$\begin{aligned} P_{\overline{\mathbf{Y}}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}(\overline{\mathbf{y}}' \mid \overline{\mathbf{y}}, \overline{a}) \stackrel{\text{def}}{=} & P_{\overline{S}_{k+1}|\overline{S}_k,\overline{A}_k}(\overline{s}' \mid \overline{s}, \overline{a}) \\ & \times P_{\overline{W}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}(\overline{w}' \mid \overline{\mathbf{y}}, \overline{a}), \end{aligned} \tag{3.7}$$

where $\overline{\mathbf{y}}$ and $\overline{\mathbf{y}}'$ are in $\mathbb{Y}$, and $\overline{a}$ is in $\overline{\mathbb{A}}_{\overline{w}}$. Subject to these action constraints, the right-hand terms of (3.7) are defined, in connection with $\mathbf{X}$, as follows:

$$P_{\overline{S}_{k+1}|\overline{S}_k,\overline{A}_k}(\overline{s}' \mid \overline{s}, \overline{a}) \stackrel{\text{def}}{=} P_{S_{k+1}|S_k,A_k}(\overline{s}' \mid \overline{s}, \overline{a}) \tag{3.8}$$

$$P_{\overline{W}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}(\overline{w}' \mid \overline{\mathbf{y}}, \mathcal{W}_i) \stackrel{\text{def}}{=} \begin{cases} \mu(\overline{s}, i) & \text{if } \overline{w}' = \mathcal{A} \\ \\ 1 - \mu(\overline{s}, i) & \text{if } \overline{w}' = \mathcal{B}_i \end{cases} \tag{3.9a}$$

$$P_{\overline{W}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}(\overline{w}' \mid \overline{\mathbf{y}}, \mathcal{R}) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \overline{w}' = \mathcal{A} \\ \\ 0 & \text{o.w.} \end{cases} \tag{3.9b}$$

- **A relation between the transition probabilities of X and $\overline{\mathbf{Y}}$:** From the definition above and (3.5), we can deduce the following equality: for all $q_i \geq 1$ $i \in \mathbb{T}$,

$$\begin{aligned} &P_{\overline{W}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}\big(\overline{w}' \mid \overline{\mathbf{y}}, \mathcal{W}_i\big) \\ &= \sum_{q_1'=0}^{\infty} \cdots \sum_{q_m'=0}^{\infty} P_{W_{k+1},\mathbf{Q}_{k+1}|\mathbf{X}_k,A_k}\big((\overline{w}', \mathbf{q}') \mid (\ \overline{\mathbf{y}}, \mathbf{q}), \mathcal{W}_i\big), \end{aligned} \tag{3.10}$$

which holds for any $\overline{w}'$ in $\mathbb{W}$ and $\overline{\mathbf{y}}$ in $\mathbb{Y}$. Notice that the right-hand side (RHS) of (3.10) does not change when we vary $\mathbf{q}$ across the positive integers. From this, in conjunction with (3.4), (3.7) and (3.8), we have, for all $q_i \geq 1$ and $i \in \mathbb{T}$,

$$\begin{aligned} &P_{\overline{\mathbf{Y}}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}\big(\overline{\mathbf{y}}' \mid \overline{\mathbf{y}}, \mathcal{W}_i\big) \\ &= \sum_{q_1'=0}^{\infty} \cdots \sum_{q_m'=0}^{\infty} P_{\mathbf{X}_{k+1}|\mathbf{X}_k,A_k}\big((\overline{\mathbf{y}}', \mathbf{q}') \mid (\overline{\mathbf{y}}, \mathbf{q}), \mathcal{W}_i\big). \end{aligned} \tag{3.11}$$

The equality in (3.11) indicates that $P_{\overline{\mathbf{Y}}_{k+1}|\overline{\mathbf{Y}}_k,\overline{A}_k}$ also characterizes the transition probabilities of the server state $\mathbf{Y}_k = (S_k, W_k)$ in $\mathbf{X}$ when the current queue sizes are positive. This is consistent with our earlier viewpoint that $\overline{\mathbf{Y}}$ can be considered the server state in $\mathbf{X}$ initialized with infinite queue sizes at the beginning. We will explore this relationship in Section 4.2, where we use $\overline{\mathbf{Y}}$ to prove Theorems 4.1 and 4.2.

3.4.1 Stationary policies of $\overline{\mathbf{Y}}$

Analogously to the MDP $\mathbf{X}$, we only consider stationary randomized policies for $\overline{\mathbf{Y}}$, which are defined below.

Definition 3.6 (Stationary randomized policies for $\overline{\mathbf{Y}}$)**.** *We restrict our attention to stationary randomized policies acting on $\overline{\mathbf{Y}}$, which are specified by a mapping $\phi : \mathbb{Y} \to [0,1]^m$, as follows:*

$$P_{\overline{A}_k|\overline{\mathbf{Y}}_k,\dots,\overline{\mathbf{Y}}_0}(\mathcal{W}_i|\overline{\mathbf{y}}_k,\dots,\overline{\mathbf{y}}_0) = \phi(\overline{\mathbf{y}}_k)_i$$

$$P_{\overline{A}_k|\overline{\mathbf{Y}}_k,\dots,\overline{\mathbf{Y}}_0}(\mathcal{R}|\overline{\mathbf{y}}_k,\dots,\overline{\mathbf{y}}_0) = 1 - \sum_{i=1,2} \phi(\overline{\mathbf{y}}_k)_i$$

for every k in $\mathbb{N}$ and $\overline{\mathbf{y}}_k,\dots,\overline{\mathbf{y}}_0$ in $\mathbb{Y}$. The set of all stationary randomized policies for $\overline{\mathbf{Y}}$ which honor (3.6) is defined to be Φ_R.

3.4.2 Stationary PMFs of $\overline{\mathbf{Y}}^{\phi}$

The MDP $\overline{\mathbf{Y}}$ subject to a policy ϕ in Φ_R is a *finite-state* time-homogeneous MC and is denoted by $\overline{\mathbf{Y}}^{\phi} \stackrel{\text{def}}{=} \{\overline{\mathbf{Y}}^{\phi}_k;\ k \in \mathbb{N}\}$. Because $\mathbb{Y}$ is finite, for any policy ϕ in Φ_R, $\overline{\mathbf{Y}}^{\phi}$ has a positive recurrent communicating class and a stationary distribution [84]. In fact, there are at most two positive recurrent communicating classes as explained below.

Define a mapping $\mathcal{T} : \Phi_R \to \mathbb{S} \cup \{0\}$, where

$$\mathcal{T}(\phi) \stackrel{\text{def}}{=} \max\{\overline{s} \in \mathbb{S} \mid \sum_{j \in \mathbb{T}} \phi(\overline{s}, \mathcal{A})_j = 1\}, \ \phi \in \Phi_R.$$

We assume that $\mathcal{T}(\phi) = 0$ if the set on the RHS is empty.

<u>*Case 1.* $\phi(1, \mathcal{A})_i > 0$ *for an* $i \in \mathbb{T}$*:*</u> First, from the definition of $\mathcal{T}(\phi)$, clearly all states $(\overline{s}, \overline{w})$ with $\overline{s} \geq \mathcal{T}(\phi)$ communicate with each other, but none of these states communicates with any other state $(\overline{s}', \overline{w}')$ with $\overline{s}' < \mathcal{T}(\phi)$ because $\sum_{j \in \mathbb{T}} \phi(\mathcal{T}(\phi), \mathcal{A})_j = \phi(\mathcal{T}(\phi), \mathcal{B}_i)_i = 1$. Second, because $\phi(1, \mathcal{A})_i > 0$ by assumption, all states $(\overline{s}', \overline{w}')$ with $\overline{s} < \mathcal{T}(\phi)$ communicate with states $(\overline{s}, \overline{w})$ with $\overline{s} \geq \mathcal{T}(\phi)$. Together with the first observation, this implies that these states $(\overline{s}', \overline{w}')$ with $\overline{s}' < \mathcal{T}(\phi)$ are transient. Therefore, there is only one positive recurrent communicating class given by

$$\mathbb{Y}^{\phi} \stackrel{\text{def}}{=} \{(\overline{s}, \overline{w}) \in \mathbb{Y} \mid \overline{s} \geq \mathcal{T}(\phi)\}. \tag{3.12}$$

<u>*Case 2.* $\phi(1, \mathcal{A})_i = 0$ *for all* $i \in \mathbb{T}$*:*</u> In this case, it is clear that $(1, \mathcal{A})$ is an absorbing state and forms a positive recurrent communicating class by iself. Hence, if $\mathcal{T}(\phi) = 0$, as all other states communicate with $(1, \mathcal{A})$, the only positive recurrent communicating class is $\{(1, \mathcal{A})\}$ and all other states are transient. On the other hand, if $\mathcal{T}(\phi) > 1$, for the same reason explained in the first case, $\mathbb{Y}^{\phi}$ gives rise to a second positive recurrent communicating class, and all other states $(\overline{s}', \overline{w}')$ with $\overline{s}' < \mathcal{T}(\phi)$, except for $(1, \mathcal{A})$, are transient.

In our study, we often limit our discussion to randomized policies ϕ in Φ_R with

$\phi(1, \mathcal{A})_i > 0$ for at least an $i \in \mathbb{T}$. For this reason, for notational convenience, we define the set of randomized policies satisfying this condition by Φ_R^+. The reason for this will be explained in the subsequent section.

The following proposition is an immediate consequence of the above observation.

Corollary 3.1. *For any policy ϕ in Φ_R^+, $\overline{\mathbf{Y}}^\phi$ has a unique stationary PMF, which we denote by $\overline{\boldsymbol{\pi}}^\phi = (\overline{\pi}^\phi(\mathbf{y});\ \mathbf{y} \in \mathbb{Y})$.*

3.4.3 Policies Mapping Relation between $\mathbf{X}$ and $\overline{\mathbf{Y}}$

One of key facts which we will make use of in our analysis is that, for any stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$, we can find a policy ϕ in Φ_R^+ which achieves the same steady-state distribution of server state. To this end, we first define, for each $\overline{\mathbf{y}}$ in $\mathbb{Y}$,

$$\mathbb{Q}^{\overline{\mathbf{y}}} \stackrel{\text{def}}{=} \{\mathbf{q} \in \mathbb{N}^m \mid (\overline{\mathbf{y}}, \mathbf{q}) \in \mathbb{X}\}.$$

Definition 3.7 (Policy projection map $\mathscr{Y}$)**.** *We define a mapping $\mathscr{Y} : \Theta_S(\boldsymbol{\lambda}) \mapsto \Phi_R$, where*

$$\mathscr{Y}(\theta) \stackrel{def}{=} \phi^\theta,\ \theta \in \Theta_S(\boldsymbol{\lambda}),$$

with

$$\phi^\theta(\overline{\mathbf{y}})_i \stackrel{def}{=} \frac{\sum_{\mathbf{q}\in\mathbb{Q}^{\overline{\mathbf{y}}}} \theta(\overline{\mathbf{y}},\mathbf{q})_i \pi^\theta(\overline{\mathbf{y}},\mathbf{q})}{\sum_{\mathbf{q}\in\mathbb{Q}^{\overline{\mathbf{y}}}} \pi^\theta(\overline{\mathbf{y}},\mathbf{q})}, \ \overline{\mathbf{y}} \in \mathbb{Y}, \ i \in \{1,\ldots,m\}. \tag{3.13}$$

We first present a lemma that proves useful in our analysis.

Lemma 3.2. *For every stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$, we have $\phi^\theta(1,\mathcal{A})_i > 0$ for an $i \in \mathbb{T}$.*

Proof. First, note that every state in $\mathbb{C}_\theta$ communicates with some state of the form $(1,\mathcal{A},\mathbf{q})$, which must lie in $\mathbb{C}_\theta$ as well; this can be seen from the fact that, after each epoch the server works (resp. rests), the one of the queue size (resp. the activity state) decreases by one with positive probability. As a result, $\mathbf{X}^\theta$, starting at $\mathbf{x} = (s, w, \mathbf{q})$ in $\mathbb{C}_\theta$, can reach a state $(1,\mathcal{A},\mathbf{q}')$ for some $\mathbf{q}'$ in $\mathbb{N}^m$, after at most $(s-1) + \sum_{i\in\mathbb{T}} q_i$ epochs with positive probability.

Since $(1,\mathcal{A},\mathbf{q}')$ is in $\mathbb{C}_\theta$ (because $\mathbb{C}_\theta$ is closed [84]) and communicates with all states $(1,\mathcal{A},\tilde{\mathbf{q}})$, $\tilde{\mathbf{q}} \geq \mathbf{q}'$ element-wise, they also belong to $\mathbb{C}_\theta$. This in turn means that there exists $\mathbf{q}^\star$ such that (a) $(1,\mathcal{A},\mathbf{q}^\star)$ is in $\mathbb{C}_\theta$, hence $\pi^\theta(1,\mathcal{A},\mathbf{q}^\star) > 0$, and (b) $\theta(1,\mathcal{A},\mathbf{q}^\star)_i > 0$ for an $i \in \mathbb{T}$; otherwise, the states $(1,\mathcal{A},\tilde{\mathbf{q}})$, $\tilde{\mathbf{q}} \geq \mathbf{q}'$, must be transient and cannot belong to $\mathbb{C}_\theta$, leading to a contradiction.

□

An obvious implication of the lemma is that $\mathscr{Y}(\theta)$ belongs to Φ_R^+ for every θ in $\Theta_S(\boldsymbol{\lambda})$, and there exists a unique stationary PMF for $\overline{\mathbf{Y}}^{\mathscr{Y}(\theta)}$, namely $\overline{\boldsymbol{\pi}}^{\mathscr{Y}(\theta)}$.

The following lemma shows that the steady-state distribution of the server state in $\mathbf{X}$ under policy θ in $\Theta_S(\boldsymbol{\lambda})$ is identical to that of $\overline{\mathbf{Y}}$ under policy $\mathscr{Y}(\theta)$.

Lemma 3.3. *Suppose that $\theta \in \Theta_S(\boldsymbol{\lambda})$. Then, we have*

$$\overline{\pi}^{\mathscr{Y}(\theta)}(\overline{\mathbf{y}}) = \sum_{\mathbf{q}\in\mathbb{Q}^{\overline{\mathbf{y}}}} \pi^{\theta}(\overline{\mathbf{y}}, \mathbf{q}), \;\; \overline{\mathbf{y}} \in \mathbb{Y}. \tag{3.14}$$

Proof. For notational convenience, let $\phi = \mathscr{Y}(\theta)$. Taking advantage of the fact that there is a unique stationary PMF of $\overline{\mathbf{Y}}^{\phi}$, it suffices to show that the distribution given in (3.14) satisfies the definition of stationary PMF:

$$\overline{\pi}^{\phi}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}') \; \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \;\text{ for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{3.15}$$

where $\overline{\mathbf{P}}^{\phi}$ denotes the one-step transition matrix of $\overline{\mathbf{Y}}^{\phi}$.

- **Right-hand side of (3.15):** Using the policy ϕ in place,

$$\begin{aligned}
&\sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}') \; \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
&= \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}')\big(\sum_{i=1}^{m} \phi(\overline{\mathbf{y}}')_i \overline{\mathbf{P}}^{\mathcal{W}_i}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} + (1 - \sum_{i=1}^{m} \phi(\overline{\mathbf{y}}')_i) \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \big) \\
&= \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}')\Big(\overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} + \sum_{i=1}^{m} \phi(\overline{\mathbf{y}}')_i (\overline{\mathbf{P}}^{\mathcal{W}_i}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} - \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}) \Big),
\end{aligned} \tag{3.16}$$

where $\overline{\mathbf{P}}^{\mathcal{R}}$ (resp. $\overline{\mathbf{P}}^{\mathcal{W}_i}$) denotes the one-step transition matrix of $\overline{\mathbf{Y}}$ under a policy that always rests (resp. works on a new task) when available.

Substituting (3.13) for $\phi(\overline{\mathbf{y}}')$ in (3.16) and using the given expression $\overline{\pi}^{\phi}(\overline{\mathbf{y}}') =$

$\sum_{q \in \mathbb{Q}^{\overline{\mathbf{y}}'}} \pi^\theta(\overline{\mathbf{y}}', q)$ in (3.14), we obtain

$$(3.16) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \sum_{i=1}^{m} \sum_{\mathbf{q}' \in \mathbb{Q}^{\overline{\mathbf{y}}'}} \theta(\overline{\mathbf{y}}', \mathbf{q}')_i \pi^\theta(\overline{\mathbf{y}}', \mathbf{q}') \big(\overline{\mathbf{P}}^{\mathcal{W}_i}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}} - \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}}\big) + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}'}} \pi^\theta(\overline{\mathbf{y}}', \mathbf{q}) \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}}$$
$$= \sum_{i=1}^{m} \sum_{\mathbf{x}' \in \mathbb{X}} \theta(\mathbf{x}')_i \pi^\theta(\mathbf{x}') \big(\overline{\mathbf{P}}^{\mathcal{W}_i}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}} - \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}}\big) + \sum_{\mathbf{x}' \in \mathbb{X}} \pi^\theta(\mathbf{x}') \overline{\mathbf{P}}^{\mathcal{R}}_{\overline{\mathbf{y}}', \overline{\mathbf{y}}}. \tag{3.17}$$

- **Left-hand side of (3.15):** Using (3.14), we get

$$\overline{\pi}^\phi(\overline{\mathbf{y}}) = \sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}}} \pi^\theta(\overline{\mathbf{y}}, \mathbf{q}). \tag{3.18}$$

For notational ease, we denote $(\overline{\mathbf{y}}, \mathbf{q})$ on the RHS of (3.18) simply by $\mathbf{x}$. Since $\boldsymbol{\pi}^\theta$ is the unique stationary PMF of $\mathbf{X}^\theta$, we have

$$\pi^\theta(\overline{\mathbf{y}}, \mathbf{q}) = \sum_{\mathbf{x}' \in \mathbb{X}} \pi^\theta(\mathbf{x}') \mathbf{P}^\theta_{\mathbf{x}', \mathbf{x}} \tag{3.19}$$
$$= \sum_{\mathbf{x}' \in \mathbb{X}} \sum_{i=1}^{m} \pi^\theta(\mathbf{x}') \theta(\mathbf{x}')_i \big(\mathbf{P}^{\mathcal{W}_i}_{\mathbf{x}', \mathbf{x}} - \mathbf{P}^{\mathcal{R}}_{\mathbf{x}', \mathbf{x}}\big) + \sum_{\mathbf{x}' \in \mathbb{X}} \pi^\theta(\mathbf{x}') \mathbf{P}^{\mathcal{R}}_{\mathbf{x}', \mathbf{x}},$$

where $\mathbf{P}^\theta$ is the one-step transition matrix of $\mathbf{X}^\theta$, and $\mathbf{P}^{\mathcal{R}}$ (resp. $\mathbf{P}^{\mathcal{W}_i}$) is the one-step transition matrix under a policy that always rests (resp. assigns a new task) when the server is available and at least one task is waiting for service.

Substituting (3.19) in (3.18) and rearranging the summations, we obtain

$$\overline{\pi}^{\phi}(\overline{\mathbf{y}}) = \sum_{i=1}^{m} \sum_{\mathbf{x}' \in \mathbb{X}} \pi^{\theta}(\mathbf{x}') \theta(\mathbf{x}')_i \sum_{q \in \mathbb{Q}^{\overline{\mathbf{y}}}} \left(\mathbf{P}^{\mathcal{W}_i}_{\mathbf{x}',\mathbf{x}} - \mathbf{P}^{\mathcal{R}}_{\mathbf{x}',\mathbf{x}} \right) + \sum_{\mathbf{x}' \in \mathbb{X}} \pi^{\theta}(\mathbf{x}') \sum_{q \in \mathbb{Q}^{\overline{\mathbf{y}}}} \mathbf{P}^{\mathcal{R}}_{\mathbf{x}',\mathbf{x}}. \tag{3.20}$$

By comparing (3.17) and (3.20), in order to prove (3.15), it suffices to show

$$\overline{\mathbf{P}}^{a}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}}} \mathbf{P}^{a}_{(\overline{\mathbf{y}}',\mathbf{q}'),(\overline{\mathbf{y}},\mathbf{q})}, \ \ a \in \mathbb{A}.$$

Note that

$$\sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}}} \mathbf{P}^{a}_{(\overline{\mathbf{y}}',\mathbf{q}'),(\overline{\mathbf{y}},\mathbf{q})} = \mathbf{P}_{\mathbf{Y}_{k+1}|(\mathbf{Y}_k,\mathbf{Q}_k),A_k}(\overline{\mathbf{y}} \mid (\overline{\mathbf{y}}',\mathbf{q}'),a). \tag{3.21}$$

Clearly, conditional on $\{(\mathbf{Y}_k, A_k) = (\overline{\mathbf{y}}', a)\}$, $\mathbf{Y}_{k+1}$ does not depend on the queue size at epoch k. As a result, the RHS of (3.21) does not depend on $\mathbf{q}'$ and is equal to $P_{\mathbf{Y}_{k+1}|\mathbf{Y}_k,A_k}(\overline{\mathbf{y}} \mid \overline{\mathbf{y}}', a) = \overline{\mathbf{P}}^{a}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$. □

3.5 Summary

We present our queuing server framework in this section. The model is made up of several first-in-first-out queues that store incoming tasks. The assigned tasks are handled by a non-preemptive server with history-dependent efficiency. A scheduler assigns tasks base on system status to the server. We also layout an auxiliary process with the queuing server framework, which shares some stochastic properties. In particular, we defined a projection mapping from a policy for the original system to auxiliary CMC. We proved that the PMF on the server state is preserved after the mapping. This additional process greatly simplified our analysis from infinite state space to finite state space and is especially helpful in the later chapters to prove our theorems.

Chapter 4: Queuing Server with One Type of Tasks

In this chapter, we restrict our attention to the case where only one type of task exists ($m = 1$). We present our main results that address two main challenges for designing policies, which are stability and utilization rate. We first show that there exists an upper-bound on the arrival rate such that the system can still be stable, and the value can be computed efficiently. Moreover, we prove the existence of a threshold policy that keeps the queue stable whenever doing so is possible using some scheduler. Then, we give the formal definition of utilization rate. We find the minimum utilization rate given an arrival rate and propose policies that achieve near-optimal utilization rate while keeping the queue stable by relating two CMCs, $\mathbf{X}$ and $\mathbf{Y}$.

The chapter is organized as follows. We begin by presenting the stability results follow by the proofs of two Theorems. In Section 4.3, we give the formal definition of utilization rate for a policy on CMC $\mathbf{X}$. In Section 4.4, we introduce the utilization rate for a policy on $\mathbf{Y}$. In Section 4.5, we identified a tractable method to compute the infimum of all utilization rates achievable by a stabilizing scheduling policy and design policies whose utilization rate is arbitrarily close to the fundamental limit.

For notational convenience, we simplify our representation by not presenting the type of the task which we denote by i in the model definition because we focus on single type case in this chapter.

4.1 Stability Results

Our answers for Problem 3.1 and Problem 3.2 when there is only one type of task (m=1) are Theorems 4.1 and 4.2, where we state that a soon-to-be defined quantity $\overline{\nu}^*$, which can be computed efficiently, is the least upper bound of all arrival rates for which there exists a stabilizing policy in Θ_R (see Definition 3.4). The theorems also assert that, for any arrival rate λ less than $\overline{\nu}^*$, there is a stabilizing deterministic *threshold* policy in Θ_R with the following structure:

$$\theta_\tau(s, w, q) \stackrel{\text{def}}{=} \begin{cases} \phi_\tau(s, w) & \text{if } q > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where τ lies in $\mathbb{S} \cup \{n_s + 1\}$, and ϕ_τ is a threshold policy that acts as follows:

$$\phi_\tau(s, w) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } s \geq \tau \text{ and } w = \mathcal{A}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.2}$$

Notice that, when the server is available and the queue is not empty, θ_τ assigns a new task only if s is less than the threshold τ and lets the server rest otherwise.

Next, we provide the intuition behind the value $\overline{\nu}^*$. In Section 3.4, we introduce an auxiliary CMC with *finite* state space $\mathbb{Y}$, which can be viewed as the server state in $\mathbf{X}$ when the queue size Q_k is always positive. Using the fact that $\mathbb{Y}$ is finite, we demonstrate that, for every $\phi \in \Phi_R^+$ (Φ_R^+ is the set of stationary randomized policies for the auxiliary CMC), the auxiliary CMC subject to ϕ has a unique stationary PMF, which we denote by $\overline{\boldsymbol{\pi}}^{\phi}$. Then, we show that, for any stable system $\mathbf{X}^{\theta}$ under some policy θ in Θ_R, we can find a policy ϕ for the auxiliary CMC, which achieves the same long-term departure rate of completed tasks as in $\mathbf{X}^{\theta}$. As a result, the maximum long-term departure rate of completed tasks in the auxiliary CMC among all randomized policies serves as an upper bound on the arrival rate λ for which we can hope to find a stabilizing policy θ in Θ_R. Finally, we show the maximum departure rate among all randomized policies and the maximum departure rate among all threshold policies ϕ_τ with τ in $\mathbb{S} \cup \{n_s + 1\}$ are identical.

Making use of this observation and denoting the unique stationary PMF of auxiliary CMC subject to ϕ_τ by $\overline{\boldsymbol{\pi}}^{\phi_\tau}$, we define the following important quantity:

$$\overline{\nu}^* \overset{\text{def}}{=} \max_{\tau \in \mathbb{S} \cup \{n_s+1\}} \left(\sum_{(\overline{s},\overline{w}) \in \mathbb{Y}} \overline{\boldsymbol{\pi}}^{\phi_\tau}(\overline{s},\overline{w}) \; \phi_\tau(\overline{s},\overline{w}) \; \mu(\overline{s}) \right) \tag{4.3}$$

From the definition of the stationary PMF $\overline{\boldsymbol{\pi}}^{\phi_\tau}$, $\overline{\nu}^*$ can be interpreted as the maximum long-term departure rate of completed tasks under any threshold policy of the form in (4.1), *assuming that the queue is always non-empty.*

The following are the main results of this section.

Theorem 4.1. *(Necessity) If, for a given arrival rate λ, there exists a stabilizing policy in Θ_R, then $\lambda \leq \overline{\nu}^*$.*

Theorem 4.2. *(Sufficiency) Let τ^* be a maximizer of (4.3). If the arrival rate λ is strictly less than $\overline{\nu}^*$, then θ_{τ^*} stabilizes the system.*

Proof. Please see Section 4.2.1 and Section 4.2.2 for the proofs. □

Remark 2. *The following important observations are direct consequences of (4.3) and Theorems 4.1 and 4.2:*

- *The computation of $\overline{\nu}^*$ in (4.3) along with a maximizing threshold τ^* relies on a* finite *search that can be carried out efficiently.*
- *The theorems are valid for* any *choice of service rate function μ that takes values in $(0, 1)$. In particular, μ could be multi-modal, increasing or decreasing respect to activity state.*
- *The search that yields $\overline{\nu}^*$ and an associated τ^* does not require knowledge of λ.*

We point out two key differences between our study and [43, 44] . The model employed by Savla and Frazzoli assumes that the service time function is convex, which is analogous to our service rate function being unimodal. In addition, a threshold policy is proved to be maximally stabilizing only for identical task workload. In our study, however, we do not impose any assumption on the service rate function, and the workloads of tasks are modeled using i.i.d. random variables.[1]

[1] To be more precise, our assumptions correspond to the case with exponentially distributed workloads. However, as mentioned earlier, this assumption can be relaxed to allow more general workload distributions.

4.2 Proofs of Stability Results

In this section, we begin with a comment on the long-term average departure rate of completed tasks when the system is stable. Then, in order to prove Theorem 4.1, we make use of a similar notion of *long-term service rates* of $\overline{\mathbf{Y}}^{\phi}$, which can be viewed in most cases as the average number of completed tasks per epoch. We establish that, for every stabilizing policy θ, we can find a *related* policy ϕ in Φ_R whose long-term service rate equals that of θ or, equivalently, the arrival rate λ. Finally, we provide a useful Lemma that argues that the policies should always choose to work at $s = 1$ before proving Theorem 4.1.

Remark 3. *Recall from our discussion in Section 3.3 that, under a stabilizing policy θ in $\Theta_S(\lambda)$, there exists a unique stationary PMF $\boldsymbol{\pi}^{\theta}$. Consequently, the average number of completed tasks per epoch converges almost surely as k goes to infinity. In other words,*

$$\lim_{k \to \infty} \frac{\sum_{t=0}^{k-1} \mathcal{I}\left\{\textit{a task is completed at epoch } t \textit{ in } \mathbf{X}^{\theta}\right\}}{k}$$
$$= \sum_{\mathbf{x} \in \mathbb{X}} \mu(s) \pi^{\theta}(\mathbf{x}) \theta(\mathbf{x}) \overset{def}{=} \nu^{\theta} \quad \textit{with probability 1,}$$

where s, w and q are the coordinates of $\mathbf{x} = (s, w, q)$. We call ν^{θ} the long-term service rate *of θ (for the given arrival rate $\lambda > 0$). Moreover, because θ is assumed to be a stabilizing policy, we have $\nu^{\theta} = \lambda$.*

4.2.1 Necessity

In order to prove Theorem 4.1, we make use of a similar notion of *long-term service rate* of $\overline{\mathbf{Y}}^{\phi}$, which can be viewed in most cases as the average number of completed tasks per epoch. **(Step 1)** We first establish that, for every stabilizing policy θ, we can find a *related* policy ϕ in Φ_R^+ whose long-term service rate equals that of θ or, equivalently, the arrival rate λ. **(Step 2)** We prove that $\overline{\nu}^*$ in (4.3) equals the maximum long-term service rate achievable by any policy in Φ_R. Together, they tell us $\lambda \leq \overline{\nu}^*$.

• **Long-term service rates of $\overline{\mathbf{Y}}^{\phi}$:** The *long-term service rates* associated with $\overline{\mathbf{Y}}^{\phi}$ under policy ϕ in Φ_R is defined as follows. First, for each ϕ in Φ_R, let $\overline{\mathbf{\Pi}}(\phi)$ be the set of stationary PMFs of $\overline{\mathbf{Y}}^{\phi}$. Clearly, by Corollary 3.1, for any ϕ in Φ_R^+, there exists a unique stationary PMF and $\overline{\mathbf{\Pi}}(\phi)$ is a singleton. The long-term service rate of ϕ in Φ_R is defined to be

$$\overline{\nu}^{\phi} \stackrel{\text{def}}{=} \sup_{\overline{\boldsymbol{\pi}} \in \overline{\mathbf{\Pi}}(\phi)} \Big(\sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}) \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}}) \Big). \tag{4.4}$$

Recall that $\overline{\mathbf{y}}$ is the pair $(\overline{s}, \overline{w})$ taking values in $\mathbb{Y}$.

Step 1: The following lemma illustrates that the long-term service rates achieved by $\mathscr{Y}(\theta)$ in Φ_R^+ equals that of θ.

Lemma 4.1. *Suppose that θ is a stabilizing policy in $\Theta_S(\boldsymbol{\lambda})$. Then, $\overline{\nu}^{\mathscr{Y}(\theta)} = \nu^{\theta} = \lambda$.*

Proof. First, note

$$\begin{aligned}
\overline{\nu}^{\mathscr{Y}(\theta)} &\stackrel{(a)}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s})\ \overline{\pi}^{\mathscr{Y}(\theta)}(\overline{\mathbf{y}})\ \phi(\overline{\mathbf{y}}) \\
&\stackrel{(b)}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}) \Big(\sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}}} \pi^{\theta}(\overline{\mathbf{y}}, \mathbf{q}) \Big) \theta(\overline{\mathbf{y}}, \mathbf{q}) \\
&\stackrel{(c)}{=} \sum_{\mathbf{x} \in \mathbb{X}} \mu(s) \pi^{\theta}(\mathbf{x}) \theta(\mathbf{x}) \stackrel{(d)}{=} \nu^{\theta},
\end{aligned}$$

where (b) follows from Lemma 3.3, and (c) results from rearranging the summations in terms of $\mathbf{x} = (\mathbf{y}, \mathbf{q})$. Finally (a) and (d) hold by definition. The lemma follows from Remark 3 that ν^{θ} is equal to λ. □

Step 2: We shall prove that $\overline{\nu}^*$ in (4.3) equals the maximum long-term service rate achievable by any policy in Φ_R^+. Together with Lemma 4.1, they tell us $\lambda \leq \overline{\nu}^*$. Since $\mathscr{Y}(\theta)$ belongs to Φ_R^+ as explained earlier, Lemma 4.1 implies

$$\lambda = \overline{\nu}^{\mathscr{Y}(\theta)} \leq \max_{\phi \in \Phi_R^+} \overline{\nu}^{\phi} \stackrel{\text{def}}{=} \overline{\nu}^{**}. \tag{4.5}$$

We shall prove that $\overline{\nu}^* = \overline{\nu}^{**}$ in three steps. First, we show that we can restrict our search from Φ_R^+ to Φ_R^{++} which is a set of policies that always choose to work when $\overline{\mathbf{y}} = (1, \mathcal{A})$. Next, we establish that there is a stationary *deterministic* policy in Φ_R^{++} that achieves $\overline{\nu}^{**}$. Then, we show that, for any stationary deterministic policy, we can find a deterministic *threshold* policy that achieves the same long-term service rate, thereby completing the proof of Theorem 4.1.

Lemma 4.2. *For a policy $\phi \in \Phi_R^+$, there exists a policy $\phi' \in \Phi_R^{++} \stackrel{def}{=} \{\phi \in \Phi_R^+ : \phi(1, \mathcal{A}) = 1\}$ such that,*

$$\overline{\nu}^{\phi'} \geq \overline{\nu}^{\phi}.$$

Proof. A proof for the Lemma in the context of general m types of tasks is provided in Section 5.5.4. □

By Lemma 4.2, it is clear that,

$$\overline{\nu}^{**} = \max_{\phi \in \Phi_R^+} \overline{\nu}^{\phi} = \max_{\phi \in \Phi_R^{++}} \overline{\nu}^{\phi}. \tag{4.6}$$

Let us define Φ_D^+ to be a subset of Φ_R^{++}, which consists only of stationary *deterministic* policies for $\overline{\mathbf{Y}}$. In other words, if $\phi \in \Phi_D^+$, then $\phi(\mathbf{y}) \in \{0, 1\}$ for all $\mathbf{y} \in \mathbb{Y}$ except $(1, \mathcal{A})$ and $\phi(1, \mathcal{A}) = 1$. Theorem 9.1.8 in [85, p. 451] tells us that if (i) the state space is finite and (ii) the set of available actions is finite for every state, there exists a deterministic stationary optimal policy. Thus,

$$\overline{\nu}^{**} = \max_{\phi \in \Phi_R^{++}} \overline{\nu}^{\phi} = \max_{\phi \in \Phi_D^+} \overline{\nu}^{\phi}. \tag{4.7}$$

While the equality in (4.7) simplifies the computation of the maximum long-term service rate achievable by some ϕ in Φ_R, it requires a search over a set of 2^{n_s} deterministic policies in the worst case. Thus, when n_s is large, it can be computationally expensive. As we show shortly, it turns out that the maximum long-term service rate on the RHS of (4.7) can always be achieved by a deterministic *threshold* policy of the form in (4.2).

Definition 4.1. *Recall from (4.2) that, for a given* τ *in* $\mathbb{S} \cup \{n_s + 1\}$, ϕ_τ *is the following deterministic threshold policy for* $\overline{\mathbf{Y}}$*:*

$$\phi_\tau(\overline{\mathbf{y}}) = \begin{cases} 0 & \text{if } \overline{s} \geq \tau \text{ and } \overline{w} = \mathcal{A} \\ 1 & \text{otherwise.} \end{cases}$$

The following lemma shows that, for each deterministic policy ϕ satisfying $\phi(1, \mathcal{A}) = 1$, there is a deterministic *threshold* policy with the same long-term service rate. Note that we define $\mathcal{T}$ in Section 3.4.2.

Lemma 4.3. *Suppose that* ϕ *is a policy in* Φ_D^+*. Then,* $\overline{\nu}^{\phi} = \overline{\nu}^{\phi_{\tau'}}$*, where* $\tau' = \mathcal{T}(\phi) + 1$*.*

Proof. We begin with the following facts that will be utilized in the proof.

F1. The postulation that the server is non-preemptive, which we formally impose in (3.6), means that after the sever initiates work on a task, it will be allowed to rest only after the task is completed. This implies that any policy ϕ in Φ_D satisfies $\phi(\overline{s}, \mathcal{B}) = 1$ for all $\overline{s}$ in $\mathbb{S}$.

F2. From (3.3) and (3.8), we know that S_{k+1} is never less than S_k while the server is working.

From **F1** and **F2** stated above, we conclude that the following holds for any σ in $\mathbb{S}$:

$$\phi(\sigma, \mathcal{A}) = 1 \implies \Pr(\overline{S}_{k+1}^{\phi} \geq \sigma \mid \overline{S}_k^{\phi} = \sigma) = 1 \tag{4.8}$$

Here, we recall that $\overline{\mathbf{Y}}_k^\phi = (\overline{S}_k^\phi, \overline{W}_k^\phi)$ represents the state of $\overline{\mathbf{Y}}^\phi$ at epoch k.

The implication in (4.8) leads us to the following important observation: suppose that a deterministic policy ϕ in Φ_D^+ satisfies $\phi(\sigma, \mathcal{A}) = 1$ for some σ greater than 1. Then, all states $(\overline{s}, \overline{w})$ with $\overline{s}$ less than σ are transient and, therefore,

$$\overline{\pi}^\phi(\overline{s}, \overline{w}) = 0 \text{ if } \overline{s} < \sigma. \tag{4.9}$$

The reason for this is that (i) because $\phi(1, \mathcal{A}) = 1$, all states $(\overline{s}, \overline{w})$ with $\overline{s} < \sigma$ communicate with every state $(\overline{s}', \overline{w}')$ with $\overline{s}' \geq \sigma$, and (ii) none of the states $(\overline{s}', \overline{w}')$ with $\overline{s}' \geq \sigma$ communicates with any state $(\overline{s}, \overline{w})$ with $\overline{s} < \sigma$ since $\phi(\sigma, \mathcal{A}) = \phi(\sigma, \mathcal{B}) = 1$.

F3. The above observation means that, given a deterministic policy ϕ in Φ_D, every state $(\overline{s}, \overline{w})$ with $\overline{s} < \mathcal{T}(\phi)$ is transient and $\overline{\pi}^\phi(\overline{s}, \overline{w}) = 0$.

F4. Moreover, the remaining states $(\overline{s}, \overline{w})$ in $\mathbb{Y}^\phi$ with $\overline{s} \geq \mathcal{T}(\phi)$ communicate with each other and their period is one (because it is possible to transition from $(\mathcal{T}(\phi), \mathcal{A})$ to itself. Since $\mathbb{Y}^\phi$ is finite, it forms an aperiodic, positive recurrent communicating class of $\overline{\mathbf{Y}}^\phi$.

We will complete the proof of Lemma 4.3 with the help of following lemma.

Lemma 4.4. *Suppose that ϕ and $\tilde{\phi}$ are two deterministic policies in Φ_D satis-*

fying $\phi(1, \mathcal{A}) = \tilde{\phi}(1, \mathcal{A}) = 1$. *Then,*

$$\mathcal{T}(\tilde{\phi}) = \mathcal{T}(\phi) \implies \boldsymbol{\pi}^{\tilde{\phi}} = \boldsymbol{\pi}^{\phi} \tag{4.10}$$

Proof. If $\mathcal{T}(\tilde{\phi}) = \mathcal{T}(\phi)$, **F3** states that, for any state $(\overline{s}, \overline{w})$ with $\overline{s} < \mathcal{T}(\phi)$, we have $\overline{\pi}^{\phi}(\overline{s}, \overline{w}) = \overline{\pi}^{\tilde{\phi}}(\overline{s}, \overline{w}) = 0$. Furthermore, **F4** tells us that the positive recurrent communicating classes are identical, i.e., $\mathbb{Y}^{\phi} = \mathbb{Y}^{\tilde{\phi}}$. From **F1** and the definition of mapping $\mathcal{T}$, we conclude that, for all states $(\overline{s}, \overline{w})$ in $\mathbb{Y}^{\phi}$, $\tilde{\phi}(\overline{s}, \overline{w}) = \phi(\overline{s}, \overline{w})$. This in turn means that, for all $(\overline{s}, \overline{w})$ in $\mathbb{Y}^{\phi}$, we have $\overline{\pi}^{\phi}(\overline{s}, \overline{w}) = \overline{\pi}^{\tilde{\phi}}(\overline{s}, \overline{w})$. □

Let us continue with the proof of Lemma 4.3. Select $\tilde{\phi} = \phi_{\tau'}$ with $\tau' = \mathcal{T}(\phi) + 1$. Then, Lemma 4.4 tells us that $\boldsymbol{\pi}^{\phi} = \boldsymbol{\pi}^{\tilde{\phi}}$. From the definition of $\overline{\nu}^{\phi}$ in (4.4), Lemma 4.3 is now a direct consequence of this observation and **F4**. □

Proceeding with the proof of the theorem, Lemma 4.3 tells us that,

$$\overline{\nu}^{**} = \max_{\phi \in \Phi_D^+} \overline{\nu}^{\phi} = \max_{\tau \in \mathbb{S} \cup \{n_s+1\}} \overline{\nu}^{\phi_\tau} = \overline{\nu}^{*}. \tag{4.11}$$

Together with (4.5), we have $\lambda < \overline{\nu}^{*}$.

4.2.2 Sufficiency

In this section, we shall prove our proposed policies indeed stabilize any stabilizable arrival rates by utilizing Foster-Lyapunov type argument. We begin with a Proposition which is the keystone for building the Lyapunov functions. Then, we proceed with the proof of the Theorem.

Proposition 4.1. *Consider the CMC* $\overline{\mathbf{Y}}$ *with finite states space* $\mathbb{Y}$ *and a reward function* $r : \mathbb{Y} \times \Phi_R \to \mathbb{R}$ *that calculates a reward for each state given a policy.*[a] *Suppose the CMC* $\overline{\mathbf{Y}}$ *is equipped with a stationary randomized policy* ϕ *such that the resulting Markov Chain* $\overline{\mathbf{Y}}_k^{\phi}$ *has only one positive recurrent communicating class and the stationary PMF* $\overline{\pi}^{\phi}$ *exists. Then, there exists a non-negative potential-like function* $f : \mathbb{Y} \to \mathbb{R}+$ *such that*

$$r(\overline{\mathbf{y}}, \phi) - \mathbb{E}\left[f(\overline{\mathbf{Y}}_{k+1}^{\phi}) - f(\overline{\mathbf{Y}}_k^{\phi}) \mid \overline{\mathbf{Y}}_k^{\phi} = \overline{\mathbf{y}}\right] = r_{avg}^{\phi}, \tag{4.12}$$

for each states $\overline{\mathbf{y}} \in \mathbb{Y}$ *where* $\mathbb{E}$ *stands for expected value and* r_{avg}^{ϕ} *is the average reward for* $\overline{\mathbf{Y}}_k^{\phi}$*, i.e.,*

$$r_{avg}^{\phi} = \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} r(\overline{\mathbf{y}}, \phi)\overline{\pi}^{\phi}(\overline{\mathbf{y}}).$$

[a] The statement in Proposition 4.1 is true for any finite states CMC. We state it using $\overline{\mathbf{Y}}$ to simplify our notation.

Proof. We reindex $\mathbb{Y}$ and assume $\mathbb{Y} = \{1, \ldots, n\}$. Let us construct a temporary f' function by first assigning $f'(n) = 0$ (WOLG assume state n is positive recurrent).

Next, for each $\overline{\mathbf{y}} \in \{1, \ldots, n-1\}$, the Proposition statement can be rewritten as

$$r(\overline{\mathbf{y}}, \phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}\right] = r^{\phi}_{avg},$$

$$-\mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}\right] = r^{\phi}_{avg} - r(\overline{\mathbf{y}}, \phi),$$

$$-\mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1})|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}\right] - f'(\overline{\mathbf{y}}) = r^{\phi}_{avg} - r(\overline{\mathbf{y}}, \phi),$$

$$-\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} f'(\overline{\mathbf{y}}') \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = \overline{\mathbf{y}}' \mid \overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}) + f'(\overline{\mathbf{y}}) = r^{\phi}_{avg} - r(\overline{\mathbf{y}}, \phi).$$

Since $f'(n) = 0$, the constraints can be written as,

$$\big(1 - \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = \overline{\mathbf{y}} \mid \overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}})\big) f'(\overline{\mathbf{y}}) - \sum_{\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{\overline{\mathbf{y}}, n\}} f'(\overline{\mathbf{y}}') \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = \overline{\mathbf{y}}'|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}) = r^{\phi}_{avg} - r(\overline{\mathbf{y}}, \phi). \quad (4.13)$$

We collect all the constraints (4.13) for each $\overline{\mathbf{y}} \in \{1, \ldots, n-1\}$ and form a system of linear equations. The equations can be represented in matrix form as follow.

$$\begin{bmatrix} \alpha_1 & \beta_{1,2} & \beta_{1,3} & \ldots & \beta_{1,n-1} \\ \beta_{2,1} & \alpha_2 & \beta_{2,3} & \ldots & \beta_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \beta_{n-1,3} & \ldots & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} f'(1) \\ f'(2) \\ \vdots \\ f'(n-1) \end{bmatrix} = \begin{bmatrix} r^{\phi}_{avg} - r(1, \phi) \\ r^{\phi}_{avg} - r(2, \phi) \\ \vdots \\ r^{\phi}_{avg} - r(n-1, \phi) \end{bmatrix}$$

where $\alpha_i = 1 - \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = \overline{\mathbf{y}} \mid \overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}})$ and $\beta_{i,j} = -\Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = j|\overline{\mathbf{Y}}^{\phi}_{k} = i)$.

This matrix is a Weakly Chained Diagonally Dominant (WCDD) matrix [86] by the following observation. First, the matrix is weakly diagonally dominant since

$$
\begin{aligned}
|\alpha_i| &= |1 - \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = i | \overline{\mathbf{Y}}^{\phi}_{k} = i)| \\
&= \left| \sum_{j \in \{1,\ldots,n\} \neq i} \Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = j | \overline{\mathbf{Y}}^{\phi}_{k} = i) \right| \\
&= \sum_{j \in \{1,\ldots,n\} \neq i} |\Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = j | \overline{\mathbf{Y}}^{\phi}_{k} = i)| \\
&\geq \sum_{j \in \{1,\ldots,n-1\} \neq i} |-\Pr(\overline{\mathbf{Y}}^{\phi}_{k+1} = j | \overline{\mathbf{Y}}^{\phi}_{k} = i)| \\
&= \sum_{j \in \{1,\ldots,n-1\} \neq i} |\beta_{i,j}|
\end{aligned}
$$

Furthermore, for any state ℓ that has positive probability to transit to state n, row ℓ is strong diagonally dominant(SDD)

$$
|\alpha_\ell| > \sum_{j \in \{1,\ldots,n-1\} \neq \ell} |\beta_{\ell,j}|
$$

The last step is to show that for each row i that is not SDD, there exists a walk from state i to a state ℓ in the directed graph associated with the matrix such that row ℓ is SDD. Since state n is in the positive recurrent communicating class, from any state, there exists a walk to the state n. Hence, for all the states that have zero probability to transit to state n (which are WDD), there exist a walk to the states that have positive probability to transit to state n (which are SDD). This satisfies the definition for weakly chained diagonally dominant matrix.

Since WCDD matrix is nonsingular [86], there exists a solution to the equation. Now we have assign the value to the potential function f'. The final step is to show this potential function satisfied the Lemma statement at state n.

Let us consider the following value,

$$r_{avg2}^{\phi} \stackrel{\text{def}}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \left(r(\overline{\mathbf{y}}, \phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}_{k+1}^{\phi}) - f'(\overline{\mathbf{Y}}_{k}^{\phi}) | \overline{\mathbf{Y}}_{k}^{\phi} = \overline{\mathbf{y}} \right] \right).$$

Suppose the process start from state n. Let $R_k = r\left(\overline{\mathbf{Y}}_k^{\phi}, \phi\right) - \left(f'(\overline{\mathbf{Y}}_k^{\phi}) - f'(\overline{\mathbf{Y}}_{k-1}^{\phi})\right)$, $R_0 = 0$. The weak law of large number tells us that

$$\frac{\sum_{k=1}^{T} R_k}{T} \to r_{avg2}^{\phi} \quad \text{with probability 1},$$

when T goes to infinite. However, we shall show that $\frac{\sum_{k=1}^{T} R_k}{T}$ converges to r_{avg}^{ϕ} as well,

$$\begin{aligned}
\frac{\sum_{k=1}^{T} R_k}{T} &= \frac{\sum_{k=1}^{T} r\left(\overline{\mathbf{Y}}_k^{\phi}, \phi\right) - \left(f'(\overline{\mathbf{Y}}_k^{\phi}) - f'(\overline{\mathbf{Y}}_{k-1}^{\phi})\right)}{T} \\
&= \frac{f'(\overline{\mathbf{Y}}_T^{\phi}) - \overbrace{f'(\overline{\mathbf{Y}}_0^{\phi})}^{f'(n)=0} + \sum_{k=1}^{T} r\left(\overline{\mathbf{Y}}_k^{\phi}, \phi\right)}{T} \\
&= \frac{f'(\overline{\mathbf{Y}}_T^{\phi}) + \sum_{k=1}^{T} r\left(\overline{\mathbf{Y}}_k^{\phi}, \phi\right)}{T} \to r_{avg}^{\phi} \text{ with probability one},
\end{aligned}$$

when T goes to infinity because $f'(\overline{\mathbf{Y}}_T^{\phi})$ is upper-bunded. Hence, we have $r_{avg2}^{\phi} =$

r^{ϕ}_{avg}, and,

$$\begin{aligned}
r^{\phi}_{avg} = r^{\phi}_{avg2} &= \sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \left(r(\overline{\mathbf{y}},\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}\right]\right) \\
&= \overline{\pi}^{\phi}(n) \left(r(n,\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = n\right]\right) \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus n} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \left(r(\overline{\mathbf{y}},\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = \overline{\mathbf{y}}\right]\right) \\
&= \overline{\pi}^{\phi}(n) \left(r(n,\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = n\right]\right) \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus n} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) r^{\phi}_{avg},
\end{aligned}$$

where the last equality comes from the fact that f' satisfies (4.13) for every $\overline{\mathbf{y}} \neq n$. Moving the second term on the RHS to the LHS, we obtain

$$\begin{aligned}
&\left(1 - \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus n} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) r^{\phi}_{avg}\right) r^{\phi}_{avg} = \overline{\pi}^{\phi}(n) r^{\phi}_{avg} \\
&= \overline{\pi}^{\phi}(n) \left(r(n,\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = n\right]\right)
\end{aligned}$$

Thus, we have $\left(r(n,\phi) - \mathbb{E}\left[f'(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f'(\overline{\mathbf{Y}}^{\phi}_{k})|\overline{\mathbf{Y}}^{\phi}_{k} = n\right]\right) = r^{\phi}_{\text{avg}}$ because $\pi^{\phi}(n) > 0$. Finally, define the potential-like function $f(\overline{\mathbf{y}}) = f'(\overline{\mathbf{y}}) - \min_{\overline{\mathbf{y}}'\in\mathbb{Y}} f'(\overline{\mathbf{y}}')$. Then, this f is non-negative and satisfies all the constraints in the Proposition statement.

□

One can think of this f function as a potential-like function that adjusts reward such that the expected one step reward at every state is identical.

We shall prove the single queue result by using Foster's theorem [87]. (**Step 1.**) We argue that the Markov Chain $\mathbf{X}^{\theta_{\tau^*}}$ is irreducible. (**Step 2.**) We construct a Lyapunov function V by utilizing Proposition 4.1. (**Step 3.**) Then, we show that our Lyapunov function V satisfies

$$\mathbb{E}\left[V\left(\mathbf{X}_{k+1}^{\theta_{\tau^*}}\right) \mid \mathbf{X}_k^{\theta_{\tau^*}} = \mathbf{x}\right] \leq M \qquad \text{for all } \{\mathbf{x} \in \mathbb{X} : q = 0\}, \tag{4.14a}$$

$$\mathbb{E}\left[V\left(\mathbf{X}_{k+1}^{\theta_{\tau^*}}\right) \mid \mathbf{X}_k^{\theta_{\tau^*}} = \mathbf{x}\right] - V(\mathbf{x}) \leq \lambda - \overline{\nu}^* \qquad \text{for all } \{\mathbf{x} \in \mathbb{X} : q > 0\}. \tag{4.14b}$$

Finally, by Foster's theorem and $\lambda < \overline{\nu}^*$, we conclude that our CMC equips with θ_{τ^*} is stable.

Step 1. The fact that $\mathbf{X}^{\theta_{\tau^*}}$ is irreducible comes from following two observations.

- If the chain starts from any state $\mathbf{x} \in \mathbb{X}$, there exists a path with positive probability where the chain goes to $(1, \mathcal{A}, 0)$. First, we assume there is no arrival, every service successes with only one time step and no increment of activity state, and every rest reduces the activity state by 1. It is clear that either activity state reduces by 1 or the queue length reduces by 1 for every time step. Together with the fact that $\theta_{\tau^*}(1, \mathcal{A}, q_1) = 1$ for every $q \neq 0$, the process will arrive $(1, \mathcal{A}, 0)$ after finite steps. Thus, there exists a path with positive probability that starts from $\mathbf{x} \in \mathbb{X}$ and goes to $(1, \mathcal{A}, 0)$.

- Suppose the chain start from $(1, \mathcal{A}, 0)$, we shall show that there exists a path with positive probability to go to any state $\mathbf{x}' \in \mathbb{X}$. Consider two cases:

  - $\mathbf{x}' = (s', \mathcal{B}, q')$: For this case, it is clear that the following path has positive probability.

$$(1, \mathcal{A}, 0) \to (1, \mathcal{B}, 1) \to \ldots \to (s', \mathcal{B}, 1) \to \ldots \to (s', \mathcal{B}, q')$$

First, a task arrives and the server starts working on it. Then, the server keep working on the task and the activity state increase to s'. Finally, the server still working on the task and $q' - 1$ of tasks arrive.

– $\mathbf{x}' = (s', \mathcal{A}, q')$: For this case, it is clear that the following path has positive probability.

$$(1, \mathcal{A}, 0) \to (1, \mathcal{B}, 1) \to \ldots \to (s', \mathcal{B}, 1) \to \ldots \to (s', \mathcal{B}, q'+1) \to (s', \mathcal{A}, q')$$

The first three steps are exactly the same with previous case except that q' of tasks arrive. The final step is that the server complete the task it is working on and w goes to $\mathcal{A}$.

Combing these two observation, it is clear that for every $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$, there exist a path with positive probability from $\mathbf{x}$ to $\mathbf{x}'$ under policy θ_{τ^*} which implies that MC $\mathbf{X}^{\theta_{\tau^*}}$ is irreducible.

Step 2. We proceed our proof by considering the auxiliary CMC $\overline{\mathbf{Y}}$ and the threshold policy ϕ_{τ^*}. By Proposition 4.1, if our reward function r is given by

$$r(\overline{\mathbf{y}}, \phi) = \mu(\overline{s})\phi(\overline{\mathbf{y}}),$$

there exist a non-negative function f such that, for every $\overline{\mathbf{y}} \in \mathbb{Y}$,

$$\mu(\overline{s})\phi_{\tau^*}(\overline{\mathbf{y}}) - \mathbb{E}\left[f(\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k+1}) - f(\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k})|\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k} = \overline{\mathbf{y}}\right] = r^{\phi_{\tau^*}}_{avg} = \overline{\nu}^*, \tag{4.15}$$

where $r^{\phi_{\tau^*}}_{avg} = \sum_{\mathbf{y}\in\mathbb{Y}} \mu(\overline{s})\phi_{\tau^*}(\overline{\mathbf{y}})\overline{\pi}^{\phi_{\tau^*}}(\overline{\mathbf{y}})$ equals to λ^* by definition. To this end, we define our Lyapunov function $V : \mathbb{X} \to \mathbb{R}+$ to be the sum of the queue length and the potential function f, i.e.,

$$V(\mathbf{x}) \stackrel{\text{def}}{=} q + f(\mathbf{y}).$$

Step 3. In this step, we shall work on the Markov chain $\mathbf{X}^{\theta_{\tau^*}}$ (where we use $\mathbf{X}'$ to represent for reserving space) with a fixed arrival rate $\lambda < \overline{\nu}^*$ and prove that (4.14) holds.

(a)$\{\mathbf{x} \in \mathbb{X} : q = 0\}$ We can directly prove (4.14a) by expending the expected value.

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{X}'_{k+1}\right) \mid \mathbf{X}'_k = (\mathbf{y}, 0)\right] \\
&= \mathbb{E}\left[Q'_{k+1} + f(\mathbf{Y}'_{k+1}) \mid Q'_k = 0, \mathbf{Y}'_k = \mathbf{y}\right] \\
&\leq \lambda + \max_{\overline{\mathbf{y}}\in\mathbb{Y}} f(\overline{\mathbf{y}}) = M.
\end{aligned}$$

The expected queue length upper-bounded by λ since the expected number of arrival is λ. The maximum of $f(\overline{\mathbf{y}})$ exists because there are only finite many elements in $\mathbb{Y}$.

(b)$\{\mathbf{x} \in \mathbb{X} : q > 0\}$ The proof for (4.14b) is similar. Let us expend the expected value.

$$
\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{X}'_{k+1}\right) \mid \mathbf{X}'_k = \mathbf{x}\right] - V(\mathbf{x}) \\
&= \mathbb{E}\left[Q'_{k+1} + f(\mathbf{Y}'_{k+1}) \mid \mathbf{X}'_k = \mathbf{x}\right] - q - f(\mathbf{y}) \\
&= \mathbb{E}\left[Q'_{k+1} - Q'^{1}_{k} \mid \mathbf{X}'_k = \mathbf{x}\right] + \mathbb{E}\left[f(\mathbf{Y}'_{k+1}) - f(\mathbf{Y}'_k) \mid \mathbf{X}'_k = \mathbf{x}\right]. \qquad (4.16)
\end{aligned}
$$

The first term is the expected queue length change which can be represented by the expected arrival minus expected service, i.e.,

$$
\begin{aligned}
&\mathbb{E}\left[Q'_{k+1} - Q'_k \mid \mathbf{X}'_k = \mathbf{x}\right] \\
&= \lambda - \mu(s)\theta_{\tau^*}(\mathbf{x}) \\
&= \lambda - \mu(s)\phi_{\tau^*}(\mathbf{y}),
\end{aligned}
$$

where we use an important observation that the policy θ_{τ^*} is identical to ϕ_{τ^*} when queue length is not zero. By using the same fact, the second term can be written as follows:

$$
\mathbb{E}\left[f(\mathbf{Y}'_{k+1}) - f(\mathbf{Y}'_k) \mid \mathbf{X}'_k = \mathbf{x}\right] = \mathbb{E}\left[f(\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k+1}) - f(\overline{\mathbf{Y}}^{\phi_{\tau^*_1}}_{k}) | \overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k} = \mathbf{y}\right]
$$

Together with (4.15) tells us that

$$
\begin{aligned}
(4.16) &= \lambda - \mu(s)\phi_{\tau^*}(\mathbf{y}) + \mathbb{E}\left[f(\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k+1}) - f(\overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k}) | \overline{\mathbf{Y}}^{\phi_{\tau^*}}_{k} = \mathbf{y}\right] \\
&= \lambda - \overline{\nu}^* < 0.
\end{aligned}
$$

Therefore, the Markov Chain $\mathbf{X}^{\theta_{\tau^*}}$ [87] is stable by Foster's theorem and the assumption that λ is strictly smaller than $\overline{\nu}^*$.

4.3 Utilization Rate: Definition and Infimum

To answer Problem 3.3 and Problem 3.4, we first give the formal definition of the utilization rate.

Definition 4.2. (Utilization rate function) *The function that determines the utilization rate in terms of a given stabilizable arrival rate λ and a stabilizing policy θ, is defined as:*

$$\mathscr{U}(\lambda, \theta) := \sum_{\mathbf{x} \in \mathbb{X}} \pi^\theta(\mathbf{x})\theta(\mathbf{x}), \quad \lambda \in (0, \overline{\nu}^*), \ \theta \in \Theta_S(\lambda) \tag{4.17}$$

The utilization rate quantifies the proportion of the time in which the server is working. Notably, the expected utilization rate $\mathscr{U}(\lambda, \theta)$, computed for $\mathbf{X}$ with arrival late λ and stabilized by θ, coincides with the probability limit of the utilization rate, as defined for instance in [88] (with $\mathbb{U} = \{0, 1\}$), when the averaging horizon tends to infinity. Using our notation, the aforesaid probability limit can be stated as follows:

$$\operatorname*{plim}_{N \to \infty} \frac{\sum_{k=0}^{N} \mathcal{I}_{A_k = \mathcal{W}}}{N+1} = \mathscr{U}(\lambda, \theta), \quad \lambda \in (0, \overline{\nu}^*), \ \theta \in \Theta_S(\lambda)$$

where $\mathcal{I}_{A_k = \mathcal{W}}$ is 1 when $A_k = \mathcal{W}$ and 0 otherwise.

Definition 4.3. *The infimum utilization rate for a given stabilizable arrival rate λ*

is defined as:

$$\mathscr{U}^*(\lambda) := \inf_{\theta \in \Theta_S(\lambda)} \mathscr{U}(\lambda, \theta), \quad \lambda \in (0, \overline{\nu}^*) \tag{4.18}$$

4.4 Service and Utilization Rate of $\overline{\mathbf{Y}}$

Recall that we define the service rate of $\overline{\mathbf{Y}}^{\phi}$ for a given policy ϕ in Φ_R^+:

$$\overline{\nu}^{\phi} \stackrel{\text{def}}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}) \phi(\overline{\mathbf{y}}) \overline{\pi}^{\phi}(\overline{\mathbf{y}}).$$

The maximal service rate $\overline{\nu}^*$ for $\overline{\mathbf{Y}}$ is defined below.

$$\overline{\nu}^* \stackrel{\text{def}}{=} \sup_{\phi \in \Phi_R^+} \overline{\nu}^{\phi}$$

As stated Theorem 4.1 and Theorem 4.2, any arrival rate λ lower than $\overline{\nu}^*$ is stabilizable.

Definition 4.4. *We define the map $\mathscr{X} : \Phi_R^+ \to \Theta_R^+$ as follows:*

$$\mathscr{X}(\phi) \stackrel{def}{=} \vartheta^{\phi}, \quad \phi \in \Phi_R^+$$

where

$$\vartheta^{\phi}(\mathbf{x}) \stackrel{def}{=} \begin{cases} \phi(\mathbf{y}) & \text{if } q > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{x} \in \mathbb{X} \tag{4.19}$$

It follows from its definition that $\mathscr{X}$ yields a policy for $\mathbf{X}$ that acts as the given ϕ in Φ_R^+ when the queue is not empty and imposes rest otherwise.

Convention We reserve $\overline{\nu}$, without a superscript, to denote a design parameter. It acts as a constraint in the definition of the following policy sets.

Definition 4.5. (Policy sets $\Phi_R^{\epsilon}(\overline{\nu})$ and $\Phi_R^+(\overline{\nu})$) *Given $\overline{\nu}$ in $(0, \overline{\nu}^*)$, we define the following policy sets:*

$$\Phi_R^+(\overline{\nu}) \overset{def}{=} \{\phi \in \Phi_R^+ \mid \overline{\nu}^{\phi} = \overline{\nu}\}$$

$$\Phi_R^{\epsilon}(\overline{\nu}) \overset{def}{=} \{\phi \in \Phi_R^{\epsilon} \mid \overline{\nu}^{\phi} = \overline{\nu}\}, \quad \epsilon \in [0,1]$$

where Φ_R^{ϵ} is defined as:

$$\Phi_R^{\epsilon} := \{\phi \in \Phi_R \mid \phi(1, \mathcal{A}) \geq \epsilon\}, \quad \epsilon \in [0,1]$$

We also define the following class of policies generated from $\Phi_R^+(\overline{\nu})$ and $\Phi_R^{\epsilon}(\overline{\nu})$ through $\mathscr{X}$:

$$\mathscr{X}\Phi_R^{\epsilon}(\overline{\nu}) \overset{\text{def}}{=} \{\mathscr{X}(\phi) \mid \phi \in \Phi_R^{\epsilon}(\overline{\nu})\}, \quad \overline{\nu} \in (0, \overline{\nu}^*),\ \epsilon \in (0,1]$$

$$\mathscr{X}\Phi_R^+(\overline{\nu}) \overset{\text{def}}{=} \{\mathscr{X}(\phi) \mid \phi \in \Phi_R^+(\overline{\nu})\}, \quad \overline{\nu} \in (0, \overline{\nu}^*)$$

The following proposition establishes important stabilization properties for the policies in $\mathscr{X}\Phi_R^+(\overline{\nu})$.

Proposition 4.2. *Let the arrival rate λ in $(0, \overline{\nu}^*)$ be given. If $\overline{\nu}$ is in $(\lambda, \overline{\nu}^*)$ then $\mathbf{X}^\theta$ is stable, irreducible and aperiodic for any θ in $\mathscr{X}\Phi_R^+(\overline{\nu})$.*

Proof. Stability of $\mathbf{X}^\theta$ can be established using the same method adopted in Section 4.2.2 to prove Theorem 4.2. □

An immediate consequence of Proposition 4.2 is that $\{\mathscr{X}(\phi) | \phi \in \Phi_R^+(\overline{\nu})\}$ is a nonempty subset of $\Theta_S(\lambda)$ when $\lambda < \overline{\nu} \leq \overline{\nu}^*$. This implies that, as far a stabilizability is concerned, there is no loss of generality in restricting our analysis to policies with the structure in (4.19). More interestingly, from Theorem 4.3, which will be stated and proved later on in Section 4.5, we can conclude that restricting our methods for solving Problem 3.4 to policies of the form (4.19) also incurs no loss of generality.

Recall that we define a policy projection map $\mathscr{Y}$ in Definition 3.7. Notice that although the map $\mathscr{Y}$ depends on λ, for simplicity of notation, we chose not to denote that explicitly. It is worthwhile to note that the map $\mathscr{Y}$, for a given λ less than $\overline{\nu}^*$, allows us to establish the following remark comparing the service rate notions for $\mathbf{X}$ and $\overline{\mathbf{Y}}$.

Remark 4. *Given λ in $(0, \overline{\nu}^*)$ and $\overline{\nu}$ in $(\lambda, \overline{\nu}^*)$, the following hold:*

$$\lambda \overset{(i)}{=} \nu^\theta \overset{(ii)}{=} \overline{\nu}^{\mathscr{Y}(\theta)} \leq \overline{\nu}^*, \qquad \theta \in \Theta_S(\lambda) \tag{4.20a}$$

$$\lambda \overset{(iii)}{=} \nu^{\mathscr{X}(\phi)} < \overline{\nu} \leq \overline{\nu}^*, \qquad \phi \in \Phi_R^+(\overline{\nu}) \tag{4.20b}$$

Notably, (i) and (ii) follow from Lemma 4.1. Using a similar argument, (iii) follows *from the fact that $\mathscr{X}(\phi)$ is stabilizing, as guaranteed by Proposition 4.2 when $\overline{\nu}$ is in $(\lambda, \overline{\nu}^*)$.*

4.4.1 Utilization rate of $\overline{\mathbf{Y}}$ and computation via LP

We now proceed to defining the utilization rate of $\overline{\mathbf{Y}}^{\phi}$ for a given ϕ in Φ_R. Subsequently, we will define and propose a linear programming approach to computing the infimum of the utilization rates attainable by any policy for $\overline{\mathbf{Y}}$ subject to a given service rate.

Definition 4.6. *Given a policy ϕ in Φ_R^+, the following function determines the utilization rate of $\overline{\mathbf{Y}}^{\phi}$:*

$$\bar{\mathscr{U}}(\phi) := \sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \overline{\pi}^{\phi} \phi(\overline{\mathbf{y}}) \tag{4.21}$$

Definition 4.7. (Infimum utilization rate $\bar{\mathscr{U}}_R^+$ and $\bar{\mathscr{U}}_R^{\epsilon}$) *The infimum utilization rate of $\overline{\mathbf{Y}}$ for a given departure rate $\overline{\nu}$ is defined as:*

$$\bar{\mathscr{U}}_R^+(\overline{\nu}) := \inf_{\phi \in \Phi_R^+(\overline{\nu})} \sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}}) \tag{4.22}$$

We also define the following approximate infimum utilization rates:

$$\bar{\mathscr{U}}_R^{\epsilon}(\overline{\nu}) := \inf_{\phi \in \Phi_R^{\epsilon}(\overline{\nu})} \sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}}) \tag{4.23}$$

Notice that the infimum that determines $\bar{\mathscr{U}}_R^+$ and $\bar{\mathscr{U}}_R^{\epsilon}$ is well-defined because there is a unique stationary PMF $\overline{\pi}^{\phi}$ for each policy ϕ in Φ_R^+.

Remark 5. *Notice that since* $\Phi_R^+(\overline{\nu}) = \bigcup_{\epsilon\in(0,1]} \Phi_R^\epsilon(\overline{\nu})$*, we conclude the that following holds:*

$$\bar{\mathscr{U}}_R^+(\overline{\nu}) = \lim_{\epsilon\to 0^+} \bar{\mathscr{U}}_R^\epsilon(\overline{\nu}) \tag{4.24}$$

We now proceed to outlining efficient ways to compute $\bar{\mathscr{U}}_R^+$, which is relevant because, as Corollary 4.1, we can use it to compute $\mathscr{U}^*(\lambda)$ when $\lambda < \overline{\nu}^*$. Hence, below we follow the approach in [89, Chapter 4] to construct approximate versions of $\bar{\mathscr{U}}_R^\epsilon$ that are computable using a finite-dimensional linear program (LP). Subsequently, we will obtain the policies in Φ_R^+ corresponding to solutions of the LP, as is done in [89, Chapter 4]. The policies obtained in this way will form a set for each ϵ in $(0, 1)$ that will be useful later on.

Definition 4.8. (ϵ-LP utilization rate $\bar{\mathscr{U}}_{\mathbb{L}}^\epsilon(\overline{\nu})$)

Let ϵ *be a given constant in* $[0, 1]$ *and* $\overline{\nu}$ *be a pre-selected departure rate in* $(0, \overline{\nu}^*)$*. The* ϵ*-LP utilization rate* $\bar{\mathscr{U}}_{\mathbb{L}}^\epsilon(\overline{\nu})$ *is defined as:*

$$\bar{\mathscr{U}}_{\mathbb{L}}^\epsilon(\overline{\nu}) := \min_{\ell\in\mathbb{L}} \sum_{\overline{\mathbf{y}}\in\overline{\mathbb{Y}}} \ell_{\overline{\mathbf{y}},\mathcal{W}} \tag{4.25a}$$

(4.25b)-(4.25e)

where the minimization is carried out over the following set:

$$\mathbb{L} := \Pi_{\overline{a}\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}},\overline{\mathbf{y}}\in\mathbb{Y}}\{\ell_{\overline{\mathbf{y}},\overline{a}} \geq 0\}$$

Every solution is subject to the following constraints and is compactly represented *as* $\ell := \Pi_{\overline{a}\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}},\overline{\mathbf{y}}\in\mathbb{Y}}\{\ell_{\overline{\mathbf{y}},\overline{a}}\}$*:*

$$(1-\epsilon)\ell_{(1,\mathcal{A}),\mathcal{W}} \geq \epsilon\ell_{(1,\mathcal{A}),\mathcal{R}} \tag{4.25b}$$

$$\sum_{\{\overline{\mathbf{y}}\in\overline{\mathbb{Y}}|\mathcal{W}\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}}\}} \mu(\overline{s})\ell_{\overline{\mathbf{y}},\mathcal{W}} = \overline{\nu} \tag{4.25c}$$

$$\sum_{\overline{\mathbf{y}}\in\overline{\mathbb{Y}}}\sum_{\overline{a}\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}}} \ell_{\overline{\mathbf{y}},\overline{a}} = 1 \tag{4.25d}$$

and the equality below guarantees that every solution will be consistent with $\overline{\mathbf{Y}}$*:*

$$\sum_{\overline{\mathbf{y}}'\in\overline{\mathbb{Y}}}\sum_{\overline{a}'\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}'}} \ell_{\overline{\mathbf{y}}',\overline{a}'} \Pr\left(\overline{\mathbf{Y}}_{t+1} = \overline{\mathbf{y}} \middle| \overline{\mathbf{Y}}_t = \overline{\mathbf{y}}', \overline{A}_t = \overline{a}'\right) = \sum_{\overline{a}\in\overline{\mathbb{A}}_{\overline{\mathbf{y}}}} \ell_{\overline{\mathbf{y}},\overline{a}}, \quad \overline{\mathbf{y}} \in \overline{\mathbb{Y}} \tag{4.25e}$$

Definition 4.9. (Solution set $\mathbb{L}^{\epsilon}(\overline{\nu})$) *For each ϵ in $[0,1]$ and $\overline{\nu}$ in $(0,\overline{\nu}^*)$, we use $\mathbb{L}^{\epsilon}(\overline{\nu})$ to represent the set of solutions of the LP specified by (4.25). We adopt the convention that $\mathbb{L}^{\epsilon}(\overline{\nu})$ is empty when the LP is not feasible.*

4.4.2 LP-based policy sets

For each solution ℓ in $\mathbb{L}^{\epsilon}(\overline{\nu})$ we can obtain a corresponding policy φ_ℓ in Φ_R for $\overline{\mathbf{Y}}$ as follows:

$$\varphi_\ell(\overline{\mathbf{y}}) := \begin{cases} \frac{\ell_{\overline{\mathbf{y}},\mathcal{W}}}{\ell_{\overline{\mathbf{y}},\mathcal{W}}+\ell_{\overline{\mathbf{y}},\mathcal{R}}} & \text{if } \mathcal{R} \in \overline{\mathbb{A}}_{\overline{\mathbf{y}}} \text{ and } \ell_{\overline{\mathbf{y}},\mathcal{R}} > 0 \\ 1 & \text{otherwise.} \end{cases}, \ \overline{\mathbf{y}} \in \overline{\mathbf{Y}} \tag{4.26}$$

Remark 6. *Subject to the definition in (4.26), the constraint (4.25b) is equivalent to* $\varphi_\ell(1,\mathcal{A}) \geq \epsilon$*, which will, then, hold for every solution* ℓ *in* $\mathbb{L}^{\epsilon}(\overline{\nu})$*.*

Definition 4.10. (Policy set $\Phi^{\epsilon}_{\mathbb{L}}(\overline{\nu})$) *For each* $\overline{\nu}$ *in* $(0,\overline{\nu}^*)$ *and* ϵ *in* $[0,1]$*, we define the following set of policies* $\Phi^{\epsilon}_{\mathbb{L}}(\overline{\nu})$*:*

$$\Phi^{\epsilon}_{\mathbb{L}}(\overline{\nu}) := \{\varphi_\ell \mid \ell \in \mathbb{L}^{\epsilon}(\overline{\nu})\} \tag{4.27}$$

Here, we adopt the convention that $\Phi^{\epsilon}_{\mathbb{L}}(\overline{\nu})$ *is empty if and only if* $\mathbb{L}^{\epsilon}(\overline{\nu})$ *is empty.*

The following proposition will justify choices for ϵ we will make at a later stage to guarantee that $\Phi^{\epsilon}_{\mathbb{L}}(\overline{\nu})$ is nonempty for $\overline{\nu}$ in $(\lambda,\overline{\nu}^*)$.

Proposition 4.3. *If* ϵ^* *in* $(0,1]$ *is such that* $\Phi^{\epsilon^*}_{\mathbb{L}}(\lambda)$ *is nonempty then* $\Phi^{\bar{\epsilon}}_{\mathbb{L}}(\overline{\nu})$ *is nonempty for any* $\bar{\epsilon}$ *in* $(0,\epsilon^*]$ *and* $\overline{\nu}$ *in* $[\lambda,\overline{\nu}^*]$*.*

Proof. We start by invoking Lemma 4.2 to conclude that $\Phi^{1}_{\mathbb{L}}(\overline{\nu}^*)$ is nonempty, and consequently that $\Phi^{\epsilon^*}_{\mathbb{L}}(\overline{\nu}^*)$ is also nonempty. If ℓ_λ and $\ell_{\overline{\nu}^*}$ are in $\mathbb{L}^{\epsilon^*}(\lambda)$ and $\mathbb{L}^{\epsilon^*}(\overline{\nu}^*)$, respectively, then from (4.25) we conclude that, for any $\overline{\nu}$ in $[\lambda, \overline{\nu}^*]$, $\ell_{\overline{\nu}}$ defined below satisfies (4.25b)-(4.25e), which implies that $\mathbb{L}^{\epsilon^*}(\overline{\nu})$ is nonempty:

$$\ell_{\overline{\nu}} := \left(\frac{\overline{\nu} - \lambda}{\overline{\nu}^* - \lambda} \ell_{\overline{\nu}^*} + \frac{\overline{\nu}^* - \overline{\nu}}{\overline{\nu}^* - \lambda} \ell_\lambda \right) \tag{4.28}$$

That $\mathbb{L}^{\epsilon^*}(\overline{\nu})$ is nonempty implies that $\mathbb{L}^{\bar{\epsilon}}(\overline{\nu})$ is also nonempty for any $\bar{\epsilon}$ in $(0, \epsilon^*]$, which concludes the proof. □

Before we proceed with stating a proposition that has important implications for design, we define the following notion of dominance also used in [89].

Definition 4.11. (Policy set dominance) *Let $\overline{\nu}$ in $(0, \overline{\nu}^*)$ and any two subsets $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ of $\Phi_R^+(\overline{\nu})$ be given. We say that $\tilde{\Phi}_1$ dominates $\tilde{\Phi}_2$ if for each policy ϕ_2 in $\tilde{\Phi}_2$ there is ϕ_1 in $\tilde{\Phi}_1$ for which $\bar{\mathscr{U}}(\phi_1) \leq \bar{\mathscr{U}}(\phi_2)$.*

Proposition 4.4. *Given $\overline{\nu}$ in $(0, \overline{\nu}^*)$ and ϵ in $(0, 1]$, $\Phi_{\mathbb{L}}^\epsilon(\overline{\nu})$ dominates $\Phi_R^\epsilon(\overline{\nu})$ and the equality below holds:*

$$\bar{\mathscr{U}}_R^\epsilon(\overline{\nu}) = \bar{\mathscr{U}}_{\mathbb{L}}^\epsilon(\overline{\nu}) \tag{4.29a}$$

$$\bar{\mathscr{U}}_R^+(\overline{\nu}) = \bar{\mathscr{U}}_{\mathbb{L}}^0(\overline{\nu}) \tag{4.29b}$$

Proof. It follows immediately from [89, Theorem 4.3] that (4.29a) holds and $\Phi_{\mathbb{L}}^\epsilon(\overline{\nu})$ dominates $\Phi_R^\epsilon(\overline{\nu})$. We proceed by noticing that for any given fixed $\overline{\nu}$ in $(0, \overline{\nu}^*)$, $\bar{\mathscr{U}}_{\mathbb{L}}^\epsilon(\overline{\nu})$ is non-decreasing with respect to ϵ. Hence, continuity arguments applied to (4.25) lead to the following equality:

$$\lim_{\epsilon \to 0^+} \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}) = \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}) \tag{4.30}$$

That (4.29b) holds is a consequence of (4.24), (4.29a) and (4.30). □

4.5 Utilization Rate Results

This section starts with Theorem 4.3, which is our main result on utilization rate. Subsequently, we state Corollaries 4.1 and 4.2 that undergird our methods to tackle Problems 3.3 and 3.4, respectively.

Before stating the theorem, we define the following class of policies for **X** that can be generated from solutions of the LP:

$$\mathscr{X}\Phi_{\mathbb{L}}^{\epsilon}(\overline{\nu}) \stackrel{\text{def}}{=} \{\mathscr{X}(\phi) \mid \phi \in \Phi_{\mathbb{L}}^{\epsilon}(\overline{\nu})\}, \overline{\nu} \in (0, \overline{\nu}^*),\ \epsilon \in (0,1]$$

Theorem 4.3. *Let an arrival rate* λ *in* $(0, \overline{\nu}^*)$ *be given. For each positive gap* δ *there is a service rate* $\overline{\nu}^{\delta,\lambda}$ *in* $(\lambda, \overline{\nu}^*)$ *and* $\epsilon^{\delta,\lambda}$ *in* $(0,1]$ *for which* $\Phi_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}^{\delta,\lambda})$ *is nonempty and the following inequality holds:*

$$\mathscr{U}(\lambda, \theta) \leq \bar{\mathscr{U}}_R^{+}(\lambda) + \delta, \quad \theta \in \mathscr{X}\Phi_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}^{\delta,\lambda}) \tag{4.31}$$

Our proof of Theorem 4.3 given below relies on the continuity properties and distributional convergence results established in Section 4.5.1 and Section 4.5.2, respectively.

Proof. Since it follows Theorem 4.4 in Section 4.5.1 that $\bar{\mathscr{U}}_{\mathbb{L}}^{0}$ is continuous and nondecreasing, we know that there is $\overline{\nu}^{\dagger}$ in $(\lambda, \overline{\nu}^{*})$ such that the following inequality holds:

$$\bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}^{\dagger}) \leq \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\lambda) + \tfrac{1}{3}\delta \tag{4.32}$$

Let $\epsilon^{\dagger}$ be such that $\Phi_{\mathbb{L}}^{\epsilon^{\dagger}}(\lambda)$ is nonempty. From Proposition 4.6 we know that we can select $\epsilon^{\ddagger}$ in $(0, \epsilon^{\dagger}]$ such that the following holds:

$$\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}^{\dagger}) \leq \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}^{\dagger}) + \tfrac{1}{3}\delta, \quad \epsilon \in (0, \epsilon^{\ddagger}) \tag{4.33}$$

From Proposition 4.7 in Section 4.5.1 we know that we can select $\epsilon^{\delta,\lambda}$ in $(0, \epsilon^{\ddagger})$ such that the following holds:

$$\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}) \leq \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}^{\dagger}), \quad \overline{\nu} \in (\lambda, \overline{\nu}^{\dagger}) \tag{4.34}$$

In Section 4.5.2 we develop in sequence several results that ultimately lead to Theorem 4.5, which establishes an important distributional convergence result that takes hold when $\overline{\nu}$ in $(\lambda, \overline{\nu}^{\dagger})$ is selected as close as needed to λ. Using Corollary 4.3 stated also in Section 4.5.2, which follows immediately from Theorem 4.5, we conclude that, based on our choice of $\epsilon^{\delta,\lambda}$ above, we can select $\overline{\nu}^{\delta,\lambda}$ in $(\lambda, \overline{\nu}^{\dagger})$ such that the following inequality holds:

$$\mathscr{U}\big(\lambda, \mathscr{X}(\phi)\big) \leq \bar{\mathscr{U}}(\phi) + \tfrac{1}{3}\delta, \quad \phi \in \Phi_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}^{\delta,\lambda}) \tag{4.35}$$

Hence, using our choices for $\epsilon^{\delta,\lambda}$ and $\overline{\nu}^{\delta,\lambda}$ we infer from (4.32)-(4.35) that the following inequality holds:

$$\mathscr{U}\left(\lambda, \mathscr{X}(\phi)\right) \leq \bar{\mathscr{U}}^{0}_{\mathbb{L}}(\lambda) + \delta, \quad \phi \in \Phi_{\mathbb{L}}^{\epsilon^{\delta,\lambda}}(\overline{\nu}^{\delta,\lambda}) \tag{4.36}$$

which, together with (4.29b), leads to (4.31). □

We proceed with stating a proposition that provides an utilization-rate counterpart for (b) in (4.20a) and whose proof we omit because it follows immediately from Lemma 3.3 and Lemma 4.1.

Proposition 4.5. *Given λ in $(0, \overline{\nu}^*)$, the following equality holds for any θ in $\Theta_S(\lambda)$:*

$$\bar{\mathscr{U}}\left(\mathscr{Y}(\theta)\right) = \mathscr{U}(\lambda, \theta) \tag{4.37}$$

Corollary 4.1. *The following equality holds:*

$$\mathscr{U}^*(\lambda) = \bar{\mathscr{U}}^{+}_{R}(\lambda), \quad \lambda \in (0, \overline{\nu}^*) \tag{4.38}$$

Proof. It ensues from Proposition 4.5 and (i)-(ii) in (4.20a) that the following holds for any λ in $(0, \overline{\nu}^*)$:

$$\mathscr{U}(\lambda, \theta) = \bar{\mathscr{U}}\left(\mathscr{Y}(\theta)\right) \geq \bar{\mathscr{U}}^{+}_{R}(\lambda), \quad \theta \in \Theta_S(\lambda) \tag{4.39}$$

Since the inequality above holds for any θ in $\Theta_S(\lambda)$ we conclude that the following inequality is satisfied for any λ in $(0, \overline{\nu}^*)$:

$$\mathscr{U}^*(\lambda) \geq \bar{\mathscr{U}}_R^+(\lambda) \tag{4.40}$$

We conclude the proof by remarking that (4.40) and Theorem 4.3 imply (4.38). □

Remark 7. *Corollary 4.1 is significant because, in conjunction with Proposition 4.4, it indicates that $\mathscr{U}^*$ can be determined using the finite dimensional LP (4.25) with $\epsilon = 0$.*

The following corollary follows directly from Theorem 4.3 and Corollary 4.1.

Corollary 4.2. *Let an arrival rate λ in $(0, \overline{\nu}^*)$ be given. For each positive gap δ there is a service rate $\overline{\nu}$ in $(\lambda, \overline{\nu}^*)$ and $\bar{\epsilon}$ in $(0, 1]$ for which $\Phi_{\mathbb{L}}^{\bar{\epsilon}}(\overline{\nu})$ is nonempty and the following inequality holds:*

$$\mathscr{U}(\lambda, \theta) \leq \mathscr{U}^*(\lambda) + \delta, \quad \theta \in \mathscr{X}\Phi_{\mathbb{L}}^{\bar{\epsilon}}(\overline{\nu}) \tag{4.41}$$

While, as explained in Remark 7, $\mathscr{U}^*(\lambda)$ can be computed effectively for any stabilizable λ, Corollary 4.2 guarantees that we can address Problem 3.4 by appropriately selecting $\overline{\nu}$ and ϵ to construct policies for $\mathbf{X}$ whose utilization rate is arbitrarily close to the fundamental limit quantified by $\mathscr{U}^*(\lambda)$. The proof of Theorem 4.3 outlines a method for selecting such $\overline{\nu}$ and ϵ.

4.5.1 Continuity and monotonicity properties of ϵ-LP

We proceed with establishing three properties of $\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}$ that are needed in the proof of our main results.

The following proposition establishes that when, for a given $\overline{\nu}$ in $(0, \overline{\nu}^*)$, $\bar{\mathscr{U}}^{\epsilon}(\overline{\nu})$ is viewed as a function of ϵ it is right continuous at 0.

Proposition 4.6. *Let $\overline{\nu}$ in $(0, \overline{\nu}^*)$ be given. For any positive δ there is ϵ such that the following holds:*

$$\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}) \leq \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}) + \delta \tag{4.42}$$

Proof. Select $\overline{\nu}$ arbitrarily in $(0, \overline{\nu}^*)$. The statement of the proposition is false if and only if the following inequality holds:

$$\underline{\mathrm{d}} := \lim_{\epsilon \to 0^+} \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}) - \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}) > 0 \tag{4.43}$$

We proceed to proving the proposition by contradiction by showing that the inequality above does not hold. Take ϵ positive such that $d := \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}) - \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu})$ is in $[\underline{\mathrm{d}}, 2\underline{\mathrm{d}})$. Select ℓ^{ϵ} and ℓ^0 in $\mathbb{L}^{\epsilon}(\overline{\nu})$ and $\mathbb{L}^0(\overline{\nu})$, respectively. Define $\ell_{av} := \frac{1}{3}(\ell^{\epsilon} + 2\ell^0)$, which satisfies (4.25c)-(4.25e). Given that ϵ is positive, ℓ_{av} will also satisfy (4.25b) for some ϵ^* positive, which implies that $\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^*}(\overline{\nu}) - \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}) \leq \frac{1}{3}d \leq \frac{2}{3}\underline{\mathrm{d}}$. □

The following proposition establishes a useful monotonicity property in terms of $\overline{\nu}$.

Proposition 4.7. *Let $\overline{\nu}^{\dagger}$ and $\overline{\nu}^{\ddagger}$ in $(0, \overline{\nu}^{*})$ be given with $\overline{\nu}^{\dagger} < \overline{\nu}^{\ddagger}$. There exists a positive ϵ^{*} such that the following holds:*

$$\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}) \leq \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}^{\ddagger}), \quad \overline{\nu} \in (\overline{\nu}^{\dagger}, \overline{\nu}^{\ddagger}) \tag{4.44}$$

Proof. From (4.25a), (4.25c), and the fact that $\min_{s\in\mathbb{S}} \mu(s)$ is positive, we get

$$\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon}(\overline{\nu}) \leq \frac{1}{\min_{s\in\mathbb{S}} \mu(s)} \overline{\nu}, \quad \epsilon \in [0, 1] \tag{4.45}$$

We can find a $\overline{\nu}^{-} < \overline{\nu}^{\dagger}$ such that the following inequality holds:

$$\frac{1}{\min_{s\in\mathbb{S}} \mu(s)} \overline{\nu}^{-} \leq \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}^{\ddagger}) \tag{4.46}$$

Let ϵ^{*} be such that $\mathbb{L}^{\epsilon^{*}}(\overline{\nu}^{-})$ is not empty. Select $\ell^{\overline{\nu}^{-}}$ and $\ell^{\overline{\nu}^{\ddagger}}$ in $\mathbb{L}^{\epsilon^{*}}(\overline{\nu}^{-})$ and $\mathbb{L}^{\epsilon^{*}}(\overline{\nu}^{\ddagger})$, respectively. From (4.25) we conclude that, for any $\overline{\nu}$ in $(\overline{\nu}^{\dagger}, \overline{\nu}^{\ddagger})$, $\ell_{\overline{\nu}}$ defined below satisfies (4.25b)-(4.25e) with ϵ^{*} and $\overline{\nu}$:

$$\ell^{\overline{\nu}} := \left(\frac{\overline{\nu} - \overline{\nu}^{-}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \ell^{\overline{\nu}^{\ddagger}} + \frac{\overline{\nu}^{\ddagger} - \overline{\nu}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \ell^{\overline{\nu}^{-}} \right) \tag{4.47}$$

Furthermore, from (4.25a), (4.45), and (4.46), we obtain the following inequalities:

$$\begin{aligned}
\bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}) &\leq \frac{\overline{\nu} - \overline{\nu}^{-}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}^{\ddagger}) + \frac{\overline{\nu}^{\ddagger} - \overline{\nu}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}^{-}) \\
&\leq \frac{\overline{\nu} - \overline{\nu}^{-}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}^{\ddagger}) + \frac{\overline{\nu}^{\ddagger} - \overline{\nu}}{\overline{\nu}^{\ddagger} - \overline{\nu}^{-}} \bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}^{\ddagger}) \\
&\leq \bar{\mathscr{U}}_{\mathbb{L}}^{\epsilon^{*}}(\overline{\nu}^{\ddagger}),
\end{aligned}$$

which complete the proof. $\square$

The following theorem establishes important structural properties for $\bar{\mathscr{U}}^0_{\mathbb{L}}$. We provide a proof of the theorem in Appendix A.1.

Theorem 4.4. *The* 0*-LP utilization rate function* $\bar{\mathscr{U}}^0_{\mathbb{L}} : (0, \overline{\nu}^*) \to [0, 1]$ *is nondecreasing, piecewise affine and convex.*

4.5.2 Key Distributional Convergence Results

In this section, we provide useful distributional convergence results that are used in the proof of Theorem 4.3.

Lemma 4.5. *Let* λ *in* $(0, \overline{\nu}^*)$ *and* ϵ *in* $(0, 1)$ *be given. If* $\Phi^\epsilon_R(\lambda)$ *is nonempty then there is a positive constant* $\beta_{\lambda,\epsilon}$ *such that the following inequality holds for every* $\overline{\nu} \in (\lambda, \overline{\nu}^*)$*:*

$$\sum_{s \in \mathbb{S}} \pi^\theta_\lambda(s, \mathcal{A}, 0) \leq \frac{(\overline{\nu} - \lambda)}{\beta_{\lambda,\epsilon}}, \qquad \theta \in \mathscr{X}\,\Phi^\epsilon_R(\overline{\nu}) \tag{4.48}$$

Before we proceed with the proof of Lemma 4.5, we note that one should expect it be somewhat involved because it needs to ascertain that the inequality in (4.48) holds (uniformly) for all policies in $\mathscr{X}\,\Phi^\epsilon_R(\overline{\nu})$.

Proof. Select $\overline{\nu}$ in $(\lambda, \overline{\nu}^*)$, and let ϕ be any policy in $\Phi^\epsilon_R(\overline{\nu})$, which we know from Proposition 4.3 is nonempty, and set $\theta = \mathscr{X}(\phi)$. Henceforth, $\mathbf{X}^\theta_k$ is the state of $\mathbf{X}^\theta$, which is stable (see Proposition 4.2). In our proof we will make use of Proposition 4.1 by selecting $\overline{\mathbf{Y}}^\phi$ and $r(\mathbf{y}, \phi) = \mu(s)\phi(\mathbf{y})$, for all $\mathbf{y}$ in $\mathbb{Y}$, where we recall that

$\mathbf{y} := (s, w)$. We define $s^* := \arg\max_{s \in \mathbb{S}} f(s, \mathcal{A})$, where f is the potential-like map obtained from Proposition 4.1 for the aforementioned choices of $\overline{\mathbf{Y}}^{\phi}$ and r.

The following visit time will be central in our proof:

$$T^{\theta}_{\mathbf{x}} := \min\{k \geq 1 \mid \mathbf{X}^{\theta}_k = (s^*, \mathcal{A}, 0),\ \mathbf{X}_0 = \mathbf{x}\} \tag{4.49}$$

where we adopt the convention that $T_{\mathbf{x}}$ is infinite if $\mathbf{X}_k = (s^*, \mathcal{A}, 0)$ never occurs for $k \geq 1$. We also will use the following lower bound:

$$\underline{\mathrm{T}}^{\theta}_{\mathbf{x}} := \min\{k \geq 1 \mid \mathscr{V}(\mathbf{X}^{\theta}_k) \leq v^*,\ \mathbf{X}_0 = \mathbf{x}\} \tag{4.50}$$

where $\mathscr{V}(\mathbf{x}) := q + f(\mathbf{y})$ and $v^* := f(s^*, \mathcal{A})$. Here, we also adopt the convention that $\underline{\mathrm{T}}^{\theta}_{\mathbf{x}}$ is infinite if $\mathscr{V}(\mathbf{X}^{\theta}_k) \leq v^*$ never occurs for $k \geq 1$. Notice that since $\mathscr{V}(s^* \mathcal{A}, 0) = v^*$, the following inequality holds:

$$\underline{\mathrm{T}}^{\theta}_{\mathbf{x}} \leq T^{\theta}_{\mathbf{x}}, \quad \mathbf{x} \in \mathbb{X} \tag{4.51}$$

Subsequently, we use $T^{\theta}_{\mathbf{x}}$, $\underline{\mathrm{T}}^{\theta}_{\mathbf{x}}$ and $\mathscr{V}$ to obtain a lower bound for $\mathbb{E}[T_{(s^*, \mathcal{A}, 0)}]$ - the recurrence time of $(s^*, \mathcal{A}, 0)$ - which will ultimately lead to the proof of (4.48).

As we argue subsequently, the following lower bound for $\mathbb{E}[\underline{\mathrm{T}}_{(s^*, \mathcal{A}, 1)}]$, which we will derive later in this proof, leads to (4.48) almost immediately:

$$\mathbb{E}[\underline{\mathrm{T}}^{\theta}_{(s^*, \mathcal{A}, 1)}] \geq \frac{1}{\overline{\nu} - \lambda} \tag{4.52}$$

We start by using the law of total probability to conclude that the following inequality holds:

$$\mathbb{E}[T^{\theta}_{(s^*,\mathcal{A},0)}] \geq (1 + \mathbb{E}[T^{\theta}_{(s^*,\mathcal{A},1)}]) \Pr(\mathbf{X}^{\theta}_1 = (s^*,\mathcal{A},1) \mid \mathbf{X}_0 = (s^*,\mathcal{A},0)) \tag{4.53}$$

which after substituting (4.52) and using the fact that $\Pr(\mathbf{X}^{\theta}_1 = (s^*,\mathcal{A},1) \mid \mathbf{X}_0 = (s^*,\mathcal{A},0)) = \lambda(1-\rho_{s^*,s^*-1})$ leads to:

$$\mathbb{E}[T^{\theta}_{(s^*,\mathcal{A},0)}] \geq (1-\rho_{s^*,s^*-1})\frac{1+\overline{\nu}-\lambda}{\overline{\nu}/\lambda - 1} \tag{4.54}$$

which from [90, (3) Theorem] implies that:

$$\pi^{\theta}(s^*,\mathcal{A},0) \leq \frac{\overline{\nu}/\lambda - 1}{1-\rho_{s^*,s^*-1}} \tag{4.55}$$

At this point we intend to use the following inequality to relate $\pi^{\theta}_{\lambda}(s^*,\mathcal{A},0)$ with $\sum_{s\in\mathbb{S}} \pi^{\theta}_{\lambda}(s,\mathcal{A},0)$:

$$\pi^{\theta}(s^*,\mathcal{A},0) \geq \sum_{s\in\mathbb{S}} \pi^{\theta}(s,\mathcal{A},0) \Pr\left(\mathbf{X}^{\theta}_{k+2n_s} = (s^*,\mathcal{A},0) \;\middle|\; \mathbf{X}^{\theta}_k = (\mathbf{y},0)\right) \tag{4.56}$$

We already know from Proposition 4.2 that $\mathbf{X}^{\theta}$ is irreducible, but further analysis of the Markov chain shows that the following lower bound holds:

$$\Pr\left(\mathbf{X}^{\theta}_{k+2n_s} = (s',\mathcal{A},0) \;\middle|\; \mathbf{X}^{\theta}_k = (s,\mathcal{A},0)\right) \geq \tilde{\beta}_{\lambda,\epsilon} \tag{4.57}$$

with:

$$\tilde{\beta}_{\lambda,\epsilon} := \left((1-\lambda)\min_{s\in\mathbb{S}}\left((1-\mu(s))(1-\rho_{s,s+1})(1-\rho_{s,s-1})\right)\right)^{2n_s} \times$$
$$\epsilon\lambda(1-\lambda)^{2n_s-1}\min_{s\in\mathbb{S}}\mu(s)\times$$
$$\Pi_{s=1}^{n_s-1}(1-\mu(s))\rho_{s+1,s}\rho_{s,s+1} \quad (4.58)$$

We obtain $\tilde{\beta}_{\lambda,\epsilon}$ by multiplying the the lower bounds of probabilities of staying at $(s',\mathcal{A},0)$ for $2n_s$ step and the lower bounds to the transition probabilities across the paths that pass through $(1,\mathcal{A},0)$ and $(1,\mathcal{A},1)$ for going from any state $(s,\mathcal{A},0)$ to any other state $(s',\mathcal{A},0)$ and the probability to stay at $(s',\mathcal{A},0)$ for $2n_s$ time steps. The length of each path is no larger than $2n_s$, and the lower bound of the transition probabilities used in (4.58) must be valid irrespective of θ, so long as $\theta(1,\mathcal{A},1)\geq\epsilon$.

The proof of (4.48), then, follows from (4.55)-(4.57) after we select $\beta_{\lambda,\epsilon} := \lambda(1-\rho_{s^*,s^*-1})\tilde{\beta}_{\lambda,\epsilon}$.

Proof of (4.52) We now proceed to proving that (4.52) holds. We start with the following equalities that hold for any $\mathbf{x}$ satisfying $\mathscr{V}(\mathbf{x}) > v^*$:

$$\mathbb{E}[Q_k^\theta - q|\mathbf{X}_{k-1}^\theta = \mathbf{x}] = \lambda - \phi(s,w)\mu(s) \quad (4.59)$$

$$\mathbb{E}[f(\mathbf{Y}_k^\theta) - f(\mathbf{y})|\mathbf{X}_{k-1}^\theta = \mathbf{x}] = \mathbb{E}[f(\overline{\mathbf{Y}}_k^\phi) - f(\mathbf{y})|\overline{\mathbf{Y}}_{k-1}^\phi = \mathbf{y}] \quad (4.60)$$
$$\stackrel{(i)}{=} \phi(s,w)\mu(s) - \overline{\nu}$$

In proving (4.59)-(4.60), we used the fact that if $\mathscr{V}(\mathbf{x}) > v^*$ holds then $q \geq 1$, which, since $\theta = \mathscr{X}(\phi)$, implies that the policy ϕ is applied. In addition, we used Proposition 4.1 to establish (i), where we used the fact that, for our choices of $\overline{\mathbf{Y}}^\phi$ and r, r_{avg}^ϕ is $\overline{\nu}$. By adding the terms of (4.59) and (4.60) we can, then, arrive at:

$$\mathbb{E}[\mathscr{V}(\mathbf{X}_k^\theta) - \mathscr{V}(\mathbf{x})|\mathbf{X}_{k-1}^\theta = \mathbf{x}] = \lambda - \overline{\nu} \quad (4.61)$$

Given that $\underline{\mathrm{T}}_{(s^*,\mathcal{A},1)}^\theta \geq k$ implies that $\mathscr{V}(\mathbf{X}_{k-1}^\theta) > v^*$, we can use (4.61) to derive the following equalities:

$$\mathbb{E}[\mathscr{V}(\mathbf{X}_k^\theta) - \mathscr{V}(\mathbf{X}_{k-1}^\theta)|\underline{\mathrm{T}}_{(s^*,\mathcal{A},1)}^\theta \geq k, \mathbf{X}_0 = (s^*,\mathcal{A},1)] \quad (4.62)$$
$$= \sum_{\{\mathbf{x}:\mathscr{V}(\mathbf{x})>v^*\}} \mathbb{E}[\mathscr{V}(\mathbf{X}_k^\theta) - \mathscr{V}(\mathbf{x})|\mathbf{X}_{k-1}^\theta = \mathbf{x}]$$
$$\times \Pr(\mathbf{X}_{k-1}^\theta = \mathbf{x}|\underline{\mathrm{T}}_{(s^*,\mathcal{A},1)}^\theta \geq k, \mathbf{X}_0 = (s^*,\mathcal{A},1))$$
$$= \lambda - \overline{\nu}$$

We can further use (4.62) to arrive at the following:

$$\sum_{k=1}^{\infty} \Pr\left(\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)} \geq k \middle| \mathbf{X}_0 = (s^*, \mathcal{A}, 1)\right) \tag{4.63}$$

$$\times \mathbb{E}[\mathscr{V}(\mathbf{X}^{\theta}_{k}) - \mathscr{V}(\mathbf{X}^{\theta}_{k-1}) | \underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)} \geq k, \mathbf{X}_0 = (s^*, \mathcal{A}, 1)]$$

$$= (\lambda - \overline{\nu})\, \mathbb{E}[\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}]$$

where we also used the fact that the equality $\Pr\left(\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)} \geq k \middle| \mathbf{X}_0 = (s^*, \mathcal{A}, 1)\right) = \Pr\left(\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)} \geq k\right)$ holds, which follows from the definition of $\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}$.

We also remark that (4.63) leads to:

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\left(\mathscr{V}(\mathbf{X}^{\theta}_{k}) - \mathscr{V}(\mathbf{X}^{\theta}_{k-1})\right) \mathcal{I}_{\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)} \geq k} \middle| \mathbf{X}_0 = (s^*, \mathcal{A}, 1)\right] \tag{4.64}$$

$$= (\lambda - \overline{\nu})\, \mathbb{E}[\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}]$$

Since $\mathbf{X}^{\theta}$ is positive recurrent, we conclude that $\mathbb{E}[\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}]$ is bounded and that (4.64) converges absolutely. Hence, we can exchange the summation and expectation in (4.64) to obtain:

$$\mathbb{E}\left[\mathscr{V}\left(\mathbf{X}^{\theta}_{\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}}\right) - \mathscr{V}(\mathbf{X}_0) \;\middle|\; \mathbf{X}_0 = (s^*, \mathcal{A}, 1)\right] \tag{4.65}$$

$$= (\lambda - \overline{\nu})\, \mathbb{E}[\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}]$$

which leads to the desired equality in (4.52) once we realize that the following inequality holds:

$$\begin{aligned}
&\mathbb{E}\left[\mathscr{V}\left(\mathbf{X}^{\theta}_{\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}}\right) - \mathscr{V}(\mathbf{X}_0) \;\middle|\; \mathbf{X}_0 = (s^*,\mathcal{A},1)\right] \qquad (4.66)\\
&\quad = \mathbb{E}\left[\mathscr{V}\left(\mathbf{X}^{\theta}_{\underline{\mathrm{T}}^{\theta}_{(s^*,\mathcal{A},1)}}\right) \;\middle|\; \mathbf{X}_0 = (s^*,\mathcal{A},1)\right] - (v^*+1)\\
&\quad \le v^* - (v^*+1) = -1
\end{aligned}$$

□

Theorem 4.5. *Let λ in $(0,\overline{\nu}^*)$ and ϵ in $(0,1)$ be given. If $\Phi^{\epsilon}_R(\lambda)$ is nonempty then there is a positive constant η_ϵ such that the following inequality holds for every $\overline{\nu} \in (\lambda,\overline{\nu}^*)$:*

$$\sum_{\mathbf{y}\in\mathbb{Y}} \left| \overline{\pi}^{\phi}(\mathbf{y}) - \sum_{q>0} \pi^{\mathscr{X}(\phi)}(\mathbf{y},q) \right| \le \frac{\beta_{\lambda,\epsilon}+\eta_\epsilon}{\beta_{\lambda,\epsilon}}(\overline{\nu}-\lambda)^{\frac{1}{2}} + \frac{3}{\beta_{\lambda,\epsilon}}(\overline{\nu}-\lambda), \qquad \phi \in \Phi^{\epsilon}_R(\overline{\nu}) \quad (4.67)$$

We make use of the following Lemmas to complete the proof of the Theorem 4.5. We denote the one-step transition matrix of $\overline{\mathbf{Y}}^{\phi}$ by $\overline{\mathbf{P}}^{\phi}$. Note that we use $\overline{\boldsymbol{\pi}}$ to denote the stationary distribution in a row vector form and recall that we defined Φ^{ϵ}_R to be the set of $\phi \in \Phi_R$ such that $\phi(1,\mathcal{A}) \ge \epsilon$.

Lemma 4.6. *There exists a positive constant η_ϵ such that, for any distribution*

p *over* $\mathbb{Y}$, *we have*

$$\sum_{r=1}^{\infty} \left\| \mathbf{p} \left(\overline{\mathbf{P}}^{\phi}\right)^{r} - \overline{\boldsymbol{\pi}}^{\phi} \right\|_1 \leq \eta_\epsilon, \qquad \phi \in \Phi_R^\epsilon.$$

Proof. Then, for every $\phi' \in \Phi_R^\epsilon$, the following lower bound holds:

$$\Pr\left(\overline{\mathbf{Y}}_{k+2n_s}^{\phi'} = (n_s, \mathcal{B}) \mid \overline{\mathbf{Y}}_k^{\phi'} = \mathbf{y}\right) \geq \tilde{\alpha}_\epsilon \tag{4.68}$$

with:

$$\tilde{\alpha}_\epsilon := (1-\mu(n_s))^{2n_s} \times \epsilon \Pi_{s=1}^{n_s-1}(1-\mu(s))\rho_{s+1,s}\rho_{s,s+1} \tag{4.69}$$

We obtain $\tilde{\alpha}_\epsilon$ by multiplying lower bounds to the transition probabilities across the paths that pass through $(1, \mathcal{A})$ for going from any state $\mathbf{y}$ to state $(n_s, \mathcal{B})$ and the probability to stay at $(n_s, \mathcal{B})$ for $2n_s$ time steps. The length of each path is no larger than $2n_s$, and the lower bound of the transition probabilities used in (4.69) must be valid irrespective of ϕ', so long as $\phi'(1, \mathcal{A}) \geq \epsilon$.

Next, we follow an analysis that is similar with the proof of Theorem 4.16 of [91]. We define a function $\tau_1 : \mathbb{R}^{n \times n} \to \mathbb{R}^+$ as

$$\tau_1(\mathbf{P}) = \frac{1}{2} \max_{i,j} \sum_{\ell=1}^{n} |p_{i\ell} - p_{j\ell}|$$

where $p_{i\ell}$ is the $\{i, \ell\}$ element of matrix $\mathbf{P}$.

We observe that every element in the column of $\left(\overline{\mathbf{P}}^{\phi}\right)^{2n_s}$ corresponding to $(n_s, \mathcal{B})$ is lower-bounded by $\tilde{\alpha}_\epsilon$ because of (4.68) and the fact that $\phi \in \Phi_R^\epsilon$. The equation (4.6) of [91] tells us that

$$\tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{2n_s}\Big) \le 1 - \tilde{\alpha}_\epsilon.$$

Proceeding with the proof, for every $r \ge 2n_s$, $k = \lfloor r/2n_s \rfloor$,

$$\begin{aligned}\tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{r}\Big) &= \tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{r-2kn_s}\Pi_{i=1}^{k}\big(\overline{\mathbf{P}}^{\phi}\big)^{2n_s}\Big)\\ &\le \tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{r-2kn_s}\Big)\Pi_{i=1}^{k}\tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{2n_s}\Big)\\ &\le \Big(1-\tilde{\alpha}_\epsilon\Big)^{\lfloor r/2n_s \rfloor}\\ &\le \Big(1-\tilde{\alpha}_\epsilon\Big)^{\frac{r}{2n_s}-1} = K_\epsilon \sigma_\epsilon^r,\end{aligned}$$

where $K_\epsilon = (1-\tilde{\alpha}_\epsilon)^{-1}$ and $\sigma_\epsilon = (1-\tilde{\alpha}_\epsilon)^{1/2n_s}$. The first inequality follows from $\tau_1(\mathbf{P}_1\mathbf{P}_2) \le \tau_1(\mathbf{P}_1)\tau_1(\mathbf{P}_2)$ as shown in [91, Lemma 4.3]. The second inequality follows from $\tau_1(\mathbf{P}) \le 1$ for any stochastic matrix $\mathbf{P}$ and $\Big(1-\tilde{\alpha}_\epsilon\Big) < 1$ leads to the final inequality. Combining with Lemma 4.3 of [91] and the fact that sum of all elements of $\mathbf{p} - \overline{\boldsymbol{\pi}}^{\phi}$ equals zero, we know, for every $r \ge 2n_s$

$$\begin{aligned}\Big\|\mathbf{p}\ \big(\overline{\mathbf{P}}^{\phi}\big)^{r} - \overline{\boldsymbol{\pi}}^{\phi}\Big\|_1 &= \Big\|\mathbf{p}\ \big(\overline{\mathbf{P}}^{\phi}\big)^{r} - \overline{\boldsymbol{\pi}}^{\phi}\big(\overline{\mathbf{P}}^{\phi}\big)^{r}\Big\|_1\\ &\le \tau_1\Big(\big(\overline{\mathbf{P}}^{\phi}\big)^{r}\Big)\ \big\|\mathbf{p} - \overline{\boldsymbol{\pi}}^{\phi}\big\|_1\\ &\le 2K_\epsilon\sigma_\epsilon^r.\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{r=1}^{\infty} \left\| \mathbf{p} \left(\overline{\mathbf{P}}^{\phi}\right)^{r} - \overline{\boldsymbol{\pi}}^{\phi} \right\|_1 \\
&= \sum_{r=1}^{2n_s} \left\| \mathbf{p} \left(\overline{\mathbf{P}}^{\phi}\right)^{r} - \overline{\boldsymbol{\pi}}^{\phi} \right\|_1 + \sum_{r=2n_s+1}^{\infty} \left\| \mathbf{p} \left(\overline{\mathbf{P}}^{\phi}\right)^{r} - \overline{\boldsymbol{\pi}}^{\phi} \right\|_1 \\
&\leq 4n_s + \sum_{r=2n_s+1}^{\infty} 2K_\epsilon \sigma_\epsilon^r = 4n_s + \frac{2K_\epsilon \sigma_\epsilon^{2n_s+1}}{1-\sigma_\epsilon} =: \eta_\epsilon .
\end{aligned}$$

□

For notational convenience, we denote the unique stationary distribution of $\mathbf{X}^{\mathscr{X}(\phi)}$ on server state $\mathbb{Y}$ by $\boldsymbol{\varrho}^{\mathscr{X}(\phi)}$ and

$$\varrho^{\mathscr{X}(\phi)}(\mathbf{y}) = \sum_{q\in\mathbb{Q}^{\overline{\mathbf{y}}}} \pi^{\mathscr{X}(\phi)}(\mathbf{y}, q) \quad \forall \mathbf{y} \in \mathbb{Y}.$$

Lemma 4.7. *For every $r \in \mathbb{N}$, we have*

$$\left\| \boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)} \left(\overline{\mathbf{P}}^{\phi}\right)^{r} \right\|_1 \leq 2r \frac{(\overline{\nu}-\lambda)}{\beta_{\lambda,\epsilon}}, \qquad \phi \in \Phi_R^{\epsilon}(\overline{\nu}).$$

Proof. Let $\overline{\mathbf{P}}^{\mathcal{R}}$ be the one-step transition matrix of $\overline{\mathbf{Y}}$ under a policy that always chooses $\mathcal{R}$ when the server is available. We denote the row of $\overline{\mathbf{P}}^{\phi}$ (resp. $\overline{\mathbf{P}}^{\mathcal{R}}$) corresponding to the server state $\mathbf{y} = (s, w) \in \mathbb{Y}$ by $\overline{\mathbf{P}}^{\phi}_{\mathbf{y}}$ (resp. $\overline{\mathbf{P}}^{\mathcal{R}}_{\mathbf{y}}$).

By the fact that $\boldsymbol{\varrho}^{\mathscr{X}(\phi)}$ remains the same after one step transition and using the equality $\varrho^{\mathscr{X}(\phi)}(\mathbf{y}) = \sum_{q\in\mathbb{Q}^{\overline{\mathbf{y}}}} \pi^{\mathscr{X}(\phi)}(\mathbf{y}, q)$, we can rewrite $\boldsymbol{\varrho}^{\mathscr{X}(\phi)}$ as

$$\begin{aligned}
\boldsymbol{\varrho}^{\mathscr{X}(\phi)} &= \sum_{s\in\mathbb{S}} \Big[\pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0)\ \overline{\mathbf{P}}^{\mathcal{R}}_{(s,\mathcal{A})} \\
&\qquad + \sum_{w\in\mathbb{W}} \Big(\sum_{q=1}^{\infty} \pi^{\mathscr{X}(\phi)}(s,w,q)\Big)\overline{\mathbf{P}}^{\phi}_{(s,w)}\Big] \\
&= \sum_{s\in\mathbb{S}} \Big[\pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0)\ \overline{\mathbf{P}}^{\mathcal{R}}_{(s,\mathcal{A})} + \varrho^{\mathscr{X}(\phi)}(s,\mathcal{B})\ \overline{\mathbf{P}}^{\phi}_{(s,\mathcal{B})} \\
&\qquad + \Big(\varrho^{\mathscr{X}(\phi)}(s,\mathcal{A}) - \pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0)\Big)\overline{\mathbf{P}}^{\phi}_{(s,\mathcal{A})}\Big] \\
&= \sum_{s\in\mathbb{S}} \Big[\pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0)\big(\overline{\mathbf{P}}^{\mathcal{R}}_{(s,\mathcal{A})} - \overline{\mathbf{P}}^{\phi}_{(s,\mathcal{A})}\big)\Big] \\
&\qquad + \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\ \overline{\mathbf{P}}^{\phi}.
\end{aligned} \tag{4.70}$$

Define $\boldsymbol{\gamma}^{\phi} \stackrel{\text{def}}{=} \sum_{s\in\mathbb{S}} \big[\pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0)\big(\overline{\mathbf{P}}^{\mathcal{R}}_{(s,\mathcal{A})} - \overline{\mathbf{P}}^{\phi}_{(s,\mathcal{A})}\big)\big]$. Applying (4.70) iteratively, we obtain

$$\boldsymbol{\varrho}^{\mathscr{X}(\phi)} = \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\big(\overline{\mathbf{P}}^{\phi}\big)^{r} + \boldsymbol{\gamma}^{\phi}\sum_{\tau=1}^{r}\big(\overline{\mathbf{P}}^{\phi}\big)^{\tau-1}. \tag{4.71}$$

Subtracting the first term on the RHS of (4.71) from both sides and taking the norm,

$$\begin{aligned}
\Big\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\big(\overline{\mathbf{P}}^{\phi}\big)^{r}\Big\|_1 &= \Big\|\boldsymbol{\gamma}^{\phi}\sum_{\tau=1}^{r}\big(\overline{\mathbf{P}}^{\phi}\big)^{\tau-1}\Big\|_1 \\
\leq \big\|\boldsymbol{\gamma}^{\phi}\big\|_1 \sum_{\tau=1}^{r}\Big\|\big(\overline{\mathbf{P}}^{\phi}\big)^{\tau-1}\Big\|_\infty &= r\,\big\|\boldsymbol{\gamma}^{\phi}\big\|_1
\end{aligned}$$

Substituting the expression for γ^{ϕ} and using the inequality $\Big\|\overline{\mathbf{P}}^{\mathcal{R}}_{\mathbf{y}} - \overline{\mathbf{P}}^{\phi}_{\mathbf{y}}\Big\|_1 \leq 2$ for all

$\mathbf{y} \in \mathbb{Y}$, we get

$$r \left\|\boldsymbol{\gamma}^{\phi}\right\|_1 \leq 2r\Big(\sum_{s\in\mathbb{S}} \pi^{\mathscr{X}(\phi)}(s, \mathcal{A}, 0)\Big). \tag{4.72}$$

Thus, we get

$$\begin{aligned}\left\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\big(\overline{\mathbf{P}}^{\phi}\big)^{r}\right\|_1 &\leq 2r\Big(\sum_{s\in\mathbb{S}} \pi^{\mathscr{X}(\phi)}(s, \mathcal{A}, 0)\Big)\\ &\leq 2r\frac{(\overline{\nu} - \lambda)}{\beta_{\lambda,\epsilon}}.\end{aligned}$$

The last inequality follows from Lemma 4.5. □

Lemma 4.8. *For every $N \in \mathbb{N}$, we have*

$$\left\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \overline{\boldsymbol{\pi}}^{\phi}\right\|_1 \leq \frac{\alpha_\epsilon}{N} + \frac{(N+1)(\overline{\nu}-\lambda)}{\beta_{\lambda,\epsilon}},\ \phi \in \Phi_R^{\epsilon}(\overline{\nu}).$$

Proof. We start by duplicating the difference by N times and add and subtract a term into each difference,

$$\begin{aligned}&\left\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \overline{\boldsymbol{\pi}}^{\phi}\right\|_1\\ &= \left\|\frac{\sum_{r=1}^{N} \boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \overline{\boldsymbol{\pi}}^{\phi}}{N}\right\|_1\\ &= \left\|\frac{\sum_{r=1}^{N} \boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\big(\overline{\mathbf{P}}^{\phi}\big)^{r} + \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\big(\overline{\mathbf{P}}^{\phi}\big)^{r} - \overline{\boldsymbol{\pi}}^{\phi}}{N}\right\|_1.\end{aligned} \tag{4.73}$$

Then, we can bound the one norm of the difference by

$$
\begin{aligned}
(4.73) \le & \frac{1}{N}\sum_{r=1}^{N}\left\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\left(\overline{\mathbf{P}}^{\phi}\right)^{r}\right\|_1 \\
& + \frac{1}{N}\sum_{r=1}^{N}\left\|\boldsymbol{\varrho}^{\mathscr{X}(\phi)}\left(\overline{\mathbf{P}}^{\phi}\right)^{r} - \overline{\boldsymbol{\pi}}^{\phi}\right\|_1 \\
\le & \frac{(N+1)(\overline{\nu}-\lambda)}{\beta_{\lambda,\epsilon}} + \frac{\alpha_\epsilon}{N},
\end{aligned} \tag{4.74}
$$

by Lemma 4.6 and Lemma 4.7. □

Proof of Theorem 4.5: We have

$$
\begin{aligned}
& \sum_{\mathbf{y}\in\mathbb{Y}}\left|\overline{\pi}^{\phi}(\mathbf{y}) - \sum_{q>0}\pi^{\mathscr{X}(\phi)}(\mathbf{y},q)\right| \\
& \le \left\|\overline{\boldsymbol{\pi}}^{\phi} - \boldsymbol{\varrho}^{\mathscr{X}(\phi)}\right\|_1 + \sum_{\mathbf{y}\in\mathbb{Y}}\left|\varrho^{\mathscr{X}(\phi)}(\mathbf{y}) - \sum_{q>0}\pi^{\mathscr{X}(\phi)}(\mathbf{y},q)\right| \\
& = \left\|\overline{\boldsymbol{\pi}}^{\phi} - \boldsymbol{\varrho}_{\lambda}^{\mathscr{X}(\phi)}\right\|_1 + \sum_{s\in\mathbb{S}}\pi^{\mathscr{X}(\phi)}(s,\mathcal{A},0) \\
& \le \frac{\alpha_\epsilon}{N} + \frac{(N+1)(\overline{\nu}-\lambda)}{\beta_{\lambda,\epsilon}} + \frac{\overline{\nu}-\lambda}{\beta_{\lambda,\epsilon}},
\end{aligned}
$$

where the final inequality follows from Lemma 4.8 and Lemma 4.5. Let $N = \left\lceil \frac{\eta_\epsilon}{(\overline{\nu}-\lambda)^{\frac{1}{2}}} \right\rceil$ and we get

$$
\begin{aligned}
& \sum_{\mathbf{y}\in\mathbb{Y}}\left|\overline{\pi}^{\phi}(\mathbf{y}) - \sum_{q>0}\pi^{\mathscr{X}(\phi)}(\mathbf{y},q)\right| \\
& \le (\overline{\nu}-\lambda)^{\frac{1}{2}} + \frac{\left(\frac{\eta_\epsilon}{(\overline{\nu}-\lambda)^{\frac{1}{2}}} + 1 + 1\right)(\overline{\nu}-\lambda)}{\beta_{\lambda,\epsilon}} + \frac{\overline{\nu}-\lambda}{\beta_{\lambda,\epsilon}} \\
& \le \frac{\beta_{\lambda,\epsilon}+\eta_\epsilon}{\beta_{\lambda,\epsilon}}(\overline{\nu}-\lambda)^{\frac{1}{2}} + \frac{3}{\beta_{\lambda,\epsilon}}(\overline{\nu}-\lambda).
\end{aligned}
$$

Corollary 4.3. *Let λ in $(0, \overline{\nu}^*)$ and ϵ in $(0,1)$ be given. If $\Phi_R^\epsilon(\lambda)$ is nonempty then there is a positive constant η_ϵ such that the following inequality holds for every $\overline{\nu} \in (\lambda, \overline{\nu}^*)$:*

$$\left|\bar{\mathscr{U}}(\phi) - \mathscr{U}(\lambda, \mathscr{X}(\phi))\right| \leq \frac{\beta_{\lambda,\epsilon} + \eta_\epsilon}{\beta_{\lambda,\epsilon}}(\overline{\nu} - \lambda)^{\frac{1}{2}} + \frac{3}{\beta_{\lambda,\epsilon}}(\overline{\nu} - \lambda), \qquad \phi \in \Phi_R^\epsilon(\overline{\nu})$$

Proof. Theorem 4.5 and the fact that $\mathscr{X}(\phi)(\mathbf{y}, q)$ equals zero when $q = 0$ tell us that

$$\begin{aligned}
&\left|\bar{\mathscr{U}}(\phi) - \mathscr{U}(\lambda, \mathscr{X}(\phi))\right| = \left|\sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \overline{\pi}^\phi \phi(\overline{\mathbf{y}}) - \sum_{\mathbf{x} \in \mathbb{X}} \pi^{\mathscr{X}(\phi)}(\mathbf{x}) \mathscr{X}(\phi)(\mathbf{x})\right| \\
&= \left|\sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \overline{\pi}^\phi \phi(\overline{\mathbf{y}}) - \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \sum_{q>0} \pi^{\mathscr{X}(\phi)}(\overline{\mathbf{y}}, q) \phi(\overline{\mathbf{y}})\right| \\
&\leq \sum_{\overline{\mathbf{y}} \in \overline{\mathbb{Y}}} \phi(\overline{\mathbf{y}}) \left|\overline{\pi}^\phi - \sum_{q>0} \pi^{\mathscr{X}(\phi)}(\overline{\mathbf{y}}, q)\right| \\
&\leq \frac{\beta_{\lambda,\epsilon} + \eta_\epsilon}{\beta_{\lambda,\epsilon}}(\overline{\nu} - \lambda)^{\frac{1}{2}} + \frac{3}{\beta_{\lambda,\epsilon}}(\overline{\nu} - \lambda).
\end{aligned}$$

□

4.6 Simulation Result

To evaluate the performance of our propose policies, we run the simulation for 10^8 time steps and record the utilization rate and the average queue length. The ρ

that governs the transition probabilities for action-dependent states are defined as

$$\begin{aligned}\rho_{s,s+1} &\stackrel{\text{def}}{=} \frac{1}{5}\cdot\left(1-\frac{s-1}{n_s-1}\right)\\ \rho_{s,s-1} &\stackrel{\text{def}}{=} \frac{1}{5}\cdot\left(\frac{s-1}{n_s-1}\right)\end{aligned}$$

The motivation for this ρ is from [44] where Salva view the server state as utilization ratio. The ρ here is a probabilistic proxy for the differential equation that describes the server state evolution in [44]. The number of action-dependent state n_s is set to be 7. The server efficiency function μ is set to be $[0.01, 0.5, 0.2, 0.2, 0.5, 0.05, 0.01]$. By (4.3), the upper bound for stabilizable arrival rate $\overline{\nu}^*$ is 0.1683.

For this experiment, we fix our arrival rate $\lambda = 0.14$. At this arrival rate, the minimum utilization rate $\mathscr{U}^*(\lambda) = 0.4212$. We test the policies with $\epsilon = 0.001$ and different design departure rate ν.

The result is shown in Fig 4.6. As a result of Corollary 4.2, we observe that the utilization rate $\mathscr{U}(\lambda, \mathscr{X}(\phi^{\overline{\nu}}))$ drops closer to the optimal value as the design departure rate $\overline{\nu}$ approaches arrival rate λ. However, the average queue length increase exponentially. Thus, it might not be wise to choose $\overline{\nu}$ arbitrarily close to λ for a lower utilization rate without considering average queue length or delay. It remains an open question on how to address this trade-off.

4.7 Summary

We investigated the stability problem of designing a task scheduling policy when the efficiency of the server is allowed to depend the past utilization, which is

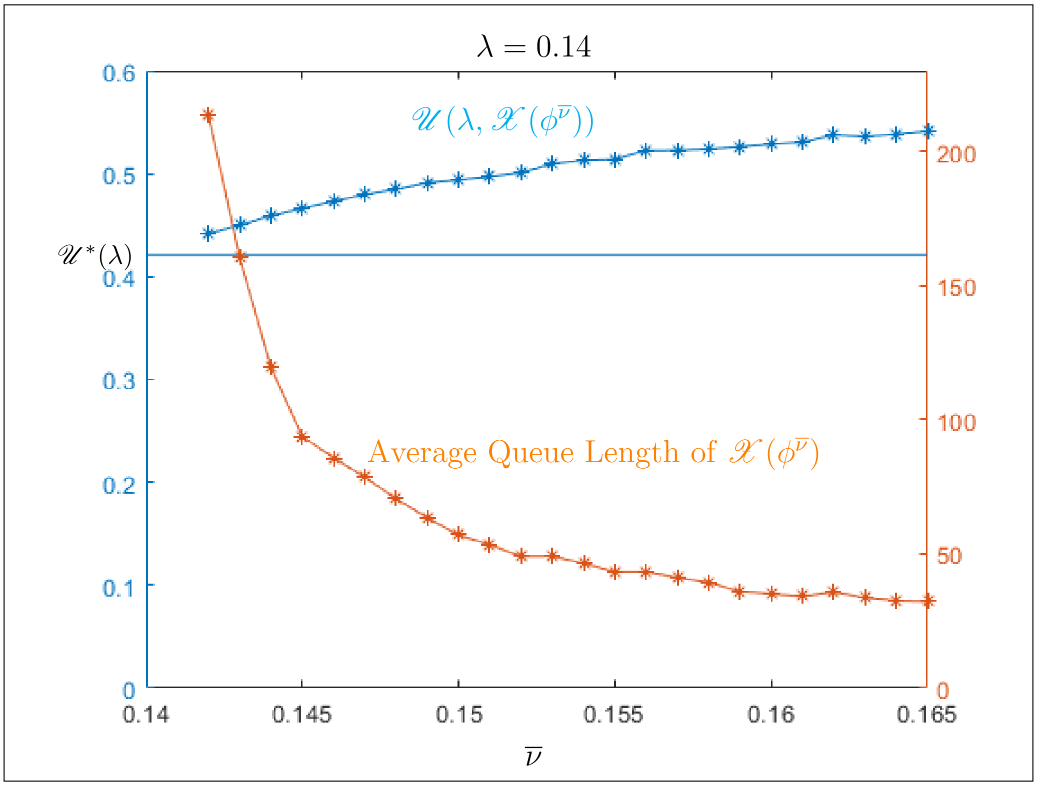

Figure 4.1: Simulation Results

modeled using an internal state of the server. Making use of the new framework, we characterized the set of task arrival rates for which there exists a stabilizing stationary scheduling policy. Moreover, finding this set can done by solving a simple optimization problem over a finite threshold value. We identified an optimal threshold policy that stabilizes the system whenever the task arrival rate lies in the interior of the aforementioned set for which there is a stabilizing policy.

In addition to the problem of stability, we also identified a tractable way (i.e., linear programming) of calculating the minimum of all utilization rates that can be achieved through a stabilizing scheduling policy. Such a fundamental limit is vital in determining how effective the utilization rate of any given policy is. Furthermore, we were able to use this linear programming to design policies whose utilization rate is arbitrarily close to the fundamental limit.

Chapter 5: Queuing Server with Multiple Types of Tasks

In this chapter, we relax the single type of task constraint and study the stability problem for the system with multiple types of jobs. The region for stable arrival rates is recognized. We suggest a strategy that would stabilize the queues whenever the rate of arrival is within the stability region. Furthermore, the calculation of the policy does not depend on the understanding of the rate of arrival. It only depends on m threshold values, while each of the threshold values can be computed by considering a single type system and applying the analysis in the previous chapter.

The chapter is organized as follows. We begin by presenting the stability results for the system with two types of tasks ($m = 2$) and follow by the proofs of the stability results. In Section 5.3, we briefly discuss how the Theorems can be generalized to m types of tasks and provide the proofs in Appendix A.

5.1 Stability Results for Two Types

Before presenting our results for system with two types of tasks, we need to introduce some necessary quantities for our discussion. To this end, we first consider two related systems with a single queue, which is studied in Chapter 4: $\mathbf{X}^i$, $i \in \mathbb{T} = \{1, 2\}$, is a system only with type i tasks and, hence, only a single queue is needed to hold type i tasks. Define

$$\Lambda_i = \{\lambda_i \in (0, \infty) \mid \text{there exists a stationary policy that stabilizes the system } \mathbf{X}^i\}, \ i \in \mathbb{T},$$

and let $\overline{\nu}_i^* \stackrel{def}{=} \sup \Lambda_i$. In Chapter 4, we showed that, for each $\mathbf{X}^i$, there exists a threshold policy on the activity state, which can stabilize the system for all $\lambda_i < \overline{\nu}_i^*$. We denote this threshold policy for $\mathbf{X}^i$ by $\tilde{\theta}_i^{\text{th}}$, $i \in \mathbb{T}$.

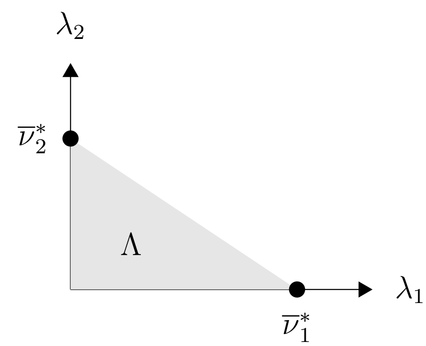

Figure 5.1: Stability Region for the system with both types of tasks

Our main results on the system with two different types of tasks consist of two parts – necessity and sufficiency – and show that, in order for the system to be stable, the arrival rates $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ must lie in the shaded triangle Λ in Fig. 5.1, where

$$\Lambda \stackrel{def}{=} \left\{(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}_+^2 \mid \frac{\tilde{\lambda}_1}{\overline{\nu}_1^*} + \frac{\tilde{\lambda}_2}{\overline{\nu}_2^*} \leq 1\right\}.$$

We point out that, although our findings are intuitively satisfying, their proofs are far from obvious; a straightforward time-sharing between two optimal (threshold) policies for single-queue systems is not applicable. This is due to the coupling between the two single-queue systems, which is introduced via the activity state. This coupling renders the analysis of our system far more challenging.

The first theorem states that any arrival rates for which we can find a stabilizing policy must lie in Λ.

Theorem 5.1 (Necessity)**.** *Suppose that there exists a stabilizing policy in* Θ_R *for arrival rates* (λ_1, λ_2)*. Then, they satisfy* $\frac{\lambda_1}{\nu_1^*} + \frac{\lambda_2}{\nu_2^*} \leq 1$*.*

A natural question that arises is whether or not there exists a scheduling policy in Θ_R, which can stabilize the system for any arrival rates $\boldsymbol{\lambda}$ in Λ. In order to answer this question, we now turn our attention to the problem of designing a single policy with simple structure which can stabilize the system $\mathbf{X}$ for any arrival rates $\boldsymbol{\lambda}$ that satisfy $\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\nu_i^*} < 1$, i.e., the arrival rates lie below the hypotenuse of the shaded triangle Λ in Fig. 5.1.

Before we present our second main result, we shall digress a little, in order to bring to light an interesting point that highlights the key difference between our study and many existing studies on the stability of queues. Based on our earlier study of single-queue systems, one may suspect that the well-known max-weight policy may be able to stabilize the system for any arrival rates satisfying the strict inequality.

Suppose that i_k is the type with a larger weighted queue size at epoch k, i.e., $i_k \in \arg\max_{i \in \mathbb{T}} \omega_i Q_k^i$, where $\omega_i > 0$ is the weight for queue i size. Consider the policy θ^{MW} that schedules a task of type i_k in accordance with the policy $\tilde{\theta}_{i_k}^{\text{th}}$ when the server is available. It turns out that this max-weight policy is not optimal in that, in some cases, we can find arrival rates $\boldsymbol{\lambda}$ for which θ^{MW} cannot stabilize the system, even though there exists a stabilizing policy in Θ_R.

The intuition behind this is as follows: suppose (i) the threshold of $\tilde{\theta}_i^{\text{th}}$, $i \in \mathbb{T}$, on the activity state is τ_i^*, and τ_1^* and τ_2^* are not close and (ii) the probability that the activity state changes its value (after either resting or servicing a task) is small so that the dynamics of the activity state is relatively slow compared to the departures of completed tasks. In this case, when the queue sizes remain near the decision boundary where $\omega_1 q_1 \approx \omega_2 q_2$, as long as the activity state remains below $\min(\tau_1^*, \tau_2^*)$, the policy θ^{MW} will alternate between the two types of tasks frequently. Recall that the policies $\tilde{\theta}_i^{\text{th}}$, $i \in \mathbb{T}$, are designed to maximize the throughput only with a single type of tasks. Therefore, when the aforementioned switching between the two types of tasks happens sufficiently often (i.e., a non-negligible fraction of time), it causes a significant change in the stationary PMF of the activity state, compared to those of $\mathbf{X}^i$ under $\tilde{\theta}_i^{\text{th}}$, $i \in \mathbb{T}$. This in turn leads to inefficiency for at least one type of tasks and a drop in achievable (maximum) long-term service rate.

We now proceed to present a policy with simple structure which can stabilize the system for any arrival rates satisfying $\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\nu_i^*} < 1$. Without loss of generality, we assume $\tau_1^* \geq \tau_2^*$. Moreover, since the server is assumed non-preemptive, we only specify the scheduling decision when the server is available, i.e., $w = \mathcal{A}$.

Theorem 5.2 (Sufficiency)**.** *Assume (i) the arrival rates* (λ_1, λ_2) *satisfy* $\frac{\lambda_1}{\overline{\nu}_1^*} + \frac{\lambda_2}{\overline{\nu}_2^*} < 1$ *and (ii)* $\tau_1^* \geq \tau_2^*$*. Then, the following policy* θ^{opt} *stabilizes the system* $\mathbf{X}$*.*

$$\theta^{\text{opt}}(s, \mathcal{A}, \mathbf{q}) = \begin{cases} (0,1) & \textit{if (i) } q_2 > 0 \textit{ and } s < \tau_2^* - 1, \\ & \quad \textit{(ii) } q_2 > 0, q_2 \geq q_1, s = \tau_2^* - 1 \\ & \quad\quad \textit{and } \tau_1^* = \tau_2^*, \textit{ or} \\ & \quad \textit{(iii) } q_2 > 0, s = \tau_2^* - 1 \textit{ and } \tau_1^* > \tau_2^*, \\ (1,0) & \textit{if (i) } q_1 > 0, q_2 = 0 \textit{ and } s < \tau_1^* - 1, \textit{ or} \\ & \quad \textit{(ii) } q_1 > 0, q_1 > q_2 \textit{ and } s = \tau_1^* - 1, \\ (0,0) & \textit{otherwise.} \end{cases}$$

Note that the proposed policy θ^{opt} assigns a new type 1 task for service only if either (i) there is no type 2 task to service and the activity state is below $\tau_1^* - 1$ or (ii) the length of queue 1 is greater than or equal to that of queue 2 and the activity state is equal to $\tau_1^* - 1$. More notably, it gives a higher priority to type 2 tasks when the activity state is less than $\tau_2^* - 1$.

5.2 Proofs of Stability Results

Similar to the single queue case, we begin with a comment on the long-term average departure rate of completed tasks when the system is stable. Note that we use $\theta(\mathbf{x})_i$ to denote the i-th element of vector $\theta(\mathbf{x})$.

Remark 8. *Recall from our discussion in Section 3.3 that, under a stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$, there exists a unique stationary PMF $\boldsymbol{\pi}^\theta$. Consequently, the average number of completed type i tasks per epoch converges almost surely as k goes to infinity. In other words,*

$$\lim_{k\to\infty} \frac{\sum_{\tau=0}^{k-1} \mathcal{I}\left\{\text{a type } i \text{ task is completed at epoch } \tau \text{ in } \mathbf{X}^\theta\right\}}{k} = \sum_{\mathbf{x}\in\mathbb{X}} \mu(s,i)\pi^\theta(\mathbf{x})\theta(\mathbf{x})_i \stackrel{def}{=} \nu_i^\theta \quad \text{with probability 1,}$$

where s, w and $\mathbf{q}$ are the coordinates of $\mathbf{x} = (s, w, \mathbf{q})$. We call ν_i^θ the long-term type i task service rate *of θ (for the given arrival rate $\lambda_i > 0$). Moreover, because θ is assumed to be a stabilizing policy, we have $\nu_i^\theta = \lambda_i$.*

The long-term service rate of type i tasks in $\overline{\mathbf{Y}}$ can be defined in an analogous manner: for each ϕ in Φ_R, let $\overline{\boldsymbol{\Pi}}(\phi)$ be the set of stationary PMFs of $\overline{\mathbf{Y}}^\phi$. We define the long-term service rate of type i tasks for ϕ in Φ_R to be

$$\overline{\nu}_i^\phi \stackrel{\text{def}}{=} \sup_{\overline{\boldsymbol{\pi}}\in\overline{\boldsymbol{\Pi}}(\phi)} \Big(\sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \mu(\overline{s}, i)\overline{\pi}(\overline{\mathbf{y}})\phi(\overline{\mathbf{y}})_i \Big). \tag{5.1}$$

When the policy ϕ belongs to Φ_R^+, Corollary 3.1 tells us that there exists a unique stationary PMF $\overline{\boldsymbol{\pi}}^{\phi}$. Hence, $\overline{\nu}_i^{\phi}$ is given by

$$\overline{\nu}_i^{\phi} \stackrel{\text{def}}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, i) \overline{\pi}^{\phi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}})_i. \tag{5.2}$$

5.2.1 Necessity

We are now ready to proceed with the proof of Theorem 4.1. The theorem will be proved with the help of the following three lemmas.

Lemma 5.1. *For every stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$, there exists a policy ϕ in Φ_R^+ for $\overline{\mathbf{Y}}$ such that, for all $i \in \mathbb{T}$,*

$$\nu_i^{\theta} = \overline{\nu}_i^{\phi} = \lambda_i.$$

Proof. Please see Section 5.5.1 for a proof. □

Lemma 5.2. *For every policy ϕ in Φ_R^+, the throughput $\overline{\nu}_i^{\phi}$ cannot exceed $\overline{\nu}_i^*$ for all $i \in \mathbb{T}$.*

Proof. A proof is provided in Section 5.5.2. □

Lemmas 5.1 and 5.2 tell us that $\overline{\nu}_1^*$ and $\overline{\nu}_2^*$ serve as upper bounds on achievable long-term service rates for type 1 tasks and type 2 tasks, respectively, by any stabilizing policy θ in $\Theta_S(\boldsymbol{\lambda})$. Thus, if $\Theta_S(\boldsymbol{\lambda})$ is non-empty, then the arrival rates $\boldsymbol{\lambda}$ must belong to the rectangular region $[0, \overline{\nu}_1^*] \times [0, \overline{\nu}_2^*]$.

Lemma 5.3. *Suppose $\lambda_2 \leq \overline{\nu}_2^*$. If a policy ϕ in Φ_R achieves $\overline{\nu}_2^\phi$ equal to the arrival rate λ_2, then $\overline{\nu}_1^\phi$ is upper bounded by $\overline{\nu}_1^* \frac{\overline{\nu}_2^* - \lambda_2}{\overline{\nu}_2^*} \geq 0$.*

Proof. Please see Section 5.5.3 for a proof. □

Lemma 5.3 implies that, in order for the system to be stable, the arrival rates $\boldsymbol{\lambda}$ must also satisfy

$$\frac{\lambda_1}{\overline{\nu}_1^*} + \frac{\lambda_2}{\overline{\nu}_2^*} \leq 1.$$

If this inequality is violated, we have $\lambda_1 > \overline{\nu}_1^*(\overline{\nu}_2^* - \lambda_2)/\overline{\nu}_2^*$, which contradicts the claim in Lemma 5.3. This completes the proof of the theorem.

5.2.2 Sufficiency

Without loss of generality, we assume $\tau_1^* \geq \tau_2^*$. We use the Proposition 4.1 in the previous Chapter again for building the Lyapunov function. In the remainder of the proof, let f be a potential-like function that satisfies (4.12) in Proposition 4.1 with the policy $\varphi^{\tau_1^*}$ and the following reward function $r : \mathbb{Y} \times \Phi_R \to \mathbb{R}$:

$$r(\mathbf{y}, \phi) = \sum_{i \in \mathbb{T}} \frac{\mu(s,i)\phi(\mathbf{y})_i}{\overline{\nu}_i^*}, \ (\mathbf{y}, \phi) \in \mathbb{Y} \times \Phi_R \tag{5.3}$$

Define a function $V : \mathbb{X} \to \mathbb{R}_+$, where

$$V(\mathbf{x}) = a \left(\sum_{i \in \mathbb{T}} \frac{q_i}{\overline{\nu}_i^*} + f(\mathbf{y}) \right), \tag{5.4}$$

where

$$a = \frac{T}{(1 - \sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*})}, \tag{5.5}$$

and T is some positive constant to be explained shortly (Lemma 5.4 below). Similarly, let $g : \mathbb{X} \to \mathbb{N}$ with

$$g(\mathbf{x}) = \begin{cases} T & \text{if } q_2 = 0 \text{ and } V(\mathbf{x}) > N, \\ 1 & \text{otherwise,} \end{cases} \tag{5.6}$$

where

$$N = 2a \frac{T}{\min(\overline{\nu}_1^*, \overline{\nu}_2^*)} + a \cdot f_{\max}, \tag{5.7}$$

and $f_{\max} \stackrel{\text{def}}{=} \max_{\mathbf{y} \in \mathbb{Y}} f(\mathbf{y})$. Finally, define

$$M = 1 + a\left(\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max}\right). \tag{5.8}$$

Lemma 5.4. *Suppose* $\mathbf{X}$ *is the CMC under the policy* θ^{opt}*. Then, there exists* *finite T such that the functions V and g in (5.4) and (5.6), respectively, satisfy*

$$\begin{aligned} &\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\ &\leq -g(\mathbf{x}) + M \cdot \mathbb{I}(V(\mathbf{x}) \leq N) \\ &= \begin{cases} -g(\mathbf{x}) + M & \text{if } V(\mathbf{x}) \leq N \\ -g(\mathbf{x}) & \text{otherwise} \end{cases} \end{aligned} \tag{5.9}$$

for every $\mathbf{x} \in \mathbb{X}$.

Proof. A proof of the lemma is provided in Section 5.5.8. □

We now proceed with the proof of the theorem with the help of Theorem 1 of [92]: suppose that (i) $V' : \mathbb{X} \to \mathbb{R}_+$, (ii) $h_1 : \mathbb{X} \to \mathbb{R}$, and (iii) $h_2 : \mathbb{X} \to \{1, 2, \ldots\}$ are functions that satisfy the following.

L1. $\inf_{\mathbf{x} \in \mathbb{X}} h_1(\mathbf{x}) > -\infty$, i.e., h_1 is bounded below.

L2. There is a compact subset $\mathbb{X}_0 \subsetneq \mathbb{X}$ such that $h_1(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{X} \setminus \mathbb{X}_0$.

L3. For all $N' > 0$, $\sup_{\mathbf{x} \in \mathbb{X}: V'(\mathbf{x}) \leq N'} h_2(x) < \infty$.

L4. For all sequences $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ such that $V'(\mathbf{x}_l) \to \infty$ as $l \to \infty$, $\limsup_{l \in \mathbb{N}} h_2(\mathbf{x}_l)/h_1(\mathbf{x}_l) < \infty$.

Theorem 1 of [92]: Consider a function $V' : \mathbb{X} \to \mathbb{R}_+$, and let $h_1 : \mathbb{X} \to \mathbb{R}$ and $h_2 : \mathbb{X} \to \mathbb{N}$ be functions that satisfy conditions L1 through L4. Suppose that, for all $\mathbf{x} \in \mathbb{X}$, the drift of V' in $h_2(\mathbf{x})$ steps satisfies

$$\mathbb{E}[V'(\mathbf{X}_{h_2(\mathbf{x})}) - V'(\mathbf{X}_0) \mid \mathbf{X}_0 = \mathbf{x}] \leq -h_1(\mathbf{x}).$$

For $N' > 0$, define

$$\tau_{N'} \overset{def}{=} \inf\{k > 0 \mid V'(\mathbf{X}_k) \leq N'\}.$$

Then, there exists $N'_0 > 0$ such that, for all $N' \geq N'_0$ and every $\mathbf{x} \in \mathbb{X}$, we have $\mathbb{E}[\tau_{N'} \mid \mathbf{X}_0 = \mathbf{x}] < \infty$.

First, note that, under the policy θ^{opt}, every state in $\mathbb{X}$ communicates with state $(1, \mathcal{A}, \mathbf{0})$ and the state $(1, \mathcal{A}, \mathbf{0})$ communicates with every other state in $\mathbb{X}$. Thus, $\mathbf{X}$ is irreducible. Choose $h_1(\mathbf{x}) = g(\mathbf{x}) - M \cdot \mathbb{I}(V(\mathbf{x}) \leq N)$ and $h_2(\mathbf{x}) = g(\mathbf{x})$. Then, by Lemma 5.4, all the conditions in Theorem 1 of [92] are satisfied and, hence, $\mathbf{X}$ is positive recurrent. This completes the proof of the theorem.

5.3 Stability Results for Multiple Types

Our results on the system with two different types of tasks can be easily generalize to m types of tasks. Theorem 5.3 summarizes the necessity condition for stability, which shares a very similar structure as Theorem 5.1.

Theorem 5.3 (Necessity)**.** *Suppose that there exists a stabilizing policy in* Θ_R *for arrival rates* $(\lambda_1, \ldots, \lambda_m)$*. Then, they satisfy* $\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} \leq 1$.

The proof of the Theorem 5.3 is similar to that of the Theorem 5.1 and consists of three Lemmas. Although we omit the proof for Lemma 5.1 and Lemma 5.2 in the general form due to similarities, we provide the proof for Lemma 5.3 for m types tasks in the Appendix A.2 for interested readers.

We now proceed to present the general version of Theorem 5.2.

Theorem 5.4 (Sufficiency)**.** *Assume (i) the arrival rates* $\boldsymbol{\lambda}$ *satisfy* $\sum_{i\in\mathbb{T}}\frac{\lambda_i}{\nu_i^*}\leq 1$ *and (ii)* $\tau_1^*\geq\ldots\geq\tau_m^*$*. Then, the following policy* θ^{opt} *stabilizes the system* $\mathbf{X}$*.*

$$\theta^{\text{opt}}(s,\mathcal{A},\mathbf{q})$$
$$=\begin{cases}\mathbf{e}_m & \text{if (i) } q_m>0 \text{ and } s<\tau_m^*-1,\ \text{or}\\ & \text{(ii) } q_m>0, q_m=\max_{j:\tau_j^*=\tau_m^*}q_j,\ \text{and } s=\tau_m^*-1,\\ \mathbf{e}_i & \text{if (i) } q_i>0, \sum_{j=i+1}^{m}q_j=0,\ \text{and } s<\tau_i^*-1,\ \text{or}\\ & \text{(ii) } q_i>0, q_i=\max_{j:\tau_j^*\leq\tau_i^*}q_j, q_i>\max_{j:j>i}q_j,\ \text{and } s=\tau_i^*-1,\\ \mathbf{0} & \text{otherwise,}\end{cases}$$

where $\mathbf{0}$ *is a vector with all elements equal zero and* $\mathbf{e}_i$ *is a vector with* i*-th element equals one and all other elements equal zero.*

Note that the proposed policy follows the same structure as result in Theorem 5.2. The proposed policy assigns a new type i task for service only if either (i) there is no type j task for j from $i+1$ to m to service and the activity state is below τ_i^*-1 or (ii) the length of queue i is greater than that of queue j for all j from $i+1$ to m and the activity state is equal to τ_i^*-1.

The proof for Theorem 5.4 is almost identical to Theorem 5.2 with slight changes for $g(\mathbf{x})$. We provide the V, g and the proof of Lemma 5.4 in the general case in Appendix A.3. The rest of the proof is the same as Section 5.2.2 which follow directly from Theorem 1 of [92].

5.4 Summary

We relax the single type constraint in the Chapter 4 and investigate system with two or more types of tasks. First, we identified the collection of task arrival rates for which there is a stabilizing stationary scheduling strategy by showing the service rate of any policy with one randomizing state is the convex combination of service rates of two threshold policies. In addition, finding this set can be achieved by solving the simple optimization problem for each type that we studied in Chapter 4. By exploring a potential-like function that redistribute the reward as we did in Chapter 4 and using a general version of Foster's Theorem, we identified an optimal threshold policy that stabilizes the system whenever the task arrival rate lies in the interior of the aforementioned set for which there is a stabilizing policy.

5.5 Proofs of Lemmas

5.5.1 A Proof of Lemma 5.1

Suppose that θ is a stabilizing policy in $\Theta_S(\boldsymbol{\lambda})$. Recall from Lemma 3.2 that $\mathscr{Y}(\theta)$ belongs to Φ_R^+. The following lemma illustrates that the long-term service rates achieved by $\mathscr{Y}(\theta)$ are identical to those of θ.

Lemma 5.5. *Suppose that θ is a stabilizing policy in $\Theta_S(\boldsymbol{\lambda})$. Then, $\overline{\nu}_i^{\mathscr{Y}(\theta)} = \nu_i^\theta = \lambda_i$ for all $i \in \mathbb{T}$.*

Proof. Let $\phi = \mathscr{Y}(\theta)$. First, note

$$\begin{aligned}
\overline{\nu}_i^{\phi} &\overset{(a)}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, i)\ \overline{\pi}^{\phi}(\overline{\mathbf{y}})\ \phi(\overline{\mathbf{y}})_i \\
&\overset{(b)}{=} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, i) \Big(\sum_{\mathbf{q} \in \mathbb{Q}^{\overline{\mathbf{y}}}} \pi^{\theta}(\overline{\mathbf{y}}, \mathbf{q}) \theta(\overline{\mathbf{y}}, \mathbf{q})_i \Big) \\
&\overset{(c)}{=} \sum_{\mathbf{x} \in \mathbb{X}} \mu(\overline{s}, i) \pi^{\theta}(\mathbf{x}) \theta(\mathbf{x})_i \overset{(d)}{=} \nu_i^{\theta},
\end{aligned}$$

where (b) follows from Lemma 3.3 and the equality in (3.13), and (c) is obtained by rearranging the double summations in terms of $\mathbf{x} = (\overline{\mathbf{y}}, \mathbf{q})$. Finally (a) and (d) hold from their definitions. The lemma now follows from the fact that ν_i^{θ} is equal to λ_i for all $i \in \mathbb{T}$ when the system is stable. ☐

5.5.2 A Proof of Lemma 5.2

First, consider the following optimization problem.

$$\underset{\phi \in \Phi_R^+}{\text{maximize}} \qquad \overline{\nu}_1^{\phi} \tag{5.10}$$

In order to prove that the long-term service rate of type i tasks cannot exceed $\overline{\nu}_i^*$ for any policy ϕ in Φ_R^+, we first introduce the following lemma.

Lemma 5.6. *There exists an optimal policy that solves (5.10) in a closed subset*

$$\Phi_R^{++} \overset{def}{=} \Big\{ \phi \in \Phi_R^+ \ \Big|\ \sum_{i \in \mathbb{T}} \phi(1, \mathcal{A})_i = 1 \Big\}.$$

The proof of the lemma is similar to that of Lemma 5.8 in Section 5.5.3 below, and we omit it here.

The intuition behind the lemma is that when the server state is $(1, \mathcal{A})$ and the server rests, the server's new state is $(1, \mathcal{A})$. This suggests that the server wasted an epoch without contributing to long-term service rates. Therefore, when the server state is $(1, \mathcal{A})$, the server should be required to work on a task with probability one, in order to increase the long-term service rates.

First, note that $\overline{\mathbf{Y}}^{\phi}$ is a unichain for all ϕ in Φ_R^{++}. In other words, $\mathbf{Y}^{\phi}$ is a finite-state Markov chain with a single recurrent communicating class and, possibly, transient states. Since there is no (explicit) constraint in the optimization problem of (5.10), Theorem 4.4 of [89] tells us that there exists a deterministic optimal policy that solves (5.10).

Second, suppose that ϕ_*^1 is an optimal deterministic policy in Φ_R^{++}. Because ϕ_*^1 belongs to Φ_R^{++}, $\mathcal{T}(\phi_*^1) \geq 1$. In addition, we must have $\phi_*^1(\mathcal{T}(\phi_*^1), \mathcal{A}) = (1, 0)$; otherwise, $\overline{\nu}_1^{\phi_*^1} = 0$ because the unique PRCC is given by $\{(\overline{s}, \overline{w}) \in \mathbb{Y} \mid \overline{s} \geq \mathcal{T}(\phi_*^1), \overline{w} \in \{\mathcal{A}, \mathcal{B}_2\}\}$. Therefore, it is clear that the optimal value of (5.10) is equal to the maximum long-term service rate achieved by a threshold policy on the activity state with only type 1 tasks, namely $\overline{\nu}_1^*$. Similarly, the largest long-term service rate of type 2 tasks which can be achieved by any policy ϕ in Φ_R^+ is equal to $\overline{\nu}_2^*$.

5.5.3 A Proof of Lemma 5.3

Consider the following optimization problem, which is related to (5.10), with a constraint on the long-term service rate of type 2 tasks. Since the case with $\lambda_2 = 0$

reduces to a single-queue case studied in Chapter 4 (and discussed in Step 2 above), we assume $\lambda_2 > 0$.

$$\begin{aligned} \underset{\phi \in \Phi^\star}{\text{maximize}} \quad & \overline{\nu}_1^\phi \\ \text{subject to} \quad & \overline{\nu}_2^\phi \geq \lambda_2 \end{aligned} \tag{5.11}$$

where $\Phi^\star$ is some subset of Φ_R^+. We denote the optimal value of (5.11) by $\overline{\nu}^*(\Phi^\star)$.

We shall prove that the optimization problem with $\Phi^\star = \Phi_R^+$ has an optimal value $\overline{\nu}_1^* \frac{\overline{\nu}_2^* - \lambda_2}{\overline{\nu}_2^*}$ and there exists is an optimal policy ϕ^* in Φ_R^+ which achieves the optimal value. To this end, we consider (5.11) with a sequence of decreasing subsets of Φ_R^+ and show that the optimal value does not decrease as the subset of policies we allow shrinks (Lemmas 5.7 through 5.9).

Since $\Phi_R^{++} \subsetneq \Phi_R^+$, we have $\overline{\nu}^*(\Phi_R^{++}) \leq \overline{\nu}^*(\Phi_R^+)$. Furthermore, because Φ_R^{++} is closed, an optimal solution to (5.11) with $\Phi^\star = \Phi_R^{++}$ exists, i.e., the optimal value $\overline{\nu}^*(\Phi_R^{++})$ is achievable.

Lemma 5.7. $\overline{\nu}^*(\Phi_R^{++}) = \overline{\nu}^*(\Phi_R^+)$.

Proof. Please see Section 5.5.4 for a proof. □

Define $\Phi^\dagger$ to be set of policies in Φ_R^{++} which are deterministic except for at most at one state where the policy randomizes between two admissible actions. In other words,

$$\Phi^\dagger \stackrel{\text{def}}{=} \Big\{\phi \in \Phi_R^{++} \;\Big|\; \phi(\overline{s}, \overline{w}) \in \{0,1\}^2 \text{ for all } \overline{s} \in \mathbb{S}_\phi^D \subseteq \mathbb{S}$$
$$\text{such that (a) } |\mathbb{S} \setminus \mathbb{S}_\phi^D| \le 1 \text{ and (b) at a state in}$$
$$\mathbb{S} \setminus \mathbb{S}_\phi^D,\ \phi \text{ randomizes between two actions}\Big\}$$

In general, $\Phi^\dagger$ is a strict subset of Φ_R^{++} and $\overline{\nu}^*(\Phi^\dagger) \le \overline{\nu}^*(\Phi_R^{++})$. The following lemma, however, tells us that the equality holds.

Lemma 5.8. $\overline{\nu}^*(\Phi^\dagger) = \overline{\nu}^*(\Phi_R^{++})$.

Proof. Recall that, for every ϕ in Φ_R^{++}, the corresponding $\overline{\mathbf{Y}}^\phi$ is a unichain. Therefore, the optimization problem (5.11) gives rise to a unichain MDP problem. Since there is only one constraint in (5.11), Theorem 4.4 of [89] tells us that there exists an optimal policy with at most one randomization, i.e., it either (i) is deterministic (at every state) or (ii) randomizes between two admissible actions at exactly one state and is deterministic at every other state. Therefore, this optimal policy belongs to $\Phi^\dagger$. □

We introduce two families of threshold policies on activity state – φ^τ and ψ^τ, $\tau \in \mathbb{S}^+ \stackrel{\text{def}}{=} \mathbb{S} \cup \{s_{\max}+1\}$:

$$\varphi^\tau(\overline{s}, \mathcal{A}) = \begin{cases} (1,0) & \text{if } \overline{s} < \tau, \\ (0,0) & \text{otherwise.} \end{cases}$$

and

$$\psi^{\tau}(\overline{s}, \mathcal{A}) = \begin{cases} (0,1) & \text{if } \overline{s} < \tau, \\ (0,0) & \text{otherwise.} \end{cases}$$

When the server is available to take on a new task, φ^{τ} (resp. ψ^{τ}) asks the server to service a type 1 task (resp. type 2 task) only if the activity state is less than τ.

Finally, we define $\Phi^{\ddagger}$ to be the subset of policies in $\Phi^{\dagger}$ of the following forms: suppose $\boldsymbol{\tau} \stackrel{\text{def}}{=} (\tau_1, \tau_2) \in \mathbb{S}^+ \times \mathbb{S}^+$ and $\gamma \in [0,1]$.

f1. $\tau_1 \leq \tau_2$

$$\zeta^{\boldsymbol{\tau},\gamma}(\overline{s}, \overline{w}) = \begin{cases} (1-\gamma)\varphi^{\tau_1}(\overline{s}, \overline{w}) + (0,\gamma) & \text{if } \overline{s} = \tau_2 - 1, \overline{w} = \mathcal{A} \\ \varphi^{\tau_1}(\overline{s}, \overline{w}) & \text{otherwise} \end{cases} \tag{5.12}$$

f2. $\tau_1 > \tau_2$

$$\xi^{\boldsymbol{\tau},\gamma}(\overline{s}, \overline{w}) = \begin{cases} (1-\gamma)\psi^{\tau_2}(\overline{s}, \overline{w}) + (\gamma,0) & \text{if } \overline{s} = \tau_1 - 1, \overline{w} = \mathcal{A} \\ \psi^{\tau_2}(\overline{s}, \overline{w}) & \text{otherwise} \end{cases} \tag{5.13}$$

Clearly, these policies randomize between two admissible actions only at a single state and belong to $\Phi^{\dagger}$.

Lemma 5.9. $\overline{\nu}^*(\Phi^\ddagger) = \overline{\nu}^*(\Phi^\dagger)$.

Proof. Please see Section 5.5.5 for a proof. □

Let us consider the policies of the form in (5.12) for some given $\tau_i \in \mathbb{S}^+$, $i \in \mathbb{T}$, satisfying $\tau_1 \leq \tau_2$ and $\gamma \in [0, 1]$. We rewrite γ in (5.12) as

$$\gamma = \frac{\alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\tau_2 - 1, \mathcal{A})}{\alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\tau_2 - 1, \mathcal{A}) + (1 - \alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2 - 1, \mathcal{A})} \tag{5.14}$$

for some $\alpha \in [0, 1]$. Note that, for every $\gamma \in [0, 1]$, we can find an appropriate $\alpha \in [0, 1]$ that satisfies (5.14) because $\overline{\pi}^{\varphi^{\tau_1}}(\tau_2 - 1, \mathcal{A}) > 0$ and $\overline{\pi}^{\psi^{\tau_2}}(\tau_2 - 1, \mathcal{A}) > 0$ from the assumption $\tau_1 \leq \tau_2$.

By solving the global balance equations for $\overline{\mathbf{Y}}$ under the policy $\zeta^{\boldsymbol{\tau},\gamma}$, we get the following stationary PMF. Its derivation is provided in Section 5.5.6: for every $\overline{\mathbf{y}}$ in $\mathbb{Y}$,

$$\overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}) = (1 - \alpha)\overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}) + \alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}) \tag{5.15}$$

The long-term service rate of type 1 tasks can be obtained using the stationary PMF.

$$\overline{\nu}_1^{\zeta^{\boldsymbol{\tau},\gamma}} = \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, 1)\ \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}})\ \zeta^{\boldsymbol{\tau},\gamma}(\overline{\mathbf{y}})_1$$

Substituting the RHS of (5.15) for $\overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}})$, we obtain

$$\begin{aligned}
\overline{\nu}_1^{\zeta^{\boldsymbol{\tau},\gamma}} &= \sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \Big(\mu(\overline{s},1)\big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}) + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}})\big)\zeta^{\boldsymbol{\tau},\gamma}(\overline{\mathbf{y}})_1\Big) \\
&= \mu(\tau_2-1,1)\big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\tau_2-1,\mathcal{A}) \\
&\qquad\qquad + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2-1,\mathcal{A})\big)\zeta^{\boldsymbol{\tau},\gamma}(\tau_2-1,\mathcal{A})_1 \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \Big(\mu(\overline{s},1)\big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}) + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}})\big)\zeta_1^{\boldsymbol{\tau},\gamma}(\overline{\mathbf{y}})_1\Big). \qquad (5.16)
\end{aligned}$$

Using the definition of $\zeta^{\boldsymbol{\tau},\gamma}$ in (5.12),

$$\begin{aligned}
(5.16) &= \mu(\tau_2-1,1)\big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\tau_2-1,\mathcal{A}) + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2-1,\mathcal{A})\big) \\
&\qquad \times(1-\gamma)\varphi^{\tau_1}(\tau_2-1,\mathcal{A})_1 \qquad (5.17) \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \Big(\mu(\overline{s},1)\big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}) + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}})\big)\varphi^{\tau_1}(\overline{\mathbf{y}})_1\Big). \qquad (5.18)
\end{aligned}$$

First, using the expression in (5.14) for γ in the first term, we get

$$(5.17) = \mu(\tau_2-1,1)(1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2-1,\mathcal{A})\varphi^{\tau_1}(\tau_2-1,\mathcal{A})_1.$$

Second, we show $\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}})\varphi^{\tau_1}(\overline{\mathbf{y}})_1 = 0$ for all $\overline{\mathbf{y}} \in \mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}$ by considering the following four cases.

- If (i) $s \geq \tau_2$ and $w = \mathcal{A}$ or (ii) $w = \mathcal{B}_2$, we have $\varphi^{\tau_1}(\overline{s},\overline{w})_1 = 0$ from the definition of φ^{τ_1}.
- If (iii) $s < \tau_2 - 1$ or (iv) $w = \mathcal{B}_1$, then $\overline{\pi}^{\psi^{\tau_2}}(\overline{s},\overline{w}) = 0$.

As a result,

$$(5.18) = \sum_{\overline{\mathbf{y}} \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}} \Big(\mu(\overline{s}, 1)(1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}) \varphi^{\tau_1}(\overline{\mathbf{y}})_1 \Big).$$

Summing (5.17) and (5.18), we get

$$\begin{aligned} \overline{\nu}_1^{\zeta^{\boldsymbol{\tau}}, \gamma} &= \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \Big(\mu(\overline{s}, 1)(1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}) \varphi^{\tau_1}(\overline{\mathbf{y}})_1 \Big) \\ &= (1-\alpha) \overline{\nu}_1^{\varphi^{\tau_1}} \leq (1-\alpha) \overline{\nu}_1^*. \end{aligned} \tag{5.19}$$

Following similar steps, we can show

$$\overline{\nu}_2^{\zeta^{\boldsymbol{\tau}}, \gamma} = \alpha\, \overline{\nu}_2^{\psi^{\tau_2}} \leq \alpha\, \overline{\nu}_2^*. \tag{5.20}$$

Together with the constraint $\overline{\nu}_2^{\zeta^{\boldsymbol{\tau}}, \gamma} \geq \lambda_2$, (5.20) yields $\alpha \geq \lambda_2 / \overline{\nu}_2^*$. Using this inequality in (5.19) gives us

$$\overline{\nu}_1^{\zeta^{\boldsymbol{\tau}}, \gamma} \leq (1-\alpha) \overline{\nu}_1^* \leq \left(1 - \frac{\lambda_2}{\overline{\nu}_2^*} \right) \overline{\nu}_1^*.$$

The same inequality can be proved in a similar fashion, starting with a policy of the form in (5.13).

5.5.4 A Proof of Lemma 5.7

Given some $\phi \in \Phi_R^+$, consider the following policy $\phi^* \in \Phi_R^{++}$.

$$\phi^*(\mathbf{y}) = \begin{cases} \frac{\phi(\mathbf{y})}{\sum_{i\in\mathbb{T}} \phi(\mathbf{y})_i} & \text{if } \mathbf{y} = (1, \mathcal{A}) \\ \phi(\mathbf{y}) & \text{otherwise} \end{cases} \tag{5.21}$$

Lemma 5.10. *The stationary PMF $\overline{\pi}^{\phi^*}$ is related to the stationary PMF $\overline{\pi}^{\phi}$ as follows:*

$$\overline{\pi}^{\phi^*}(\mathbf{y}) = \begin{cases} \frac{\overline{\pi}^{\phi}(\mathbf{y}) \sum_{i\in\mathbb{T}} \phi(\mathbf{y})_i}{\alpha} & \textit{if } \mathbf{y} = (1, \mathcal{A}), \\ \frac{\overline{\pi}^{\phi}(\mathbf{y})}{\alpha} & \textit{otherwise}, \end{cases} \tag{5.22}$$

where $\alpha = 1 - \overline{\pi}^{\phi}(1, \mathcal{A})(1 - \sum_{i\in\mathbb{T}} \phi(1, \mathcal{A})_i) \leq 1$.

Proof. A proof is provided in Section 5.5.7. □

Using the stationary PMF $\overline{\pi}^{\phi^*}$ in Lemma 5.10, we get

$$\begin{aligned}
\overline{\nu}_i^{\phi^*} &= \sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \mu(\overline{s}, i)\overline{\pi}^{\phi^*}(\overline{\mathbf{y}})\phi^*(\overline{\mathbf{y}})_i \\
&= \mu(1, i)\overline{\pi}^{\phi^*}(1, \mathcal{A})\phi^*(1, \mathcal{A})_i \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus(1,\mathcal{A})} \mu(\overline{s}, i)\overline{\pi}^{\phi^*}(\overline{\mathbf{y}})\phi^*(\overline{\mathbf{y}})_i \\
&= \mu(1, i)\frac{\overline{\pi}^{\phi}(1, \mathcal{A})\sum_{i\in\mathbb{T}}\phi(1, \mathcal{A})_i}{\alpha}\frac{\phi(1, \mathcal{A})_i}{\sum_{i\in\mathbb{T}}\phi(1, \mathcal{A})_i} \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}\setminus(1,\mathcal{A})} \mu(\overline{s}, i)\frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})}{\alpha}\phi(\overline{\mathbf{y}})_i \\
&= \frac{1}{\alpha}\sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \mu(\overline{s}, i)\overline{\pi}^{\phi}(\overline{\mathbf{y}})\phi(\overline{\mathbf{y}})_i \\
&= \frac{\overline{\nu}_i^{\phi}}{\alpha} \geq \overline{\nu}_i^{\phi} \geq \lambda_i.
\end{aligned}$$

Hence, $\overline{\nu}_2^{\phi^*} \geq \lambda_2$ and ϕ^* is a feasible solution.

We also know that $\overline{\nu}_1^{\phi^*} = \frac{\overline{\nu}_1^{\phi}}{\alpha}$. This proves that, for every policy ϕ in Φ_R^+ which satisfies the constraint on the long-term service rate of type 2 tasks, we can find a policy ϕ^* in Φ_R^{++} that satisfies the same constraint and achieves the long-term service rate of type 1 tasks which is greater than or equal to that of ϕ.

5.5.5 A Proof of Lemma 5.9

In order to prove the lemma, we show that, for every feasible policy ϕ in $\Phi^\dagger$ which (a) satisfies the constraint $\overline{\nu}_2^{\phi} \geq \lambda_2$ and (b) achieves positive $\overline{\nu}_1^{\phi}$, we can find a feasible policy ϕ' in $\Phi^\ddagger$ with (i) the identical stationary PMF as ϕ and (ii) the same scheduling decision at all states in the unique PRCC of $\overline{\mathbf{Y}}^{\phi}$.

Choose a feasible policy ϕ in $\Phi^{\dagger}$. Note that $\mathcal{T}(\phi) \geq 1$ because ϕ belongs to Φ_R^{++}. Let

$$\mathbb{T}^{\phi} \stackrel{def}{=} \{i \in \mathbb{T} \mid \phi(\mathcal{T}(\phi), \mathcal{A})_i > 0\}$$

be the set of types of tasks the policy ϕ schedules for service with positive probability at state $(\mathcal{T}(\phi), \mathcal{A})$. Recall that the unique PRCC is a subset of $\mathbb{Y}^{\phi}$ as explained in Section 3.4.1. Moreover, it is clear that a feasible policy ϕ cannot be deterministic with $\mathbb{T}^{\phi} = \{1\}$; the long-term service rate $\overline{\nu}_2^{\phi}$ of the policy would be zero, contradicting the assumption that it is a feasible policy with $\overline{\nu}_2^{\phi} \geq \lambda_2 > 0$.

We consider two possible cases.

C1. ϕ is a deterministic policy with $\mathbb{T}^{\phi} = \{2\}$ – In this case, $\phi(\overline{s}, \mathcal{A}) = (0, 0)$ for all $\overline{s} > \mathcal{T}(\phi)$ and the threshold policy $\psi^{\mathcal{T}(\phi)+1}$ has the same unique PRCC with the identical stationary PMF $\overline{\boldsymbol{\pi}}^{\phi}$. Since $\phi(\overline{\mathbf{y}}) = \psi^{\mathcal{T}(\phi)+1}(\overline{\mathbf{y}})$ for all $\overline{\mathbf{y}} \in \mathbb{Y}^{\phi}$, it is clear that they yield the same long-term service rates for both types.

C2. ϕ randomizes at one state $\overline{\mathbf{y}}^{\star} = (\overline{s}^{\star}, \mathcal{A})$ – There are three possibilities. First, assume $\overline{s}^{\star} < \mathcal{T}(\phi)$. This is similar to case C1, and the threshold policy $\psi^{\mathcal{T}(\phi)+1}$ achieves the same long-term service rates as ϕ.

Second, suppose $\overline{s}^{\star} = \mathcal{T}(\phi)$. This implies that $\mathbb{T}^{\phi} = \mathbb{T}$ and $\phi(\overline{\mathbf{y}}^{\star}) = (p_1, p_2) > (0, 0)$ with $p_1 + p_2 = 1$. Then, as mentioned in Section 3.4.1, the unique PRCC is $\mathbb{Y}^{\phi}$. In this case, one can verify that the policy $\zeta^{(\overline{s}^{\star}+1, \overline{s}^{\star}+1), p_2}$ has the same stationary

PMF and long-term service rates.

Third, suppose $\overline{s}^\star > \mathcal{T}(\phi)$ and $\phi(\overline{\mathbf{y}}^\star) = (p_1, p_2)$. Note that only one of p_1 and p_2 can be positive and they satisfy $0 < p_1 + p_2 < 1$. Let $t^\dagger$ be the type selected for service with positive probability, i.e., $p_{t^\dagger} > 0$. Then, in order to satisfy the constraints $\overline{\nu}_i^\phi > 0$ for all $i \in \mathbb{T}$, $t^\dagger$ cannot be in $\mathbb{T}^\phi$; otherwise, since $\mathbb{T}^\phi$ is a singleton and $\phi(s, \mathcal{A}) = (0, 0)$ for all $s \in \{\mathcal{T}(\phi) + 1, \ldots, \overline{s}^\star - 1\}$, only the long-term service rate of type $t^\dagger$ tasks can be positive and that of the other type must be zero. Here, we assume $\mathbb{T}^\phi = \{1\}$ and $t^\dagger = 2$ with $\phi(\overline{\mathbf{y}}^\star) = (0, p_2)$. The other case can be handled similarly. Then, the policy $\zeta^{(\mathcal{T}(\phi)+1, \overline{s}^\star+1), p_2}$ yields the same stationary PMF and long-term service rates.

5.5.6 Derivation of Stationary PMF in (5.15)

In order to prove (5.15) is the correct stationary PMF, it suffices to show that the given PMF satisfies the following global balance equations:

$$\overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}') \; \overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \text{ for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{5.23}$$

where $\overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}$ is the one-step transition matrix of $\overline{\mathbf{Y}}^{\zeta^{\boldsymbol{\tau},\gamma}}$. To this end, we shall demonstrate that the RHS of (5.15) is equal to the RHS of (5.23).

First, we break the RHS of (5.23) into two terms.

$$\sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

$$= \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\tau_2-1,\mathcal{A})\overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} \tag{5.24}$$

$$+ \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \tag{5.25}$$

We then rewrite each term on the RHS: from (5.15) and (5.12), we have

$$(5.24) = \big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\tau_2-1,\mathcal{A}) + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2-1,\mathcal{A})\big)$$
$$\times\big((1-\gamma)\overline{\mathbf{P}}^{\varphi^{\tau_1}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} + \gamma\overline{\mathbf{P}}^{\psi^{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}}\big)$$

Substituting the expression for γ in (5.14),

$$(5.24) = (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\tau_2-1,\mathcal{A})\overline{\mathbf{P}}^{\varphi^{\tau_1}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}}$$
$$+\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\tau_2-1,\mathcal{A})\overline{\mathbf{P}}^{\psi^{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}}$$

Second, from (5.15)

$$(5.25) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \big(\alpha\cdot\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}') + (1-\alpha)\overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}')\big)$$
$$\times\overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}.$$

From (5.12), for all $\overline{\mathbf{y}}' = (\overline{s}', \overline{w}') \in \mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}$, we have $\zeta^{\boldsymbol{\tau},\gamma}(\mathbf{y}') = \varphi^{\tau_1}(\mathbf{y}')$ and $\overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\varphi^{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$. Moreover, because ψ^{τ_2} is a deterministic policy with a threshold on the activity state of the server, $\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}') = 0$ for all $\overline{\mathbf{y}}' = (\overline{s}', \overline{w}')$ with $\overline{s}' < \tau_2 - 1$ or $\overline{w}' = \mathcal{B}_1$. Hence, for all $\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{(\tau_2 - 1), \mathcal{A})\}$ with $\overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}') > 0$, together with the assumption $\tau_1 \leq \tau_2$, we have

$$\varphi^{\tau_1}(\overline{\mathbf{y}}') = \psi^{\tau_2}(\overline{\mathbf{y}}') = \begin{cases} (0,0) & \text{if } \overline{s}' \geq \tau_2 \text{ and } \overline{w}' = \mathcal{A} \\ (1,0) & \text{if } \overline{w}' = \mathcal{B}_1 \\ (0,1) & \text{if } \overline{w}' = \mathcal{B}_2 \end{cases}$$

and, consequently, $\overline{\mathbf{P}}^{\varphi^{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\psi^{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$. Therefore,

$$\begin{aligned}(5.25) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}} \Big(& \alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}') \overline{\mathbf{P}}^{\psi^{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\ & + (1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}') \overline{\mathbf{P}}^{\varphi^{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \Big)\end{aligned}$$

Substituting the new expressions for (5.24) and (5.25), we obtain

$$\begin{aligned}&\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\zeta^{\boldsymbol{\tau},\gamma}}(\overline{\mathbf{y}}') \ \overline{\mathbf{P}}^{\boldsymbol{\tau},\gamma}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\ &= (1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\tau_2 - 1, \mathcal{A}) \overline{\mathbf{P}}^{\varphi^{\tau_1}}_{(\tau_2 - 1, \mathcal{A}), \overline{\mathbf{y}}} \\ &\quad + \alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\tau_2 - 1, \mathcal{A}) \overline{\mathbf{P}}^{\psi^{\tau_2}}_{(\tau_2 - 1, \mathcal{A}), \overline{\mathbf{y}}} \\ &\quad + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}} \Big(\alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}') \overline{\mathbf{P}}^{\psi^{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\ &\qquad + (1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}') \overline{\mathbf{P}}^{\varphi^{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \Big) \\ &= \alpha \cdot \overline{\pi}^{\psi^{\tau_2}}(\overline{\mathbf{y}}) + (1-\alpha) \overline{\pi}^{\varphi^{\tau_1}}(\overline{\mathbf{y}}),\end{aligned}$$

where the last equality follows from the fact that $\overline{\pi}^{\varphi^{\tau_1}}$ and $\overline{\pi}^{\psi^{\tau_2}}$ are the stationary PMFs of $\overline{\mathbf{Y}}^{\varphi^{\tau_1}}$ and $\overline{\mathbf{Y}}^{\psi^{\tau_2}}$, respectively.

5.5.7 A Proof of Lemma 5.10

From the definition of a stationary PMF, we know

$$\overline{\pi}^{\phi}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \ \text{ for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{5.26}$$

where $\overline{\mathbf{P}}^{\phi}$ denotes the one-step transition matrix of $\overline{\mathbf{Y}}^{\phi}$. Starting with this equality, we shall show that the distribution in (5.22) satisfies

$$\overline{\pi}^{\phi^*}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \ \text{ for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{5.27}$$

where $\overline{\mathbf{P}}^{\phi^*}$ denotes the one-step transition matrix of $\overline{\mathbf{Y}}^{\phi^*}$. To this end, we consider the two cases in (5.22).

- <u>*Case 1:* $\overline{\mathbf{y}} = (1, \mathcal{A})$</u> – First, we can write the given expression for $\overline{\pi}^{\phi^*}(\overline{\mathbf{y}})$ in a more convenient form, using the assumed value of α.

$$\begin{aligned} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}) &= \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}}) - (1-\alpha)}{\alpha} \\ &= \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}}) \sum_{i \in \mathbb{T}} \phi(\overline{\mathbf{y}})_i}{\alpha}. \end{aligned} \tag{5.28}$$

Second, the RHS of (5.27) can be shown to be equal to the above expression as follows.

$$\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \tag{5.29}$$
$$= \overline{\pi}^{\phi^*}(\overline{\mathbf{y}})\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}\setminus\{\overline{\mathbf{y}}\}} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

Using the provided expression from (5.22) for $\overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')$ in the second term, we get

$$(5.29) = \overline{\pi}^{\phi^*}(\overline{\mathbf{y}})\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} + \frac{1}{\alpha} \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}\setminus\{\overline{\mathbf{y}}\}} \overline{\pi}^{\phi}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

Note from (5.26) that the summation in the second term is equal to $\overline{\pi}^{\phi}(\overline{\mathbf{y}})\big(1-\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}}\big)$. Together with (5.28), this gives us

$$(5.29) = \overline{\pi}^{\phi^*}(\overline{\mathbf{y}})\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} + \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})(1-\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}})}{\alpha}$$
$$= \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i}{\alpha}\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} + \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})(1-\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}})}{\alpha} \tag{5.30}$$

The transition probability $\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}}$ is equal to the probability that $\mathbf{Y}^{\phi}$ would transition from $\overline{\mathbf{y}}$ back to itself conditional on that the action chosen by the scheduler is not $\mathcal{R}$, i.e., it assigns either a type 1 task or a type 2 task to the server. Thus, the transition probability is equal to

$$\overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} = \frac{\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} - \big(1-\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i\big)}{\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i}. \tag{5.31}$$

Note that the numerator is equal to the probability that the server is asked to work on a task and then the server state returns to the same state $\overline{\mathbf{y}}$.

Substituting (5.31) in (5.30), we obtain

$$
\begin{aligned}
(5.29) = & \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})(\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i)}{\alpha}\left(\frac{\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}} - \left(1-\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i\right)}{\sum_{i\in\mathbb{T}}\phi(\overline{\mathbf{y}})_i}\right) \\
& + \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})(1-\overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}},\overline{\mathbf{y}}})}{\alpha} \\
= & \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})\sum_{i\in\mathbb{T}}\phi_i(\overline{\mathbf{y}})}{\alpha}.
\end{aligned}
$$

It is clear that this is equal to the LHS of (5.27) as shown in (5.28).

- <u>*Case 2:* $\overline{\mathbf{y}} \neq (1, \mathcal{A})$</u> – Following similar steps used in the first case, we first

rewrite the RHS of (5.27).

$$
\begin{aligned}
& \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
& = \overline{\pi}^{\phi^*}(1,\mathcal{A})\overline{\mathbf{P}}^{\phi^*}_{(1,\mathcal{A}),\overline{\mathbf{y}}} + \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus\{(1,\mathcal{A})\}} \overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi^*}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}
\end{aligned} \tag{5.32}
$$

Substuting (5.28) and (5.22) for $\overline{\pi}^{\phi^*}(1,\mathcal{A})$ and $\overline{\pi}^{\phi^*}(\overline{\mathbf{y}}')$, respectively, we get

$$
\begin{aligned}
(5.32) = & \frac{\overline{\pi}^{\phi}(1,\mathcal{A})\sum_{i\in\mathbb{T}}\phi(1,\mathcal{A})_i}{\alpha}\left(\frac{\overline{\mathbf{P}}^{\phi}_{(1,\mathcal{A}),\overline{\mathbf{y}}}}{\sum_{i\in\mathbb{T}}\phi(1,\mathcal{A})_i}\right) \\
& + \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus(1,\mathcal{A})} \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}}')}{\alpha}\ \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
= & \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}}')}{\alpha}\ \overline{\mathbf{P}}^{\phi}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
= & \frac{\overline{\pi}^{\phi}(\overline{\mathbf{y}})}{\alpha}.
\end{aligned}
$$

5.5.8 A Proof of Lemma 5.4

For notational simplicity, we omit the dependence on the policy θ^{opt} and denote the conditional expected value of the difference in potential function, $\mathbb{E}\left[f(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f(\overline{\mathbf{Y}}^{\phi}_{k}) \mid \overline{\mathbf{Y}}^{\phi}_{k} = \mathbf{y}\right]$, by $\Delta f(\overline{\mathbf{Y}}^{\phi}; \mathbf{y})$. In addition, r^{ϕ}_{avg} denotes the average reward in $\overline{\mathbf{Y}}^{\phi}$ when there is a unique PRCC under a policy ϕ in Φ_R.

Consider the CMC $\overline{\mathbf{Y}}^{\psi^{\tau_2^*}}$ equipped with the policy $\psi^{\tau_2^*}$. Assume that f is a potential function that satisfies the equality in (4.12) of Proposition 4.1 for $\overline{\mathbf{Y}}^{\psi^{\tau_2^*}}$, with the reward function in (5.3). Define $\mathbb{Y}_{\tau_1^*} \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{Y} \mid s \geq \tau_1^* - 1\}$.

Lemma 5.11. *For every $\mathbf{y} \in \mathbb{Y}_{\tau_1^*}$, we have*

$$r(\mathbf{y}, \varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}; \mathbf{y}) = 1. \tag{5.33}$$

Proof. Please see Section 5.5.9 for a proof. □

Consider CMC $\mathbf{X}^{\theta^{\text{opt}}}$ that starts at some state $\mathbf{x}_0$ with $q_2 = 0$ and $q_1 \geq 2T'$, where T' is some positive integer. Then, for all $k \in \{0, 1, \ldots, T'-1\}$, (i) $Q^1_k \geq Q^2_k$ and (ii) $Q^1_k > 0$. These imply that, when $\mathbf{Y}_k = (\tau_1^* - 1, \mathcal{A})$ for some $k \in \{0, 1, \ldots, T'-1\}$, $\theta^{\text{opt}}(\mathbf{Y}_k, \mathbf{Q}_k) = (1, 0)$ and a new type 1 task is scheduled for service.

Let us take a look at the server state $\mathbf{Y}_k$ for $k \in \{0, 1, \ldots, T'-1\}$. First, if $\mathbf{Y}_{k^\dagger} \in \mathbb{Y}_{\tau_1^*}$ for some $k^\dagger \in \{0, 1, \ldots, T'-2\}$, then $\mathbf{Y}_k \in \mathbb{Y}_{\tau_1^*}$ for all $k \in \{k^\dagger, \ldots, T'-1\}$ under θ^{opt}. Second, if $q_1 \geq n_s$, starting with $\{\mathbf{Y}_0 = \mathbf{y}\}$ for some $\mathbf{y} \in \mathbb{Y}^C_{\tau_1^*} \stackrel{\text{def}}{=} \mathbb{Y} \setminus \mathbb{Y}_{\tau_1^*} = \{(s, w) \in \mathbb{Y} \mid s < \tau_1^* - 1\}$, the server state $\mathbf{Y}_k$ will reach a state in $\mathbb{Y}_{\tau_1^*}$ with positive probability after at most $\tau_1^* - 2$ epochs: for each $i \in \mathbb{T}$, define $\alpha^i_{\min} \stackrel{\text{def}}{=} \min\{\rho^i_{s,s+1};\ s \in \mathbb{S} \setminus \{n_s\}\}$ and $\beta^i_{\min} \stackrel{\text{def}}{=} \min\{\rho^i_{s+1,s};\ s \in \mathbb{S} \setminus \{n_s\}\}$. Then, because $\theta^{\text{opt}}(s, \mathcal{A}, \mathbf{q}) \in \{(1,0),(0,1)\}$ when $s < \tau_2^* - 1$ and $\mathbf{q} \neq (0,0)$, the probability of reaching a state in $\mathbb{Y}_{\tau_1^*}$ after at most $2n_s$ is lower bounded by $\delta \stackrel{def}{=} \min(\alpha^1_{\min}, \alpha^2_{\min})^{n_s} \min(\beta^1_{\min}, \beta^2_{\min})^{n_s}(1 - \min_s \mu(s))^{n_s}$. Consequently, for all $\mathbf{y} \in \mathbb{Y}^C_{\tau_1^*}$ and $q_1 \geq 2n_s$,

$$\Pr\big(\mathbf{Y}_{k'} \in \mathbb{Y}_{\tau_1^*} \text{ for some } k' = k+1, \dots, k+2n_s \\ \mid \mathbf{Y}_k = \mathbf{y}, \mathbf{Q}_k = (q_1, 0)\big) \geq \delta.$$

Using this bound, we can upper bound the probability that the server state does not belong to $\mathbb{Y}_{\tau_1^*}$ at epoch $2jn_s$ for all $j \geq 1$ (for $q_1 \geq 2jn_s$) as follows.

$$\Pr\big(\mathbf{Y}_{2jn_s} \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{X}_0 = (\mathbf{y}, q_1, 0)\big) \\ \leq (1-\delta) \Pr\big(\mathbf{Y}_{2(j-1)n_s} \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{X}_0 = (\mathbf{y}, q_1, 0)\big)$$

Thus, the probability $\Pr(\mathbf{Y}_{2jn_s} \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{Y}_0 \in \mathbb{Y}^C_{\tau_1^*}, \mathbf{Q}_0 = (q_1, 0))$ can be made arbitrarily small by choosing sufficiently large q_1. In addition, it is clear $\Pr(\mathbf{Y}_k \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{Y}_0 \in \mathbb{Y}^C_{\tau_1^*}, \mathbf{Q}_0 = (q_1, 0))$ is non-increasing in k, assuming that queue 1 remains non-empty.

Next, we study the following T'-step drift with $q_1 \geq T'$.

$$\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{T'} - Q^i_0}{\overline{\nu}^*_i} + f(\mathbf{Y}_{T'}) - f(\mathbf{Y}_0) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0\right]$$
$$= \sum_{k=0}^{T'-1} \mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{k+1} - Q^i_k}{\overline{\nu}^*_i} + f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0\right] \tag{5.34}$$

From the Lindley's equation in (3.1),

$$\begin{aligned}
&\mathbb{E}\left[Q^i_{k+1} - Q^i_k \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \mathbb{E}\left[B^i_k + D^i_k \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \lambda_i + \mathbb{E}\left[\mu(S_k, i)\mathbb{I}(A_k = \mathcal{W}_i) \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \lambda_i + \mathbb{E}\left[\mu(S_k, i)\theta^{\text{opt}}_i(\mathbf{X}_k) \mid \mathbf{X}_0 = \mathbf{x}_0\right].
\end{aligned} \tag{5.35}$$

Substituting (5.35) in (5.34), we obtain

$$\begin{aligned}
(5.34) = \sum_{k=0}^{T'-1} \Bigg(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}^*_i} + \mathbb{E}\Bigg[-\sum_{i\in\mathbb{T}} \frac{\mu(S_k, i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}^*_i} \\
+ f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\Bigg|\; \mathbf{X}_0 = \mathbf{x}_0 \Bigg]\Bigg).
\end{aligned} \tag{5.36}$$

We upper bound the conditional expected value using the sum two terms by conditioning on whether or not $\mathbf{Y}_k$ belongs to $\mathbb{Y}_{\tau_1^*}$.

$$\begin{aligned}
&\mathbb{E}\left[-\sum_{i\in\mathbb{T}}\frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*}+f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\;\middle|\;\mathbf{X}_0=\mathbf{x}_0\right]\\
&\le \Pr(\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*}\mid\mathbf{X}_0=\mathbf{x}_0)\;\mathbb{E}\Bigg[-\sum_{i\in\mathbb{T}}\frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*} \qquad (5.37)\\
&\qquad +f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\;\Bigg|\;\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*},\mathbf{X}_0=\mathbf{x}_0\Bigg]\\
&\qquad +\Pr(\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*}^C\mid\mathbf{X}_0=\mathbf{x}_0)f_{\max}.
\end{aligned}$$

By further conditioning on the server state at epoch k,

$$\begin{aligned}
(5.37) &= \Pr(\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*}\mid\mathbf{X}_0=\mathbf{x}_0)\\
&\qquad\times\sum_{\mathbf{y}\in\mathbb{Y}_{\tau_1^*}}\Pr(\mathbf{Y}_k=\mathbf{y}\mid\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*},\mathbf{X}_0=\mathbf{x}_0)\\
&\qquad\times\mathbb{E}\Big[-\sum_{i\in\mathbb{T}}\frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*}+f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\\
&\qquad\qquad\Big|\;\mathbf{Y}_k=\mathbf{y},\mathbf{X}_0=\mathbf{x}_0\Big] \qquad (5.38)\\
&\quad+\Pr(\mathbf{Y}_k\in\mathbb{Y}_{\tau_1^*}^C\mid\mathbf{X}_0=\mathbf{x}_0)f_{\max}.
\end{aligned}$$

When $\mathbf{Y}_k = \mathbf{y}$ for some $\mathbf{y}\in\mathbb{Y}_{\tau_1^*}$ and $k < T'$, $\theta^{\text{opt}}(\mathbf{y},\mathbf{q}) = \varphi^{\tau_1^*}(\mathbf{y})$ and $\mathbb{E}[f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\mid\mathbf{Y}_k=\mathbf{y},\mathbf{X}_0=\mathbf{x}_0] = \mathbb{E}[f(\overline{\mathbf{Y}}_{k+1}^{\varphi^{\tau_1^*}})-f(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}})\mid\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}=\mathbf{y}]$ because $Q_k^1 > 0$ from the assumption $q_1\ge T'$. Thus, using the reward function in (5.3), the conditional expected value in (5.38) is equal to $-r(\mathbf{y},\varphi^{\tau_1^*})+\Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y})$, which is equal to -1 by Lemma 5.11. This gives us

$$(5.38) = -\Pr(\mathbf{Y}_k \in \mathbb{Y}_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0) \\ +P(\mathbf{Y}_k \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max}. \tag{5.39}$$

From (5.34) - (5.39), we have

$$\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{T'} - Q^i_0}{\overline{\nu}^*_i} + f(\mathbf{Y}_{T'}) - f(\mathbf{Y}_0) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0\right] \\ \leq \sum_{k=0}^{T'-1} \left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}^*_i} - \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0) \right. \\ \left. + \Pr(\mathbf{Y}_k \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max} \right). \tag{5.40}$$

Recall $\sum_{i\in\mathbb{T}}(\lambda_i/\overline{\nu}^*_i) < 1$. In addition, $\Pr(\mathbf{Y}_k \in \mathbb{Y}_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0)$ converges to 1 (and, hence, $\Pr(\mathbf{Y}_k \in \mathbb{Y}^C_{\tau_1^*} \mid \mathbf{X}_0 = \mathbf{x}_0)$ goes to 0) as $k \to \infty$ (as long as T' grows accordingly) from our earlier discussion. Thus, for all sufficiently large T', the sum of the terms inside the parentheses is negative. This implies that, as $T' \to \infty$, (5.40) goes to $-\infty$. As a result, we can find finite T such that, for every state $\mathbf{x}_0$ with $q_2 = 0$ and $q_1 \geq T$,

$$\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_T - Q^i_0}{\overline{\nu}^*_i} + f(\mathbf{Y}_T) - f(\mathbf{Y}_0) \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\ \leq \sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}^*_i} - 1. \tag{5.41}$$

We are ready to prove that the functions V and g satisfy (5.9) when the parameter T is chosen to honor (5.41). To this end, we consider following five cases separately.

Case 1: $V(\mathbf{x}) \leq N$ – From the given function g in (5.6), when $V(\mathbf{x}) \leq N$, $g(x) = 1$. Thus, from the assumed Lyapunov function in (5.4),

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&\leq a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max}\right) \\
&= -1 + \left[1 + a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max}\right)\right] \\
&= -g(\mathbf{X}) + M.
\end{aligned}$$

Case 2: $V(\mathbf{x}) > N$**,** $q_2 > 0$ **and** $q_2 \geq q_1$ – In this case, $\theta^{\text{opt}}(\mathbf{x}) = \psi^{\tau_2^*}(\mathbf{y})$. Furthermore, $g(\mathbf{x}) = 1$ because $q_2 > 0$.

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - \sum_{i\in\mathbb{T}} \frac{\mu(s,i)\psi_i^{\tau_2^*}(\mathbf{y})}{\overline{\nu}_i^*} + \Delta f(\mathbf{Y};\mathbf{y})\right) \\
&= a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - r(\mathbf{y}, \psi^{\tau_2^*}) + \Delta f(\mathbf{Y};\mathbf{y})\right) \qquad (5.42)
\end{aligned}$$

where the last equality follows directly from the assumed reward function in (5.3). Note that the sum of the last two terms inside the parentheses is equal to -1 from

(5.44). Using (5.5), we obtain

$$(5.42) = a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1\right) = -T \le -1 = -g(\mathbf{x}).$$

Case 3: $V(\mathbf{x}) > N$**,** $q_2 > 0$**,** $q_2 < q_1$**, and** $\mathbf{y} \neq (\tau_1^* - 1, \mathcal{A})$ – In this case, $\theta^{\text{opt}}(\mathbf{x}) = \psi^{\tau_2^*}(\mathbf{y})$ again. The proof is similar to that of the previous case, and we omit it here.

Case 4: $V(\mathbf{x}) > N$**,** $q_2 > 0$**,** $q_2 < q_1$**, and** $\mathbf{y} = (\tau_1^* - 1, \mathcal{A})$ – Under the given condition, we have $\theta^{\text{opt}}(\mathbf{x}) = \varphi^{\tau_1^*}(\mathbf{y})$ and $g(\mathbf{x}) = 1$.

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - \sum_{i\in\mathbb{T}} \frac{\mu(s,i)\varphi_i^{\tau_1^*}(\mathbf{y})}{\overline{\nu}_i^*} + \Delta f(\mathbf{Y};\mathbf{y})\right) \\
&= a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - r(\mathbf{y}, \varphi^{\tau_1^*}) + \Delta f(\mathbf{Y};\mathbf{y})\right) \qquad (5.43)
\end{aligned}$$

From (5.45) and (5.33), the sum of the last two terms inside the parentheses is equal to -1. Therefore,

$$(5.43) = a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1\right) = -T \le -1 = -g(\mathbf{x}).$$

Case 5: $V(\mathbf{x}) > N$ **and** $q_2 = 0$ – Because $V(\mathbf{x}) > N$, from the assumed Lyapunov function in (5.4) and value of N in (5.7), we have

$$\sum_{i \in \mathbb{T}} \frac{q_i}{\overline{\nu}_i^*} \geq 2 \frac{T}{\min_i \overline{\nu}_i^*},$$

which implies $q_1 \geq 2T$ since $q_2 = 0$. Also, from the assumed function g, we have $g(\mathbf{x}) = T$.

From the inequality in (5.41),

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+T}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\, \mathbb{E}\left[\sum_{i \in \mathbb{T}} \frac{Q^i_{T+k} - Q^i_k}{\overline{\nu}_i^*} + f(\mathbf{Y}_{T+k}) - f(\mathbf{y}) \mid \mathbf{X}_k = \mathbf{x} \right] \\
&\leq a \left(\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1 \right) \\
&= -T \leq -g(\mathbf{x}).
\end{aligned}$$

5.5.9 A Proof of Lemma 5.11

From the choice of the potential function, we know that, for all $\mathbf{y}$ in $\mathbb{Y}$,

$$r(\mathbf{y}, \psi^{\tau_2^*}) - \Delta f(\overline{\mathbf{Y}}^{\psi^{\tau_2^*}}; \mathbf{y}) = r_{\text{avg}}^{\psi^{\tau_2^*}} = 1. \tag{5.44}$$

The second equality follows from the observation that the long-term service rate of type 2 tasks under $\psi^{\tau_2^*}$ equals $\overline{\nu}_2^*$ and the average reward is equal to the long-term service rate of type 2 tasks normalized by $\overline{\nu}_2^*$.

Our proof relies on the following set of observations. First, compare CMC $\overline{\mathbf{Y}}^{\psi^{\tau_2^*}}$ to CMC $\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}$. An important observation is that, for all $\mathbf{y} = (s, \mathcal{A})$ with $s \geq \tau_1^*$, we have $\varphi^{\tau_1^*}(\mathbf{y}) = \psi^{\tau_2^*}(\mathbf{y}) = (0,0)$ as both policies choose to rest the server. At the state $(\tau_1^* - 1, \mathcal{A})$, however, $\varphi^{\tau_1^*}$ and $\psi^{\tau_2^*}$ select different actions; the former chooses a type 1 task, whereas the latter either rests if $\tau_1^* > \tau_2^*$ or selects a type 2 task when $\tau_1^* = \tau_2^*$. This observation tells us that, for all $\mathbf{y} = (s, w)$ with $s \geq \tau_1^* - 1$, except for the state $(\tau_1^* - 1, \mathcal{A})$, because both policies choose the same action, from (5.44), the following holds.

$$r(\mathbf{y}, \varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}; \mathbf{y}) = 1 \tag{5.45}$$

Second, define

$$r^{\varphi^{\tau_1^*}} \stackrel{\text{def}}{=} \sum_{\mathbf{y} \in \mathbb{Y}_{\tau_1^*}} \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}) \Big(r(\mathbf{y}, \varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}; \mathbf{y}) \Big),$$

where $\overline{\boldsymbol{\pi}}^{\varphi^{\tau_1^*}}$ denotes the unique stationary PMF of $\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}$, and $\mathbb{Y}_{\tau_1^*} \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{Y} \mid s \geq \tau_1^* - 1\}$. Consider the CMC $\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}}$ starting at state $(\tau_1^* - 1, \mathcal{A})$ at epoch $k = 0$. For notational simplicity, we shall denote $(\tau_1^* - 1, \mathcal{A})$ by $\mathbf{y}_0$. Define

$$R_k \stackrel{\text{def}}{=} r\big(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}, \varphi^{\tau_1^*}\big) - \left(f(\overline{\mathbf{Y}}_{k+1}^{\varphi^{\tau_1^*}}) - f(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}) \right). \tag{5.46}$$

From the Weak Law of Large Number,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} R_k}{N} = r^{\varphi^{\tau_1^*}} \quad \text{with probability 1.} \tag{5.47}$$

Third, from (5.46),

$$\sum_{k=1}^{N} R_k = \sum_{k=1}^{N} r\big(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}, \varphi^{\tau_1^*}\big) - \sum_{k=1}^{N} \Big(f(\overline{\mathbf{Y}}_{k+1}^{\varphi^{\tau_1^*}}) - f(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}) \Big) .$$

We can simplify the telescoping sum in the second term.

$$\sum_{k=1}^{N} R_k = \sum_{k=1}^{N} r\big(\overline{\mathbf{Y}}_k^{\varphi^{\tau_1^*}}, \varphi^{\tau_1^*}\big) - \Big(f(\overline{\mathbf{Y}}_{N+1}^{\varphi^{\tau_1^*}}) - f(\overline{\mathbf{Y}}_1^{\varphi^{\tau_1^*}}) \Big) . \tag{5.48}$$

Using (5.48) and the fact that $f(\overline{\mathbf{Y}}_{N+1}^{\varphi^{\tau_1^*}}) - f(\overline{\mathbf{Y}}_1^{\varphi^{\tau_1^*}})$ is bounded, the Weak Law of Large Number tells us

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} R_k}{N} = r_{\text{avg}}^{\varphi^{\tau_1^*}} \quad \text{with probability 1.} \tag{5.49}$$

However, the average reward $r_{\text{avg}}^{\varphi^{\tau_1^*}}$ is equal to one from the employed reward function and the definition of the policy $\varphi^{\tau_1^*}$; the long-term service rate of type 1 tasks under $\varphi^{\tau_1^*}$ is equal to $\overline{\nu}_1^*$.

From (5.47) and (5.49), we have $r_{\text{avg}}^{\varphi^{\tau_1^*}} = r^{\varphi^{\tau_1^*}} = 1$ with probability 1 under $\varphi^{\tau_1^*}$.

Thus, from the definition of $r^{\varphi^{\tau_1^*}}$, we have

$$
\begin{aligned}
1 &= \sum_{\mathbf{y}\in\mathbb{Y}_{\tau_1^*}} \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y})\Big(r(\mathbf{y},\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y})\Big) \\
&= \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}_0)\Big(r(\mathbf{y}_0,\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}_0)\Big) \\
&\quad + \sum_{\mathbf{y}\in\mathbb{Y}_{\tau_1^*}\setminus\{\mathbf{y}_0\}} \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y})\Big(r(\mathbf{y},\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y})\Big).
\end{aligned}
$$

Recall from (5.45) that, for all $\mathbf{y}$ in $\mathbb{Y}_{\tau_1^*}\setminus\{\mathbf{y}_0\}$, we have $r(\mathbf{y},\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}) = 1$. Thus,

$$
\begin{aligned}
1 &= \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}_0)\Big(r(\mathbf{y}_0,\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}_0)\Big) \\
&\quad + \sum_{\mathbf{y}\in\mathbb{Y}_{\tau_1^*}\setminus\{\mathbf{y}_0\}} \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}).
\end{aligned}
\tag{5.50}
$$

Moving the second term on the RHS of (5.50) to the other side, we get

$$
\begin{aligned}
1 &- \sum_{\mathbf{y}\in\mathbb{Y}_{\tau_1^*}\setminus\{\mathbf{y}_0\}} \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}) \\
&= \overline{\pi}^{\varphi^{\tau_1^*}}(\mathbf{y}_0)\Big(r(\mathbf{y}_0,\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}_0)\Big)
\end{aligned}
$$

or, equivalently, $1 = r(\mathbf{y}_0,\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}_0)$. As a result, together with (5.44) and (5.45), we have, for all $\mathbf{y}\in\mathbb{Y}_{\tau_1^*}$,

$$
1 = r(\mathbf{y},\varphi^{\tau_1^*}) - \Delta f(\overline{\mathbf{Y}}^{\varphi^{\tau_1^*}};\mathbf{y}). \tag{5.51}
$$

Chapter 6: Remote State Estimation Across An Activity Packet-Drop Link

In this chapter, we model and study the usage dependent efficiency effect for remote estimation. We focus on a single agent setup where the link quality is affected by past transmission history. Similar to the previous chapter where we study when the server should work, the question that we ask here is, "When should the agent transmit?". This problem leads us to some structural insights for scheduling policies that guarantee the stability for estimation. The work presented in this chapter has partly appeared in [93].

The chapter is organized as follows. We begin by presenting an overview of our setting. In Section 6.2, we introduce the notations and the statistical properties for the so-call activity packet-drop link. Notice that we define a new set of notations which is uncorrelated with the previous chapter. We then provide a way to determine whether a scheduling policy exists for stabilizing the estimation and the structural features of maximal stabilizing strategies in Section 6.3. Finally, we give the proofs for our result with a highlight on finding an intriguing property of the spectral radius of matrices with a unique structure in Section 6.4.

6.1 Introduction

Remote wireless sensing has gained increasing interest in recent years in the infrastructure, environment, and human-body monitoring. Wireless technologies bring some clear benefits, such as reducing wiring material and installation costs. For instance, embedded sensors inside the human body allow continuous monitoring without exposing wire to the patients. Remote bridges vibration sensors provide advantages like no requirement for a nearby power source. It is readily apparent that such sensors are preferable in the hard-to-reach locations or on moving objects.

However, wireless sensing presents some unique challenges. Comparing to their wired counterparts, which have stable power sources, they need to rely on energy harvesting technologies that obtain electricity from the surrounding environments in the form of thermal, solar, or mechanical. Furthermore, the batteries' capacities on these sensors are likely to have strict limits due to weight, size, and cost constraints. The trade-off between using the energy to transmit or store it for later use emerges. Moreover, in the context of the embedded sensor inside the human body, we observe similar trade-off not only on the power usage but also on the temperature fluctuation of the surrounding area due to transmission. The scheduler chooses between to transmit but with the risk that the rising temperature will deteriorate the channel quality or wait for the neighboring area to cool down.

We consider the discrete-time remote estimation system portrayed in Fig. 6.1 that tries to capture this trade-off, wherein an estimator seeks to estimate the state of a non-collocated plant that is persistently exited by process noise. Upon requests

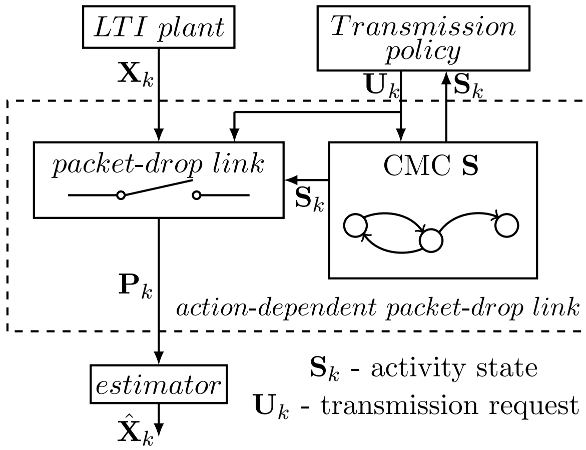

Figure 6.1: Basic system architecture.

issued by a transmission policy, a packet-drop link attempts to relay the state of the plant to the estimator. When a transmission request is made, the state of the plant[1] is either unerringly transmitted to the estimator or a packet-drop event occurs. The estimator receives a "no-transmission" symbol when either there is no request or the transmission fails due to a packet-drop.

6.1.1 Activity State: Discussion and Motivation

The probability of a packet-drop event, given a transmission request, is governed by the so-called activity state $\mathbf{S}_k$, which takes values in a finite interval of integers $\mathbb{S}$. A controlled Markov chain, denoted as CMC $\mathbf{S}$ in Fig. 6.1, models how the history of requests governs the probabilistic evolution of $\mathbf{S}_k$. The packet-drop link and the CMC $\mathbf{S}$ form what we call the activity packet-drop link.

When the communication is wireless, packet-drops typically occur if the transmission power does not suffice to overcome interference. In the context of wireless

[1] See Remark 9 for the case in which the plant's state is not available at the transmitter.

communication powered by a battery that is recharged via energy harvesting, we can use the activity state to model the following cases:

- **Case I:** Each value in $\mathbb{S}$ could represent the highest power level[2] that the battery could provide for transmission. In this case, $\mathbf{S}_k$ would determine the power available for transmission at time k, and the probability of packet-drop would decrease when $\mathbf{S}_k$ increases. The CMC $\mathbf{S}$ could model the charge dynamics of the battery, which would be continually recharged by an energy harvesting module. The model should account for the fact that frequent transmission requests tend to deplete the battery, which would be gradually recharged during periods of inactivity.

- **Case II:** In addition, we may also consider the so-called capture effect, in which somewhat frequent transmissions may temporarily silence the interference sources and consequently contribute to reducing the probability of packet-drops. In the context of the activity state of Case I, we can regard the available power as a proxy for the frequency of recent transmission requests. According to this, in the presence of the capture effect, the probability of drop may no longer be decreasing on the available power. There may be a sweet-spot in which transmissions are frequent enough to guarantee capture but not too frequent that the battery would be depleted.

[2]The typical discharge curve of a battery specifies how the power it can deliver decreases as it loses charge.

6.1.2 Objectives and Outline of Main Results

We seek to design stationary randomized policies that use the activity state to decide when to request a transmission. More specifically, we wish to propose necessary and sufficient conditions in terms of the plant and the CMC $\mathbf{S}$ for the existence of a stationary randomized policy that stabilizes the estimation error. In addition, we seek to obtain one such policy when it exists. The following results address these challenges:

- In Theorem 6.1, we show that it suffices to search the finite set of deterministic policies for a stabilizing solution. The theorem also gives necessary and sufficient conditions for the existence of a stabilizing solution in terms of an inequality that extends previous results.

- When the probability of drop is a non-decreasing function of the activity state, Theorem 6.2 states that we can further narrow the search to the set of threshold policies. As we argue in Remark 13, this incurs a significant complexity reduction when testing the necessary and sufficient condition of Theorem 6.1, and subsequently determining a stabilizing solution.

6.2 Framework and Problem Formulation

We proceed with defining the main components of our framework, and in Section 6.2.4 we discuss our problem statement.

Definition 6.1. (Plant) *The plant is an n-dimensional discrete-time linear time-invariant system. We assume it is excited by white process noise, which leads to the following recursion for the state:*

$$\mathbf{X}_{k+1} = A\mathbf{X}_k + \mathbf{V}_k, \quad k \geq 0, \tag{6.1}$$

where $\mathbf{X}_k$ is the state of the plant, the entries of $\mathbf{X}_0$ are zero and A is a matrix in $\mathbb{R}^{n \times n}$. Here, $\mathbf{V}$ is a zero-mean independent, identically distributed process with finite second-moment.

6.2.1 Activity packet-drop link

We consider that information is transmitted to the estimator across a link whose internal state is represented by $\mathbf{S}_k$ and takes values in $\mathbb{S} \stackrel{def}{=} \{1, \ldots, n_s\}$. We also refer to $\mathbf{S}_k$ as activity state to emphasize its dependence on current and past transmission requests. The following describes the dynamics of $\mathbf{S}$ as a CMC.

Definition 6.2 (CMC S). *Let $\mathbf{U}_k$ represent a channel input at time k, which is* 1 *when there is a request for transmission, and is* 0 *otherwise. The evolution of $\mathbf{S}_k$ is* *modeled by the following transition probabilities for all s and s' in $\mathbb{S}$.*

$$\mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_k,\mathbf{U}_k}(s' \mid s, 1) = \begin{cases} \alpha_{s,s+1} & \text{if } s < n_s \text{ and } s' = s+1, \\ 1 - \alpha_{s,s+1} & \text{if } s < n_s \text{ and } s' = s, \\ 1 & \text{if } s = n_s \text{ and } s' = n_s, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2a}$$

$$\mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_k,\mathbf{U}_k}(s' \mid s, 0) = \begin{cases} \alpha_{s,s-1} & \text{if } s > 1 \text{ and } s' = s-1, \\ 1 - \alpha_{s,s-1} & \text{if } s > 1 \text{ and } s' = s, \\ 1 & \text{if } s = 1 \text{ and } s' = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2b}$$

where the parameters $\alpha_{s,s'}$, which take values in $(0,1)$, model the likelihood that the activity state will transition to a greater or lesser value, depending on whether there is a transmission request or not. Transmission requests have a "tendency" to cause an increment of $\mathbf{S}_k$, while no requests have the opposite effect.

Definition 6.3 (Link Status). *The link has two possible statuses termed "effective" and "ineffective". The process $\mathbf{L}$ indicates whether the status is "effective" or "ineffective" depending on whether $\mathbf{L}_k$ is 1 or 0, respectively. It is modeled proba-*

bilistically as follows:

$$\mathcal{P}_{\mathbf{L}_k|\mathbf{S}_k}(l|s) \stackrel{def}{=} \begin{cases} d(s) & \text{if } l = 0, \\ 1 - d(s) & \text{if } l = 1. \end{cases}, \quad s \in \mathbb{S},\ k \geq 1 \tag{6.3}$$

where $d : \mathbb{S} \to [0, 1]$ *indicates how the state influences the status of the link.*

Assumption 6.1. *We assume that* $\mathbf{V}$ *is independent of the pair of processes* $(\mathbf{S}, \mathbf{L})$*.*

Although we have not yet described how $\mathbf{U}$ is generated, the following assumption imposes that, conditioned on $\mathbf{S}_k$, the three randomizations that generate $\mathbf{L}_k$, $\mathbf{U}_k$ and $\mathbf{S}_{k+1}$ are independent among themselves and across time. The class of transmission policies defined in Section 6.2.4 will be consistent with this.

Assumption 6.2. *For every* $k \geq 0$*, we assume that the following holds:*

- $\mathbf{L}_k$ *is independent of* $\left(\{\mathbf{L}_j\}_{j=1}^{k-1}, \{\mathbf{U}_j\}_{j=1}^{k}\right)$ *when conditioned on* $\mathbf{S}_k$*.*
- $\mathbf{U}_k$ *is independent of* $\left(\{\mathbf{L}_j\}_{j=1}^{k}, \{\mathbf{U}_j\}_{j=1}^{k-1}\right)$ *when conditioned on* $\mathbf{S}_k$*.*
- $\mathbf{S}_{k+1}$ *is independent of* $\left(\{\mathbf{L}_j\}_{j=1}^{k}, \{\mathbf{U}_j\}_{j=1}^{k-1}\right)$ *when conditioned on* $\mathbf{S}_k$ *and* $\mathbf{U}_k$*.*

As we clarify in the following definition, given that a transmission has been requested, d determines the probability of packet-drop in terms of $\mathbf{S}_k$.

Definition 6.4 (**Packet-drop Event**)**.** *We say that a packet-drop event occurs when* $\mathbf{U}_k = 1$ *and* $\mathbf{L}_k = 0$*. That is to say that a packet-drop occurs when the link is ineffective at the same time that a transmission request is received by the link. More specifically* $\Pr(packet - drop|\mathbf{S}_k = s, \mathbf{U}_k = 1) = d(s)$*.*

Definition 6.5 (**Activity packet-drop link).** *A given CMC* $\mathbf{S}$ *and a map* $d : \mathbb{S} \to [0,1]$ *define, what we call, an* <u>*activity packet-drop link*</u>. *The inputs to the link are* $\mathbf{X}_k$ *and* $\mathbf{U}_k$*, which represent the state of the plant and indicate if there is a transmission request, respectively. The output is denoted with* $\mathbf{P}_k$ *and takes values in* $\mathbb{P} \overset{def}{=} \mathbb{R}^n \cup \{\mathfrak{E}\}$*, where* $\mathfrak{E}$ *indicates that no transmission has occurred. We consider that* $\mathbf{S}_k$*, governed by the CMC* $\mathbf{S}$*, is the internal state of the link,* $\mathbf{L}_k$ *is its status and, together with the input, they determine the output as follows:*

$$\mathbf{P}_k \overset{def}{=} \begin{cases} \mathbf{X}_k & \text{if } \mathbf{L}_k \mathbf{U}_k = 1 \\ \mathfrak{E} & \text{otherwise} \end{cases} \tag{6.4}$$

This indicates that a successful transmission occurs only when it is requested and the link is effective.

Remark 9. (When $\mathbf{X}_k$ is not available for transmission) *To simplify our notation throughout the article, we assume that* $\mathbf{X}_k$ *is available for transmission. However, we would like to stress that our results may remain valid even when the transmitter can only access noisy measurements of the plant's output, as in [94], provided that it can construct a state estimate* $\mathbf{X}_k^E$*. More specifically, if* $\mathbf{X}_k^E - \mathbf{X}_k$ *is second-moment stable then Theorems 6.1 and 6.2 remain valid when* $\mathbf{X}_k$ *is replaced with* $\mathbf{X}_k^E$ *in (6.4).*

6.2.2 Estimator, Estimation Error and System State

We proceed to defining the estimator and the estimation error.

Definition 6.6. (Estimator) *The estimator has the following structure:*

$$\hat{\mathbf{X}}_k \overset{def}{=} \begin{cases} A\hat{\mathbf{X}}_{k-1} & \text{if } \mathbf{P}_k = \mathfrak{E} \\ \mathbf{P}_k & \text{otherwise} \end{cases}, \quad k \geq 1 \tag{6.5}$$

where the entries of $\hat{\mathbf{X}}_0$ are zero. Notice that, according to our assumptions so far, (6.5) is the recursive implementation of the following minimum expected mean-squared error estimator:

$$\hat{\mathbf{X}}_k = \mathbb{E}[\mathbf{X}_k \mid \mathbf{P}_1, \ldots, \mathbf{P}_k], \quad k \geq 1$$

It follows that the estimation error can be represented as:

$$\mathbf{E}_{k+1} \overset{def}{=} \mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1} = (1 - \mathbf{U}_{k+1}\mathbf{L}_{k+1})(A\mathbf{E}_k + \mathbf{V}_k), \quad k \geq 0 \tag{6.6}$$

6.2.3 Overall System State and CMC $\mathbf{Y}$

We use $\mathbf{Y}_k \overset{\text{def}}{=} (\mathbf{S}_k, \mathbf{E}_{k-1})$ to denote the overall state of the estimation system and $\mathbb{Y} \overset{\text{def}}{=} \mathbb{S} \times \mathbb{R}^n$ represents the associated set of possible values. The following is an outline for how the transition probability kernel for $\mathbf{Y}$ can be obtained from what we introduced so far.

We start by noticing that it is a consequence of Assumption 6.1 that the transition kernel of $\mathbf{Y}$ can be decomposed as follows:

$$\begin{aligned}&\Pr\left(\mathbf{Y}_{k+1} \in \{s\} \times \mathbb{H} \mid \mathbf{Y}_k = y', \mathbf{U}_k = u\right) \\ &= \Pr\left(\mathbf{E}_k \in \mathbb{H} \mid \mathbf{Y}_k = y', \mathbf{U}_k = u\right) \Pr\left(\mathbf{S}_{k+1} = s \mid \mathbf{S}_k = s', \mathbf{U}_k = u\right).\end{aligned} \tag{6.7}$$

for every Lebesgue measurable subset $\mathbb{H}$ of $\mathbb{R}^n$, y' in $\mathbb{Y}$ and u in $\mathbb{U}$. Notice that we use an abuse of notation according to which y' represents the pair (s', e').

The first term on the RHS of (6.7) can be expressed as:

$$\begin{aligned}&\Pr\left(\mathbf{E}_k \in \mathbb{H} \mid \mathbf{Y}_k = y', \mathbf{U}_k = u\right) \\ &= \Pr\left((1 - u\mathbf{L}_k)(Ae' + \mathbf{V}_{k-1}) \in \mathbb{H} \mid \mathbf{Y}_k = y', \mathbf{U}_k = u\right)\end{aligned}$$

Finally, Assumption 6.2 can be used to establish that:

$$\begin{aligned}&\Pr\left(\mathbf{E}_k \in \mathbb{H} \mid \mathbf{Y}_k = y', \mathbf{U}_k = u\right) \\ &= \Pr\left((1 - u\mathbf{L}_k)(Ae' + \mathbf{V}_{k-1}) \in \mathbb{H} \mid \mathbf{S}_k = s'\right)\end{aligned}$$

which can be computed using (6.3) and the probabilistic description of $\mathbf{V}$.

At this point, we have introduced all the necessary concepts needed to characterize the overall system as follows.

Definition 6.7. *(**CMC $\mathbf{Y}$**) The CMC with input $\mathbf{U}$ and state $\mathbf{Y}$, whose probabilistic model is now completely defined, is denoted with CMC $\mathbf{Y}$. (See Table 6.1 for a summary of the notation for CMC $\mathbf{Y}$.)*

$\mathbb{S}$	set of activity states $\{1, \ldots, n_s\}$
$\mathbf{S}_k$	activity state at time k (takes values in $\mathbb{S}$)
$\mathbb{R}^n$	n dimensional real vector
$\mathbf{X}_k$	LTI plant state at time k (takes values in $\mathbb{R}^n$)
$\mathbf{U}_k$	action chosen at time k (takes value 0 or 1).
$\mathbf{L}_k$	link status at time k (takes value 0 or 1)
$\mathbb{P} \overset{\text{def}}{=} \mathbb{R}^n \cup \{\mathfrak{E}\}$	possible output from link
$\mathbf{P}_k$	link output at time k (takes value in $\mathbb{P}$)
$\hat{\mathbf{X}}_k$	estimation of plant state at time k (takes values in $\mathbb{R}^n$)
$\mathbb{Y}$	state space formed by $\mathbb{S} \times \mathbb{R}^n$
$\mathbf{Y}_k \overset{\text{def}}{=} (\mathbf{S}_k, \mathbf{E}_{k-1})$	system state at time k (takes values in $\mathbb{Y}$)

Table 6.1: A summary of notation describing CMC $\mathbf{Y}$.

Remark 10. *It follows from the discussion that specifying the CMC* $\mathbf{Y}$ *requires a description of: i) the plant, which is determined by the matrix A and the probabilistic description of* $\mathbf{V}$*; and ii) the activity packed drop link, which is determined by the CMC* $\mathbf{S}$ *and the map d.*

6.2.4 Transmission Policies, Stability and Problem Statement

Henceforth, we describe the class of stationary randomized transmission policies adopted here and how they generate the request process $\mathbf{U}$. We then proceed with specifying the notion of stability we use to qualify the effect that a transmission policy has on the CMC $\mathbf{Y}$.

Definition 6.8. *A stationary randomized policy is specified by a mapping $\theta : \mathbb{S} \rightarrow [0, 1]$. It determines the probability that a transmission is attempted based only on* *the activity state, as follows:*

$$\mathcal{P}_{\mathbf{U}_k|\mathbf{Y}_k,\ldots,\mathbf{Y}_0}(1|y_k,\ldots,y_0) = \theta(s_k) \text{ and}$$

$$\mathcal{P}_{\mathbf{U}_k|\mathbf{Y}_k,\ldots,\mathbf{Y}_0}(0|y_k,\ldots,y_0) = 1-\theta(s_k),$$

where the randomization of $\mathbf{U}_k$ is such that Assumption 6.2 holds.

Definition 6.9 (**Stationary Randomized Policies**). *The set of stationary randomized policies is denoted by Θ_R.*

Definition 6.10 (**Mean-square stabilization**). *For a given CMC $\mathbf{Y}$, a policy θ in Θ_R is said to be mean-square stabilizing if the following holds:*

$$\limsup_{k\to\infty} \mathbb{E}\left[\mathbf{E}_k^T\mathbf{E}_k\right] < \infty \tag{6.8}$$

We can now state the main problem addressed in this article.

Problem 6.1. *We seek to obtain a method that ascertains, for any given CMC $\mathbf{Y}$, whether Θ_R contains a stabilizing policy. We also intend to characterize structural properties of such stabilizing policies in terms of the parameters defining the CMC $\mathbf{Y}$. Whenever possible, we would like to use these properties to streamline the search for a stabilizing policy in Θ_R.*

6.3 Second Moment Stability Results

Our preliminary results are Theorem 6.1 and Theorem 6.2, which characterize the existence and structure of stabilizing policies under varying assumptions on d.

Before we state the theorems, we proceed with defining key preliminary concepts.

Definition 6.11 (**Deterministic Policies**). *The set of deterministic policies Θ_D is defined as follows:*

$$\Theta_D \overset{def}{=} \{\theta \in \Theta_R \mid \theta(s) \in \{0,1\},\ s \in \mathbb{S}\} \tag{6.9}$$

Definition 6.12 (**Threshold Policy**). *A member of Θ_D is said to be a threshold policy if it can be expressed as:*

$$\theta_\tau(s) \overset{def}{=} \begin{cases} 0 & \text{if } s \geq \tau, \\ 1 & \text{otherwise.} \end{cases}, \quad s \in \mathbb{S} \tag{6.10}$$

where τ is a threshold taking values in $\{1, \ldots, n_s + 1\}$.

Definition 6.13 ($\mathcal{H}^{\mathbf{S}}(\theta)$). *Given a CMC $\mathbf{S}$, the map $\mathcal{H}^{\mathbf{S}} : \Theta_R \to \mathbb{R}^{n_s \times n_s}$ is defined as:*

$$\mathcal{H}^{\mathbf{S}}_{ij}(\theta) \overset{def}{=} d(i)\theta(i)\mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_k,\mathbf{U}_k}(j \mid i, 1) + (1 - \theta(i))\mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_k,\mathbf{U}_k}(j \mid i, 0),$$

for $\theta \in \Theta_R,\ i, j \in \mathbb{S}$.

In addition, we define the following quantities:

Definition 6.14 ($\lambda^{\mathbf{S}}$, $\lambda^{\mathbf{S}}_D$ and $\lambda^{\mathbf{S}}_T$)**.** *Given a CMC* $\mathbf{S}$*, we define the following contraction rates:*

$$\lambda^{\mathbf{S}} \stackrel{def}{=} \min_{\theta \in \Theta_R} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta)\big) \tag{6.11a}$$

$$\lambda^{\mathbf{S}}_D \stackrel{def}{=} \min_{\theta_D \in \Theta_D} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) \tag{6.11b}$$

$$\lambda^{\mathbf{S}}_T \stackrel{def}{=} \min_{1 \le \tau \le n_s+1} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_\tau)\big) \tag{6.11c}$$

Now, we are ready to state our main results.

Theorem 6.1. *Given a CMC* $\mathbf{Y}$*, it holds that* $\lambda^{\mathbf{S}} = \lambda^{\mathbf{S}}_D$*. In addition, the following holds:*

- *There is a stabilizing policy in* Θ_R *only if* $\rho(A)^2\lambda^{\mathbf{S}} < 1$*.*
- *If* $\rho(A)^2\lambda^{\mathbf{S}} < 1$ *then any policy* θ^*_D *in* Θ_D *for which* $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta^*_D)\big) = \lambda^{\mathbf{S}}$ *holds is stabilizing.*

Proof. A proof of Theorem 6.1 is provided in Section 6.4.1. □

Remark 11. *Notice that Theorem 6.1 not only gives a necessary and sufficient condition for stabilizability, but it also implies that we only need to consider deterministic policies.*

Remark 12. *Consider the uncontrolled case in which* $n_s = 2$ *and the CMS* $\mathbf{S}$ *is* *modeled as follows:*

$$\mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_k,\mathbf{U}_k}(s' \mid s, u) = Q_{s's}, \quad u \in \{0,1\},\ s, s' \in \mathbb{S} \tag{6.12}$$

where Q is a 2-dimensional stochastic matrix. Furthermore, we assume that $d(2) = 1$ and $d(1) = 0$ to indicate that the link is always effective when $\mathbf{S}_k = 2$ and ineffective otherwise. In this case $\lambda^S = Q_{22}$ and the necessary and sufficient condition for stabilizability of Theorem 6.1 becomes $\rho(A)^2 Q_{22} < 1$, which recovers a well-known result from [63] when applied to our context.

Unfortunately, Theorem 6.1 does not solve the problem that, when n_s is large, the cardinality of Θ_D may render the computation of $\lambda_D^{\mathbf{S}}$ intractable. However, as we state in the following theorem, this issue disappears when d is non-decreasing.

Theorem 6.2. *Consider that a CMC $\mathbf{Y}$ for which d is non-decreasing is given. It holds that $\lambda^{\mathbf{S}} = \lambda_D^{\mathbf{S}} = \lambda_T^{\mathbf{S}}$. In addition, if τ^* in $\{1, \ldots, n_s + 1\}$ is such that $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta_{\tau^*})\big) = \lambda^{\mathbf{S}}$ then θ_{τ^*} is stabilizing.*

Section 6.4.2 gives a proof sketch of Theorem 6.2.

Remark 13. *It follows from Theorem 6.2 that it suffices to consider threshold policies when d is non-decreasing. For this case, the theorem also states that $\lambda^{\mathbf{S}}$ is equal to $\lambda_T^{\mathbf{S}}$, which can be efficiently computed by minimizing a function over a set of thresholds that has $n_s + 1$ elements. It also states that any minimizing threshold yields a stabilizing threshold policy.*

6.4 Proofs of Main Results

We start our proof with a lemma that relates the spectral radius of $\mathcal{H}^{\mathbf{S}}(\theta)$ and stability.

Lemma 6.1. *Given a CMC* $\mathbf{Y}$*, the following holds:*

- *There is a stabilizing policy in* Θ_R *only if* $\rho(A)^2\lambda^{\mathbf{S}} < 1$.
- *If* $\rho(A)^2\lambda^{\mathbf{S}} < 1$ *then any policy* θ^* *in* Θ_R *for which* $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta^*)\big) = \lambda^{\mathbf{S}}$ *holds is stabilizing.*

Proof. A proof is provided in Section 6.4.3 □

By Lemma 6.1, the remaining proof for Theorem 6.1 is to establish that $\lambda^{\mathbf{S}} = \lambda^{\mathbf{S}}_D$.

6.4.1 A Proof of Theorem 6.1

First, we introduce another Lemma, which offers an insight into the spectral radius of matrices with a unique structure and is essential for the proof of the Theorem. Let $\mathcal{F}(t)$ be an n_s-by-n_s matrix with constant non-negative elements except for the kth row. The kth row of $\mathcal{F}(t)$ is a convex combination of two non-negative row vectors. More specifically, the kth row is equal to $(1-t)\mathbf{a}^T + t\mathbf{b}^T$, where $\mathbf{a}$ and $\mathbf{b}$ are non-negative n_s-by-1 real vectors.

Lemma 6.2. *The minimum of the spectral radius of* $\mathcal{F}(t)$ *for* $t \in [0,1]$ *is at one* *of the endpoints, i.e.,*

$$\min_{t\in[0,1]} \rho\big(\mathcal{F}(t)\big) = \min\{\rho\big(\mathcal{F}(0)\big), \rho\big(\mathcal{F}(1)\big)\}.$$

Proof. We first define $\mathcal{F}_\epsilon(t)$,

$$\mathcal{F}_\epsilon(t) \stackrel{\text{def}}{=} \mathcal{F}(t) + \epsilon O,$$

where ϵ is a positive real number and O is an n_s-by-n_s matrix with all entries equal to one. The purpose of introducing this matrix is to ensure that we can work on a matrix with only positive elements.

Proceeding with the proof of the Lemma, we will show that the spectral radius of $\mathcal{F}_\epsilon(t)$ is greater than or equal to the minimum of the spectral radius of $\mathcal{F}_\epsilon(0)$ and $\mathcal{F}_\epsilon(1)$ for all $t \in [0, 1]$ by contradiction.

Suppose there exist a t^* such that $\rho\big(\mathcal{F}_\epsilon(t^*)\big)$ is less than $\rho\big(\mathcal{F}_\epsilon(0)\big)$ and $\rho\big(\mathcal{F}_\epsilon(1)\big)$. We use $\mathbf{v}$ to denote one of the eigenvectors of $\mathcal{F}_\epsilon(t^*)$ corresponding to eigenvalue $\rho\big(\mathcal{F}_\epsilon(t^*)\big)$. Since $\mathcal{F}_\epsilon(t^*)$ has only positive elements, all elements of $\mathbf{v}$ are also positive. From Theorem 8.1.26 in [95], we have, for any t in the closed interval of zero and one,

$$\rho\big(\mathcal{F}_\epsilon(t)\big) \le \max_{i\in\mathbb{S}} \frac{\sum_{j=1}^{n_s} \mathcal{F}_\epsilon(t)_{ij} v_j}{v_i}, \tag{6.13}$$

where v_i is the ith element of $\mathbf{v}$, and the equality holds when t is equal to t^*. For notational convenience, we use $\beta_i(t)$ to denote $\frac{1}{v_i}\sum_{j=1}^{n_s}\mathcal{F}_\epsilon(t)_{ij}v_j$ for the rest of the proof.

Notice that only row k of the matrix $\mathcal{F}_\epsilon(t)$ is affected by parameter t. Thus, for any $i \neq k$ and any $t \in [0,1]$, row i of $\mathcal{F}_\epsilon(t)$ is the same as row i of $\mathcal{F}_\epsilon(t^*)$. Therefore,

$$\beta_i(t) = \frac{\sum_{j=1}^{n_s}\mathcal{F}_\epsilon(t)_{ij}v_j}{v_i} = \frac{\sum_{j=1}^{n_s}\mathcal{F}_\epsilon(t^*)_{ij}v_j}{v_i} = \rho\big(\mathcal{F}_\epsilon(t^*)\big),$$

for all $i \neq k$ and $t \in [0,1]$. For i equal to k,

$$\beta_k(t) = \frac{\sum_{j=1}^{n_s}\mathcal{F}_\epsilon(t)_{kj}v_j}{v_k} = \frac{\big((1-t)\mathbf{a}^T + t\mathbf{b}^T + \epsilon\mathbf{1}^T\big)\mathbf{v}}{v_k},$$

Notice that $\beta_k(t)$ is a linear function of t. Thus, for any t in $[0,1]$,

$$\beta_k(t) \geq \min\{\beta_k(0), \beta_k(1)\}.$$

With these observations, let us examine (6.13) for $t = 0$ and $t = 1$ to lower bound $\beta_k(0)$ and $\beta_k(1)$.

Case 1. $t = 0$: From (6.13), we know that the maximum of $\beta_i(0)$ over i is greater than or equal to $\rho\big(\mathcal{F}_\epsilon(0)\big)$. However, for all $i \neq k$, $\beta_i(0)$ is equal to $\rho\big(\mathcal{F}_\epsilon(t^*)\big)$ which is strictly less than $\rho\big(\mathcal{F}_\epsilon(0)\big)$ by our assumption. The maximum must be achieved by k and we have

$$\beta_k(0) \geq \rho\bigl(\mathcal{F}_\epsilon(0)\bigr).$$

<u>*Case 2. $t = 1$:*</u> By a similar argument, we have

$$\beta_k(1) \geq \rho(\mathcal{F}_\epsilon(1)).$$

Thus, $\beta_k(t)$ is greater than or equal to the minimum of $\rho\bigl(\mathcal{F}_\epsilon(0)\bigr)$ and $\rho\bigl(\mathcal{F}_\epsilon(1)\bigr)$ for all $t \in [0, 1]$. However, this contradicts the fact that $\beta_k(t^*)$ equals $\rho\bigl(\mathcal{F}_\epsilon(t^*)\bigr)$ which is less than $\rho\bigl(\mathcal{F}_\epsilon(0)\bigr)$ and $\rho\bigl(\mathcal{F}_\epsilon(1)\bigr)$ by our assumption. Therefore,

$$\rho\bigl(\mathcal{F}_\epsilon(t)\bigr) \geq \min\{\rho\bigl(\mathcal{F}_\epsilon(0)\bigr), \rho\bigl(\mathcal{F}_\epsilon(1)\bigr)\} \tag{6.14}$$

for any $t \in [0, 1]$ and $\epsilon > 0$.

Next, we will show that the inequality holds even with ϵ equal to zero by contradiction.

Suppose that there exists a t' such that $\rho\bigl(\mathcal{F}(t')\bigr) < \min\{\rho\bigl(\mathcal{F}(0)\bigr), \rho\bigl(\mathcal{F}(1)\bigr)\}$. Define δ to be the difference between $\min\{\rho\bigl(\mathcal{F}(0)\bigr), \rho\bigl(\mathcal{F}(1)\bigr)\}$ and $\rho\bigl(\mathcal{F}(t')\bigr)$. Since the spectral radius is a continuous function of each element of the matrix, there exists an $\epsilon' > 0$ such that $\rho\bigl(\mathcal{F}_{\epsilon'}(t')\bigr) - \rho\bigl(\mathcal{F}(t')\bigr) < \delta$. (Note that $\rho\bigl(\mathcal{F}_{\epsilon'}(t')\bigr) \geq \rho\bigl(\mathcal{F}(t')\bigr)$ by

Theorem 8.1.18 in [95]). Hence,

$$
\begin{aligned}
&\rho\big(\mathcal{F}_{\epsilon'}(t')\big)\\
&< \rho\big(\mathcal{F}(t')\big) + \delta\\
&= \rho\big(\mathcal{F}(t')\big) + \min\{\rho\big(\mathcal{F}(0)\big), \rho\big(\mathcal{F}(1)\big)\} - \rho\big(\mathcal{F}(t')\big)\\
&= \min\{\rho\big(\mathcal{F}(0)\big), \rho\big(\mathcal{F}(1)\big)\}\\
&\le \min\{\rho\big(\mathcal{F}_{\epsilon'}(0)\big), \rho\big(\mathcal{F}_{\epsilon'}(1)\big)\},
\end{aligned}
$$

where we use Theorem 8.1.18 in [95] again in the last inequality. This contradicts (6.14). Thus, $\rho\big(\mathcal{F}(t^*)\big) \ge \min\{\rho\big(\mathcal{F}(0)\big), \rho\big(\mathcal{F}(1)\big)\}$ for all $t \in [0,1]$. □

With this lemma, we can now prove that $\lambda^{\mathbf{S}} = \lambda^{\mathbf{S}}_D$. Let $\theta \in \Theta_R$ be a stationary randomized policy. Define a set $\mathcal{S}_\theta \overset{\text{def}}{=} \{s \in \mathbb{S} : \theta(s) \notin \{0,1\}\}$ to be the set of the activity states where the policy chooses randomly. We denote all states in $\mathcal{S}_\theta$ by $\{s_1, s_2, \ldots, s_m\}$. Then, we investigate a set of policies that are parameterized by $t \in [0,1]$:

$$
\theta^t(s) = \begin{cases} \theta(s) & \text{if } s \ne s_1, \\ t & \text{if } s = s_1. \end{cases}
$$

An important observation is that the matrix $\mathcal{H}^{\mathbf{S}}(\theta^t)$ satisfies the matrix format in Lemma 6.2. Hence, either $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta^0)\big)$ or $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta^1)\big)$ is less than or equal to $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta)\big)$. We find a policy θ^{new} (θ^0 or θ^1) such that it chooses the action deterministically at the state s_1 and results in a lower or identical spectral radius. We take θ^{new} as θ and apply this procedure with $\{s_2, \ldots, s_m\}$. Notice that the number of the activity states at which the policy chooses randomly decreases by one after each usage of Lemma 6.2. Since the set of the activity states $\mathbb{S}$ is finite, the procedure will eventually terminate, and we have a deterministic policy θ_D with $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) \leq \rho\big(\mathcal{H}^{\mathbf{S}}(\theta)\big)$.

We have now shown that, for any randomized policy, we can find a deterministic policy which results in a lower or identical spectral radius. Thus,

$$\min_{\theta \in \Theta_R} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta)\big) = \min_{\theta_D \in \Theta_D} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big).$$

Combining with Lemma 6.1, the proof of Theorem 6.1 is complete.

6.4.2 A Proof of Theorem 6.2

In this subsection, we will show that, if d is non-increasing, $\lambda_D^{\mathbf{S}}$ is equal to $\lambda_T^{\mathbf{S}}$, which is sufficient for proving Theorem 6.2.

Proof. Suppose that $\theta_D \in \Theta_D$ is a deterministic policy. We define a mapping $\mathcal{T} : \Theta_D \to \mathbb{S} \cup \{n_s + 1\}$, where

$$\mathcal{T}(\theta) \stackrel{\text{def}}{=} \min\{\overline{s} \in \mathbb{S} \mid \theta(\overline{s}) = 1\},\ \theta \in \Theta_D.$$

We assume that $\mathcal{T}(\theta) = n_s + 1$ if the set on the RHS is empty. First, the definition of the mapping $\mathcal{T}$ tells us that θ_D is a threshold policy if $\mathcal{T}(\theta_D) = n_s+1$ or $\mathcal{T}(\theta_D) = n_s$. Thus, without loss of generality, we assume that $\mathcal{T}(\theta_D)$ does not equal to $n_s + 1$ or n_s. We use s^* to denote $\mathcal{T}(\theta_D)$ in the rest of the proof for notational convenience.

Next, we will show that $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) \geq \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_{s^*})\big)$ where θ_{s^*} is a threshold policy with the threshold value s^*.

The matrix $\mathcal{H}^{\mathbf{S}}(\theta_D)$ has following structure,

$$\mathcal{H}^{\mathbf{S}}(\theta_D) = \begin{bmatrix} J & \mathbf{0} \\ L & M \end{bmatrix},$$

where M is an $(n_s - s^*)$-by-$(n_s - s^*)$ square matrix, J is an s^*-by-s^* square matrix, L is an $(n_s - s^*)$-by-s^* matrix, and $\mathbf{0}$ is an s^*-by-$(n_s - s^*)$ matrix with all elements equal to zero. From the lower triangular like structure, we know that the spectral radius of $\mathcal{H}^{\mathbf{S}}(\theta_D)$ is lower bounded by the spectral radius of the top left matrices. Therefore,

$$\rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) \geq \rho(J). \tag{6.15}$$

Proceeding with the proof of the Theorem, we consider the matrix $\mathcal{H}^{\mathbf{S}}(\theta_{s^*})$ with the threshold policy θ_{s^*},

$$\mathcal{H}^{\mathbf{S}}(\theta_{s^*}) = \begin{bmatrix} J & \mathbf{0} \\ L' & M' \end{bmatrix},$$

where M' is an $(n_s - s^*)$-by-$(n_s - s^*)$ diagonal matrix with the diagonal elements

$d(i)(1-\alpha_{i,i-1})$ for i from (s^*+1) to n_s, L' is an $(n_s - s^*)$-by-s^* matrix, and other parts of matrix identical to $\mathcal{H}^{\mathbf{S}}(\theta_D)$. Next, We take a deeper look at matrix J,

$$J = \begin{bmatrix} \ulcorner & & \urcorner & 0 \\ & K & & \vdots \\ \llcorner & & \lrcorner & \alpha_{s^*-1,s^*} \\ 0 & \dots & d(s^*)\alpha_{s*,s^*-1} & d(s^*)(1-\alpha_{s*,s^*-1}) \end{bmatrix}.$$

Theorem 8.1.22 in [95] tells us that the spectral radius of a matrix is lower bounded by the smallest row sum. The row sum of each row in J is equal to one except for the s^*th row, and the row sum of the s^*th row is equal to $d(s^*)$ which is less than or equal to 1. Thus,

$$\rho\left(J\right) \geq d(s^*).$$

The spectral radius of the matrix $\mathcal{H}^{\mathbf{S}}(\theta_{s^*})$ equals the maximum of the diagonal elements of M' and $\rho(J)$. Since $\rho(J)$ is greater than or equal to $d(s^*)$ which is greater than or equal to each of the diagonal elements of M' by the assumption that d is non-increasing, the spectral radius of the matrix $\mathcal{H}^{\mathbf{S}}(\theta_{s^*})$ is equal to $\rho(J)$. Finally, by (6.15), we have

$$\rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) \geq \rho\left(J\right) = \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_{s^*})\big),$$

which completes the proof, since, for any deterministic policy, we can find a threshold policy that results in a spectral radius which does not exceed the original one. Thus,

$$\min_{\theta_D \in \Theta_D} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_D)\big) = \min_{1 \le \tau \le n_s+1} \rho\big(\mathcal{H}^{\mathbf{S}}(\theta_\tau)\big).$$

□

6.4.3 A Proof of Lemma 6.1

Under a fixed policy θ, we will prove the Lemma by showing that evolution of the second moment of the estimation error follow a Markov Jump Linear System [96]. This system is stable if and only if $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta)\big)\rho(A)^2 < 1$.

Proof. We start our proof by defining $\mathbf{D}_k^i$ which is a decomposition of the second moment of estimation error at time k condition on the activity state at time $k+1$.

$$\mathbf{D}_k^i \overset{\text{def}}{=} \mathbb{E}[\mathbf{E}_k \mathbf{E}_k^T | \mathbf{S}_{k+1} = i] \Pr(\mathbf{S}_{k+1} = i).$$

Notice that the error can be represented by sum of $\mathbf{D}_k^i$ now.

$$\mathbb{E}[\mathbf{E}_k \mathbf{E}_k^T] = \sum_{i=1}^{n_s} \mathbf{D}_k^i.$$

Next, we try to figure out the evolution equation for $\mathbf{D}_k^i$. We decompose $\mathbf{D}_{k+1}^i$ by conditioning on $\mathbf{S}_{k+1}$,

$$\begin{aligned}\mathbf{D}_{k+1}^{i} &= \sum_{j=1}^{n_s} \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+2} = i, \mathbf{S}_{k+1} = j] \mathcal{P}_{\mathbf{S}_{k+1}|\mathbf{S}_{k+2}}(j|i) \mathcal{P}_{\mathbf{S}_{k+2}}(i) \\ &= \sum_{j=1}^{n_s} \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+2} = i, \mathbf{S}_{k+1} = j] \mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1}}(i|j) \mathcal{P}_{\mathbf{S}_{k+1}}(j), \end{aligned} \tag{6.16}$$

where we use the Bayes Rule in the second equality. The expected value term can be rewritten as,

$$\begin{aligned}&\mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+2} = i, \mathbf{S}_{k+1} = j] \\ &= \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+2} = i, \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1} = 0] \mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(0|i,j) \\ &\quad + \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+2} = i, \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1} = 1] \mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(1|i,j) \\ &= \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1} = 0] \mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(0|i,j) \\ &\quad + \mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1} = 1] \mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(1|i,j), \end{aligned} \tag{6.17}$$

where we condition on $\mathbf{U}_{k+1}$ in the first equality, and use the fact that $\mathbf{S}_{k+2}$ is independent of $\mathbf{E}_{k+1}$ given $\mathbf{S}_{k+1}$ and $\mathbf{U}_{k+1}$ in the second equality (Assumption 6.2). Then, we use the fact that $\mathbf{E}_{k+1} = (1 - \mathbf{U}_{k+1}\mathbf{L}_{k+1})(A\mathbf{E}_k + \mathbf{W}_k)$ from (6.6). The expected value term in (6.17) can be written as,

$$\begin{aligned}&\mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T | \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1}] \\ &= \mathbb{E}[(1 - \mathbf{U}_{k+1}\mathbf{L}_{k+1})^2 (A\mathbf{E}_k + \mathbf{W}_k)(A\mathbf{E}_k + \mathbf{W}_k)^T \mid \mathbf{S}_{k+1} = j, \mathbf{U}_{k+1}] \\ &= \mathbb{E}[(1 - \mathbf{U}_{k+1}\mathbf{L}_{k+1})^2 | \mathbf{S}_{k+1} = j](A\,\mathbb{E}[\mathbf{E}_k\mathbf{E}_k^T | \mathbf{S}_{k+1} = j] A^T + R_w),\end{aligned}$$

where $R_w \stackrel{\text{def}}{=} \mathbb{E}[\mathbf{W}_k\mathbf{W}_k^T]$. We use $\mathbb{E}[\mathbf{E}_k\mathbf{W}_k^T] = \mathbb{E}[\mathbf{E}_k]\,\mathbb{E}[\mathbf{W}_k^T] = 0$, our assumption that the link status $\mathbf{L}_{k+1}$ is independent of $(\mathbf{U}_{k+1}, \mathbf{E}_k, \mathbf{W}_k)$ given $\mathbf{S}_{k+1}$, and $\mathbf{U}_{k+1}$ is independent of $(\mathbf{E}_k, \mathbf{W}_k)$ given $\mathbf{S}_{k+1}$ (Assumption 6.2) in the second equality. We apply this to (6.17) and recognize that $\mathbb{E}[(1-\mathbf{L}_{k+1})^2|\mathbf{S}_{k+1}=j] = d(j)$ to get,

$$
\begin{aligned}
&\mathbb{E}[\mathbf{E}_{k+1}\mathbf{E}_{k+1}^T|\mathbf{S}_{k+2}=i,\mathbf{S}_{k+1}=j] \\
&= \left(\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(0|i,j) + d(j)\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(1|i,j)\right)(A\,\mathbb{E}[\mathbf{E}_k\mathbf{E}_k^T|\mathbf{S}_{k+1}=j]A^T + R_w).
\end{aligned}
$$

Combining this with (6.16) tells us that,

$$
\begin{aligned}
\mathbf{D}_{k+1}^i &= \sum_{j=1}^{n_s} \mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1}}(i|j)\mathcal{P}_{\mathbf{S}_{k+1}}(j)\left(\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(0|i,j) + d(j)\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(1|i,j)\right) \\
&\qquad \times(A\,\mathbb{E}[\mathbf{E}_k\mathbf{E}_k^T|\mathbf{S}_{k+1}=j]A^T + R_w) \\
&= \sum_{j=1}^{n_s} \Big(\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(0|i,j)\mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1}}(i|j) + d(j)\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+2},\mathbf{S}_{k+1}}(1|i,j)\mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1}}(i|j)\Big) \\
&\qquad \times(A\,\mathbb{E}[\mathbf{E}_k\mathbf{E}_k^T|\mathbf{S}_{k+1}=j]\mathcal{P}_{\mathbf{S}_{k+1}}(j)A^T + \mathcal{P}_{\mathbf{S}_{k+1}}(j)R_w) \\
&= \sum_{j=1}^{n_s} \Big(\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+1}}(0|j)\mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1},\mathbf{U}_{k+1}}(i|j,0) + d(j)\mathcal{P}_{\mathbf{U}_{k+1}|\mathbf{S}_{k+1}}(1|j)\mathcal{P}_{\mathbf{S}_{k+2}|\mathbf{S}_{k+1},\mathbf{U}_{k+1}}(i|j,1)\Big) \\
&\qquad \times(A\,\mathbb{E}[\mathbf{E}_k\mathbf{E}_k^T|\mathbf{S}_{k+1}=j]\mathcal{P}_{\mathbf{S}_{k+1}}(j)A^T + \mathcal{P}_{\mathbf{S}_{k+1}}(j)R_w) \\
&= \sum_{j=1}^{n_s} \left(\mathcal{H}^{\mathbf{S}}(\theta)\right)_{j,i}(A\mathbf{D}_k^jA^T + \mathcal{P}_{\mathbf{S}_{k+1}}(j)R_w),
\end{aligned}
$$

where we use Bayes rule in the third equality and the definition of matrix $\mathcal{H}^{\mathbf{S}}(\theta)$ and

$\mathbf{D}$ in the last equality. At this point, we successfully acquire the evolution equation for $\mathbf{D}$. However, it is unclear how to directly analyze the stability for $\mathbf{D}$ because it is a matrix. Fortunately, the matrix can be vectorized and [97, Theorem 13.26] $vec(ABC) = (C^T \otimes A)vec(B)$ tells us that,

$$vec(\mathbf{D}_{k+1}^i) = \sum_{j=1}^{n_s} \left(\mathcal{H}^{\mathbf{S}}(\theta)\right)_{j,i} (A \otimes A) vec(\mathbf{D}_k^j) + \sum_{j=1}^{n_s} \left(\mathcal{H}^{\mathbf{S}}(\theta)\right)_{j,i} \mathcal{P}_{\mathbf{S}_{k+1}}(j) vec(R_w). \tag{6.18}$$

Stack the vector $vec(\mathbf{D}_{k+1}^i)$ from $i = 1$ to n_s, and we obtain that the dynamic system recursion is governed by the matrix $(\mathcal{H}^{\mathbf{S}}(\theta)^T \otimes I_n)(I_{n_s} \otimes (A \otimes A))$. Thus, if the spectral radius of this matrix is less than one, $\limsup_{k\to\infty} \mathbb{E}\left[\mathbf{E}_k^T \mathbf{E}_k\right]$ is bounded.

Utilizing the mixed-product property of the Kronecker product, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and the fact that $\rho(A \otimes B) = \rho(A)\rho(B)$, we conclude that,

$$\begin{aligned}
&\rho\left((\mathcal{H}^{\mathbf{S}}(\theta)^T \otimes I_n)(I_{n_s} \otimes (A \otimes A))\right) \\
&= \rho\left((\mathcal{H}^{\mathbf{S}}(\theta)^T I_{n_s}) \otimes (I_n(A \otimes A))\right) \\
&= \rho\left(\mathcal{H}^{\mathbf{S}}(\theta)^T \otimes (A \otimes A)\right) \\
&= \rho\left(\mathcal{H}^{\mathbf{S}}(\theta)^T\right) \rho\left(A \otimes A\right) \\
&= \rho\left(\mathcal{H}^{\mathbf{S}}(\theta)\right) \rho\left(A\right)^2.
\end{aligned}$$

Hence, θ is a stabilizing policy if and only if $\rho\left(\mathcal{H}^{\mathbf{S}}(\theta)\right) \rho\left(A\right)^2 < 1$. By the fact that we can find a θ^* such that $\rho\big(\mathcal{H}^{\mathbf{S}}(\theta^*)\big) = \lambda^{\mathbf{S}}$, the proof of Lemma 6.1 is complete. □

6.5 Summary

We proposed a formulation comprising an LTI plant persistently excited by white process noise, a remote estimator and a packet-drop link that attempts to relay the state of the plant to the estimator only when it receives a request. Unlike most existing work, the link has an (activity) state that is influenced by the history of current and past requests. A controlled Markov chain models this dependence, and a given function determines the packet-drop probability in terms of the state of the link. We allow for randomized stationary transmission policies that use the state of the link to determine when to issue a request. Our goal is to determine the existence of a policy that stabilizes the estimation error in the second-moment sense. By exploring quasi-concavity properties of the spectral radius, with respect to a certain parametrized family of matrices, we show in Theorem 6.1 that it suffices to search for a deterministic stabilizing policy. The theorem also introduces a necessary and sufficient condition for the existence of a stabilizing policy. As we indicate in Remark 12, if the link is uncontrolled then the aforementioned condition recovers well-known existing results. When the packet-drop probability is non-increasing on the link's state, Theorem 6.2 states that the search can be further reduced to threshold policies. We also describe how these results impact the complexity of evaluating the proposed necessary and sufficient condition for stability.

Chapter 7: Conclusion and Future Directions

We explored the issue of developing a task scheduling policy when a queuing server's efficiency can be dependent on the previous usage, which is modeled using the server's internal state. A new controlled Markov chain framework was proposed to study the system's queue length stability.

We used the new framework to define the collection of task arrival rates for which a stabilizing stationary scheduling strategy exists. For a single type server, we established that there exists an upper-bound for the stabilizable arrival rate. This upper-bound can be computed efficiently by searching through finite threshold values. Moreover, we defined an optimal threshold policy that stabilizes the process whenever the arrival frequency of the task is within the specified range for which a stabilizing policy exists. The computation of the optimal policy does not depend on the information of the arrival rate. For a multiple type server, we also identified the set of all stabilizable arrival rates. While the coupling between activity state and queue length prevents the well-known max-weight policy to stabilize the system, we proposed a policy with a simple structure that keeps the process stable. The parameters of the policy can be found by searching the optimal threshold values for each type.

Besides the problem of stability, for the single type case, we have found a tractable way (i.e., linear programming) to determine the minimum of all utilization rates that can be reached through a stabilizing scheduling strategy. Such a fundamental limit is vital in determining the effectiveness of any policy's utilization rate. In addition, we could use this linear programming to build policies with arbitrarily close utilization rates to the fundamental limit. While the proof of the single type infimum utilization rate problem established a reduction from optimization on infinite state space CMC $\mathbf{X}$ to finite state space CMC $\mathbf{Y}$, it remains an open question whether such reduction exists for the multiple types or not.

In addition to stability and utilization rate problem, the expected queue length and delay are often considered for researchers. Some empirical results on such properties were obtained by numerical simulation. For example, the expected queue length and delay grow when we designed a policy with a lower utilization rate by the method in Chapter 4. Future directions for this queuing system models include studying theoretical structure or bounds on the queue behavior of policies.

We have also proposed a remote estimation framework consisting of an LTI plant, a remote estimator, and packet drop-link that attempts, when a request is received, to relay the state of the plant to the estimator. By exploring quasi-concavity properties of the spectral radius, with respect to a certain parametrized family of matrices, we showed that it suffices to search for a deterministic stabilizing policy, and the search can be further reduced to threshold policies under non-decreasing drop-rate condition. Future directions include investigating the strategies that minimize the variance of the LTI plant state of the remote control system with an activity

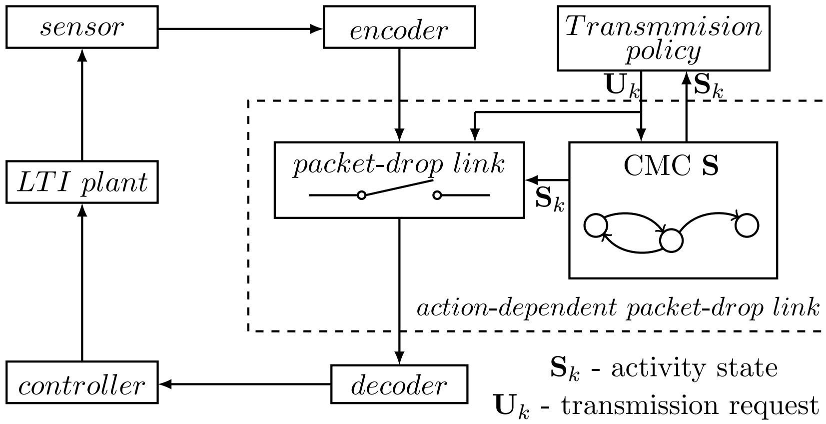

Figure 7.1: History-Dependent Estimator in A Controlled Loop

packet drop link in Fig. 7.1. Gupta et al. [63] established that, if the server state evolved according to a Markov chain and the transmitter always request to transmit, the optimal control policies and encoding scheme could be designed separately. For our model, where the server sate evolved according to a controlled Markov chain, it remains an open problem to identify the transmission policies, control policies, and encoding scheme combinations such that the variance of the LTI plant state is minimized.

Appendix A: Appendix

A.1 A Proof of Theorem 4.4: Structure of Optimal Utilization Rate

Define $\Phi^{\dagger}$ to be set of policies in Φ_R which are deterministic except for at most at one state where the policy randomizes between two admissible actions. In other words,

$$\Phi^{\dagger} \overset{\text{def}}{=} \Big\{ \phi \in \Phi_R \;\Big|\; \phi(\overline{s}, \overline{w}) \in \{0,1\} \;\text{ for all } \overline{s} \in \mathbb{S}_{\phi}^{D} \subseteq \mathbb{S} \\ \text{such that (a) } |\mathbb{S} \setminus \mathbb{S}_{\phi}^{D}| \leq 1 \text{ and (b) at a state in} \\ \mathbb{S} \setminus \mathbb{S}_{\phi}^{D},\; \phi \text{ randomizes between two actions} \Big\}$$

Lemma A.1. *Given $\Phi_{\mathbb{L}}^{0}(\overline{\nu})$ with $\overline{\nu} \in (0, \overline{\nu}^{*}]$, there exists a $\phi \in \Phi_{\mathbb{L}}^{0}(\overline{\nu})$ such that ϕ is also an element in $\Phi^{\dagger}$.*

Proof. With $\epsilon = 0$, we can drop the inequality constraint (4.25b) in LP (4.25a). It is sufficient to consider the optimization problem over the occupation measure $\ell \in \mathbb{L}$ that are generated by the policies with at most one randomization as shown in [89, Theorem 4]. □

For each ϕ in Φ_R, let $\overline{\mathbf{\Pi}}(\phi)$ be the set of stationary PMFs of $\overline{\mathbf{Y}}^{\phi}$. With

Lemma A.1, we can rewrite $\bar{\mathscr{U}}_{\mathbb{L}}^{0}$ as

$$\bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu}) = \min_{\boldsymbol{\pi} \in \overline{\boldsymbol{\Pi}}(\phi), \phi \in \Phi^{\dagger}} \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \overline{\pi}(\overline{\mathbf{y}})\phi(\overline{\mathbf{y}}) \tag{A.1a}$$

(A.1b)

$$\sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s})\overline{\pi}(\overline{\mathbf{y}})\phi(\overline{\mathbf{y}}) = \overline{\nu} \tag{A.1b}$$

We shall further divide $\Phi^{\dagger}$ into three subset and consider the linear programming problem (A.1a) on each of the subset. Before we proceed with the proof, we restate definition of threshold policies.

We define a threshold policy ϕ_{τ} as

$$\phi_{\tau}(s, w) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } s \geq \tau \text{ and } w = \mathcal{A}, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma A.2. *For every $\phi \in \Phi^{\dagger}$ with $\phi(1, \mathcal{A}) = 1$, there exists $\tau_1, \tau_2 \in \mathbb{S} \cup \{n_s + 1\}$ and $\alpha \in [0, 1]$ such that*

$$\overline{\nu}^{\phi} = (1 - \alpha)\overline{\nu}^{\phi_{\tau_1}} + \alpha\overline{\nu}^{\phi_{\tau_2}},$$

$$\bar{\mathscr{U}}(\phi) = (1 - \alpha)\bar{\mathscr{U}}(\phi_{\tau_1}) + \alpha\bar{\mathscr{U}}(\phi_{\tau_2}).$$

Proof. We define the mapping $\mathcal{T} : \Phi_R \to \mathbb{S} \cup \{0\}$, where

$$\mathcal{T}(\phi) \stackrel{\text{def}}{=} \max\{\overline{s} \in \mathbb{S} \mid \phi(\overline{s}, \mathcal{A}) = 1\}, \ \phi \in \Phi_R.$$

We assume that $\mathcal{T}(\phi) = 0$ if the set on the RHS is empty. We first observe that $\mathcal{T}(\phi) \geq 1$ in this case since $\phi(1, \mathcal{A}) = 1$ and the only positive recurrent communicating class is $\{\overline{\mathbf{y}} \in \mathbb{Y} : \overline{s} \geq \mathcal{T}(\phi)\}$. It is clear the following ϕ' has the same long-term service rate and utilization rate with ϕ,

$$\phi'(\overline{\mathbf{y}}) = \begin{cases} \phi(\overline{\mathbf{y}}) & \text{if } \overline{s} \geq \mathcal{T}(\phi) \\ 1 & \text{otherwise,} \end{cases}$$

because both policies have the same positive recurrent communicating class and the policies inside the class are identical. Furthermore, since there exists only one state $\overline{s}'$ where ϕ choose randomly between two actions, ϕ' can be express in the following form,

$$\phi'(\overline{\mathbf{y}}) = \begin{cases} \gamma & \text{if } \overline{w} = \mathcal{A} \text{ and } \overline{s} = s' \\ \phi_{\mathcal{T}(\phi)+1}(\overline{\mathbf{y}}) & \text{otherwise,} \end{cases} \tag{A.2}$$

where $\phi_{\mathcal{T}(\phi)+1}$ is the threshold policy with threshold $\mathcal{T}(\phi) + 1$ and we assume $s' > \mathcal{T}(\phi)$. If $s' < \mathcal{T}(\phi)$, ϕ' is just a threshold policy. Suppose that $\tau_1 = \mathcal{T}(\phi) + 1$ and $\tau_2 = s' + 1$.

We rewrite γ in (A.2) as

$$\gamma = \frac{\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A})}{\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A}) + (1 - \alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A})} \tag{A.3}$$

for some $\alpha \in [0, 1]$. Note that, for every $\gamma \in [0, 1]$, we can find an appropriate $\alpha \in [0, 1]$ that satisfies (A.3) because $\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A}) > 0$ and $\overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A}) > 0$

from the assumption $\tau_1 \leq \tau_2$.

By solving the global balance equations for $\overline{\mathbf{Y}}$ under the policy ϕ', we get the following stationary PMF. Its derivation is provided in Appendix A.1.1: for every $\overline{\mathbf{y}}$ in $\mathbb{Y}$,

$$\overline{\pi}^{\phi'}(\overline{\mathbf{y}}) = (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}) + \alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}) \tag{A.4}$$

The long-term service rate can be obtained using the stationary PMF.

$$\overline{\nu}^{\phi'} = \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s})\ \overline{\pi}^{\phi'}(\overline{\mathbf{y}})\ \phi'(\overline{\mathbf{y}})$$

Substituting the RHS of (A.4) for $\overline{\pi}^{\phi'}(\overline{\mathbf{y}})$, we obtain

$$\begin{aligned}
\overline{\nu}^{\phi'} &= \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \Big(\mu(\overline{s}) \big(\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}) \\
&\qquad + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}) \big) \phi'(\overline{\mathbf{y}}) \Big) \\
&= \mu(\tau_2 - 1) \big(\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A}) \\
&\qquad + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_1 - 1, \mathcal{A}) \big) \phi'(\tau_2 - 1, \mathcal{A}) \\
&\quad + \sum_{\overline{\mathbf{y}} \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}} \Big(\mu(\overline{s}) \big(\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}) \\
&\qquad + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}) \big) \phi' \Big).
\end{aligned} \tag{A.5}$$

Using the definition of ϕ',

$$
\begin{aligned}
(\text{A.5}) = \mu(\tau_2 - 1)\big(&\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A}) \\
&+(1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A})\big) \\
&\times\big((1-\gamma)\phi_{\tau_1}(\tau_2 - 1, \mathcal{A}) \\
&+\gamma\phi_{\tau_2}(\tau_2 - 1, \mathcal{A})\big) \qquad \text{(A.6)} \\
+\sum_{\overline{\mathbf{y}} \in \mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \Big(&\mu(\overline{s})\big(\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}) \\
&+(1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}})\big)\phi_{\tau_1}(\overline{\mathbf{y}})\Big). \qquad \text{(A.7)}
\end{aligned}
$$

First, using the expression in (A.3) for γ in the first term, we get

$$
\begin{aligned}
(\text{A.6}) = \mu(\tau_2 - 1)\Big(&(1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A})\phi_{\tau_1}(\tau_2 - 1, \mathcal{A}) \\
&+\alpha\overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A})\phi_{\tau_2}(\tau_2 - 1, \mathcal{A})\Big).
\end{aligned}
$$

Second, we show $\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}})\phi_{\tau_1}(\overline{\mathbf{y}}) = \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}})\phi_{\tau_2}(\overline{\mathbf{y}})$ for all $\overline{\mathbf{y}} \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}$ by considering the following three cases.

- If $s \geq \tau_2$ and $w = \mathcal{A}$, we have $\phi_{\tau_1}(\overline{s}, \overline{w}) = \phi_{\tau_2}(\overline{s}, \overline{w}) = 0$ from the definition of ϕ_{τ_1} and ϕ_{τ_2}.
- If $s < \tau_2 - 1$, then $\overline{\pi}^{\phi_{\tau_2}}(\overline{s}, \overline{w}) = 0$.
- If $w = \mathcal{B}$, then $\phi_{\tau_1}(\overline{s}, \overline{w}) = \phi_{\tau_2}(\overline{s}, \overline{w}) = 1$.

As a result,

$$(A.7) = \sum_{\overline{\mathbf{y}} \in \mathbb{Y} \setminus \{(\tau_2 - 1, \mathcal{A})\}} \mu(\overline{s}) \Big((1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}})\phi_{\tau_1}(\overline{\mathbf{y}}) + \alpha\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}})\phi_{\tau_2}(\overline{\mathbf{y}}) \Big).$$

Summing (A.6) and (A.7), we get

$$\begin{aligned} \overline{\nu}^{\phi'} &= \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}) \Big((1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}})\phi_{\tau_1}(\overline{\mathbf{y}}) + \alpha\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}})\phi_{\tau_2}(\overline{\mathbf{y}}) \Big) \\ &= (1-\alpha)\overline{\nu}^{\phi_{\tau_1}} + \alpha\overline{\nu}^{\phi_{\tau_2}}. \end{aligned} \tag{A.8}$$

Following similar steps, we can show

$$\bar{\mathscr{U}}(\phi) = (1-\alpha)\bar{\mathscr{U}}(\phi_{\tau_1}) + \alpha\bar{\mathscr{U}}(\phi_{\tau_2}). \tag{A.9}$$

□

Lemma A.3. *For every $\phi \in \Phi^{\dagger}$ with $\phi(1, \mathcal{A}) \in (0, 1)$, there exists $\tau_2 \in \mathbb{S} \cup \{n_s + 1\}$ and $\beta \in [0, 1]$ such that*

$$\overline{\nu}^{\phi} = \beta\overline{\nu}^{\phi_{\tau_2}},$$

$$\bar{\mathscr{U}}(\phi) = \beta\bar{\mathscr{U}}(\phi_{\tau_2}).$$

Proof. Since there can be only one state that choose randomly which is $(1, \mathcal{A})$ in this case, ϕ is deterministic on all other states. Without loss of generality we assume

$\mathcal{T}(\phi) = 0$. If not, the ϕ would have same long-term service rate and utilization rate as threshold policy $\phi_{\mathcal{T}(\phi)+1}$.

$$\phi(\overline{\mathbf{y}}) = \begin{cases} \gamma & \text{if } \overline{\mathbf{y}} = (1, \mathcal{A}) \\ 1 & \text{if } \overline{w} = \mathcal{B} \\ 0 & \text{otherwise,} \end{cases}$$

The rest of the proof is identical to Lemma A.2 by replacing ϕ_{τ_2} with ϕ_2, and ϕ_{τ_1} with ϕ_1 which is a policy that always rest and $\overline{\nu}^{\phi_{\tau_1}} = \bar{\mathscr{U}}(\phi_{\tau_1}) = 0$. □

Before we state the final Lemma, note that, when $\phi(1, \mathcal{A}) = 0$, the process $\overline{\mathbf{Y}}^{\phi}$ could have two positive recurrent communicating classes. The $\bar{\mathscr{U}}$ is not well defined on such ϕ. Thus, we define a set of service rate and utilization rate pair for ϕ that choose to rest at state $(1, \mathcal{A})$.

$$\mathbf{S\bar{U}}(\phi) \stackrel{\text{def}}{=} \left\{ \Big(\sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}) \overline{\pi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}}),\ \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \overline{\pi}(\overline{\mathbf{y}}) \phi(\overline{\mathbf{y}}) \Big) : \ \boldsymbol{\overline{\pi}} \in \overline{\boldsymbol{\Pi}}(\phi) \right\}$$

Lemma A.4. *For every $\phi \in \Phi^{\dagger}$ with $\phi(1, \mathcal{A}) = 0$, there exists $\tau_1, \tau_2 \in \mathbb{S} \cup \{n_s + 1\}$*

and $\alpha \in [0,1]$ *such that*

$$\mathbf{S\bar{U}}(\phi) = \Big\{\beta\Big((1-\alpha)\overline{\nu}^{\phi_{\tau_1}} + \alpha\overline{\nu}^{\phi_{\tau_2}},$$

$$(1-\alpha)\bar{\mathscr{U}}(\phi_{\tau_1}) + \alpha\bar{\mathscr{U}}(\phi_{\tau_2})\Big) : \ \beta \in [0,1]\Big\}$$

Proof. If $\mathcal{T}(\phi) = 0$ which implies that the policy always rest, it is clear that $(1, \mathcal{A})$ is an absorbing state and the service rate and the utilization ratio are both zero. If $\mathcal{T}(\phi) > 0$, we can represent ϕ as

$$\phi(\overline{\mathbf{y}}) = \begin{cases} 0 & \text{if } \overline{\mathbf{y}} = (1, \mathcal{A}) \\ \phi'(\overline{\mathbf{y}}) & \text{otherwise,} \end{cases}$$

where ϕ' has the same form as (A.2). The MC now have two positive recurrent communicating classes and the stationary PMF can be any convex combination of stationary PMF of ϕ' and ϕ_1 (always rest policy). This is also true for utilization rate and service rate. □

Proof of Theorem 1: By Lemmas A.2-A.4, the optimization problem (A.1a) can be transform into an optimization problem over $\alpha, \beta \in [0,1]$ and $\tau_1, \tau_2 \in \mathbb{S} \cup \{n_s+1\}$.

$$\bar{\mathscr{U}}^0_{\mathbb{L}}(\overline{\nu}) = \min_{\alpha, \beta, \tau_1, \tau_2} \beta\Big((1-\alpha)\bar{\mathscr{U}}(\phi_{\tau_1}) + \alpha\bar{\mathscr{U}}(\phi_{\tau_2})\Big)$$

(A.10a)

$$\beta\Big((1-\alpha)\overline{\nu}^{\phi_{\tau_1}} + \alpha\overline{\nu}^{\phi_{\tau_2}}\Big) = \overline{\nu} \tag{A.10a}$$

If we plot $(0,0)$ and $\left(\overline{\nu}^{\phi_\tau}, \bar{\mathscr{U}}(\phi_\tau)\right)$ on the x-y plane for all $\tau \in \mathbb{S} \cup \{n_s+1\}$, the $\bar{\mathscr{U}}_{\mathbb{L}}^{0}(\overline{\nu})$ is the lower bound of the convex hull of all the points which is non-decreasing, piece-wise affine and convex for $\overline{\nu} \in [0, \overline{\nu}^*]$.

A.1.1 Derivation of Stationary PMF in (A.4)

In order to prove (A.4) is the correct stationary PMF, it suffices to show that the given PMF satisfies the following global balance equations:

$$\overline{\pi}^{\phi'}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi'}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \ \text{ for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{A.11}$$

where $\overline{\mathbf{P}}^{\phi'}$ is the one-step transition matrix of $\overline{\mathbf{Y}}^{\phi'}$. To this end, we shall demonstrate that the RHS of (A.4) is equal to the RHS of (A.11).

First, we break the RHS of (A.11) into two terms.

$$\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi'}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

$$= \overline{\pi}^{\phi'}(\tau_2 - 1, \mathcal{A}) \overline{\mathbf{P}}^{\phi'}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} \tag{A.12}$$

$$+ \sum_{\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{(\tau_2-1,\mathcal{A})\}} \overline{\pi}^{\phi'}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \tag{A.13}$$

We then rewrite each term on the RHS: from (A.4) and (A.2), we have

$$\text{(A.12)} = \left(\alpha \cdot \overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A}) + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A})\right)$$

$$\times \left((1-\gamma)\overline{\mathbf{P}}^{\phi_{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} + \gamma \overline{\mathbf{P}}^{\phi_{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}}\right)$$

Substituting the expression for γ in (A.3),

$$(\text{A.12}) = (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2-1,\mathcal{A})\overline{\mathbf{P}}^{\phi_{\tau_1}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} + \alpha\cdot\overline{\pi}^{\phi_{\tau_2}}(\tau_2-1,\mathcal{A})\overline{\mathbf{P}}^{\phi_{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}}$$

Second, from (A.4)

$$(\text{A.13}) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \left(\alpha\cdot\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}') + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}')\right) \times\overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}.$$

From (A.2), for all $\overline{\mathbf{y}}' = (\overline{s}',\overline{w}') \in \mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}$, we have $\phi'(\mathbf{y}') = \phi_{\tau_1}(\mathbf{y}')$ and $\overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\phi_{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$. Moreover, because ϕ_{τ_2} is a deterministic policy with a threshold on the activity state of the server, $\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}') = 0$ for all $\overline{\mathbf{y}}' = (\overline{s}',\overline{w}')$ with $\overline{s}' < \tau_2 - 1$. Hence, for all $\overline{\mathbf{y}}' \in \mathbb{Y}\setminus\{(\tau_2-1),\mathcal{A})\}$ with $\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}') > 0$, together with the assumption $\tau_1 \leq \tau_2$, we have

$$\phi_{\tau_1}(\overline{\mathbf{y}}') = \phi_{\tau_2}(\overline{\mathbf{y}}') = \begin{cases} 0 & \text{if } \overline{s}' \geq \tau_2 \text{ and } \overline{w}' = \mathcal{A} \\ 1 & \text{if } \overline{w}' = \mathcal{B} \end{cases}$$

and, consequently, $\overline{\mathbf{P}}^{\phi_{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\phi_{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$. Therefore,

$$(\text{A.13}) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus\{(\tau_2-1,\mathcal{A})\}} \Big(\alpha\cdot\overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\phi_{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\phi_{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}\Big)$$

Substituting the new expressions for (A.12) and (A.13), we obtain

$$\begin{aligned}
&\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\phi'}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\phi'}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
&= (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\tau_2 - 1, \mathcal{A})\overline{\mathbf{P}}^{\phi_{\tau_1}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} \\
&\quad + \alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\tau_2 - 1, \mathcal{A})\overline{\mathbf{P}}^{\phi_{\tau_2}}_{(\tau_2-1,\mathcal{A}),\overline{\mathbf{y}}} \\
&\quad + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y} \setminus \{(\tau_2-1,\mathcal{A})\}} \Big(\alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\phi_{\tau_2}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
&\qquad\qquad + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\phi_{\tau_1}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \Big) \\
&= \alpha \cdot \overline{\pi}^{\phi_{\tau_2}}(\overline{\mathbf{y}}) + (1-\alpha)\overline{\pi}^{\phi_{\tau_1}}(\overline{\mathbf{y}}),
\end{aligned}$$

where the last equality follows from the fact that $\overline{\pi}^{\phi_{\tau_1}}$ and $\overline{\pi}^{\phi_{\tau_2}}$ are the stationary PMFs of $\overline{\mathbf{Y}}^{\phi_{\tau_1}}$ and $\overline{\mathbf{Y}}^{\phi_{\tau_2}}$, respectively.

A.2 Lemma 5.3 for m types of tasks

Lemma A.5. *Suppose $\lambda_i \leq \overline{\nu}_i^*$ for all $i \in \{2 \ldots, m\}$. If a policy ϕ in Φ_R achieves $\overline{\nu}_i^{\phi}$ equal to the arrival rate λ_i for all $i \in \{2 \ldots, m\}$, then $\overline{\nu}_1^{\phi}$ is upper bounded by $\overline{\nu}_1^*\left(\sum_{i=2}^{m} \frac{\overline{\nu}_i^* - \lambda_i}{\overline{\nu}_i^*} \right) \geq 0$.*

Consider the following optimization problem with a constraint on the long-term service rate of type i tasks for $i = 2, \ldots, m$. Since the case with one of the $\lambda_i = 0$ reduces to a multiple-queues case with $m-1$ queues and the case where all $\lambda_i = 0$ for $i = 2, \ldots, m$ reduces to a single-queue case, here we assume $\lambda_i > 0$ for all

$i = 2, \ldots, m$.

$$\begin{aligned} &\underset{\phi \in \Phi^*}{\text{maximum}} && \overline{\nu}_1^{\phi} \\ &\text{subject to} && \overline{\nu}_i^{\phi} \geq \lambda_i \quad \forall i \in \{2, \ldots, m\}. \end{aligned} \tag{A.14}$$

where Φ^* is some subset of Φ_R^+. We denote the optimal value of (A.14) by $\overline{\nu}^*(\Phi^*)$.

We shall prove that the optimization problem with $\Phi^* = \Phi_R^+$ has an optimal value $\overline{\nu}_1^{*\left(\sum_{i=2}^{m} \frac{\overline{\nu}_i^* - \lambda_i}{\overline{\nu}_i^*} \right)}$ and there exists an optimal policy ϕ^* in Φ_R^+ which achieves the optimal value. To this end, we consider (A.14) with a sequence of decreasing subsets of Φ_R^+ and show that the optimal value does not decrease as we reduce the set of policies we allow.

Lemma A.6. *For a policy $\phi \in \Phi_R^+$, there exists a policy $\phi' \in \Phi_R^{++} \stackrel{def}{=} \{\phi \in \Phi_R^+ : \sum_{i \in \mathbb{T}} \phi(1, \mathcal{A})_i = 1\}$ such that,*

$$\overline{\nu}_i^{\phi'} \geq \overline{\nu}_i^{\phi},$$

for all i from 1 to m.

Proof. The proof is omited due to the similarity with the proof of Lemma 5.7 in Section 5.5.4. □

Lemma A.7. *The optimal value of (A.14) remains the same when we allow only the policies in Φ_R^{++}, i.e., $\overline{\nu}^*(\Phi_R^{++}) = \overline{\nu}^*(\Phi_R^+)$.*

Proof. This follow directly from Lemma A.6. □

The intuition behind Lemma A.7 is that when the server state is $(1, \mathcal{A})$ and the server rests, the server's new state is $(1, \mathcal{A})$. This suggests that the server wasted an epoch without contributing to long-term service rates. Therefore, when the server state is $(1, \mathcal{A})$, the server should be required to work on a task with probability one, in order to increase the long- term service rates.

Define $\Phi^{\dagger}$ to be set of policies that have at most $m-1$ randomizations. To be more specifically,

$$\Phi^{\dagger} \overset{\text{def}}{=} \left\{ \phi \in \Phi_R^{++} \mid \sum_{\overline{s} \in \mathbb{S}} \left(\mathcal{I} \left\{ \sum_{i=1}^{m} \phi(\overline{s}, \mathcal{A})_i < 1 \right\} + \sum_{i=1}^{m} \mathcal{I} \left\{ \phi(\overline{s}, \mathcal{A})_i > 0 \right\} - 1 \right) \leq m - 1 \right\},$$

where we call the number of randomizations at each state $\overline{\mathbf{y}}$ to be the number of possible actions chosen by ϕ minus one and the number of randomization for the policy is the sum of the number at each state.

Lemma A.8. *We have* $\overline{\nu}^*(\Phi_R^{++}) = \overline{\nu}^*(\Phi^{\dagger})$.

Proof. Recall that, for every ϕ in Φ_R^{++}, the corresponding $\overline{\mathbf{Y}}$ is a unichain, i.e., a finite-state Markov chain with a single recurrent communication class and, possibly, transient states. Therefore, the optimization problem (A.14) gives rise to a unichain MDP problem. Since there is only $m-1$ constraints in (A.14), Theorem 4.4 of [89] tells us that there exists an optimal policy with at most $m-1$ randomization. $\square$

Denote by $\psi^{i,\tau}$, $i \in \{1, \ldots, m\}$ and $\tau \in \mathbb{S}^+ \overset{\text{def}}{=} \mathbb{S} \cup n_s + 1$, a threshold policy on activity state with threshold τ which only chooses type i tasks when available. In other words,

$$\psi^{i,\tau}(\overline{s}, \mathcal{A}) = \begin{cases} \mathbf{e}_i & \text{if } \overline{s} < \tau, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Hence, when the server is available to take on a new task, $\psi^{i,\tau}$ asks the server to service a type i task only if the action- dependent state is less than τ.

Finally, we define $\Phi^{\ddagger}$ to be the subset of policies in $\Phi^{\dagger}$ of the following forms: suppose $\boldsymbol{\tau} \stackrel{\text{def}}{=} (\tau_1, \ldots, \tau_m) \in (\mathbb{S}^+)^m$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-1}) \in [0,1]^{m-1}$, and $\xi : \mathbb{T} \to \mathbb{T}$ is a one-to-one function such that $\tau_{\xi(1)} \geq \ldots \geq \tau_{\xi(m)}$, we shall define a sequence of policies $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ from $\ell = m$ to $\ell = 1$.

$$\zeta_m^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) = \psi^{\xi(m),\tau_{\xi(m)}}(\overline{s}, \overline{w})$$

$$\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) = \begin{cases} (1-\gamma_\ell)\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) + \gamma_\ell \mathbf{e}_{\xi(\ell)} & \text{if } \overline{s} = \tau_{\xi(\ell)} - 1, \overline{w} = \mathcal{A} \\ \zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) & \text{otherwise.} \end{cases}$$

$$\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) = \begin{cases} (1-\gamma_1)\zeta_2^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) + \gamma_1 \mathbf{e}_{\xi(1)} & \text{if } \overline{s} = \tau_{\xi(1)} - 1, \overline{w} = \mathcal{A} \\ \zeta_2^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \overline{w}) & \text{otherwise.} \end{cases}$$

Notice that all ζ are in $\Phi^{\dagger}$ and the set of $\zeta_1^{(\tau,\gamma,\xi)}$ is what we call $\Phi^{\ddagger}$. The intuition behind this build process is to first have a threshold policy on type $\xi(m)$ (the type with lowest τ) tasks. Then, choose the state with $\overline{s} = \tau_{\xi(m-1)}$ and $\overline{w} = \mathcal{A}$ to randomize between working on type $\xi(m-1)$ tasks or adopted the original policy. Keep doing this $m-1$ times and we have a policy in $\Phi^{\ddagger}$. Clearly, these policies have at most $m-1$ randomization and belong to $\Phi^{\dagger}$.

Lemma A.9. $\overline{\nu}^*(\Phi^{\ddagger}) = \overline{\nu}^*(\Phi^{\dagger})$

Proof. In order to prove the lemma, we show that, for a feasible policy ϕ in $\Phi^{\dagger}$ which (a) satisfies the constraint $\overline{\nu}_i^{\phi} \geq \lambda_i$ for i from 2 to m and (b) achieves positive $\overline{\nu}_1^{\phi}$, we can find a policy ϕ' in $\Phi^{\ddagger}$ with (i) the identical stationary PMF as ϕ and (ii) the same scheduling decision at all states in the unique positive recurrent communicating class.

Choose a feasible policy ϕ in $\Phi^{\dagger}$. Note that $\mathcal{T}(\phi) \geq 1$ because $\phi \in \Phi_R^{++}$. We denote the unique positive recurrent communicating class is $\mathbb{Y}^{\phi} \in \mathbb{Y}$ where $\overline{s} \geq \mathcal{T}(\phi)$. Since ϕ in $\Phi^{\dagger}$, we have,

$$\sum_{\overline{s} \geq \mathcal{T}(\phi)} \left(\mathcal{I}\left\{ \sum_{i \in \mathbb{T}} \phi(\overline{s}, \mathcal{A})_i < 1 \right\} + \sum_{i \in \mathbb{T}} \mathcal{I}\left\{ \phi(\overline{s}, \mathcal{A})_i > 0 \right\} - 1 \right) \leq m - 1. \quad \text{(A.15)}$$

From the definition of $\mathcal{T}$, $\sum_{i \in \mathbb{T}} \phi(\overline{s}, \mathcal{A})_i < 1$ for all $\overline{s} > \mathcal{T}(\phi)$. Thus, we rewrite (A.15) into,

$$\sum_{\overline{s} \geq \mathcal{T}(\phi)} \left(\sum_{i \in \mathbb{T}} \mathcal{I}\left\{ \phi(\overline{s}, \mathcal{A})_i > 0 \right\} \right) \leq m. \quad \text{(A.16)}$$

Moreover, for every $i \in \mathbb{T}$, it is clear that there must exist a state in $\mathbb{Y}^\phi$ with the form $(\overline{s}, \mathcal{A})$ such that $\phi(\overline{s}, \mathcal{A})_i > 0$. Otherwise, $\overline{\nu}_i^\phi = 0$. Together with (A.16) tells us that, for every $i \in \mathbb{T}$, there is exactly one state with $\overline{s} \geq \mathcal{T}(\phi)$ such that $\phi(\overline{s}, \mathcal{A})_i > 0$.

With the observations, we let our τ_i to be the state $\overline{s}$ where $\phi(\overline{s}, \mathcal{A})_i > 0$ plus 1,

$$\tau_i = \{\overline{s} \in \mathbb{S} \mid \overline{s} \geq \mathcal{T}(\phi), \phi(\overline{s}, \mathcal{A})_i > 0\} + 1.$$

We find one ξ function such that $\tau_{\xi(1)} \geq \ldots \geq \tau_{\xi(m)}$, and define $\boldsymbol{\gamma}$,

$$\begin{aligned}
\gamma_{m-1} &= \frac{\phi(\tau_{\xi(m-1)} - 1, \mathcal{A})_{\xi(m-1)}}{1 - \sum_{j=1}^{m-2} \phi(\tau_{\xi(m-1)} - 1, \mathcal{A})_{\xi(j)}} \\
\gamma_\ell &= \frac{\phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)}}{1 - \sum_{j=1}^{\ell-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}} \\
\gamma_1 &= \phi(\tau_{\xi(1)} - 1, \mathcal{A})_{\xi(1)}
\end{aligned}$$

We plug the $(\boldsymbol{\tau}, \boldsymbol{\gamma}, \xi)$ back to our iterative definition for $\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ and finish the proof with following Lemma.

Lemma A.10. *$\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ and ϕ are identical for every state with $\overline{s} \geq \mathcal{T}(\phi)$. Therefore, both policies have exactly the same stationary PMF and the same scheduling decision at all states in the unique positive recurrent communicating class.*

Proof. Please see Section A.2.1 for a proof. □

□

Let us consider the policies in ζ for some given $(\boldsymbol{\tau}, \boldsymbol{\gamma}, \xi)$. We rewrite each γ_ℓ as

$$\gamma_\ell = \frac{\alpha_\ell \cdot \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)} - 1, \mathcal{A})}{\alpha_\ell \cdot \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)} - 1, \mathcal{A}) + (1 - \alpha_\ell) \cdot \overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)} - 1, \mathcal{A})} \tag{A.17}$$

for some $\alpha_\ell \in [0, 1]$ and $\ell \in \{1, \ldots, m-1\}$. Note that, for every $\gamma_\ell \in [0, 1]$, we can find an appropriate α_ℓ that satisfies (A.17) because $\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)} - 1, \mathcal{A}) > 0$ for all ℓ. By solving the balance equations, we get the following stationary PMF:

Lemma A.11. *(Distribution Split)*

$$\begin{aligned}
&\overline{\pi}^{\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) \\
&= \alpha_1 \overline{\pi}^{\psi^{\xi(1),\tau_{\xi(1)}}}(\overline{\mathbf{y}}) + \sum_{\ell=2}^{m-1} \alpha_\ell \prod_{j=1}^{\ell-1}(1-\alpha_j)\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) + \prod_{j=1}^{m-1}(1-\alpha_j)\overline{\pi}^{\psi^{\xi(m),\tau_{\xi(m)}}}(\overline{\mathbf{y}}).
\end{aligned}$$

Proof. A proof is provided in Section A.2.2 □

For notational convenience, we define

$$\beta_1 \stackrel{\text{def}}{=} \alpha_1, \qquad \beta_\ell \stackrel{\text{def}}{=} \alpha_\ell \prod_{j=1}^{\ell-1}(1-\alpha_j), \qquad \beta_m \stackrel{\text{def}}{=} \prod_{j=1}^{m-1}(1-\alpha_j).$$

Clearly, the sum of all β equals to one by an simple observation that $\sum_{j=\ell}^m \beta_j = \prod_{j=1}^{\ell-1}(1-\alpha_j)$. By the similar approach, we have the following Lemma.

Lemma A.12.

$$\overline{\nu}_{\xi(\ell)}^{\zeta_1^{(\tau,\gamma,\xi)}} = \beta_\ell \overline{\nu}_{\xi(\ell)}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}},$$

for all ℓ from 1 to m.

Proof. A proof is provided in Section A.2.3. □

Before we proof our final result, we re-index β into β' such that $\beta'_i = \beta_{\xi^{-1}(i)}$. Notice that ξ is an one-to-one function and is invertible. The Lemma A.12 can be rewritten into

$$\overline{\nu}_i^{\zeta_1^{(\tau,\gamma,\xi)}} = \beta'_i \overline{\nu}_i^{\psi^{i,\tau_i}},$$

for all i from 1 to m. Considering our objective which is the long-term service rate of type 1 task, by the fact that sum of all β equals one,

$$\overline{\nu}_1^{\zeta_1^{(\tau,\gamma,\xi)}} = \beta'_1 \overline{\nu}_1^{\psi^{1,\tau_1}} = \left(1 - \sum_{i=2}^{m} \beta'\right) \overline{\nu}_1^{\psi^{1,\tau_1}}.$$

In order to upper bound $\overline{\nu}_1^{\zeta_1^{(\tau,\gamma,\xi)}}$, we shall upper bound $\overline{\nu}_1^{\psi^{1,\tau_1}}$ and lower bound each β'. The previous step tells us that $\overline{\nu}_1^{\psi^{1,\tau_1}} \leq \overline{\nu}_1^*$ since $\overline{\nu}_1^*$ is the maximum long-term departure rate for single type of queue. Moreover, with the constraints in the optimization problem that $\overline{\nu}_i^{\zeta_1^{(\tau,\gamma,\xi)}} \geq \lambda_i$ for all $i \in \{2, \ldots, m\}$, Lemma A.12 tells us that,

$$\lambda_i \leq \overline{\nu}_i^{\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} = \beta_i' \overline{\nu}_i^{\psi^{i,\tau_i}} \leq \beta_i' \overline{\nu}_i^*, \qquad \beta_i' \geq \frac{\lambda_i}{\overline{\nu}_i^*},$$

Combining all the observations, we have,

$$\overline{\nu}_1^{\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} = \left(1 - \sum_{i=2}^{m} \beta'\right) \overline{\nu}_1^{\psi^{1,\tau_1}} \leq \left(1 - \sum_{i=2}^{m} \frac{\lambda_i}{\overline{\nu}_i^*}\right) \overline{\nu}_1^*.$$

A.2.1 A Proof of Lemma A.10

We shall show that $\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \mathcal{A}) = \sum_{j:\tau_j=\overline{s}} \phi(\overline{s}, \mathcal{A})_j \mathbf{e}_j$ for every $\overline{s} \geq \mathcal{T}(\phi)$. It is clear from the definition that $\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s}, \mathcal{A})$ is non-zero if and only if $\overline{s}$ equals to some $\tau_i - 1$. Thus, it is sufficient to show that $\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_i - 1, \mathcal{A}) = \sum_{j:\tau_j=\tau_i} \phi(\tau_i - 1, \mathcal{A})_j \mathbf{e}_j$ for all i.

Case 1: Suppose τ_i is distinct such that there is no other τ_j that has the same value as τ_i and $\xi(m) = i$, we know from the constraint of ξ that $\tau_{\xi(m-1)} > \tau_{\xi(m)}$ and

$$\begin{aligned}
\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_i - 1, \mathcal{A}) &= \zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(m)} - 1, \mathcal{A}) \\
&= \psi^{\xi(m),\tau_{\xi(m)}}(\tau_{\xi(m)} - 1, \mathcal{A}) \\
&= \phi(\tau_\xi(m) - 1, \mathcal{A})_{\xi(m)} \mathbf{e}_{\xi(m)} = \phi(\tau_i - 1, \mathcal{A})_i \mathbf{e}_i,
\end{aligned}$$

Suppose $\xi(\ell) = i$, we know from the constraint of ξ that $\tau_{\xi(\ell-1)} > \tau_{\xi(\ell)} > \tau_{\xi(\ell+1)}$

and

$$
\begin{aligned}
\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_i - 1, \mathcal{A}) &= \zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&= \zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&= (1-\gamma_\ell)\cancel{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})} + \gamma_\ell \mathbf{e}_{\xi(\ell)} \\
&= \frac{\phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)}}{1 - \cancel{\sum_{j=1}^{\ell-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}}} \mathbf{e}_{\xi(\ell)} = \phi(\tau_i - 1, \mathcal{A})_i \mathbf{e}_i,
\end{aligned}
$$

where $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})$ equals to zeros since all $(\tau_{\xi(\ell+1)}, \ldots, \tau_{\xi(m)})$ is less than $\tau_\xi(\ell)$. $\phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}$ equals to zero for all j from 1 to $\ell - 1$ since $\tau_\xi(j) > \tau_{\xi(\ell)}$.

Case 2: Suppose there exist n duplication of τ_i, without loss of generality, we can assume $\xi(\ell) = i$ and $\tau_{\xi(\ell-1)} > \tau_{\xi(\ell)} = \tau_{\xi(\ell+1)} = \ldots = \tau_{\xi(\ell+n)} > \tau_{\xi(\ell+n+1)}$.

$$
\begin{aligned}
&\zeta_1^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_i - 1, \mathcal{A}) \\
&= \zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&= \gamma_\ell \mathbf{e}_{\xi(\ell)} + (1-\gamma_\ell)\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&= \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)} \mathbf{e}_{\xi(\ell)} \\
&\quad + \left(1 - \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)}\right)\left(\gamma_{\ell+1}\mathbf{e}_{\xi(\ell+1)} + (1-\gamma_{\ell+1})\zeta_{\ell+2}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})\right) \\
&= \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)} \mathbf{e}_{\xi(\ell)} \\
&\quad + \cancel{\left(1 - \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell)}\right)} \frac{\phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell+1)}}{\cancel{1 - \sum_{j=1}^{\ell} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}}} \mathbf{e}_{\xi(\ell+1)} \\
&\quad + \left(1 - \sum_{j=\ell}^{\ell+1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}\right)\zeta_{\ell+2}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&= \sum_{j=\ell}^{\ell+n-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} \mathbf{e}_{\xi(j)} \\
&\quad + \left(1 - \sum_{j=\ell}^{\ell+n-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}\right)\zeta_{\ell+n}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A}),
\end{aligned}
\tag{A.18}
$$

where the cancel out is because $\phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} = 0$ for j from 1 to $\ell - 1$. Now, consider two cases where $\ell + n = m$ and $\ell + n < m$. If $\ell + n = m$,

$$
\begin{aligned}
(\text{A.18}) = &\sum_{j=\ell}^{m-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} \mathbf{e}_{\xi(j)} \\
&\quad + \left(1 - \sum_{j=\ell}^{m-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}\right)\left(\psi^{\xi(m),\tau_{\xi(m)}}(\tau_{\xi(\ell)} - 1, \mathcal{A})\right) \\
= &\sum_{j=\ell}^{m} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} \mathbf{e}_{\xi(j)},
\end{aligned}
$$

where the last equality is from the facts that $\mathcal{T}(\phi) = \tau_{\xi(m)} - 1 = \tau_{\xi(\ell)} - 1$ and $\sum_{j=\ell}^{m} \phi(\mathcal{T}(\phi), \mathcal{A})_{\xi(j)} = 1$. If $\ell + n < m$,

$$
\begin{aligned}
(\text{A.18}) = &\sum_{j=\ell}^{\ell+n-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} \mathbf{e}_{\xi(j)} \\
&\quad + \left(1 - \sum_{j=\ell}^{\ell+n-1} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)}\right)\left(\gamma_{\ell+n} \mathbf{e}_{\xi(\ell+n)} + (1 - \gamma_{\ell+n}) \cancel{\zeta_{\ell+n+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})}\right) \\
= &\sum_{j=\ell}^{\ell+n} \phi(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(j)} \mathbf{e}_{\xi(j)},
\end{aligned}
$$

where $\zeta_{\ell+n+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})$ equals to zeros since all $(\tau_{\xi(\ell+n+1)}, \ldots, \tau_{\xi(m)})$ is less than $\tau_\xi(\ell)$.

A.2.2 A Proof of Lemma A.11

Notice that it is sufficient to show that

$$
\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) = \alpha_\ell \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) + (1 - \alpha_\ell) \overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})
$$

and the rest follow from the induction from ℓ equals $m-1$ to 1.

Recap

$$\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s},\overline{w}) = \begin{cases} (1-\gamma_\ell)\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s},\overline{w}) + \gamma_\ell \mathbf{e}_{\xi(\ell)} & \text{if } \overline{s} = \tau_{\xi(\ell)} - 1, \overline{w} = \mathcal{A} \\ \zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{s},\overline{w}) & \text{otherwise.} \end{cases}$$

$$\gamma_\ell = \frac{\alpha_\ell \cdot \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})}{\alpha_\ell \cdot \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A}) + (1-\alpha_\ell)\cdot \overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})}$$

•We shall show that

$$\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) = \alpha_\ell \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})$$

satisfies the balance equation of $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$,

$$\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\; \overline{\mathbf{P}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} \quad \text{for all } \overline{\mathbf{y}} \in \mathbb{Y}. \tag{A.19}$$

Utilize the balance equations of $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ and $\psi^{\xi(\ell),\tau_{\xi(\ell)}}$,

$$\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}} \overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\; \overline{\mathbf{P}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} \quad \text{for all } \overline{\mathbf{y}} \in \mathbb{Y}, \tag{A.20}$$

$$\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) = \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \quad \text{for all } \overline{\mathbf{y}} \in \mathbb{Y}. \tag{A.21}$$

Proof. • **Left-hand side of (A.19):**

$$\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) = \alpha_\ell \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})$$

• **Right-hand side of (A.19):**

$$\begin{aligned}
&\sum_{\overline{\mathbf{y}}' \in \mathbb{Y}} \overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
&= \overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})} \overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} \\
&= (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} + \alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} \\
&\quad + \sum_{\overline{\mathbf{y}}' \in \mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})} \left(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}\right) \\
&\quad \text{by (A.22) and (A.23)} \\
&= \alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) \quad \text{by (A.20) and (A.21).}
\end{aligned}$$

$$\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})}$$

$$= \left(\alpha_\ell \overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A}) + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\right)$$

$$\times\left((1-\gamma_\ell)\overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} + \gamma_\ell \overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})}\right)$$

$$= (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} + \alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})\overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{(\tau_{\xi(\ell)}-1,\mathcal{A}),(\overline{\mathbf{y}})} \tag{A.22}$$

$$\sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})} \overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\ \overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

$$= \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})} \left(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}') + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\right)\ \overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$$

$$= \sum_{\overline{\mathbf{y}}'\in\mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})} \left(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} + (1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}')\overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}\right) \tag{A.23}$$

The last equality is from the fact that $\overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$ for all $\overline{\mathbf{y}}'\in\mathbb{Y} : \overline{s}'\geq\tau_{\xi(\ell)}$ such that $\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}')>0$, and $\overline{\mathbf{P}}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}} = \overline{\mathbf{P}}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}_{\overline{\mathbf{y}}',\overline{\mathbf{y}}}$ for all $\overline{\mathbf{y}}'\in\mathbb{Y}\setminus(\tau_{\xi(\ell)}-1,\mathcal{A})$. $\square$

A.2.3 A Proof of Lemma A.12

Notice that it is sufficient to show that, for all ℓ, $\ell^+ > \ell$, and $\ell^- < \ell$,

$$\overline{\nu}_{\xi(\ell^-)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} = 0, \tag{A.24a}$$

$$\overline{\nu}_{\xi(\ell)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} = \alpha_\ell \overline{\nu}_{\xi(\ell)}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}, \tag{A.24b}$$

$$\overline{\nu}_{\xi(\ell^+)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} = (1-\alpha_\ell)\overline{\nu}_{\xi(\ell^+)}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}, \tag{A.24c}$$

and the rest follow from the induction from ℓ equals $m-1$ to 1.

Case (A.24a):

This is obvious because $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ does not choose to work on any type $\xi(\ell^-)$ tasks when the server is available ($\overline{w} = \mathcal{A}$). The long-term service rate of type $\xi(\ell^-)$ tasks equals zero.

Case (A.24b):

First, we identified that $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ choose to work on type $\xi(\ell)$ task with positive probability if and only if $\overline{\mathbf{y}} = (\tau_{\xi(\ell)}-1, \mathcal{A})$ or $\overline{\mathbf{y}} \in \mathbb{Y} : \overline{w} = \mathcal{B}_{\xi(\ell)}$

$$\begin{aligned}
\overline{\nu}_{\xi(\ell)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} &= \sum_{\overline{\mathbf{y}}\in\mathbb{Y}} \mu(\overline{s},\xi(\ell))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) \\
&= \mu(\tau_{\xi(\ell)}-1,\xi(\ell))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A}) \\
&\quad + \sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{w}=\mathcal{B}_{\xi(\ell)}} \mu(\overline{s},\xi(\ell))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})
\end{aligned} \tag{A.25}$$

We apply the definition of $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ and the distribution split in the previous section to the first-term on the RHS of (A.25),

$$
\begin{aligned}
&\mu(\tau_{\xi(\ell)}-1,\xi(\ell))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})\\
&=\mu(\tau_{\xi(\ell)}-1,\xi(\ell))\\
&\qquad\cdot\Big((1-\gamma_\ell)\cancel{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell)}}+\gamma_\ell\Big)\\
&\qquad\cdot\Big(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\xi(\ell))+(1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\xi(\ell))\Big)\\
&=\mu(\tau_{\xi(\ell)}-1,\xi(\ell))\\
&\qquad\cdot\left(\frac{\alpha_\ell\cdot\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})}{\alpha_\ell\cdot\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})+(1-\alpha_\ell)\cdot\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})}\right)\\
&\qquad\cdot\Big(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\xi(\ell))+(1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\xi(\ell))\Big)\\
&=\alpha_\ell\cdot\mu(\tau_{\xi(\ell)}-1,\xi(\ell))\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A}).
\end{aligned}
$$

We proceed with the second-term on the RHS of (A.25) and use the distribution split.

$$
\begin{aligned}
&\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{w}=\mathcal{B}_{\xi(\ell)}}\mu(\overline{s},\xi(\ell))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})\\
&=\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{w}=\mathcal{B}_{\xi(\ell)}}\mu(\overline{s},\xi(\ell))\cdot 1\cdot\left(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}})+(1-\alpha_\ell)\cancel{\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})}\right)\\
&=\alpha_\ell\cdot\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{w}=\mathcal{B}_{\xi(\ell)}}\mu(\overline{s},\xi(\ell))\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}),
\end{aligned}
$$

where $\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})=0$ for all $\overline{\mathbf{y}}\in\mathbb{Y}:\overline{w}=\mathcal{B}_{\xi(\ell)}$ because the policy $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ never choose to work on type $\xi(\ell)$ tasks when server is available. Together with (A.25), we have

$$
\begin{aligned}
\overline{\nu}_{\xi(\ell)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} &= \alpha_\ell \cdot \mu(\tau_{\xi(\ell)} - 1, \xi(\ell))\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&\quad + \alpha_\ell \cdot \sum_{\overline{\mathbf{y}} \in \mathbb{Y}: \overline{w} = \mathcal{B}_{\xi(\ell)}} \mu(\overline{s}, \xi(\ell))\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) \\
&= \alpha_\ell \cdot \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, \xi(\ell))\psi^{\xi(\ell),\tau_{\xi(\ell)}}(\overline{\mathbf{y}})_{\xi(\ell)}\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}}) \\
&= \alpha_\ell \overline{\nu}_{\xi(\ell)}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}.
\end{aligned}
$$

Case (A.24c): First, we use the definition of the long-term service rate.

$$
\begin{aligned}
\overline{\nu}_{\xi(\ell^+)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} &= \sum_{\overline{\mathbf{y}} \in \mathbb{Y}} \mu(\overline{s}, \xi(\ell^+))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) \\
&= \mu(\tau_{\xi(\ell)} - 1, \xi(\ell^+))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell^+)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&\quad + \sum_{\overline{\mathbf{y}} \in \mathbb{Y}: \overline{\mathbf{y}} \neq (\tau_{\xi(\ell)} - 1, \mathcal{A})} \mu(\overline{s}, \xi(\ell^+))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}). \qquad \text{(A.26)}
\end{aligned}
$$

We apply the definition of $\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}$ and the distribution split in the previous section to the first-term on the RHS of (A.26),

$$\mu(\tau_{\xi(\ell)}-1,\xi(\ell^+))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell^+)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})$$

$$=\mu(\tau_{\xi(\ell)}-1,\xi(\ell^+))$$

$$\cdot\big((1-\gamma_\ell)\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell^+)}\big)$$

$$\cdot\big(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\xi(\ell))+(1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\xi(\ell))\big)$$

$$=\mu(\tau_{\xi(\ell)}-1,\xi(\ell^+))$$

$$\cdot\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell^+)}$$

$$\cdot\left(\frac{(1-\alpha_\ell)\cdot\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})}{\alpha_\ell\cdot\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\mathcal{A})+(1-\alpha_\ell)\cdot\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A})}\right)$$

$$\cdot\big(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\tau_{\xi(\ell)}-1,\xi(\ell))+(1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\xi(\ell))\big)$$

$$=(1-\alpha_\ell)\mu(\tau_{\xi(\ell)}-1,\xi(\ell^+))\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)}-1,\mathcal{A})_{\xi(\ell^+)}\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)}-1,\mathcal{A}).$$

We proceed with the second-term on the RHS of (A.26) and use the distribution split and the definition of ζ.

$$\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{\mathbf{y}}\neq(\tau_{\xi(\ell)}-1,\mathcal{A})}\mu(\overline{s},\xi(\ell^+))\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\overline{\pi}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})$$

$$=\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{\mathbf{y}}\neq(\tau_{\xi(\ell)}-1,\mathcal{A})}\mu(\overline{s},\xi(\ell^+))\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\left(\alpha_\ell\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}})+(1-\alpha_\ell)\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}})\right)$$

$$=(1-\alpha_\ell)\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{\mathbf{y}}\neq(\tau_{\xi(\ell)}-1,\mathcal{A})}\mu(\overline{s},\xi(\ell^+))\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}),$$

where the last equality comes from the fact that $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\cdot\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}})$ equals to zero for all $\overline{\mathbf{y}}\in\mathbb{Y}:\overline{\mathbf{y}}\neq(\tau_{\xi(\ell)}-1,\mathcal{A})$. This is because $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}$ is non-zero only when $\overline{w} = \mathcal{B}_{\xi(\ell^+)}$ or $\overline{\mathbf{y}} = (\tau_{\xi(\ell^+)} - 1, \mathcal{A})$. Without loss of generality, we can assume $\tau_{\xi(\ell^+)} < \tau_{\xi(\ell)}$ since $\overline{\mathbf{y}} = (\tau_{\xi(\ell)} - 1, \mathcal{A})$ is excluded from the set. Then, it is clear that $\overline{\pi}^{\psi^{\xi(\ell),\tau_{\xi(\ell)}}}(\overline{\mathbf{y}})$ equals zero for all these $\overline{\mathbf{y}}$ such that $\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}$ is non-zero. Together with (A.26), we have,

$$\begin{aligned}
\overline{\nu}_{\xi(\ell^+)}^{\zeta_\ell^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}} &= (1-\alpha_\ell)\mu(\tau_{\xi(\ell)} - 1, \xi(\ell^+))\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\tau_{\xi(\ell)} - 1, \mathcal{A})_{\xi(\ell^+)}\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\tau_{\xi(\ell)} - 1, \mathcal{A}) \\
&\quad +(1-\alpha_\ell)\sum_{\overline{\mathbf{y}}\in\mathbb{Y}:\overline{\mathbf{y}}\neq(\tau_{\xi(\ell)}-1,\mathcal{A})} \mu(\overline{s}, \xi(\ell^+))\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}(\overline{\mathbf{y}})_{\xi(\ell^+)}\overline{\pi}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}(\overline{\mathbf{y}}) \\
&= (1-\alpha_\ell)\overline{\nu}_{\xi(\ell^+)}^{\zeta_{\ell+1}^{(\boldsymbol{\tau},\boldsymbol{\gamma},\xi)}}.
\end{aligned}$$

A.3 Lemma 5.4 for m types of tasks

Without loss of generality, we assume $\tau_1^* \geq \ldots \geq \tau_m^*$. We use the Proposition 4.1 in the previous Chapter again for building the Lyapunov function. In the remainder of the proof, let f be a potential-like function that satisfies (4.12) in Proposition 4.1 with the policy $\varphi^{\tau_m^*}$ and the following reward function $r : \mathbb{Y} \times \Phi_R \to \mathbb{R}$:

$$r(\mathbf{y}, \phi) = \sum_{i\in\mathbb{T}} \frac{\mu(s,i)\phi(\mathbf{y})_i}{\overline{\nu}_i^*},\ \ (\mathbf{y}, \phi) \in \mathbb{Y} \times \Phi_R \tag{A.27}$$

Define a function $V : \mathbb{X} \to \mathbb{R}_+$, where

$$V(\mathbf{x}) = a\left(\sum_{i\in\mathbb{T}} \frac{q_i}{\overline{\nu}_i^*} + f(\mathbf{y})\right), \tag{A.28}$$

where

$$a = \frac{T}{(1 - \sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*})}, \tag{A.29}$$

and T is some positive constant to be explained shortly (Lemma A.13 below). Similarly, let $g : \mathbb{X} \rightarrow \mathbb{N}$ with

$$g(\mathbf{x}) = \begin{cases} T & \text{if } q_m = 0 \text{ and } V(\mathbf{x}) > N, \\ 1 & \text{otherwise,} \end{cases} \tag{A.30}$$

where

$$N = 2a \frac{m(m+1)T}{\min_{i \in \mathbb{T}}(\overline{\nu}_i^*)} + a \cdot f_{\max}, \tag{A.31}$$

and $f_{\max} \stackrel{\text{def}}{=} \max_{\mathbf{y} \in \mathbb{Y}} f(\mathbf{y})$. Finally, define

$$M = 1 + a \left(\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max} \right). \tag{A.32}$$

Lemma A.13. *Suppose* $\mathbf{X}$ *is the CMC under the policy* θ^{opt}*. Then, there exists finite* T *such that the functions* V *and* g *in* (A.28) *and* (A.30)*, respectively,* *satisfy*

$$\begin{aligned} &\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\ &\leq -g(\mathbf{x}) + M \cdot \mathbb{I}(V(\mathbf{x}) \leq N) \\ &= \begin{cases} -g(\mathbf{x}) + M & \text{if } V(\mathbf{x}) \leq N \\ -g(\mathbf{x}) & \text{otherwise} \end{cases} \end{aligned} \tag{A.33}$$

for every $\mathbf{x} \in \mathbb{X}$.

Proof. For notational simplicity, we omit the dependence on the policy θ^{opt} and denote the conditional expected value of the difference in potential function, $\mathbb{E}\left[f(\overline{\mathbf{Y}}^{\phi}_{k+1}) - f(\overline{\mathbf{Y}}^{\phi}_{k}) \mid \overline{\mathbf{Y}}^{\phi}_{k} = \mathbf{y}\right]$, by $\Delta f(\overline{\mathbf{Y}}^{\phi}; \mathbf{y})$. In addition, r^{ϕ}_{avg} denotes the average reward in $\overline{\mathbf{Y}}^{\phi}$ when there is a unique PRCC under a policy ϕ in Φ_R.

Consider the CMC $\overline{\mathbf{Y}}^{m,\psi^{\tau^*_m}}$ equipped with the policy ψ^{m,τ^*_m}. Assume that f is a potential function that satisfies the equality in (4.12) of Proposition 4.1 for $\overline{\mathbf{Y}}^{\psi^{m,\tau^*_m}}$, with the reward function in (A.27). Define $\mathbb{Y}_{\tau^*_i} \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{Y} \mid s \geq \tau^*_i - 1\}$.

Lemma A.14. *For every* $\mathbf{y} \in \mathbb{Y}_{\tau^*_i}$ *and* $i \in \{2, \ldots, m\}$, *we have*

$$r(\mathbf{y}, \psi^{i,\tau^*_i}) - \Delta f(\overline{\mathbf{Y}}^{\psi^{i,\tau^*_i}}; \mathbf{y}) = 1. \tag{A.34}$$

Proof. We omit the proof because it is identical with the proof of Lemma 5.11 in Section 5.5.9 for a proof. ☐

First, we define σ_ℓ for $\ell = 1, \ldots, L$ to be all distinct optimal threshold values τ^*_i for queues (WLOG assume σ_ℓ is an increasing sequence). Consider CMC $\mathbf{X}^{\theta^{\text{opt}}}$

that starts at some state $\mathbf{x}_0$ with $q_m = 0$, all the type i queues with $\tau_i^* < \sigma_\ell$ for a $\ell = \{1, \ldots, L\}$ satisfy $\max_{i:\tau_i^* = \sigma_{\ell'}} q_i \leq 2\ell' T'$ for all $1 \leq \ell' < \ell$ and , and all queues with τ_j^* equals σ_ℓ satisfy $\max_{j:\tau_j^* = \sigma_\ell} q_j > 2\ell T'$ where T' is a positive integer. Then, for all $k \in \{0, 1, \ldots, T'-1\}$, (i) $\max_{j:\tau_j^* = \sigma_\ell} Q_k^j \geq \max_{i:\tau_i^* < \sigma_\ell} Q_k^i$ and (ii) $\max_{j:\tau_j^* = \sigma_\ell} Q_k^j > 0$. These imply that, when $\mathbf{Y}_k = (\sigma_\ell - 1, \mathcal{A})$ for some $k \in \{0, 1, \ldots, T'-1\}$, $\theta^{\text{opt}}(\mathbf{Y}_k, \mathbf{Q}_k) \in \{\mathbf{e}_j : \tau_j = \sigma_\ell\}$ and a new type j task is scheduled for service.

Let us take a look at the server state $\mathbf{Y}_k$ for $k \in \{0, 1, \ldots, T'-1\}$. First, if $\mathbf{Y}_{k^\dagger} \in \mathbb{Y}_{\sigma_\ell}$ for some $k^\dagger \in \{0, 1, \ldots, T'-2\}$, then $\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}$ for all $k \in \{k^\dagger, \ldots, T'-1\}$ under θ^{opt}. Second, if $q_i \geq 2n_s$ for a i such that $\tau_i^* = \sigma_\ell$, starting with $\{\mathbf{Y}_0 = \mathbf{y}\}$ for some $\mathbf{y} \in \mathbb{Y}_{\sigma_\ell}^C \stackrel{\text{def}}{=} \mathbb{Y} \setminus \mathbb{Y}_{\sigma_\ell} = \{(s, w) \in \mathbb{Y} \mid s < \sigma_\ell - 1\}$, the server state $\mathbf{Y}_k$ will reach a state in $\mathbb{Y}_{\sigma_\ell}$ with positive probability after at most $2n_s$ epochs. The probability of reaching a state in $\mathbb{Y}_{\sigma_\ell}$ after at most $2n_s$ is lower bounded by a $\delta > 0$. Consequently, for all $\mathbf{y} \in \mathbb{Y}_{\sigma_\ell}^C$ and $q_i \geq 2n_s$,

$$\Pr\big(\mathbf{Y}_{k'} \in \mathbb{Y}_{\sigma_\ell} \text{ for some } k' = k+1, \ldots, k+2n_s$$
$$\mid \mathbf{Y}_k = \mathbf{y}, \mathbf{Q}_k = \mathbf{q}\big) \geq \delta.$$

Using this bound, we can upper bound the probability that the server state does not belong to $\mathbb{Y}_{\sigma_\ell}$ at epoch $2jn_s$ for all $j \geq 1$ (for $q_i \geq 2jn_s$) as follows.

$$\Pr\big(\mathbf{Y}_{2jn_s} \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = (\mathbf{y}, \mathbf{q})\big)$$
$$\leq (1-\delta) \Pr\big(\mathbf{Y}_{2(j-1)n_s} \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = (\mathbf{y}, \mathbf{q})\big)$$

Thus, the probability $\Pr(\mathbf{Y}_{2jn_s} \in \mathbb{Y}^C_{\sigma_\ell} \mid \mathbf{Y}_0 \in \mathbb{Y}^C_{\sigma_\ell}, \mathbf{Q}_0 = \mathbf{q})$ can be made arbitrarily small by choosing sufficiently large j. In addition, it is clear $\Pr(\mathbf{Y}_k \in \mathbb{Y}^C_{\sigma_\ell} \mid \mathbf{Y}_0 \in \mathbb{Y}^C_{\sigma_\ell}, \mathbf{Q}_0 = \mathbf{q})$ is non-increasing in k, assuming that queue 1 remains non-empty.

Next, we study the following T'-step drift.

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{T'} - Q^i_0}{\overline{\nu}^*_i} + f(\mathbf{Y}_{T'}) - f(\mathbf{Y}_0) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \sum_{k=0}^{T'-1} \mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{k+1} - Q^i_k}{\overline{\nu}^*_i} + f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0\right]
\end{aligned} \tag{A.35}$$

From the Lindley's equation in (3.1),

$$\begin{aligned}
&\mathbb{E}\left[Q^i_{k+1} - Q^i_k \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \mathbb{E}\left[B^i_k + D^i_k \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \lambda_i + \mathbb{E}\left[\mu(S_k, i)\mathbb{I}(A_k = \mathcal{W}_i) \mid \mathbf{X}_0 = \mathbf{x}_0\right] \\
&= \lambda_i + \mathbb{E}\left[\mu(S_k, i)\theta^{\text{opt}}_i(\mathbf{X}_k) \mid \mathbf{X}_0 = \mathbf{x}_0\right].
\end{aligned} \tag{A.36}$$

Substituting (A.36) in (A.35), we obtain

$$\begin{aligned}
(\text{A.35}) = \sum_{k=0}^{T'-1} \Bigg(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}^*_i} + \mathbb{E}\Bigg[-\sum_{i\in\mathbb{T}} \frac{\mu(S_k, i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}^*_i} \\
+ f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\Bigg|\; \mathbf{X}_0 = \mathbf{x}_0\Bigg]\Bigg).
\end{aligned} \tag{A.37}$$

We upper bound the conditional expected value using the sum two terms by conditioning on whether or not $\mathbf{Y}_k$ belongs to $\mathbb{Y}_{\sigma_\ell}$.

$$\mathbb{E}\left[-\sum_{i\in\mathbb{T}} \frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*} + f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\middle|\; \mathbf{X}_0 = \mathbf{x}_0 \right]$$
$$\leq \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell} \mid \mathbf{X}_0 = \mathbf{x}_0)\ \mathbb{E}\left[-\sum_{i\in\mathbb{T}} \frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*} \right. \tag{A.38}$$
$$\left. + f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k) \;\middle|\; \mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}, \mathbf{X}_0 = \mathbf{x}_0 \right]$$
$$+ \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max}.$$

By further conditioning on the server state at epoch k,

$$(\text{A.38}) = \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell} \mid \mathbf{X}_0 = \mathbf{x}_0)$$
$$\times \sum_{\mathbf{y}\in\mathbb{Y}_{\sigma_\ell}} \Pr(\mathbf{Y}_k = \mathbf{y} \mid \mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}, \mathbf{X}_0 = \mathbf{x}_0)$$
$$\times \mathbb{E}\Big[-\sum_{i\in\mathbb{T}} \frac{\mu(S_k,i)\theta^{\text{opt}}(\mathbf{X}_k)_i}{\overline{\nu}_i^*} + f(\mathbf{Y}_{k+1}) - f(\mathbf{Y}_k)$$
$$\Big| \ \mathbf{Y}_k = \mathbf{y}, \mathbf{X}_0 = \mathbf{x}_0 \Big] \tag{A.39}$$
$$+ \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max}.$$

We shall prove that the expected value term for all $\mathbf{y} \in \mathbb{Y}_{\sigma_\ell}$ equals -1. Recall that we use $\psi^{i,\tau}$ to represent the threshold policy on type i task with threshold value τ as defined in (A.15).

Case 1: $\mathbf{y} : w \neq \mathcal{A}$

In this case, $\theta^{\text{opt}}(\mathbf{X}_k) = \psi^{m,\tau_m^*}(\mathbf{y})$. Therefore, the evolution of server state is exactly the same as $\overline{\mathbf{Y}}^{\psi^{m,\tau_m^*}}$, and

$$
\begin{aligned}
&\mathbb{E}\left[-\sum_{j\in\mathbf{T}}\frac{\mu(S_k,j)\theta^{\text{opt}}(\mathbf{X}_k)_j}{\overline{\nu}_j^*}+f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\mid \mathbf{X}_0=\mathbf{x}_0,\mathbf{Y}_k=\mathbf{y}\right]\\
&=\mathbb{E}\left[-\sum_{j\in\mathbf{T}}\frac{\mu(s,j)\psi^{m,\tau_m^*}(\mathbf{y})_j}{\overline{\nu}_j^*}+f(\overline{\mathbf{Y}}_{k+1}^{\psi^{m,\tau_m^*}})-f(\overline{\mathbf{Y}}_k^{\psi^{m,\tau_m^*}})\mid \overline{\mathbf{Y}}_k^{\psi^{m,\tau_m^*}}=\mathbf{y}\right]\\
&=-r(\mathbf{y},\psi^{m,\tau_m^*})+\Delta f(\overline{\mathbf{Y}}^{\psi^{m,\tau_m^*}};\mathbf{y})\\
&=-1,\quad \text{by definition of } f \text{ and Proposition 4.1.}
\end{aligned}
$$

Case 2: $\mathbf{y}: w=\mathcal{A}, s\neq\sigma_{\ell'}-1\quad \forall\ell'\geq\ell$

In this case, the policy choose to rest and $\theta^{\text{opt}}(\mathbf{X}_k)=\psi^{m,\tau_m^*}(\mathbf{y})$. The proof is exactly the same as previous step.

Case 3: $\mathbf{y}: w=\mathcal{A}, s=\sigma_{\ell'}-1\quad \ell'\geq\ell$

In this case, θ^{opt} can either choose to rest (same as Case 2) or work on queue with threshold value $\sigma_{\ell'}$. For the later case, (WLOG assume work on a type i task with $\tau_i^*=\sigma_{\ell'}$), the evolution of server state is exactly the same as $\overline{\mathbf{Y}}^{\psi^{i,\tau_i^*}}$ Then, from Lemma A.14,

$$
\begin{aligned}
&\mathbb{E}\left[-\sum_{j\in\mathbb{T}}\frac{\mu(S_k,j)\theta^{\text{opt}}(\mathbf{X}_k)_j}{\overline{\nu}_j^*}+f(\mathbf{Y}_{k+1})-f(\mathbf{Y}_k)\mid \mathbf{X}_0=\mathbf{x}_0,\mathbf{Y}_k=\mathbf{y}\right]\\
&=\mathbb{E}\left[-\sum_{j\in\mathbb{T}}\frac{\mu(\tau_i^*-1,j)\psi^{i,\tau_i^*}(\mathbf{y})_j}{\overline{\nu}_j^*}+f(\overline{\mathbf{Y}}_{k+1}^{\psi^{i,\tau_i^*}})-f(\overline{\mathbf{Y}}_k^{\psi^{i,\tau_i^*}})\mid \overline{\mathbf{Y}}_k^{\psi^{i,\tau_i^*}}=\mathbf{y}\right]\\
&=-r(\mathbf{y},\psi^{i,\tau_i^*})+\Delta f(\overline{\mathbf{Y}}^{\psi^{i,\tau_i^*}};\mathbf{y})\\
&=-1.
\end{aligned}
$$

We plug the result back in (A.39) to get

$$(A.39) = -\Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell} \mid \mathbf{X}_0 = \mathbf{x}_0) + P(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max}. \tag{A.40}$$

From (A.35) - (A.40), we have

$$\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q_{T'}^i - Q_0^i}{\overline{\nu}_i^*} + f(\mathbf{Y}_{T'}) - f(\mathbf{Y}_0) \,\Big|\, \mathbf{X}_0 = \mathbf{x}_0\right] \leq \sum_{k=0}^{T'-1}\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell} \mid \mathbf{X}_0 = \mathbf{x}_0) + \Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = \mathbf{x}_0) f_{\max}\right). \tag{A.41}$$

Recall $\sum_{i\in\mathbb{T}}(\lambda_i/\overline{\nu}_i^*) < 1$. In addition, $\Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell} \mid \mathbf{X}_0 = \mathbf{x}_0)$ converges to 1 (and, hence, $\Pr(\mathbf{Y}_k \in \mathbb{Y}_{\sigma_\ell}^C \mid \mathbf{X}_0 = \mathbf{x}_0)$ goes to 0) as $k \to \infty$ (as long as T' grows accordingly) from our earlier discussion. Thus, for all sufficiently large T', the sum of the terms inside the parentheses is negative. This implies that, as $T' \to \infty$, (5.40) goes to $-\infty$. As a result, for every $\ell \in \{1, \ldots, L\}$, we can find a finite T_ℓ such that, for every state $\mathbf{x}_0^\ell$ with $q_m = 0$, all the type i queues with $\tau_i^* < \sigma_\ell$ satisfy $\max_{i:\tau_i^*=\sigma_{\ell'}} q_i \leq 2\ell' T_\ell$ for all $1 \leq \ell' < \ell$ and , and all queues with τ_j^* equals σ_ℓ satisfy $\max_{j:\tau_j^*=\sigma_\ell} q_j > 2\ell T_\ell$ such that

$$\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q_K^i - Q_0^i}{\overline{\nu}_i^*} + f(\mathbf{Y}_K) - f(\mathbf{Y}_0) \mid \mathbf{X}_0 = \mathbf{x}_0^\ell\right] \leq \sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1 \text{ for all } K \geq T_\ell. \tag{A.42}$$

We are ready to prove that the functions V and g satisfy (A.33) when the parameter T is chosen to be $\max_\ell T_\ell$. To this end, we consider following cases separately.

- **For $\mathbf{x} : V(\mathbf{x}) \leq N$:** From the given function g in (A.30), when $V(\mathbf{x}) \leq N$, $g(x) = 1$. Thus, from the assumed Lyapunov function in (A.28),

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&\leq a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max}\right) \\
&= -1 + \left[1 + a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} + f_{\max}\right)\right] \\
&= -g(\mathbf{X}) + M.
\end{aligned}$$

- **For $\mathbf{x} : V(\mathbf{x}) > N, q_m > 0$**

In this set of states, $\theta^{\text{opt}}(\mathbf{x})$ either follows $\psi^{m,\tau_m^*}(\mathbf{y})$ or it can also work on a type i ($i \neq m$) task if $s = \tau_i^* - 1$. Furthermore, $g(\mathbf{x}) = 1$ because $q_m > 0$.

Case $\theta^{\text{opt}}(\mathbf{x}) = \psi^{m,\tau_m^*}(\mathbf{y})$

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\left(\sum_{j\in\mathbb{T}} \frac{\lambda_j}{\overline{\nu}_j^*} - \sum_{j\in\mathbb{T}} \frac{\mu(s,j)\psi_j^{m,\tau_m^*}(\mathbf{y})}{\overline{\nu}_j^*} + \Delta f(\mathbf{Y};\mathbf{y})\right) \\
&= a\left(\sum_{j\in\mathbb{T}} \frac{\lambda_j}{\overline{\nu}_j^*} - r(\mathbf{y}, \psi^{m,\tau_m^*}) + \Delta f(\mathbf{Y};\mathbf{y})\right) \qquad \text{(A.43)}
\end{aligned}$$

where the last equality follows directly from the assumed reward function in (A.27). Note that the sum of the last two terms inside the parentheses is equal to -1 from definition of f and Proposition 4.1. Using (A.29), we obtain

$$(\text{A.43}) = a\bigg(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1\bigg) = -T \leq -1 = -g(\mathbf{x}).$$

Case $\theta^*(\mathbf{x}) = \psi^{i,\tau_i^*}(\mathbf{y})$ **and** $s = \tau_i^* - 1$

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+1}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\bigg(\sum_{j\in\mathbb{T}} \frac{\lambda_j}{\overline{\nu}_j^*} - \sum_{j\in\mathbb{T}} \frac{\mu(s,j)\psi_j^{i,\tau_i^*}(\mathbf{y})}{\overline{\nu}_j^*} + \Delta f(\mathbf{Y};\mathbf{y})\bigg) \\
&= a\bigg(\sum_{j\in\mathbb{T}} \frac{\lambda_j}{\overline{\nu}_j^*} - r(\mathbf{y}, \psi^{i,\tau_i^*}) + \Delta f(\mathbf{Y};\mathbf{y})\bigg) \qquad (\text{A.44})
\end{aligned}$$

From Lemma A.14, the sum of the last two terms inside the parentheses is equal to -1. Therefore,

$$(\text{A.44}) = a\bigg(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1\bigg) = -T \leq -1 = -g(\mathbf{x}).$$

- **For** $\mathbf{x} : V(\mathbf{x}) > N$**,** $q_m = 0$ Recall that we assume we have L distinct τ_i^*. Also, σ_ℓ for $\ell = 1, \ldots, L$ denote all distinct τ_i^* for queues (WLOG assume σ_ℓ is increasing sequence). We partition the case where $\mathbf{x} : V(\mathbf{x}) > N$, $q_m = 0$ into L cases. First, for $\ell \in \{1, \ldots, L-1\}$,

Case ℓ: For $\mathbf{x} : V(\mathbf{x}) > N$, $q_m = 0$, $\max_{j:\tau_j^* = \sigma_{\ell'}} q_j \leq 2\ell' T$ for all $1 \leq \ell' < \ell$, and $\max_{j:\tau_j^* = \sigma_\ell} q_j > 2\ell T$ **:** From the given function g in (A.30), $g(x) = T$. From the inequality in (A.42),

$$
\begin{aligned}
&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= \mathbb{E}[V(\mathbf{X}_{k+T}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x}) \\
&= a\,\mathbb{E}\left[\sum_{i \in \mathbb{T}} \frac{Q_{T+k}^i - Q_k^i}{\overline{\nu}_i^*} + f(\mathbf{Y}_{T+k}) - f(\mathbf{y}) \mid \mathbf{X}_k = \mathbf{x}\right] \\
&\leq a\left(\sum_{i \in \mathbb{T}} \frac{\lambda_i}{\overline{\nu}_i^*} - 1\right) \\
&= -T \leq -g(\mathbf{x}).
\end{aligned}
$$

Case L: For $\mathbf{x} : V(\mathbf{x}) > N$, $q_m = 0$, $\max_{j:\tau_j^* = \sigma_{\ell'}} q_j \leq 2\ell' T$ for all $\ell' < L$

First, from the fact that $V(\mathbf{x}) > N$, we know

$$
\begin{aligned}
a\left(\sum_{j \in \mathbb{T}} \frac{q_j}{\overline{\nu}_j^*} + f(\mathbf{y})\right) &> 2a\frac{m(m+1)T}{\min_i \overline{\nu}_i^*} + a \max_{\mathbf{y} \in \mathbb{Y}} f(\mathbf{y}), \\
\sum_{j \in \mathbb{T}} \frac{q_j}{\overline{\nu}_j^*} &> 2\frac{m(m+1)T}{\min_i \overline{\nu}_i^*}, \\
\sum_{j \in \mathbb{T}} q_j &> 2m(m+1)T,
\end{aligned}
$$

We expand the summation by considering their τ_j^*,

$$\begin{aligned}\sum_{j:\tau_j^*<\sigma_L} q_j + \sum_{j:\tau_j^*=\sigma_L} q_j &> 2m(m+1)T,\\ \sum_{j:\tau_j^*=\sigma_L} q_j &> 2m(m+1)T - \sum_{j:\tau_j^*<\sigma_L} q_j,\\ \sum_{j:\tau_j^*=\sigma_L} q_j &> 2m^2T,\end{aligned}$$

because maximum of q_j with $\tau_j^* < \sigma_L$ is less than $2LT \leq 2mT$ by the assumption of this case. Therefore,

$$\max_{j:\tau_j^*=\sigma_L} q_j > 2mT \geq 2LT$$

Finally, From the given function g in (A.30), $g(x) = T$. From the inequality in (A.42),

$$\begin{aligned}&\mathbb{E}[V(\mathbf{X}_{k+g(\mathbf{x})}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x})\\ &= \mathbb{E}[V(\mathbf{X}_{k+T}) \mid \mathbf{X}_k = \mathbf{x}] - V(\mathbf{x})\\ &= a\,\mathbb{E}\left[\sum_{i\in\mathbb{T}} \frac{Q^i_{T+k} - Q^i_k}{\overline{\nu}^*_i} + f(\mathbf{Y}_{T+k}) - f(\mathbf{y}) \mid \mathbf{X}_k = \mathbf{x}\right]\\ &\leq a\left(\sum_{i\in\mathbb{T}} \frac{\lambda_i}{\overline{\nu}^*_i} - 1\right)\\ &= -T \leq -g(\mathbf{x}).\end{aligned}$$

□

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