# ABSTRACT

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Conventional risk analysis and assessment tools rely on the use of probability to represent and quantify uncertainties. Modeling complex engineering problems with pure probabilistic approach can encounter challenges, particularly in cases where contextual knowledge and information are needed to define probability distributions or models. For the study and assessment of risks associated with complex engineering systems, researchers have been exploring augmentation of pure probabilistic techniques with alternative, non-fully, or imprecise probabilistic techniques to represent uncertainties. This exploratory research applies an alternative probability theory, quantum probability and the associated tools of quantum mechanics, to investigate their usefulness as a risk analysis and assessment tool for engineering

problems. In particular, we investigate the application of the quantum framework to study complex engineering systems where the tracking of states and contextual knowledge can be a challenge. This study attempts to gain insights into the treatment of uncertainty, to explore the theoretical implication of an integrated framework for the treatment of aleatory and epistemic uncertainties, and to evaluate the use of quantum probability to improve the fidelity and robustness of risk system models and risk analysis techniques.

# ON ENGINEERING RISKS MODELING
# IN THE CONTEXT OF QUANTUM PROBABILITY.

By

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Dissertation submitted to the Faculty of the Graduate School of the
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# Dedication

This dissertation is dedicated to the loving memory of my father,

Joseph Yuk Lau Lee.

His lifelong pursuit of knowledge has been

and continues to be my inspiration.

# Acknowledgements

First and foremost, I would like to thank my advisor, Professor Gregory Baecher, for taking a chance on me and my ideas. Professor Baecher has patiently worked with me through all my strange ideas, joined me with the wild rides and took the various detours with me along the way. Without his guidance over the years, I would probably still be going around in circles. Though the years, I have learned a tremendous amount from him. Professor Baecher, thanks for sticking with me.

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My family has always served as my inspiration. To my parents, I am forever grateful for their unwavering support over the years while I switched back and forth between different pursuits. My brother, Patrick, and his family, for always being there during ups and downs. My son, Adrian, for letting me see things through different angles and perspectives. Witnessing my son growing up into a young adult has been my most rewarding experience. He has now matured into a fine young man, charting his own path at Brown. I would like to let him know how proud we are.

Finally, and most importantly, this work cannot be accomplished without the unconditional love and support from my wife. Thanks, Grace, for being my partner in life, sharing with me this and other fantastic journeys.  Let’s go and start again a new chapter!

# Executive Summary

***Introduction***

The representation of uncertainty in risk assessments often relies on the use of probabilistic methods. Most modern probabilistic methods follow the Kolmogorov formalism where the Kolmogorov axioms define the probability mathematical framework. Researchers have been exploring augmentation of pure probabilistic techniques with alternative, non-fully, or imprecise probabilistic techniques to represent uncertainties. Recognizing that contextual information, an essential component of the risk assessment framework, cannot be fully captured by pure probabilistic techniques, the risk research community has been continuously looking for ways to enhance our ability to capture more information and contextual knowledge in the Probabilistic Risk Assessment (PRA) process. This raises the challenge of how to reconcile or combine pure and imprecise probabilities.

For the study of risks associated with complex engineering systems, the exploration of different probability theories can be of great interest since probability plays an important role in most risk assessment techniques. PRA or quantitative risk assessment techniques have been used for 30 plus years. While PRA techniques based on conventional probabilistic treatments, such as Kolmogorov axiomatic, Frequentist, or Bayesian treatments, have been widely and successfully used to investigate and model many complex engineering problems, the diversity in the type of engineering problems, problem conditions, and characteristics often require tailoring different specialized techniques. Modeling complex engineering problems with pure probabilistic approach can encounter challenges, particularly in cases where contextual knowledge and information are needed to define probability distributions or models.

In this dissertation, we applied an alternative probability theory, *quantum probability* and the associated tools of quantum mechanics, to investigate their usefulness as a risk analysis and assessment tool for engineering problems. In particular, we would like to investigate the application of the quantum framework to study complex engineering systems where the tracking of states and contextual knowledge can be a challenge. *In this research, we attempt to gain insights into the treatment of uncertainty with an alternative probability framework. The framework is based on geometric objects to model systems, events, and uncertainties. This dissertation explores the use of quantum probability and the tools of quantum mechanics to model complex engineering systems for risk assessments.*

***Organization of the Dissertation***

Chapter 2 starts with a general overview of PRA methodologies, from conventional pure probabilistic frameworks to their extension to incorporate imprecise probabilities into current techniques. An alternative probabilistic framework, quantum (von Neumann) probability, is introduced as a possible bridge between pure and imprecise probabilities. In Chapter 3, using the quantum framework, a simple experimental model is constructed to represent a complex engineering system, namely a levee-flood wall storm protection system. Chapter 4 provides a short primer on quantum probability and quantum mechanics. Chapter 5 takes simple models from the earlier chapter and extends them to create composite models to describe complex engineering systems. The models are then utilized to explore how this quantum framework offers new capabilities and insights to augment existing PRA methodologies. In Chapter 6, the quantum techniques are compared to a number of current techniques such as fragility curves and event trees for new insights. Chapter 7 takes a look on the interpretation of the quantum approach in modeling engineering systems and assessing risks. In Chapter 8, 9, and 10, these chapters explore how the quantum framework can be applied to a variety of risk related problems, from the modeling of heterogeneous engineering systems, the combination of probabilities for concurrent failure modes, to modeling scheduling risks. Finally, future directions are discussed in the closing chapter of this dissertation (Figure E1).

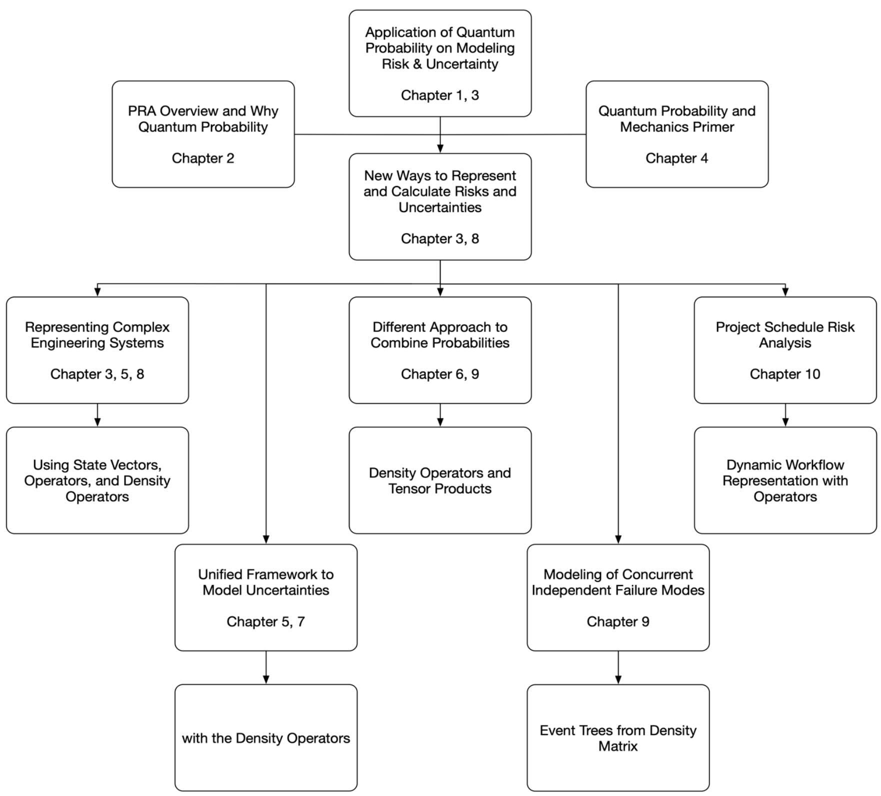


Figure E1. Roadmap for this dissertation.

***The Role of Probability in PRA and the Current Probabilistic Approach in PRA***

Current PRA techniques generally consist of the following three elements: 1) sets of scenarios, which can be physics or probability models to represent events and engineering systems (simple or composite), 2) the frequency of occurrences of the events associated with the scenarios, usually in the form of probabilistic models for events, and 3) the consequences associated with the occurrences of the events, which can be numerical values, event triggers, event sequences, or impact statements. Uncertainties are quantified using probabilistic distributions and models. Contextual knowledge and information about the events, the sequences, the configurations, the interactions, the physical processes, and the parameter spaces are essential ingredients for the quantification of uncertainties, serving as the glue that connect the probabilities together.

One of the critical PRA elements is the use of probability to characterize event occurrences, describe physical processes, and to represent and model uncertainties. Those physical processes have intrinsic variability (randomness). The use of probabilistic models to describe stochastic processes (referred to as "aleatory" uncertainty) is a central element of PRA. Beyond that, risk scenarios used in the representation of the physical problem introduce yet another type of uncertainty, referred to as epistemic uncertainty, which is a reflection of the completeness of our knowledge about system behavior, models, and modeling parameter. The epistemic uncertainty is a quantification of the degree of knowledge or the state-of-knowledge of the fidelity of the models, modeling parameters, and assumptions in representing the reality of the relevant physical processes and the systems' behaviors as defined by the conditions. The challenge is on the mapping or association of uncertainty models to a probability distribution. Contextual information and knowledge play critical roles, but the amount of information that can be encoded in the probabilistic models is limited by the mathematical and probability frameworks.

The concept of probability itself is not absolute or definitive but subject to interpretation. There are a number of interpretations of probability and a number of different theories of probabilities, from axiomatic formulations such as Kolmogorov probability, to Cox's logical probability, and to imprecise probability such as that of Dempster-Shafer. The choice of which probability interpretation and theory to use in modeling uncertainty can greatly affect the outcome of the PRA results and conclusions (Table E1). Since the states of the system being modeled can change as a function of time, there is also a differentiation between static and dynamic techniques.

Quantitative techniques rely on probability models. Conventional probabilistic methods may encounter limitations in the capturing and incorporation of contextual knowledge in risk analysis, which can impose limitations when dealing with ambiguity, defined as the possibility of different interpretations for a result based on the availability of contextual knowledge. If analysis scenarios yield identical probabilities in an ambiguous state, the proper interpretation of the results might require additional contextual information and knowledge because the scalar probabilities do not capture or later retain the information necessary to support complex decision processes.

Table E1: Different treatments of probability theory.

| Treatment/ Interpretation | Representative works by | Principles |
|---|---|---|
| Classical | Cardano, Pascal, Fermat, Bernoulli, Laplace | Principle of indifference, assign equal probability to events |
| Frequentist | Mills, Ellis, Cournot, Fries, Venn, Bernoulli, Gauss, Laplace, Fisher, Neyman, Pearson. | Assign event probability based on the frequency of occurrence in a large number of trials. |
| Subjective/ Evidential/ Bayesian | Bayes, Laplace, de Finetti, Jeffreys, Wald, Savage, Ramsey | Bayes Theorem; Bayesian updates; subjective and a reflection of degree of confidence on the occurrence of events; degree of belief |
| Physical/ Propensity | Pierce, Popper | Physical disposition or propensities of events |
| Modern Axiomatic | Kolmogorov, Cox, Jaynes | Kolmogorov Axioms, Measure Theory, Cox Theorem and postulates - propositional logic based; Plausible reasoning |
| Logical Interpretation | Johnson, Keynes, Jeffreys, Carnap | Degree of confirmation based on empirical evidence leading to a proposition |
| Measure-theoretic | Borel, Lebesgue | Mixing of discrete and continuous probability distribution for event assignments. Instead of working with cumulative probability distributions, works with probability measures, which is based on Measure Theory |
| Quantum/Dirac-von Neumann | Dirac, von Neumann | von Neumann noncommutative measure theory, Noncommutative analog of Kolmogorov Probability, Dirac quantum mechanics |
| Information Geometry | Amari, Nagaoka | Application of differential geometry to model probability distributions by mapping the distributions to Riemann manifold, resulting in the creation of a statistical manifold. |
| Imprecise | Boole, Keynes, Walley, Dempster, Shafer | Introduce non-fully probabilistic ideas and frameworks, the use of Dempster-Shafer theory of belief functions, fuzzy sets, evidence theory, possibility theory, interval probabilities, probability-boxes (p-boxes), etc. |

Further complications arise when the problem involves complex decision chains, where ambiguous upstream decisions affect the downstream outcomes (Figure E2). The workflow is a reduction process, where contextual information is reduced as a result of the parametric abstraction and the computation of numerical probabilities. Elements of uncertainty and ambiguity are introduced along the way. The risk analysis workflow removes information, and uncertainty may not be reduced.

This limitation affects the formulation of risk questions and scenarios, affecting the interpretation of results. The information loss also introduces additional uncertainties into the process. One can see that the reduction of contextual information, which is a form of knowledge, can introduce additional epistemic uncertainties.  The consequence can lead to the increase of uncertainty and risk. New techniques that can counter the information loss may improve the quality of the analysis results.

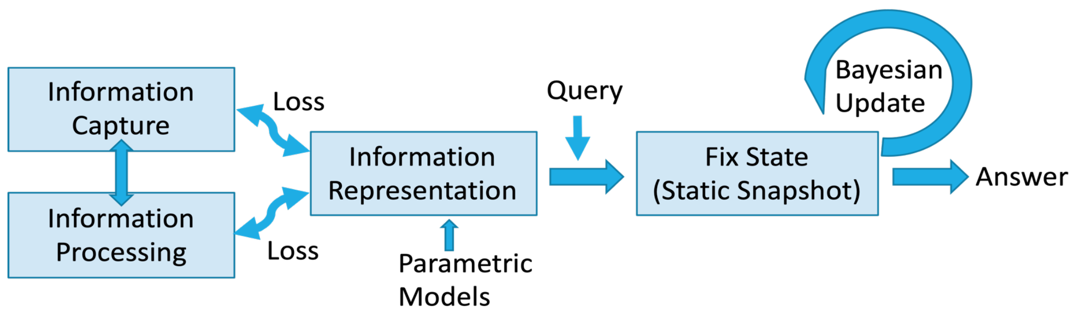


Figure E2. The general risk analysis workflow.

To summarize, current PRA techniques incorporate knowledge by overlaying additional structures on top of probabilities (e.g. event tree, decision tree, fault tree, Bayesian network, etc.). These techniques limit what and how much information can be encoded in the model. Furthermore, at each state selection or transition, information loss occurs. These limitations can potentially be addressed by having additional structures built into the probability framework itself to encode additional information.

Such a framework may align better with how a system behaves and how we track system states. Information can be captured internal to the structure itself and incorporate ignorance. Information loss can be reduced, uncertainty and ambiguity can be handled, computational efficiency can be enhanced. *This dissertation explores* *the use of quantum probability and the tools of quantum mechanics to increase the information encoding in risk models and to improve risk analysis techniques.*

***A Short Quantum Primer***

In the quantum framework, a system can be represented by the *superposition* of all possible system states, in the form of a vector. While for example a system can be working or not working, in the quantum framework it is described as the superposition of both states. If the working state is represented by the symbol $|1\rangle$ and the not working state is represented by the symbol $|0\rangle$, then the system, S, can be described by a *quantum state vector*, labelled by the symbol $|S\rangle$, as a vector sum (linear combination) of $|0\rangle$ and $|1\rangle$, where $|0\rangle$ and $|1\rangle$ are the two basis states. Probabilities of finding the system in the different states connect the states via probability amplitudes (to be explained below), which are the components of the state vector.

If system L is a levee that can either be working $|1\rangle$ or not working $|0\rangle$ at a given time, the state vector for L is expressed as:

$$|L\rangle = l_0|0\rangle + l_1|1\rangle \qquad \text{(E.1)}$$

where $l_0$ and $l_1$ are the probability amplitudes, and the magnitude (length) of $|L\rangle$ is 1. In the quantum framework, probability is given by the square of the probability amplitudes which is known as the Born Rule. The probability of finding the system L in a working state is denoted by $P(|1\rangle)$ and the probability of finding it not working is

$P(|0\rangle)$ which is equal to $1-P(|1\rangle)$; the total probability is conserved. As an example to provide numerical context, let the probability of finding system L not working, $P(|0\rangle)$, be 0.01. Correspondingly, the probability of finding L working is $1-P(|0\rangle)$, which is 0.99. The probability amplitudes, $l_0$ (for $|0\rangle$) and $l_1$ (for $|1\rangle$), are therefore the square root of the probabilities. For system L, $|L\rangle$: $|L\rangle = l_0|0\rangle + l_1|1\rangle$ where $l_0 = \sqrt{0.01}$, $l_1 = \sqrt{0.99}$ ■

As a vector, many different types of vector operations can be performed on these quantum state vectors. The operations can: 1) change the state of a system, 2) create new systems by combining different systems (they then become subsystems of the new system), 3) extract information about system states, and 4) extract information about measurable system parameters. Additional vector properties are available to describe complex events and systems. Vectors when expressed as matrices, carry matrix properties among which is the non-commutative nature of matrix operations. When applying the sequence of operations "A and B" on a state vector, the resulting event might be different than those obtained from the sequence of operations "B and A". In this example, A and B are operators, representing events and actions, which act on state vectors to transition those vectors from one state to another. The operation A on a system S is expressed as $|S'\rangle = \mathbf{A}|S\rangle$, where **A** is the operator acting on the system and $|S'\rangle$ is the system after the operation. Different types of operators can represent a variety of actions, such as modeling a measurement operation, extracting information from a state vector, or an event causing the system to transition into another state. The quantum framework provides a rich set of tools and operations to support different modeling needs.

The quantum framework has additional features that can describe systems based on the amount of knowledge (full or partial) one possesses. One possesses full knowledge about a system when one knows the state it is in. The state vector describing such a fully known system is called a *pure state*. When one possesses only partial knowledge, the state vector is called a *mixed state*. In this condition the system is a *mixture* of the different possible states. As one does not have enough information to precisely identify the state, it can only be described by the range of possible states in which the system could be found.

The *density operator* is a linear combination of the different possible states of the system. The density operator is used to construct the mixed state quantum representation. The density operator $\rho$ is expressed as:

$$\rho = \sum_i p_i |S_i\rangle\langle S_i| \qquad \text{(E.2)}$$

where $|S_i\rangle$ is the i$^{th}$ state (vector) of the system S, and $p_i$ is the probability of finding the system S in its i$^{th}$ state. $|S_i\rangle\langle S_i|$ is a special case of the tensor product, called the *outer product*, a product of two vectors forming a linear operator; based on the context of what it is constructed for, this type of operator can be utilized to encode information about a system (the density operator), to extract information from a system (the projector), or to alter the state of a system. The density operator can be expressed as a matrix with a specified basis, and as such, the term density operator and density matrix are often used interchangeably.

The density operator formalism is effective in describing a wide range of system properties and the amount of information one has about the system. The pure state is simply a density operator with a single term: $\rho = |S_1\rangle\langle S_1|$ as the system is known precisely to be in this state, and the probability of finding the system in that state is 1.0. The mixed state system is described as a statistical sum:

$$\rho = p_1|S_1\rangle\langle S_1| + p_2|S_2\rangle\langle S_2| + p_3|S_3\rangle\langle S_3| + \dots, \qquad \text{(E.3)}$$

which reflects lack of precision and therefore the description is a mixture of possible states, $|S_1\rangle, |S_2\rangle, \dots, |S_n\rangle$. The number of components can provide a measure of how much or how little one knows about the state of the system. The longer the sum, the larger the possible number of states, and the larger the uncertainty. However, if one cannot fully establish the completeness of the system states, as in the case of deep uncertainty with "unknown-unknowns", then the expression might take the different form:

$$\rho = p_1|S_1\rangle\langle S_1| + p_2|S_2\rangle\langle S_2| + p_3|S_3\rangle\langle S_3| + \dots + p_n|D\rangle\langle D|, \qquad \text{(E.4)}$$

where $p_n|D\rangle\langle D|$ corresponds to the unknown-unknowns. The density operator formalism opens up new opportunity to model risk problems.

The density operator formalism increases the amount of information that can be encoded in the system models. In current approaches, uncertainties are encoded in a single set of probabilities. In the density operator formalism, uncertainties are encoded with two sets of probabilities. State vectors (quantum) contain one set of probabilities in the form of probability amplitudes, while a second set of probabilities (classical) describes the distribution of the states. This formalism provides additional bandwidth in encoding system information and more information can be captured and thereby potentially increase fidelity.

The two sets of probabilities are interpreted to represent aleatory and epistemic uncertainties, respectively (Table E2). Aleatory uncertainty deals with the inherent, intrinsic random stochastic variations associated with a physical system. Epistemic uncertainty reflects the lack of knowledge and information about some properties and characteristics of a system. Human judgement and belief reflect the use of subjective knowledge and judgement (or bias) in the formulation of designs, opinions, and decisions, which can increase or decrease the accuracy regarding the "true" representation of a system.

Table E2. Quantum representations for the different types of uncertainty.

| **Uncertainty** | **Quantum Representation** |
|---|---|
| Aleatory | $\vert S\rangle = s_0\vert 0\rangle + s_1\vert 1\rangle \Leftrightarrow \rho = \vert S\rangle\langle S\vert$ |
| Aleatory + Epistemic (physics) | $\vert S'\rangle = \mathbf{H}\vert S\rangle \Leftrightarrow \rho = \vert S'\rangle\langle S'\vert$ |
| Aleatory + Epistemic (physics)<br>+ Epistemic (beliefs) | $\rho = \sum_i p_i \vert S_i\rangle\langle S_i\vert$ |

The terms in the density operator $\rho = \sum_i p_i \mathbf{H}_i \vert S_i\rangle\langle S_i\vert \mathbf{H}_i^\dagger$ can be interpreted and mapped to the three types of uncertainties according to the following:

$\vert S_i\rangle$ — The probabilities encoded by the probability amplitudes within this *i-th* state vector represent the aleatory uncertainties. The probabilities correspond to the inherent intrinsic random stochastic variations associated with a physical system's states.

$\mathbf{H}_i$ — The operator, itself a model representing some physical processes, represents the moderator or modulators of *uncertainties (epistemic)*. The operator acts on the state vector, resulting in a change of state in the form of a change of the probability amplitudes; the probabilities derive from the state vectors post operation encapsulate both the aleatory uncertainties and the model based epistemic uncertainties. The contextual knowledge about the sequential behaviors of the system can be incorporated into a coherent mathematical framework.

$p_i$ — This term represents the probability of finding the system in the *i-th* state. If we have full knowledge of the system, then there is only one term with $p = 1$, and the density operator contains a single state vector. In the case with partial knowledge, the $p_i$ will be a probability distribution satisfying the conservation of total probability ($\sum_i p_i = 1$). This term is another component of *epistemic uncertainties* corresponding to the certainty of knowing the precise state of the system.

***Risk in the Quantum Context***

Risk is often defined as the probability of the risk event multiplied by the magnitude of the consequence or impact: Risk = Probability × Consequence. In this definition, risk is an event associated with a probability and a consequence. The uncertainty is on the occurrence of the event and a probability value is assigned to the event with the specific outcome. The total risk is the expected loss due to the risk event. Instead of asking for the probability of the occurrence of event A, the quantum approach now asks for the probability of finding the system in a particular state. This reframing of the question leads to the following definition of risk states, risk, and the risk system:

- A *risk state* is a system state that can potentially impact, positively or negatively (risk vs. opportunity), the outcome of a system event.
- *Risks* are the probabilities of finding the system in risk states, couple with the significance of the consequence associated with those states.
- *The risk system* is that collection of possible system states that can have impacts (usually negative) to the outcome as specified by the risk analyst, who then derives some quantities to represent the degree of severity for those states.

Risk assessment obtains the probability of finding certain risk states (relevant to the question) out of all possible states (the complete set of states), assign to them scalable factors to represent the degree of significance, to be used for making decisions. The numerical valuation or the magnitude of a risk is contextually driven and can be subjective.

These concepts are mapped to elements of the quantum framework according to the following. Associated with each risk system is an observable called the *risk value* denoted by the operator **Ri**.

- Performing the measurement **Ri** on a system's basis state, yields the scalar *risk value* representing the significance of the consequence or impact.

If $\left|\lambda_i\right\rangle$ is a failure state, the result of the measurement is **Ri** on the basis state is $\lambda_i$ corresponding to a numerical value representing the significance of the consequence. This measurement is denoted by

$$\mathbf{Ri}\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle. \qquad \text{(E.5)}$$

For example, if a risk analyst assigns a real number value of 5 to the most significant failure state and 1 to the least significant state, then for the trivial case of a system with a binary working $\left|1\right\rangle$ and not working $\left|0\right\rangle$ states, $\mathbf{Ri}\left|0\right\rangle = 5\left|0\right\rangle$ and $\mathbf{Ri}\left|1\right\rangle = 1\left|1\right\rangle$.

Recall that risk, that is, the probability of finding a system in a certain state $\left|\psi\right\rangle$ follows the Born Rule:

$$P(\left|\psi\right\rangle,\left|\phi\right\rangle) = \left|\left\langle\psi\middle|\phi\right\rangle\right|^2 \qquad \text{(E.6)}$$

where $\left|\phi\right\rangle$ is the given system state. The probability of the system $\left|A\right\rangle$ in a non-working state can be obtained with

$$P(\left|0\right\rangle) = \left\langle A\middle|0\right\rangle\left\langle 0\middle|A\right\rangle = \left|\left\langle A\middle|0\right\rangle\right|^2 = a_0^2. \qquad \text{(E.7)}$$

The risk value is the expectation of the observable *Ri*, for a system in state $\left|A\right\rangle$ it is denoted by $\left\langle Ri\right\rangle$ which is given by

$$\left\langle Ri\right\rangle = \left\langle A\right|\mathbf{Ri}\left|A\right\rangle. \qquad \text{(E.8)}$$

The density operator ($\rho$ from Table E2) represents the system, capturing the combination of aleatory, epistemic, and belief uncertainties.

- Risk analysis is the process of identifying corresponding risk states from the density matrix, calculating their chances of occurrence, and associating with them values of consequence.

To obtain the risk value, a measurement **Ri** can be performed to obtain the expectation:

$$\langle Ri \rangle = tr\left(\rho \mathbf{Ri}\right) \tag{E.9}$$

in which $\rho$ is the density operator and the righthand side is simply the trace of the product of two matrices $\rho$ and **Ri**.

***Modeling Different Types of Problems***

In this research, the quantum framework was applied to model different types of engineering system and problem. Various quantum probability and quantum mechanical constructs were introduced to build foundation concepts: concepts such as how to represent and model a single system in binary states, how to perform measurements, how to construct complex systems, and how these concepts can be applied to model real engineering.

*Simple (homogeneous) systems* — Simple levee and flood wall systems were constructed using state vector representations. Operators modeling different events were constructed to describe situations that can alter the state of the systems. Composite systems of levee + flood wall were constructed using tensor product operations to combine different individual systems together. Composite models and operators were then used to derive and reproduce results from conventional probabilistic techniques, such as fragility curves and event trees. Comparing the results of conventional and quantum techniques, besides reproducing the results as the conventional approach, the results were already encoded in the model itself. Moreover, the rule to compute probabilities, the Born Rule, was itself encoded in the density matrix.

*Heterogeneous systems* — A heterogeneous system, in this case the Hurricane Protection System (HPS), was modeled. A model suitable for the investigation of this complex heterogeneous system was constructed using the density operator quantum framework. The HPS (before Katrina) and not the HSDRSS (post Katrina) was chosen as a reference system for the model since analyses done by the Interagency Performance Evaluation Task (IPET) Force and others are readily available for comparison. Given the scope of this dissertation, a direct one-to-one comparison between the quantum model with the HPS studies by IPET is not feasible. Nonetheless, using a simplified HPS baseline and focusing on key attributes and properties, comparisons at the macroscopic level highlighted the differences between the techniques. The objectives for this case study were to illustrate how to model a heterogeneous system using the density operator formalism, to demonstrate how the quantum approach offers additional information about system risks over classic methods, and to compare and evaluate how the two approaches assess risks.

This case study evaluated the IPET and quantum productions of the fragility curves. At the macroscopic level, the fragility curves produced by the two approaches were compared. The reference IPET model with classic techniques provides a snapshot of the system in time and the quantum model traces the system evolution over time. Essentially, the two different fragility curves reflect the time behaviors of two populations of levees, with the IPET model similar to a collection of a homogeneous (or with less variations) population of levees and flood walls, and the quantum model a collection of heterogeneous (with variations over time) population. The quantum framework extends the classic framework by capturing and modeling various temporal events, which were not performed in the IPET models. The quantum fragility curves were constructed by extracting state information from the density matrices, with the system first subjected to various events and conditions. The density matrices tracked the system states over time via operators; the system was transitioned to the different "states" at different times with event operators, and the failure probabilities were thus "updated" after each transition. The fragility curve constructed for the period in question, therefore, took into account the evolution history of the system.

In comparisons, classic methods are similar to the modeling of a pure state, and the quantum approach models the HPS as a mixed state system. In the case of a pure state model, the failure probabilities are derived from a single state vector: $\rho = p_1 |S_1\rangle\langle S_1|$ ; whereas for a mixed state model, the failure probabilities are derived from a mixture: $\rho = p_1 |S_1\rangle\langle S_1| + p_2 |S_2\rangle\langle S_2| + p_3 |S_3\rangle\langle S_3| + \ldots$. Over time, various events altered different parts of the HPS, and introduced different mix of system states. The mixed state density operator acknowledges this and models the system as a mixture of states. For the case with the HPS, the historical events introduced new or altered existing system states with higher failure probabilities, such as degraded systems, new systems built below specifications, and systems weakened by annual weather events. When the different populations of states with higher failure probabilities were fully taken into account with the mixed state model, as expected, has a higher failure probability and the total failure probability rose earlier and faster. The incorporation of the historical events and earlier failure states shifts the quantum fragility curve according to the event-driven behaviors of the population. As the system evolves over time, uncertainty increases as the system move further and further away from pure state; the system ensemble population (a group of identical or highly similar levee-flood wall) changes over time, turning more and more into a statistical ensemble (a collection of groupings of levee-flood wall).

The quantum framework takes the classic techniques as a starting point and takes steps further by refining the risk questions and drilling further down beyond the system parameters and configurations. The $p_i$ terms are classic probabilities. The second set of probabilities in the state vectors extends the capability to encode information into the mathematical construct - the density operator. Furthermore, the construction of the density operator captures additional contextual information in the form of the different terms, such as $p_2|S_2\rangle\langle S_2|, S_3|S_3\rangle\langle S_3|, \ldots p_n|S_n\rangle\langle S_n|$, for the different possible states. This refinement drill deeper into the system behaviors and time evolution scenarios via event operators that model environmental effects, such as weather events, and further explore areas where the contextual questions can actively re-shape the modeling processes, such as the change of system specifications. The quantum models are sensitive to the amount of information and knowledge available, and how much one would like to apply (e.g. the number of terms in the density operator) to formulate the model at a sufficient level of details to arrive at the answers one seeks. Such framework that has the flexibility to handle both precise (pure states) and imprecise scenarios (mixed states) helps to extend the risk analysis and assessment process on the HPS. The density operator formalism is shown to be a capable framework that can be utilized to provide a coherent and concise mathematical structure to represent the risk states of complex engineering systems.

*Combining probabilities for concurrent failure modes* — One of the current challenges in risk assessment is the computation of total failure probability for a complex system with many different failure mechanisms. In engineering risk analysis, frequently we might be working with systems with more than a single failure mode, such as seismic activities can subject a spillway monolith to lateral forces of various magnitudes, leading to structural failures such as cracking, sliding, or overturning. Calculating the total probability of failure for engineering systems with multiple failure modes having order-dependent probabilities (e.g., overtopping and overturning) can be challenging. Assumptions and approximations have to be made to obtain total probabilities. In the study of dam failures for example, different failure modes under a single hazard for the different dam sections can make the calculation of the total failure probability challenging.

This problem about finding the total probability for systems with concurrent failure modes was analyzed with the quantum framework. The quantum formulation allows the integration of complex models and physical models in the form of state vectors, which reflect our knowledge about the relationships between the failure modes. Additional information, knowledge, and physical models are expressed in terms of the operators, connecting the physical problem with the change and evolution of failure probabilities.

The quantum framework provides a set of tools to differentiate the characteristics of system beyond a simple probability framework. Binary event trees can be constructed from the state vectors using tensor products. The quantum framework does not predefine the paths; rather, it maintains that all states are possible as a result of superposition, and the system evolves as events unfold. Probabilities are extracted at the end of the system evolution. The quantum framework will result in total probabilities consistent with the law of probability when framed properly. The quantum calculation relies on projection to extract information; one does not just sum but select what to "sum" via a purposeful extraction.

*Scheduling risk for software development* — Besides modeling physical engineering systems, the quantum framework was applied to analyze other classes of problems. A significant number of space missions planned by the (US) National Aeronautics and Space Administration (NASA) did not develop and launch on schedule, and software development has been identified as a major contributor to schedule delays and cost overruns which became risk drivers. Standard approaches to analyze schedule risks focus on the identification, quantification, and probabilistic representations of cost and schedule uncertainties. Often, current frameworks and techniques lack comprehensive perspectives on the incorporation of the continuous evolution of uncertainties, the transitions of the "system" from states to states, and the contextual information about the activities into the uncertainty models. The quantum approach realigns the scheduling risk modeling better to represent scheduling lifecycle behaviors, reflecting how different software development workflow methodologies and processes can dynamically change scheduling risks.

To explore this concept, a simplified and generalized software development scenario derived from the development lifecycle for a subsystem that is part of a current NASA flagship mission ground system was used as a case scenario. This software subsystem, refer to as W, consists of three components: A, B, and C. These components are developed by three separate teams from three different organizations at different geographical locations. In this scenario, the development of component A and B are in parallels, and the development of component C depends on the completion of A and B. This development process can be represented by the schedule network (Figure E3).

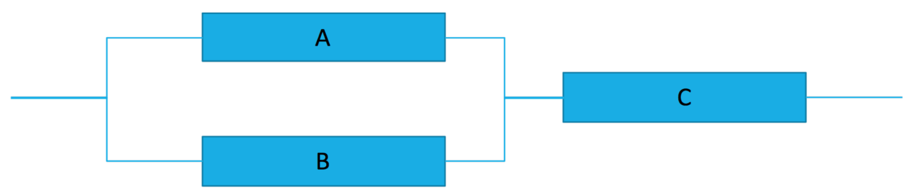


Figure E3: Schedule network for a software subsystem development scenario.

With the quantum approach, the schedule network can be thought of as a complex system with many different states where each state represents the developmental maturity of a software component. The quantum model represents the various possible states in which the schedule network can be found, and the quantum operators model the behaviors of the system, the change of states, and the dynamic schedule behaviors of the system components.

This analysis modelled different development lifecycle activities (methodologies), such as the development maturity lifecycle (typically follows a S-curve profile), agile type development lifecycle, and software fault rework with a simple operator tracing the time behavior for development maturity. Density operators representing the development states of component A, B, and C were constructed. A hybrid development approach, where different methodologies are mixed together, was incorporated into the construction of mixed states density operators. The density operators for the components were then used to form the schedule network density operator: $\rho_S = \rho_A \otimes \rho_B \otimes \rho_C$.

The model was analyzed to answer three different scenario questions, showcasing it robustness in handling a variety of situations and lines of queries.

- *All components have to be fully operational at launch. What is the probability of finding the system ready for launch?*
- *Only components A and C have to be fully operational at launch. B needs actual flight data at the stationary orbit to complete calibration for operation. What is the probability of finding the system ready for launch?*
- *Only components A and C have to be fully operational at launch. B needs actual flight data at the stationary orbit to complete calibration for operation, but B has to be 80% complete. What is the probability of finding the system ready for launch?*

The density matrix represents and captures all possible states of the system, which means all of the scenarios are simultaneously represented in the density matrix, and the system can be queried on multiple scenarios and perspectives at the same time, with the benefits of gaining procedural and computational efficiencies.

The quantum model supports more intricate modeling of the actual processes, events and activities. The software development methodologies can be directly modeled to evaluate how they can impact the chance of realizing the system on schedule. The density matrix representation allows some degree of impreciseness, which makes it possible to derive useful results without specifying precisely the exact type of methodology the development team employs. The model can capture a specific and well-defined workflow, but it also has the flexibility to handle hybrid development workflow methodologies.

The activity operator models the development lifecycle and is essentially modeling the development methodology. How the development methodology affects the schedule has never been fully incorporated in other schedule risk analysis model in this fashion; the most it does is to capture the effects of the methodology in affecting the probability distributions modeling the schedule duration. The change of perspectives enables the incorporation of additional elements such as workflow models. Workflow models to describe different development methodologies can be directly integrated into the process to "evolve" the probabilities. This allows additional degrees of freedom to test and evaluate constraints to improve schedule risk assessment. The direct incorporation of the "questions" in shaping the analyses aligns and reflects better with the dynamic nature of complex engineering systems, where traditional deterministic flow might not necessarily be true (e.g. event driven workflow).

***Conclusion and Future Directions***

In this dissertation, the theory of quantum probability and the associated tools of quantum mechanics were applied to investigate their usefulness as a risk analysis and assessment approach for engineering problems. This research investigated the application of the quantum framework to study complex engineering systems where the tracking of states and contextual knowledge can be challenging. This dissertation laid out the case for why the quantum framework provides new vantages with which to capture and encapsulate information in the risk system models; how quantum tools can monitor, change and evolve the system; and how these additional information and tools can support risk analysis and assessments.

This research demonstrated that the quantum framework can have robust and broad applications to different types of problems, beyond the study of risk for engineering systems. This research is a demonstration of concept, focusing on setup of a risk model with quantum probability and the quantum mechanics apparatus. The quantum framework is rich with features, and as a foundation framework, it has the potential to be used for modeling a wide range of problem types.

To realize the potential, significant additional research will need to take place beyond this dissertation. Follow on research might pursue any of three directions. One is to experiment with the above models to see how well they make predications for complex modern storm protection systems. A second is to follow up on the application of the quantum framework to study problems from other engineering domains, such as agile software engineering workflows and practices. The third is to continue to map out the theoretical quantum constructs and explore how they can be used to perform risk modeling.

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# Chapter 1: Introduction

*Uncertainty is inevitable at the frontiers of knowledge.*

*- Joel Achenbach 2015*

There have been many explorations for alternative probabilistic models in the representation of uncertainty in risk assessments. Most modern probabilistic methods follow the Kolmogorov formalism, where the Kolmogorov axioms define the probability mathematical framework. However, explorations on other probability formalisms, beyond Kolmogorov probabilities, have been rising in recent years in various communities, in risk and others. Researchers have been exploring other ways to represent uncertainty, from exploring other probabilistic interpretations (Flage, Aven, Zio, & Baraldi, 2014) to other new mathematical framework such as information geometry (Amari & Nagaoka, 2007; Amari, 2016) where probabilities are derived from differential geometry techniques. Others have explored the use of quantum (or Dirac-von Neumann) probability (Dirac, 1930, 1958; von Neumann, 1932, 1955), which itself came from the study of quantum mechanics in physics, in areas such as cognitive science (Busemeyer, Wang, & Townsend, 2006; Pothos & Busemeyer, 2013; Bruza, Wang, & Busemeyer, 2015). Indeed, over the past several decades there have been a wide range of interests in applying quantum probability in other disciplines such as finance, economics, game theory, decision support, psychology, and civil engineering (Piotrowski & Sladkowski, 2004; Darbyshire, 2005;

Khrennikov, 2010; Cheon & Tsutsui, 2006; Guo, Zhang, & Koehler, 2008; Pothos & Busemeyer, 2009, 2013; Lozada Aguilar, et al., 2017).

For the study of risks associated with complex engineering systems, the exploration of different probability theories can be of great interests since probability plays an important role in most risk assessment techniques. Probabilistic risk assessment (PRA) or quantitative risk assessment techniques have been used for 30 plus years, used by many such as the U.S. Nuclear Regulatory Commission (USNRC) (1975, 2009, 2018), the National Aeronautics and Space Administration (NASA) (Dezfuli, 2010; Stamatelatos, 2011), Bureau of Reclamation, US Department of the Interior, and the US Army Corps of Engineers (USDIBR/USACE) (2015) in assessing engineering risks. While PRA techniques based on conventional probabilistic treatments, such as Kolmogorov axiomatic, Frequentist, or Bayesian treatments, have been widely and successfully used to investigate and model many complex engineering problems, the diversity in the type of engineering problems, problem conditions, and characteristics often require tailoring different specialized techniques. Modeling complex engineering problems with pure probabilistic approach can encounter challenges, particularly in cases where contextual knowledge and information are needed to define probability distributions or models (Pedroni, Zio, Pasanisi, & Couplet, 2017).

Researchers have been exploring augmentation of pure probabilistic techniques with alternative, non-fully, or imprecise probabilistic techniques to represent uncertainties; Pedroni, et al. (2017) provides a review of the state of current probabilistic and non-fully probabilistic techniques in uncertainty treatment for engineering risk assessment. Recognizing that contextual information, an essential component of the risk assessment framework, cannot be fully captured by pure probabilistic techniques, the risk research community has been continuously looking for ways to enhance our ability to capture more information and contextual knowledge into the PRA process to help connecting the probabilities. The exploration of non-pure probabilistic approaches, as highlighted by Pedroni et al. (2017), points to one of several directions.

Between pure probability theories such as Kolmogorov probability and imprecise probability theories such as the Dempster-Shafer theory of belief functions (Dempster, 1967; Shafer, 2002), there exists a spectrum of different alternative probability theories and different interpretations of probability. In this dissertation, we applied an alternative probability theory, *quantum probability* and the associated tools of quantum mechanics, to investigate their usefulness as a risk analysis and assessment tool for engineering problems. In particular, we would like to investigate the application of the quantum framework to study complex engineering systems where the tracking of states and contextual knowledge can be a challenge.

Why quantum probability and quantum mechanics? Probability theories without interpretations are simply mathematical concepts. In order to apply these abstract mathematical concepts to model physical problems, they must be interpreted according to the context and the characteristics of the problem at hand. Researchers in a number of disciplines, notably in the area of cognitive and decision science (Busemeyer & Bruza, 2012), and engineering risk assessment (Pedroni et al., 2017) have identified a number of analysis problems where characteristics of the system in question cannot be fully captured by Kolmogorov probability. Quantum probability, with a different mathematical structure, can potentially offer different interpretive pathways and provides additional tools for modeling engineering risks.

From another perspective, Kolmogorov probability can be regarded as a scalar theory with probabilities map to scalar values and functions; whereas quantum or von Neumann probability can be regarded as a geometric theory with probabilities map to vectors and operators. *While people might be familiar with the concept of the risk vector, where the event probabilities and consequences are represented in the form of vectors, this is not to be confused with the geometric representation of probability itself that we are exploring here*.

Why adopting a geometric approach? The geometric approach introduces additional dimensions beyond scalar theories, adding new degrees of freedom and expanding the size of the parameter space available to capture additional information about the systems. There are properties and capabilities not available in common theories that can potentially improve and enhance current techniques in the treatment of order events in time, imprecisions, and ambiguities. The geometric approach has the potential to open new research avenues for exploring new techniques.

*In this research, we attempt to gain insights into the treatment of uncertainty with an alternative probability framework. The framework is based on geometric objects to model systems, events, and uncertainties. This dissertation explores the use of quantum probability and the tools of quantum mechanics to model complex engineering systems for risk assessments, and to increase the information encoding in risk models and to improve risk analysis techniques.*

The next chapter (Chapter 2) starts with a general overview of PRA methodologies, from conventional pure probabilistic frameworks to their extension to incorporate *imprecise probabilities* into current techniques. An alternative probabilistic framework, quantum (von Neumann) probability, is introduced as a possible bridge between pure and imprecise probabilities. In Chapter 3, using the quantum framework, a simple experimental model is constructed to represent a complex engineering system, namely a levee-flood wall storm protection system. Chapter 4 provides a short primer on quantum probability and quantum mechanics. Chapter 5 takes simple models from the earlier chapter and extend them to create composite models to describe complex engineering systems. The models are then utilized to explore how this quantum framework offers new capabilities and insights to augment existing PRA methodologies. In Chapter 6, the quantum techniques are compared to a number of current techniques such as fragility curves and event trees for new insights. Chapter 7 takes a look on the interpretation of the quantum approach in modeling engineering systems and assessing risks. In Chapter 8, 9, and 10, these chapters explore how the quantum framework can be applied to a variety of risk related problems, from the modeling of heterogeneous engineering systems, the combination of probabilities for concurrent failure modes, to modeling scheduling risks. Finally, future directions are discussed in the closing chapter of this dissertation (Figure 1).

Before closing this introductory chapter, the following section provides two brief case studies to illustrate why exploring different probability theories can be useful in the quest of improving risk management.

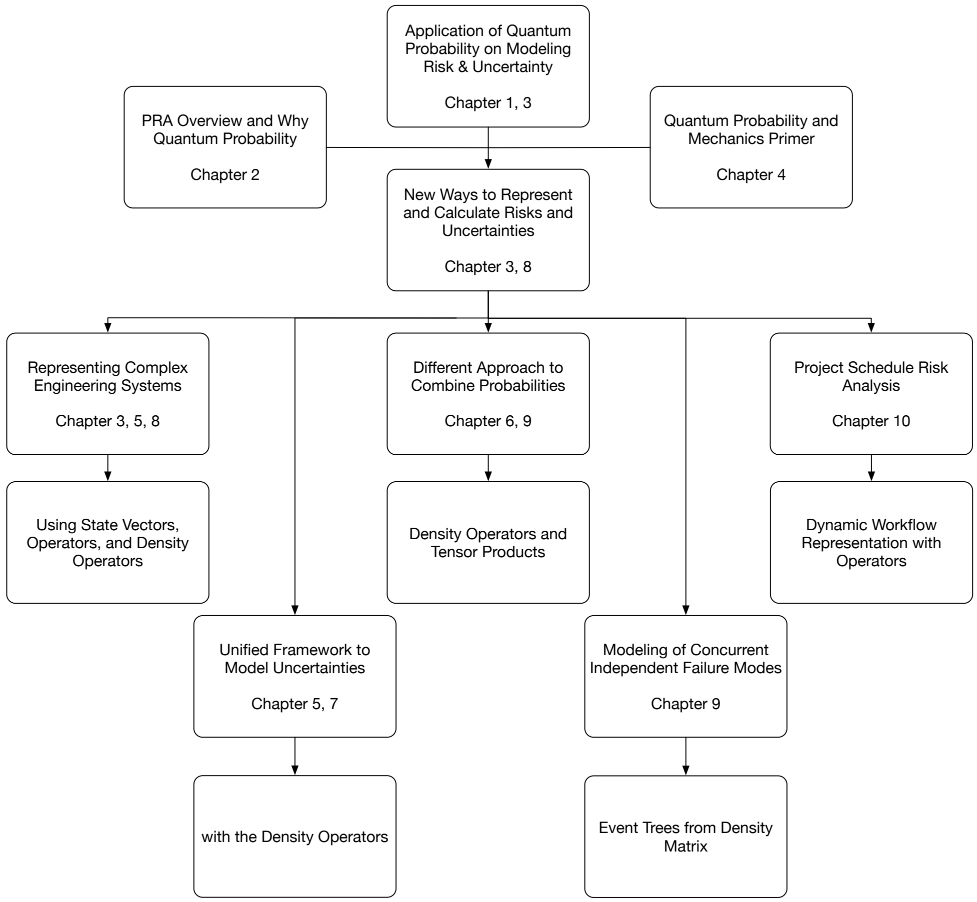


Figure 1: The dissertation road map.

## *1.1 Two Case Studies*

To illustrate the challenge of performing risk assessment with current techniques on these systems and how the assessments can underestimate the risk exposures, we will consider the failure case of the New Orleans Hurricane Protection System when Hurricane Katrina struck the area in 2005, and the James Webb Space Telescope Program.

*Case Study #1: Hurricane Katrina and the Hurricane Protection System* — The geography of the New Orleans region, which is a marshland below sea level surrounded by the Mississippi River, necessitates the construction of a system of levees and flood walls to protect the area from periodic high water and flooding. This system of levees and flood walls, built over a span of more than 50 years, forms the New Orleans Hurricane Protection System (HPS), now known as the Hurricane and Storm Damage Risk Reduction System (HSDRRS). On August 29, 2005, Hurricane Katrina struck New Orleans, resulting in severe flooding due to the overtopping and breaching of levees and flood walls. The catastrophe resulted in significant destructions of the region and the loss of more than 1800 human lives. Subsequent investigations and studies point to three areas of failure: 1) the engineering of the HPS; 2) the management of the project, risk, and uncertainties; 3) the human decision and communication processes.

Many investigations into the failure of the HPS during Hurricane Katrina identified a number of critical failure scenarios. One of the more comprehensive risk analyses was conducted by IPET over several years, and they issued a final report in 2009. In the comprehensive eight volumes report (IPET, 2009), the team utilized a number of techniques in their risk analysis methodology, including: 1) a Probabilistic Risk Model (PRM), 2) Event Trees, and 3) Risk Quantification (see Volume VIII of the IPET, 2009 report for full details). Probabilistic risk analysis was used as the foundation methodology, linking the different analysis elements together, to analyze the risk associated with the HPS.

Risk was misestimated for the HPS before Katrina, not because of a single reason, but a multitude of factors connecting and interacting with each other in intricate fashions that contribute to the mischaracterization of the state of the system. Prior to Hurricane Katrina, formal quantification of the risk for the HPS was never fully conducted, which is a significant problem by itself (ASCE, 2007). Imperfect knowledge about the HPS engineering system also played a role that can lead to ambiguous results when applying PRA methodologies to analyze the risk associated with the HPS.

The HPS levee system was built piece-wise over a long duration. As such, the HPS was a collection of heterogeneous subsystems. At the time of the storm, the levee/I-wall subsystems (for simplicity we will simply refer them as the levee system) were all in different states since they were all built at different times with different histories, design criteria, and specifications. Over time, the individual levee subsystem was subjected to different physical elements, and the collective system was difficult to model given the diverse evolutionary histories and the imperfect/incomplete knowledge of the subsystems (ASCE, 2007; IPET, 2009).

The core of this HPS problem can be illustrated with this simplified hypothetical example. Assuming that for a perfect homogeneous system with 10 levees all constructed at the same time, which we will call this system a "pure" state system, we have 10 identical levee systems. Within this population, they all have the same failure probability of 5%. Suppose now we have a heterogeneous system: half of levees (five of them) have 5% failure probability and the other five were damaged by previous storms, and their failure probability is 25%. For this system, we will call it a “mixed” state system since we now have a mixture of levees with different failure characteristics. A simplistic calculation for the probability of failure will give a higher number for the mixed state system over the pure state. The failure to account for the heterogeneous system states led to the underestimate of risks for the integrated system. How much knowledge we possess about the system and the knowledge about the evolutionary history of this system can affect the failure estimates.

One can argue that since a quantitative computational process, like the one described above, is available, one should be able to improve the assessments by simply develop a better model with the incorporation of more parameters, components, and conditions. This argument only works when the modelers have adequate knowledge of the situation, allowing her or him to fully parameterize and characterize the situation with the model, reducing the uncertainty to only those of the aleatory kind. However, the problem with “deep uncertainty” (Cox, 2012) challenges the notion of our abilities to truly reduce and eliminate epistemic uncertainties. Often when one cannot, one fills in the knowledge gaps with expert opinions and beliefs, which themselves further increase the uncertainties. Moreover, while the above simple thought experiment can be modeled, a realistic complex situation requires modelers to take into account system dynamics, changing system states, and more importantly changing knowledge, and the process is neither simple nor straightforward. Conventional PRA frameworks might not handle these situations comprehensively since probabilities do not tell the whole story (Cox, 2009, 2012; Dezfuli et al., 2011; Fenton & Neil, 2013; Aven & Zio, 2014; Flage et al., 2014; Aven & Cox, 2016; Pedroni et al., 2017).

To further compound the problem, risk also changes with time. In the case of HPS, the systems and the environment went through significant changes over the extended construction period. The subsidence of the floodwalls over time, identified as a key failure HPS failure mode, was not fully incorporated into the risk models; the design specifications did not provide temporal information about their long-term behaviors. As pointed out in the ASCE 2007 report, “The level of risk also changes with time, depending on changes in the natural and man-made environment. Therefore, the risk analyses need to be updated as new information becomes available.” While some of the latest probabilistic modeling methodologies, such as Bayesian Network which allows the incorporation of new data into the risk models, are addressing some aspects of extending the concept of uncertainty by incorporating a temporal dimension, fundamental concern about the ability to use probability to fully characterize the uncertainties associated with a complex dynamic system remains.

*Case Study #2: The James Webb Space Telescope Program* — For the second illustration, we consider the development of the James Webb Space Telescope (JWST) program. The JWST program is the current NASA flagship mission to succeed the highly successful Hubble Space Telescope (HST) Project. The JWST program is an ambitious mission to use some of the latest technologies to study the beginning of the universe. The telescope is scheduled to launch in 2021 and will fly to the 2nd Sun-Earth Lagrange point (L2). The telescope engineering system is complex, with many physical deployable components such as sun shields, 18 mirror segments, and adaptive optics. Furthermore, advanced event-driven artificial intelligence scheduling software systems are deployed on the ground and on the spacecraft for operations. A key challenge in the development and deployment of the system is the completeness of the testing of individual components and the integrated system, which is another example of a process dealing with incomplete knowledge, insufficient data, and long test cycle in order to assess the different scenarios and combinations for event driven operations.

The JWST software system consists of many parts and components working together to ensure the smooth operation and functioning of the space observatory. Modern software systems are complex, event-driven, multi-component, and highly interactive. For event-driven operations, there are many possible and different combinations of paths as well as many complex interactions between the components. Exhaustive testing of all the system pathways in the traditional sense can be impossible, since there are far too many possible perturbations and permutations to check out over finite time. Statistical sampling and simulation techniques such as Bayesian Inferences and System Dynamics are often employed to provide confidence limits to support risk-informed decision making (Dulac et al., 2007; Dezfuli et al., 2009; Dezfuli et al., 2011; Stamatelatos & Dezfuli, 2011). Similar challenges can also be found in other types of software systems and not limited to space systems.

Another challenging aspect of testing modern software systems has to do with the lack of test data. Space systems are highly integrated, and the interactions between the subsystems can only be tested with the conditions on the ground or with simulated space conditions. Until the spacecraft is in space and at the operational location, high fidelity test data are not available. The system will not be fully tested until the spacecraft goes in orbit, into the actual operational environment, when meaningful and representative data can be collected to fully test the systems. But then, how do we make the risk-based launch decision? The go-no-go decision is a risk-based decision process. How do we determine the risk level when precise information are not available? Are the systems good enough to launch?

While scientists and engineers can make good educated decisions as to what to expect at the extreme space environments, more often than not we will encounter unknowns. In this case, how do we represent system risks, and how do we evaluate a system that can be constantly evolving, both in states and as a system? Can we develop probabilistic risk models with incomplete and evolving knowledge (the unknowns)? Current techniques focus mostly in the modeling of the system in time slices. The dynamic Bayesian network, for example, handles time evolution by simple propagation of the system states over time slices. What if the system itself changes, like when a component failed?

Risk assessments for this type of engineering problem are often limited by the availability of data and information. In the case of event driven interactive systems, the completeness of the testing scenarios is often limited by the availability of test data, testing resources, and the ability to enumerate the many possible paths the system can follow. The degree of completeness becomes a tracer of risk! The higher the completeness, the lower the risk, and vice versa. Since event driven systems are highly non-linear and the key challenge in the assessment of the system risks are related to our ability to identify and specify all the states. The completeness of our knowledge directly affects the accuracy of the risk estimates; if we are not aware of the existence of some other failure states, assessments made with incomplete knowledge will certainly result in the underestimation of risks.

Furthermore, the non-linear nature of the event driven processes is changing the states of the system dynamically, and in some cases completely changed the system. For example, Hubble Space Telescope (HST) has six onboard gyroscopes and three working ones are needed to allow the telescope to point accurately. These gyroscopes have limited service lifetimes, and eventually all of them will fail. Several years ago, HST lost four of the six gyroscopes, and the observatory went into safe mode since there were only two working gyroscopes and the observatory cannot be fully functional. Not ready to give up, some brilliant scientists and engineers devised a method to allow the observatory to continue to operate in a 2-gyro mode with some limitations. The control laws for the spacecraft were reworked and updated. The system was changed into a new one; the system itself changed and the system states changed!

Risks can be misestimated because we do not know precisely all the system states, their evolutions, and changes. Our estimates are limited by the knowledge we possess, and our ignorance can potentially result in the misestimate of risks. The consequence of it could be the failure of subsystems, or a catastrophic failure leading to loss of mission, and for human spaceflight, the loss of lives.

*Probabilities do not tell the whole story* — A simple pattern emerged from the above two examples. Incomplete knowledge and information, subjective opinions, and the dynamic evolution of a complex engineering system can affect the accuracy of the risk analysis process, resulting in the misestimate of risk. Risk decisions often have to be made under challenging circumstances with limited, incomplete, subjective and changing knowledge. The current probability risk assessment frameworks might not be robust enough to handle certain cases with changing system states and incomplete knowledge, which could lead to ambiguities (see Johansen & Rausand, 2015 and Aven & Cox, 2016 for an overview of the problem of ambiguity associated with general and probabilistic risk assessments). Researchers have pointed out that a framework that takes into account changing systems and incomplete knowledge is presently lacking in the area of risk research (Aven & Zio, 2014). Many new techniques are being developed and refined, but perhaps a different way to think about the problem can take us down a new path. The quest is to look for other modeling frameworks that can better handle imperfect knowledge and ambiguities.

# Chapter 2: Probabilistic Risk Assessments with Quantum Probability and Mechanics

*Take calculated risks. That is quite different from being rash.*

*- George S. Patton (1885-1945)*

Probabilistic Risk Assessment (PRA) has been used by many in many fields, and they have been very successful in treating broad classes of problems. This Chapter starts with a general overview of the basic PRA methodology and how PRA frameworks model complex engineering systems. The role of probability in PRA and the current probabilistic approach in PRA are examined. A brief overview of how others are exploring different alternative probability theories to enhance current methodologies follows, highlighting areas that can benefit from the incorporation of other techniques beyond pure probabilistic approaches. Quantum probability and quantum mechanics are introduced as a framework for more in-depth exploration with this dissertation.

## *2.1 General Overview*

Current PRA techniques generally consist of the following three elements: 1) sets of scenarios, which can be physics or probability models to represent events and engineering systems (simple or composite), 2) the frequency of occurrences of the events associated with the scenarios, usually in the form of probabilistic models for events, and 3) the consequences associated with the occurrences of the events, which can be numerical values, event triggers, event sequences, or impact statements. PRA methodologies can be grouped under static or dynamic approaches. Static PRA typically model system states in discrete time slices, essentially snapshots of system states in time. Event trees, fault trees, and Bayesian networks are examples of static techniques (Rasmussen, 1975; Swaminathan & Smidts, 1999; Ericson, 1999, Pearl, 1988). While these techniques handle engineering systems with static behaviors well, many engineering systems do exhibit dynamic behaviors and static PRA methodologies might be less effective in modeling dynamic engineering systems (Siu, 1994). Developed around early 1990s, Dynamic PRA (DPRA) techniques model system states according how the system transition from states to states over time, essentially an event driven approach. DPRA takes PRA techniques and adds to them simulation elements and stochastic tools to incorporate and reflect dynamic changes of the system (see for example, Mandelli et al, 2013, 2019 for simulation-based techniques, and Varuttamaseni, 2011 for a short summary for dynamic Bayesian networks). These dynamic techniques found applications in complex engineering risk assessments and the modeling of complex systems with dynamic interactions and dependencies between components, where static logic structure-based approaches might have limitations (Varuttamaseni, 2011; Mandelli et al., 2017; Modarres et al., 2017; Jankovsky, Haskin, & Denman, 2018). Table 1 lists a sample of current static and dymanic PRA methodologies, and Appendix A provides a general overview on current PRA techniques.

Table 1: A sample of current PRA methodologies.

| Type | Methodology |
|---|---|
| Static -<br>Model is a snapshot of a system state in time | • Risk Matrices<br>• Event Tree of Event Sequence Diagrams<br>• Decision Tree Analysis<br>• Fault Tree Analysis<br>• Event-Chain Model<br>• Bayesian Interference<br>• Bayesian Network and Bayesian Belief Networks |
| Dynamic -<br>Model includes the transition of system states over time | • System Dynamics<br>• Dynamic Event Tree<br>• Dynamic Fault Tree<br>• Discrete Event Transition models<br>• Monte Carlo Simulation<br>• Dynamic Bayesian Networks |

Probability frameworks, such as Kolmogorov probability and Bayesian probabilistic techniques, are often used to model events and quantify uncertainties in PRAs. The successful execution of PRA depends on the association of events and the quantification of uncertainties with probability distributions. Critical to the association and the quantification processes is the need for contextual knowledge and information about the events, the sequences, the configurations, the interactions, the physical processes, the parameter spaces in order to create scenarios. The more comprehensive is the knowledge, the better the support for the quantification of uncertainties with probabilistic distributions and models. Contextual knowledge is an important ingredient since it often serves as the glue connecting the probabilities together. For engineering system failure analyses and in reliability analyses, a very common approach is to capture contextual knowledge and information in event and system models, such as event trees and fault trees. The event tree, which is a logical model built from events and systems knowledge, connects the events together; the tree can be seen as two separate components where the knowledge is contained within the tree structure, connecting the probabilities and the probability distributions. Probabilities model the occurrences of events with the event tree structure connects the events and the probabilities together. Contextual information, which forms the scenarios, shapes the tree.

The other half of the PRA process, the quantification of uncertainty, has been a subject where many brilliant minds are currently working to improve existing methodologies in order to achieve higher modeling fidelity. While the applications of traditional and Bayesian techniques have been highly successful in the treatment of many problems, contextual knowledge is not something that can be easily integrated within a probabilistic framework, especially for the conventional "pure probabilistic" frameworks, e.g. Kolmogorov and Bayesian (Pedroni, Zio, Pasanisi, & Couplet, 2017). There are other areas where improvements are being sought by researchers as well (Aven & Zio, 2014; Zio, 2018).

### 2.1.1 Representing Complex Engineering Systems.

To build the scenarios for event tree and fault tree, the process begins with the logical modeling of the engineering system and the associated events. The PRA process starts with the definition of the problem and the identification of the assessment objectives. The system in question is then examined to determine key characteristics and properties relevant to the objectives. Some commonly used frameworks (for example by NASA) develop scenarios with the use of techniques such as master logic diagrams to identify the initiating events (IE), the engineering system's behavior, the different scenarios that can lead to system failures and accidents, and the consequences associated with the events (Stamatelatos and Dezfuli, 2011). Then the states of the engineering system in question are constructed and traced out.

The information is then used for the construction of event sequence diagrams and event trees, starting with the initiating event, and branching out to subsequent sequential pivotal events, progressing step by step to the final outcome, which can be a simple binary one, such as good or bad, working or not working. Figure 2 is an example of a typical sequence diagram.

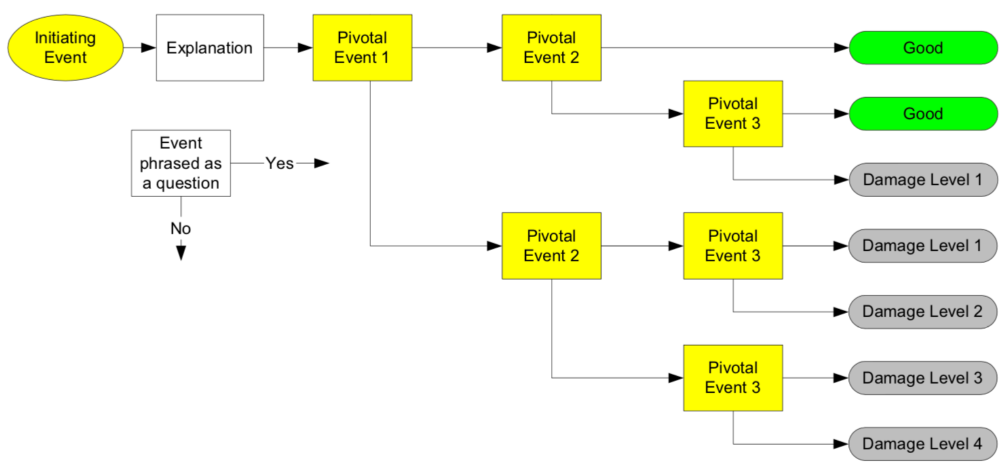


Figure 2: An example of a typical sequence diagram, from Stamatelatos and Dezfuli, 2011.

Once the pivotal events have been sequenced, detailed parametric models are developed to the level that is sufficient to support the quantification of the system and the interacting environment (the physics models). For the physical modeling of complex engineering systems, a variety of techniques are used based on the characteristics and properties of the problem and objectives. For typical linearly sequencing scenarios, simple sequencing logical models, dynamic flow graph methodology could be used. Physical models representing engineering systems and structures can be derived from finite element analysis, or simulation techniques such as structural analysis with Monte Carlo and systems dynamics. Restructuring of the logical tree structures can be accomplished via common cause analysis or the identification of cut sets (see Stamatelatos and Dezfuli, 2011 for further expositions).

The basic events are then assigned probabilities or frequencies as part of the quantification process, which also incorporate the treatment of uncertainties depending on the completeness of the knowledge about the events and the physical processes. Finally, the integrated scenario model (the cut sets) are put together from all the different elements. With the integrated scenarios, risk analysts can perform a variety of analyses, such as sensitivity analysis, uncertainty analysis, and importance ranking. The results from the analyses are then interpreted, leading to conclusions.

### 2.1.2 The Role of Probability

One of the critical PRA elements is the use of probability to characterize event occurrences, physical processes, to represent and model uncertainties. Those physical processes have intrinsic variability (randomness). The use of probabilistic models to describe stochastic processes (referred to as "aleatory" uncertainty) is a central element of PRA. Beyond that, risk scenarios used in the representation of the physical problem introduce yet another type of uncertainty, referred to as epistemic uncertainty, which is a reflection of the completeness of our knowledge about system behavior, models, and modeling parameter. The epistemic uncertainty is a quantification of the degree of knowledge or the state-of-knowledge of the fidelity of the models, modeling parameters, and assumptions in representing the reality of the relevant physical processes and the systems' behaviors as defined by the conditions. The challenge is on the mapping or association of uncertainty models to a probability distribution.

The degree of success for the association of uncertainty models to probability distributions depends on our knowledge about the problem, from the availability of information and their quality, to statistical data such as event frequency and occurrences. Contextual information and knowledge play critical roles, but the amount of information that can be encoded in the probabilistic models is limited by the mathematical and probability frameworks. Events are not as rare when one takes into consideration different contextual knowledge about the systems and their evolutionary histories. Yet, knowledge, evolutionary history, and the dynamic behaviors can be difficult to quantify objectively and incorporate into probability based quantitative risk analysis (Aven & Zio, 2014).

The concept of probability itself is not absolute or definitive but subject to interpretation. There are a number of interpretations of probability and a number of different theories of probabilities, from axiomatic formulations such as Kolmogorov probability, to Cox's logical probability (Cox, 1946), to imprecise probability such as that of Dempster-Shafer, and to subjective probability (e.g. Beer, Ferson, & Kreinovich, 2013). A thorough review of the subject can be found in Hájek, 2012. The choice of which probability interpretation and theory to use in the modeling of uncertainty can greatly affect the outcome of the PRA results and conclusions. Since the states of the system being modeled can change as a function of time, there is also a differentiation between static and dynamic techniques (for Systems Dynamics, see for example Forrester, 1961; Sterman, 2000; Leveson, 2006, 2011; Dulac, 2007; Mohaghegh, 2010; Varuttamaseni, 2011).

### 2.1.3 Different Probability Theories

In looking for ways to enhance PRA frameworks, some researchers have been focusing on the step associated with the assignment of probabilities and the use of probability to quantify uncertainties. Researchers have pointed out that the use of pure probabilistic modeling approach in PRA techniques, while successful in treating many problems, might not be sufficient in fully characterizing the problem (Pedroni et al. 2017). Pure probability refers to probability models based on conventional probability framework, as in Kolmogorov and Bayesian probabilities. Numerous examples were cited where the use of pure probability can lead to inconclusive or ambiguous results (Aven, 2010; Dubois & Guyonnet, 2011; Aven and Zio, 2014; Pedroni et al., 2017), sometimes refer to as the ambiguity problem (Renn, Klinke, & van Asselt, 2011; Johansen & Rausand, 2015) which will be further elaborated in next section. The scalability of the conventional probabilistic techniques is also a question when dealing with highly complex integrated engineering systems, particular in the area of model fidelity and computational efficiency.

Other probability theories and different interpretations of them exist at various stages of maturities (Hájek, 2002, 2012). Besides Kolmogorov probability theory on the “conventional” end and the above-mentioned imprecise probability theories on the “subjective” end of the spectrum, there are other probability theories in between. Table 2 summarizes the different kinds of probability theories, from classical to modern ones.

Table 2: Different treatments of probability theory.

| Treatment/ Interpretation | Representative works by | Principles |
|---|---|---|
| Classical | Cardano, Pascal, Fermat, Bernoulli, Laplace | Principle of indifference, assign equal probability to events |
| Frequentist | Mills, Ellis, Cournot, Fries, Venn, Bernoulli, Gauss, Laplace, Fisher, Neyman, Pearson. | Assign event probability based on the frequency of occurrence in a large number of trials. |
| Subjective/ Evidential/ Bayesian | Bayes, Laplace, de Finetti, Jeffreys, Wald, Savage, Ramsey | Bayes Theorem; Bayesian updates; subjective and a reflection of degree of confidence on the occurrence of events; degree of belief |
| Physical/ Propensity | Pierce, Popper | Physical disposition or propensities of events |
| Modern Axiomatic | Kolmogorov, Cox, Jaynes | Kolmogorov Axioms, Measure Theory, Cox Theorem and postulates - propositional logic based; Plausible reasoning |
| Logical Interpretation | Johnson, Keynes, Jeffreys, Carnap | Degree of confirmation based on empirical evidence leading to a proposition |
| Measure-theoretic | Borel, Lebesgue | Mixing of discrete and continuous probability distribution for event assignments. Instead of working with cumulative probability distributions, works with probability measures, which is based on Measure Theory |
| Quantum/Dirac-von Neumann | Dirac, von Neumann | von Neumann noncommutative measure theory, Noncommutative analog of Kolmogorov Probability, Dirac quantum mechanics |
| Information Geometry | Amari, Nagaoka | Application of differential geometry to model probability distributions by mapping the distributions to Riemann manifold, resulting in the creation of a statistical manifold. |
| Imprecise | Boole, Keynes, Walley, Dempster, Shafer | Introduce non-fully probabilistic ideas and frameworks, the use of Dempster-Shafer theory of belief functions, fuzzy sets, evidence theory, possibility theory, interval probabilities, probability-boxes (p-boxes), etc. |

Aguirre et al. (2013) highlights some of the current research directions on alternative uncertainty theories:

- Probability theory: this is to extend classical probability with the inclusion of subjective probabilities, “where a probability measure represents a degree of belief of an agent about the occurrence of an event A.” (Aguirre et al., 2013)
- Fuzzy set theory: True or False or either (true or false). This is in essence an extension of the traditional truth table with the inclusion of an additional “either” state (Zadeh, 1965).
- Possibility theory: extending the fuzzy set with the additional condition of normalization. (Zadeh, 1978; Dubois & Prade,1988, 2001; Dubois & Prade, 2009).
- Belief functions theory: the Dempster-Shafer theory - theory of evidence (Dempster, 1967; Shafer, 1976). This approach models the degree of belief using mass function, belief function, or plausibility function, and relaxes Kolmogorov’s additivity axiom.
- Imprecise probabilities (Walley, 1991): a framework that admits imprecision in probability models and introduces probability bounds.

The term *imprecise probability* describes a class of non-probabilistic frameworks that have been applied in the treatment of uncertainty in engineering risk assessments. Pedoni et al. (2017) summarized the branch of investigation on the use of non-fully probabilistic models to complement pure probabilistic techniques.

#### 2.1.4 Beyond Conventional Probability

Quantitative techniques rely on probability models. Conventional probabilistic methods may encounter limitations in the capturing and incorporation of contextual knowledge in risk analysis, which can impose limitations when dealing with ambiguity (Cox, 2009, 2012; Fenton & Neil, 2013; Aven & Zio, 2014; Aven & Cox, 2016). Ambiguity can be defined as the possibility of different interpretations for a result based on the availability of contextual knowledge (Renn et al., 2011; Johansen & Rausand, 2015). If analysis scenarios yield identical probabilities in an ambiguous state, the proper interpretation of the results might require additional contextual information and knowledge because the scalar probabilities do not capture or later retain the information necessary to support complex decision processes.

To incorporate additional information, techniques such as various network-based framework (e.g. decision tree, event tree, Bayesian networks) capture knowledge and information into the probabilistic models by overlaying additional structures on top of probabilities (Figure 3). Knowledge and probabilities are separated by the structures; in event trees, the probabilities are associated with the nodes and the tree structures capture the knowledge. At each decision points, a path is chosen at the expense of information loss. For example, at the node for state 1, a path has to be chosen to go to state 2a or 2b. After a path has been chosen and followed, part of the information available at state 1 will no longer be carried forward, similar to a Markov process. Information loss occurs when a path is chosen down the analysis workflow.

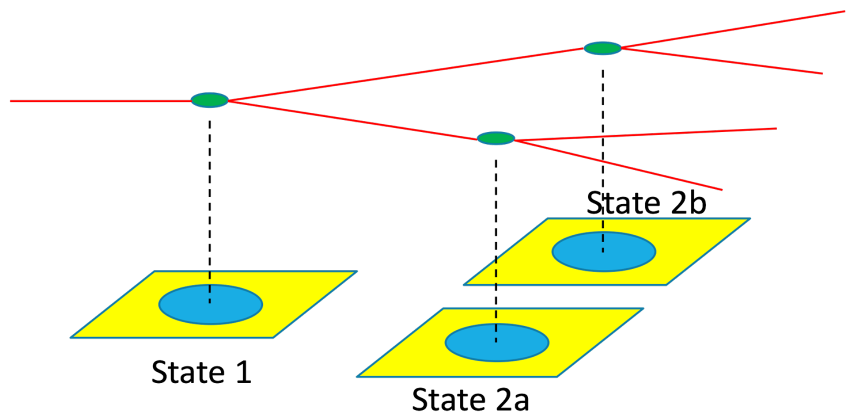


Figure 3: Decision tree as a structure external to probabilities.

Further complications arise when the problem involves complex decision chains, where ambiguous upstream decisions affect the downstream outcomes. While conventional Kolmogorov axioms' based representations place bounds for how much information can be captured in the risk models, the risk analysis workflow itself can introduce additional information loss as well.

The workflow to determine the probability of a risk event involves a number of steps:

1. Capture information
2. Process information
3. Identify the risk events, develop impact assessments
4. Risk Model representing the information/knowledge based on the risk events
5. Compute the probabilities for particular outcomes
6. If new data is available, perform Bayesian update on the probabilities, and then re-compute the outcomes

7. At the end you get a scalar value, a number between 0 and 1 for the probability

Figure 4 illustrates the general risk analysis workflow based on the above steps. This workflow suffers information loss along the way, from the initial data processing to the arrival of the probabilistic values, which inadvertently increase ambiguity and uncertainty. The workflow is a reduction process, where contextual information is reduced as a result of the parametric abstraction and the computation of numerical probabilities. Elements of uncertainty and ambiguity are introduced along the way. The risk analysis workflow removes information, and uncertainty may not be reduced.

This limitation affects the formulation of risk questions and scenarios, affecting the interpretation of results. The information loss also introduces additional uncertainties into the process. One can see that the reduction of contextual information, which is a form of knowledge, can introduce additional epistemic uncertainties.  The consequence can lead to the increase of uncertainty and risk. New techniques that can counter the information loss may improve the quality of the analysis results.

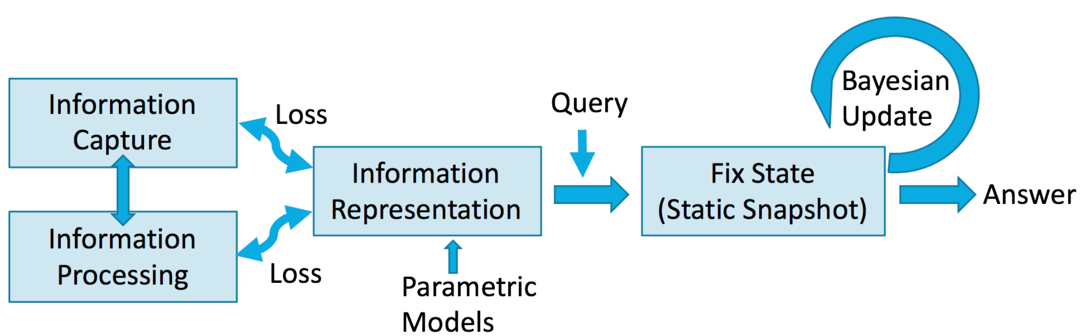


Figure 4: The general risk analysis workflow.

One might argue that current approaches of refining our methodologies and models by connecting different ones and adding more modeling elements work for a lot of problems, but we also see from the previous examples with Katrina and JWST that these frameworks do have their weaknesses.  Any framework that do not adequately incorporate historical information, knowledge, expert opinions, contextual information, results in ambiguity and an increase of uncertainty. The notion where we can keep on refining the models by adding more parts will bound to fail, when we reach the boundaries of our knowledge domains. Alternative frameworks must have the option to handle ignorance, incomplete knowledge, increase the information capacity in the models. We need a new framework robust enough to handle missing or changing data, changing states, changing system, and ignorance.

The present challenge can be restated as the following:

- Results from risk analyses and assessments are only as good as the fidelity of the system models, which are limited by how much information encapsulated in them.
- To increase modeling information capacity, a framework that can allow us to represent fully the many different risk states, encapsulate more information and with enhanced computation mechanism to manipulate, process, evolve and extract results is desirable.
- An internally consistent mechanism to incorporate the uncertainties into a single framework and use it to perform objective assessments on the states of

a system to evaluate the risk would reduce ambiguity and improve decision outcomes.

- The framework should allow risk analyst to perform both qualitative and quantitative risk modeling and analysis. The quantitative results provide concrete values for decisions to be made, and the qualitative representation facilitate high level decision making and communication to the stakeholders.
- The risk systems in consideration need to be event driven and dynamic. Set theory-based probability framework is limited in its ability to deal with incomplete information and subjective knowledge. The mathematical framework needs to have the apparatus to capture more than just scalar values, but it must also be capable to capture and embed contextual information.
- Probabilities do not tell the whole story.

To summarize, current PRA techniques incorporate knowledge by overlaying additional structures on top of probabilities (e.g. event tree, decision tree, fault tree, Bayesian network, etc.). These models constrain what and how much information can be encoded in the model. Further, at each state selection or transition, information loss may occur. These limitations can potentially be addressed by having additional structures built into the probability framework to encode additional information.

Such a framework may align better with how a system behaves and how we track system states. Information can be captured internal to the structure itself and incorporate ignorance. Information loss can be reduced, uncertainty and ambiguity can be handled, computational efficiency can be enhanced. Since conventional probabilities do not tell the whole story, changing the probability framework could be one approach to improve the situation. As it turns out, quantum probability might be the answer. *This dissertation explores the use of quantum probability and the tools of quantum mechanics to increase the information encoding in risk models and to improve risk analysis techniques.*

### 2.2 From Kolmogorov to von Neumann

A new branch of probability theory began to emerge around 1930s to explain the new development in physics at the time, namely the "new" quantum theories (Schrödinger's wave mechanics, Heisenberg's matrix mechanics, and others). The independent pioneering works of Paul Dirac (1930, 1958) and John von Neumann (1932, 1955) unified these quantum theories with a mathematical formulation based on the concept of Hilbert space, an infinite-dimensional complex vector or functional space. Central to this formulation are the Dirac-von Neumann axioms, leading to the modern formalism of quantum probability (see for example Chang, 2015), a non-commutative generalization of the Kolmogorvian modern axiomatic probability theory. Historically, quantum mechanics came about first in the 1930s. The Dirac von-Neumann Axioms of quantum mechanics serves as the foundation for quantum physics. The branch of quantum probability was developed in the 1980s to help establishing a firmer mathematical foundation for the physical theory.

Whereas Kolmogorov probability can be regarded as a scalar theory based on the concepts of set theory, measure theory, probability space and Boolean logic, quantum probability can be viewed as a geometric theory based on vectors. *While people might be familiar with the concept of the risk vector, where the event probabilities and consequences are represented in the form of vectors, this is not to* *be confused with the geometric representation of probability itself that we are exploring here. The term quantum probability is used here in alignment with the quantum mechanics/physicist's viewpoint instead of the mathematician's perspective. Also, we are interested in applying the quantum framework to model engineering systems; it is more natural to approach from the physicist perspective than the mathematician perspective. Quantum Mechanics contains both a theory of probability as well as a comprehensive toolset to describe physical systems.* Table 3 summarizes and compares the basic concepts between Kolmogorov and quantum probabilities.

Table 3: Comparisons between Kolmogorov and Quantum Probabilities.

| Probability Framework | Characteristics |
|---|---|
| Kolmogorov Probability | • Scalar theory<br>• Set theory based within the traditional Boolean logic<br>• Event A, the probability of the occurrence of A, P(A), is a scalar value<br>• Commutative operation: union and intersection of sets; P(A)P(B)=P(B)P(A)<br>• System S is characterized by its states<br>• A state is represented by the scalar value: e.g. P(A) = 0.5<br>• Operations perform on the sample, event, and function spaces |
| Quantum (Dirac-von Neumann) Probability | • Geometric theory<br>• Complex Hilbert Space, a N-dimensional C* vector space<br>• Event A is a point in the vector space, corresponding to a system state<br>• A non-commutative analog of Kolmogorov's; noncommutative operations<br>• System S, characterized by its states, observables, operations, and measurements (expectations)<br>• A state is represented by a state vector<br>• Operations perform on the vector, changing the system from one state to another<br>• Probabilities are the squared magnitudes of the vectors |

The choice of quantum probability for this study stems from the similarity in the lines of inquiry for both fields. The questions asked by quantum mechanics: "A quantum system (such as a photon) was prepared with an initial state. We want to know the probability of finding the system in a certain state after subjecting the system to certain events." The questions asked by risk analysts: "An engineering system (such as a dam) was built to an initial configuration. We want to know the probability of finding the system in a certain state after it goes through certain events." The fundamental questions ask by both share similar basis, suggesting that quantum probability could potentially be used to answer risk questions.

Quantum probability offers at a minimum a number of interesting possibilities when applied to the formulation of a theory of uncertainty. Quantum probability is a generalization of the Kolmogorvian probability and therefore it is compatible with existing uncertainty frameworks. Second, the generalization with the additional mathematical vector space dimensions will open up new possibilities for additional or alternative representations. Third, quantum theory originated from the study of real physical phenomena, and it was developed to enable us to describe what we observed in physics. Quantum probability might be a more suitable and natural way to describe reality and might allow us to develop better "models of the world." In other words, the inadequacy of classical physics to describe quantum physics serves as a parallel for the inadequacy of classical probability to describe epistemic uncertainty.

This work evaluates the use of quantum probability and elements of quantum mechanics to model risks for engineering systems. *We want to see if we can gain new insights in the treatment of uncertainty with an alternative probability framework* *based on quantum probability and quantum mechanics to model events and uncertainties.*

# Chapter 3: A Quantum Model: Part 1

*Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly, it's a wonderful problem because it doesn't look so easy.*

*- Richard Feynman 1981*

What is quantum probability and quantum mechanics? For the purpose of developing risk assessment models with quantum probability and the tools of quantum mechanics, this Chapter provides a short introduction with examples on how to construct basic models to represent engineering systems using quantum probability. The notion is to provide a few concrete examples before introducing the more abstract notion of quantum probability and quantum mechanics in Chapter 4. This Chapter will start with introducing a few basic concepts on quantum probability derived from the original von Neumann treatment in his mathematical treatise on quantum mechanics (Neumann, 1955). *The approach is more from the physicists' perspective than from the mathematicians' perspective.* Based on those concepts, several example models will be constructed to illustrate how quantum probability can be used to describe the probability of finding a system in a certain state. A more comprehensive primer on quantum probability and mechanics is presented in Chapter 4.

For the rest of the dissertation, the following physicist notations will be used:

- Probability of event A: P(A) maps to a non-negative real number (a scalar)

- A 3d vector in traditional notation is given by:

$$\mathbf{A} = \vec{A} = \sum_{i=1}^{N} A_i \hat{e}_i = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}^T \quad (3.1)$$

where $\hat{e}_i$ is a unit basis vector.

- The same vector in Quantum notation, known as the Dirac bra-ket notation, is given by:

$$\mathbf{A} = |A\rangle = A_1|e_1\rangle + A_2|e_2\rangle + A_3|e_3\rangle \quad (3.2)$$

- The state vector $|A\rangle$ is call a "ket" and the dual vector $\langle A|$, the Hermitian conjugate (conjugate transpose) of $|A\rangle$, is call a "bra", which corresponds to:

$$\langle A| = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \quad (3.3)$$

*The Absolute Minimal Fundamentals* — The state of a system can be described by a *state vector*, *S*, with unit vectors representing the various possible unique basis states the system can be found in. For a system with three basis states $\hat{s}_1 \left(\equiv |s_1\rangle\right), \hat{s}_2, \hat{s}_3$, the vector S is given by:

$$\mathbf{S} = |S\rangle = s_1|s_1\rangle + s_2|s_2\rangle + s_3|s_3\rangle \quad (3.4)$$

The components of the vector, $s_1, s_2, s_3$ are called the *probability amplitudes*. The probability of finding a system in state $|s_i\rangle$ is *the squared of the probability amplitude*. For example, the probability of finding a system in state $|s_1\rangle$ is simply $s_1^2$.

The *density operator ($\hat{\rho}$) or density matrix ($\rho$)*:

$$\hat{\rho} = \sum_i p_i |S_i\rangle\langle S_i| \tag{3.5}$$

is the equivalent to the state vector for the description of the statistical ensemble states of a quantum system, and where $p_i$ the probability of finding an element of the ensemble in state $|S_i\rangle$. The term density operator and density matrix are often used interchangeably; operator is used when a specific basis is not specified, and matrix is when the operator is associated with a specific basis. If the system ensemble state can be fully described by a single state vector, such as $\hat{\rho} = |S\rangle\langle S|$, it is called a *pure state*; if the system ensemble state is a statistical mixture of several possible state vectors, such as $\hat{\rho} = p_1|S_1\rangle\langle S_1| + p_2|S_2\rangle\langle S_2|$ then it is called a *mixed state*. These fundamental concepts will be further explained in Chapter 4.

### <u>*3.1 A Simple Example*</u>

*The quantum representation of a system state* — A system state is first expressed in the form of a state vector, which is a mapping of the vector and its components to a state associated with the system being modeled. The vector space spans by the state vectors describe the systems. In general, the state vectors are Hilbert space vectors, which are complex. *For this research, we will restrict to the real part of the Hilbert space, focusing on the demonstration of the principles and concepts*.

To help illustrating the concepts, we will use the quantum mathematical apparatus to model a simple system: a levee system. A levee is considered to be a system that when it is working, water is prevented to go from one side to the other. If the levee is built in a desert, then the consequence of it failing has little implication to the risk of flooding since contextually the notion of flooding in a desert - the consequence - is negligible. However, if the levee is built in an area below sea level, then the consequence of failure would be highly significant. The risk analysis process thus focuses on assessing the occurrence of those states, and then based on the context, assign a scaling factor to reflect the significance of the risk.

For simplicity, it is assumed for this exercise that a levee can either be working or not working at a given time. Certainly, the system can have more states, but for the present purpose, it is treated as a two states system. We will label this levee system L, and the working state is labelled as $|1\rangle$ and the not working state is labelled as $|0\rangle$, with $|0\rangle$ and $|1\rangle$ as the basis state vectors for this system L. The probability of finding the system L in a working state is denoted by $P(|1\rangle)$ and the probability of finding it not working is $P(|0\rangle)$ which is equal to $1-P(|1\rangle)$. Figure 5 is a simplified generic representation of the flood wall system with 2 risk states: working or not working, with the state vector denoted by $|L\rangle$ for the specific system L. Recall that the probability is given by the square of the probability amplitudes (the basis components), in this example the probabilities of finding the system in either states is 50/50, and the probability amplitudes are therefore $1/\sqrt{2}$ .

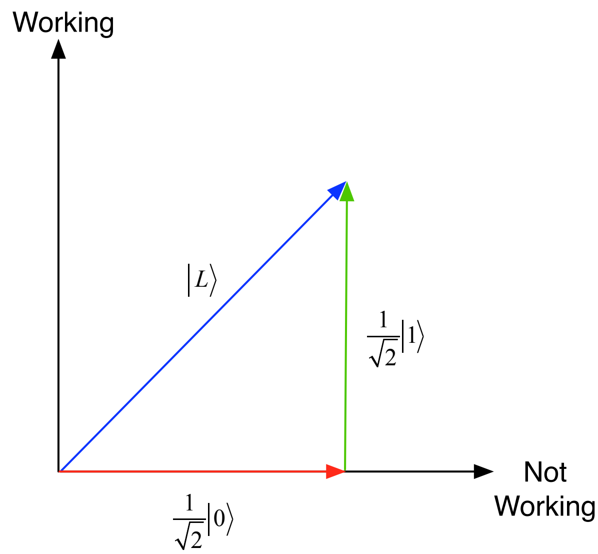


Figure 5: A simple vector representing a system state for a levee.

To provide additional numerical context, for this next example let the probability of finding system L not working, $P(|0\rangle)$, be 0.01. Correspondingly, the probability of finding L working is $1-P(|0\rangle)$, which is 0.99. The probability amplitudes, $l_0$ (for $|0\rangle$) and $l_1$ (for $|1\rangle$), are therefore the square root of the probabilities. Expressing these in the Dirac notations, with system L denoted by $|L\rangle$:

$$|L\rangle = l_0|0\rangle + l_1|1\rangle \quad (3.6)$$

where $l_0 = \sqrt{0.01}$ , $l_1 = \sqrt{0.99}$ ■

The probability of system failure is therefore finding the system in state $|0\rangle$, which is $l_0^2 = 0.01$. If a quantitative risk measure is desirable, a scalar "scale" value can be multiplied to the probability to reflect and project significance.

### 3.2 Quantum Model I: Hurricane Protection System

In this and later sections, a basic "experimental" model is constructed using the quantum framework to represent a complex engineering system, namely the levee-flood wall storm protection system. The model is then utilized to explore how this quantum framework can offer new capabilities and insights to augment existing PRA methodologies and on the treatment of uncertainty.

*The Reference Scenario for the Models* — To facilitate the development of the framework, a reference scenario based on the New Orleans Hurricane Protection System (HPS), now known as the Hurricane and Storm Damage Risk Reduction System (HSDRRS), will be used for this series of papers. The HPS is chosen as a reference system because extensive studies and risk analyses were performed on the HPS after hurricane Katrina and comparison results are readily available (IPET, 2007, 2009). Furthermore, the actual HPS system exhibits many heterogeneous properties and characteristics, and heterogeneity leads to variations and introduces uncertainties into the system. This provides a good platform and rich environment to test different configurations, scenarios, conditions, and ideas with the new framework. Ultimately, the critical challenge is to find a framework that can support the modeling of the many different and complex scenarios in a coherent fashion. The following is a description of the "experimental" reference scenario:

> *A reach consisting of a levee with an I-wall structure, underwent periodic seasonal water/flooding events, with sediment deposit and erosion at the front side (facing the river). Occasionally, the flooding overtopped the structure, resulting with backside erosion*

*and the weakening of the general structure. Over time, some of the backside erosions were repaired, and portion of the structure were upgraded to bring it closer to the modern design specification/construction code. More significant storms occurred, resulting in significant overflow and backside erosion. Finally, a 100-200 year hurricane hit the region, with catastrophic overflow and the destruction of the structure due to backside failure, resulting in a breaching event.*

This reference modeling scenario further specifies the parts, applicable terminologies, and defines a number of boundary conditions:

- The reach is a subsystem and serves as a unit of uniform characteristics and properties.
- There are 138 reaches, in 37 sub-basins, and in 8 basins for the New Orleans and Southeast Louisiana HPS.
- A reach section can be a levee (L), or a levee with a flood wall (L+W).
- A sub-basin can consist of *m* number of L or *n* number of (L+W), and a basin can be made up of integral units of *m*L and *n*(L+W).
- A generic system is described by a state vector. The levee is denoted by the state vector $|L\rangle$ and the flood wall is denoted by the state vector $|W\rangle$.
- Composite systems are formed by tensor products (Chapter 4.4) between the state vectors. The composite (L+W) system is denoted by the composite state vector.
- An equivalent formalism, the density matrix (or the density operator, Chapters 4.3.3, 7, 8), can be used to represent a system and a composite system. This

density matrix extends the capability to allow the representation of ensemble systems. To represent the HPS system with heterogeneous components and in heterogeneous states requires the use of the density matrix formalism with the ensemble treatment.

- System evolutions are driven by event operators (Chapter 5.2). Operators can alter the state vector probability amplitudes or change the system entirely. The physics are incorporated into the operators.
- While the quantum formalism is in the complex space, this work will restrict it to the real space. This restriction is adopted to simplify the computation to focus on illustrating the key principles and concepts, with the advantage of making the system easily visualized to better illustrate the system and the problem. Future work can remove this restriction to explore other capabilities.

### 3.2.1 The Levee/Floodwall: Basic Models

The first step begins with the development of the basic steady state system model following a bottoms-up approach: 1) starts with a single individual system, a Levee (L) and a Flood wall (W), 2) then combine them forming the composite Levee + Flood wall (L+W), and 3) finally combining the composite systems together forming the HPS (a combination of (L+W), L, W, etc).

As defined earlier, a system in a working state will be represented by the $\left|1\right\rangle$ state vector, and a system in a non-working state will be represented by the $\left|0\right\rangle$ state vector. The corresponding matrix notations are:

$$\left|0\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \left|1\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

A note on notations: The systems will be denoted by capital letters A, B, C. The states of the system will be denoted by numerals 1, 2, 3. General parameters will be in Greek letters and lowercase letters, i, j, k, for indices. Following this convention, a system is written in general as: $S^{System}_{state,\left|basis\right\rangle}$ or $S^{System}_{state,basis}$ . For example, a levee system A in the initial state 0, is represented as:

$$L^A_0 : \left|L^A_0\right\rangle = l^A_{0,\left|0\right\rangle}\left|0\right\rangle + l^A_{0,\left|1\right\rangle}\left|1\right\rangle = l^A_{0,0}\left|0\right\rangle + l^A_{0,1}\left|1\right\rangle. \tag{3.8}$$

*Representation for an Individual System* - *The Levee* — The Levee is a structure, usually raised and compacted earthen embankment, built between the protected area from a body of water such a river (Figure 6). The control of the water is achieved by blocking or diverting the flow of the water. Over time, levees can fail due to natural erosion and material degradation, or they can fail as a result of external events, such as seasonal flooding and hurricanes, resulting in the rise of water level which can overtop or breach the levee. For this work, a levee system and other systems are assumed to be 2-states systems, in either the working $\left|1\right\rangle$ or the not working $\left|0\right\rangle$ states. The not working state can be a result of flooding due to overtopping, breaching events, natural degradation, or other events that can render the system no being able to control the water level.

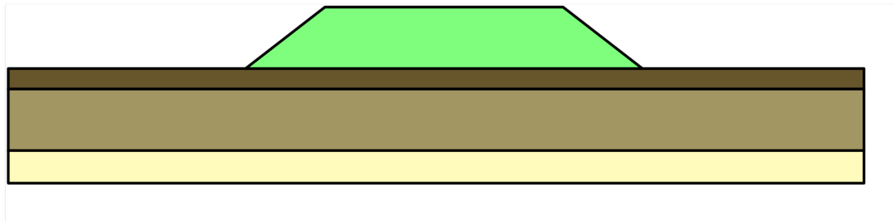

Figure 6: A basic levee.

The Levee state vector $\left|L\right\rangle$ is expressed as: $\left|L\right\rangle = l_{|0\rangle}\left|0\right\rangle + l_{|1\rangle}\left|1\right\rangle$, where $l_{|0\rangle}$ is the probability amplitude of finding L in the not working state $\left|0\right\rangle$, and $l_{|1\rangle}$ is the probability amplitude of finding L in the working state $\left|1\right\rangle$. The squared magnitude of $l_{|0\rangle}$ and $l_{|1\rangle}$ represent the intrinsic aleatory uncertainty associated with the working and failing states respectively. $\left|l_{|0\rangle}\right|^2$ is the intrinsic failure probability, which reflects the stochastic failure probability for the levee. If we have full knowledge and history of the construction of the Levee, the proper inspection and quality processes, then the failure probability of the levee (the probability of finding the newly constructed levee in a defective state) should be restricted to the intrinsic stochastic failure probability. If the exact state of the Levee is precisely known to the aleatory limit, the state vector $\left|L\right\rangle$ is pure. The corresponding density matrix for the levee is:

$$\rho_L = \left|L\right\rangle\left\langle L\right|. \tag{3.9}$$

Model 3-M001 Scenario: A simple levee

*A levee was built at time* $t_0$ *with an intrinsic stochastic failure probability of 0.03.*

Let the levee state be $\left|L_1\right\rangle$. The state vectors representing the levee state is given by:

$$\left|L_1\right\rangle = l_{1,\left|0\right\rangle}\left|0\right\rangle + l_{1,\left|1\right\rangle}\left|1\right\rangle = \sqrt{0.03}\left|0\right\rangle + \sqrt{0.97}\left|1\right\rangle \tag{3.10}$$

In matrix form:

$$\left|L_1\right\rangle = \sqrt{0.03}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{0.97}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.03} \\ \sqrt{0.97} \end{pmatrix} \blacksquare \tag{3.11}$$

Up to this point, this model represents a perfectly built levee, with only the intrinsic uncertainties, evolving in time and transitioning from states to states. However, in reality, multiple transition pathways exist and given two levees starting out exactly the same initially, at some future time $t_1$, the states of the two levees can be the same or they can be different. For example, two levees with the same built quality were installed in two separate areas; one area receives a higher annual rainfall than the other. The erosion rates could potentially be very different between the sites. When one inspects the two levees at some other future time $t_2$, the two levees could be in very different states due to the different evolutionary paths. This particular scenario illustrates how history data and knowledge have to be an integral part of the model representing the states of the system.

Model 3-M002 Scenario: Different possible system states due to different rate of degradation.

*A group of levees was built at time* $t_0$ *and they gradually degrade over time. The levees degrade differently due to the different materials, with some* *degrade more rapidly than others. There is a 60% chance one would find the levee at a slower degradation rate, resulting in a failure probability of 0.03 at time t. There is a 40% chance finding the levee at a faster degradation rate, resulting in a failure probability of 0.08 at time t. Describe a levee at time t in this group.*

Let the levee state with a slower degradation rate be state 1, $|L_1\rangle$ , and the faster rate be state 2, $|L_2\rangle$ . The state vectors representing the possible levee states are given by:

$$|L_1\rangle = l_{1|0\rangle}|0\rangle + l_{1|1\rangle}|1\rangle = \sqrt{0.03}|0\rangle + \sqrt{0.97}|1\rangle \tag{3.12}$$

$$|L_2\rangle = l_{2|0\rangle}|0\rangle + l_{2|1\rangle}|1\rangle = \sqrt{0.08}|0\rangle + \sqrt{0.92}|1\rangle \tag{3.13}$$

The corresponding equivalent representation of the levee system (could be state 1, or could be state 2) in the density matrix form:

$$\rho_L = p_1|L_1\rangle\langle L_1| + p_2|L_2\rangle\langle L_2| = 0.6|L_1\rangle\langle L_1| + 0.4|L_2\rangle\langle L_2| \tag{3.14}$$

In matrix form:

$$|L_1\rangle = \sqrt{0.03}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{0.97}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.03} \\ \sqrt{0.97} \end{pmatrix} \tag{3.15}$$

$$|L_2\rangle = \sqrt{0.08}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{0.92}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.08} \\ \sqrt{0.92} \end{pmatrix} \tag{3.16}$$

$$\rho_L = 0.6\begin{pmatrix} \sqrt{0.03} \\ \sqrt{0.97} \end{pmatrix}\begin{pmatrix} \sqrt{0.03} & \sqrt{0.97} \end{pmatrix} + 0.4\begin{pmatrix} \sqrt{0.08} \\ \sqrt{0.92} \end{pmatrix}\begin{pmatrix} \sqrt{0.08} & \sqrt{0.92} \end{pmatrix}$$

$$= 0.6\begin{pmatrix} 0.03 & 0.17 \\ 0.17 & 0.97 \end{pmatrix} + 0.4\begin{pmatrix} 0.08 & 0.27 \\ 0.27 & 0.92 \end{pmatrix}$$

$$= \begin{pmatrix} 0.05 & 0.21 \\ 0.21 & 0.95 \end{pmatrix}$$

■ (3.17)

Model 3-M003 Scenario: Partial knowledge.

*There is 50% chance that the Levee will have a manufacturing defect or several defects, resulting in a higher chance of failure. Let's say for a standard Levee system with good craftsmanship, we will call it Levee type 1 (or state 1),* $|L_1\rangle$*, their failure probability is 0.01. Let's say the manufacturing defect will increase the failure rate of a standard Levee system. We will call those Levee type 2 (or state 2),* $|L_2\rangle$*, with a failure probability at 0.02.*

The density matrix representing the levee system (could be type 1, or could be type 2) takes the following form:

$$|L_1\rangle = l_{1|0\rangle}|0\rangle + l_{1|1\rangle}|1\rangle = \sqrt{0.01}|0\rangle + \sqrt{0.99}|1\rangle \quad (3.18)$$

$$|L_2\rangle = l_{2|0\rangle}|0\rangle + l_{2|1\rangle}|1\rangle = \sqrt{0.02}|0\rangle + \sqrt{0.98}|1\rangle \quad (3.19)$$

$$\rho_L = p_1|L_1\rangle\langle L_1| + p_2|L_2\rangle\langle L_2| = 0.5|L_1\rangle\langle L_1| + 0.5|L_2\rangle\langle L_2| \quad (3.20)$$

■

One of the failure modes is the natural degradation of the levee over time. For example, cyclical period of dry spells can introduce cracks into the earthen structure, and water seepage into the cracks can widen them and weaken the Levee structure. This failure mode is a time-evolutionary process, and in the quantum framework, the process is being modeled with an operator that operates on the state vector, transitioning it from one state to another state, resulting in a change of the probability amplitudes. We will denote this operator as **H**, which can be a static or time-development operator, and the operator $\mathbf{H}^\dagger$ which is obtained from taking the complex conjugate of the transpose of **H,** $\mathbf{H}^\dagger = (\mathbf{H}^*)^T$. The formulation of the **H** operator will be fully explored in next Chapter. The transition of the levee from one state to another state is expressed as: $|L'\rangle = \mathbf{H}|L\rangle$, or in density matrix form:

$$\rho_{L'} = \mathbf{H}|L\rangle\langle L|\mathbf{H}^\dagger . \tag{3.21}$$

Given that the evolutionary history plays a role in the state transition, one can ask the following question: "If one inspects a levee (a measurement, an observation), what state would one find? Using the two levees example, perhaps there is a 60% chance one would find the levee to be at a state corresponding to a slower degradation rate, and 40% chance the levee will be at a state with a higher degradation rate. This is a first indication that we might only have partial knowledge about the system, since we are not 100% sure which state we might find a levee to be in. The levee in question is therefore in a mixed state, a mixture of two possible states. The system can no longer be represented by state vectors, and we can only represent it with the density matrix, where $p_i$ is the probability of finding the system in the $i^{th}$ state:

$$\rho_{L'} = \sum_i p_i \mathbf{H} \left| L_i \right\rangle \left\langle L_i \right| \mathbf{H}^\dagger . \tag{3.22}$$

Keen observers will notice that several different system configurations can be mapped into the same density matrix. A possible interpretation is to associate this property to the notion of ambiguity, which can further expand into the use of density operators to represent epistemic uncertainties. More on this will be discussed in later Chapters.

*The Flood Wall* — The flood wall is a structure providing additional flood control by raising the height of an earthen bank or a levee higher to reduce the chance of overtopping (Figure 7). Over time, flood walls can fail due to natural degradation, or they can fail as a result of external events such as hurricanes, resulting in the rise of water level which can overtop or breach the flood walls. For this work, the flood wall system is also assumed to be 2-states systems, in either the working $\left|1\right\rangle$ or the not working $\left|0\right\rangle$ states. The not working state can be a result of flooding due to overtopping, breaching events, natural degradation, or other events that can render the system no being able to control the water level.

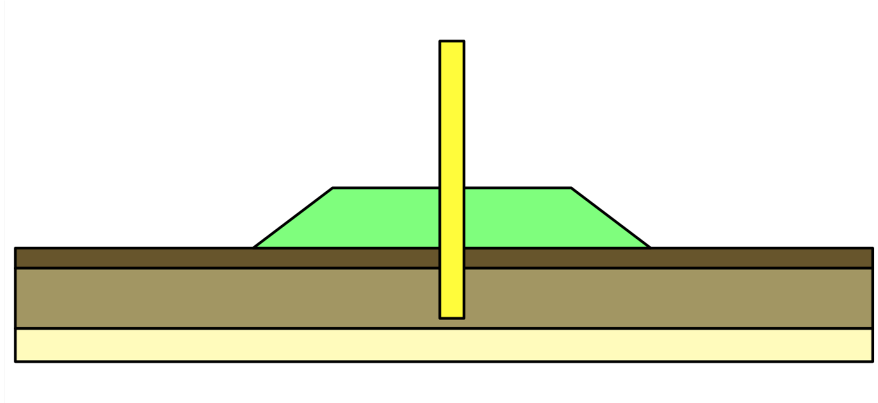

Figure 7: A basic flood wall.

The formulation of the flood wall model is essentially the same as the levee. The flood wall state vector $\left|W\right\rangle$ can be expressed as:

$$\left|W\right\rangle = w_{|0\rangle}\left|0\right\rangle + w_{|1\rangle}\left|1\right\rangle, \tag{3.23}$$

where $w_{|0\rangle}$ is the probability amplitude of finding W in the not working state $\left|0\right\rangle$, and $w_{|1\rangle}$ is the probability amplitude of finding W in the working state $\left|1\right\rangle$. The squared magnitude of $w_{|0\rangle}$ and $w_{|1\rangle}$ represent the intrinsic aleatory uncertainty associated with the working and failing states respectively. $\left|w_{|0\rangle}\right|^2$ is the intrinsic failure probability, which reflects the aleatory failure rate for the flood wall. If we have full knowledge and history of the construction of the flood wall, the proper inspection and quality processes, then the failure probability of the flood wall (the probability of finding the newly constructed flood wall in a defective state) should be restricted to the intrinsic stochastic probability. If the exact state of the flood wall is precisely known to the aleatory limit, the state vector $\left|W\right\rangle$ is pure. The corresponding density matrix for the flood wall is:

$$\rho_w = \left|W\right\rangle\left\langle W\right| . \qquad (3.24)$$

Similar to the Levee, the flood wall will also experience processes that can transition it from one state to another. We will denote this operator as **H**, which can be a static or time-development operator. The transition of the flood wall from one state to another state is expressed as:

$$\left|W'\right\rangle = \mathbf{H}\left|W\right\rangle , \qquad (3.25)$$

or in density matrix form:

$$\rho_{W'} = \mathbf{H}\left|W\right\rangle\left\langle W\right|\mathbf{H}^{\dagger} . \qquad (3.26)$$

The flood wall can also have multiple transition pathways, and the mixed state density matrix is given by:

$$\rho_{W'} = \sum_i q_i \mathbf{H}\left|W_i\right\rangle\left\langle W_i\right|\mathbf{H}^{\dagger} , \qquad (3.27)$$

where $q_i$ is the probability of finding the system in the $i^{th}$ state.

Model 3.M004 Scenario: A levee from an *ensemble* of levees.

*Two contractors are responsible for building 100 I-walls, with each building 50. The excellent craftsmanship of contractor A results in I-walls with failure probability of 0.01. The I-walls from contractor B might have several manufacturing defects resulting in a failure probability of 0.02. How do we represent a I-wall one might encounter within this pool of 100 I-walls?*

For a standard I-wall system with good craftsmanship from contractor A, with failure rate at 1%, we will call it I-wall A. Systems from contractor B with the manufacturing defect will have a higher failure rate. We will call those I-wall B with a failure rate at 2%. The density matrix representing the system takes the following form:

$$\left|W_0^A\right\rangle = w_{0,|0\rangle}^A\left|0\right\rangle + w_{0,|1\rangle}^A\left|1\right\rangle = \sqrt{0.01}\left|0\right\rangle + \sqrt{0.99}\left|1\right\rangle \quad (3.28)$$

$$\left|W_0^B\right\rangle = w_{0,|0\rangle}^B\left|0\right\rangle + w_{0,|1\rangle}^B\left|1\right\rangle = \sqrt{0.02}\left|0\right\rangle + \sqrt{0.98}\left|1\right\rangle \quad (3.29)$$

$$\rho_W = q_0^A\left|W_0^A\right\rangle\left\langle W_0^A\right| + q_0^B\left|W_0^B\right\rangle\left\langle W_0^B\right| = 0.5\left|W_0^A\right\rangle\left\langle W_0^A\right| + 0.5\left|W_0^B\right\rangle\left\langle W_0^B\right| \quad (3.30)$$

■

Notice a subtle distinction here, that the mathematical structure is similar to earlier scenarios, we are describing systems and not just different states of a single system. These are crucial distinctions that will come together in later discussions, playing an important role in the discussion of uncertainty and ignorance.

The above simple models and examples illustrate how an engineering system, the system states, and the probability of finding a system in a particular state can be represented by a framework based on quantum probability and mechanics. In next chapter, a more comprehensive introduction to quantum principles is provided.

# Chapter 4: A Short Quantum Primer: The Probability and Tools of Quantum Probability and Quantum Mechanics

*If you want to understand quantum mechanics, just do the math.*

*- Freeman Dyson, 2007*

What is quantum probability and quantum mechanics? For the purpose of developing risk assessment models with the tools of quantum probability and quantum mechanics, this Chapter provides a short primer on basic quantum probability originates from the works of Dirac (1958) and von Neumann (1955) treatments on quantum mechanics. This section only covers the essential mathematical frameworks and results necessary for use by this dissertation. In 4.1, the postulates of quantum mechanics are presented to introduce a number of basic definitions and terminologies. Section 4.2, 4.3, and 4.4 dive into some of the basic definitions, concepts, and operations. A general discussion on the properties of the quantum framework and how they can potentially reshape engineering system risk analysis (4.5 and 4.6) conclude this primer section.

*This primer approaches the subject more from the physicist's perspective than from the mathematician's perspective.* Readers are suggested to consult the following reference materials and texts for more in-depth treatment. Bruza, Wang, and Busemeyer (2015) provides a short overview and comparison between classic and quantum probability theory. Busemeyer & Bruza (2012) contains a more extensive introduction to quantum probability. Texts suitable for beginners: Susskind & Friedman (2015). At the intermediate level: Miller (2008), and at graduate level: Sakurai (1993). For mathematicians: Chang (2015). For general introduction to quantum computation: Perry (2012); the definitive text: Nielsen & Chuang (2011). For those who are interested in where it all begins: Dirac (1932, 1958), von Neumann (1938, 1955). Finally, on the application of quantum probability and quantum mechanics to other disciplines, such as finance: Khrennikov (2010), Rebentrost, et al., (2018); in psychology, and cognitive science: Khrennikov (2010), Busemeyer & Bruza (2012), Pothos and Busemeyer (2009, 2013), and Bruza et al. (2015).

### *4.1 Quantum Probability & Mechanics: The Postulates*

Quantum mechanics is a theory and mathematical framework developed in the early 20th century for the studying of the physics of the small, the atoms and the subatomic particles. Quantum probability is a more recent mathematical theory, developed in the 1980s to establish the mathematical foundation for quantum mechanics, particular in the statistical interpretations of the theory. The quantum probability framework complements the conventional probability framework to include non-commutative operator algebra known as the von Neumann algebra (Meyer, 1995; Redei and Summers, 2006; Chang, 2015). The framework reformulates the concept of probability based on a different set of axioms (or postulates) known as the Dirac-von Neumann axioms (from quantum mechanics) that is based on the theory of complex inner product vector space, the Hilbert space, and quantum operators. The original set of Dirac-von Neumann axioms were not developed specifically for quantum probability but to serve as the foundation principles for describing quantum mechanical systems in physics; Kronz and Lupher (2012) provide an interesting history and perspective on the different approaches and formalisms between Dirac and von Neumann. The postulates of quantum probability and mechanics can be formulated in a number of ways, and this section provides a synopsis of the postulates.

### 4.1.1 The Postulates.

The postulates of quantum mechanics:

*Postulate 1: State Space*

A quantum system is described by a state vector, a unit vector in a complex inner product vector space (a Hilbert space) called the space of states (state space). A quantum state vector $|\psi\rangle$, a unit vector, satisfies

$$\|\psi\|^2 = \langle\psi|\psi\rangle = 1\,, \qquad (4.1)$$

and where $\langle\psi|\psi\rangle$ or $\langle\psi|\phi\rangle$ denotes the inner product of two vectors.

*Postulate 2: Observables*

The measurable quantities and properties of a quantum system are called the *observables*. They are represented with self-adjoint linear operators. In order words, an observable O, a measurable quantity, is associated with an operator **O** that acts on the vectors in state space.

*Postulate 3a: Measurement*

The results from the measurement of observables are eigenvalues of the observable operator. If a measurement is performed on a system in a state

$|\lambda_i\rangle$ , the result of the measurement is $\lambda_i$ . $\lambda_i$ is the eigenvalue and $|\lambda_i\rangle$ is the eigenvector. The measurement is denoted by

$$\mathbf{O}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle . \quad (4.2)$$

Let $|\psi\rangle$, $|\phi\rangle$ denote 2 different quantum states of a system.

i. The probability of finding $|\psi\rangle$ in state $|\phi\rangle$ is given by the probability P where

$$P(|\psi\rangle,|\phi\rangle) = |\langle\psi|\phi\rangle|^2 . \quad (4.3)$$

This is known as the Born Rule.

ii. If a system is described by the state vector $|\psi\rangle$, and associated with this an observable O with $\mathbf{O}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ , then the probability of a measurement with outcome value $\lambda_i$ is

$$P(\lambda_i) = \langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle = |\langle\psi|\lambda_i\rangle|^2 \quad (4.4)$$

*Postulate 3b: Expectation Value*

The expectation value of an observable (operator) O, denoted by $\langle O\rangle$ , for a quantum system in state $|\psi\rangle$ is given by the inner product

$$\langle O\rangle = (\psi, O\psi) = \langle\psi|\mathbf{O}|\psi\rangle . \quad (4.5)$$

*Postulate 4: Evolution*

For a closed system, the evolution of system, the change from one state to another state is described by a unitary transformation, **U**: $|\psi'\rangle = \mathbf{U}|\psi\rangle$ . If the evolution is a function of time, then the unitary operator can vary with time:

$$\left|\psi(t_2)\right\rangle = \mathbf{U}(t_1,t_2)\left|\psi(t_1)\right\rangle, \qquad (4.6)$$

where $t_1$ and $t_2$ are points in time.

While these postulates are abstract in nature, the next section will put them into context and perspective.

## *4.2 Conceptual Summary and Basic Definitions*

The mathematical machinery for quantum mechanics consists of mathematical operators acting on objects in a complex inner product vector space known as a Hilbert space. Objects in Hilbert space are vectors, and events are outcomes which are points in this vector space. The states of a quantum system are described by a Hilbert space vector, which is simply called a state vector. The vector evolves by unitary operators, operations that preserve the magnitudes of the vector, and Hermitian operators, operations that ensure a physical measurement have real expectation values (and eigenvalues). The physical measurements return values that follow probability distributions; hence, the measured results are expectations. The probabilistic interpretation of quantum mechanics, the Copenhagen interpretation, was based on the work of Bohr and Heisenberg, and the statistical interpretation of quantum measurements was originated from Max Born, with the cornerstone law known as the Born rule, which is one of the fundamental postulates for quantum mechanics (see Hájek, 2012 for a general overview).  A system fully represented by a state vector is called a pure state system; a system represented by statistical mixing of the state vectors is called a mixed state system. These concepts are described in greater details below.

### 4.2.1 Inner Product Vector Space

The *inner product space* is a vector space where a number of specific structures and operations, such as the notion of length, the angle between vectors, and the inner product between vectors are defined. The inner product can be thought of as a generalization of the Euclidean space scalar product or dot product to multi-dimensional vector space (can be complex and infinite dimension). The inner product operation maps a pair of vectors to a scalar, and in bra-ket notations the inner product for the vectors $\left|A\right\rangle$ and $\left|B\right\rangle$ is written as: $\left\langle B\middle|A\right\rangle$. Inner product satisfies the following conditions:

1) Linearity: $\left\langle C\right|\left\{\left|A\right\rangle+\left|B\right\rangle\right\}=\left\langle C\middle|A\right\rangle+\left\langle C\middle|B\right\rangle$

2) Conjugation: $\left\langle B\middle|A\right\rangle=\left\langle A\middle|B\right\rangle^{*}$

3) The normalized vector: $\left\langle A\middle|A\right\rangle=1$

4) Orthogonality condition: $\left\langle B\middle|A\right\rangle=0$

5) Positive definite: $\left\langle A\middle|A\right\rangle\geq 0$ and $\left\langle A\middle|A\right\rangle=0$ iff $\left\langle A\middle|A\right\rangle=0$

### 4.2.2 Events

Intuitively, events are “points in spacetime” where something happened leading to a condition, an outcome, or a result. For example, a risk event is the occurrence of “something” within the system in consideration, resulting in some outcome (usually negative) that has an impact on the system or others. A catastrophic event, for example, can be the destruction of a system such as a spacecraft. In classical terms, the flipping of a coin leads to an event, resulting with one of the coin face facing up, and the outcome is either head or tail. Thus, the events form the set that contains all of the possible outcomes.

Within the quantum framework, events are points in a complex inner product vector space known as Hilbert space; the vector space contains the complete set of possible outcomes. A particular collection of events forms a subspace in Hilbert space, and many subspaces can be formed within the Hilbert space. For example, the events resulting from the rolling of a 6-faces dice form a subspace, as do the events resulting from the flipping of a coin.  A coin with its head facing up is the event (outcome) of the system (the coin) after the occurrence of a measurement (the observation). The subspace spanned by all the possible system states forms the state space for the system.

Furthermore, in this framework each elementary outcome is a basis vector in the subspace, and as a consequence, a vector in Hilbert space. These basis vectors span the vector space and orthogonal to each other. Different state vectors can be formed from the linear combination of these basis vectors. The classical set theory analogy of the subspace concept is the subset: Subset (classical) → Subspace (quantum). This is only an analogy, since the logic of subspace follows the axioms and principle of quantum mechanics and does not follow the axioms of set theory.

Finally, symmetries in operations, such as measurements, might not be preserved for events in Hilbert space. The order of events, such as the order of measurements, can result in different end outcomes.

### 4.2.3 The Quantum System

Consider a physical experiment with a system. The experimental process consists of two steps: the first step involves setting up the system to generate possible outcomes, and the second step is to extract the outcome value by performing a measurement (with an apparatus for example). The quantum framework provides the mathematical tools and objects to model this process. Instead of possible outcomes and values, in the quantum framework they are replaced by the probabilities and statistical measurements of outcomes.

A quantum system, therefore, is described by two types of mathematical objects. The first one characterizes the various possible configurations for the system in terms of probabilities, which is call the *state* of the system. The second describes the parameters and values one can measure about the system in terms of statistical outcomes, which are call the *observables*. A quantum system is characterized by it states, observables, and outcomes form measurements, which is call expectation value (average value of a measurement). Associated with these properties are operations describing the behaviors of them. The dynamic behaviors of these properties are described by operations on the state vector. The state vector contains all the information (organized facts) and knowledge (relationships and patterns about facts) about the possible states of the system. This state vector is a superposition of all the possible states the system can be in (the concept of superposition is further discussed in section 4.5). At any given time, the state of the system is described by the state vector (next section). To determine the state of a quantum system at an instance, a measurement is performed on the system for an observable event.

#### 4.2.4 State Vector

The collection of all possible events defines the quantum state space and the state of a quantum system is specified by a vector in the system's state space called the state vector. State vectors completely specify the properties of quantum systems. Mathematically, a state vector $|\Psi\rangle$ is expressed as the *superposition* of the basis states $|\psi_i\rangle$, which are the orthonormal basis vectors in the system's state space, and it is expressed as:

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle \tag{4.7}$$

where $c_i$ are the probability amplitudes, and the squares of the amplitudes $c_i^2$ gives the probability of finding the system in state $|\psi_i\rangle$. Earlier in Chapter 3, the expression for the levee system as $|L\rangle = l_{|0\rangle}|0\rangle + l_{|1\rangle}|1\rangle$ represents a system with two basis states, $|0\rangle$ and $|1\rangle$, and the squared magnitude of $l_{|0\rangle}$ and $l_{|1\rangle}$ represent the intrinsic aleatory uncertainty associated with the working and failing states respectively.

The state vector encapsulates all the information about the probabilities for the occurrence of events, what can happen, and how systems evolve. This strikes in contrast to classical states, where a state (a scalar) is a probability function connecting events to probabilities. Classical event probability is obtained from the linear summation of the elementary event probabilities, which has a deterministic outcome, and follows the law of total probability. In contrast, since quantum states are Hilbert space vectors, event probabilities are captured within the state vectors, and their evolutions are defined by vector operations on them.

#### 4.2.5 Measurements and Observables

A system can be found in any of the possible system states, but the precise state of the system is undetermined until a measurement is performed on it. Reality becomes real at the occurrence of events and outcomes. After flipping a coin, one must look at it to determine which side is facing up, which is a measurement of the outcome of an observable.

A measurement is an operation performed on a system resulting in an outcome. The quantities that measurements can be performed on are called the physical observables, or observables in short. In the quantum framework, physical observables are represented by linear operators. Care must be given to distinguish between a measurement of observable O vs. the operator **O** operating on a state vector. A measurement of the observable O will result in a value for the system's state. The operator **O** operating on a state vector results in a state vector. Observables and operators are connected to each other by probability and expectation values.

This point was made to distinguish between the measurement of an observable O and the operator **O** operating on a vector. In quantum mechanics, an observable does not have a precise value, but a range or a spectrum of values. When a measurement is performed on the observable, the results can be any one of the values within the range or spectrum belonging to the observable. The result of a measurement is essentially random, and the act of performing a measurement collapses the system into an eigenvector. If the exact measurement is to perform right after the first one, the same result will be observed since the system has already been collapsed into a final state.

4.2.6 Probabilities and Expectation Values

The relationship between probability, state, and probability amplitudes is given by the Born Rule, which simply states that the probability of finding an outcome $P(x)$ is equal to the squared of the probability amplitude $a_x$, corresponding to that outcome:

$$P(x) = |a_x|^2 \qquad (4.8)$$

For a system $|A\rangle$, the probability of finding it in the "i" state is therefore

$$P(i) = \langle A|i\rangle\langle i|A\rangle = |a_i|^2 = a_i^* a_i \qquad (4.9)$$

The total probability for a quantum system, in order to preserve unity, requires that the squared sum of the probability amplitudes to be equal to one.

$$\sum_i a_i^* a_i = 1 \qquad (4.10)$$

This is equivalent to saying that the state vector is normalized to a unit vector

$$\langle A|A\rangle = 1 \qquad (4.11)$$

In the quantum regime, the law of total probability is different in the following way. Since the state of a system is defined by the state vector, total probability is thus defined by the normalization condition itself.

In summary, the state of a quantum system is represented by normalized unit vector in the state space. The squared values of the probability amplitudes, which correspond to the squared magnitude of the components of the basis vectors, give us the probabilities of finding the system in those states. A measurement collapses the state vector into a particular state, and the projection is the event occurrence probability.

The expectation value of an observable, meanwhile, can be thought of as the average or mean value of the measured outcomes of an observable. If O is an observable in a quantum system, then we denote the expectation value of the observable with $\langle O \rangle$. The expectation, or the average, is therefore define by

$$\langle O \rangle = \sum_i \lambda_i P(\lambda_i) \tag{4.12}$$

and can be expressed in state vector notation as

$$\langle O \rangle = \langle A | \mathbf{O} | A \rangle . \tag{4.13}$$

This is essentially the same as the classical definition of expectation in traditional statistics. Besides using it to compute averages, the expectation value will have important roles in dealing with mixed states system and the quantitative representation of risk.

### *4.3 Operators*

Earlier, the concept of operators was introduced as an operation that takes an input state vector (like the initial state of a system) to another state vector (the system at a different state). Mathematically, we define an operator **M** by its action on a vector, where **M** acts on the vector $|A\rangle$ to give $|B\rangle$:

$$\mathbf{M} | A \rangle = | B \rangle . \tag{4.14}$$

The operators are essentially "models of the world" that govern how the system states can be changed. In the quantum operator framework, there are several types of operators. This study will focus on a subset of the operators: state transformation operators and projectors. State transformation operators include Unitary operators, Projection operators (Projectors), and Density operators, and they are defined below.

#### 4.3.1 Unitary Operators

Unitary operators keep the unit vector unchanged in magnitude. In other words, while the system might have changed to a new state, the system itself has not, keeping the same set of basis vectors and unit magnitude for the state vector. This operator can be a useful tool to model evolutionary effects. For example, consider again the flood wall. At $t = 0$, when the system is new, the failure probability could be 1%. However, as time goes by, materials weaken due to environmental effects: wind, rain, erosion, etc., and the probability of finding the system at a failing state (the failure probability) would be expected to increase. A unitary operator can be used to model the "operations" which in this example would be an environmental effect, changing the state from one to another while preserving the total probability and hence the system. An example of a unitary operator is the rotation operator, which rotate the state vector and hence changing the probability amplitudes but preserving the length of the vector.

#### 4.3.2 Projection Operators or Projectors

Projectors are operators that can select a certain state or a set of states from the system and project them into a subspace, resulting with a state vector with a different set of basis. For example, given a state vector $|\psi\rangle = a_1|a_1\rangle + a_2|a_2\rangle$ and another vector $|\lambda\rangle = |a_2\rangle$, a projection operator can be constructed to project $|\psi\rangle$ onto the direction specified by $|\lambda\rangle$ and return with a vector along $|\lambda\rangle$. One can think of the basic projector as "asking the question" or "making a query" about a particular state or states of the system and returning with a state vector "answering" the question. The projector can be used to query a system for information about a particular state, such as calculating the probability of observing the system in a particular state.

The abstract mathematical notation for a projector is denoted by $|a_i\rangle\langle a_i|$ in the bra-ket notation. Examples of projectors that select a subspace: $|0\rangle\langle 0|$ projects out the $|0\rangle$ component, $|1\rangle\langle 1|$ projects out the $|1\rangle$ component, and in general $|a_i\rangle\langle a_i|$ projects out the $i^{th}$ component. The $|0\rangle\langle 0|$ can be interpreted as asking the question, "what is the risk state vector representing the failing state?" This operation projects out the failing state vector together with the corresponding probability amplitude. To answer the question "What is the risk of failure?" one can then compute the probability of the state vector for the failure state, which is $a_i^2$. Certainly, one can also directly calculate the probability of the failure state by the direction application of $P(\lambda_i) = \langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle = |\langle\psi|\lambda_i\rangle|^2$. Note that it is a projector that is sandwiched between the state vector.

Consider again the levee L (Section 3.2.1). When a builder delivers a levee to a customer, the customer can ask the question "what is the expected failure rate of the levee," and that can be recast into the question "what is the probability of finding the levee in a failed state." The manufacturer states that the levee was made to the customer's specifications (the system is in a pure state, since the full knowledge about the system is available). Given that the customer has maximum knowledge (their specifications) about the system at the time of delivery, the question of what is the expected failure rate can be obtained by projecting out the $|0\rangle$ state vector, $|0\rangle\langle 0|L\rangle = l_0|0\rangle$, and the failure probability is simply the squared of the probability amplitude, $l_0^2$. Equivalently, one can arrive at the same result from:

$$P(|0\rangle) = \langle L|0\rangle\langle 0|L\rangle = l_0^2. \tag{4.15}$$

The projection operator can also be applied as part of an operation to change the system (the mathematical model) into the one as observed (a projective measurement). When the projector is applied as a measurement, such as a projective measurement, the projector selects the subset of the basis as observed and construct a new normalized state vector. For example, when a hurricane destroyed a levee, the system is in a 100% failing state. The projective measurement is the application of the projector to transition the system from the initial state (before the hurricane hit) to the observed state (after the hurricane hit).

This second use of the projector "reconfigures" the system model, the state vector, to match what is measured and observed. In the case where a system changed due to some event resulting in the loss of a basis, the projection operation is used to transition the system from the "before observation" state to the "after observation" state, matching the observed result. The new state vector is normalized to reset the total probability to unity for the system. Consider the trivial scenario where there is a hurricane, the levee is damaged and is no longer working. Physically, one observed that the levee has been leveled, which means the system can only be in the $|0\rangle$ state. To represent the state of the system after the hurricane, one first performs the projective measurement to project out the observed state, and subsequently normalize the state vector so that it now reflects the final "new" state of a whole system.

Mathematically, the first step to project out the failure state is done by:

$$|0\rangle\langle 0|L\rangle = l_0|0\rangle, \tag{4.16}$$

and then follow with the normalization of the state vector for the "new" system, L':

$$|L'\rangle = \frac{l_0|0\rangle}{\left\| l_0|0\rangle \right\|} = \frac{l_0|0\rangle}{\left|\langle 0|l_0^2|0\rangle\right|} = \frac{l_0|0\rangle}{\sqrt{l_0^2}} = |0\rangle \tag{4.17}$$

This is the final state since this is the only real state. The probability of finding the system in that state therefore is 1, which one recovers from the renormalization process. The state vector $|L'\rangle$ is now the description of the actual observed system.

4.3.3 Density Operators, a Preview

The density operator ($\hat{\rho}$) or matrix ($\rho$) is the equivalent to the state vector when use to describe a system, including composite systems such as one formed from two or more individual subsystems, or a system formed from a mixed ensemble of subsystems (large number of subsystems, but with many of them in similar states and configurations). All the information about the composite quantum system is encoded in the density matrix (a density operator with a chosen basis).

The density operator is the counterpart of the observable operator. Recall that if O is the observable for state $|\lambda_i\rangle$ ($\mathbf{O} = |\lambda_i\rangle\langle\lambda_i|$), the probability of finding a system in state $|\lambda_i\rangle$ for a given state vector $|B\rangle$ is given by $P(\lambda_i) = \langle B|\lambda_i\rangle\langle\lambda_i|B\rangle$. In a parallel fashion, the density operator, $\hat{\rho}$, is formulated to give the statistical average of the probability of finding an ensemble system or complex composite systems in a particular state. The trace of the density operator, $Tr(\hat{\rho})$ , is 1 which corresponds to the total probability of the system states. But before going further, a few concepts on constructing complex composite systems with tensor operations must first be introduced.

*4.4 Tensor Products*

To create composite systems, one needs mathematical operations to connect different state vectors, forming new composite state vectors with new states. *Tensor product* is such a tool to create new (bigger) vector spaces from some existing vector spaces or to create new linear operators. Earlier, the *Inner product* was introduced as an operation that maps a pair of vectors to a scalar. *Tensor product* (⊗) is the product of tensors (of which vectors are subsets), and the result is an expanded space formed from combining vector spaces together. For example, the tensor product of two vectors **A** and **B** is denoted by: $\mathbf{A}\otimes\mathbf{B}$ . General tensor product in matrix notation (the Kronecker product) is given by:

$$\mathbf{A}\otimes\mathbf{B}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\otimes\mathbf{B}=\begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}=\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

(4.18)

Example of the *Kronecker* (tensor) product ($\left|A\right\rangle\otimes\left|B\right\rangle$) of two 3D vectors, **A** and **B**, resulting in a vector with a new basis:

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{AB} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} A_1 B_1 \\ A_1 B_2 \\ A_1 B_3 \\ A_2 B_1 \\ A_2 B_2 \\ A_2 B_3 \\ A_3 B_1 \\ A_3 B_2 \\ A_3 B_3 \end{pmatrix} \tag{4.19}$$

4.4.1 Outer Product

The *Outer product* is a special case of the tensor product. *Outer product* is the product of two vectors forming an operator. The outer product is a linear operator $\mathbf{Op} \equiv |\psi\rangle \otimes \langle\phi| \Leftrightarrow |\psi\rangle\langle\phi|$ defines by its operation on another vector $|A\rangle$:

$$\mathbf{Op}|A\rangle = |\psi\rangle\langle\phi||A\rangle \tag{4.20}$$

and following the bra-ket computation rule,

$$|\psi\rangle\langle\phi||A\rangle = |\psi\rangle\langle\phi|A\rangle \tag{4.21}$$

where $\langle\phi|A\rangle$ is the inner product of $\langle\phi|$ and $|A\rangle$. Since $\langle\phi|A\rangle$ is simply a scalar value, it can be switched to the left side, leading to the following:

$$\left|\psi\right\rangle\left\langle\phi\right|A\rangle=\left\langle\phi\right|A\rangle\left|\psi\right\rangle. \qquad (4.22)$$

To summarize, outer product is defined by the following linear operation:

$$\left(\left|u\right\rangle\left\langle v\right|\right)\left(\left|w\right\rangle\right)=\left|u\right\rangle\left\langle v\right|w\rangle=\left\langle v\right|w\rangle\left|u\right\rangle \qquad (4.23)$$

In a similar fashion, the operator **Op** can operate on a bra vector $\left\langle B\right|$ by taking the inner product of $\left\langle B\right|$ and $\left|\psi\right\rangle$:

$$\left\langle B\right|\left|\psi\right\rangle\left\langle\phi\right|\equiv\left\langle B\right|\psi\rangle\left\langle\phi\right|. \qquad (4.24)$$

Assuming $\left|\psi\right\rangle$ is a normalized vector, one can construct a special *outer product* $\left|\psi\right\rangle\left\langle\psi\right|$, called the *projection operator*. When applied to another vector, $\left|A\right\rangle$, it projects the vector onto the direction of $\left|\psi\right\rangle$:

$$\begin{aligned}\left|\psi\right\rangle\left\langle\psi\right|\left|A\right\rangle &\equiv\left|\psi\right\rangle\left\langle\psi\right|A\rangle\\ &=\left\langle\psi\right|A\rangle\left|\psi\right\rangle\end{aligned} \qquad (4.25)$$

In matrix notation, an example of the *Outer* (tensor) product ($\left|A\right\rangle\otimes\left\langle B\right|$) of two 3D vectors, **A** and **B**, resulting in an operator (a matrix):

$$\mathbf{A}=\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix},\mathbf{B}=\begin{pmatrix}B_1 & B_2 & B_3\end{pmatrix}$$

$$\mathbf{A}\otimes\mathbf{B}=\mathbf{AB}^T=\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}\begin{pmatrix}B_1 & B_2 & B_3\end{pmatrix}=\begin{pmatrix}A_1B_1 & A_1B_2 & A_1B_3\\A_2B_1 & A_2B_2 & A_2B_3\\A_3B_1 & A_3B_2 & A_3B_3\end{pmatrix}$$

(4.26)

A useful theorem on how to calculate the expectation value using the outer product operator:

For any observable O, the expectation value of it at a given state $|\psi\rangle$ can be calculated by finding the trace of the projection of the observable:

$$\langle\psi|\mathbf{O}|\psi\rangle = Tr|\psi\rangle\langle\psi|\mathbf{O}. \qquad (4.27)$$

4.4.2 Density Operators

The *density operator* is a linear combination of the different possible states of the system, and it is used to construct the mixed state quantum representation. The density operator ( $\hat{\rho}$ ) is the equivalent to the state vector for a describing a system and the density matrix is simply a density operator with a chosen basis. Since the density operator can be expressed as a matrix with a specified basis, and as such, the term density operator and density matrix are often used interchangeably. With tensor products, the density operator ( $\hat{\rho}$ ) or matrix ( $\rho$ ) first mentioned in 4.3.3 can now be established. Density operators are constructed from the outer product of state vectors. In its most general form, the density operator $\hat{\rho}$ is expressed as:

$$\widehat{\rho} = \sum_i p_i|\chi_i\rangle\langle\chi_i|, \qquad (4.28)$$

where the set of $|\chi_i\rangle$ are normalized state vectors (yes, state vectors), and $p_i$ is the probability of finding the system described by state vector $|\chi_i\rangle$, with $0 \le p_i \le 1, \sum_i p_i = 1$. The density operator allows the representation of a system even when one does not have full and precise knowledge about the system.

What does it mean by having full knowledge about the system? One has full knowledge about the system when one can clearly describe the system with precisely a single state vector. For example, ten flood walls were built at the same time with the same specification, and the flood walls can all be described with the same state vector, say $|W_1\rangle$. Knowing that, if one randomly walks up to any one of them, one can be certain that $|W_1\rangle$ represents that flood wall; in other words, the density operator only projects onto one state, $|W_1\rangle$. The density operator is therefore:

$$\hat{\rho} = |W_1\rangle\langle W_1|, \tag{4.29}$$

and this is called a *pure state*.

One does not have full knowledge about the system when one cannot describe the system with a single state vector; alternatively, this can be interpreted as one lacking enough knowledge to isolate the state vector to a single one. If the ten flood walls earlier were built to not one but two different specifications, seven were built according to specification “1” and three were built according to specification “2”. Some of them will then be described by the state vector $|W_1\rangle$ and some by $|W_2\rangle$. Now if one randomly walks up to any one of them, the flood wall cannot be purely described by $|W_1\rangle$ or by $|W_2\rangle$, but by a mixture of the two. In other words, the density operator must consist of several operators and maps to the two states. This *mixed state* density operator, for this flood wall example, is given by:

$$\hat{\rho} = p_1|W_1\rangle\langle W_1| + p_2|W_2\rangle\langle W_2|, \tag{4.30}$$

where $p_1 = 0.7$ and $p_2 = 0.3$, corresponding to the probabilities of encounter $\left|W_1\right\rangle$ and $\left|W_2\right\rangle$ respectively.

*Expectation Value for Density Operator* — The expectation value of an observable can be calculated directly with density operators. Recall earlier the definition of *Expectation Value (*or just *Expectation),* the expectation value $\left\langle \mathrm{O} \right\rangle$ is given by the inner product $\left\langle \mathrm{O} \right\rangle = (\psi, O\psi) = \left\langle \psi \right| \mathbf{O} \left| \psi \right\rangle$ (equation 4.5), and the expectation can also be obtained from taking the trace (equation 4.27): $\left\langle \mathrm{O} \right\rangle = \left\langle \psi \right| \mathbf{O} \left| \psi \right\rangle = Tr \left| \psi \right\rangle \left\langle \psi \right| \mathbf{O}$. For an observable O of the ensemble system, the ensemble average, the expectation $\left\langle \mathrm{O} \right\rangle$, is given by

$$\left\langle \mathrm{O} \right\rangle = \sum_i p_i \left\langle \psi_i \right| \mathbf{O} \left| \psi_i \right\rangle . \tag{4.31}$$

This can be re-expressed as $\left\langle \mathrm{O} \right\rangle = Tr(\rho \mathbf{O})$ since:

$$\left\langle \mathrm{O} \right\rangle = \sum_i p_i \left\langle \psi_i \right| \mathbf{O} \left| \psi_i \right\rangle = Tr\left( \sum_i p_i \left| \psi_i \right\rangle \left\langle \psi_i \right| \mathbf{O} \right) = Tr\left( \rho \mathbf{O} \right) \ \blacksquare \tag{4.32}$$

The expectation value of the observable for a composite system with mixed states is the weighted sum of the expectation values of O for the mixed states. Moreover, there is a handy way to compute the expectation value, which is simply the trace of the product of the density matrix and the matrix representation of the observable O.

## *4.5 Features*

The quantum framework extends the Kolmogorov framework in directions for which there are no Kolmogorov analogues. For example:

1. *Non-commutative algebra*: operators are matrices in the quantum framework. As a result, quantum operators inherit matrix properties, and one of it is the non-commutative nature of matrix operations. When applying the sequence of operations “A and B” on a state vector, the resulting event might be different than those obtained from the sequence of operations “B and A”.

2. *Superposition*: when a system is described by a state vector, such as the earlier flood wall example, $\left|L\right\rangle = l_{|0\rangle}\left|0\right\rangle + l_{|1\rangle}\left|1\right\rangle$, the vector is formed from the linear combination of two states, the working $\left|1\right\rangle$ and non-working $\left|0\right\rangle$ states. The flood wall $\left|L\right\rangle$ in this case is in a state other than working or not working. Any linear combination of $\left|0\right\rangle$ and $\left|1\right\rangle$ is an acceptable state for the system, and these states are called *superposition* states.

3. *Pure & Mixed States*: traditional approach to build a description a system and its states relies on gaining maximum and complete knowledge about the system. A state vector that is constructed with full knowledge is therefore pure; the pure state density operator $\rho = \left|S_1\right\rangle\left\langle S_1\right|$ projects only to a single state. In the case where full knowledge is not available, a pure state vector cannot be constructed for a system since it cannot be described precisely; however, the density operator formalism does allow the construction of a quantum model using a mixture of possible states. The mixed state density operator $\rho = p_1\left|S_1\right\rangle\left\langle S_1\right| + p_2\left|S_2\right\rangle\left\langle S_2\right| + p_3\left|S_3\right\rangle\left\langle S_3\right| + \ldots$ projects onto the number of possible states a system can be found in.

4. *Entanglement*: quantum entanglement describes the quantum state of a joint

system, such as a pair of particles, where the entangled quantum state describes joint system completely, but the quantum state of the individual particle can be partially or entirely unknown. At maximum entanglement, the state of the joint system is completely known, yet the individual states are complete unknown. A consequence of the framework is the ability to have full knowledge of a composite system, based on the combination of subsystems for which we only have partial knowledge.

5. Taking the concepts of mixed state and entanglement further, the density operator introduces the notion of making predictions and with partial knowledge and ignorance. Probabilities and expectation values can be obtained from the density operator via the diagonal entries of $\rho$ and $\langle O \rangle = Tr(\rho\mathbf{O})$.

The density operator formalism is effective in describing a wide range of system properties and the amount of information one has about the system. The pure state is simply a density operator with a single term: $\rho = |S_1\rangle\langle S_1|$ as the system is known precisely to be in this state, and the probability of finding the system in that state is 1.0. The mixed state system is described as a statistical sum:

$$\rho = p_1|S_1\rangle\langle S_1| + p_2|S_2\rangle\langle S_2| + p_3|S_3\rangle\langle S_3| + \ldots, \tag{4.33}$$

which reflects lack of precision and therefore the description is a mixture of possible states, $|S_1\rangle, |S_2\rangle, \ldots, |S_n\rangle$. The number of components can provide a measure of how much or how little one knows about the state of the system. The longer the sum, the larger the possible number of states, and the larger the uncertainty. However, if one cannot fully establish the completeness of the system states, as in the case of deep uncertainty with “unknown-unknowns”, then the expression might take the different form:

$$\rho = p_1 \left| S_1 \right\rangle \left\langle S_1 \right| + p_2 \left| S_2 \right\rangle \left\langle S_2 \right| + p_3 \left| S_3 \right\rangle \left\langle S_3 \right| + \ldots + p_n \left| D \right\rangle \left\langle D \right|, \quad (4.34)$$

where $p_n \left| D \right\rangle \left\langle D \right|$ corresponds to the unknown-unknowns. The density operator formalism opens up new opportunity to model risk problems.

The density operator formalism increases the amount of information that can be encoded in the system models. In current approaches, uncertainties are encoded in a single set of probabilities. In the density operator formalism, uncertainties are encoded with two sets of probabilities. State vectors (quantum) contain one set of probabilities in the form of probability amplitudes, while a second set of probabilities (classical) describes the distribution of the states. This formalism provides additional bandwidth in encoding system information and more information can be captured and thereby potentially increase fidelity. These features are explored in this research for the modeling and analysis of risk scenarios, and they represent a subset of what quantum probability adds to the risk modeling toolbox beyond conventional ones.

### *4.6 Risk Analysis with Elements of Quantum Probability and Mechanics*

In this section, concepts such as state vectors, operators, density operators, and projectors will be introduced in the context to reframe the discussion of the risk states and their evolution behaviors for complex dynamic systems. The concept of risk will be refined to align with the quantum approach for an engineering system as defined next.

An engineering system is described by state vectors, formed from the linear combination of the basis states the system can be found in. The probability of finding the system in a particular state is the squared of the scalar component(s) of the basis state(s) – the probability amplitudes. Composite systems (with 2 or more subsystems) can be assembled from the tensor products of the state vectors. The fidelity of the engineering system model with the state vectors is a direct reflection of the amount of knowledge and the degree of ignorance (as defined in Aven & Steen, 2010) about the system.

The engineering system evolves according to events, which are represented by mathematical operators acting on the state vectors:

1. The operators contain the physics that change the effective uncertainties about the states, such as how a load or other tracer parameters can affect the failure probabilities.
2. The operators can preserve the number of system states, or can fundamentally change the system by altering the number of states, resulting in a new system.
3. Interaction with the environment can alter the operator/system, etc.

The above properties and characteristics are fully captured by the density operator:

$$\rho = \sum_i p_i \left| L_i \right\rangle \left\langle L_i \right|, \tag{4.35}$$

and the transition of the system from states to states are given by:

$$\rho_{L'} = \mathbf{H} \rho_L \mathbf{H}^\dagger = \sum_i p_i \mathbf{H} \left| L_i \right\rangle \left\langle L_i \right| \mathbf{H}^\dagger \tag{4.36}$$

To summarize, an engineering system can be found in different states, and these states are represented by the state vectors. The collection of these state vectors forms the engineering system and can be expressed in the form of the density operator. Complex system models can be constructed from combining components and subsystems together via their tensor products, a process that aligns well with basic system engineering principles in the decomposition and construction of an engineering system. The engineering system evolves in time, transitioning from states to states, and the mathematical abstractions representing the transitions are the event operators.

*Risk in the quantum context* — Risk is often defined as the probability of the risk event multiplied by the magnitude of the consequence or impact: Risk = Probability × Consequence. In this definition, risk is an event associated with a probability and a consequence. The uncertainty is on the occurrence of the event and a probability value is assigned to the event with the specific outcome. Mathematically, the consequence term is a quantitative value representing the significance of the outcome. The total risk is the expected loss due to the risk event.

Often, risk assessments start with asking the question: "What is the probability of the occurrence of event A?" This question will now be reframed to align with the quantum mindset. Instead of asking for the probability of the occurrence of event A, we now ask for the probability of finding the system in a particular state. This reframing of the question leads to the following definition of risk states, risk, and the risk system:

- A *risk state* is a system state that can potentially impact, positively or negatively (risk vs. opportunity), the outcome of a system event.
- *Risks* are the probabilities of finding the system in risk states, couple with the significance of the consequence associated with those states.
- *The risk system* is that collection of possible system states that can have impacts (usually negative) to the outcome as specified by the risk analyst, who then derives some quantities to represent the degree of severity for those states.

Risk assessment obtains the probability of finding certain risk states (relevant to the question) out of all possible states (the complete set of states) and assigns to them scalable factors to represent the degree of significance for making decisions. The numerical valuation or the magnitude of a risk is contextually driven and can be subjective.

These concepts are mapped to elements of the quantum framework according to the following. The system for risk analysis, referred to as the risk system in this study, is represented by a state vector, say $|A\rangle = a_0|0\rangle + a_1|1\rangle$ (recall the example in 3.1), formed from the linear combination of the possible states the system can be in. To accommodate incomplete knowledge, i.e. uncertainty about the completeness of the system state space, the general system will be represented by the density operator, which can represent both pure state (the normal state vector) and mixed states. Associated with each risk system is an observable called the *risk value* denoted by the operator **Ri**.

- Performing the measurement **Ri** on a system's basis state, yields the scalar *risk value* representing the significance of the consequence or impact.

If $|\lambda_i\rangle$ is a failure state, the result of the measurement **Ri** on the basis state is $\lambda_i$ corresponding to a numerical value representing the significance of the consequence, then the measurement is denoted by

$$\mathbf{Ri}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle . \tag{4.37}$$

For example, if a risk analyst assigns a value of 5 to the most significant failure state and 1 to the least significant state, then for the trivial case of a system with a binary working $|1\rangle$ and not working $|0\rangle$ states, $\mathbf{Ri}|0\rangle = 5|0\rangle$ and $\mathbf{Ri}|1\rangle = 1|1\rangle$.

The probability of finding a system in a certain state follows the Born Rule: $P(|\psi\rangle,|\phi\rangle) = |\langle\psi|\phi\rangle|^2$. The probability of the system in a non-working state can be obtained with $P(0) = \langle A|0\rangle\langle 0|A\rangle = |\langle A|0\rangle|^2 = a_0^2$. The value of risk is the expectation value of the observable Ri, for a system in state $|A\rangle$ it is denoted by $\langle Ri\rangle$ which is given by

$$\langle Ri\rangle = \langle A|\mathbf{Ri}|A\rangle . \tag{4.38}$$

*Another Trivial Example* — Returning to the earlier example in Chapter 3.1, where the levee-floodwall risk state vector for the system was denoted by $|A\rangle = a_0|0\rangle + a_1|1\rangle$. Consider again the flood wall example. Flood walls are built by a contractor, and each of the flood walls is made to certain specifications. Let's say for a high quality flood wall A, their failure probability is 0.01, i.e. the fundamental chance of finding a failing flood wall is 1 in 100. Let's say there is another class of flood wall B, not built as well, with failure probability of 0.02. To represent that:

Flood wall A: $|A\rangle = a_0|0\rangle + a_1|1\rangle$, $a_0 = \sqrt{0.01}$, $a_0 = \sqrt{0.01}$

Flood wall B: $|A\rangle = a_0|0\rangle + a_1|1\rangle$, $b_0 = \sqrt{0.02}$, $b_1 = \sqrt{0.98}$

Probability of finding A in failure state = $a_0^2 = 0.01$

Probability of finding B in failure state = $b_0^2 = 0.02$

Assigning a value of 5 to the most significant failure state and 1 to the least significant state, the risk of failure for:

A: $\langle R \rangle = \langle A | Ri | A \rangle = 1.04$

B: $\langle Ri \rangle = \langle B | Ri | B \rangle = 1.09$

On a scale of 1 to 5, the risk of failure is low for both A and B, and B is at a slightly higher risk. ■

The density operator represents the system, capturing the combination of aleatory, epistemic, and belief uncertainties.

- Risk analysis is the process of identifying corresponding risk states from the density matrix, calculating their chances of occurrence, and associating with them values of consequence.

To obtain the risk value, a measurement **Ri** can be performed to obtain the expectation:

$$\langle Ri \rangle = tr\left(\rho \mathbf{Ri}\right) \tag{4.39}$$

in which $\rho$ is the density operator and the righthand side is simply the trace of the product of two matrices $\rho$ and **Ri**.

# Chapter 5: A Quantum Model: Part 2

*Readers probably haven't heard much about it yet, but they will.*

*Quantum technology turns ordinary reality upside down.*

*- Michael Crichton*

Earlier Chapters have been focused on describing simple single systems, such as levee and floodwall. Here, composite engineering systems will be constructed from simple single systems.

## *5.1 Composite (engineering system)*

Composite systems are formed from component state vectors through the use of tensor product (Chapter 4.4). The (Levee + Flood Wall) system, for which we use the notion (L+W), consists of an earthen levee and vertical flood wall. Figure 8 is a schematic depiction of how the addition of a flood wall to a levee can help increase the vertical height of the protection system without increasing the horizontal width of the structure.

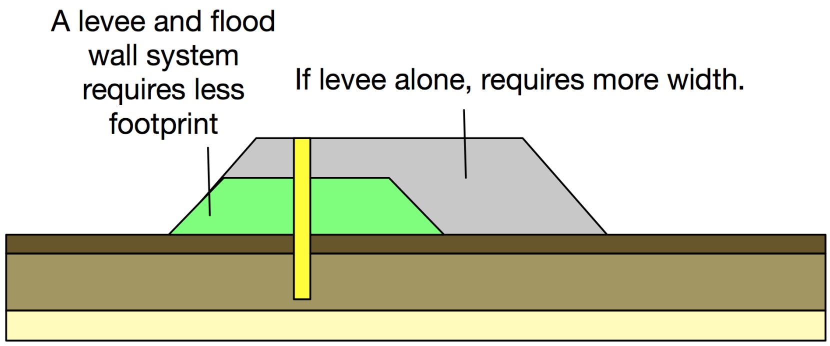


Figure 8: Levee and flood wall system

To describe this system using the quantum framework, one begins with the construction of the composite (L+W) system state vector, denoted by $|LW\rangle$, from the tensor product of the individual levee and flood wall state vectors. Let the levee state vector be $|L\rangle$ and the flood wall state vector be $|W\rangle$. The composite state vector is found from the tensor product of the state vectors:

$$|LW\rangle = |L\rangle \otimes |W\rangle = \begin{pmatrix} l_{|0\rangle} \\ l_{|1\rangle} \end{pmatrix} \otimes \begin{pmatrix} w_{|0\rangle} \\ w_{|1\rangle} \end{pmatrix} = \begin{pmatrix} l_{|0\rangle} w_{|0\rangle} \\ l_{|0\rangle} w_{|1\rangle} \\ l_{|1\rangle} w_{|0\rangle} \\ l_{|1\rangle} w_{|1\rangle} \end{pmatrix} \qquad (5.1)$$

In state vector form:

$$|LW\rangle = l_{|0\rangle} w_{|0\rangle} |00\rangle + l_{|0\rangle} w_{|1\rangle} |01\rangle + l_{|1\rangle} w_{|0\rangle} |10\rangle + l_{|1\rangle} w_{|1\rangle} |11\rangle \qquad (5.2)$$

This is the generic form of the state vector for the (L+W) system. Contextual information from different scenarios are then applied to configure the state vector into models for risk analyses. A few scenarios are presented below to serve as examples.

5.1.1 Scenario: Levee + Flood wall (L+W) Composite System

The scenario for the simple (L+W) composite system.

Model 5.M005 Scenario: Levee + Flood wall (L+W) Composite System

*The levee has an intrinsic failure probability of 0.03. An I-wall was built into the levee at the same time and its intrinsic failure probability is 0.01. The state vector describing this composite system is given by:*

$$\left|L_0^A\right\rangle = l_{0,|0\rangle}^A\left|0\right\rangle + l_{0,|1\rangle}^A\left|1\right\rangle = \sqrt{0.03}\left|0\right\rangle + \sqrt{0.97}\left|1\right\rangle \tag{5.3}$$

$$\left|W_0^A\right\rangle = w_{0,|0\rangle}^A\left|0\right\rangle + w_{0,|1\rangle}^A\left|1\right\rangle = \sqrt{0.01}\left|0\right\rangle + \sqrt{0.99}\left|1\right\rangle \tag{5.4}$$

We will drop the index for the system A since we are only considering the different states for the composite system in this example. The composite state vector is:

$$\left|LW\right\rangle = \left|L\right\rangle \otimes \left|W\right\rangle = \begin{pmatrix} l_{|0\rangle} \\ l_{|1\rangle} \end{pmatrix} \otimes \begin{pmatrix} w_{|0\rangle} \\ w_{|1\rangle} \end{pmatrix} = \begin{pmatrix} l_{|0\rangle} w_{|0\rangle} \\ l_{|0\rangle} w_{|1\rangle} \\ l_{|1\rangle} w_{|0\rangle} \\ l_{|1\rangle} w_{|1\rangle} \end{pmatrix} = \begin{pmatrix} \sqrt{0.03}\sqrt{0.01} \\ \sqrt{0.03}\sqrt{0.99} \\ \sqrt{0.97}\sqrt{0.01} \\ \sqrt{0.97}\sqrt{0.99} \end{pmatrix} = \begin{pmatrix} 0.0173 \\ 0.1723 \\ 0.0985 \\ 0.9799 \end{pmatrix}$$

■ (5.5)

The system can also be represented by a density matrix. Starting with the pure state density matrices representing the levee and the flood wall: $\rho_L = \left|L\right\rangle\left\langle L\right|$ and $\rho_W = \left|W\right\rangle\left\langle W\right|$, the density matrix for $\left|LW\right\rangle$, denoted $\rho_{LW}$, is:

$$
\begin{aligned}
\rho_{LW} &= \rho_L \otimes \rho_W \\
&= \left| LW \right\rangle \left\langle LW \right| \\
&= \begin{pmatrix} l_{|0\rangle} w_{|0\rangle} \\ l_{|0\rangle} w_{|1\rangle} \\ l_{|1\rangle} w_{|0\rangle} \\ l_{|1\rangle} w_{|1\rangle} \end{pmatrix} \begin{pmatrix} l_{|0\rangle} w_{|0\rangle} & l_{|0\rangle} w_{|1\rangle} & l_{|1\rangle} w_{|0\rangle} & l_{|1\rangle} w_{|1\rangle} \end{pmatrix} \\
&= \begin{pmatrix} 0.0173 \\ 0.1723 \\ 0.0985 \\ 0.9799 \end{pmatrix} \begin{pmatrix} 0.0173 & 0.1723 & 0.0985 & 0.9799 \end{pmatrix} \\
&= \begin{pmatrix} 0.0003 & 0.0030 & 0.0017 & 0.0170 \\ 0.0030 & 0.0297 & 0.0170 & 0.1689 \\ 0.0017 & 0.0170 & 0.0097 & 0.0965 \\ 0.0170 & 0.1689 & 0.0965 & 0.9603 \end{pmatrix}
\end{aligned}
\tag{5.6}
$$

■

Thus, the composite system's density matrix elements are the products of the probability amplitudes. The diagonal elements, $\rho_{LW,ii}$, of the density matrix correspond to the probabilities of the system states. In this example, the probability of the entire system failing, $P(|00\rangle)$, is 0.0003. The off-diagonal elements, $\rho_{LW,ij}$, when interpreted according to quantum principles, correspond to the interference between the probability amplitudes contributed from the basis states, the $|0\rangle$ and $|1\rangle$. The quantum formalism interprets this as coherences between the two states. In the full quantum formulation with the full Complex Hilbert space (Section 4.1.1), state vectors also contain time dependent components and phase information. The phases can evolve over time as the system interacts with the environment. As more and more mixtures are introduced into the system over time, the phases gradually spread out and become decoherent, and the interference terms approach zeros.

For this work on the experimental application of the quantum framework to represent engineering systems, due to the consideration of only the real components of the vector space, the interpretation of the off-diagonal terms for the corresponding density matrix has not been fully established. If one were to simply consider the edge cases, the off-diagonal terms have zero as their values when one of the probability amplitudes is zero, and that can only be true if the precise state of the system is available. The off-diagonal terms have maximum values when the probability amplitudes are equal. The off-diagonal terms can simply be interpreted as a quantitative indicator on the uncertainty of the states; it is a measurement of our completeness of knowledge about the system, about our ignorance. These concepts will be elaborated further in Chapter 7, Section 7.5.2.

#### 5.1.2 Scenario: Composite system with mixed states

For the mixed state (Section 4.4.2) density matrices representing the (L+W) system, there are two possible forms, where the first one is more restrictive and the second one is more general. The more restrictive form applies to composite systems where the components or subsystems do not interact; the composite system is formed from simple direct product of the component states. In this case, the composite system can be factor back into separate subsystem states. The state vector for composite system formed from product states takes the form of Equation 5.2 above, and the density matrix representation for a single state is simply:

$$\rho = |LW\rangle\langle LW| \tag{5.7}$$

The mixed state density matrix for this case takes the form:

$$\rho_{LW} = \sum_i p_i |L_i W_i\rangle\langle L_i W_i| \tag{5.8}$$

which is a simple weighted sum of the different joint states:

$$\rho_{LW} = p_1 |L_1 W_1\rangle\langle L_1 W_1| + p_2 |L_2 W_2\rangle\langle L_2 W_2| + \dots \tag{5.9}$$

In the more general setting, where the levee subsystem and the flood wall subsystem can behave separately, with their own separate states such as $L_1$, $L_2$, $W_1$, $W_2$, …, and with the assumption that the individual subsystems are tracked (i.e., full knowledge), the density matrix takes the general form:

$$\rho_{LW} = \sum_{i,j} p_i q_j |L_i W_j\rangle\langle L_i W_j| \tag{5.10}$$

where $p_i$ is the statistical weight for the mixed state levee subsystem and $q_i$ is the statistical weights for the mixed state flood wall subsystem. The matrix elements of the mixed state density matrix are more tedious to compute. The following is the 1-1 element (top left of the 4×4) of the matrix:

$$\begin{aligned}
\rho_{LW,00} &= \sum_{i,j} p_i q_j \left(l_{i,0} l_{i,0} w_{j,0} w_{j,0}\right) \\
&= \sum_i p_i q_1 \left(l_{i,0} l_{i,0} w_{1,0} w_{1,0}\right) + \sum_i p_i q_2 \left(l_{i,0} l_{i,0} w_{2,0} w_{2,0}\right) \\
&= p_1 q_1 \left(l_{1,0} l_{1,0} w_{1,0} w_{1,0}\right) + p_2 q_1 \left(l_{2,0} l_{2,0} w_{1,0} w_{1,0}\right) + p_1 q_2 \left(l_{1,0} l_{1,0} w_{2,0} w_{2,0}\right) + p_2 q_2 \left(l_{2,0} l_{2,0} w_{2,0} w_{2,0}\right)
\end{aligned} \tag{5.11}$$

While it can be tedious, the numerical computation can be done in a straightforward manner with Mathematica™, MatLab™, or in Python.

To illustrate the concept of building the mixed state density matrix, we consider a simple scenario with a composite system formed with two separable subsystems with two states each. In this case, the subsystems do not behave in a correlated fashion, which is modeled with the general mixed case representation. Subsystems that exhibit correlated behaviors will be discussed in next section, with further elaboration in Chapter 7.

Model 5.M006 Scenario: Composite system with mixed states

*The levee was built at time* $t_0$ *, but the quality of the construction varies, resulting in two possible levee states at time t, state 1 and 2. For state 1, the intrinsic (aleatory) failure probability is 0.03, and for state 2, the intrinsic failure probability is 0.08. The probability (epistemic) of finding the levee in state 1 is 0.65, and the probability of finding the levee in state 2 is 0.35. An I-wall was built into the levee at the same time, and it too suffers from quality issues, also resulting in two I-wall states, state 1 and 2 at time t. For state 1, the intrinsic failure probability is 0.01, and for state 2, the intrinsic failure probability is 0.04. The probability of finding the levee in state 1 is 0.25. The probability of finding the levee in state 2 is 0.75.*

The mixed states density matrix describing this system is constructed according to the following steps:

For the levee:

$$\left|L_1^A\right\rangle = l^A_{1,|0\rangle}\left|0\right\rangle + l^A_{1,|1\rangle}\left|1\right\rangle = \sqrt{0.03}\left|0\right\rangle + \sqrt{0.97}\left|1\right\rangle \tag{5.12}$$

$$\left|L_2^A\right\rangle = l^A_{2,|0\rangle}\left|0\right\rangle + l^A_{2,|1\rangle}\left|1\right\rangle = \sqrt{0.08}\left|0\right\rangle + \sqrt{0.92}\left|1\right\rangle \tag{5.13}$$

For the wall:

$$\left|W_1^A\right\rangle = w_{1,|0\rangle}^A\left|0\right\rangle + w_{1,|1\rangle}^A\left|1\right\rangle = \sqrt{0.01}\left|0\right\rangle + \sqrt{0.99}\left|1\right\rangle \tag{5.14}$$

$$\left|W_2^A\right\rangle = w_{2,|0\rangle}^A\left|0\right\rangle + w_{2,|1\rangle}^A\left|1\right\rangle = \sqrt{0.04}\left|0\right\rangle + \sqrt{0.96}\left|1\right\rangle \tag{5.15}$$

The density matrix for the levee:

$$\begin{aligned}\rho_L &= p_1\left|L_1\right\rangle\left\langle L_1\right| + p_2\left|L_2\right\rangle\left\langle L_2\right| \\ &= 0.65\left|L_1\right\rangle\left\langle L_1\right| + 0.35\left|L_2\right\rangle\left\langle L_2\right| \\ &= \begin{pmatrix} 0.0475 & 0.2058 \\ 0.2058 & 0.9525 \end{pmatrix}\end{aligned} \tag{5.16}$$

The density matrix for the wall:

$$\begin{aligned}\rho_w &= q_1\left|W_1\right\rangle\left\langle W_1\right| + q_2\left|W_2\right\rangle\left\langle W_2\right| \\ &= 0.25\left|W_1\right\rangle\left\langle W_1\right| + 0.75\left|W_2\right\rangle\left\langle W_2\right| \\ &= \begin{pmatrix} 0.0325 & 0.1718 \\ 0.1718 & 0.9675 \end{pmatrix}\end{aligned} \tag{5.17}$$

The composite density matrix constructed with tensor product:

$$\begin{aligned}\rho_{LW} &= \rho_L \otimes \rho_W \\ &= \begin{pmatrix} 0.0475 & 0.2058 \\ 0.2058 & 0.9525 \end{pmatrix} \otimes \begin{pmatrix} 0.0325 & 0.1718 \\ 0.1718 & 0.9675 \end{pmatrix} \\ &= \begin{pmatrix} 0.0015 & 0.0082 & 0.0067 & 0.0354 \\ 0.0082 & 0.0460 & 0.0354 & 0.1991 \\ 0.0067 & 0.0354 & 0.0310 & 0.1637 \\ 0.0354 & 0.1991 & 0.1637 & 0.9215 \end{pmatrix}\end{aligned} \tag{5.18}$$

■

5.1.3 Scenario: Composite system with entanglement (subsystems no longer treated as separable)

The composite system above consists of "product states," where the individual components are formed by simple products of the probability amplitudes. The composite system with product states can be factored back into the separate individual subsystems. The ability to decompose the system can be interpreted as having full knowledge of the individual subsystems, their characteristics and behaviors.

The more general form applies to composite systems where the components or subsystems exhibit joint or correlated behaviors (statistical correlation and not necessarily causal). In this case, the composite system state vector can be generalized further. Note that equation:

$$\left|LW\right\rangle = l_{|0\rangle} w_{|0\rangle} \left|00\right\rangle + l_{|0\rangle} w_{|1\rangle} \left|01\right\rangle + l_{|1\rangle} w_{|0\rangle} \left|10\right\rangle + l_{|1\rangle} w_{|1\rangle} \left|11\right\rangle \tag{5.19}$$

can have a more generalized form:

$$\left|LW\right\rangle = \alpha_{|00\rangle} \left|00\right\rangle + \alpha_{|01\rangle} \left|01\right\rangle + \alpha_{|10\rangle} \left|10\right\rangle + \alpha_{|11\rangle} \left|11\right\rangle \tag{5.20}$$

What does this general form with $\alpha_{|ij\rangle}$ represent? One possible interpretation is that the two subsystems are connected in such a way that they operate as one, and instead of two separate parameters $l_{|0\rangle}$ and $w_{|0\rangle}$, the probability amplitude associated with the $\left|00\right\rangle$ state, for example, is represented by a single parameter, $\alpha_{|00\rangle}$. Note that $\left|00\right\rangle$ is a single state, a basis state (see 4.4). This represents us as having full knowledge of how the composite system operates, with the behaviors of the subsystems correlated and the individual subsystem's behavior not fully known. A system satisfying this general form is in an entangled state, exhibiting correlation behaviors.

Take the (L+W) system as an example, if the levee is weakened by a water-filled gap, the flood wall will also be weakened due to a weakened foundation and an increase in pressure gradient on the flood wall. In this scenario, the exact state of the flood wall might not need to be fully characterized, but if the levee's failure state is imminent, the flood wall will probably fail as well (while this is a casual process, the model for the operator could be derived from statistics). The independent (or separable) $L_i$ and $W_i$ are not effective representations for this system.

Model 5.M007 Scenario: Composite system with entanglement (subsystems no longer treated as separable, which can be due to correlation or simply incompleteness of knowledge about their behaviors)

*For a (L+W) system, the subsystems' behaviors can be correlated and cannot be considered as separable. In this case, something happens to one of the subsystems will result in the change of the combined system. If the levee has been eroded, the flood wall structure can be weakened as a result. An example would be the failure case where the over-topping of the levee resulting in the erosion of the inner side of the levee, causing the tilting of the flood wall, increasing the probability of the catastrophic failure of the joint system, leading to a breaching event.*

In this scenario, one can no longer model the joint system with separable probability amplitudes, as in:

$$\left|LW\right\rangle = l_{|0\rangle}w_{|0\rangle}\left|00\right\rangle + l_{|0\rangle}w_{|1\rangle}\left|01\right\rangle + l_{|1\rangle}w_{|0\rangle}\left|10\right\rangle + l_{|1\rangle}w_{|1\rangle}\left|11\right\rangle \quad (5.21)$$

and the system must be represented in the more general form:

$$\left|LW\right\rangle = \alpha_{|00\rangle}\left|00\right\rangle + \alpha_{|01\rangle}\left|01\right\rangle + \alpha_{|10\rangle}\left|10\right\rangle + \alpha_{|11\rangle}\left|11\right\rangle \quad (5.22)$$

In this case, $\alpha_{ij} \neq l_i w_j$, and the state vector $\left|LW\right\rangle$ has four parameters and can be relabeled as:

$$\left|LW\right\rangle = \alpha_0\left|00\right\rangle + \alpha_1\left|01\right\rangle + \alpha_2\left|10\right\rangle + \alpha_3\left|11\right\rangle \quad (5.23)$$

The individual components will need to be determined empirically through observations and measurements with similar existing systems or test systems.

A note of clarification: $\alpha_{ij}$ and $l_i w_j$ are probability amplitudes, and the product of probability amplitudes is also a probability amplitude, since the product is still a component of the state vector. Only taking the square of the probability amplitudes will give probabilities, as prescribed by the Born Rule (see Section 4.1.1, Postulate 3a). Chapter 7 will revisit this concept of entanglement. ■

The density matrix for more complex systems can also be formed with tensor products of the composite subsystems. Using the shorthand, $\rho^A \equiv \rho^{(A)}$ , the density matrix for a composite system with many subsystems can be formed according to:

$$\rho^D = \rho^A \otimes \rho^B \otimes \rho^C \otimes \cdots \quad (5.24)$$

Since the HPS consisting of many (L+W) systems, the composite representation takes the form:

$$\rho_{HPS} = \rho_{L^A W^A} \otimes \rho_{L^B W^B} \otimes \rho_{L^C W^C} \otimes \cdots$$
$$= \left| L^A W^A \right\rangle \left\langle L^A W^A \right| \otimes \left| L^B W^B \right\rangle \left\langle L^B W^B \right| \otimes \left| L^C W^C \right\rangle \left\langle L^C W^C \right| \otimes \cdots \tag{5.25}$$

The above tensor product form is great if the detail knowledge and specifications about the individual systems and states are available, but that is often not the case. For example, a system with five subsystems will have 1024 matrix elements. Each subsystem can have multiple states. If each subsystem can be in 3 states, the system will have $\sim 10^9$ matrix elements, and tracking them, if they are all available, can be daunting. Fortunately, an alternative interpretation and usage of the density matrix might offer a solution.

#### 5.1.4 Scenario: Composite systems treated as an ensemble, incorporating ignorance and uncertainties

The density matrix can be interpreted in three different ways:

- as a fundamental framework (a mathematical construct, section 4.3),
- as a system of subsystems (subsystems can be similar or different, such as a flood protection system consists of a levee and a flood wall, section 5.1.1), or
- as an ensemble, which is a large collection of identical or near identical systems (e.g. a storm protection system consists of 25 identical levees is an ensemble of levees).

Earlier scenarios made use of the first two interpretations. By taking the ensemble interpretation, requiring the assumption that the subsystems are in similar states, the density matrix representation can be simplified to only capture and track the statistical behaviors of the ensemble. In doing so, we sacrifice the exact knowledge of the individual components (no longer keeping track of every states for every subsystem) and just track the ensemble states:

$$\rho_{HPS} = \sum_i p_i^S \left| L_i^S W_i^S \right\rangle \left\langle L_i^S W_i^S \right| \tag{5.26}$$

where the index "i" tracks not the individual (L+W) system but the state where a number of the (L+W) systems can be found in. The term $p_i$ is the probability of finding (L+W) systems in the state.

This ensemble interpretation also introduces a new perspective into the modeling process. Recall that for the earlier models, the composite representation (such as equation 5.25) is exact and complete; all available information are incorporated into the formulation of the density matrix. With the ensemble view, the composite representation is not exact and not all available information or knowledge are used in the formulation of the density matrix. By choice, ignorance is introduced into the model formulation. The term ignorance is generally referring to the lack of information and knowledge, an absence of awareness of missing knowledge, or it can also describe the state where information and knowledge are deliberately discarded or ignored. The ensemble interpretation leads to an incomplete description of the system, and in doing so, introduces uncertainty (a measure of completeness and preciseness) into the model. These concepts will be elaborated further in Chapter 7.

For very complex systems, this alternative interpretation of the density matrix can simplify the computational complexity, but at the cost of precision (by the introduction of the concept of ignorance).

Model 5.M008 Scenario: Composite systems treated as an ensemble, incorporating ignorance and uncertainties

*There are 138 reaches for the New Orleans HSDRRS. The levees and flood walls were constructed under three projects over three different periods. For this example, the projects are referred to as Project A, Project B, and Project C. x reaches were constructed during Project A. y reaches were constructed during Project B. And z reaches were constructed during Project C. For simplicity, assume that the contractors built to the specifications and only consider pure state composite systems.*

The density matrix representing the Project A (L+W) system: $\rho_{L^A W^A} = \left| L^A W^A \right\rangle \left\langle L^A W^A \right|$

The density matrix representing the Project B (L+W) system: $\rho_{L^B W^B} = \left| L^B W^B \right\rangle \left\langle L^B W^B \right|$

The density matrix representing the Project C (L+W) system: $\rho_{L^C W^C} = \left| L^C W^C \right\rangle \left\langle L^C W^C \right|$

The density matrix representing the HSDRRS:

$$\begin{aligned}\rho_{HSDRRS} &= \sum_i p_i^S \left| L_i^S W_i^S \right\rangle \left\langle L_i^S W_i^S \right| \\ &= p_A \left| L^A W^A \right\rangle \left\langle L^A W^A \right| + p_B \left| L^B W^B \right\rangle \left\langle L^B W^B \right| + p_C \left| L^C W^C \right\rangle \left\langle L^C W^C \right| \\ &= \frac{x}{138} \left| L^A W^A \right\rangle \left\langle L^A W^A \right| + \frac{y}{138} \left| L^B W^B \right\rangle \left\langle L^B W^B \right| + \frac{z}{138} \left| L^C W^C \right\rangle \left\langle L^C W^C \right|\end{aligned} \tag{5.27}$$

Note that this is a mixed state (Section 4.4.2).

■

Consider the four density matrices from the above four scenarios, equation (5.6, 5.18, 5.25, and 5.27), a trend emerges regarding the structure of the density matrix formalism. Statistical data from experiments and observations are collected to formulate physical operator models and derive probabilities for making predictions. The density matrix contains two sets of probabilities: the probabilities derivable from the probability amplitudes, the $l_i w_j$, $\alpha_{ij}$ terms (quantum), and the probabilities describing the distribution of possible states, the $p_i$ terms (conventional). The first density matrix (Equation 5.6) utilized the least amount of statistics and probabilities, only the aleatory components have probability amplitudes; this is consistent with the notion that maximum knowledge about the system is available. As information and knowledge about the system became less, statistical structures are mixed in with quantum structures (for example, Equation 5.27); pure state representation gradually evolved to mixed state representations. Ignorance is incorporated into the model via the use of the mixed state density matrix; the risk analyst would need to make a conscious choice to use ensembles over tensor products, and hence potentially ignoring some available information in the formulation of the density matrix. Later in Chapter 7, this concept will be further explored and elaborated.

Meanwhile, regardless of a density matrix's status as pure or mixed, one can obtain the probabilities of the states and the expectation values of the observables following the same general computational rules.

5.1.5 Summary

The above scenarios illustrate that

1. The risk states of a composite system can be represented by a density matrix. The density operator formalism is capable of capturing both aleatory and epistemic uncertainties

2. When one possesses perfect knowledge about a system, only aleatory uncertainty (inherent randomness) remains. Such a system will be described with a pure state density matrix:

$$\rho_{LW} = \rho_L \otimes \rho_W = \left| LW \right\rangle \left\langle LW \right| \tag{5.28}$$

3. If we know everything about a complex composite system, then the pure state density matrix has the form:

$$\begin{aligned} \rho_{HPS} &= \rho_{L^A W^A} \otimes \rho_{L^B W^B} \otimes \rho_{L^C W^C} \otimes \cdots \\ &= \left| L^A W^A \right\rangle \left\langle L^A W^A \right| \otimes \left| L^B W^B \right\rangle \left\langle L^B W^B \right| \otimes \left| L^C W^C \right\rangle \left\langle L^C W^C \right| \otimes \cdots \end{aligned} \tag{5.29}$$

4. When our knowledge about the system is incomplete, epistemic uncertainty, the uncertainty about the models and parameters, comes into play. Many states of incomplete knowledge exist, and they can be described with the mixed state density matrix. A general mixed state composite system is given by:

$$\rho_{HPS} = \sum_{i,j} p_i^S q_j^S \left| L_i^S W_j^S \right\rangle \left\langle L_i^S W_j^S \right| \tag{5.30}$$

5. In the case where the subsystems are entangled (correlated), and we have knowledge about the composite but not the individual subsystems as they are entangled and not separable. The probability amplitudes cannot be expressed as separate products:

$$\left|LW\right\rangle = \alpha_{|00\rangle}\left|00\right\rangle + \alpha_{|01\rangle}\left|01\right\rangle + \alpha_{|10\rangle}\left|10\right\rangle + \alpha_{|11\rangle}\left|11\right\rangle, \text{ where } \alpha_{|00\rangle} \neq l_{|0\rangle}^{(A)} w_{|0\rangle}^{(A)} \tag{5.31}$$

The numerical values for the probability amplitudes are derived from empirical observations and measurements.

6. When we have a large ensemble of similar subsystems, we can choose to keep track of every subsystems by calculating the tensor products:

$$\begin{aligned} \rho_{HPS} &= \rho_{L^A W^A} \otimes \rho_{L^B W^B} \otimes \rho_{L^C W^C} \otimes \cdots \\ &= \left|L^A W^A\right\rangle\left\langle L^A W^A\right| \otimes \left|L^B W^B\right\rangle\left\langle L^B W^B\right| \otimes \left|L^C W^C\right\rangle\left\langle L^C W^C\right| \otimes \cdots \end{aligned} \tag{5.32}$$

Or we can choose to "ignore" a portion of our knowledge, and only track the ensemble distribution:

$$\rho = \sum_i p_i \left|L_i W_i\right\rangle\left\langle L_i W_i\right| \tag{5.33}$$

Notice that they all share the same fundamental mixed state density matrix form. This is precisely the characteristics we are looking for, a coherent framework that can incorporate different types of uncertainty into the model. Furthermore, taking the ensemble perspective can simplify the computational complexity, but at the cost of precision (Chapter 7.1 will elaborate on this further).

This current model is at its most rudimentary state. The model incorporated basic aleatory uncertainties, plus judgmental uncertainties (incomplete information + ensemble statistics). Epistemic uncertainties, contextual information, the physics models, the evolution models, and knowledge changes have not been fully captured. In the next section, we will show how the operator model can be used to represent all of these, and how these operators when applied to the state vectors and density matrices will alter the system probabilities.

> *The epistemic uncertainty can be view as an operation that modifies the uncertainty of a system, resulting in a "effective" uncertainty for the system as a whole.*

The resulting effective probabilities, we will argue, represent an integrated expression of the different types of uncertainties: aleatory, epistemic, and subjective beliefs.

*5.2 Operations — Change of States*

State transitions could be due to the occurrence of external events, and they are modeled with *operators* acting on state vectors, modulating the probability amplitudes. In the previous section, the steady-state state vectors for the levee, the flood wall, and the (L+W) composite system were developed. The pure state density matrix precisely describes the system and portrays the aleatory randomness of the system. The mixed state density matrix incorporates epistemic uncertainty at the cost of precision but could potentially gain some computational efficiency. This section applies the operators to take systems from one state to another similar to conventional analysis where a "load" can drive a system to a new state.

Consider the (L+W) system from the last section again, two different evolutionary paths could lead to different configurations. This system could be constructed all at once, or it could be constructed over two separate periods. In the latter case, the original levee was constructed first, but after some time, a raising water level required the increase of the levee's height to control floods. Instead of increasing the height of the levee, a flood wall was built to increase the system's effective height. While their configurations might look similar, the system models could be quite different to account for the different evolutionary pathways they took. In reality, a system evolves over time and transitioning from one risk state to another.

Let **U** be a unitary operator representing events or system behaviors leading to a change of the state vector. The system evolution is described by the operator **U** operating on the initial state vector, $|L_1\rangle$ , and transitioning it to a new state, $|L_2\rangle$ (i.e., the "stressed" system). Mathematically, the new state is related to the old state according to:

$$|L_2\rangle = \mathbf{U}|L_1\rangle \tag{5.34}$$

The evolution is also given by:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger \tag{5.35}$$

The operator **U** contains the physics model describing how events modify the state vectors. Taking the ensemble interpretation, different states in the ensemble can have different **U** operators, representing different evolutionary pathways leading to different subsequent states. The density operator, when interpreted this way, is a sum of different possible outcome states due to different evolutionary paths.

This operator concept augments how we model a system's risk and its risk states (defined earlier in 4.6). The approach combines the modifications of two elements of the risk equation into a single operation using an operator: 1) the probability of finding the system in a certain state as a result of some physical processes and events, and 2) the risk states, which constitute the modified state vector $\left|L_2\right\rangle$.

### 5.2.1 Properties of Operators

Operators represent events that can change a system in two ways.

- First, an event can change the state of a system, but conserve the total number of system states.
- Second, an event can change the state of system, but not conserve the total number of system states.

The second event fundamentally change the system by collapsing it to a different subspace reducing the total number of system states. In such case the number of system states after the event is less that the number of system states before; certain states are no longer available or permissible for the system to transition into, such as the case where a destroyed flood wall for a Levee+Flood Wall (LW) system is deemed to be permanently irreparable. As a result, the total probability is not conserved with that change in the system. The final system has to be re-normalized to re-establish the conservation of total probability before measurements can be performed on the system.

*5.2.1.1 Unitary Operators.* Events that only change the system states and not the system itself are represented by **unitary** operators. Unitary operators conserve total probability. For example, an earthen levee might gradually decompose or degrade over time, resulting in an increase of the system failure probability. The levee system itself does not change since there is no addition or subtraction of the number of possible system states; only a property of the system, the failure probability, changes over time (a simple example would be that of an incandescent light bulb; the longer it has been on and used, the higher the chance it would fail). Such state changes are represented by a unitary operator **U** that acts on the levee state vector, transitioning it to a different state, as given by:

$$\left| L_2 \right\rangle = \mathbf{U} \left| L_1 \right\rangle \tag{5.36}$$

**U** can be a function of time. For example, one can start with the initial state of a levee from the time of construction, t = 0, transitioning it to a different state at a future time *t*:

$$\left| L_2(t) \right\rangle = U(t) \left| L_1(0) \right\rangle \tag{5.37}$$

The operator U operates on the density matrix according to:

$$\rho_{L_2} = U \rho_{L_1} U^{\dagger} \tag{5.38}$$

If an event causes a fundamental change to the system, then certain states might no longer available and the resulting system is therefore described by a different state vector. For example, a levee can be damaged by a hurricane and no longer work. The system is in a non-working state, $\left|0\right\rangle$, all the time. The working state, $\left|1\right\rangle$, is no longer available as a possibility for the levee to be found in. The new state vector can be constructed by performing a *projection operation* on the old system, selecting (projecting) the relevant states (in this case only $\left|0\right\rangle$) and collapsing them into the new system (a new subspace) with the single new state $\left|0\right\rangle$. Once it has been projected into the new state, the probability amplitudes need to be renormalized so that the total probability equals unity.

An example of a projection operator for the hurricane is the $P = |0\rangle\langle 0|$. The new state vector is therefore:

$$|L_2\rangle = P|L_1\rangle = |0\rangle\langle 0|L_1\rangle = l_{1,|0\rangle}|0\rangle \tag{5.39}$$

And the final system state vector, after normalization with the "length" of $|L_2\rangle$, is:

$$|L_3\rangle = \frac{|L_2\rangle}{\||L_2\rangle\|} = |0\rangle \tag{5.40}$$

The final density matrix for the system is:

$$\rho_{L_3} = \frac{P\rho_{L_1}P^\dagger}{tr\left(P\rho_{L_1}P^\dagger\right)} \tag{5.41}$$

*5.2.1.2 Composite Operators.* For a composite system, the composite operator is formed from the tensor product of the subsystem operators and operates on the composite system state vectors. If the operator **A** operates on the levee, and the operator **B** operates on the flood wall, the operator operating on the composite (L+W) system is the tensor product of **A** and **B**:

$$\mathbf{A} \otimes \mathbf{B} \tag{5.42}$$

which operates on the (L+W) state vector according to:

$$\begin{aligned}(\mathbf{A} \otimes \mathbf{B})|LW\rangle &= (\mathbf{A} \otimes \mathbf{B})\left(|L\rangle \otimes |W\rangle\right) \\ &= \mathbf{A}|L\rangle \otimes \mathbf{B}|W\rangle\end{aligned} \tag{5.43}$$

The following shorthand notation is used for a single operator operating on one of the subsystems:

$$\begin{aligned}\mathbf{A}|LW\rangle &= (\mathbf{A} \otimes \mathbf{I})\left(|L\rangle \otimes |W\rangle\right) \\ &= \mathbf{A}|L\rangle \otimes \mathbf{I}|W\rangle\end{aligned} \tag{5.44}$$

where **I** is the identity operator.

*5.2.1.3 An Example.* To illustrate this concept, consider this simple example where an event A increased the failure probability of the levee from the initial 0.05 to 0.10, but the failure probability of the flood wall was not affected.  Let the initial state vectors for L and W be:

$$\left|L\right\rangle = l_{\left|0\right\rangle}\left|0\right\rangle + l_{\left|1\right\rangle}\left|1\right\rangle = \sqrt{0.05}\left|0\right\rangle + \sqrt{0.95}\left|1\right\rangle = \begin{pmatrix} 0.2236 \\ 0.9747 \end{pmatrix} \quad (5.45)$$

$$\left|W\right\rangle = w_{\left|0\right\rangle}\left|0\right\rangle + w_{\left|1\right\rangle}\left|1\right\rangle = \sqrt{0.05}\left|0\right\rangle + \sqrt{0.95}\left|1\right\rangle = \begin{pmatrix} 0.2236 \\ 0.9747 \end{pmatrix} \quad (5.46)$$

Since event only change the failure probability of the levee, Equation 5.44 applies to this case. The process on how to construct the operator **A** will be detailed in the following sections. For this example, to take the levee from the initial failure state (failure probability = 0.05) to the new failure state (failure probability = 0.10) requires A to be:

$$\mathbf{A} = \begin{pmatrix} 0.9954 & 0.0958 \\ -0.0958 & 0.9954 \end{pmatrix} \quad (5.47)$$

The new state for the levee is therefore:

$$\mathbf{A}\left|L\right\rangle = \begin{pmatrix} 0.9954 & 0.0958 \\ -0.0958 & 0.9954 \end{pmatrix}\begin{pmatrix} 0.2236 \\ 0.9747 \end{pmatrix} = \begin{pmatrix} 0.3159 \\ 0.9488 \end{pmatrix} \quad (5.48)$$

As a check, the square of 0.3159 is 0.10, which corresponds to the increase of the failure probability to 0.10 correctly. The new state for the levee and flood wall system is therefore:

$$\begin{aligned}\mathbf{A}\left|LW\right\rangle &= \left(\mathbf{A}\otimes\mathbf{I}\right)\left(\left|L\right\rangle\otimes\left|W\right\rangle\right)\\ &= \mathbf{A}\left|L\right\rangle\otimes\mathbf{I}\left|W\right\rangle\\ &= \begin{pmatrix}0.3159\\0.9488\end{pmatrix}\otimes\begin{pmatrix}0.2236\\0.9794\end{pmatrix}\\ &= \begin{pmatrix}0.0706\\0.3094\\0.2122\\0.9293\end{pmatrix}\end{aligned} \tag{5.49}$$

■

5.2.2 Operators to Model Evolution of the System Due to Load or Time

Operators can perform a number of functions such as simple changes to the probability amplitudes of the state vectors or select a particular component and operate on it. For a composite system, operators can selectively operate on the components of the system, such as when a system can function partially with semi-working components.

Let **M** be the physical model expressed with an operator that would change the probabilities such that the failure probabilities changes as a function of modeling parameters. For example, the model can represent failure probabilities as a function of load (akin to a fragility curve); or it can represent failure probabilities as a function of time (modeling gradual degradation). The state transition as a function of load is expressed as:

$$\left|A_{final}\right\rangle = \mathbf{M}(load)\left|A_{initial}\right\rangle \tag{5.50}$$

The operator model for the HPS levee and flood wall systems assumes the following characteristics:

1) The system has two states, $|0\rangle$ failure and $|1\rangle$ no failure. As the scope of this study is restricted to real state space (Chapter 3), a system state vector is a unit vector on the real plane, and observable results restricted to the first quadrant (Figure 9). In general, the state vector can have complex values but not here.

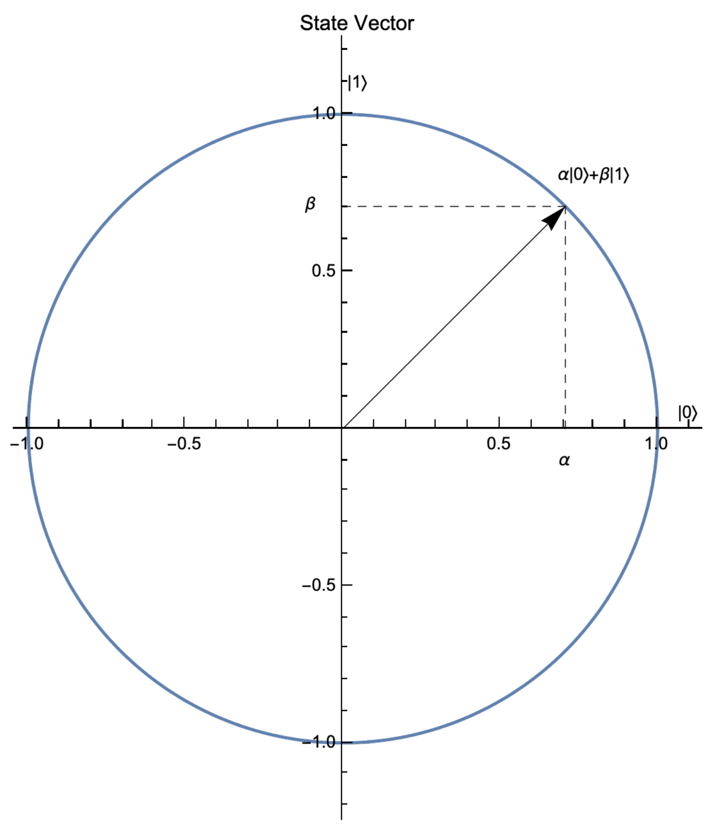


Figure 9: Visualization of the system state vector on the real plane.

2) The choice of the functional form of the operator reflects the nature, behaviors, and characteristics of the system. For a single system with two states, the rotation operator **R** can be a suitable choice to model the change of states. **R** is a unitary operator and there are other unitary operators one can construct to model different system behaviors. A working system will start

with the unit vector pointing close to the $|1\rangle$ axis. The rotation operator **R** can rotate the state vector clockwise and counterclockwise, taking the system from one state to another state (Figure 10). For example, to model the increase in failure probability over time, the state vector rotates from the $|1\rangle$ axis towards the $|0\rangle$ axis, and a rotation operator **R** (clockwise rotation) acting on the state vector can take the following matrix form:

$$\mathbf{R}^{+} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (5.51)$$

where $\theta$ is the function representing the physical behavior.

At time t, the state of the system is given by:

$$|L_{final}\rangle = \mathbf{R}(\theta)|L_{initial}\rangle \qquad (5.52)$$

where $\theta$ is a function of time.

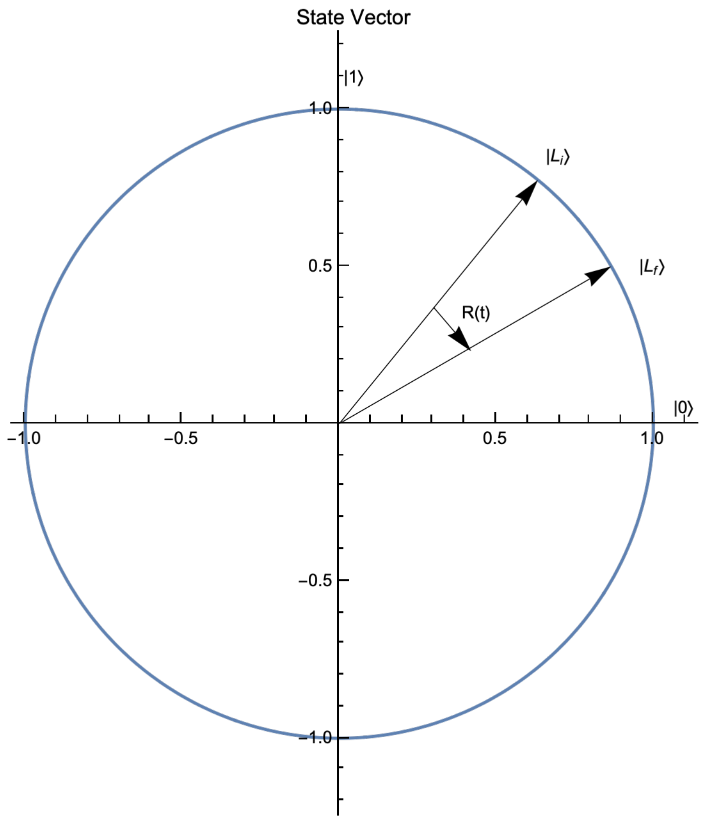


Figure 10: The operator R transition $\left|L_i\right\rangle$ to $\left|L_f\right\rangle$.

3) The rotation angle, $\theta$, in the rotation operator reflects the physics model and how it changes the probability amplitudes of the state vector and the state of the system. The rotation angle $\theta$ encapsulates the physical model into a quantitative operation. $\theta$ models and parametrizes the physical processes describing the systems behaviors, and we interpret this to represent epistemic uncertainties. When an operator acts on a state vector, it changes the probability amplitudes and therefore the probabilities of finding the system in the different states. Epistemic uncertainty deals with the uncertainty of the model and the uncertainties in the model parameters. We argue that part of the epistemic uncertainty is captured via the effects the operator has on the state

vector. If there are two viable physical models (which also encapsulate parametric variances) A and B, each has its $\theta$ function, $\theta_A$ and $\theta_B$. The difference between the $\theta$ functions provides a quantitative measurement of the model uncertainty. This concept becomes more apparent with the mixed density operator formalism (see Equation 5.8, 5.9, and section 5.1.5) where the different states are weighted by $p_A$ and $p_B$ in $\rho_{LW} = p_A|L_AW_A\rangle\langle L_AW_A| + p_B|L_BW_B\rangle\langle L_BW_B|$.

4) To model the individual physical behaviors and events, the rotation angle $\theta$ can take on different functional forms with different parameters (e.g. x, t), such as: a) Linear (e.g. $\theta \sim mx + c$), b) Cubic (e.g. $\theta \sim x^3$), c) Exponential (e.g. $\theta \sim 2^x$), d) Periodic (e.g. $\theta \sim \theta(x + np)$), e) Ratio (e.g. $\theta \sim \left(\frac{f(t)}{Max(f(t))}\right)$), or other functions such as sigmoid.
5) Expectation values are calculated from the final system state after the application of the operator to the state vector or the density matrix (see Section 4.1 and 4.2 for discussion on expectation, Section 4.4.2 on expectation value for density operator, and 4.6 on expectations and risk values).

*The modeling functions for $\theta$: the $\theta$ functions* — The rotation operator **R** can be thought of as a “dial” that turns the system from the working to the non-working state or vice versa. The operators change the probability of finding the system in state $|0\rangle$ and $|1\rangle$. For a simple 2 states system, since system starts at functioning $|1\rangle$ state, measuring from the $\left|1\right\rangle$ axis the $\theta$ angle will rotate clockwise towards the $\left|0\right\rangle$ axis. The physical and logical models and associated parameters, such as the probabilities of the physical events, are all encapsulated in the rotation angle, $\theta$, embedded within a rotation operator. **R** operates on the state vector (the aleatory part) and arrive at an effective uncertainty. In the case where there are alternative models (different functional form of $\theta$) that can lead to the same outcome (in the case of uncertainty of the model itself, the epistemic part), the treatment of the different alternative models can be represented by the probability terms in the density operator which captures and represents incomplete knowledge plus expert opinions. *The $\theta$ term, therefore, represents the evolutionary profile of the probability amplitudes as a function of some characteristics (which the failure probability track and trace) associated with the system, capturing the physics driving the behaviors. The functional form of the $\theta$ term will be referred to as the $\theta$ function in the models.* The $\theta$ function is constructed to model the expected evolutionary behavior of the failure probability, map to the rotation angle $\theta$ and scale to the corresponding lower and upper bounds or 0 and $\pi/2$ (in Radians).

The rotation operator in 2-dimension acting on a 2-states system takes the form:

$$\mathbf{R}^{-} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ for clockwise rotation, and}$$

$$\mathbf{R}^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ for counterclockwise rotation} \qquad (5.53)$$

where the + and - denote clockwise and counterclockwise rotation respectively. For composite systems, higher dimension rotation matrices will be needed to represent the extended basis. For example, the (L+W) system has 4 basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, and the system can be viewed as "living" in a four-dimensional Euclidean space with the four basis vectors. Rotations about a point in such four-dimensional space are studied in Group theory, which has rotation matrices applicable for developing operators for the (L+W) system. The set of 4-dimensional rotation matrix belongs to the Special Orthogonal group of order 4, SO(4), consisting of

$$\mathbf{R}^{-} = \left\{ \begin{array}{c} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \end{array} \right\} \tag{5.54}$$

can be used to model the operations applicable to the composite system. In this case for the (L+W) system, there could be a total of 6 different rotation operators that can operate on the $\left|LW\right\rangle$ state vector. The rotation corresponds to rotating the vector with one of the rotating planes fixed.

Furthermore, two types of 4-dimensional operator can be constructed for the composite system formed out of two 2-states subsystems. The first one represents operators acting on individual subsystems, and this is applicable to system in product states. The second one represents operators acting on the composite system as a whole, and this is applicable to system in entangled states. To form the operator for the first kind, individual operators for the subsystems will need to be specified, and then the composite operator is formed by computing the tensor product of the individual operators.

$$\mathbf{R}_L^+ = \begin{pmatrix} \cos\theta_L & \sin\theta_L \\ -\sin\theta_L & \cos\theta_L \end{pmatrix}, \ \mathbf{R}_W^+ = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \tag{5.55}$$

$$\begin{aligned} \mathbf{R}_{LW}^+ &= \mathbf{R}_L^+ \otimes \mathbf{R}_W^+ \\ &= \begin{pmatrix} \cos\theta_L & \sin\theta_L \\ -\sin\theta_L & \cos\theta_L \end{pmatrix} \otimes \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_L \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} & \sin\theta_L \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \\ -\sin\theta_L \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} & \cos\theta_L \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \end{pmatrix} \end{aligned} \tag{5.56}$$

To form the operator for the second kind, the operator is a 4-d rotation operator acting on the 4 components of the basis vectors with a single function and R operate on the entangled states.

$$\mathbf{R}_{LW}^{-}=\left\{\begin{matrix}\begin{pmatrix}\cos\theta_{LW} & -\sin\theta_{LW} & 0 & 0\\ \sin\theta_{LW} & \cos\theta_{LW} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, & \begin{pmatrix}\cos\theta_{LW} & 0 & -\sin\theta_{LW} & 0\\ 0 & 1 & 0 & 0\\ \sin\theta_{LW} & 0 & \cos\theta_{LW} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, & \begin{pmatrix}\cos\theta_{LW} & 0 & 0 & -\sin\theta_{LW}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sin\theta_{LW} & 0 & 0 & \cos\theta_{LW}\end{pmatrix},\\ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_{LW} & -\sin\theta_{LW} & 0\\ 0 & \sin\theta_{LW} & \cos\theta_{LW} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, & \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_{LW} & 0 & -\sin\theta_{LW}\\ 0 & 0 & 1 & 0\\ 0 & \sin\theta_{LW} & 0 & \cos\theta_{LW}\end{pmatrix}, & \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\theta_{LW} & -\sin\theta_{LW}\\ 0 & 0 & \sin\theta_{LW} & \cos\theta_{LW}\end{pmatrix}\end{matrix}\right\} \quad (5.57)$$

This research work explores the use of the rotation operator **R** to represent events acting on system states. At the simplest basic form, a simple two-dimensional rotation (with only two basis vectors), the operation is unitary and thus preserves the magnitude of the state vector. In the case with complex composite systems, the associated operators acting on the composite systems will be of higher dimensions, leading to the possibility of non-commutative operations since higher dimensional rotation operators do not necessarily commute with each other. In short, the order of application of the operators matters in the determination of the eventual state of the system.

There are a number of possible interpretations for non-commutative operations. One can potentially model casually connected activities where the sequence of their occurrences can lead to different outcomes. For example, at a levee site, flooding before a destructive earthquake (destruction of the levee) and flooding after a destructive earthquake can be drastically different. Another interpretative application of non-commutative operations can represent incompatible events where events do not share a common basis, which means the operations will act differently on different systems.

### 5.2.3 S-curve with sigmoid function

To construct the operators to describe the physical events and processes that change and alter the states of the system, one begins with the selection of the functional form of $\theta$. For the HPS, the following failure events and failure models have been identified as the critical failure modes in the IPET report (2007, 2009):

1. Pressure (P) build up on one side of the wall due to water volume.
2. Overtopping (O) of the structures as a result of storm surges or seasonal water level changes.
3. Erosion (E) due to water flow forming water fill gaps or wash-aways, both the front and back sides of the structure.
4. Material (M) degradation over time, such as vegetation overgrowth on the levee led to the weakening of the structure, and structural degradation over time.
5. System argumentation and repair (R), such as additional reinforce structure and the replacement of older or failing structures.
6. System level changes (X), such as the catastrophic destruction of the structure due to hurricane.

These 6 operations form a basic set for the different failure modes (see Figures 11 for some examples), and they can be chained together to form more complex operations to model the many different combinations of failure condition, such as erosion by overtopping. In the following discussions, “failure mode” is referred to the state of the system and “event” is a mechanism causing failure.

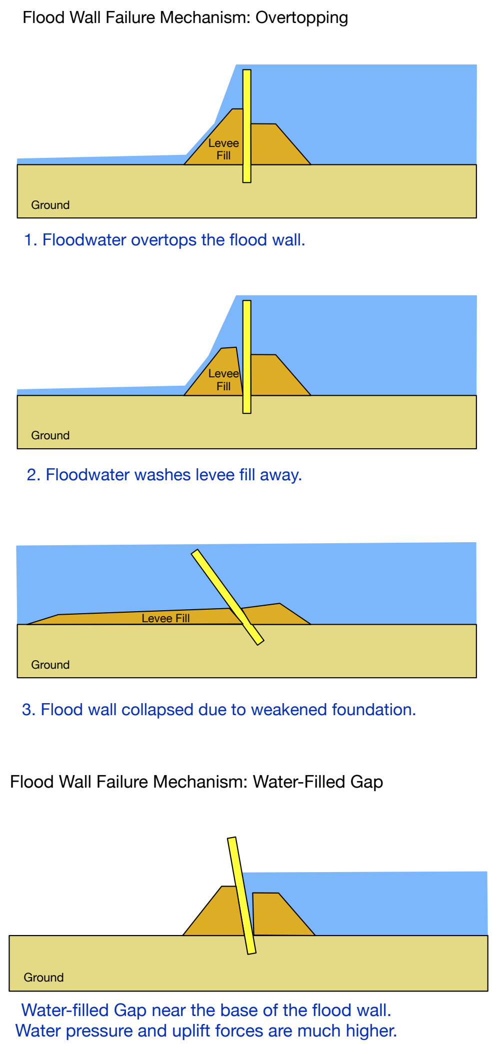


Figure 11: Some examples of Levee + I-wall failure mechanisms and modes, adapted from ASCE, 2007.

The choice of the functional form reflects how the system behaves. For example, consider the case with the *pressure build up on one side of the wall due to water volume and flow (event)*. The increase of water volume on the exterior structure (water facing side) during storm and flow events elevate the underneath soil water pressure, which can exert lateral and vertical forces against the structure (IPET 2007, 2009, FEMA 2012), increasing the system failure probability. Moreover, the pressure increase in the foundation materials beyond the soil's strength can result in the shifting or lifting of the materials, displacing the above and surrounding structures, which further increase the probability of a foundation failure. This building up of water pressure is found to lag behind the rise of the water level; the pressure effect trails behind other effects that trace directly with the water level (IPET, 2007). To construct the operator, the proper functional form must be chosen to model the behavior of the system.

For this initial study, the scope will be limited to two functional forms that are commonly seen and observed in physical situations. One of the functional forms, the sigmoid function or commonly refers to as the S-curve (Figure 12), is observed in many physical systems, effects and events, such as dam failure probability as a function of external force (load) (see Figure 12 for comparison between the shape of a s-curve and fragility curves). Another often used functional form takes the shape of a Weibull curve (skewed to the right), a Gumbel curve (skewed to the left), or a Gamma distribution ($\propto x^{\alpha-1}e^{-\beta x}$), derived from curve fitting data obtained from observations and measurements, can be used to model environmental events such material degradation or river level as a function of time. Figure 13 and Figure 14 for a comparison between Weibull and water level. One should note the similarities of the shape of the curves between the hydrography and the Weibull curves, which makes Weibull or similar type curves suitable for modeling storm surges. Depending on the risk questions and scenarios, different functions can be utilized to represent the situation. For example, if one is interested in looking at the working/not-working state of a system as a function of load, the sigmoid function can be a viable choice. If one is interested in looking at the system's behavior over time, such as water level, a Weibull function could be used.

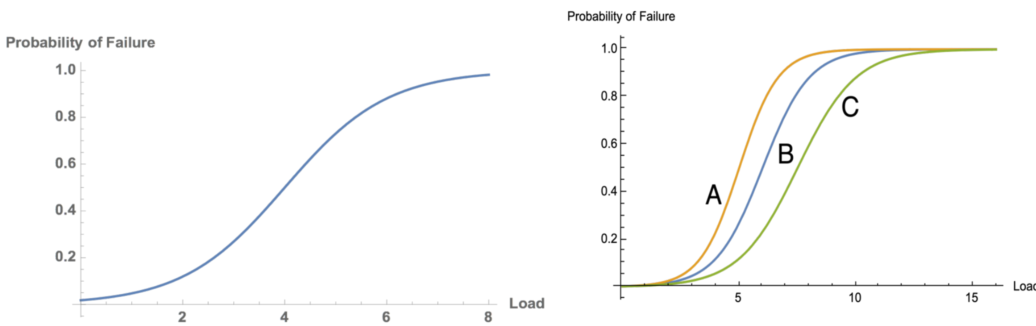


Figure 12: A S curve (left) and fragility curves for different modes of failure (right).

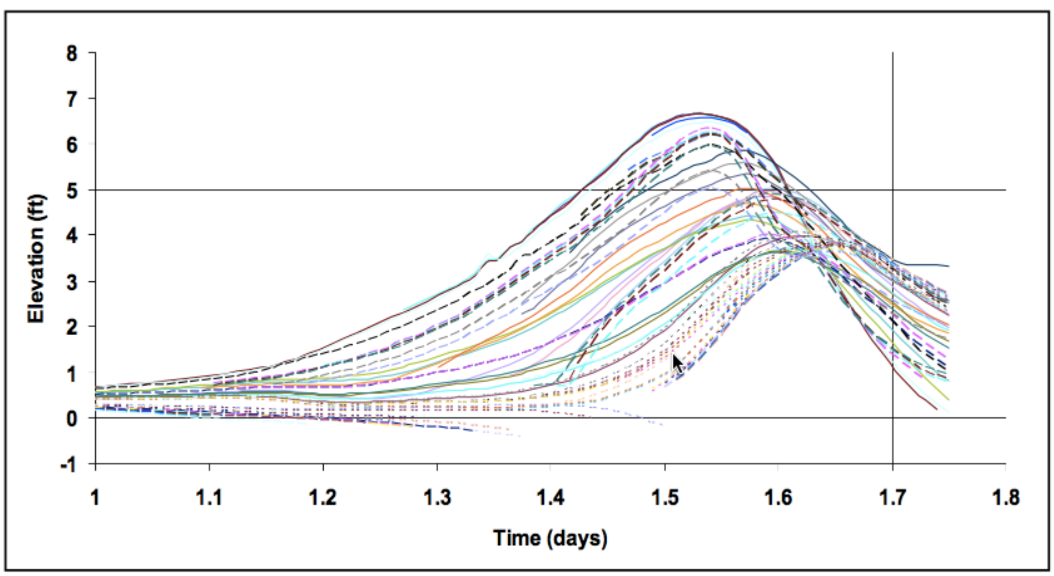


Figure 13: Sample hydrographs from locations surrounding the HSDRSS during a single storm, from Vol. I of the IPET report (IPET, 2009).

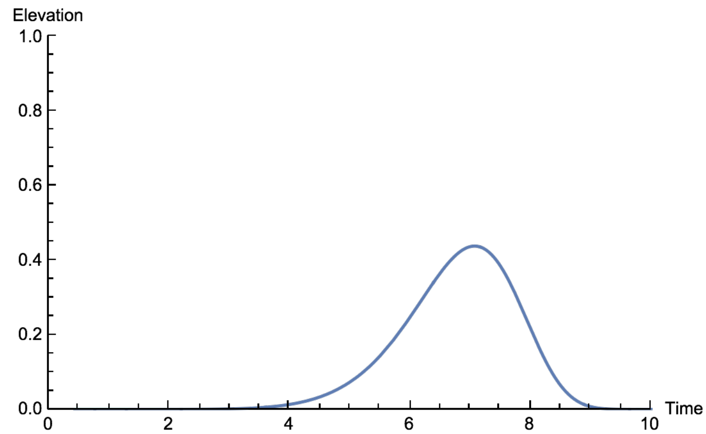

Figure 14: A Weibull or similar type curves can be constructed to model water elevation over time.

In the following example, we will illustrate the construction of the $\theta$ function for the operator using the sigmoid function. The objective is to allow us to illustrate how we construct operators to model the physical effects, and how the basic concept and idea work.

Recall that, to model the increase in failure probability over time, a rotation operator **R** (clockwise rotation) acting on the state vector can take the following matrix form:

$$\mathbf{R}^{+} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{5.58}$$

where $\theta$ is the function representing the physical behavior. At time t, the state of the system is given by:

$$\left|A(t)\right\rangle = \mathbf{R}(\theta(t))\left|A(0)\right\rangle \tag{5.59}$$

where $\theta$ is a function of time. Let *f(x)* be the function modeling the change, and the $\theta$ function for these events will share the basic common form:

$$\sin\theta = \left(f(x)\right)^{\frac{1}{2}}$$
$$\cos\theta = \left(1-\sin^2\theta\right)^{\frac{1}{2}} \quad , \tag{5.60}$$
$$\theta = \arcsin(f(x)^{\frac{1}{2}})$$

where $\theta$ is interpreted simply as the rotation of the state vector corresponding to the change event. This form models the behavior of the failure probability of an event. Consider the simple levee state vector $\left|L\right\rangle = l_{|0\rangle}\left|0\right\rangle + l_{|1\rangle}\left|1\right\rangle$, and let $\theta$ be the angle between the state vector and the $\left|1\right\rangle$ axis, then $\sin\theta = l_{|0\rangle}$. Recall that the probability is compute by taking the squared of the probability amplitude, the failure probability is $l_{|0\rangle}^2$. If the model function is $p(x)$ corresponding to the failure probability function, then:

$$\sin\theta = l_{|0\rangle} = p(x)^{\frac{1}{2}}, \tag{5.61}$$

where $p(x)$ is a function modeling the failure probability of a system driven by an event.

Model 5.M009 Scenario: Breaching of a section of I-wall due to foundation displacement (IPET 2007, I-45)

*The storm event begins at time t. At time t during the storm, the water level surged to height h ~ 10 ft. No overtopping occurred as the height of the levee is 15 ft. The water pressure from the surge cause significant lateral movement of the flood wall, resulting in catastrophic failure leading to a breach of a section of the wall.*

Technically, this scenario has two operations: 1) the weakening of the underneath soil, increasing the failure probability, and then 2) the full hydrostatic pressure pushing the foundation structure, resulting in a lateral movement the I-wall with catastrophic failure. The scenario example here focuses on the second stage.

The increase of water volume on the exterior structure (water facing side) during storm and flow events elevate the underneath soil water pressure, which can exert lateral and vertical forces against the structure, increasing the system failure probability. Moreover, the pressure increase in the foundation materials beyond the soil's strength can result in the shifting or lifting of the materials, displacing the above and surrounding structures, which further increase the probability of a foundation failure. This building up of water pressure is found to lag behind the rise of the water level; the pressure effect trails behind other effects that trace directly with the water level (IPET, 2007, 2009). The function for the pressure operator, therefore, must assume a time-dependent functional form that can represent these physical system behaviors.

The curve constructed using the Weibull function, given by

$$f(t)=\left(\frac{\alpha}{\beta}\right)\left(\frac{t-\mu}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t-\mu}{\beta}\right)^{\alpha}} \quad \text{where} \quad \begin{aligned} &\alpha = \text{shape parameter} \\ &\beta = \text{scale parameter} \\ &\mu = \text{location parameter} \end{aligned} \tag{5.62}$$

is chosen as the function to model the profile for the expected failure probability as a function of time describing a system with the front water side facing the hydraulic pressure. Since hydrostatic pressure is proportional to the density of the fluid and the depth, $p = \rho g h$, the pressure profile is tied to the water height profile.

The choice of the function is influence by the risk question, which in this case is the behavior of the system over time. The choice of using the Weibull function reflects how you parameterize the behavior of the system, which in case example we choose to parameterize the behavior as a function of the pressure (load) since it is the pressure that cause the lateral movement.

Since we are interested in how the system state transition and evolve over time, the operator will need to be a function of time. A corresponding tracer parameter will need to be identify, which in this case, would be the elevation (h) of the river. The pressure (p) exerted by the water is directly proportional to the elevation, since $p \sim \rho g h$. The time behavior of the system is therefore proportional to the water level over time (Figure 15).

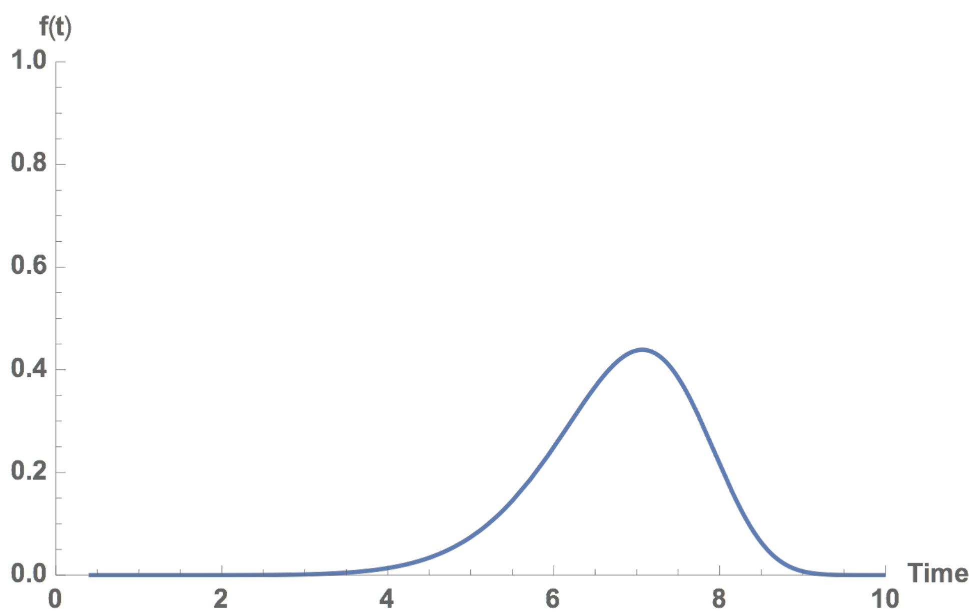


Figure 15: Time behavior of the system corresponds to the water level behavior over time.

The $\theta$ function takes the following form:

$$\sin\theta = \left(\frac{f(t)}{N}\right)^{\frac{1}{2}} \tag{5.63}$$

where *f(t)* is water height as a function of time, and the N is a normalization factor reflecting the intensity of the event. In the case of pressure, N could be the maximum water pressure the system can handle before failure.

If one is simply asking the question about the failure probability of the levee as a function of pressure (load) then the sigmoid function can be used. The sigmoid function might be represented by a logistic function for convenience:

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1} \tag{5.64}$$

is used to model the behavior of the system. In this case, risk relates to the probability of finding the system in a failure state,

$$\left|L(l)\right\rangle = P_{load}{}^{+}(l)\left|L(l_0)\right\rangle \tag{5.65}$$

If one computes all the states corresponding to different loads, compute the expectations for the $\left|0\right\rangle$ states, and plots out the failure probability vs. load, one obtains the fragility curve. ■

5.2.4 Different event operators

Returning to the six key system event types introduced earlier in 5.2.3, this section further explores the Pressure (P), Overflow (O), Erosion (E), Material (M), Repair (R), and Extreme (X) operators. These six operations form a basic set for the different failure modes, and they can be chained together to form more complex operations to model the many different failure conditions, such as erosion by overtopping. Figure 16, 17, 18, 19 and 20 from the ASCE Hurricane Katrina External Review Panel report (ASCE, 2007) are failure mechanisms identified to be the contributing factors leading to the failure of the HPS during Katrina. The function for these events will share the basic common form:

$$\sin\theta = (\text{change function})^{\frac{1}{2}} \quad . \tag{5.66}$$

Specific operators for the following failure events and failure models will be developed since they are identified as the critical failure modes in the IPET report (IPET 2009):

1) Pressure build up on one side of the wall due to water volume.
2) Overtopping of the structures as a result of storm surges or seasonal water level changes.
3) Erosion due to water flow forming water fill gaps or wash-aways, both the front and back sides of the structure.
4) Material degradation over time, such as vegetation overgrowth on the levee led to the weakening of the structure, and structural degradation over time.
5) System argumentation and repair, such as additional reinforce structure and the replacement of older or failing structures.
6) System level changes, such as the catastrophic destruction of the structure due to hurricane.

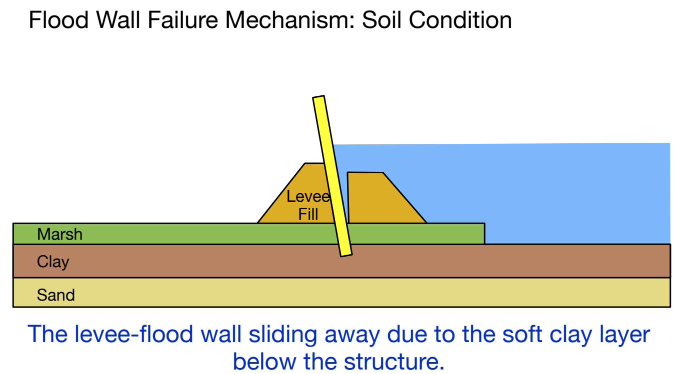


Figure 16: Failure mechanism due to soil condition at the structure, adapted from ASCE, 2007.

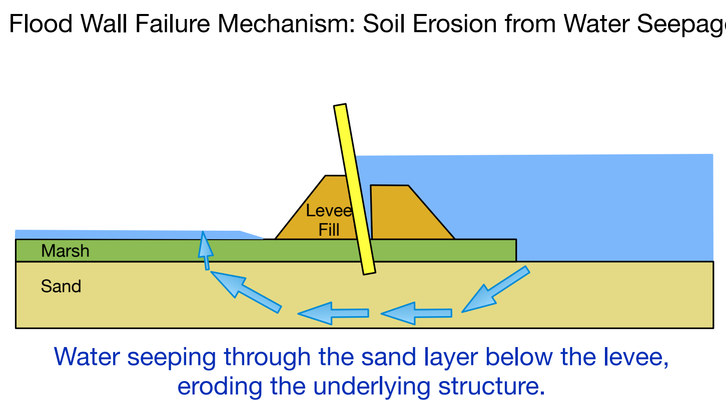


Figure 17: Failure mechanism due to soil erosion as a result of water seepage, adapted from ASCE, 2007.

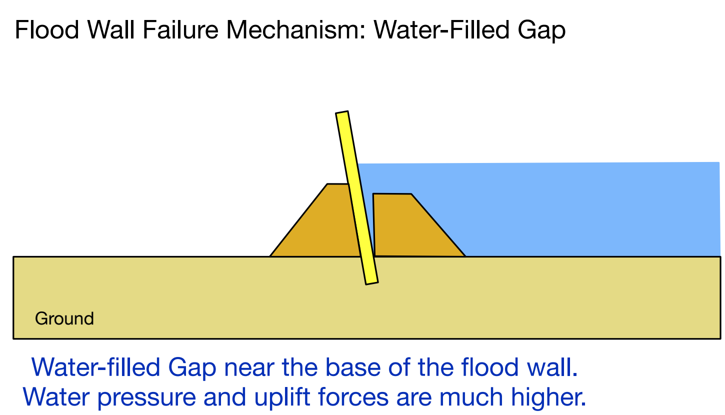


Figure 18: Failure mechanism due to a water-filled gap, adapted from ASCE, 2007.

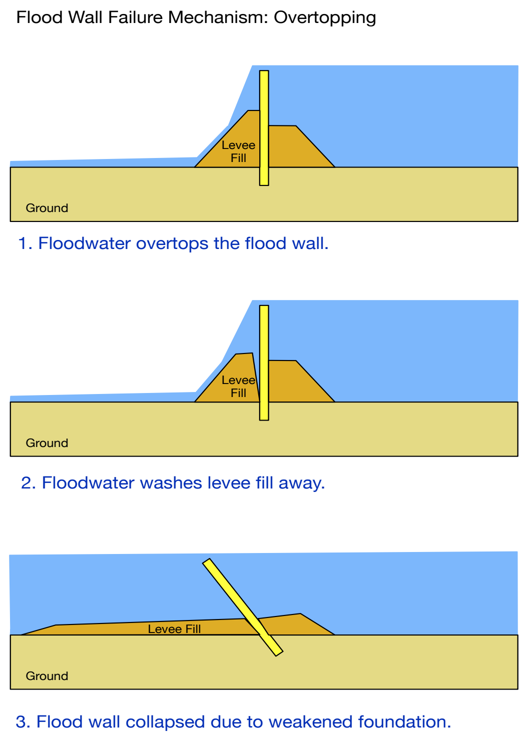


Figure 19: Failure mechanisms due to overtopping of a levee with a flood wall, adapted from ASCE, 2007.

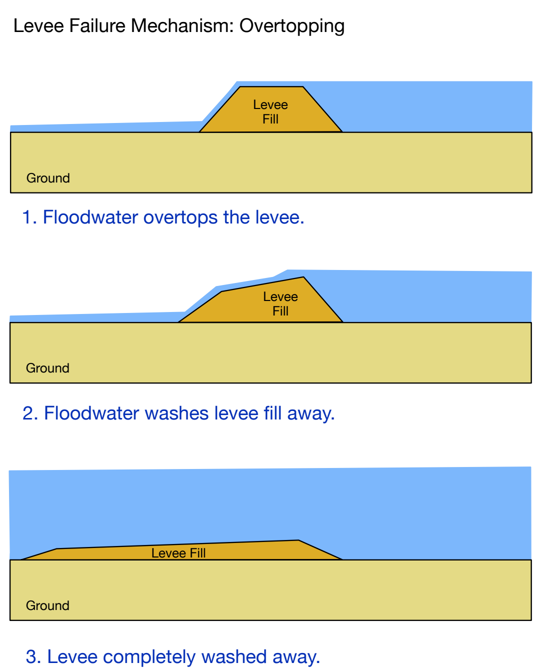


Figure 20: Failure mechanisms due to overtopping of a levee, adapted from ASCE, 2007.

One should note that operators can be grouped into different types according to how they track and trace the temporal behaviors of the system, reflecting the physical evolutions over time and the effect timescales. Long timescale operators trace the evolution behavior of the system over its lifetime; whereas short timescale operators trace the behavior of the systems during short term events with short elapsed time. For the purpose of this research, we will focus on operators tracking a system's lifetime behaviors and short timescale (days) elapsed time events.

*Pressure (P) - Pressure build up on one side of the wall due to water volume and flow (event)* — Besides the basic pressure operator introduced in the last section, other pressure operators can be constructed to describe different event behaviors, such as the pressure changes over a time range. For example, seasonal patterns can be modeled with periodic functions, where the failure probabilities will reflect changes according to the periodic seasonal conditions, such as increase in failure as water pressure increase during rainy seasons and decrease in failure when water pressure decrease back to nominal as the season changes. Furthermore, the repetitive cycles over the years as well as the gradual raising sea levels and the subsidence of the ground will result in the cumulative increase of the overall baseline failure probabilities; the front side water-filled gap for example, will gradually worsen over repeated events. As the pressure events repeats over time, one would expect the weakening of the physical structure. Therefore, the basal "continuum" profile will continue to increase over time, and the profile for the expected failure probability as a function of time takes the following form (Figure 21):

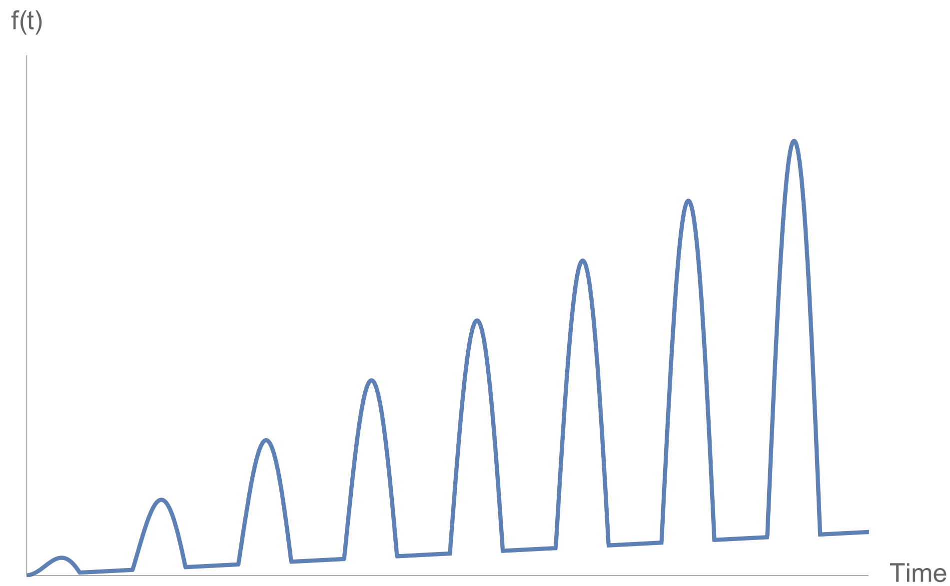

Figure 21: The function for P over long period of time.

*Overtopping (O) - Overtopping of the structures as a result of storm surges or seasonal water level changes (event)* — Overtopping occurs when water level rises above the top of the structure due to some external events. Results from overtopping can be as simple as a localized flooding that dissipates at the end of the event, or the catastrophic destruction of the levee or the flood wall structure. Overtopping does not automatically lead to the destruction of the levee/flood wall structure but can cause backside structural erosions. If the height of the wall is $h_w$, as long as the height of the water level is below it, the system will function normally. As the water level rises, the probability of finding the system in the not working state, the $|0\rangle$ state, increases. Once the water reach $h_w$, the system will be in the $|0\rangle$ state as overflowing occurs when $h > h_w$. However, it is important to point out that the state vector describing the system still consist of the $|0\rangle$ state and the $|1\rangle$ state vectors. The overtopping does not change the system as the physical system is unchanged and still functioning. Overtopping changes the state of the system and not the system itself, and the state vector for the overtopped system is $1|0\rangle + 0|1\rangle$.

The Weibull curve, or other appropriate parameterized function, can be used to model the profile for the expected failure probability as a function of time describing a system with the front side facing the raise in the water level. The general functional form of the $\theta$ function for the O operator:

$$\sin\theta = \left(\frac{h_{water}(t)}{h_w}\right)^{\frac{1}{2}} \tag{5.67}$$

where $h_{water}(t)$ takes the form of a Weibull curve. The profile of this Weibull curve is chosen to trace the height of the water level against the height of the levee and/or flood wall structure. The exact physical dimension is not critical, only the ratio is:

$$\frac{h_{water}(t)}{h_{wall}} \tag{5.68}$$

where $h_{water}$ is the water level and $h_{wall}$ is the height of the levee and/or flood wall structure. This is a powerful statement. The actual physical dimensions of the levee and/or flood wall are not the key parameters, but the profile shape of the rise of water level over time is.

The parameter shape parameters will need to be chosen appropriately to reflect the behaviors of the event. The parameters are chosen to reflect that: 1) the failure probability rise and fall symmetrically, 2) the function trace the water level profile, and 3) the effect track the rise of the water level directly without any lags. The corresponding $\theta$ function assumes that the rotation is proportional to the ratio profile of the water level to the height of the structure (levee or the levee-plus-wall). The closer the water level to the height of the wall, the more likely that the system will fail, which corresponds to a higher probability of finding the system in the $|0\rangle$ state.

For a single overflow event, such as the rise of water level during a storm event, the profile would take the general form (Figure 22):

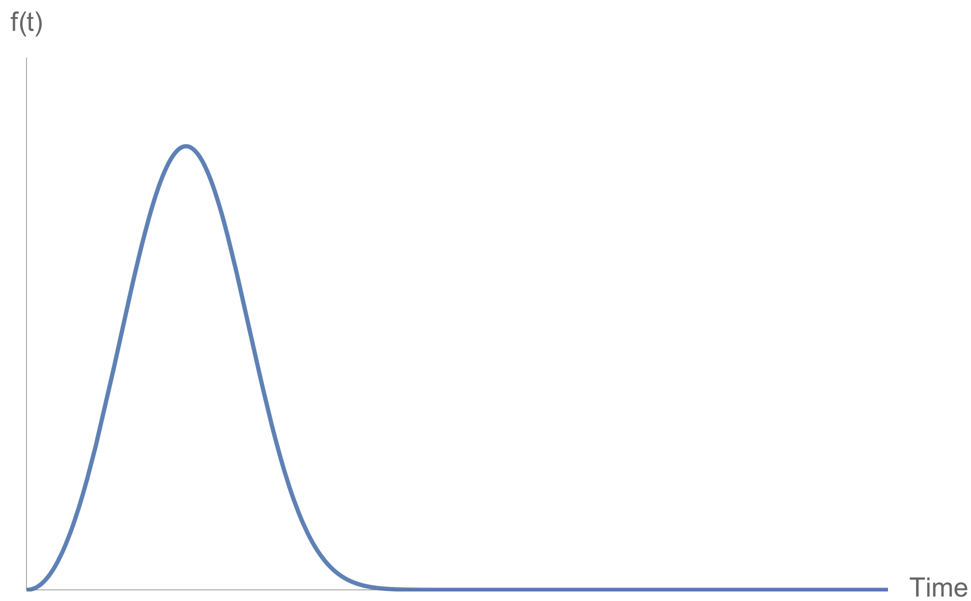


Figure 22: A general profile for the function tracing the change of water level on the water facing side of the structure.

*Model 5.M010 Scenario: Overtopping of a section of a levee due to storm water raising above the top of the structure (IPET 2007, I-45)*

*The storm event begins at time* $t_1$. *At time t during the storm, the water level surged to height h ~ 10 ft. The ratio* $h_{water}/h_{wall} \geq 1$, *overtopping occurs. The foundation of the structure holds and incur no damage.*

In this simple scenario, the state of the system simply reflects the behavior of a system that works according to the design specifications. The risk of failure increases nominally, tracking the water level. After the passing of the storm, the system returns back to the original state.

$$\left|L(t)\right\rangle = O^{+}(t-t_1)\left|L(t_0)\right\rangle \tag{5.69}$$

The O operator takes the following form

$$\mathbf{O}^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.70}$$

for clockwise rotation

$$\sin\theta = \left(\frac{f(t)}{Max(f(t))}\right)^{\frac{1}{2}} \tag{5.71}$$

■

*Erosion (E) - Erosion due to water flow forming water fill gaps or wash-aways, both the front and back sides of the structure (event)* — Environmental erosions can be due to many factors and the type of structural design can affect the erosion scenarios and erosion rates. This study focuses on the water-filled gap and under-the-structure seepage (IPET, 2007, I-45). Erosion events can be the result of the nominal water flow on the water facing side, leading to the formation of the water-filled gap, or water seepage under the structure resulting in the weakening of the foundation at the backside, or the overtopping event, which can lead to the backside erosion of the levee. Under certain circumstances the erosion event can be catastrophic resulting in the destruction of the entire structure. For this study the seepage ($\mathbf{E}_s$), the front side

($\mathbf{E}_f$) and the back side ($\mathbf{E}_b$) failure modes are modeled to trace the soil behaviors over time. Since the systems are subjected to repeated events over time, the gap conditions are modeled to gradually worsen for long term behaviors.

*Front side ($\mathbf{E}_f$)* — The exponential curve is chosen as the function to model the profiles for the erosion event for the front side (water facing side). The initial erosion of the levee should be coarsely proportional to the flow volume, flow rate, and other factors such as vegetative growth and seasonal weather variations. The magnitudes of the front side erosion events are assumed to be smoothed out over a long timescale. The profile to represent erosion would behave initially as simple linear growth; however, once substantial erosion occurred to the structure past certain threshold, the structural deterioration would accelerate (such as in the case where the erosion opens up the foundation, increasing the chance of a structural displacement), and the failure probability will grow exponentially (Figure 23). The adoption of this simple function reflects a choice of timescale and the granularity of types of event under considerations, including with it a choice of the degree of ignorance. In choosing the simple function, the individual short timescale events are ignored, while long term systemic events dominate and drive the evolutionary behaviors. This maps to the behavior of the water-filled gap which will gradually worsen over time and the basal failure risk will continue to increase.

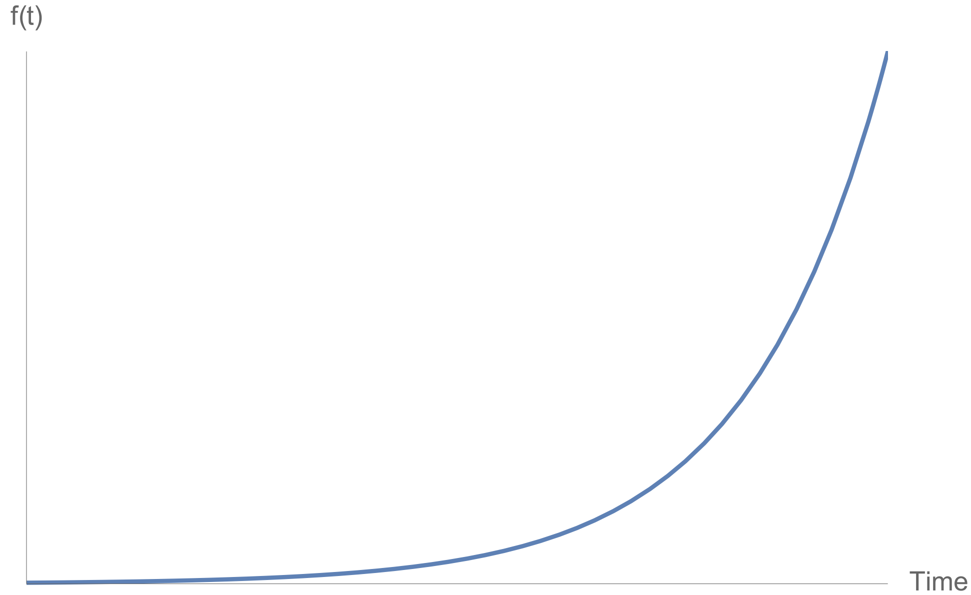


Figure 23: The function for $\mathbf{E}_f$.

The general functional form of the $\theta$ function for the $\mathbf{E}_f$ operator:

$$\sin\theta = \left(\frac{f(t)}{Max(f(t))}\right)^{\frac{1}{2}} \qquad (5.72)$$

where *f(t)* is an exponential function.

*Back side (*$\mathbf{E}_b$*)* — The basic behavior for the back side erosion event is similar to that of the front side, except the fact that the triggering point is not the water flow/storm surge, but the overflow event and seepage. One can argue that there are two components for this, once is the basal low level small magnitude erosion due to seepage, and then a more rapid larger magnitude erosion as a result of the overtopping event. The curve will have a profile of a raised basal level and then a rapid exponential rise at the time when overtopping begins.

The general functional form of the $\theta$ function for the $\mathbf{E}_b$ operator:

$$\sin\theta = \left( \frac{f_{\exp}(t) + f_{seepage}(t)}{Max(f_{\exp}(t) + f_{seepage}(t))} \right)^{\frac{1}{2}} \tag{5.73}$$

where *f(t)* is a two-stage function, where the first stage reflects the design of the drainage condition on the land side, and the second stage is the exponential behavior for the erosion event.

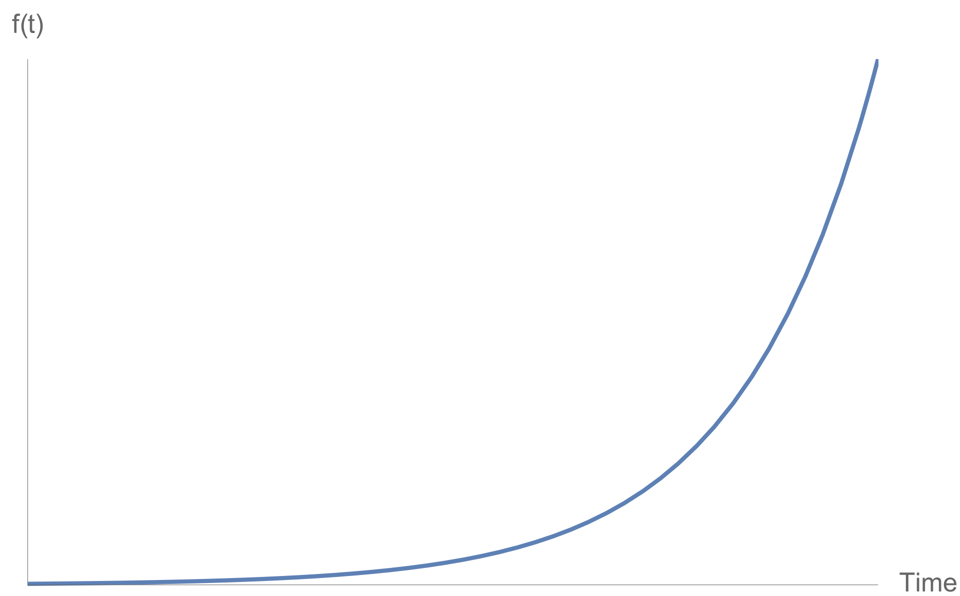


Figure 24: The function for $\mathbf{E}_b$ .

*Seepage (*$\mathbf{E}_s$*)* — Seepage is a long-term event where the land side soil is weakened over time as a result of water seeping through below the sand underneath the levee, building up pressure on the land side of the structure causing the near service earthen structure to crack. Water flows through the cracks resulting in soil erosion, gradually weakens the levee. This is being modeled with a simple linear function that grows in time.

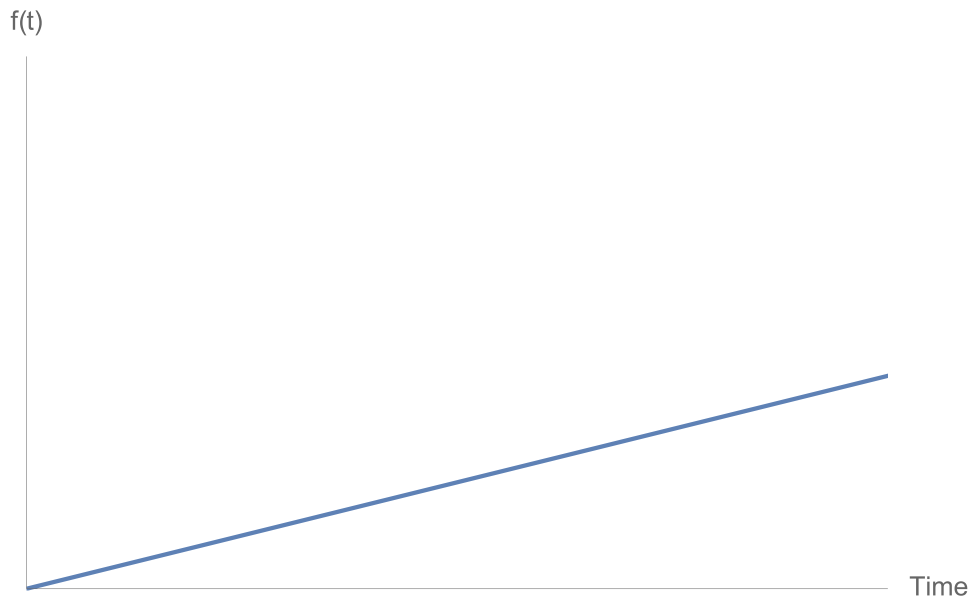


Figure 25: The function for $\mathbf{E}_s$.

The general functional form of the $\theta$ function for the $\mathbf{E}_s$ operator:

$$\sin\theta = \left(\frac{f(t)}{Max(f(t))}\right)^{\frac{1}{2}} \quad (5.74)$$

where f(t) is a simple linear function.

The E operator takes the following form:

$$\mathbf{E}^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad (5.75)$$

for clockwise rotation. However, in this case, since erosion is a one way process, the $\mathbf{E}^{-}$ does not represent any real physical operations.

$$\sin\theta = \left(\frac{f(t)}{Max(f(t))}\right)^{\frac{1}{2}} \quad (5.76)$$

where

$$f(t) = mx + c\,. \quad (5.77)$$

*Model 5.M011 Scenario: Front side erosion resulting in seepage and the erosion of the back side (IPET 2007, I-45)*

This is a simple example of the combination of the 3 erosion operators in this order: $\mathbf{E}_b \, \mathbf{E}_s \, \mathbf{E}_f$. More about the construction of multi-steps operation will be discussed in section 5.2.4. Note that the order of the operation is significant since matrix operations do not necessary commute. In this scenario, the water facing front side erodes over time, resulting in cracks in the soil substructure, allowing water to seep through to the back side of the structure, resulting in the erosion of the back side. Also note that there is a time lag between the front side erosion and the back side. ■

*General material and structure Fatigue (M) - Material degradation over time, such as vegetation overgrowth on the levee led to the weakening of the structure, and structural degradation over time (lifetime)* — General material fatigue happens when a material is subjected to cyclic loading over period of times, resulting in the weakening of the materials. Typically, the material fatigue curves are modeled with a variety of distribution functions, such as the log-normal, the extreme value, Birnbaum-Saunders, and the Weibull distribution. In the present analysis, we are interested in modeling how the material fatigue changes the failure probabilities over time, and not the failure rate (normally defined as the total number of failures in a population over time) and the hazard rate.

In choosing the $\theta$ function, the physical system determines the choice of the functional form. In the case of the HPS, the 138 reaches can be treated as a population to the first order. For the levee and flood wall systems, the number of systems failing over time increase exponentially and a simple exponential function can be considered for modeling the change of the failure probability.  In this case, we will use an exponential grow function to represent the grow of the failure probability. There could be other degradation mechanisms as well, and one would build those models into the function. Given that these type of material and structural do not occur overnight, this is considered as a lifetime event.

If the failure probably grows exponentially over time, then the change function takes the form of an exponential function:

$$f(t) = e^{kt} \tag{5.78}$$

where t is the time since construction and k is the growth constant, then the general functional form of the $\theta$ function for the M operator can be expressed as:

$$\sin\theta = \left( \frac{e^{kt}}{e^{kt_{life}}} \right)^{\frac{1}{2}} = e^{\frac{k(t-t_{life})}{2}} \tag{5.79}$$

where the denominator $e^{kt_{life}}$ is the expected failure at the end of life. The ratio $e^{kt} / e^{kt_{life}}$ is the normalization of the change against the expected end of life rate. Figure 26 plotted the $\theta$ function over time with different growth constants.

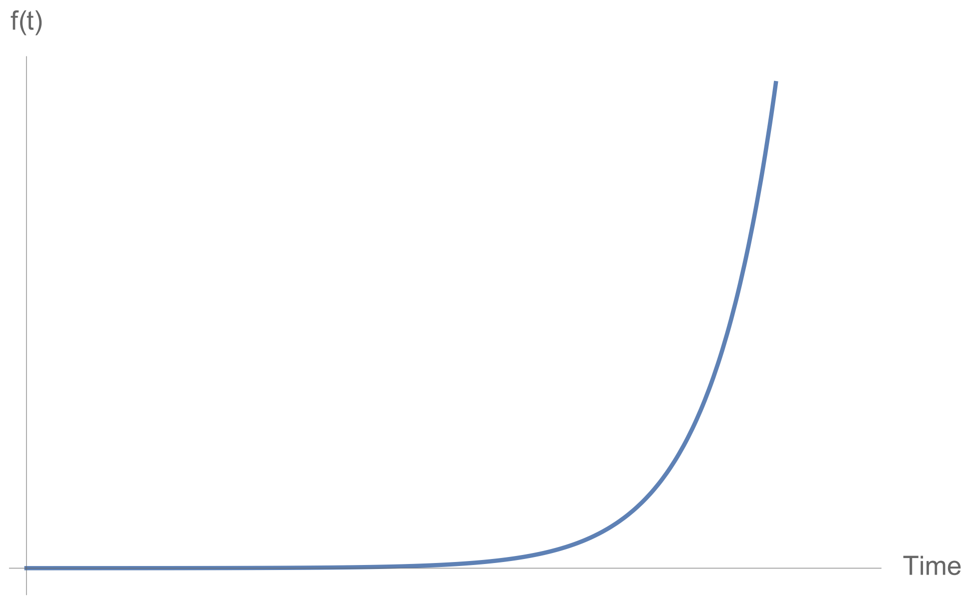

Figure 26: The function for **M**.

*Model 5.M012 Scenario: Vegetation on levee, resulting in structural cracks over time (IPET 2007, I-45)*

The state of the system simply reflects the material degradation over time as a result of vegetative growth. The degradation traces an exponential profile:

$$|L(t)\rangle = M^{+}(t-t_1)|L(t_0)\rangle \tag{5.80}$$

The M operator takes the following form:

$$\mathbf{M}^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.81}$$

for clockwise rotation, where

$$\sin\theta = \left(\frac{e^{kt}}{e^{kt_{life}}}\right)^{\frac{1}{2}} \tag{5.82}$$

Similar to the case of erosion, since material degradation is a one-way process, the $\mathbf{M}^{-}$ does not represent any real physical operations. ■

*Repairs (R) - System argumentation and repair, such as additional reinforce structure and the replacement of older or failing structures* — The HPS system was constructed in multiple phases, with additional reinforce structures (e.g. I-wall) added on top of existing structure, and some failed structure replaced with new ones. These types of events can either be a "reset" of the system, which can be modeled with a unitary operation, or a fundamental change in the system. If the operation is simply a repair of an existing component of a system, the unitary operation acts as "turning the dial back" for the system, rotating the particular component of the state vector back to the $|1\rangle$ state. For example, if the I-wall was damaged by a storm and was subsequently repaired, since the component was already captured in the state vector, the repair operator would reset that component back to the initial working state.

The **R** operator takes the following form if there is no fundamental changes to the system itself and just a reset to the initial states:

$$\mathbf{R}^{-} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{5.83}$$

for counterclockwise rotation.

In our convention, repair operation reverts the system back to the initial operational state, which means the operation will rotate the state vector towards the $|1\rangle$ state, a counterclockwise rotation. The $\theta$ function for the **R** operator is simply the delta angle between the initial state and the current state: $\theta = \theta_0 - \theta_t$

*Model 5.M013 Scenario: The repair of a levee.*

*Repair work was done to repair a levee.*

At time $t_1$ , the levee is in a failing state, with a 90% chance of failure, $\sin\theta_t = (0.9)^{\frac{1}{2}}$. Assuming that the initial state of the system is $\sin\theta_0 = (0.05)^{\frac{1}{2}}$. The repair resets the system back to the initial state, corresponding to $\theta = \theta_0 - \theta_t = -1.0245$. Note: $\theta$ is in Radians, and negative sign means the rotation is counterclockwise.

$$\left|L(t_2)\right\rangle = R^{-}\left|L(t_1)\right\rangle \quad \blacksquare \tag{5.84}$$

For a fundamental change in the system, the events are modeled not with a unitary operator but with either a projection operator for a system with a reduction of states, or with the tensor product if the event resulted in the construction of a new system, such as the addition of a flood wall on top of a levee.

If the repair results in the fundamental change of the system, such as a) the removal of a component, or b) the addition of a component, then the repair operation should be represented by:

a) a projection operator: $\mathbf{R} = \left|substates\right\rangle\left\langle substates\right|$, or

b) a tensor product: $\left|system\right\rangle = \left|old\right\rangle \otimes \left|new\right\rangle$

*Model 5.M014 Scenario: An existing levee is retrofitted with a new flood wall*

*The levee was built at time* $t_0$*, and a flood wall was built on top of the levee at a later time* $t_1$.

To model the state of the composite system at time $t_2$ requires a two stage process. The first stage is to bring the state of the levee from $t_0$ to $t_1$, and the second stage is to perform the tensor product to form the new composite system. Assuming that the levee went through a simple material degradation process from $t_0$ to $t_1$, the state of the levee at $t_1$ is given by:

$$\left|L(t_1)\right\rangle = \mathbf{M}(t_1)\left|L(t_0)\right\rangle \tag{5.85}$$

Then the final composite system is formed by taking the tensor product of the levee and the flood wall:

$$\left|LW(t_1)\right\rangle = \left|L(t_1)\right\rangle \otimes \left|W(t_1)\right\rangle \quad \blacksquare \tag{5.86}$$

*Extreme events such as earthquake or hurricane (X) - System level changes, such as the catastrophic destruction of the structure due to hurricane* — Extreme or rare events, like hurricanes and earthquakes, can fundamentally alter the state compositions of the system. In the case of a catastrophic event, where the system is completely destroyed, the probability of finding the system in the $\left|1\right\rangle$ state is zero. The general state vector representation of the system will no longer include the $\left|1\right\rangle$ basis state, since it is no longer a real option. Instead of the rotation operator, the operator to model these events will simply be a projector selecting the remaining possible states. For events that alter the system itself, we can model them with projectors to project the system to the end state (such as the complete destruction of the system): $\mathbf{X} = |\text{end state}\rangle\langle\text{end state}|$. For a single system, it will simply be the $|0\rangle\langle 0|$ projector. For a composite system with 2 subsystems, it could be the $|00\rangle\langle 00|$, $|01\rangle\langle 01|$, $|10\rangle\langle 10|$, or I (the identity operator, where nothing changed).

*Model 4.3.M015 Scenario: A levee was breached and destroyed.*

For an event that alter a single system, the projector $P = |0\rangle\langle 0|$ project the system to the end state (such as the complete destruction of the system), and the matrix form of it is:

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.87}$$

One should also note that X is not unitary as this operation is fundamentally changing the system. Unitary matrix preserves the system, and thus preserve the law of conservation or total probability. By changing the system, total probability cannot be conserved, which account for the fact that we will need to re-normalized the new system to re-establish the conservation of total probability.

$$|L(t_1)\rangle = \mathbf{X}|L(t_0)\rangle \ \blacksquare \tag{5.88}$$

### 5.2.5 Applying the Operators to the Composite System

The "basic" operators were introduced above as building blocks to support the modeling of the HPS. An actual event is often formed out of a combination of these basic building blocks, changing the system from one state to another. Consider the example of the overtopping of a levee during Katrina. The breaching event is the end result corresponding to the end state for the system. A number of events occurred to take the system from one state to another, and finally to the end state. In this example, the breaching was the result of the following chain of events: (i) gradual degradation of the levee structure (**E**), model with a sigmoid function, and (ii) overtopping (**O**), model with a Weibull function.

The modeling of an event or a chain of complex events over time with the operators, therefore, follows this event chain construction workflow:

1) Construct the event operator $H_t$ from the unit operators, such as $H_t = OE$.

   The grouping rule is that composite operations can be formed only if they are occurring during the same time intervals where the state transition occurs. An event such as a hurricane can be a collection of different individual events, as determined by the periods of occurrence.

2) Construct other event operators based on various knowledge, including historical context: $H_1$, $H_2$, $H_3$, …

3) Based on historical knowledge, construct event chain $H_c = H_n ... H_2 H_1$, where $n$ is the $n^{th}$ event, which describe the change of the system states over time.

4) Construct risk evaluation scenarios based on the event chain, add knowledge-based variations, usually in the form of projection operators. Different risk

questions can be asked about the system at different times.

5) Evolve the system to the desired time based on the event chains. Compute projectors at the end of the evolution.

6) Compute the probabilities from the state vector or the density matrix.

*Applying operators to a composite system: Pure States* — Operators formed from the tensor products of individual subsystem operators work great if the system is factorable into individual product states. In this case we have maximum knowledge about the make-up of the system. To form the tensor product, we need to know the individual operations applicable to the individual subsystems. The resulting composite operator therefore will track the individual operations. In the case where there exist correlations between the subsystems, or that we don't have full knowledge of how the individual subsystem works, tensor product-based operators cannot be formed. While it presents limitation as to how much we know about the individual states of the subsystems, we can still have operators working on the joint system, accepting our ignorance about the system's inner working.

The composite operator acting on the pure state vector for the (L+W) system can be expressed as:

$$\left|LW_2\right\rangle = \mathbf{H}_{LW}\left|LW_1\right\rangle \tag{5.89}$$

Recall from earlier that the composite system is constructed from the tensor product of the individual subsystems:

$$\left|LW\right\rangle = \left|L\right\rangle \otimes \left|W\right\rangle \tag{5.90}$$

and the composite operator is the tensor product of the operators for the subsystems:

$$\left|LW_2\right\rangle = \mathbf{H}_{LW}\left|LW_1\right\rangle \tag{5.91}$$

The density operator representation for the composite system can be formed from tensor product of the subsystems, and the operator acts on the density matrix according to:

$$\rho_{LW_2} = \mathbf{H}_{LW}\rho_{LW_1}\mathbf{H}^{\dagger}{}_{LW} \tag{5.92}$$

This pure state representation of the (L+W) composite system require us to have all the information about the different subsystems that make up the composite system, and furthermore, in order for this model to work all parameters must be tracked. The operator acting on this density matrix will be formed also from tensor products of all the individual operators acting on the individual subsystems.

*Model 5.M017 Scenario: Applying the H operator on the (L+W) composite system*

*A levee was constructed first. After 25 years, the levee was degraded by vegetation and has a 50/50 chance of failure. A flood wall was installed then, with an intrinsic failure probability of 1%. A storm event resulted in erosion on the front side of the levee, creating a water gap. The state vector describing this composite system is given by:*

$$\left|L_0^A\right\rangle = l_{0\left|0\right\rangle}^A\left|0\right\rangle + l_{0\left|1\right\rangle}^A\left|1\right\rangle = \sqrt{0.5}\left|0\right\rangle + \sqrt{0.5}\left|1\right\rangle \tag{5.93}$$

$$\left|W_0^A\right\rangle = w_{0\left|0\right\rangle}^A\left|0\right\rangle + w_{0\left|1\right\rangle}^A\left|1\right\rangle = \sqrt{0.01}\left|0\right\rangle + \sqrt{0.99}\left|1\right\rangle \tag{5.94}$$

Recall that the state vector and the density matrix for the (L+W) composite system are given by:

$$\left|LW\right\rangle = \left|L\right\rangle \otimes \left|W\right\rangle = \begin{pmatrix} l_{|0\rangle} \\ l_{|1\rangle} \end{pmatrix} \otimes \begin{pmatrix} w_{|0\rangle} \\ w_{|1\rangle} \end{pmatrix} = \begin{pmatrix} l_{|0\rangle} w_{|0\rangle} \\ l_{|0\rangle} w_{|1\rangle} \\ l_{|1\rangle} w_{|0\rangle} \\ l_{|1\rangle} w_{|1\rangle} \end{pmatrix} = \begin{pmatrix} \sqrt{0.5}\sqrt{0.01} \\ \sqrt{0.5}\sqrt{0.99} \\ \sqrt{0.5}\sqrt{0.01} \\ \sqrt{0.5}\sqrt{0.99} \end{pmatrix} = \begin{pmatrix} 0.0707 \\ 0.7036 \\ 0.0707 \\ 0.7036 \end{pmatrix} \tag{5.95}$$

$$\begin{aligned} \rho_{LW} &= \rho_L \otimes \rho_W \\ &= \left|LW\right\rangle\left\langle LW\right| \\ &= \begin{pmatrix} 0.0707 \\ 0.7036 \\ 0.0707 \\ 0.7036 \end{pmatrix} \begin{pmatrix} 0.0707 & 0.7036 & 0.0707 & 0.7036 \end{pmatrix} \\ &= \begin{pmatrix} 0.0049 & 0.0497 & 0.0049 & 0.0497 \\ 0.0497 & 0.4951 & 0.0497 & 0.4951 \\ 0.0049 & 0.0497 & 0.0049 & 0.0497 \\ 0.0497 & 0.4951 & 0.0497 & 0.4951 \end{pmatrix} \end{aligned} \tag{5.96}$$

The above gives the state of the system right before the storm event. After the storm event, the levee was eroded and a water gap was formed, resulting in an increase of failure probability for the system. The transition operator:

$$\begin{aligned}
\mathbf{H}_{LW} &= \mathbf{H}_L \otimes \mathbf{H}_W \\
&= \mathbf{E}_{f|L\rangle} \otimes \mathbf{I} \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}
\end{aligned} \tag{5.97}$$

For this example, we assume that the $\theta$ function is a simple Weibull function. To transition the system to the state after the storm event, the transition operator is applied on the density matrix:

$$\rho_{LW'} = \mathbf{H}_{LW}\rho_{LW}\mathbf{H}^{\dagger}{}_{LW} \tag{5.98}$$

The above operation can be generalized to the HPS consist of many different (L+W) subsystems, A, B, C, …:

$$\rho^N = \rho^A \otimes \rho^B \otimes \rho^C \otimes \cdots \tag{5.99}$$

For example, the HPS can be represented as:

$$\begin{aligned}
\rho_{HPS} &= \rho_{L_1^A W_1^A} \otimes \rho_{L_1^B W_1^B} \otimes \rho_{L_1^C W_1^C} \otimes \cdots \\
&= \left|L_1^A W_1^A\right\rangle\left\langle L_1^A W_1^A\right| \otimes \left|L_1^B W_1^B\right\rangle\left\langle L_1^B W_1^B\right| \otimes \left|L_1^C W_1^C\right\rangle\left\langle L_1^C W_1^C\right| \otimes \cdots
\end{aligned} \tag{5.100}$$

The corresponding operator is constructed with tensor products of the subsystem operators:

$$\mathbf{H}_{LW}^{HPS} = \mathbf{H}_{LW}^{A} \otimes \mathbf{H}_{LW}^{B} \otimes \mathbf{H}_{LW}^{C} \otimes \cdots \tag{5.101}$$

■

*Applying operators to a composite system: Mixed States* — If there are more than one possible state for the composite (L+W) system at a given time, then the system has to be represented by the mixed state density operator:

$$\rho_{LW} = \sum_i p_i \left| L_i W_i \right\rangle \left\langle L_i W_i \right| \tag{5.102}$$

The density operator representation for the composite system can be formed from tensor product of the subsystems, and operator acts on the density matrix according to:

$$\rho_{LW'} = \mathbf{H}_{LW} \rho_{LW} \mathbf{H}^{\dagger}{}_{LW} \tag{5.103}$$

*Model 4.3.M018 Scenario: Applying the **H** operator on a mixed state (L+W) composite system - in the case where the knowledge about the system state is limited*

*Consider the scenario from the previous model. Assuming that we do not have full knowledge of the degree of erosion, and there is a 30% chance that the erosion is more severe than the standard model.*

In this scenario, the system in question has two possible states, but definitive knowledge is not available about the exact state the system is in. Hence, the system can only be described by a mixed state density matrix.

$$\rho_{LW} = \sum_i p_i \left| L_i W_i \right\rangle \left\langle L_i W_i \right| = 0.7 \left| L_1 W_1 \right\rangle \left\langle L_1 W_1 \right| + 0.3 \left| L_2 W_2 \right\rangle \left\langle L_2 W_2 \right|$$

(5.104)

The state transition equation becomes:

$$\rho_{LW'} = 0.7\mathbf{H}_{LW}\left|L_1W_1\right\rangle\left\langle L_1W_1\right|\mathbf{H}^{\dagger}{}_{LW} + 0.3\mathbf{H}_{LW}\left|L_2W_2\right\rangle\left\langle L_2W_2\right|\mathbf{H}^{\dagger}{}_{LW} \tag{5.105}$$

■

*Model 4.3.M019 Scenario: Applying the **H** operator on a mixed state ensemble (L+W) composite system, with 10 (L+W) subsystems forming the ensemble*

*Consider the scenario from the previous model. Assuming that the HPS consists of 10 (L+W) subsystems. 3 out of the 10 subsystems were eroded more severely than the rest due to the geographic locations. The 3 subsystems are found to be in identical erosion state.*

In this scenario, the HPS is a single system comprised of 10 subsystems. While it is perfectly fine to construct a state vector of the entire system via 10 tensor products, we can also treat the system as an ensemble of subsystems. Instead of tracking individual states of the subsystems, we can choose to ignore the individual characteristics and focus on the statistical characteristics of the ensemble. The system can be described by a density matrix:

$$\rho_{HPS} = \sum_i p_i\left|L_iW_i\right\rangle\left\langle L_iW_i\right| = 0.7\left|L_1W_1\right\rangle\left\langle L_1W_1\right| + 0.3\left|L_3W_3\right\rangle\left\langle L_3W_3\right| \;\blacksquare \tag{5.106}$$

This has the identical form as the earlier mixed state density matrix. The similarity in form is a reflection of the incorporation of ignorance when we chose to ignore individual characteristics and adopt the ensemble viewpoint. The 2 scenarios are a reflection of incomplete knowledge. The density matrix can represent two different possible system configurations.

*Summary* — In this and the last sections, the framework to represent the risk states with state vectors and density operators was developed, and operators were developed to describe the change of states of a system. Complex events can be modeled with sequences of operators, and the non-commutative nature of the operation preserves the ordering of events. Regardless of the degree of completeness of knowledge, a system can be represented by the density matrix framework, in pure state or mixed states, incorporating statistical knowledge as well as ignorance into the fold, incorporating different types of uncertainties in a unifying structure.

The density operator is an effective tool to represent system when we do have complete knowledge. The density matrix is great to describe: 1) an ensemble, 2) system formed from subsystems, 3) fundamental. In the last section, two different types of operators that operate on composite systems were presented. Operators formed from the tensor products of individual subsystem operators work great if the system is factorable into individual product states, since in this case we have maximum knowledge about the make-up of the system. To form the tensor product, we need to know the individual operations applicable to the individual subsystems. The operator therefore can track the individual operations.

In the case where there exist correlations between the subsystems, or that we don't have full knowledge of how the individual subsystem works, tensor product-based operators cannot be formed. While it presents limitation as to how much we know about the individual states of the subsystems, we can still have operators working on the joint system, accepting our ignorance about the inner working.

# Chapter 6: Fragility, Epistemic Probability, and Event Trees

*Knowledge about risk is knowledge about lack of knowledge.*

*- Hansson 2014*

In the previous chapter a basic quantum model was developed for a simple storm protection system for the overtopping scenario under environmental conditions that change over time. In this chapter we explore this quantum model on how it produces two commonly used PRA products: fragility curves and event trees. The resulting products will be compared and contrasted with the current risk assessment approaches.

## *6.1 Fragility Curves from State Vectors*

> *"Fragility curves are functions that describe the conditional probability of system failure over the full range of loads to which that system might be exposed. In contrast to nominal failure probabilities estimated from reliability indices, fragility curves provide a richer, much more comprehensive perspective on system reliability because they are functions rather than points and because they are interpreted in terms of absolute probabilities rather than nominal probabilities, implying knowledge of the underlying probability distributions."*
>
> — *Schultz et al. 2010.*

The use of fragility curves in representing failure probabilities is common in dam safety studies (Schultz et al., 2010). A typical workflow in dam risk evaluation begins with the generation of fragility curves to model the engineering system. Often, a family of fragility curves is produced for a range of confidence intervals, probabilistic or statistical distributions, or different physical models, providing an envelope to quantify uncertainties for the different engineering risk model scenarios. Here, using the quantum model, the equivalents to fragility curves are produced from the overtopping scenario for the system. The resulting products are then compared and contrasted with the current approach using fragility curves for risk assessments.

*Generating a fragility curve* — a fragility curve usually plots the failure probability as a function of some independent variable (or parameter) such as load and time. To generate a fragility curve from the quantum models, one starts with the initial state vector. Each point on a fragility curve is a projection of a state from a state vector. The points for the fragility curve are obtained directly from the projections of all system states over the range of the independent parameter. The system states are derived from operators acting on the state vector, transitioning it from states to states. The operators represent the physics models that alter the state vector resulting in a change of the dependent variable, in this case the probability amplitudes. The collection of all the failure probabilities from the states forms the fragility curve.

To generate the fragility curve, one begins with the initial state vector, projects out the failure probability, transitions the system to the next state with operators, projects out the failure probability, and repeats. Let $\left|L_0\right\rangle$ be the initial state vector for a system as before with $\left|0\right\rangle$ and $\left|1\right\rangle$ as the basis states, and $\mathbf{H}$ be the load operator that is a function of the load parameter, $l_i$. Consider the simplest scenario where the $i^{th}$ state of the system is given by:

$$\left|L_i\right\rangle = \mathbf{H}(l_i)\left|L_0\right\rangle \quad (6.1)$$

the $i^{th}$ failure probability is obtained from the projector $|0\rangle\langle 0|$:

$$P_i(|0\rangle) = \langle L_i|0\rangle\langle 0|L_i\rangle \tag{6.2}$$

Since operators represent the physical models, different fragility curves can be constructed with different operators with different underlying physical models. If a system has three different failure modes A, B, and C represented by the operators, **A**, **B**, and **C**. Each will be described by its own vector equation:

$$\begin{aligned} |L_f\rangle &= \mathbf{A}|L_i\rangle \\ |L_f\rangle &= \mathbf{B}|L_i\rangle \\ |L_f\rangle &= \mathbf{C}|L_i\rangle \end{aligned} \tag{6.3}$$

The collection of the failure probabilities from all the possible states for a given mode, e.g. A, becomes a fragility curve, and different failure modes (A, B, and C) produce different fragility curves (Figure 27).

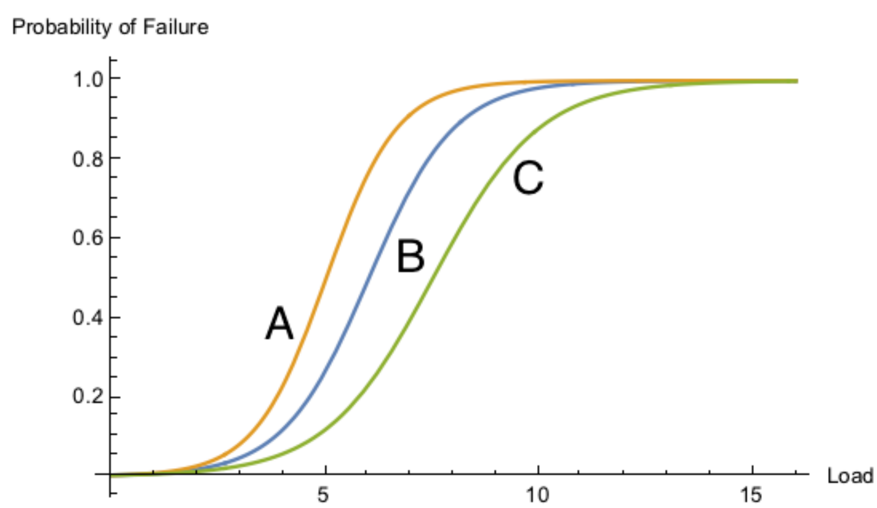


Figure 27: Fragility curves produced by different failure mode operators. Each curve corresponds to a different failure mode.

*On aleatory and epistemic uncertainties* — The state vector itself contains the aleatory components, the $l_0$ and $l_1$, intrinsic to the system:

$$|L\rangle = l_0|0\rangle + l_1|1\rangle \tag{6.4}$$

and they are not a function of the independent parameters, such as load and time. However, the operators contain the physical models that can alter the probability amplitudes of the state vector, and they are functions of the independent parameters. Thus, operators play a role in the modeling and representing epistemic uncertainties. Whereas in conventional approaches, families of fragility curves corresponding to probability distributions (e.g. PDFs) are used to model epistemic uncertainties, the quantum framework models epistemic uncertainties directly with the construction of operator sets to generate the families of curves. Chapter 7 will further detail and explore how epistemic uncertainties are represented by the quantum objects, the state vectors, the operators, and the density operator.

The operator sets can be constructed in a number of ways, depending on the availability, the breadth, and depth of contextual knowledge. Contextual knowledge, such as how one formulates the risk question, influence the choice of the measurement parameters as well as the type of operators. In previous chapter, the rotation operator was chosen to model the state transition for the $|0\rangle$ and $|1\rangle$ basis, but other operators can be selected for other problem formulations based on the nature of the basis. An operator set can be based entirely on physics, in which different operators represent different or sometimes competing physical processes (e.g. different failure modes), or it can be empirically based on observational data and measurements via curve fitting functions, or it can be based on expert opinions and contextual knowledge.

The conventional spread of fragility curves for a given loading reflects the epistemic characteristics and properties of the problem, where the family of fragility curves is produced from statistical distributions or confidence intervals. With the quantum approach, the state vectors reflect the understanding of the number of possible states the system possesses. The operators replace the conventional statistical based approach to produce the family of fragility curves. Each fragility curve is the product of the application of models representing physical processes, workflows, or system behaviors. The epistemic uncertainty corresponds to the range of the operators; the lack of precise knowledge about the states and their behaviors will result in a collection of different operator sets. Chapter 7 will provide examples to further illustrate and explore this perspective.

The quantum approach converts the conventional fragility curves into a measurement of the different population of states and how those states evolve. This characteristic opens up many new possibilities, such as a different approach in the computation of the total failure probability for a complex system with many different failure modes; a topic to be developed further in Chapter 9. From a different perspective, this can be viewed as another way to measure the number of states and track their behaviors, which can potentially lead to other concepts such as entropy, which characterize of the microstates of a system. The connection to entropy concepts presents an exciting opportunity for future research.

### *6.2 Event Trees from Tensor Products*

The event tree and other related network tree methodologies are commonly used PRA techniques. A description of event tree logic and structure can be found in Appendix A.3.2. In brief, an event tree is an inductive logical and chronological decomposition of an event into a progressive series of events leading to some subsequent outcomes, consequences, or end states. The decomposed events, system elements, and steps are represented as a branching tree graph or flowchart, and Boolean logic serves as the connectors or nodes between them. Probabilities for the chance of occurrences of the events can be associated with the nodes. Each event tree represents different scenarios formulated by the risk analysts to describe the various event paths leading to various outcomes. The event tree itself is usually the output product from the PRA modeling process and from which numerical values of risk events probabilities are computed.

The quantum framework can also generate event trees. The basic event tree leaf nodes can be generated from the tensor products when forming the composite system model. Since the output from the tensor product is a state vector, operators can act on it to model a variety of scenarios and system behaviors; operators make the model a "living" (i.e. dynamic) event tree that can evolve in time.

Consider the following model for two concurrent failure modes E (erosion) and O (overtopping), introduced in Sections 5.2.3, 5.2.4. Failure modes E and O are considered as independent from each other: E is a function of time, O is a function of the height of the levee. A system that is subjected to these failure modes, concurrent or not, can be constructed from the tensor product:

$$\left|\psi_L\right\rangle = \left|\psi_E\right\rangle \otimes \left|\psi_O\right\rangle = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \otimes \begin{pmatrix} o_0 \\ o_1 \end{pmatrix} = \begin{pmatrix} e_0 o_0 \\ e_0 o_1 \\ e_1 o_0 \\ e_1 o_1 \end{pmatrix} \tag{6.5}$$

$$\left|\psi_L\right\rangle = e_0 o_0 \left|00\right\rangle + e_0 o_1 \left|01\right\rangle + e_1 o_0 \left|10\right\rangle + e_1 o_1 \left|11\right\rangle \tag{6.6}$$

Since E and O are independent, the subsystems are not correlated. The probability amplitudes are separable and can be factorized into separate products, and the subsystems are not correlated. The state vector $\left|\psi_L\right\rangle$ is equivalent to the regular binary event tree (Figure 28).

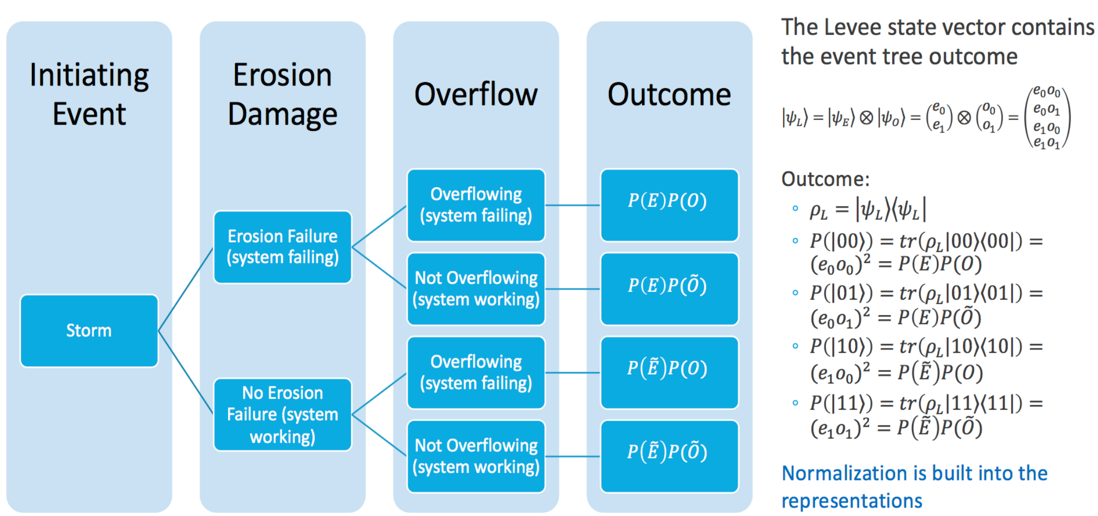

Figure 28: The binary event tree a system with independent E and O failure modes.

The probabilities can be obtained via the projection operator **P** on the density matrix $\rho_L = \left|\psi_L\right\rangle\left\langle\psi_L\right|$ , $\left\langle \mathbf{P} \right\rangle = tr\left(\rho_L \mathbf{P}\right)$. If one computes the outcome of the states from the state vector $|\psi_L\rangle$ (Equation 6.5) or from the density matrix $\rho_L = |\psi_L\rangle\langle\psi_L|$, one obtains:

$$P(|00\rangle) = tr(\rho_L|00\rangle\langle 00|) = (e_0 o_0)^2 = P(E)P(O) \qquad (6.7)$$

$$P(|01\rangle) = tr(\rho_L|01\rangle\langle 01|) = (e_0 o_1)^2 = P(E)P(\tilde{O}) \qquad (6.8)$$

$$P(|10\rangle) = tr(\rho_L|10\rangle\langle 10|) = (e_1 o_0)^2 = P(\tilde{E})P(O) \qquad (6.9)$$

$$P(|11\rangle) = tr(\rho_L|11\rangle\langle 11|) = (e_1 o_1)^2 = P(\tilde{E})P(\tilde{O}) \qquad (6.10)$$

The binary event tree outcome is now contained within the state vector $|\psi_L\rangle$!

### <u>*6.3 Discussion*</u>

In the above two case examples — the reproduction of the fragility curves and the construction of the leaf results from the event tree — important properties of the quantum model emerge:

1) *The quantum state vector becomes the fundamental object for describing risk in an engineering system for risk studie*s.

The first case example demonstrated how one can generate conventional risk analysis diagnostic tools, the fragility curves, from the state vectors. The use of the state vector representation recast the problem in a different way, connecting it to a new set of tools and opening up new possibilities and perspectives. In the example, the fragility curves are products from the application of operators and projectors to alter the probabilities based on engineering model; these new tools introduce dynamic elements into the modeling process.

2) *The quantum operators model the various behaviors of the system, and by operating on the state vectors change the system states*.

Using the density operator representation, we can investigate new ways to compute the total probability of failure for a complex system with many different failure modes, such as independent or concurrent failure modes. Calculating the total failure probability for engineering systems with multiple failure modes, especially if they can occur simultaneously and not mutually exclusive or not independent, can be challenging. Simple linear additive combinations of the failure mode probabilities can only be applicable under a restrictive set of conditions. Often assumptions and approximations have to be made to combine the probabilities and obtain usable total probabilities.

When the assumptions and approximations no longer hold true, the law of probability can be violated, and in certain cases the total probability can be greater than 1. As we approach the limits of the approximations, total probability calculated from the application of conventional probability rules can lead to inconsistent and incorrect results, such as the value of the total probability exceed unity (>1), violating the basic conservation of probability, leading to the mis-estimation of risk. The concept of re-normalization was introduced to address this problem. However, the validity of this approach has been questioned by some as it is seen more as a mathematical manipulation than basing off from sound principles and engineering methods. The quantum framework can provide an alternative framework to support investigation from a different direction. This topic will be explored further in Chapter 9.

3) *Various information extract from the state vectors via projections give results that can be used for the construction of various risk analysis and assessment*

*scenarios for evaluation and investigations, supporting risk-informed decisions*.

In the second example, the event tree, conventional approach has one set up and configured the event tree based on available information and knowledge. Once the tree is set, only a few operations can be performed on it without requiring a revision of the tree; for example, when the environment and properties change, one will need to rebuild the tree. For a static world, that approach will be more than sufficient. However, the notion of risk analysis and assessment is about looking at systems in dynamic settings. Dynamic treatments using event trees can be complicated, requiring regular updates and modifications to the tree structures. New techniques such as Bayesian updates and dynamic decision/event trees are promising, but the quantum framework can potentially offer an alternative starting point. With the quantum framework, the event tree can be automatically built from the construction of the system state vectors using tensor product. The resulting state vector becomes the starting point, packing extensive amount of information within the object itself, ready for further operations to evolve the system.

Both of these rudimentary examples affirm the notion that with the adoption of a different probability theory in the representation of an engineering system and the associated uncertainties, we are not just changing how the systems are represented, but also the types of operation and toolset that can be used to model the problem. Before using these new tools to model the HPS, in the next Chapter the density operator approach will be examined from the perspective of how the interpretation of the mathematical construct can open up new ways to model uncertainty and risk.

# Chapter 7: Interpreting the Quantum Framework

*[…] Willful ignorance is the central concept that underlies mathematical probability. In a nutshell, the idea is to deal effectively with an uncertain situation, we must filter out, or ignore, much of what we know about it. In short, we must simplify our conceptions by reducing ambiguity.*

*- Herbert Weisberg 2014*

Many methodologies, techniques, and frameworks have been developed over the years to customize the treatment for the different types of uncertainty. Often, the different types of uncertainty require separate and specialized framework to address problem specific conditions. Quantum probability with density operator framework offers yet another approach; however, using the quantum density operator formalism to describe a system, the terms in the density matrix are mapped and interpreted as different types of uncertainties, bringing them under a single framework. The density matrix formalism recognizes the reality that these uncertainties are not isolated, and they do connect and relate to each other in some ways, changing a system's uncertain states. For example, a system's intrinsic aleatory uncertainty itself might not change, the application of a state transition or transformation operator can alter the system's states and the probability amplitudes, changing the uncertainty of the system.

The density operator formalism is a system model framework that can capture the *effective uncertainty* for the system. The numerical probabilities calculated from the density matrix represent effective uncertainties associated with a system, reflecting the dynamic changes and evolutions of the system states. The probability term, $p_i$, provides a mechanism to encode ignorance into the mathematical model. Not only does it allow us to reflect incomplete knowledge, the construction of the probability term can itself be a process to weight subjective judgement and belief. Lastly, since the $p_i$ term follow conventional statistical probability, conventional probability operations can be applied to it, such as Bayesian updates. The robustness of the quantum framework can open up new paths to model risks and uncertainties.

The quantum framework adds the following to the risk analyst's toolkit:

- Additional dimensions for capturing contextual information and knowledge are incorporated and extend beyond the conventional statistical and probabilistic frameworks; contextual information and knowledge are directly captured in the mathematical structure.
- The operator algebra captures knowledge, and particularly information about sequential operations, which is possible due to the non-commutative nature of the mathematical structure; this, in addition, aids the modeling of event driven scenarios.
- System models can dynamically evolve via projective measurements. Instead of building new models or creating time bins, the quantum models have the dynamics built into system models. Instead of traditional Boolean logic and conditionals (such as if-then-else in the graphs and networks), operators and

their sequence of operations can provide more robust means to select and alter the timelines.

- Finally, the quantum framework provides a mechanism to address the notion of dealing with unknown-unknowns. The uncertainty about a system model is limited by our knowledge. We cannot be totally certain about the validity of a model given that we do not know what we do not know! In this framework, we acknowledge that we do not truly quantify the uncertainties about the models, but only the effective uncertainties. The effective uncertainties, in turn, provide a quantitative measure. We can compare the effective uncertainties against observations, and while we do not fully quantify the model itself, we can quantify its effects on the uncertainties. Analyzing and comparing the effective uncertainties against observations, one can potentially yield information that can be used to update and refine the model, a Bayesian like approach!

To analyze a problem, a model has to be instantiated with boundary conditions and parameters based on the physical situations. The various mathematical elements of the model must be interpreted so that they can be mapped to the parameters. This rest of this chapter will focus on interpreting the quantum framework in terms of uncertainty and risk.

## *7.1 The Quantum Engineering System and Risk*

We define a quantum engineering system as one exhibiting properties as defined by the following postulates:

1. Engineering systems exhibit behaviors and characteristics that can be

described by quantum probability.

    a. They are “quantum-like” system with characteristics similar to physical quantum systems, following similar mathematical structures, but they are not quantum mechanical systems as in subatomic particle physics.

2. The states of the system are represented by state vectors, or in a more general equivalent formulation, the density operator.
    a. The fundamental aleatory uncertainties about the system states are contained within the state vectors in the form of probability amplitudes and the superpositions of the basis states. The individual state vector is *pure*.
    b. The system’s epistemic uncertainties from the physics models are captured in, quantified by, and modulated with operators acting on the state vectors. The resulting uncertainties determined from the system model represent effective uncertainties for the system as a whole.
3. Different forms of epistemic uncertainties are captured by the probability terms $\left(p_i\right)$ in the density matrix, interpreted as statistical ensemble of possible system states, a mixture formed from pure states.
4. Subjective beliefs and expert opinions can be incorporated into the density matrix, using the probabilities as weighting functions.

As stated in Section 4.6, an engineering system can be found in different states, and these states are represented by state vectors formed from the linear combination of the basis states the system can be found in. The collection of these state vectors forms the engineering system and can be expressed in the form of the density operator:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{7.1}$$

Composite systems are assembled from the tensor products of the state vectors or density operators. The probability of finding the system in a particular state is the squared of the probability amplitudes of the state vectors or from the trace of the density matrix. The engineering system evolves according to events, which are represented by mathematical operators acting on the state vectors, and the transition of the system from states to states are given by:

$$\rho_{S'} = \mathbf{U}\rho_S\mathbf{U}^\dagger = \sum_i p_i \mathbf{U}|\psi_i\rangle\langle\psi_i|\mathbf{U}^\dagger \tag{7.2}$$

Complex system models can be constructed from combining components and subsystems together via their tensor products, a process that aligns well with basic system engineering principles in the decomposition and construction of an engineering system. The engineering system evolves in time, transitioning from states to states, and the mathematical abstractions representing the transitions are the event operators.

The operator representation allows the tracking of the order of the events, since matrix operations are non-commutative. The sequence of operations and their ordering matters, which is a property of changing to the matrix mathematical representations. The operator can change the state of the system without modifying the system, or the operator can change the state of the system by changing the system itself. Since the density operator represent the system, different operations on it can also extract different information about the system. The projection operator or the reduced density operator (a "subarray" of the density operator), for example, "selects" particular properties of the system and can be interpreted as making a measurement to "obtain the answer to the question." These characteristics align well with what are observed in real systems. The density operator provides a general purpose mathematical framework to model and extract information from complex interacting systems, whether one possess full knowledge or partial knowledge about the system.

The density operator provides a quantitative representation of knowledge, ignorance, and uncertainty for a system. When full knowledge about a system is available, the full tensor product can be formed. The density operator from the full tensor product is an exact representation of the system, a pure state density operator (more on this in the next section). Achieving this precision requires the incurring of high computational cost associated with the tracking of large data volume and the increase in computation complexity. The purity of the state comes at the cost of the need for deep knowledge acquisition, which may or may not be possible depending on the situation. A trade off can be made where, by ignoring some of the knowledge (either do not have them or choose not to incorporate them), computation efficiency can be reclaimed at the cost of introducing or increasing uncertainties, which leads to the mixed states representation of the system. Pure states and mixed states are indications of the degree knowledge and uncertainty we have about the system. The different formulations of the density matrix allow the adaptation of this mathematical construct to reflect the degree of our knowledge about the system.

*7.2 The Interpretation of the Pure and Mixed States*

In the context of this study, whether a system is in a pure state or a mixed state is a function of how much knowledge and how precisely does one know about the system at a given time.

**A system is in a pure state** when complete and precise knowledge about it is available and known, such as right after the preparation and construction of a levee system, or after an observation and measurement. At that time, knowledge about the system is complete to the degree limited by the aleatory uncertainty. We have maximum knowledge on the states of the system, and the uncertainties about its performance and reliability are strictly based on the intrinsic stochastic uncertainties. If a levee has an intrinsic failure probability of 0.1 due to craftsmanship, materials, etc, then there is a 10% chance of finding the system in a failing state based on just this intrinsic failure probability.

**A pure state system is one that we possess full knowledge** about the system, and we can continue to track and trace the changes of the system going forward. As long as the evolution of the system can be measured and tracked, the system stays pure over time. For example, the density matrix $\rho = |L\rangle\langle L|$ where $|L\rangle = 0.3162|0\rangle + 0.9487|1\rangle$ is the pure state representation of the levee system.

**A mixed state system is when we do not know precisely the exact makeup** of the system, which means we only have knowledge about the states the system can be in. We only have statistical information about the makeup; in other words, we are uncertain about the precise and exact state of the system at a given time. Mixed state in this context can include different types of uncertainties. For example, when one comes across a (L+W) system, one cannot be totally sure of its history. A composite system could be formed out of 2 subsystems with different evolutionary paths, and we might not know all the precise information necessary to build a pure state density matrix. There could be x% chance that the system is original, or there could be y% chance that the system is one with an updated flood wall at a later time. In such scenario, the density matrix $\rho = x\left|LW_0\right\rangle\left\langle LW_0\right| + y\left|LW_1\right\rangle\left\langle LW_1\right|$ is a mixed state representation of the system, where $\left|LW_0\right\rangle$ is the pure state representation of the original L+W system, and $\left|LW_1\right\rangle$ is the state representation of the system with updated flood wall.

*7.3 Switching Views between Subsystem and Ensemble*

A system is pure when one knows and can track everything, and it is mixed when one cannot. The density operator is particularly useful in this regard since the flexible framework allows different views and representations of a system. The subsystem view (e.g. the individual levees and flood walls) aligns with systems with known characteristics and states, and the ensemble view aligns with systems with uncertain characteristics and states, or when the subsystems are treated as a statistical ensemble. The ability to treat subsystems as a statistical ensemble becomes a critical asset as the tracking of the states become more complicated and difficult when you have sizable composite systems, especially when the subsystems themselves exhibit correlated behaviors

A Principle of Equivalence:

*The ensemble view and the subsystem view are equivalent but of two different perspectives.*

Three cases below illustrate how the different views share the same underlying mathematical underpinnings. The first one illustrates the subsystem view, where we construct and track everything. The second one is the ensemble view, where we choose not to track everything or cannot track everything, and it captures uncertainties with the probability terms. The third is the case where we really don't know what's going on.

#### 7.3.1 Case 1: Full Knowledge of all Subsystem States

*Composite System evolution scenario 1a: all built at the same time. The Levee + Flood Wall (L+W) subsystems were built together at the same time, and that all the subsystems were built together at the same time and they were identical to each other.*

The system is described by a pure state density matrix with full knowledge of the subsystem states:

$$\begin{aligned}\rho_{HPS} &= \rho_{L_1^A W_1^A} \otimes \rho_{L_1^B W_1^B} \otimes \rho_{L_1^C W_1^C} \otimes \cdots \\ &= \left| L_1^A W_1^A \right\rangle \left\langle L_1^A W_1^A \right| \otimes \left| L_1^B W_1^B \right\rangle \left\langle L_1^B W_1^B \right| \otimes \left| L_1^C W_1^C \right\rangle \left\langle L_1^C W_1^C \right| \otimes \cdots\end{aligned} \quad (7.3)$$

*Composite System evolution scenario 1b: individual (L+W) subsystem built at the same time, but the system of (L+W) were built at different times. The subsystems have identical specifications and follow the same time evolution behaviors.*

The density matrix (eq. 7.3) is also applicable to subsystems built at different times if precise knowledge about the subsystems and their histories are known. For example, if system A was built at t = 0, system B was built at t = 5, system C was built at t = 10. Let **M(t)** be the operator that transition the systems from t = 0 to t = 10. The state vectors for the systems are given by:

$$\begin{aligned} \left|L_1^A(10)\right\rangle &= M(10)\left|L_1^A(0)\right\rangle, \ \left|W_1^A\right\rangle = M(10)\left|W_1^A(0)\right\rangle \\ \left|L_1^B(10)\right\rangle &= M(5)\left|L_1^B(0)\right\rangle, \ \left|W_1^B\right\rangle = M(5)\left|W_1^B(0)\right\rangle \\ \left|L_1^C(10)\right\rangle &= M(0)\left|L_1^C(0)\right\rangle, \ \left|W_1^C\right\rangle = M(0)\left|W_1^C(0)\right\rangle \end{aligned} \tag{7.4}$$

and the density matrix will have the same functional form as equation (7.3):

$$\begin{aligned} \rho_{HPS} &= \rho_{L_1^A W_1^A} \otimes \rho_{L_1^B W_1^B} \otimes \rho_{L_1^C W_1^C} \otimes \cdots \\ &= \left|L_1^A W_1^A\right\rangle\left\langle L_1^A W_1^A\right| \otimes \left|L_1^B W_1^B\right\rangle\left\langle L_1^B W_1^B\right| \otimes \left|L_1^C W_1^C\right\rangle\left\langle L_1^C W_1^C\right| \otimes \cdots \end{aligned} \tag{7.5}$$

For systems with full knowledge, all the elements of the density matrix can be computed without any uncertainty and unknowns besides the irreducible aleatory uncertainty. In this case, the state vector is pure, the density matrix is pure and fully represents the subsystems.

#### 7.3.2 Case 2: Partial Knowledge (As Given or By Choice)

*Composite System evolution scenario 2: groups of subsystems were built together, and different groups were built at different times. The (L+W) subsystems were built together and subsystems built at the same period shared the same specifications. The evolution histories were incomplete.*

*Case #2a: HPS is a merge of 3 projects, built over different times. Assuming that all subsystems built in the same project are similar, we have 3 groups of (L+W) structures. In this case, the setup is very similar to case #1, including the same computational and tracking efficiency issues. One can take advantage of the grouping and instead of tracking individual subsystems,* *groups of subsystems will be tracked. The loss of precisions results in the gain of computational efficiency.*

*Case #2b: Instead of grouping by projects, the subsystems can be grouped together according to the state vector; instead of grouping according to projects, the subsystems are group together according to population type. This is more suitable for system where a large number of subsystems were built at around similar times, but can be in various groups of states due to external factors such as environmental, geographical, etc.*

When the subsystems are constructed at different times and have different histories, tracking them individually, while possible, can be a massive undertaking. If the subsystems can be grouped according to system characteristics and the groupings can be described statistically, then a mixed state density matrix can be used to represent the system. This approach is only possible if the system can be treated as an ensemble, which then allows the adoption of statistical representations. One can consider this as a form of approximation, introduced to simplify the computational requirements. By choosing to use group properties and ignoring individual characteristics, uncertainties are introduced into the system. In the ensemble view, the $p_i$ provides the statistical distribution for the ensembles and the uncertainties are captured in the probability terms, $p_i$, in the density matrix. The only condition is that the sum of the probabilities have to be unity.

Consider the HPS with (L+W) subsystems that fall into certain statistical distribution. Instead of tracking individual (L+W) subsystem, (L+W) subsystems of similar states are grouped together. In this case, the composite state vector represents a general state where a subsystem can be found in. Individual subsystems, such as A, B, C, … as in

$$\begin{aligned}\rho_{HPS} &= \rho_{L_1^A W_1^A} \otimes \rho_{L_1^B W_1^B} \otimes \rho_{L_1^C W_1^C} \otimes \cdots \\ &= \left| L_1^A W_1^A \right\rangle \left\langle L_1^A W_1^A \right| \otimes \left| L_1^B W_1^B \right\rangle \left\langle L_1^B W_1^B \right| \otimes \left| L_1^C W_1^C \right\rangle \left\langle L_1^C W_1^C \right| \otimes \cdots\end{aligned}$$

(7.6)

are not tracked. Instead, the different group states and their distributions are differentiated and tracked:

$$\begin{aligned}\rho_{HPS} &= \sum_i p_i \left| L_i W_i \right\rangle \left\langle L_i W_i \right| \\ &= p_1 \left| L_1 W_1 \right\rangle \left\langle L_1 W_1 \right| + p_2 \left| L_2 W_2 \right\rangle \left\langle L_2 W_2 \right| + \cdots\end{aligned} \quad (7.7)$$

For example, consider the following configurations:

Subsystem A can be in state 1 and 2.

Subsystem B can be in state 2 and 3.

Subsystem C can only be in state 3.

The density matrix for this composite system formed out of subsystem A, B, and C becomes

$$\begin{aligned}\rho_{HPS} &= \sum_i p_i \left| L_i W_i \right\rangle \left\langle L_i W_i \right| \\ &= p_1 \left| L_1 W_1 \right\rangle \left\langle L_1 W_1 \right| + p_2 \left| L_2 W_2 \right\rangle \left\langle L_2 W_2 \right| + p_3 \left| L_3 W_3 \right\rangle \left\langle L_3 W_3 \right| \\ &= \frac{1}{5} \left| L_1 W_1 \right\rangle \left\langle L_1 W_1 \right| + \frac{2}{5} \left| L_2 W_2 \right\rangle \left\langle L_2 W_2 \right| + \frac{2}{5} \left| L_3 W_3 \right\rangle \left\langle L_3 W_3 \right|\end{aligned}$$

(7.8)

These types of situation are often encountered in real life. The HPS is such a scenario. The HPS was a combination of 3 different projects over 3 different time periods; the first project finished building a set of (L+W) subsystems, then the second project built another set at a later time, and finally the third project built the last one at a much later time.

7.3.3 Known-Unknowns and Unknown-Unknowns

*Composite System evolution scenario 3: different histories, information, and opinions from experts' knowledge.*

Face with insufficient knowledge, the comprehensive modeling of a system using state vectors alone becomes impossible, and thus the use of the density matrix formulation is required. The density matrix can be constructed by filling knowledge gaps through various means, from the use of statistics to expert opinions. The bigger the knowledge gap, the more subjective beliefs will have to be injected into making educated guesses about the statistics and the probability distributions. Expert opinions, analogous cases, historical data, and subjective beliefs are introduced at the price of increased uncertainties. Nonetheless, this is often how reality presents itself.

Even in this case, the mathematical form of the density matrix is still the same, except that the interpretation of the probabilities is now associated with subjective beliefs, moving away from the ensemble viewpoint.

$$\begin{aligned}\rho_{HPS} &= \sum_i p_i \left| L_i W_i \right\rangle \left\langle L_i W_i \right| \\ &= p_1 \left| L_1 W_1 \right\rangle \left\langle L_1 W_1 \right| + p_2 \left| L_2 W_2 \right\rangle \left\langle L_2 W_2 \right| + \cdots \end{aligned} \tag{7.9}$$

The system is now represented by a completely mixed state density matrix, and the probability terms, $p_i$, are no longer population statistics but weight functions reflecting subjective beliefs and expert opinions.

*7.4 The Interpretation of the Density Operator Formalism: An Integrated Representation of Three Types of Uncertainties*

The density operator formalism provides a robust framework that can be adapted to model different situations. The previous section illustrated how the framework can handle scenarios with different degree of availability and completeness of knowledge. The density operator can also support other interpretations that can potentially offer a combined description of the three types of uncertainties: aleatory, epistemic, and subjective belief.

7.4.1 Integrated Representation of Uncertainties

*Composite*

In earlier chapters, three types of uncertainties were introduced: aleatory, epistemic, and human judgement and belief. Aleatory uncertainty deals with the inherent, intrinsic random stochastic variations associated with a physical system. Epistemic uncertainty reflects the lack of knowledge and information about some properties and characteristics of a system, and it often reflects by the fidelity of the model or the model uncertainties for a system. Finally, human judgement and belief reflect the use of subjective knowledge and judgement (or bias) in the formulation of designs, opinions, and decisions, which can increase or decrease the accuracy regarding the “true” representation of a system. How does the density operator formalism unify these three uncertainties?

The general form of the density operator is expressed as:

$$\rho = \sum_i p_i \mathbf{H}_i \left| S_i \right\rangle \left\langle S_i \right| \mathbf{H}_i^\dagger \tag{7.10}$$

The right-hand side terms in this expression can be mapped to the three types of uncertainties according to the following interpretations:

$\left| S_i \right\rangle$ — The state vector for the system in the *i-th* state; this state vector contains the probability amplitudes of the basis state vectors. This is interpreted as representing the aleatory uncertainties associated with the individual basis component state vectors, in the form of probability amplitudes intrinsic to the basis states. This corresponds to the inherent, intrinsic random stochastic variations associated with a physical system's states.

$\mathbf{H}_i$ — The operator operating on the state vectors, changing the states of the system. This is interpreted as representing the moderator or modulators of the *epistemic uncertainties*. The operator itself is a model representing some physical processes, e.g., erosion, material fatigue, etc. The operator acts on the state vector, resulting in a change of state in the form of a change of the probability amplitudes; the probabilities derive from the state vectors post operation encapsulate both the aleatory uncertainties and the model based epistemic uncertainties.

Furthermore, the operator itself can be formed from a sequence of operators, i.e., $\mathbf{H}_i = \cdots \mathbf{H}_{i3}\mathbf{H}_{i2}\mathbf{H}_{i1}$ , for which their order of application matters. The operators are non-commutative, and their operations on the state vector proceed from the right to the left, starting with operator $\mathbf{H}_{i1}$, then $\mathbf{H}_{i2}$, and so on. With this mechanism, the contextual knowledge about the sequential behaviors of the system can be incorporated into a coherent mathematical framework. For example, different ordering of event sequence can lead to different outcome. An earthquake can damage a flood wall and the subsequent hurricane will result in flooding; whereas the same hurricane will not cause any flooding if it happens before the damaging earthquake. The operator sequence facilitates the representation of such sequential events.

$p_i$ — The statistical probability of finding the system in the i-th state. If we have full knowledge of the system, then there is only one system state vector. The system is in a pure state, and the statistical probability term will be unity ($p_i = 1$) for the density matrix formed by the single state vector. In the case where we only have partial knowledge about the system, or that we are uncertain as to what states the system can be in, either from the system model or subjective reasoning (such as expert opinions and beliefs), then the $p_i$ will be a probability distribution satisfying the conservation of total probability ($\sum_i p_i = 1$). This term is interpreted as another part of the *epistemic uncertainties* about the lack of precise knowledge and information about the characteristics of a system.

Finally, the human subjective bias and factors can also be incorporated into this expression via a statistical interpretation of $p_i$ as the subjective weighted distribution of expert beliefs. The algorithm to perform the weighted summation:

1) collect different experts' density operator,
2) if all opinions agree, then the normalized sum will be identical to the individual, an indication of minimal human subjective uncertainty introduced into the process,
3) if opinions differ, then perform a weighted sum (if the opinions have different weights) or an average. This will quantify the human based uncertainties.

Table 4: Quantum representations for the different types of uncertainty.

| **Uncertainty** | **Quantum Representation** |
|---|---|
| Aleatory | $\lvert S\rangle = s_0\lvert 0\rangle + s_1\lvert 1\rangle \Leftrightarrow \rho = \lvert S\rangle\langle S\rvert$ |
| Aleatory + Epistemic (physics) | $\lvert S'\rangle = \mathbf{H}\lvert S\rangle \Leftrightarrow \rho = \lvert S'\rangle\langle S'\rvert$ |
| Aleatory + Epistemic (physics)<br>+ Epistemic (beliefs) | $\rho = \sum_i p_i \lvert S_i\rangle\langle S_i\rvert$ |

7.4.2 Revisiting the Flood Wall example

To model subjective or non-fully probabilistic situations, consider again a flood wall scenario:

Flood wall A: $\lvert A\rangle = a_0\lvert 0\rangle + a_1\lvert 1\rangle$, $a_0 = \sqrt{0.01}$, $a_1 = \sqrt{0.99}$

Flood wall B: $\lvert B\rangle = b_0\lvert 0\rangle + b_1\lvert 1\rangle$, $b_0 = \sqrt{0.02}$, $b_1 = \sqrt{0.98}$

Probability of finding A in failure state = $a_0^2 = 0.01$

Probability of finding B in failure state = $b_0^2 = 0.02$

The probability of encountering a type A floor wall is 60% and a type B floor wall is 40%. The density matrix can be expressed as:

$$\rho = p_A|A\rangle\langle A| + p_B|B\rangle\langle B| = 0.6|A\rangle\langle A| + 0.4|B\rangle\langle B| \qquad (7.11)$$

Now another experienced engineer indicates that there should be more type B than type A. The mix should be 40/60. Then based on the engineer's information, the density matrix should be.

$$\rho = p_A|A\rangle\langle A| + p_B|B\rangle\langle B| = 0.4|A\rangle\langle A| + 0.6|B\rangle\langle B| \qquad (7.12)$$

As the subject expert, I think both of our opinions weight the same. The effective $p_{A,eff}$ and $p_{B,eff}$ would be the average of the 2 opinions, which have an average value: (0.4 + 0.6)/2. The subjective beliefs from different experts are now parameterized and captured. Since the $p_i$ terms are just conventional probabilities, as long as the sum of the probabilities is kept to unity, conventional probability tools such as Bayesian methods can be applied to update them!

#### 7.4.3 Analyzing Risk: The Projector and Expectation Values

The density operator represents the system capturing the aleatory, epistemic, and subjective uncertainties. Risk analysis is the process to identify the corresponding risk states in question from the density matrix, calculate their chances of occurrence, and associate with them values of consequence. The general process to obtain a risk value is to start with the initial system at the initial state S (t = 0) and apply operators

$P_{S'}$ to transition the system to the target state S' at some later time (t > 0) as specify by the assessment scenario. The density operator for S' is calculated from:

$$\rho_{S'} = \frac{P_{S'}\rho P_{S'}}{tr(P_{S'}\rho P_{S'})} \quad (7.13)$$

where $\rho_{S'}$ is the target state of the system. To obtain the risk value, a measurement **Ri** (recall Chapter 4.6) can be performed to obtain the expectation value:

$$\langle \mathbf{Ri} \rangle = tr(\rho \mathbf{Ri}) \quad (7.14)$$

which is simply the trace of the product of two matrices.

### *7.5 Exploring the Density Operator as a System Modeler for Risk and Uncertainty*

Modeling an engineering system with density operator allows us to capture and reflect the state of knowledge about the system. With full knowledge, a system can be described with a pure state density operator or density matrix (in a given basis), and only aleatory uncertainties are encoded in it. Otherwise, a system will be described by a mixed state density matrix, and both aleatory and epistemic uncertainties are encoded in it. The type of mixed state mixture provides a measure of how much knowledge one possesses for the system, or how much ignorance one can incorporate into the model. Even with full knowledge, one can choose to improve computational efficiency at the expense of precision by ignoring some of the available information and knowledge. Moving from tracking information about individual systems to tracking information about an ensemble is one form of introducing ignorance to the problem.

The density operator comes with properties that are capable to encode additional information in ways different than conventional methods, allowing the different density matrix elements to encapsulate and represent different information about the system. To fully develop and appreciate what the density operator offers, additional mathematical tools and concepts will need to be introduced beyond those established in earlier chapters. While a full comprehensive treatment is beyond the scope of this dissertation, this section will touch on additional concepts to lay down the foundation for future research.

### 7.5.1 Additional Properties of the Density Operator

The density operator has several additional properties which offer information encoding capabilities beyond those introduced thus far, making it a strong modeling tool candidate for a variety of situations. Recall the definitions of the basis states $|0\rangle$ and $|1\rangle$. In the most basic form, these two basis states correspond to the classical states for a system. The classical approach in representing a system is that it can either be in one of the two basis states; whereas, in the quantum approach, the system can either be in one of the two basis states, or that the system can be in a superposition of the two states, leading to the state vector

$$|L\rangle = l_0|0\rangle + l_1|1\rangle, \tag{7.15}$$

or in a more general form

$$|\Psi\rangle = \sum_i c_i|\psi_i\rangle. \tag{7.16}$$

These are *pure states*.

The concept of superposition is a cornerstone for the quantum theory, and it simply states that a system is defined by the "sum" of all the possible basis states. The state vector is the quantum superposition of the basis states. Up to now, the decision was made to model the systems in earlier chapters within a single quadrant of the real number space $(\mathbb{R})$ to demonstrate how the quantum theory can be used to model risk and uncertainty. If one relaxes the restrictions, additional properties and characteristics become available. Consider again the state vector $|L\rangle = l_0|0\rangle + l_1|1\rangle$, which we will relabel it as

$$|L+\rangle = l_0|0\rangle + l_1|1\rangle . \tag{7.17}$$

The probabilities for the states are given by the usual $P(|0\rangle) = l_0^2$ and $P(|1\rangle) = l_1^2$. However, one can construct a different state vector

$$|L-\rangle = l_0|0\rangle - l_1|1\rangle \tag{7.18}$$

and the probabilities for the states are also $P(|0\rangle) = l_0^2$ and $P(|1\rangle) = l_1^2$. $|L+\rangle$ and $|L-\rangle$ are related by a *relative phase*, and the $\pm$ signs indicate that phase. This phase factor is internal to the state vector and affects how the state vector evolves internally. Other additional mathematical structures come into play when the full complex $(\mathbb{C})$ space is considered since the state vector can be expressed in a different coordinate representation, and in complex space the vector can be written as:

$$|\Psi\rangle = e^{i\gamma}\left(\cos\frac{\omega}{2}|0\rangle + e^{i\varphi}\sin\frac{\omega}{2}|1\rangle\right) . \tag{7.19}$$

Additional mathematical tools and interpretations will need to be developed in order to fully examine this property in terms of risk and uncertainty. Nonetheless, taking at face value, several interesting properties and characteristics with bearings towards the interpretation of the density matrix elements can be derived.

*7.5.1.1 Pure States vs. Mixed States and Superposition vs. Mixtures.* The basis states $|0\rangle$ and $|1\rangle$, and the superposition state vector $|L\pm\rangle = l_0|0\rangle \pm l_1|1\rangle$ are pure states, which are used to describe fully isolated systems that we have full knowledge of. A *mixture* of different pure states is a mixed state. Mixed states are simply systems where their states are less than certain and the systems are described by the collection of the quantum superpositions related by probabilistic uncertainty, which can be thought of as probability distributions of the quantum superpositions ensemble: $\{p_i, L_i\} = (p_1, |L_1\rangle), \cdots, (p_i, |L_i\rangle)$. Pure and mixed states can be represented by the density operator $\rho = \sum_i p_i |L_i\rangle\langle L_i|$ or the density matrix which is simply the matrix form of the density operator with a given basis. For example, the pure state density matrix for the basis states are $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. One can also construct the density matrix for the superposition states, such as

$|L+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, as

$$|L+\rangle\langle L+| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (7.20)$$

and

$$\left|L-\right\rangle\left\langle L-\right| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \quad (7.21)$$

For mixed states, things can become a bit tricky as we are describing a *mixture* of states, which can sometimes be confused with superpositions. Consider the density matrix $\rho = \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. This represents an equal mixture of two pure basis states $\left|0\right\rangle$ and $\left|1\right\rangle$. The density matrix

$$\rho = \frac{1}{2}\left|L+\right\rangle\left\langle L+\right| + \frac{1}{2}\left|L-\right\rangle\left\langle L-\right| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad (7.22)$$

which is a mixture of two superposition states, gives the same density matrix of the mixture of the two basis states. Mixtures are different from superpositions.

*7.5.1.2 Test for Purity.* A density matrix can be tested to determine if it is pure, mixed, or somewhere in between. The tests for purity are stated below without proof; the proofs can be found in the references mentioned in the beginning of Chapter 4, such as Nielsen & Chuang (2011).

Purity Test #1:

If the density matrix is pure, it has exactly one non-zero diagonal element (one non-zero eigenvalue) equal to 1. A pure state satisfies $tr\left(\rho^2\right) = 1$ and $\rho^2 = \rho$, and a mixed state satisfies $tr\left(\rho^2\right) < 1$ and $\rho^2 \neq \rho$.

Purity Test #2

a) The pure state satisfies the relation: $\rho_{mn}\rho_{nm} = \rho_{mm}\rho_{nn}$.

b) A partially mixed state satisfies: $0 < \rho_{mn}\rho_{nm} < \rho_{mm}\rho_{nn}$.

c) A totally mixed state satisfies: $\rho_{mn} = \rho_{nm} = 0$ and $\rho_{mm}\rho_{nn} \neq 0$.

They are illustrated by the following examples.

- $\rho = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ passed Test #1. The density matrix is pure.
- $\rho = |L+\rangle\langle L+| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ passed Test #1 and #2a. The density matrix is pure.
- $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ passed Test #2c. The density is *totally mixed*.
- $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ passed Test #2c. The density is *maximally mixed*.
- $\rho = \frac{2}{3}|L+\rangle\langle L+| + \frac{1}{3}|0\rangle\langle 0| = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ passed Test #2b. The density is *partially mixed*.

A few subtle points here about the different mixed states: partially, totally, and maximally mixed. The maximally mixed state corresponds to the system behaving classically, essentially going back to conventional probability, going back to the classical coin toss. The partially mixed state corresponds to the system possessing quantum superposition behaviors. The pure state is the ultimate full coherent state, with everything determined up to the non-reducible aleatory uncertainties. The fully mixed state is the ultimate incoherent state, which means the system is totally described by a statistical distribution, and our knowledge about the state is at minimum.

*7.5.1.3 The Elements of the Density Matrix.* From the examples, one can see that the elements of the density matrix encode different type information about the system. The diagonal matrix elements give the probabilities of finding a certain state of the system. They are the probability distribution of the states with a chosen basis and the probabilities represent the uncertainties associated with the states. The diagonal elements encode information about the probabilities, but it can also be used with other elements to determine the degree of mixing.

The off-diagonal density matrix elements correspond to the interference between the probability amplitudes from the basis states due to superpositions. The elements are products of different probability amplitudes, and these are the non-classical superposition states. The off-diagonal elements alone might not be sufficient in determining the state of the system; as seen from the examples, both pure state and mixed state can have zero as values for the off diagonal elements ( $\rho_{mn} = 0$ ). These elements encode the system's "quantumness" - degree of exhibiting quantum conditions - and how the different basis states "interfere" with each other.  In the case with composite systems, such as $|LW\rangle\langle LW|$, the off-diagonal elements encode the "interference" between the subsystems' four basis states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

The off-diagonal elements do not have classical counterparts and do not directly correspond to classical outcomes. In the case of composite systems, their "existences" reflect the possibility of quantum correlation ("entanglement") between the states, which is further discussed in Section 7.5.5 below. Interactions with external events and elements over time alter the "purity" of the quantum superpositions as mixtures are introduced into the system, resulting in losing precisions. As we start losing information due to system interactions within and with the external environment, the off-diagonal elements will start approaching zero, moving more and more towards classical statistical distributions, and more into product states (losing correlation and coherence). From the ensemble interpretation (Section 7.3), a completely random ensemble is considered ***incoherent*** and maximally mixed, and a pure state is a completely ***coherent*** state. As seen earlier, an example of a pure state density matrix has the following diagonal form:

$$\rho = \begin{pmatrix} 0 & & \cdots & & 0 & 0 & 0 \\ & 0 & & & \iddots & 0 & 0 \\ & & \ddots & & & & 0 \\ \vdots & & & 1 & & & \vdots \\ 0 & & & & 0 & & \\ 0 & 0 & & \iddots & & \ddots & \\ 0 & 0 & 0 & & & \cdots & 0 \end{pmatrix} \tag{7.23}$$

and $tr(\rho^2) = 1$. On the other extreme, the completely random ensemble will have a density matrix of the following form:

$$\rho = \frac{1}{N}\begin{pmatrix} 1 & & \cdots & & 0 & 0 & 0 \\ & 1 & & & \iddots & 0 & 0 \\ & & \ddots & & & & 0 \\ \vdots & & & 1 & & & \vdots \\ 0 & & & & 1 & & \\ 0 & 0 & & \iddots & & \ddots & \\ 0 & 0 & 0 & & & \cdots & 1 \end{pmatrix} \tag{7.24}$$

where N is the number of basis states (the dimension of the matrix) and all the states are equally populated. Section 7.5.4 will return to this topic and further explore how these properties will enable a different type of modeling approach.

For this research, the interpretation of the off-diagonal terms cannot be fully established without going beyond the restriction of focusing only in the Real number space ( $\mathbb{R}$ ) and without introducing additional mathematical tools; additional properties, such as phase information, require the consideration of the full complex Hilbert space. The full quantum treatment is needed for a comprehensive investigation; further development and research will be required to establish proper interpretations in the context of uncertainty and risk. While it is premature at this time to offer any definitive interpretations regarding the off-diagonal elements, even with our current limited understanding about the properties, there are several potential applications of the density matrix in the modeling of uncertainty and risk, which we shall explore for the remaining of this chapter.

### 7.5.2 Density Operator as Tool to Model Uncertainty

Earlier section in this chapter (Section 7.4.1) presents the idea that the quantum state vector and the density operator can provide an integrated framework to represent different types of uncertainties. This ability is the result of the additional encoding capacity provided by the quantum framework. The encoding capabilities do not simply capture knowledge and information, but also how they evolve over time. The inclusion of the time dimension offers another interpretative pathway to connect the different types of uncertainties.

The quantum state vector $\left|\Psi\right\rangle = \sum_i c_i \left|\psi_i\right\rangle$ or $\left|\Psi\right\rangle\left\langle\Psi\right|$, via the concept of **superposition**, encodes the **aleatory** uncertainties associated with the system in terms of the probability amplitudes and the basis states at the start of the model construction. Any composite systems formed from the tensor products of the different subsystem state vectors are themselves state vectors, encoding with them aleatory uncertainties. The product of the probability amplitudes are themselves probability amplitudes, and thus the composite state vectors encode aleatory uncertainties associated with the composite system. The off-diagonal elements of the density matrix are the superpositions of the probability amplitudes. The initial state vector at time t = 0, which we label $\left|\Psi_{t_0}\right\rangle = \sum_i c_i \left|\psi_i\right\rangle$, encodes only the aleatory uncertainties associated with the system. This is a pure state, knowledge is at maximum, and state is purely quantum. However, the states are not static and will change unless the system is complete isolated from everything.

Different types of epistemic uncertainties enter into the expression according to the following scheme. As the system interacts with the external environment the state vector representing the system (which is the state of the system) changes. The unitary operator of an event **U**, for example, transitions the system from one state to another: $\left|\Psi_{t_1}\right\rangle = \mathbf{U}\left|\Psi_{t_0}\right\rangle$. The probability amplitudes associated with $\left|\Psi_{t_1}\right\rangle$ are different, and the probabilities obtain from $\left|\Psi_{t_1}\right\rangle$ are no longer purely aleatory. The system is still pure and quantum, continues to reflect all the knowledge available for describing the system. At this stage, while the $\left|\Psi_{t_1}\right\rangle$ contains both aleatory and elements of epistemic uncertainties (primarily parametric and physical model based), all available information have been incorporated.

The next stage introduces additional epistemic uncertainties due to the forming of mixtures as a consequence of partial knowledge. The **mixed** state density matrix $\rho = \sum_j p_j \left|\Psi_j\right\rangle\left\langle\Psi_j\right|$ is the mixture of pure basis states and pure superposition states. The mixture is necessary when one only has partial information about what states the system can be found in but does not know of the exact system state precisely. Certain quantum properties and characteristics begin to change over to classical behaviors as knowledge and information precisions reduced due to interactions or the lack of knowledge, information, or data. The density matrix can represent such system in a range of states, and the purity tests (Section 7.5.1.2) on the density matrix help to measure the degree of mixing.

A system will retain certain quantum characteristics if it is *partially* mixed, minimal quantum characteristics when it is *totally* mixed, and practically no quantum characteristics when it is *maximally* mixed. The density matrix becomes simply a fancy expression for conventional probability in the case of the maximally mixed system. This progression reflects the decrease of knowledge, or the increase of ignorance about the system; this corresponds to the increase of the epistemic uncertainty elements entering into the system, including model based, evolution based, interaction based, and subjective beliefs. At the extreme end of the spectrum, when the system is maximally mixed, the density matrix is still a valid quantum expression. As a consequence, additional quantum structures are available for density matrices to measure other properties and characteristics.

### 7.5.3 Density Operator as Tool to Model Ambiguity

The density operator provides an alternative way to represent uncertainties, in terms of pure states where one has exact knowledge and mixed states where one only has knowledge about the mixture. While the pure state density matrices derive from the basis states, such as $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, are the quantum analog of the classical states, the density operator approach can also support the development of alternative models and representations of problems, such as ambiguity, for further explorations.

The construction of a density matrix starts with the combination of different pure and mixed states. Recall that the trace of the square of the density matrix indicates whether a system is in a pure state or mixed state (Test #1 in Section 7.5.1.2). A value less than 1 indicates that it is a mixed state system. The numerical value provides a quantification of how much "mixing" is in the system, reflecting the completeness of knowledge and the epistemic uncertainty associated with the model. It is possible that the same density matrix can be formed from different initial mixed states. To illustrate this, consider the notion that there are two competing models, M1 and M2, for describing a physical system, and they both arrive at similar (or identical) results. If one believes only M1 is true, then the density operator is written as: $\rho_1 = |M1\rangle\langle M1|$. However, if one is uncertain about the models, and thinks that there is a 50/50 chance that either models can be true, then the density operator is written as: $\rho_{1\&2} = 0.5|M1\rangle\langle M1| + 0.5|M2\rangle\langle M2|$. Since in this example, M1 and M2 give the same results, $\rho_1$ will be identical to $\rho_1$. This describes a process leading to ambiguity from modeling uncertainty, a form of epistemic uncertainty!

One cannot distinguish between the mixed states if their density matrices are the same. In other words, if the different models with different mixed states have identical density matrix, one cannot differentiate between the models, which is a form of ambiguity and a model representation of epistemic uncertainty. As another example, consider the earlier discussion of a system with several possible failure modes, A, B, and C in Section 6.1. Let us further assume that failure modes A and B only occur in spring, and modes A, B, and C occur in summer. In spring, the probability of A is 0.3, and 0.7 for B. In summer, the probability of A is 0.15, 0.7 for B, and 0.15 for C, with A and C having the same failure characteristics tracing out the same fragility curves. While artificial, this configuration is chosen to provide conceptual demonstration of the inability to differentiate different density matrixes under certain situations. To model the two scenarios, one constructs the following density matrices:

Let the levee be $|L_i\rangle = l_0|0\rangle + l_1|1\rangle$, and the different failure modes for the levee be $|L_A\rangle = \mathbf{A}|L_i\rangle$, $|L_B\rangle = \mathbf{B}|L_i\rangle$, $|L_C\rangle = \mathbf{C}|L_i\rangle$. The density matrix for spring:

$$\rho_{spring} = 0.3|L_A\rangle\langle L_A| + 0.7|L_B\rangle\langle L_B| \tag{7.25}$$

The density matrix for summer:

$$\rho_{summer} = 0.15|L_A\rangle\langle L_A| + 0.7|L_B\rangle\langle L_B| + 0.15|L_C\rangle\langle L_C| \tag{7.26}$$

Since A and C have the same failure characteristics, the "numerical" values for the density matrix components for A and C are the same: $0.15|L_A\rangle\langle L_A| = 0.15|L_C\rangle\langle L_C|$. This leads to the numerical equivalence of the density matrices: $\rho_{spring} = \rho_{summer}$ .

If one were to take this further, taking into considerations the range between aleatory and epistemic uncertainties from last section, transition and time evolution operations will drive the density matrix from the ambiguous mixed state to other mixed state where tests such as those discussed in 7.5.1 or other forms of measurements can be used to separate the ambiguity.

### 7.5.4 Density Operator as Measurement of Disorder

Measurements play a major role in establishing the proper interpretation of the density matrix, in determining of the type of information encoded by it, and in deciding what information can be extracted out of it. Information extracted from the density matrix has to correspond to real world observations where measurements can be performed to gather data for comparing to the model predictions in order to establish the soundness of the model. This notion has bearings in terms of how one treats the interpretations of the different elements of the density matrix, particularly in the distinction between quantum superposition and quantum mixture.

The density matrix encodes information for both quantum superposition and mixture states. The interpretations of the density matrix focus primarily on what can be directly observed and measured, which are the probabilities, the squared of the probability amplitudes, and the expectation value for observables. The quantum superpositions, for example, are indirectly measured as results from the interference interactions between the quantum states transitions via the measurement of transition probabilities. The coherence and incoherence of the superpositions (Section 7.5.1.3) reflect the degree of quantumness one can expect the system to behave. The encoded information in the density matrix elements (e.g. the inequality $0 < \rho_{mn}\rho_{nm} < \rho_{mm}\rho_{nn}$) can be extracted to tell whether the system behaves as a classical or quantum system. The encoded information that can be directly compared to reality would be the diagonal elements, corresponding to probabilities that can be compared to measurements on actual physical systems.

The diagonal elements of the density matrix correspond to real probabilities one can measure from experiments and observations, and the Born Rule is directly encoded in the density matrix itself. As a result, the trace of the density matrix has been the primary route to extract information regarding the states of the system. It connects the mathematical concepts to things we can physically measure.

Information can be extracted from the density matrix in a different approach. From the ensemble perspectives, one can construct a quantity that can further characterize the different density matrices by measuring disorder. This was pioneered by von Neumann, where a value is associated to a system, described by a density matrix $\rho$, that correspond to the uncertainty of the states as reflected by the degree of disorder. The following derivation is based on Sakura (1993).

The von Neumann Entropy, S, is defined as:

$$S = -tr(\rho \ln \rho) \qquad (7.27)$$

and $\rho$ is expressed in terms of the basis vectors where $\rho$ is diagonal:

$$\rho = \sum_k a_k |e_k\rangle\langle e_k| \tag{7.28}$$

S can be rewritten as:

$$S = -\sum_i \rho_{ii} \ln \rho_{ii} = -\sum_k a_k \ln a_k \tag{7.29}$$

From Equation 7.23 the pure state density matrix, $S = 0$ since $\rho_{ii} = 0$ or $\ln \rho_{ii} = \ln(1) = 0$. For the completely random mixed state density matrix (Equation 7.24), where N is the dimension of the state space,

$$S = -\sum_{i=1}^{N} \frac{1}{N} \ln\left(\frac{1}{N}\right) = -N\left[\frac{1}{N}\ln\left(N^{-1}\right)\right] = \ln N \tag{7.30}$$

Without going into the detail proofs, von Neumann entropy has the following properties:

- $\rho$ is pure if S = 0
- $\rho$ is a maximally mixed state if $S = \ln N$, where N is the dimension of the state space (dimension of the density matrix)
- $S(\rho) = S\left(\mathbf{U}\rho\mathbf{U}^{\dagger}\right)$
- $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$

The density matrix, in the form of von Neumann entropy, can provide a quantitative measure of the statistical disorder associated with a system, similar to the notion in statistical mechanics that the larger entropy, the larger the uncertainty, reflecting the number of microstates a system has. The von Neumann entropy and its application are under active research, particularly in the field of quantum information theory.

### 7.5.5 Entanglement

The concept of entanglement was first introduced in Chapter 5 (Section 5.1.3). Recall the consideration of the composite system formed from subsystem A and subsystem B. Both have the $|0\rangle$ and $|1\rangle$ as the basis vectors. The composite system AB will have four basis states, $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. When the system AB is entangled, it simply means that A and B do not function separately but as a correlated system, and measurements perform a subsystem can provide information about the other. Entanglement is a generalization of the concept of correlation in the quantum sense. To illustrate this concept, consider the following non-entangled state for the composite system $|AB\rangle$:

$$|AB\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right) \tag{7.31}$$

and when expressed in the full basis:

$$|AB\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle + 0|10\rangle + 0|11\rangle \tag{7.32}$$

If one measures A, and gets $|0\rangle$ for 100% of the time, what does that tell us about B? If $|A\rangle = 1|0\rangle + 0|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in order to get the composite state above,

$|B\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. One can check that this is true by calculating

$A \otimes B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$, which is equal to Equation 7.31. The probability of getting 0 is the same as getting 1, which is 0.5. In this example, knowledge about A does not tell us anything about B.

Now consider the state vector:

$$|AB\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + 0|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \quad (7.33)$$

One cannot construct the composite state from two separate state vectors $|A\rangle$ and $|B\rangle$. However, the state vector does give a 50% probability of getting $|00\rangle$ and a 50% probability of getting $|11\rangle$. Furthermore, if one observes A to be zero, B will also be zero! There is a correlation between A and B!

The product state, as the name implies, refers to a system where the state can be factored into individual subsystem states, and operators can operate on each individual subsystem individual and independently. An example of such: a system formed from two separate subsystems, which can be two different levees. Together, they work to provide protections to the area, but a failure of one does not necessarily correlate to a failure of the other.

Correlated subsystems can be interpreted as entangled states. For example, an I-wall with an earthen levee. If the earthen levee is weakened by erosion, it can also weaken the I-wall foundation, which can increase the chance of failure for the combined system. On the opposite end, the I-wall can be weakened in some way, for example, the top portion of the flood wall was damaged, and the height of the flood wall has been effectively reduced by 3 feet. The system failure probability increases as a result, but it does not affect the earthen levee. An application of this concept of entanglement will be presented in Section 9.4.

The density matrix offers a mechanism to determine the states of the composite system, similar to the earlier purity test discussion in Section 7.5.1. In this case, the test with the density matrix help to identify the system states, whether it is a product state (no entanglement), maximum or partial entanglement. In brief, for a product state the density matrix for A will have only one nonzero eigenvalue that equals to 1, and the density matrix for B will also have only one nonzero eigenvalue that equals to 1. Maximum entanglement is when the density matrix of the composite system satisfies $tr\left(\rho^2\right)=1$ and $\rho^2=\rho$, and the subsystem density matrix satisfies $tr\left(\rho^2\right)<1$, $\rho^2\neq\rho$, and proportional to the diagonal unit matrix (Section 7.5.1.3), where the measurement outcomes are equally likely. Finally, partial entanglement is when the density matrix of the composite system satisfies $tr\left(\rho^2\right)=1$ and $\rho^2=\rho$, and the subsystem density matrix satisfies $tr\left(\rho^2\right)<1$ and $\rho^2\neq\rho$. Since the density matrix can play a role in deciphering the entanglement states of a system, the von Neumann entropy can also serve that purpose as well.

*7.6 Incorporating the Questions to Shape the Observables, the Risks, and the Results*

The quantum framework is proposed as an alternative to investigate risk problems so that more comprehensive data and information can be provided for risk analyses and assessments, leading to better evaluations. In the conventional framework, the priors play little or no role in shaping the line of risk questioning. In the quantum framework, due to how operators and quantum operations can change not only the states but also the system itself, the priors play some roles in the shaping and evolution of the risk questions. The lines of queries form the basis for identifying the observables, which also affects the construction of the model.

This also means that the questions we ask will also shape the outcome of the analysis. The right question has to be asked (the right set of risk states need to be identified) in order to get the right answers. This is coherently stated by Ray (2009) as: "Questions elicit answers (data), and together with prior expectation lead to conclusion." In the example of a levee system, the observables would correspond to the working and non-working states, derive from the question: "Is the levee system working?" and "Is the levee system non-working?" For the basic system, the main observables are the projectors: $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$; the observables represent the questions we are trying to ask. For this study, the questions we are trying to ask would be related to risks. Questions such as "What is the risk of something?" is translated into the question "What is the chance of finding the engineering system in a <certain> states that can have <certain> impacts?" The corresponding mathematical expression for this question with answers expressed in terms of risk values is therefore (recalling Chapter 4.6):

$$\langle Ri \rangle = \sum_i p_i \langle \lambda_i | \mathbf{Ri} | \lambda_i \rangle = Tr(\rho \mathbf{Ri}) \tag{7.34}$$

where **Ri** is the risk observable: $|0\rangle\langle 0|$ for the risk of system failing and $|1\rangle\langle 1|$ for the risk of the system working.

The question we are attempting to answer is the risk associated to a particular engineering system for a specific condition or event. While in general a system can have many states, the question framed the approach on how to represent the states of the system in question, namely the selection of the basis of $|0\rangle$ and $|1\rangle$, the non-working and working states. Under this consideration, the risk system consists of the collection of the possible system states related to the risk conditions, and a risk state is defined as:

1. Certain states of the system that when occur can have material impacts.
2. The system states are defined by the "question" which help shape and define the basis states needed to describe the system, how to construct the system from these basis states, what type of evolutionary processes applicable to the problem, and what makes the target states that answer the question.
3. For a simple engineering system, the system in a working state is defined by the $|1\rangle$ basis-vector, and the non-working state is defined by the $|0\rangle$ basis vector. The $|0\rangle$ and $|1\rangle$ basis vectors form the complete set for the risk engineering system.

The quantum risk model modifies the tradition risk equation by replacing the standard probability term with the quantum probability terms and expectation values, which derives from a very rich theoretical framework incorporating several forms of uncertainties, information, knowledge, and physics models. Furthermore, the concept of ignorance can be incorporated into the framework in terms of the different states of the density matrix, and subjective beliefs are now quantified in the form of the statistical distribution of the states. As a consequence, information densities in the quantum models are higher than the conventional counterparts.

### *7.7 New Tools in the Toolbox*

For challenges that conventional formalism does not address, the quantum framework provides some new insights:

- The challenge of combining the epistemic and aleatory probabilities: should the two types of uncertainties be combined into a single measure of risk? Yes, they can be combined with the density matrix. The risk states are mapped into state vectors, and the resulting density matrix representation combines both types of uncertainties into a single expression.
- The subjectivity regarding the division between aleatory and epistemic components can be reduced into the problem of the representation of the effective uncertainty in an operator algebra based mathematical framework. Specific uncertainty representations are built into the density matrix to represent the two type of distinct but interconnected parts.
- The need to utilize probability envelops of epistemic uncertainty has been reduced. The terms in the density matrix can be adjusted based on both objective and subjective (weighted statistical methods) information and knowledge. In other words, the probability envelops are built into the density matrix as one of the parameters.

- The concept on probability of probabilities, of specifying probability distributions over model parameters which are themselves probabilities, is now built into the density matrix construct. The various probabilistic terms in the construction of the state vectors and the density matrix can take on various probability distributions, drawing parallels to the quantum wave function formalism.
- Risk states can now change dynamically with operators. Unitary transformation for changes preserving the system, or projective measurements resulting in an updated system. Conditional statements of risk are now incorporated into the framework using operators and projective measurements.
- We define expert knowledge as subjective beliefs. The weighted expert assessments are reflections of the degree of expert agreement of uncertainties and certainties.
- Lastly, this formalism is based on a well-established scientific platform, namely quantum physics. The platform provides a sound and rich framework for future research and development.

With quantum probability, two probability frameworks are combined within a single mathematical construct — the density operator — where epistemic uncertainties are treated as the result of a mixture of states characterizing the aleatory uncertainties. Conventional Kolmogorov probability is used to represent that mixture. Aleatory uncertainty is represented in terms of probability amplitudes (not direct probabilities) associated with the vector object. The system is modeled as a whole with this combined framework, which offers a number of advantages. Operators modeling event and interaction scenarios act on both types of system uncertainties together. This recognizes that events can affect both types of uncertainty simultaneously, and the framework aligns well with scenarios where external events can affect both type of uncertainties often seen in the case of dynamic systems. Propagation of uncertainties is handled within the integrated model. Furthermore, the quantum framework allows the mixing of both aleatory and epistemic uncertainties, and yet it maintains the distinctions between them that analysts can identify their separate contributions to the overall uncertainty of the integrated system.

This combined framework offers an alternative mathematical platform for risk analysts to use as a theoretical research tool and as a computational tool to investigate the different coupling of aleatory and epistemic uncertainties. The quantum framework also offers an alternative investigative and experimental platform to explore different ways to incorporate additional contextual information into the mathematical models.

The quantum framework is also future forward looking. As some have pointed out that composite quantum models might encounter scalability issues when modeling systems with many components. The size of the model, as a result of the tensor products of the components, grows exponentially. Current computational platforms might have significantly difficulty in handling the computational demands. While this is certainly true with current classical computing platforms, the recent rapid advancements in quantum computing might alter this landscape. Quantum computation has the potential to significantly reduce the computational time and resource to perform complex calculations for problems properly formulated for computation with quantum computers (see for example Arute et al., 2019; Wright et al., 2019). Further exploration in the application of quantum frameworks to solve different problems, one can argue, could be our destinies.

# Chapter 8: Modeling with the Density Operator for Heterogeneous System Evolving Over Time

*The ideal engineer is a composite… He is not a scientist, he is not a mathematician, he is not a sociologist or a writer, but he may use the knowledge and techniques of any or all of these disciplines in solving engineering problems.*

*- N. W. Dougherty 1955*

Up to this point, the quantum approach has been applied to single and simple composite engineering systems. In real world applications, complex engineering systems, or system of systems, are constructed out of heterogeneous components, subsystems, and systems of various states. In such context, the notion of a *heterogeneous system* can have broad meanings. In this chapter, the quantum framework is applied to model a simplified configuration of a heterogeneous system of subsystems to demonstrate the modeling workflow. The demonstration system comprises many similar subsystems, and these subsystems are in groupings according to their different states. For example, a system can comprise 30 subsystems that were all built to the same specifications; however, the 30 subsystems were not all built at the same time. Ten of them were built initially, an additional ten were built two years later, and the final ten were built another two years later. This example is in essence a simplified version of the HPS system.

A model suitable for the investigation of this type of complex heterogeneous engineering system is constructed in this chapter using the density operator quantum framework. The HPS (before Katrina) and not the HSDRSS (post Katrina) serves as a reference system for the model since analyses done by IPET and others are readily available for comparison. Given the scope of this dissertation, a direct one-to-one comparison between the quantum model with the HPS studies by IPET is not feasible. Nonetheless, using a simplified HPS baseline and focusing on key attributes and properties, comparisons at the macroscopic level will highlight differences between the techniques. The main objectives are to illustrate how to model a heterogeneous system using the density operator formalism, to demonstrate how the quantum approach offers additional information about system risks over conventional methods, and to compare and evaluate how the two approaches assess risks.

*8.1 The Quantum Model for a Heterogeneous System Over Time*

The HPS is a collection of different subsystems acting like an ensemble, and precise information and knowledge about the subsystems are in general incomplete. The mixed state ensemble density operator formalism is well match for modeling these classes of problems. The modeling workflow begins with the development of a density operator representation for the heterogeneous states of the HPS. Stepping through time, the model incrementally incorporates different evolutionary event scenarios together to represent what the system is like at the time instance where one would like to evaluate risks. In order to derive useful risk information out of these models, such as to assess the risk of flooding as a result of the failure of the system plus other risk related information, problem and system properties over time have to be characterized, prioritized, simplified with approximation (thus introducing ignorance), and sequenced to build out the model from the past to the present.

### 8.1.1 The Characteristics of the Models

The HPS model scenario — assumptions, approximations, simplifications, generalizations, properties, and characteristics:

- Knowledge about the system and its history can be limited. This study focuses on the "present day" HPS, with the initial construction dating back to 1965 under the Flood Control Act of 1965. The construction of the HPS spanned many years, and this model will break it down to three periods: 1965-1985 (Period I), 1985-2005 (Period II), and 2005-2015 (Period III). The three periods correspond to the initial project from the Flood Control Act of 1965, the construction period after the 1984 Re-Evaluation Study by USACE, and post-Katrina. Given the vast time span, construction and change histories for the HPS are considered to be incomplete (ASCE, 2007). As such, a random pick of a "levee + flood wall" subsystem from the HPS system collective can be from any of these periods. For a more computationally manageable modeling exercise, this model makes the simple assumption that the subsystems deteriorate over time and the probability of failure increased as the subsystems aged.
- A key historical note for the subsystems built during Period 2: they were built using outdated design specification, and the height of the flood walls were 1 to 5 feet below the updated specifications.
- The HPS is treated an ensemble system formed from subsystems with similar

characteristics: The hurricane protection system current has 138 similar levee-flood wall subsystems (the reaches). Some of the subsystems were not built to the specifications, with a slightly higher failure probability.

- Changes in the system happen over different time scales. Some, such as erosion due to vegetative growth, happen over long periods of time. Others, such as overflow events, happen over short durations.
- Over the time spans of these systems, there are both long term and short term events. Given the long timescale and event granularity of the system, the time dependency will be considered as discrete.
- The temporal timeline for this exercise is divided into three periods. Figure 29 highlight the key system configurations considered for this model at the different time periods.
- The model scenario must incorporate different types of uncertainties.
- Expert assessments and opinions on the states of the system can be different. Different experts can arrive at different event occurrence probabilities for different system states. These subjective uncertainties will be incorporated as part of the density operator.

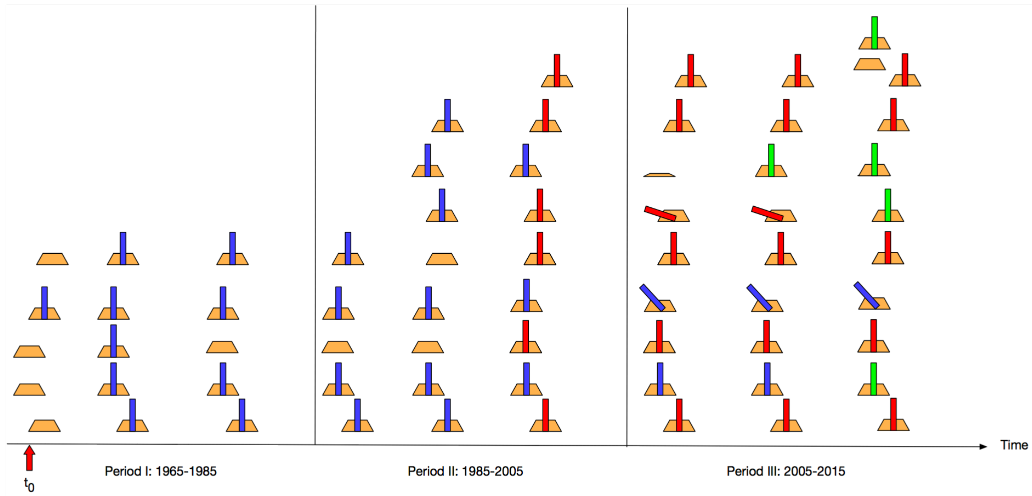

Figure 29. Schematic representation of the HPS configuration evolution over time. The color of the flood walls indicates different build generations and specifications. The degree of tilt of the floor walls indicates degree of damage.

As before, the following notation will be used: the state vector for a levee is denoted by $L^{System}_{state,|basis\rangle}$ or $L^{System}_{state,basis}$ (a notation shorthand). A levee system A in the initial state 0, is represented as: $L_0^A : \left|L_0^A\right\rangle = l^A_{0,|0\rangle}\left|0\right\rangle + l^A_{0,|1\rangle}\left|1\right\rangle = l^A_{0,0}\left|0\right\rangle + l^A_{0,1}\left|1\right\rangle$. A general mixed state composite system is, for example, represent as:

$$\rho_{HPS} = \sum_i p_i \left| L_i^A W_i^A \right\rangle \left\langle L_i^A W_i^A \right|. \tag{8.1}$$

Using the density operator, the state transition is:

$$\rho = \sum_i p_i \left| L_i^A W_i^A \right\rangle \left\langle L_i^A W_i^A \right| \xrightarrow{U} \sum_i p_i U \left| L_i^A W_i^A \right\rangle \left\langle L_i^A W_i^A \right| U^\dagger = U \rho U^\dagger \tag{8.2}$$

8.1.2 Description of the System States and Their Evolutions Over Time

The HPS is modeled as an ensemble collection of (L+W) structures with various configurations and manufacturing deviations. To construct the model, several simplifications and assumptions have to be introduced to simplify the calculations in order to illustrate the principle concepts. A levee can either be working and not working, and a flood wall can either be working or not working; hence, a subsystem is modeled to have 2 possible states, working $|1\rangle$ and not working $|0\rangle$. It is further assumed that a levee-flood wall subsystem was constructed together at the same time. The (L+W) subsystems are similar to each other and the subsystems collectively can be treated as an ensemble. The system transition from one state to another when triggered by an event. The modeling process will follow the state transitions of the HPS over the three different time periods, according to their applicable timescales (Figure 29).

*Period I* — Period I saw the initial construction of the levees and flood walls. The systematic construction of the model will start with defining the initial state vectors for the levees and flood walls, the construction of the composite levee-flood wall subsystems, their individual changes and collective changes due to events over time:

- The construction of the initial set of levees L at time $t = t_0$ and they gradually degrade over time. For the individual levee $L_0^A : |L_0^A\rangle = l_{0,0}^A|0\rangle + l_{0,1}^A|1\rangle$; for the individual flood wall $W_0^A : |W_0^A\rangle = w_{0,0}^A|0\rangle + w_{0,1}^A|1\rangle$. The ensemble collection of these levees making up the HPS, assuming that they are all identical, is a pure state density matrix:

$$\rho_{HPS} = \sum_i p_i |L_i^A\rangle\langle L_i^A| \equiv |L_i^I\rangle\langle L_i^I|. \tag{8.3}$$

- The system index is changed to I to stand for Period I. Corresponding to the simplifications and assumptions, subsystems initially with only a levee require a separate treatment. They are considered to be a different population and should model separately.

- The composite (L+W) system is represented by

$$\begin{aligned} \left|L_0^I W_0^I\right\rangle &= \left|L_0^I\right\rangle \otimes \left|W_0^I\right\rangle \\ &= l_{0,0}^I w_{0,0}^I \left|00\right\rangle + l_{0,0}^I w_{0,1}^I \left|01\right\rangle + l_{0,1}^I w_{0,0}^I \left|10\right\rangle + l_{0,1}^I w_{0,1}^I \left|11\right\rangle \end{aligned} \tag{8.4}$$

if it is a product state, or

$$\left|LW_0^I\right\rangle = \alpha_{0,00}^I \left|00\right\rangle + \alpha_{0,01}^I \left|01\right\rangle + \alpha_{0,10}^I \left|10\right\rangle + \alpha_{0,11}^I \left|11\right\rangle \tag{8.5}$$

if it is not. In general, it is assumed that the subsystems are correlated, and the entangled state (eq. 8.5) will be the focus. For example, if the earthen levee is weakened, the failure probability for the combined levee and flood wall system will increase. The corresponding density matrix is:

$$\rho_{\left|LW_0^I\right\rangle} = \left|L_0^I W_0^I\right\rangle \left\langle L_0^I W_0^I\right| = \rho_{L_0^I} \otimes \rho_{W_0^I} \tag{8.6}$$

for product state, or in shorthand $\rho_{\left|LW_0^I\right\rangle} = \left|LW_0^I\right\rangle \left\langle LW_0^I\right|$. The entangled states will be further discussed in Section 9.4.

- The (L+W) system was constructed at time $t_0$ (t = 0) and the structures (materials) gradually degraded over time. The degradation is modeled with the **M** operator. The state of the system at time t is given by:

$$\left|L_t^I W_t^I\right\rangle = \mathbf{M}^I(t) \left|L_0^I W_0^I\right\rangle \tag{8.7}$$

and the HPS system density matrix becomes:

$$\rho_{HPS} = \left|L_t^I W_t^I\right\rangle \left\langle L_t^I W_t^I\right| \ . \tag{8.8}$$

- Some (L+W) subsystems have modified flood walls with new specifications to strengthen the system at a later time. Two different ensemble populations exist as a result. The modified subsystems will have a different state vector:

$$\left|L_t^I W_0^I\right\rangle = \left|L_t^I\right\rangle \otimes \left|W_0^I\right\rangle . \tag{8.9}$$

The density matrix has to be updated to take the split populations into account:

$$\rho_{HPS} = p_1 \left|L_t^I W_t^I\right\rangle\left\langle L_t^I W_t^I\right| + p_2 \left|L_t^I W_0^I\right\rangle\left\langle L_t^I W_0^I\right| , \tag{8.10}$$

where $p_1$ is the percentage of the subsystems that are with the original population, and $p_2$ is the percentage of the subsystems that are with the new population.

- There may be periodic flooding events resulting in erosion. The number of storms per year and the severity that can result in over flooding and backside erosion are needed to compute the full history; the following illustrate the construction of a sequence.
  - The water pressure during increase in flow volume changes the failure probability according to: $\left|L_{t_f}^I W_{t_f}^I\right\rangle = \mathbf{P}\left|L_{t_i}^I W_{t_i}^I\right\rangle$, where $t_i$ and $t_f$ denote the initial and final states for this event.
  - The (L+W) structure undergoing regular seasonal events, front side erosion, and flooding events: $\left|L_{t_f}^I W_{t_f}^I\right\rangle = \mathbf{OE}_f\mathbf{P}\left|L_{t_i}^I W_{t_i}^I\right\rangle$. We will denote the composite operator as $\mathbf{H} = \mathbf{OE}_f\mathbf{P}$.
  - The (L+W) structure going through an overtopping event, with backside erosion, with and without subsequent repair.

$\left| L^I_{t_f} W^I_{t_f} \right\rangle' = \mathbf{E}_b \mathbf{O} \mathbf{E}_f \mathbf{P} \left| L^I_{t_i} W^I_{t_i} \right\rangle$.We will denote the composite operator as

$\mathbf{H} = \mathbf{E}_b \mathbf{O} \mathbf{E}_f \mathbf{P}$.

The operator **H** belongs to the class of operators reflecting our collection of knowledge and understanding of the evolutionary history of the system. We shall call this class of operator ***knowledge operators*** as they encapsulate historical and contextual knowledge about the system. The general density matrix describing a system can therefore be expressed as:

$$\rho_{\left|LW^I_{t_f}\right\rangle} = \sum_i p_i \mathbf{H} \left| LW^I_{t_i} \right\rangle \left\langle LW^I_{t_i} \right| \mathbf{H}^\dagger = \mathbf{H} \, \rho_{\left|LW^I_{t_i}\right\rangle} \mathbf{H}^\dagger \qquad (8.11)$$

where **H** is a knowledge operator and it is acting on all the possible states for the combined system. Over time, the system can undergo n-times these periodic events, and in terms of the operators acting on the density matrix, the expression can be written as:

$$\rho'_{HPS} = \left( \prod_{j=1}^{n} \mathbf{H}_j \right) \rho_{HPS} \left( \prod_{j=n}^{1} \mathbf{H}_j \right)^\dagger . \qquad (8.12)$$

The density matrix for the system as described is “complete” at this point, which means all the information and knowledge about the systems are incorporated into the state vectors and the density matrix. The physics about the individual event is encapsulated in the corresponding $\theta$ function (Chapter 5.2).

While it is possible to perform all the operations with **H**, tracking the individual data element and performing computations on them can be cumbersome. In this case where the time span between the event types can be of many orders of magnitude differences, the concept of ignorance, in which selective knowledge and information are ignored and not used, can be of help. A single operator, $\mathbf{K}_I$, can be chosen to represent the product of the operators.

$$\prod_j \mathbf{H}_j \sim \mathbf{K}_I \qquad (8.13)$$

One can think of this as the application of a smoothing function to smooth out the resolution to match the timescale. With this approximation, the HPS at the end of period will be given by:

$$\rho'_{HPS} = \mathbf{K}_I \rho_{HPS} \mathbf{K}_I^{\dagger} \qquad (8.14)$$

Further studies should be carried out to investigate how the level of granularity, time scale and range can affect the validity of this approximation.

*Period II* — Period II events resulting in the following state transitions and system changes:

- Additional levees and flood walls systems were built, but they were built with outdated specifications.
  - The Period II initial system states were carried over from Period I first generation (L+W) subsystems. These subsystems continued to evolve during Period II.
  - The Period II or the second generation (L+W) systems were built around the same time (this assumption could be wrong since the subsystems were built over a time spread; this is where the time resolution consideration is important) and the systems degraded over time. The Period II (L+W)

systems were built to the same failure probability specifications as the first generation. However, the re-analysis of the system design led to the update of the design specifications since the height of the (L+W) system needs to be 1-5 feet higher. As a consequence, the Period II systems do not meet the new specification which increase the system failure probabilities. This re-baseline of the failure probabilities is represented by an operator denoted as $\mathbf{K}_{II}$.

$$\rho'_{HPS} = \mathbf{K}_{II} \rho_{HPS} \mathbf{K}_{II}^{\dagger} . \tag{8.15}$$

- The knowledge operator $\mathbf{K}_{II}$ quantifies the fact that the systems now fall "under specification". The $\mathbf{K}_{II}$ operator is a simple rotation operator (Chapter 5.2.2), and the function captures the change of probabilities due to the change of specification.
- The density matrix for the system, $\rho'_{HPS}$, will need to be modified to account for the 2 different ensembles, with the probability $p_1$ associated with the first generation systems and $p_2$ associated with the second generation systems. The summation is over the probabilities from Period I and II.

- Further material and structural degradation

$$\rho_{HPS} = p_1 \left| L_t^I W_t^I \right\rangle \left\langle L_t^I W_t^I \right| + p_2 \left| L_t^I W_0^I \right\rangle \left\langle L_t^I W_0^I \right| + p_3 \left| L_t^{II} W_t^{II} \right\rangle \left\langle L_t^{II} W_t^{II} \right| \tag{8.16}$$

- Periodic flooding events resulting in erosion. The general operators will be similar to those of period II. However, the frequency and intensity might vary due to the changing environment, such as the increase of storm frequency due to the warming of sea water. As before, the following illustrate the construction of a sequence.

- The water pressure during increase in flow volume changes the failure probability according to: $\left|L_{t_f}^{II}W_{t_f}^{II}\right\rangle = \mathbf{P}\left|L_{t_i}^{II}W_{t_i}^{II}\right\rangle$, where $t_i$ and $t_f$ denote the initial and final states for this event.
- The (L+W) structure undergoing regular seasonal events, front side erosion, and flooding events: $\left|L_{t_f}^{II}W_{t_f}^{II}\right\rangle = \mathbf{OE}_f\mathbf{P}\left|L_{t_i}^{II}W_{t_i}^{II}\right\rangle$. We will denote the composite operator as $\mathbf{H} = \mathbf{OE}_f\mathbf{P}$
- The (L+W) structure going through an overtopping event, with backside erosion, with and without subsequent repair. $\left|L_{t_f}^{II}W_{t_f}^{II}\right\rangle' = \mathbf{E}_b\mathbf{OE}_f\mathbf{P}\left|L_{t_i}^{II}W_{t_i}^{II}\right\rangle$. We will denote the composite operator as $\mathbf{H} = \mathbf{E}_b\mathbf{OE}_f\mathbf{P}$

• Katrina: The (L+W) structure going through a catastrophic overtopping event as a result of the weakening of the backside due to prior overtopping event.

- The Post-Katrina system is derived from the application of knowledge operators on the various systems. These operations will either change the systems states or completely alter the systems, such as the application of the **X** operator; the population distributions might change as a result. A practical approach to derive the new population distributions would be to rely on observed data to simply write out the statistical ensemble. *This approach draws heavily on the use of field data and observations*. While we can simulate and calculate how many subsystems can fail during the event, the results must be reconciled with the observed data since there are other conditions that might not be built into the model.

- Given the current systems are modeled with binary states, an alternative way to model changed systems could simply be "putting" the system into a constant state, such as a constant failing state, a simple $L_0^{II} : \left| L_0^{II} W_0^{II} \right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle$ , for a failed/destroyed system.

Period II consists of 2 sub-periods: 2a) the period leading up to Hurricane Katrina, and 2b) the period during and through Hurricane Katrina. For Period 2a, the initial states are inherited from Period I, and the Period 2a starting density matrix will be: $\rho'_{HPS} = \mathbf{K}_1 \rho_{HPS} \mathbf{K}_1^{\dagger}$. Since new (L+W) structures were built, a new density matrix incorporating Period I subsystems and the new Period II subsystems would need to be constructed. The mixing of the subsystems from the two periods introduces uncertainties into the integrated system, shifting the model to resemble further as a statistical distribution. Let the density matrix for the system for period I be denoted by $\rho_I$ and period II be denoted by $\rho_{II}$, the combined system is given by:

$$\begin{aligned} \rho_{HPS} &= w_I \rho_I + w_{II} \rho_{II} \\ &= w_I \sum_i p_i \left| L_i^I W_i^I \right\rangle \left\langle L_i^I W_i^I \right| + w_{II} \sum_j p_j \left| L_j^{II} W_J^{II} \right\rangle \left\langle L_j^{II} W_J^{II} \right| \end{aligned} \qquad (8.17)$$

where $w_I, w_{II}$ are percentages of the (L+W) subsystems from the periods with respect to the total number of (L+W) subsystems and $w_I + w_{II} = 1$ . The combined system is then evolved to the point where Katrina was about to hit the region:

$$\rho'_{HPS} = \mathbf{K}_{2a} \rho_{HPS} \mathbf{K}_{2a}^{\dagger} \qquad (8.18)$$

During Katrina, the higher time resolution operators $\mathbf{H} = \mathbf{E}_b\mathbf{O}\mathbf{E}_f\mathbf{P}$ are applied. Each discrete time step will need to be checked to see if the backside erosion, $\mathbf{E}_b$, might actually cause a catastrophic failure during the overflow condition. The evolution of the system during the storm is described by the sequence of events that occurred from the beginning of the storm to the end of it, and the evolutionary histories are modeled with sequences of knowledge operators. Each application of an operator transitions the system from one state to another in a discrete manner. At each transition the system states can be evaluated against operational parameters and constraints. If certain constraints are reached, a different operator sequence can be applied. In other words, historical information and knowledge are turned into a computational algorithm consisting of operator sequences, which are applied to transition the system from states to states.

An example of a sequence can be as follow: during a hurricane, a system can undergo both short-term and long-term changes. Some operations, such as the overflow operator, might not directly result in a long-term system change. The overflow operator, by itself, only reflects the conditions where water is overtopping the structure, a short-term event, and the structure remains intact. However, the same overflow operation can lead to backside erosion, resulting in the weakening of the system. This sequence of operation is $\mathbf{H} = \mathbf{E}_b\mathbf{O}$, but the operator $\mathbf{E}_b$ only acts on the system when $\mathbf{O}$ occurs and not before that. $\mathbf{E}_b$ is a time-delayed operation that only applies when the system is at a failure state. To model such behavior requires the adoption of an algorithmic approach where one projects the component states of the systems to detect failure conditions (such as sign change for the probability amplitudes), and then choose the right operation sequence. In the case with a Hurricane Katrina level events, both $\mathbf{E}_f$ and $\mathbf{E}_b$ can leave long term effects. The model simulation will need to check for the failure of the levee by performing a projection operation, $|L\rangle\langle L|$, to obtain the failure probability of the levee. If the failure probability is 1, then the entire (L+W) system fails catastrophically, and the **X** operator will need to be applied.

The application of the **X** operation changes the population distribution for the system. Such operation fundamentally changes the density matrix; as a result, conservation rules no longer apply. The application of the **X** operator throws the system into a different ensemble configuration, and in algorithmic terms one can equate this to the throwing of an exception, resulting in the need to take a different algorithmic pathways, moving into Period 2b. Depending on the sequence combinations, a different system will emerge after the storm, and a new set of density matrices reflecting the new ensemble populations will take shape.

*Period 3* — After Katrina, the HPS/HDRSS became a more fragmented system. Period III events resulting in the following state transitions and system changes:

*a) Damaged systems fully or partially repaired, or completely replaced*

In the case of damaged (L+W) systems, there are several possible configurations: (1) only L damaged, (2) only W damaged, (3) both L & W damaged. Repairs can be done on one component, both components, or some other configurations and combinations. The repair operator is given by: $\mathbf{R}_{LW} = \mathbf{R}_L \otimes \mathbf{R}_W$.

Repairs can be done in several ways. A partial repair would use the $\mathbf{R}^{-}$ operator to rotate the vector back towards the $|1\rangle$ state. Or a projector can be utilized to extract the working component out, and then apply the tensor product with a new component: $|X_L\rangle\langle X_L||LW\rangle \otimes |W\rangle$. After the repair, the system is restored back to the initial condition; the resulting state vector will be in the form of a separable system,

$|L_0^{III}W_t^{III}\rangle = l_{0,0}^{III}w_{t,0}^{III}|00\rangle + l_{0,0}^{III}w_{t,1}^{III}|01\rangle + l_{0,1}^{III}w_{t,0}^{III}|10\rangle + l_{0,1}^{III}w_{t,1}^{III}|11\rangle$ for configuration (1),

$|L_t^{III}W_0^{III}\rangle = l_{t,0}^{III}w_{0,0}^{III}|00\rangle + l_{t,0}^{III}w_{0,1}^{III}|01\rangle + l_{t,1}^{III}w_{0,0}^{III}|10\rangle + l_{t,1}^{III}w_{0,1}^{III}|11\rangle$ for configuration (2).

These are pure state vectors comprised with mixtures of old and new parts. For configuration (3), the state vector $|LW\rangle$ is completely replaced with a new one. In essence, the ensemble population is altered with some of the members shift back to the initial state!

*b) Un-repaired damaged systems*

For these systems, the previous Period II operations already and correctly put the them in the proper "damaged" states. No further operations on them is required and they should be left to evolve! Besides repairing old systems from Period I and II, additional levees and flood walls systems were built during Period III. Together with these three possible repaired configurations, the un-repaired components further increase the complexity of the system. The density operator formalism can handle this in a systematic, consistent and effective manner.

The density matrix for the system, $\rho_{HPS}$, are modified to account for the 4 or more different ensembles, with the probability $p_i$ associated with different ensemble populations. The summation is over the population probabilities from all ensembles and not the subsystems. The construction of the state transition operators follows the same logic as before:

- Further material and structural degradation

$$\rho_{HPS} = p_1 \left| L_t^I W_t^I \right\rangle \left\langle L_t^I W_t^I \right| + p_2 \left| L_t^I W_0^I \right\rangle \left\langle L_t^I W_0^I \right| + p_3 \left| L_t^{II} W_t^{II} \right\rangle \left\langle L_t^{II} W_t^{II} \right| + p_4 \left| L_t^{III} W_t^{III} \right\rangle \left\langle L_t^{III} W_t^{III} \right| + \dots \tag{8.19}$$

- Periodic flooding events resulting in erosion. The general operators will be similar to those of Period I and Period II. However, the frequency and intensity might vary due to the changing environment, such as the increase of storm frequency due to the warming of sea water.
  - The water pressure during increase in flow volume changes the failure probability according to: $\left| L_{t_f}^{III} W_{t_f}^{III} \right\rangle = \mathbf{P} \left| L_{t_i}^{III} W_{t_i}^{III} \right\rangle$, where $t_i$ and $t_f$ denote the initial and final states for this event.
  - The (L+W) structure undergoing regular seasonal events, front side erosion, and flooding events: $\left| L_{t_f}^{III} W_{t_f}^{III} \right\rangle = \mathbf{O}\mathbf{E}_f \mathbf{P} \left| L_{t_i}^{III} W_{t_i}^{III} \right\rangle$. We will denote the composite operator as $\mathbf{H} = \mathbf{O}\mathbf{E}_f \mathbf{P}$.
  - The (L+W) structure going through an overtopping event, with backside erosion, with and without subsequent repair: $\left| L_{t_f}^{III} W_{t_f}^{III} \right\rangle' = \mathbf{E}_b \mathbf{O}\mathbf{E}_f \mathbf{P} \left| L_{t_i}^{III} W_{t_i}^{III} \right\rangle$. We will denote the composite operator as $\mathbf{H} = \mathbf{E}_b \mathbf{O}\mathbf{E}_f \mathbf{P}$.

One interesting conclusion can be drawn from the process of constructing the system model through the periods. As the system evolves, complexity increases which leads to divergence of the system states, and the system becomes more of a statistical mixture of different possible states. System uncertainties increase as a result.

8.1.3 The Integrated Model

At any given time, the integrated HPS is described by a density operator of the form: $\rho'_{HPS} = \mathbf{K}_i \rho_{HPS} \mathbf{K}_i^{\dagger}$, with the knowledge operators representing the evolutionary history of the system. The density operator itself captures and reflects the degree of ignorance one chose to accept in the construction of the model; the density operator model is the result of making trade-offs between precision and computability. Furthermore, the questions one asks of the quantum model shape the system construction and influence the paths taken to arrive at the outcome.

Following this line of thinking, there can be families of HPS models one can build based on the many different possible questioning scenarios. The questions determine what specific approach to take, what parameters to use, what operator sequences to construct. This initial attempt to apply the quantum approach to model an engineering system uses a highly simplified scenario to focus on illustrating general concepts.

In this analysis, the time scale parameter affects the construction of the integrated system models. The granularity of the time scale parameter determines what specific event operations to model. The HPS is constantly subjected to different weather events; storms of different magnitudes move through the area at different times, with different frequencies and characteristics. While in principle the highest fidelity models should track every weather event, some simplifications can be made based on whether the study is over short or long term. For example, in long term studies a frequency parameter reflecting the statistical averages over the years might be sufficient.

The construction of the integrated model begins with the question: "What is the risk of catastrophic system failure when the 2005 HPS is impacted by a 200-year storm?" To answer this question, the HPS system model will have to evolve from the initial 1965 state to the 2005 state, right before Hurricane Katrina hits the region. This time span, from 1965 to 2005, spread over 40 years. Two time-scale ranges will need to be considered in the construction of the operators: 1) long term in years, and 2) short term in range of days. The evolutionary process will need to follow the algorithmic approach below to perform discrete transition steps to take the system from one time to another, until it reaches 2005.

*The algorithm* — the process begins with the initial HPS system with only levees. The density matrix is given by $\rho_{HPS} = \sum_i p_i \left| L_i^I W_i^I \right\rangle \left\langle L_i^I W_i^I \right|$. The system goes through slow long-term degradation, and the system at a given elapsed year is given by $\rho_{HPS} = \mathbf{M}\rho_{HPS}\mathbf{M}^\dagger$. Short-term events considered for this scenario focus on the flooding events, and the event operator, **F**, is defined as $\mathbf{F} = \mathbf{E}_b\mathbf{O}\mathbf{E}_f\mathbf{P}$. The number of significant flooding event between 1965 and 1985 is eight, five from the Mississippi River and three from Hurricanes (Rogers 2008). The density matrix, therefore, will be operated on five times to account for the river flooding events,

$$\rho_{HPS} = \mathbf{F}_5\mathbf{M}_5\mathbf{F}_4\mathbf{M}_4\mathbf{F}_3\mathbf{M}_3\mathbf{F}_2\mathbf{M}_2\mathbf{F}_1\mathbf{M}_1\rho_{HPS}\mathbf{M}_1{}^\dagger\mathbf{F}_1{}^\dagger\mathbf{M}_2{}^\dagger\mathbf{F}_2{}^\dagger\mathbf{M}_3{}^\dagger\mathbf{F}_3{}^\dagger\mathbf{M}_4{}^\dagger\mathbf{F}_4{}^\dagger\mathbf{M}_5{}^\dagger\mathbf{F}_5{}^\dagger . \qquad (8.20)$$

Significant flooding events due to major hurricanes, are defined by the event operators $\mathbf{H}_i$ corresponding to the three hurricane related flooding events, and these operators will need to sandwich between the operators from the river flooding events.

During Period I, the system went through significant upgrade, with two system level changes. First, some existing levees were enhanced with the addition of flood walls. Second, additional (L+W) systems were built. The system, therefore, has to be divided into two populations, one with only levees, and the other one consists of (L+W) systems. The density matrices for the 2 populations are constructed by taking the $\rho_{HPS}$ above and splitting it into the $\rho_L = \sum_i p_i \left| L_i \right\rangle \left\langle L_i \right|$ and the $\rho_{LW} = \rho_L \otimes \rho_W$ .

The second population consists of a statistical mixture of old and new (L+W) subsystems, one population consists of totally new subsystems, and the other population consists of subsystems of old levees with new flood walls. The statistical element is captured by the probability, $p_i$, for the totally new system states and the hybrid system states. Recall from equation 8.10, the density matrix has the form

$$\rho_{HPS} = p_1 \left| L_t^I W_t^I \right\rangle \left\langle L_t^I W_t^I \right| + p_2 \left| L_t^I W_0^I \right\rangle \left\langle L_t^I W_0^I \right|. \qquad (8.21)$$

For Period II, the same algorithmic logic continues to apply. The levee population continues to evolve based on the density matrix $\rho_L$, and the $\rho_{LW}$ population from Period II continue to evolve. Additional (L+W) systems built during Period II will be incorporated into the density matrix,

$$\begin{aligned}\rho_{HPS} &= w_I \rho_I + w_{II} \rho_{II} \\ &= w_I \sum_i p_i \left| L_i^I W_i^I \right\rangle \left\langle L_i^I W_i^I \right| + w_{II} \sum_j p_j \left| L_j^{II} W_J^{II} \right\rangle \left\langle L_j^{II} W_J^{II} \right| \cdot\end{aligned} \qquad (8.22)$$

To continue to the next computational steps, the specific values associated with the various system configurations will need to be instantiated into the model. For example, the various system populations are defined by the system configuration at specific instances of time. There are 35 polders in the New Orleans region, and each has many (L+W) sections/subsystems (IPET Ch VIII 2009). These numbers changes over time tracing the completion of different projects. These numerical values will need to be compiled and entered into the model in order to arrive at the state of the HPS system right about the time when Hurricane Katrina hit the region. The risk values, in turn, can be obtained from computing the expectation values (Chapter 4.6) associated with the observables driven by the original question: "What is the risk of catastrophic system failure when the 2005 HPS is impacted by a 200-year storm?"

Figure 30 summarizes the general algorithmic flow to take a system from the initial state to the target state, ready for provide answers to the risk question.

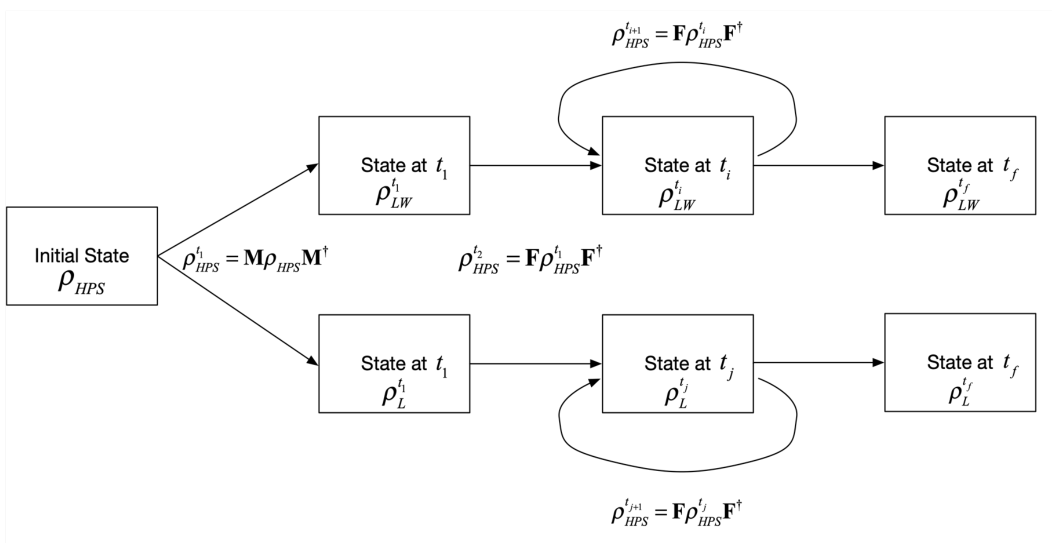

Figure 30: The evolution of the density matrix.

#### 8.1.4 Determining Risks

The determination of risks is a context driven process. The quantum density matrix approach changes the process of risk valuation. The conventional static event occurrence probability scaled with an impact value to compute a risk measure is now replaced with a knowledge based dynamic system model. Contextual information is an active participant in the shaping of the risk states and the process of extracting information out of the system model. In this case, the "risk measurements" depend on both the histories of the systems as well as the configuration of the systems.

The present risk question focuses on establishing the failure probabilities for the HPS due to various type of flood events, from seasonal to extreme ones. What constitutes a failure mode for a system must be defined so that the proper observable and measurement can be established. The flooding failure modes for a single levee system can be the overflow of the river or the breaching of the levee. The flooding failure modes for a (L+W) system could be river overtopping above the composite structure, the breaching of the flood wall, the breaching of the levee including seepage, and the breaching of the entire (L+W) structure; however, in the case of the composite system there is no overflow operation on the levee component. Note that the overflow for the single levee is different than the overflow of the (L+W) system; the overflow operators are different. This is also precisely why the operators are referred to as knowledge operators as they reflect the contextual knowledge associated with the event and cannot be blindly applied to a situation without the proper consideration of the contextual information.

The determination of risk is therefore a process of making discrete time step measurements, transitioning the system into different states over time. A classical analogy would be a decision tree, where one calculates the probabilities by going down different branches, but with one key difference: *the states of the system are not pre-determined but continuously evolve for the quantum framework*. In other words, the quantum state transition is not necessarily a Markov process as operator can be function of time, making the transition probabilities a dynamic quantity.

To obtain quantitative risk values, one would calculate the expectation values of finding the system in a particular risk state or states. For the single levee systems, there are two possible states: the non-working $|0\rangle$ state and the working $|1\rangle$ state. For the composite (L+W) subsystems, there are two possible configurations, one for "separable" subsystems and one for "correlated/entangled" subsystems. For separable subsystems, there are eight distinct states: $|0\rangle_L |0\rangle_W$, $|1\rangle_L |0\rangle_W$, $|0\rangle_L |1\rangle_W$, and $|1\rangle_L |1\rangle_W$. For correlated/entangled subsystems, there are four possible states: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. The scenario where the composite (L+W) are separable subsystems is rare since they are often subjected to event conditions together. For composite systems, we are more interested in the joint states than the separable states. In the case of the (L+W) systems, $|00\rangle$ can be interpreted as the catastrophic failure of the system, where both the levee and the flood wall failed. The $|10\rangle$ state represents the case of overtopping and with no breaching. The $|01\rangle$ state could correspond to flooding due to erosion seepage through the levee foundation. In the case of Katrina, the major flooding events were the result of the breaching of the systems, which means we are interested in finding the system in the $|00\rangle$, $|01\rangle$, and $|10\rangle$ states. The numerical values can be obtained by performing a projective measurement (a von Neumann measurement) with the following projectors (measurable): $|00\rangle\langle 00|$, $|01\rangle\langle 01|$, $|10\rangle\langle 10|$. To determine the probability of finding the (L+W) system in a breaching failure state, the projector $\mathbf{B} = |00\rangle\langle 00|$, can be used to calculate $P(|00\rangle) = \langle LW|00\rangle\langle 00|LW\rangle = |\langle LW|00\rangle|^2$. The expectation value of the measurable **Ri**, $\langle Ri\rangle = \langle LW|\mathbf{Ri}|LW\rangle$ or $\langle \mathbf{Ri}\rangle = tr(\rho\mathbf{Ri})$, gives you a measure of the risk of the system.

In principle one can also ask to quantify the risk of one of the subsystems. The notion of retrieving the state of a subsystem from the composite system joint states is not a simple concept, since it is not a mere projection of states due to the possibility of correlated states. One can extract the information about the joint state with projectors, but one cannot simply project out the individual component states for a subsystem. To determine the state of a subsystem (for example, the **Ri** for subsystem L) requires the extraction of the density matrix of the particular subsystem from the composite density matrix, using the *partial trace* operation over a subsystem.

For the composite system formed from the tensor product of subsystems A and B, the density matrix is given by: $\rho_{AB} = |AB\rangle\langle AB|$. To obtain the density matrix of subsystem B, one performs the "partial trace" over subsystem A:

$$\rho_B = \mathrm{Tr}_A\{\rho_{AB}\} \qquad (8.23)$$

where $\rho_B$ is called the reduced density matrix for subsystem B, and the partial trace of an operator (e.g. the density operator) is given by:

$$\mathrm{Tr}_A\{A \otimes B\} = \mathrm{Tr}\{A\}B \qquad (8.24)$$

For the scope of this investigation, the focus is on the risk considerations for the joint states at the system level, so we will simply state the above mathematical formalism but not discuss further in this dissertation.

In the next section, we will take a look at what this process tells us about the state of the HPS right before Hurricane Katrina hits the New Orleans area in 2005. The results from the quantum analysis will be compared to results from conventional risk analysis methods.

## *8.2 Comparisons*

The algorithmic integrated model framework developed in the previous section represents many possible HPS configuration scenarios. In this section, the integrated model scenarios and system states are compared to risk analysis results from the IPET, 2007, 2009 reports, which serve as the reference representing current standard assessment methods. This section will focus on highlighting the differences between the conventional and quantum approaches, and arguments are presented on why the new approach adds value to the risk analysis process and why further research on this new approach can be beneficial to the risk community.

### 8.2.1 The Risk Scenario Before Katrina from Conventional Methods

A comprehensive risk analysis and assessment was performed on the New Orleans HPS by IPET. Details from the analysis can be found in Volume I and VIII of the 2009 IPET final report, and the 2009 supplemental report titled “A general description of vulnerability to flooding and risk for New Orleans and Vicinity: Past, Present, and Future." A sophisticated modeling framework available at the time (2009) was utilized for the IPET study. In the reports, the team acknowledged the challenging nature of the problem: 1) the complexity of the physical costal region with substantial area below sea level, 2) the engineering system with many different interacting elements and structures increase problem complexity, and 3) known limitations of the analysis framework resulting in unresolved uncertainties. Below highlight several key aspects of the IPET analysis methodology and conclusions.

The IPET study focused on assessing the flooding vulnerability of the New Orleans area in terms of how often and how severe flooding can be. The flooding events are classified into 3 groups according to severity and occurrence probability: 2% chance of occurrence per year (average of once in 50 years or a 50-year event) resulting in significant flooding and losses, 1% chance of occurrence (average of once in 100 years or a 100-year event) resulting in extensive flooding and losses (typically due to rainfall, with minimal or no overtopping or breaching), and 0.2% chance of occurrence (average of once in 500 years or a 500-year event) resulting in major devastation to the area (hurricane). The 100-year event serves a standard engineering benchmark.

The water level from storm surges and waves, causing flooding events and in extreme cases overtop and breach the system, was identified as the main hazard. A flooding event can be the result of a combination of several interacting factors, such as the water volume, the intensity of the flow, the surge levels, the surge frequency, the overtopping probability, the system breach probability, and previous historical events.  Other key parameters include the height of the water level, the storm frequency, the size of storms and hurricanes, and the speed of the storm moving across the region (affecting frequency of surges and wave impacts). Storm surge levels by hurricane can be difficult to model since they can vary significantly due to the many different possible storm configurations and parameters, from wind speed to the storm paths in relation to the geographic location. The probability of flooding depends on how these parameters interconnect with each other. Hence, the model to compute the probabilities for the risk analysis demands careful considerations of the interactions between the complex engineering system with the geographic environment.

Sophisticated hurricane modeling programs and best practices for risk analysis were utilized in the IPET study.  The Joint Probability Method - Optimal Sampling (JPM-OS) was used, incorporating historical storm information from 1940 to 2006, including storm sizes, surge strengths, storm intensity and duration. Timing information, such as event frequency, was critical since storm surges develop over the life of the storm, the conditions before, during, and after the storm must be incorporated into the model. Two different frequency parameters were included; the frequency of storm surge during a storm, and the frequency of storm events hitting the region. Figure 31 is a compilation of different temporal profiles of storm surges used by the model.

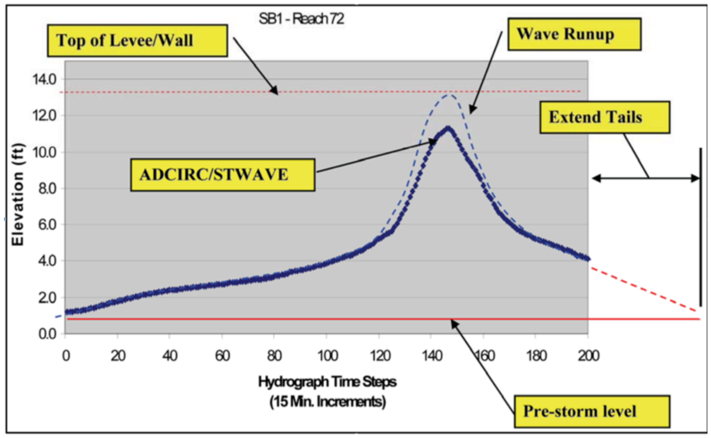


Figure 31: Temporal profile example of storm surges used in the IPET (2009) study (Figure 39 in Volume 1 of the IPET 2009 report).

To assess the probability of flooding, the geographic region was decomposed into drainage basins and sub-basins, and each was further divided into reaches, which were levees or flood walls with uniform properties (strength, elevation, etc.). 135 reaches were defined for the Pre-Katrina HPS. A system model for the engineering system, consists of levees, flood walls, gates, and pumps were systematically constructed by first analyzing the individual component's reliability and performances, which were then integrated into subsystems taking into account of the component to component dependencies and interactions. The reliability and performance of the systems (reach, subsystem, component) were captured in fragility curves. The subsystems within a reach were evaluated against the storm water volume and flow, and flows between the basins were evaluated as well. The performances of the reaches were evaluated against storm surges and waves computed from the hurricane model. The results from the analysis gave estimates on the water volume that can potentially flow into protected areas in the case where the engineering system fails (overtopping or breaching).

Event trees were developed from the reliability information from fragility curves (Figure 32), the hurricane storm surge and wave models from 76 JPM-OS hypothetical hurricanes. The probabilities of flooding within the reaches were estimated from the event tree model (Figure 33).

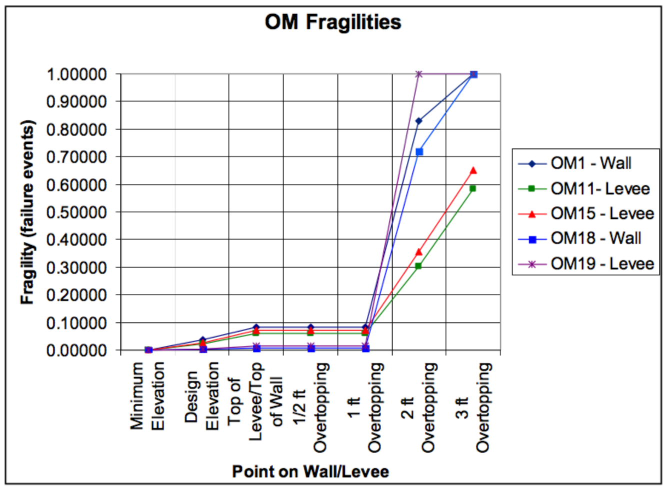


Figure 32: Fragility curves from IPET 2009 (Figure 41 in Volume 1 of the IPET 2009 report).

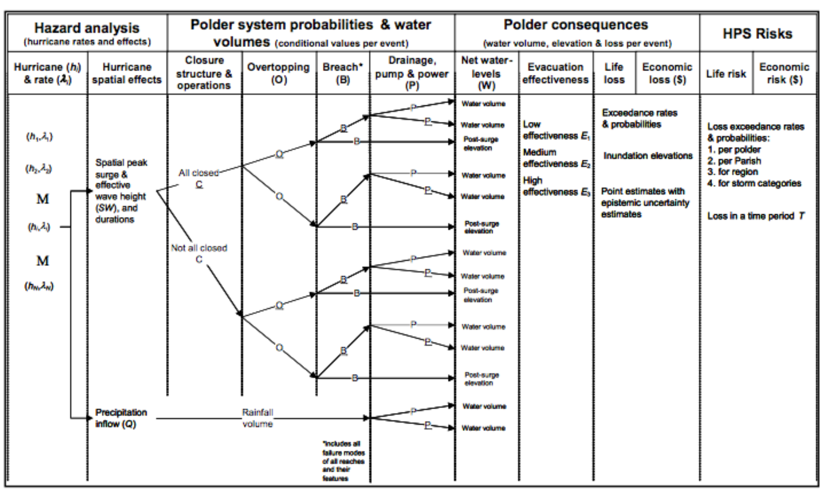


Figure 33: Event tree from IPET 2009 (Figure 42 in Volume 1 of the IPET 2009 report).

The risk analysis result boils down to:

*"Bottom line, the 100-year (or any chance flood) flood is dependent on the combined changes of experiencing a specific surge level, the chance of overtopping or breaching due to that surge, and the volume of water entering an area." (IPET 2009)*

The conventional approach identified the various parameters and dependencies that could contribute and affect the risk assessment conclusions, and it is a snapshot of what are the knowns and known-unknowns. Time behaviors of how the events could have impacted the system were not fully explored.

#### 8.2.2 The Risk Scenario from the Quantum Framework

The objective of the quantum approach is not simply to reproduce the results from the conventional method. As the last steps of the conventional techniques rely on the use of fragility curves and event trees, this comparison will focus at the macroscopic level and highlight differences between the conventional and the quantum outputs for fragility curves.

The basic process to generate fragility curves from the quantum models was introduced in Chapter 6. Before generating the fragility curves, the density matrix representing the various possible states for the system has to be instantiated. This has the equivalent effect of generating the event tree for the HPS scenario. The "leaf nodes" are all elements of the density matrix. The different operators, overtopping, erosion, etc., are then applied to the density matrix, taking the system from one state to another. For example, to model temporal profile of storm surges (Figure 33), the Weibull profile is chosen to model the shape of the profile in the construction of the overflow operator. The fragility curves are obtained by projecting out the various system state probabilities from the density matrix. The failure probabilities from the transition steps are compiled into curves to produce the quantum versions of the fragility curves. Fragility curves for various parameters can be constructed for a single structure, or for a system of systems such as a heterogeneous group of levees/flood walls consisting of systems of different states. Fragility curves can also be constructed for system of systems over time, such as the HPS system over the three development periods.

To illustrate this point, a simplified version of the model developed earlier will be used to create two curves. For computational and illustrative purposes, we will assume a simple system consists of 20 identical levees. The first curve models the entire levee system going through simple degradation over a period of time, and the degradation operator is a sigmoid function. The second curve models the same system but with half of the levees affected by some events (such as storm erosion) resulting in an increase of failure probability for that half of the population. The density matrix for the first system is given by:

$$\rho_1^{t_f} = \mathbf{M}\left|L_1\right\rangle\left\langle L_1\right|\mathbf{M}^\dagger \tag{8.25}$$

The density matrix for the second system is given by:

$$\begin{aligned}\rho_2^{t_f} &= 0.5\mathbf{M}\left|L_1\right\rangle\left\langle L_1\right|\mathbf{M}^\dagger + 0.5\mathbf{M}(t_f - t_1).\mathbf{E}.\mathbf{M}(t_1)\left|L_1\right\rangle\left\langle L_1\right|\mathbf{M}(t_1)^\dagger.\mathbf{E}^\dagger.\mathbf{M}(t_f - t_1)^\dagger \\ &= 0.5\mathbf{M}\left|L_1\right\rangle\left\langle L_1\right|\mathbf{M}^\dagger + 0.5\mathbf{K}\left|L_1\right\rangle\left\langle L_1\right|\mathbf{K}^\dagger\end{aligned} \tag{8.26}$$

where $\mathbf{K} \equiv \mathbf{MEM}$, as a short-hand. Essentially, the two curves reflect the time behaviors of two populations of levee. The first one is a homogeneous population and the second one is a heterogeneous population.

Figure 34 plots the fragility curves for a single state system and a mixed state system. These curves were constructed by extracting state information from the density matrices, with the system first subjected to the events and conditions first. The density matrices tracked the system states over time; the system was transitioned to the different "states" at different times, and the failure probabilities were projected out after each transition. The fragility curves produced as a result are only conveying a subset of information contained within the density matrices. The quantum framework extends the conventional framework by capturing and modeling temporal events, which were not performed in the IPET models.

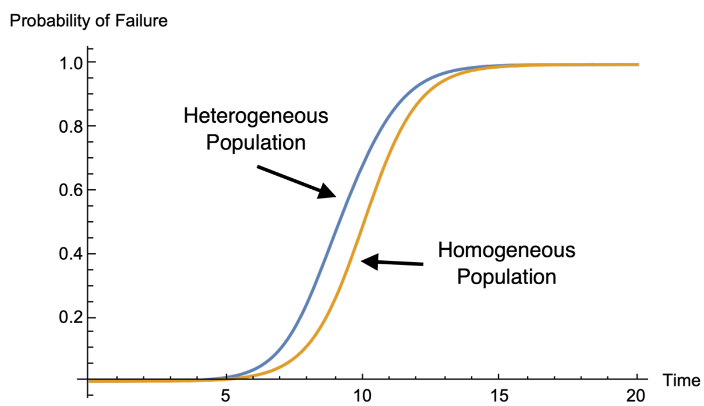


Figure 34: The curves for a homogeneous and a heterogeneous population levee system.

The curve for the heterogeneous population (the blue curve) rises faster than the curve for the homogeneous population, which indicates the system fails sooner and earlier. Certainly, if instead the second population are repaired and enhanced levees with lower failure probabilities, then one would expect the curve to shift to the right, indicating that the system will fail later. Information about the system evolution over time and the composition of the population can affect the total system failure probability. The additional second population contributes to the increase in failure probability for this scenario, as expected. The curves are essentially comparisons between pure (homogeneous population) and mixed (heterogeneous) states system.

Conventional methods are similar to the modeling of a pure state. In reality, however, most of the system are in mixed state. When the different populations of states are fully taken into account, the mixed state model, as expected, has a higher failure probability and the failure probability rise earlier and faster. The incorporation of the historical events, earlier failure states, shifts the curves to the left or to the right, depending on the behaviors of the population. As the system evolves over time, uncertainty increases as the system move further and further away from pure state; the system ensemble population changes over time, turning more and more into a statistical ensemble.

*Further thoughts…* Often there is more than one way to refine the quantum models based on contextual information and subjective interpretations of system behaviors. For example, how does one handle the different types of event spanning different timescales for the HPS scenario? What constitute system failure in one class of event could be quite different from another class. On one end, one can treat every single event as a "unit" event, and one can interpret system failure probability as the product of all "unit" probabilities. This interpretation takes the perspective that every event contributes to the failure of the system and they cannot be separated or isolated. On the other end, one can treat the system failure probability as the statistical sum of all "unit" probabilities. This interpretation takes the perspective that all events are separable and that a system failure at a given moment is due to one of the unit events;

in this case, each event has a statistical "unit" average and the total probability is the combination of all the contributing event at a given instance. For the HPS quantum model scenario, the system event timeline has to be deconstructed to separate two classes of risk event based on timescale, one for short timescale and the other for long timescale.

*Short timescale events* — The risk of the system is the cumulative product of the events. For example, the increase of water pressure increases the probability that a flood wall will collapse. The increase of flow volume also increases the probability that an overflow can occur. Further, the increase of flow volume also increases the front side erosion, which can also cause structural failure for the flood wall. The outcome can be a product of all the probabilities. This is applicable to events that happen within the same time window, such as during a hurricane.

*Long timescale events* — Over a longer timescale, for events are not "casually" connected, i.e., the events are all separable, the failure probability of the system is simply the combination of the individual event. Since each failure event is disconnected from each other, the system will have "settled" down in a stable configuration in between the events. These separable, discretized system events in time behaves like the aforementioned "unit" events. A statistical average probability can be derived and used to represent these similar discrete events, and the total probability is the sum of these "unit" probabilities associated with all the contributing events at a given instance. Typically, this can describe events with large temporal separation, such as storm events between seasons and years. This timeline also requires more “statistical” input, such as historical trending data, and “subjective” inputs from expert opinions.

The quantum risk analysis performed on the HPS is in essence an extended form of system dynamics simulation with non-Markovian elements that compute the risk states of the system at discrete time steps, with the choice of time scale granularity based on the event type. Larger time units are used for events that are not “casually” connected, and shorter time units for events that occur within the same time window. The simulation runs for the duration that takes the system from the initial state (the past) to the target state (the prediction). The quantum framework is capable to adapt to the different needs according to the problem situations.

The quantum framework takes the conclusions from the conventional techniques as the starting point and take steps further by refining the risk questions and drilling further down beyond the system parameters and configurations. The refinements drill deeper into the system behaviors and time evolution scenarios, and further explore areas where the contextual questions can actively re-shape the modeling processes. The quantum models are sensitive to the amount of information and knowledge available, and how much one would like to apply to formulate the model at a sufficient level of details to arrive at the answers one seeks. Such framework that has the flexibility to handle both precise and imprecise scenarios helps to extend the risk analysis and assessment process on the HPS. The density operator formalism has shown to be a very capable framework that can be utilized to provide a coherent and concise mathematical framework to represent the risk states of complex engineering systems. This research work only scratches the surface of a pond with great depth!

# Chapter 9: Combining Probabilities for Concurrent Failure Modes

*Probability is expectation founded upon partial knowledge. A perfect acquaintance with all the circumstances affecting the occurrence of an event would change expectation into certainty, and leave neither room nor demand for a theory of probabilities.*

*- George Boole 1854, in An Investigation of the Law of Thought*

Previous chapters introduced the basic formalism on how to obtain probabilities from the quantum models of the complex engineering systems. This chapter explores the quantum framework's additional capabilities to shed light onto the computation of probabilities for concurrent failure modes of engineering systems.

One of the many current challenges in risk assessment is the computation of the total failure probability for a complex system with many different failure modes (Hill et al., 2003; Baecher, 2012; Collier et al., 2017). In previous chapters (Ch. 5, 7, and 8), specific scenarios of multiple events sequence were explored with the use of different operator sequences. Calculating the total probability of failure for engineering systems with multiple failure modes, especially if they can occur simultaneously and not be mutually exclusive or independent, can be challenging. Simple linear additive combinations of the failure mode probabilities can only be applicable under a restrictive set of conditions. Often assumptions and approximations have to be made to combine the probabilities and obtain valid total probabilities.

When the assumptions and approximations no longer hold true, the law of probability can be violated. As we approach the limits of the approximations, total probability calculated from approximations to probability rules can lead to inconsistent and incorrect results, such as the value of the total probability exceeding unity (>1), violating the basic conservation of probability and lead to the mis-estimation of risk. While the *conservation of probability* principle is not an absolute concept, it can be applicable as a constraint for certain types of engineering problem and questions. For example, if the question asks whether a system is working or not, then the principle of conservation of probability is applicable due to the binary nature of the problem (like flipping a coin). The notion of re-mapping or re-normalizing the individual probabilities such that the sum of the probabilities equals to 1 has been suggested to address this problem (USBR/USACE, 2015). However, the validity of this approach was debated as it is seen as more a mathematical manipulation than based on sound principles probability (Hill, 2001, 2002, 2003; Baecher, 2012).

Many decisions rely on the total failure probability as a decision point. The search for new methods and techniques to obtain total failure probability from concurrent events is an active and important area of investigation as present techniques to handle concurrent events have room for improvement. Concurrent events can significantly increase the chance of failure (Collier et al., 2017), especially in the case for complex engineering systems and problems which demand new modeling approaches to deal with the increase of systems complexity. Castillo-Rodrigues et al. 2017 states that

> *Addressing and analyzing such complexity is one of the identified main concerns in the field of critical infrastructure governance, where complexity refers to the difficulty of identifying and quantifying casual links between multiple potential and specific adverse events. In this field, it is recognized the need to extend modeling in order to cope with the increasing complexity of systems.*

While others, such as the late Gary Salmon of BCHydro (NNCold 1997) expressed reservation on the ability of fault trees and event trees to treat time-dependent problems. New techniques that can handle engineering systems with complex and concurrent failure modes can play an important role in further advance the risk modeling of complex engineering systems.

### *9.1 The Problem: Calculating Total Probability of All-modes Failure*

In engineering risk analysis, frequently we are working with systems with more than a single failure mode. An example of systems with different failure modes for a single event: seismic activities can subject a spillway monolith to lateral forces of various magnitudes, leading to structural failures such as cracking, sliding, or overturning (Baecher, 2012). For complex integrated systems, individual subsystems can have several different failure modes, and when subsystems are coupled together as an integrate systems, different failure mode permutations are possible.

Further complications arise when the subsystem or component failure modes for complex engineering systems are not mutually exclusive and are not independent. The failure probabilities of the different modes cannot be combined with simple addition to arrive at the total probability of failure. In addition, changes in the system as a result of repairs, upgrades, and modifications can also result in changes in the total failure probability. The case with the HPS in earlier chapters is an example where a system, the HPS, underwent three different construction phases, leading to drastic changes to the system configurations and states.

One controversial treatment of the calculation of total probability has to deal with situations where the summation of the individual probabilities results in the total probability exceeding 1. Some treatments advocate the re-normalization of the total probability to unity, while another approach involves adjustments to the grouping or remapping of probabilities so that the final probability will not exceed unity (Hill, 2001, 2002, 2003). In these treatments, the methods are guided by algebra rather than physical principles.

Concurrent events, systems complexity, and time-dependent phenomena have posted significant challenges to risk analyses (Baecher, 2012; Collier et al., 2017; Castillo-Rodriguez et al., 2017; Salmon, 1997, USBR/USACE, 2015), and can lead to poor risk guidelines being issued by decision makers (Hill et al., 2003). Inappropriate treatment to combine probabilities can often to assessment bias, resulting in poor risk policy and decisions, and determining the proper way to combine probabilities has been identified as an important risk assessment challenge (Hill et al., 2003; Baecher, 2012; Collier et al., 2017).

## *9.2 Current Approaches*

Two methodologies commonly utilized for calculating total probability of failure for engineering systems with single event multiple concurrent failure modes are Event Trees and Fragility Curves. Their effectiveness and limitations in handling systems with non-mutually exclusive failure modes are assessed in this section.

To calculate failure probabilities, these methods rely on the assumption that failure modes are independent or mutually exclusive. With those assumptions, the total probability is simply the products or sums of the probabilities. In certain cases, if the magnitudes are small and the probabilities of the modes are almost mutually exclusive, the total probability can be "approximated" by the sum. In the case where this approximation or the assumptions are no longer valid (such as when the failure modes are not mutually exclusive), the total probability cannot be obtained from simply performing a summation of marginal probabilities.

### 9.2.1 Event Tree

An event tree is an inductive logical decomposition of an event, represented as a branching tree graph, into a progressive series of sub-events represented by event nodes, leading to some subsequent outcomes, consequences, or end states. An event tree is a representation of Sample Space and need not represent a chronology, although it is often used that way in engineering system safety studies (Hartford and Baecher, 2003). Boolean logic serves as the connectors for event nodes. Event occurrence probabilities are associated with the nodes, and these probabilities facilitate the computation of risk. The event tree represents an exhaustive enumeration and combinations of all "known" mutually exclusive events (the collection of all the branches). Each event tree represents a scenario that describes the various event paths leading to various outcomes. Figure 35 is an event tree decomposition reflecting a binary state of the system and in this case the operability of a space mission. Figure 36 is another example of an event tree with each level corresponding to conditions that can be non-binary.

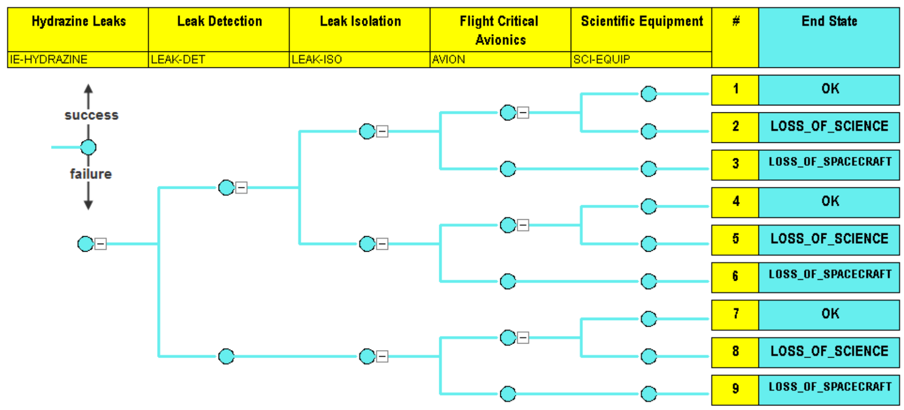


Figure 35: An example of an event tree with binary states (ET for Hydrazine Leak, Figure 3-7 in Stamatelatos & Dezfuli, 2011).

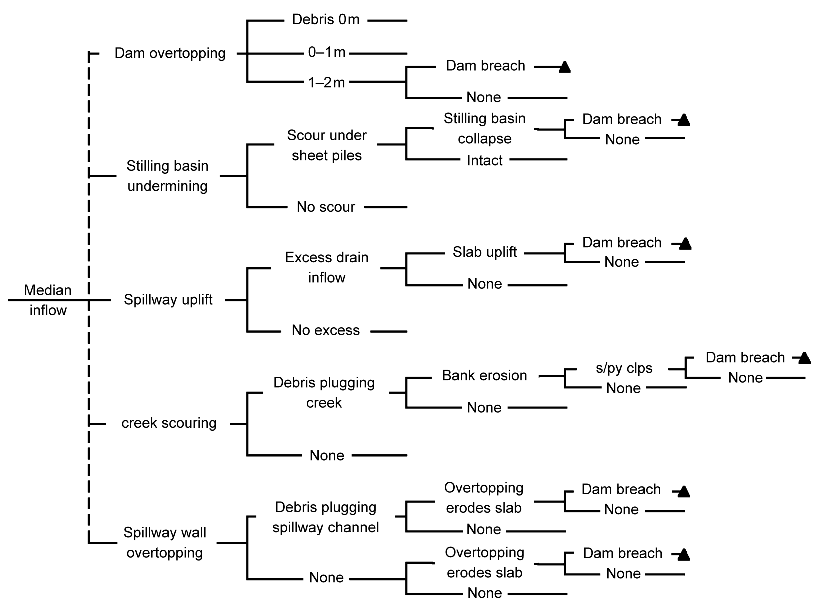


Figure 36: An example of an Event Tree (Salmon & Hartford, 1995).

The tree nodes by definition are meant to represent mutually exclusive events. By tracing down a branch and calculating the probability at the leaf node, one obtains the occurrence probability for the leaf event, which is separate from other leaf events. The collection of all the leaf probabilities represents all the possible system event states. To obtain the total probability of a particular set of events (a particular set of branch pathways), the probabilities of the leaf nodes corresponding to the set of events (the branch probabilities) are summed (additive sum of probabilities).

In the case of concurrent multiple failure modes, a single event requires tracing down several branching pathways. The total probability for failure would require the summing of the probabilities from these different paths, which corresponds to the summing of the probabilities for failure modes that occur concurrently under a particular event scenario. In the case where the fundamental assumption about non-mutually exclusive events is no longer valid, an event tree cannot be properly constructed since the branches might be "connected" at some of the nodes (Figure 37), which is not permissible due to the possibility of over counting or under counting with simple additive sum, leading to inconsistent results.

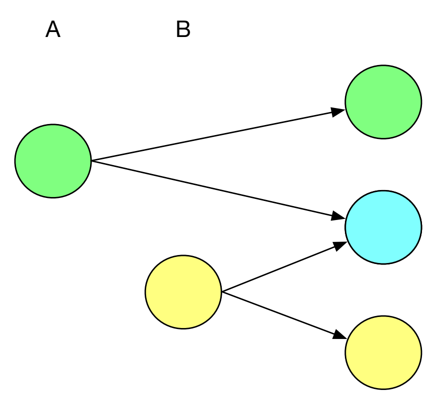


Figure 37: An example for a tree with non-mutually exclusive events.

In summary, besides the fact that the full enumeration of all possible events in an event tree can become unwieldy, the total probability obtained from tracing branches and summing leaf node probabilities can lead to inconsistent results and the mis-estimate of risk (see Chapter 5, USBR/USACE, 2015 for further details).

9.2.2 Fragility Curves

For systems like the spillway monolith of a dam, sliding, cracking, or overturning can occur when subjected to lateral seismic loads of different magnitudes. A single hazard can cause the system to fail in more than one mode. In cases like this, the use of fragility curves is a common treatment for representing and characterizing the different conditions and the conditional failure probabilities (Schultz et al., 2010).

Fragility curves are graphical plots of conditional probability of failure against the hazard condition or parameter such as the system load. Often, different probability curves corresponding to different failure modes are plotted on the same graph to show the range of failure probabilities a system can be found in due to the hazard. Figure 38 is an example of levee fragility curves from NRC, 2013, Figure I-2. Figure 39 is a set of hypothetical fragility curves from three failure modes, A, B, and C, to be used later in the chapter.

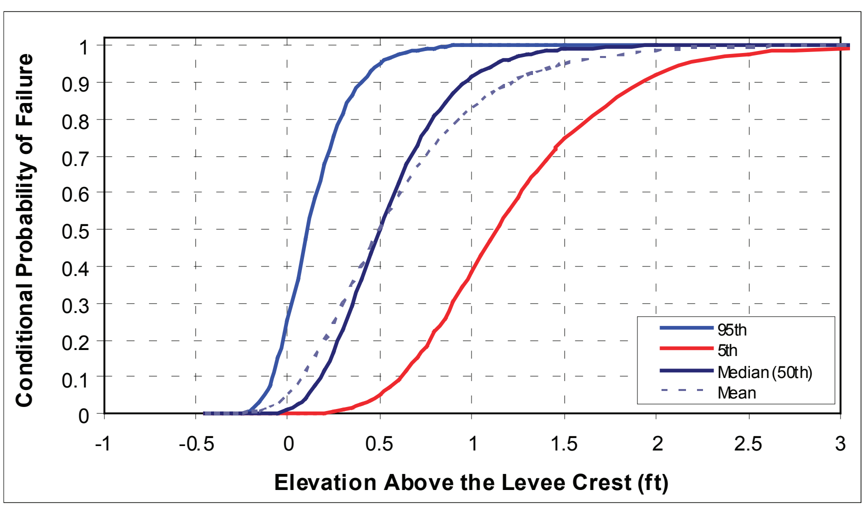


Figure 38. Examples of fragility curves, Figure I-2 from NRC (2013).

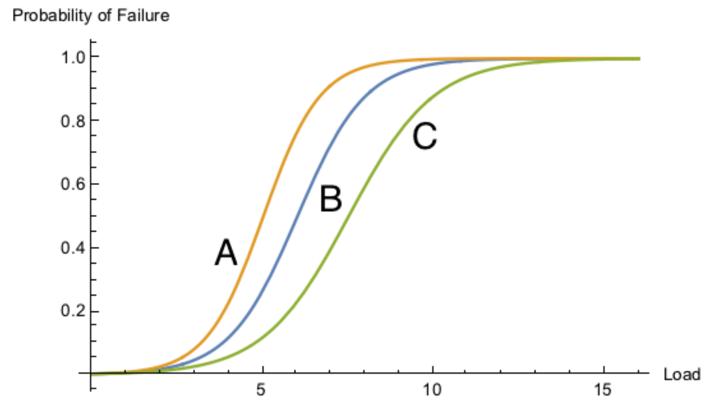


Figure 39: Fragility curves for different modes of failure A, B, C.

From the fragility curves, one can obtain several probability measures corresponding to: 1) the probability of the occurrence of *one* failure mode irrespective of the occurrence of other failure modes (i.e., the marginal probability), 2) the probability of occurrence of *multiple* failure modes simultaneously (i.e., their joint probability), and 3) the probability of the occurrence of *at least one* failure mode. For (1), the probability comes directly from the curve corresponding to the particular failure mode. For (2), the probability is the product of the probabilities corresponding to different (independent) events at the same load or hazard condition. But for (3), the probability is more complicated.

If the failure probabilities of the different modes are not independent, the law of total probability relating the marginal probabilities to conditional probabilities might not hold true, and that the total probability might not sum to unity. In the case where the failure modes are mutually independent, the total probability for the simultaneously occurrence of the failure modes is the product of their probabilities.

For *independent* modes, the probability for the occurrence of at least one of the failure modes is given by:

$$\Pr\left(\text{at least one failure}\right) = 1 - \Pr\left(\text{no failure}\right) = 1 - \prod_i \left(1 - P(i)\right) \qquad (9.1)$$

Expanding:

$$\begin{aligned}\Pr\left(\text{at least one failure}\right) &= 1 - \prod_i \left(1 - P(i)\right) \\ &= 1 - \left(1 - P(A)\right)\left(1 - P(B)\right)\left(1 - P(C)\right)\ldots \\ &= 1 - \left[1 - \left(P(A) + P(B) + P(C) + \ldots\right) + P(A)P(B) + P(B)P(C) + \ldots\right]\end{aligned} \qquad (9.2)$$

If we assume that the magnitudes for the probabilities of failure for the individual independent modes are small, we can drop the higher order terms and the probability can be approximate by this expression:

$$\Pr\left(\text{at least one failure}\right) = 1 - \prod_i \left(1 - P(i)\right) \approx \sum_i P(i) \qquad (9.3)$$

This approximation is only true if the failure probabilities of the different modes are *small*. However, in the case where the probabilities of the modes are not small, the small magnitude approximation no longer applies, and the additive sum of the probabilities can be greater than unity, violating the law of total probability.

#### 9.2.3 Challenges with the Approaches

The event tree and fragility curves methods work well when the independence assumption is applicable. Both fail when the individual failure probabilities are not small, and the independence assumption is not applicable. This leads to the following basic question: how often do we encounter engineering systems where these assumptions are not applicable?

For simple systems and rare events, which includes many types of concurrent events, such as mutually exclusive events, probabilistically independent events, correlated events, and conditional probability events, the mutually exclusive assumption are only approximations for the majority of these cases (USBR/USACE, 2015; Hill et al., 2003; Collier et al., 2017). The use of event trees and fragility curves is therefore "approximation of approximation" at best. When the assumptions no longer apply, such as when systems and the associated physical processes interact with each other (Collier et al., 2017), the approximations are no longer valid and probabilities cannot be combined via simple addition, and the methodologies will fail. As technology continues to advance and the complexity of engineering system interactions continue to grow, situations where those assumptions no longer applies are anticipated to be more of the norm than exceptions.

Researchers have been exploring ways to augment and extend the current approaches to allow them to handle situations for which the failure modes are non-mutually exclusive. Some have proposed modifications to the algorithms and logics for the construction and reconfiguration of the event tree branches to make them compatible and consistent with the rule of probability in terms of the computation of the final total probability (USBR/USACE, 2015). After the initial development of the event tree, the algorithm requires the analyst to look for common causes, then group, adjust, and reconfigure the tree branches; branches that share common cause events are group together. Probabilities associated with the reconfigured branches (which can contain multiple branches) are required to be reallocated to the remaining branches associated with the failure modes. In other words, common cause events are assigned to individual potential failure modes, and the associated probabilities are adjusted such that the total probability can be computed via the additive sum of the probabilities. This method is often illustrated using the Venn diagrams, where the algorithm reallocates overlapping regions so that the modes can be approximated to be mutually exclusive and therefore simple additive sum can be applied (Figure 40).

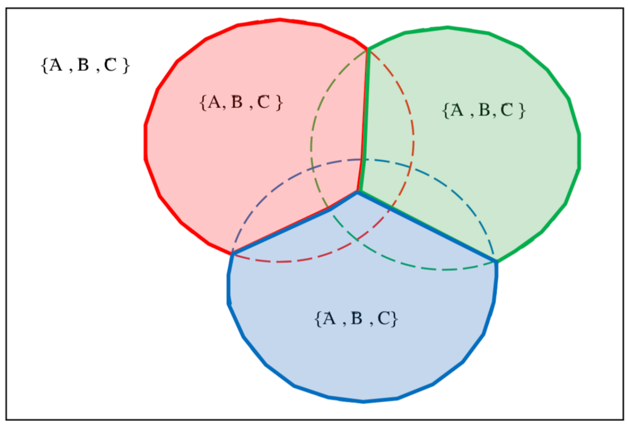


Figure 40: Allocating multiple breaches to individual potential failure modes. From USBR/USACE, 2015, Figure I-5-10.

While this approach can result in a structure conforming to $\sum P(i) \leq 1$, the adjustment of the structure and the relocation of probabilities are approximations at best, and thus the process introduces uncertainties into the model. With this approach, expert knowledge and information of risk analysts are used to modify the model's knowledge structure which is separated from the probabilities. As presented in Chapter 2, this separation between probabilities and structures reduces knowledge about the process. When coupled with the introduction of expert assessments, which are themselves inherently subjective processes, this approach introduces additional new uncertainties into the model. Such methodologies are constantly evolving, and can be superseded by different approaches. At the time of writing of this dissertation (2018-2020), the above approach has been revised and the methodology has been modified in the latest release (USBR, 2018).

In the case of fragility curves, one technique calls for the renormalization of the probabilities such that the total probability sums to unity. This approach was challenged from several fronts, from the lack of physical and engineering rationales in justifying the renormalization process, to the possibility of double counting the probabilities (Baecher, 2012). By treating the events as independent Bernoulli events, one can show that the process of performing the additive sum through enumeration can lead to the double counting of probabilities. Re-normalization of the probabilities will introduce additional weights and biases, resulting in a logical breakdown of the process.

A key conclusion, therefore, is that methodologies involving the enumeration of the various probabilities, such as event trees and fragility curves, will lead to inconsistency when the fundamental assumption about independence and mutual exclusivity are not valid. Renormalization can only work when the assumptions about independence and mutual exclusivity are true; one cannot “renormalize” when the starting point, the independent Bernoulli events assumption, is not true. The issue with additive sums and maximum consequences are examples of the breakdown of the analysis due to the invalidity of the starting assumptions such as the small order of magnitude approximation.

Recalling that in Chapter 2, an argument was put forth that if the logical structures are detached from the fundamental probability, no matter how one changes the structure, those changes cannot directly address the underlying issue with the probabilities. During the process of enumeration, knowledge and information are systematically reduced at each level of decomposition without being captured and reflected in the probabilities. While the process will result with the allocation scalar probabilities at each node, without a measure of the uncertainty arises from the knowledge degradation, the probabilities at the leaf nodes do not reflect all available information. When one computes the total probability by summing the leaf nodes, the summation only accounts for the scalar probabilities but not the missing knowledge. In essence, we are only summing partial knowledge, and the missing knowledge can result in the over or under counting the probabilities.

In closing, these modifications and adjustments of the knowledge structure can potentially introduce new problem elements. Adjustment of the knowledge structures and the reallocation probabilities based on expert assessments are inherently subjective processes. Selective adjustments of the tree structure are themselves approximations, which introduces additional uncertainties. Lacking physical and engineering justifications at times, the re-normalization of probabilities might lead to double counting probabilities, introducing additional biases and uncertainties. Risk can be mis-estimated as a result.

### *9.3 Quantum Model*

Can the quantum framework provide an alternative approach to address the weakness of the methodologies discussed in the previous section? The quantum approach is different from the conventional approach: the enumeration of the probability sets and pathways (conventional) vs. the tracking of the states of the system (quantum). In this section, the differences between the conventional and the quantum approaches, the difference in the formulation of the problem, and the difference in the questioning and querying for the answer to the questions are explored. The challenge with the combination of probabilities with non-exclusive multiple fault modes will be examined using the quantum modeling framework.

### 9.3.1 From Conventional to Quantum

Conventional methods focus on the enumeration and tracking of event probabilities. Different engineering systems, under different conditions, require different ways to combine event probabilities. Assumptions and approximations have to be made in order to compute the total probabilities in a consistent manner under various conditions, which can be a daunting task.

In the cases of event trees and fragility curves, the probabilities of the failure modes reflect conditions of the system, but information about the relationships between the modes are not fully captured. All other relational knowledge and contextual information are captured by structures connecting the probabilities. Since answers are derived mostly from the structures that map to scalar probabilities, if the structure is wrong, or the questions used to construct the structure are wrong, the answers will be wrong.

Conventional probability and quantum probability have several conceptual differences in their treatments of dependent, independent, and mutually exclusive events. In conventional approach, two events A and B are independent if these three conditions are true:

1) P(A and B) = P(A)P(B),

2) P(A|B) = P(A), and

3) P(B|A) = P(B). Events A and B are mutually exclusive if P(A and B) = 0.

In quantum probability, one can find concepts analogous to the above. However, the application of these analogous concepts does not necessarily correspond in equivalent fashions to traditional probability and risk questions. The quantum models capture and represent system states information. The state vectors are constructed from the superposition (section 4.2) of orthogonal basis vectors, with the basis vectors representing orthogonal states that are distinct and mutually exclusive from each other. This notion of superposition enables the incorporation of all possibilities in which dependent, independent, and mutually exclusive states are represented in the initial system state vector.

Consider the state vector representing a composite system such as those discussed in Chapter 5. A composite system formed from the tensor product of components A and B are represented by:

$$|AB\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle \qquad (9.4)$$

This describes the system and the associated state space, and not events or probabilities. The basis state $|11\rangle$ represents the state where both components A and B are working, but the term does not represent the probability P(A and B). The terms in equation (AB) are interpreted in this work as follows:

1) If A, B are independent, the probability amplitudes are separable (Chapter 5), $a_i$ and $b_i$ are independent from each other

2) If A, B are dependent, the probability amplitudes are not separable, i.e. $a_i b_j = k_{ij}$ (also Chapter 5)

3) If A, B are mutually exclusive, certain composite basis states are not permissible. $|00\rangle$ is not a permissible state since it corresponds to failures due to both A and B, which is not possible if they are mutually exclusive.

Note that this is not the mutual exclusivity of the orthogonality of the states mentioned above. This mutual exclusivity is a constraint applied to the system, projecting it to a subspace.

Additional information is incorporated into the construction of the system model. As discussed in earlier chapters (Chapter 7.6), the problem and questions shape the development and construction of the quantum model representing the system.

The quantum approach focuses on describing the *total states of a system* directly with a probability framework that has the capacity to incorporate meta-information. The process starts with the identification of all the possible states of the system. The system is then defined according to the collection of states or as ensembles of states. Changes to the system is permissible since we are tracking the total number of states. If certain states are no longer permissible for a system or new states are introduced into a system, then the quantum representation of the system will be updated via projections or tensor products, resulting in an updated representation, properly normalized (or renormalized). The ad hoc renormalization protocol associated with other conventional methodologies is no longer necessary.

In the quantum framework, normalization and renormalization are part of the system creation, configuration, and evolution processes (Section 5.2):

$$\rho_{S'} = \frac{U_{S'} \rho U_{S'}}{tr(U_{S'} \rho U_{S'})} \tag{9.5}$$

This, for example, is used to project a system into a subspace, implement constraints such as mutual exclusivity into the model. Instead of updating the probabilities, the quantum approach uses knowledge to update the number of states and the statistical distributions for finding those states by adjusting the statistical distributions ($p_i$) in the mixed state density matrix,

$$\rho = \sum_i p_i \left| A_i \right\rangle \left\langle A_i \right| . \tag{9.6}$$

The quantum approach focuses on the representation of the system in states. The system representation is based on the amount of knowledge we possess. In addition, as the system changes and evolves over time, additional states can be added or removed from the system model to reflect the system configuration changes over time:

1) When we have full knowledge, the system is a pure state system. With only partial knowledge, the system will be in mixed states, which also allows the incorporation of ignorance.
2) Instead of updating probabilities, the quantum approach directly updates the system model, transition it from one state to another.

9.3.2 The Quantum Model for the Engineering System

The system with three modes of failure, A, B, and C, used in the fragility curves discussion in the previous section will serve as the engineering system to be modeled here with the quantum framework. First, the problem statement (the question) will need to be reframed and recast:

> *Original problem statement*: What is the total failure probability of the system with multiple failure modes, given a particular (hazard) event (e.g. load)?

> *New problem statement*: Given a particular event (e.g. load), what is the probability of finding the engineering system in the specific state (e.g. failure state) corresponding to the event?

The new problem statement reframes the question. *The analysis will not focus on the probability of individual events, but on the probability of finding the system in a certain state or states.* The system is considered together as an integrated whole and the system as a whole is working or the system is failing. From this perspective, by definition and by construction, the total failure probability of the system will be between 0 and 1, with the sum of the total failure probability always less than or equal to 1, following the law of probability.

There is a subtle philosophical change in thinking here. The original statement asks: what is the probability of finding the particular outcomes (conditions) based on known knowledge — the event and the associated failure modes. This is driven by the event, the probability for the occurrence of events. The quantum thinking is that we do not necessarily know everything about the system, so we focus on tracking the system states. Without knowledge certainty, one describes the system by enumerating all known possible states, and the system is described by the superposition (Section 4.2) of these states. Information and knowledge serve as conditions and constraints that shape the quantum state vector (the model). Some states are eliminated as a result of the application of these knowledge constraints, and the process are done via operators and projectors (such as projection into a subspace). Therefore, the construction of the system model takes into account our lack of knowledge and our ignorance; the new problem statement simply asks for the chance for finding the system in a not working state (the outcome condition) under those conditions. This drives how one construct the quantum model where the model state space consists of states from various failure modes, A, B, and C for this exercise. The state vector represents the possible states the system can be found.

The quantum model construction process generally follows these steps:

1. Define the questions and identify the type of system states that can provide answers.
2. Describe the system in terms of states, with a focus of finding what type of states the system can be found in.
3. Build the initial system state vector from the superposition of the system states identified in step 2.

4. Identify the various modes, events, and physical processes that can change the system states. This is the knowledge base, the known-knowns and the known-unknowns.
5. Apply appropriate knowledge to configure the system model to the initial state. For example, if certain basis states are not permissible due to a physical condition, those states are removed via projections.
6. Build out the system to the appropriate initial configuration for the independent variable such as load or time. The system states are then evolved from the initial state to the desired target state with operators.
7. Construct the desire projector according to the questions (step 1). The probabilities of the relevant states can then be projected out, if one would like to determine the probabilities of finding the system in those states.
8. To perform quantitative risk measurements, one would follow the process discussed in Section 4.6. First, associate the quantitative values to consequences or impacts; they correspond to the "eigenvalues" for $\mathbf{Ri}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$. The risk value can then be computed from $\langle Ri \rangle$.

*The initial representation for the quantum system* — The starting point for the construction of the quantum model begins with the construction of a normalized engineering system, describing all the possible system states. The system can comprise independent states or a composite of states, and the individual modes are treated as components of the system. While there could be different ways to choose the basis vectors, such as choosing failure modes as basis vectors, the problem with finding the total failure probability aligns with treating the system as an ensemble of possible failure modes for a system that has two states — working and not working — as we are interested in the functional state of the system and not the state of the system failure modes. This choice of the ensemble approach aligns with the independent and mutually exclusive assumptions, plus the option to extend to address other scenarios beyond those assumptions.

Consider a system with three possible failure modes, A, B, C. In the fragility curve approach, the three modes are described by three different fragility curves with their own distributions. In the quantum approach, each fragility curve is represented by a state vector with an associated operator that changes the state vector from one state to another (Chapter 6.1). The initial state vector, $|\psi_S\rangle$, represents the basic system:

$$|\psi_S\rangle = s_0|0\rangle + s_1|1\rangle \tag{9.7}$$

A system, such as a levee (L), corresponding to the initial condition is denoted by $|\psi_L\rangle : |\psi_L\rangle = |\psi_S\rangle$. The state vectors $|\psi_A\rangle$, $|\psi_B\rangle$, and $|\psi_C\rangle$ represent the initial system states subject to failure modes A, B, and C, given by:

$$|\psi_A\rangle = \mathbf{A}|\psi_S\rangle \tag{9.8}$$

$$|\psi_B\rangle = \mathbf{B}|\psi_S\rangle \tag{9.9}$$

$$|\psi_C\rangle = \mathbf{C}|\psi_S\rangle \tag{9.10}$$

in which A, B, and C function as operators on the system state $|\psi_S\rangle$.

The state of the levee system corresponding to the particular failure mode at a particular condition (e.g. the load parameter *l*) is denoted by $\left|\psi_L\right\rangle$ where $\left|\psi_L\right\rangle = \left|\psi_A\right\rangle$ for mutually exclusive independent events. This is the state vector operated on by the load. The operators that change the states of the system corresponding to the failure modes are denoted by **A, B**, and **C**, which correspond to the physical processes governing the probability of finding the system in failure modes, A, B, and C as a function of the load parameter (*l*). The operators act on the corresponding state vector to reflect the state of the system:

$\left|\psi_L\right\rangle = \mathbf{A}\left|\psi_A\right\rangle$ is the state vector at a given load subjected to failure mode A.

$\left|\psi_L\right\rangle = \mathbf{B}\left|\psi_B\right\rangle$ is the state vector at a given load subjected to failure mode B.

$\left|\psi_L\right\rangle = \mathbf{C}\left|\psi_C\right\rangle$ is the state vector at a given load subjected to failure mode C.

The probability of failure can be obtained by projecting out the $\left|0\right\rangle$ state with the projector $\left|0\right\rangle\left\langle 0\right|$ on the state vector $\left|\psi_L\right\rangle$. This corresponds to the extraction of the event subspace. Plotting out the results from the projection as a function of "Load" will give the fragility curves corresponding to the modes of failure A, B, and C. At this point, the state vectors represent a system with a single failure mode, A, B, or C. A system that can have more than one failure modes will need to be constructed. The corresponding density matrix representation will be: $\left|\psi_L\right\rangle\left\langle\psi_L\right|$.

*The construction of a system with more than one mode of failure* — When a system is a composite of several failure modes, the state vector representing the system will need to be constructed out of the possible states using the tensor products to build up the system, or if certain failure mode component states are not applicable, projectors are used to eliminate those states from the system. If we know which specific mode is occurring, say mode A, then the system is represented by the *pure state*:

$$\left|\psi_S\right\rangle = s_0\left|0\right\rangle + s_1\left|1\right\rangle \text{ (Aleatory)} \tag{9.11}$$

$$\left|\psi_L\right\rangle = \left|\psi_A\right\rangle = \mathbf{A}\left|\psi_S\right\rangle \text{ (Aleatory + Epistemic)} \tag{9.12}$$

**A** and the state vector can represent both aleatory and epistemic uncertainties.

For a system that can have both failure modes A and B, and if we know of the specific occurrences of the multi failure modes, then the system can be represented by the pure state composite constructed from the tensor product:

$$\left|\psi_L\right\rangle = \left|\psi_A\right\rangle \otimes \left|\psi_B\right\rangle \tag{9.13}$$

In general, for a system with multiple failure modes A, B, C, …, the state vector is constructed from the tensor products of all the failure mode states:

$$\left|\psi_L\right\rangle = \left|\psi_A\right\rangle \otimes \left|\psi_B\right\rangle \otimes \left|\psi_C\right\rangle \otimes \cdots \tag{9.14}$$

The properties of the resulting composite state vector reflect the relationships between the failure modes, whether they are independent, mutually exclusive, or correlated.

Up to this point the state vector is the superposition of all possible states. Additional knowledge and constraints are then applied to configure the system. Consider the simple system with 2 failure modes A and B.

*Case 1: A and B are independent (therefore not mutually exclusive)*

If A and B are independent, then the tensor product will result in a state vector with separable probability densities. They are separable because they do not affect each other, and operator apply to one does not affect the other. The state vector is given by:

$$\left|\psi_L\right\rangle = a_0 b_0 \left|00\right\rangle + a_0 b_1 \left|01\right\rangle + a_1 b_0 \left|10\right\rangle + a_1 b_1 \left|11\right\rangle \qquad (9.15)$$

This product state represents a composite system with two totally independent subsystems. The composite probability can be factorized into separate products, and the subsystems are not correlated.

*Case 2: A and B are not independent or mutually exclusive (therefore dependent)*

But if A and B are not independent, then the state vector is given by the more general form:

$$\left|\psi_L\right\rangle = k_{00} \left|00\right\rangle + k_{01} \left|01\right\rangle + k_{10} \left|10\right\rangle + k_{11} \left|11\right\rangle \qquad (9.16)$$

This product state represents a composite system where the subsystems are correlated in some way, that the subsystems are "entangled" (Section 5.1, Section 7.5.5) to some degree. Changes due to A or B will affect the whole system.

*Case 3: A and B are mutually exclusive*

If A and B are mutually exclusive, then the state $\left|00\right\rangle$ is not a permissible state for this system since both failure state A and B cannot occur simultaneously. Therefore, the state vector is given by:

$$\left|\psi_L\right\rangle = k_{01}\left|01\right\rangle + k_{10}\left|10\right\rangle + k_{11}\left|11\right\rangle \tag{9.17}$$

Note that this is a subspace. The mutual exclusivity information about A and B acts as a constraint and change the configuration of the system. One can only consider the subspace and the target system is represented by a projection of the system into a subspace.

At this point, these are pure states. Exact information and knowledge have been used to configure the system. Additional configuration might be necessary based on the amount of knowledge one has.

*The amount of knowledge determines how we model the problem and how we construct the density matrix* — Depending on how much we know about the system, the above pure state representation might not be a comprehensive representation since the probability amplitudes might not be specified precisely. In such case, a mixed state system representing might be necessary. Consider a system with 2 failure modes A and B where the density matrix is given by $\rho_L = \left|\psi_L\right\rangle\left\langle\psi_L\right|$.

In the case where knowledge is insufficient to build out the model as prescribed above, such as for a system where we do not know precisely the occurrences of the failure modes A & B, the mixed state density matrix can be utilized to model the mixture of system states. The density matrix for this example is given by:

$$\rho_L = p_A\left|\psi_A\right\rangle\left\langle\psi_A\right| + p_B\left|\psi_B\right\rangle\left\langle\psi_B\right| \tag{9.18}$$

where $p_A$ and $p_B$ are the probabilities for the occurrences of A and B.

For example, consider the following problem statement:

*There is a 30% chance that the system can encounter failure mode A, 30% chance that the system can encounter failure mode B, and 40% chance that the system can encounter failure mode C.*

The system will in the form of a mixed state density matrix:

$$\rho = 0.3|\psi_A\rangle\langle\psi_A| + 0.3|\psi_B\rangle\langle\psi_B| + 0.4|\psi_C\rangle\langle\psi_C| \tag{9.19}$$

We can construct other variants of the density matrix corresponding to the amount of knowledge we know about the system. In the case where we do not have all the information, a mixed state density matrix will need to be used to represent the system. If we only know of the distribution of the states, then a simple density matrix with the probability distribution $p_i$ will be sufficient to represent the system:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{9.20}$$

As before, additional knowledge can be applied to constrain the system via operators acting on the density matrix (Section 4.3, 4.4, 7.5). The logic is the same as before, but instead of applying the operations on the state vector, one applies the operation on the density matrix following the established algebraic rules. The failure probability can be obtained with the usual means using projection, depending on the characteristics of the system and the question.

<u>*9.4 Quantum System Model Scenario: Concurrent Failure Modes*</u>

This section provides examples to illustrate the concepts developed thus far. A levee system (L) with state vector $|\psi_L\rangle$ will be used in this section as a scenario case study. The initial state vector, $|\psi_S\rangle$, represents the basic system with aleatory uncertainty: $\left|\psi_S\right\rangle = s_0\left|0\right\rangle + s_1\left|1\right\rangle$ . The levee system corresponding to the initial condition is denoted by $\left|\psi_L\right\rangle = \left|\psi_S\right\rangle$. The density matrix is given by: $\rho_L = \left|\psi_L\right\rangle\left\langle\psi_L\right|$.

For a single levee, $\left|\psi_L\right\rangle = l_0\left|0\right\rangle + l_1\left|1\right\rangle = \begin{pmatrix} l_0 \\ l_1 \end{pmatrix}$, the density matrix is given by:

$$\rho_L = \left|\psi_L\right\rangle\left\langle\psi_L\right| = \begin{pmatrix} l_0 l_0 & l_0 l_1 \\ l_1 l_0 & l_1 l_1 \end{pmatrix} \tag{9.21}$$

The probabilities can be obtained via the trace of the density matrix, $\left\langle\mathbf{P}\right\rangle = tr\left(\rho_L \mathbf{P}\right)$, and for the particular state or states using projectors (**P**). For example, the failing state probability is:

$$P\left(\left|0\right\rangle\right) = l_0^2 \tag{9.22}$$

If **A** is a failure mode applicable to a single simple system, such as a levee, the operator is represented as follow:

$$\mathbf{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \tag{9.23}$$

$$\left|\psi_A\right\rangle = \mathbf{A}\left|\psi_L\right\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}\begin{pmatrix} l_0 \\ l_1 \end{pmatrix} = \begin{pmatrix} A_{00} l_0 + A_{01} l_1 \\ A_{10} l_0 + A_{11} l_1 \end{pmatrix} \equiv \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{9.24}$$

To simplify, the state vectors modeling failure modes (e.g. mode A, mode B) will adopt these short hands for the rest of this chapter:

$$\left|\psi_A\right\rangle = \mathbf{A}\left|\psi_S\right\rangle = a_0\left|0\right\rangle + a_1\left|1\right\rangle \equiv \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{9.25}$$

$$\left|\psi_B\right\rangle = \mathbf{B}\left|\psi_S\right\rangle = b_0\left|0\right\rangle + b_1\left|1\right\rangle \equiv \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \tag{9.26}$$

To illustrate the concepts, several failure mode scenarios are explored, including:

1) Foundation failure due to water pressure due to flow event - Pressure (**P**), a function of water level *h* and the height of the levee
2) Overtopping due to storm surges or seasonal water changes – Overtopping (**O**), a function of water level *h* and the height of the levee
3) Erosion due to nominal material degradation over time due to environmental effects – Erosion (**E**), a function of time
4) General material and structural fatigue – Material (**M**) degradation over time such as vegetation overgrowth

Three scenarios are presented: 1) Single failure mode, 2) Concurrent independent failure modes – A & B Separable, and 3) Concurrent dependent failure modes – A & B Correlated/Entangled.

9.4.1 Scenario: Single Failure Mode – Individual Overtopping, Pressure, and Erosion Events

Before discussing the concurrent states, we first model the basic single failure mode. At a given time, only a single operator representing the failure mode is changing the state of the system. To model the overtopping event for example, recall from Section 5.2, the operator **O** is constructed from parameters that track and trace the behavior of the failure probability, and in this case the water level is a parameter of the operator, $\mathbf{O}(\theta) \sim f(h)$:

$$|\psi_O\rangle = \mathbf{O}|\psi_L\rangle = \begin{pmatrix} O_{00} & O_{01} \\ O_{10} & O_{11} \end{pmatrix}\begin{pmatrix} l_0 \\ l_1 \end{pmatrix} = \begin{pmatrix} l_0\cos\theta + l_1\sin\theta \\ l_0\sin\theta + l_1\cos\theta \end{pmatrix} \equiv \begin{pmatrix} o_0 \\ o_1 \end{pmatrix} \quad (9.27)$$

where $o_0, o_1$ are functions of water level. The density matrix is given by:

$$\rho_L = |\psi_L\rangle\langle\psi_L| = \begin{pmatrix} o_0 \\ o_1 \end{pmatrix}\begin{pmatrix} o_0 & o_1 \end{pmatrix} = \begin{pmatrix} o_0 o_0 & o_0 o_1 \\ o_1 o_0 & o_1 o_1 \end{pmatrix} \quad (9.28)$$

A fragility curve for the levee, defined as the conditional probability of failure at a given value of water level, is a plot of the water level, *h*, against the probability of failure, $o_0^2$, Figure 41.

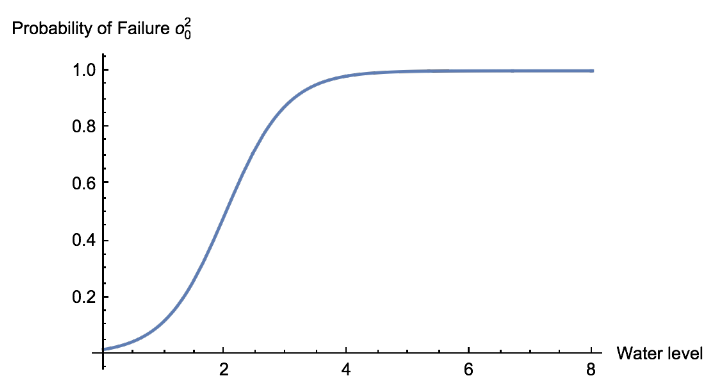


Figure 41: Fragility curve for levee failure probability as a function of water level.

In the event of a storm, the failure probability for the levee can also be represented with a different curve that plots the probability of failure against time. With the case of a storm, the water level (elevations) changes as a function of time (see Figure 13), and to model the overtopping event in this case, the operator **O** can be constructed from a different parameter that tracks the time behavior of failure probability. In this case the water level is a function of time, and in turn, a parameter of the operator, $\mathbf{O}(\theta) \sim f(t)$. With this parameterization, the plot for the failure probability becomes (Figure 42):

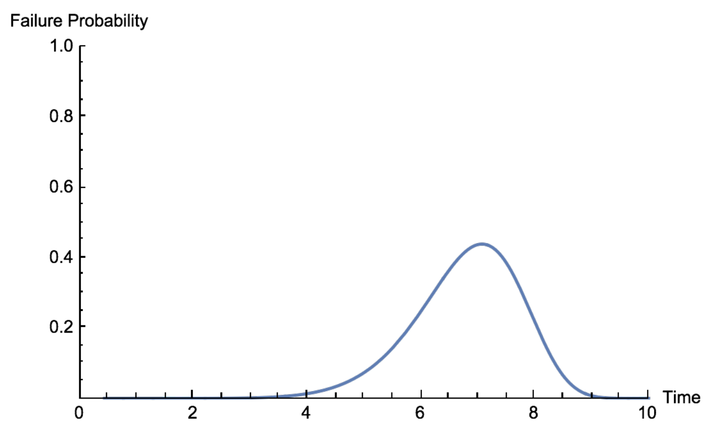


Figure 42: Failure probability for a levee as a function of time, which traces the water elevation as a storm event occurs over time.

Likewise, the state vectors for the other scenarios and the time evolution behaviors of their states are expressed with physical parameters that track and trace the failure conditions. For Pressure events, since hydrostatic pressure is proportional to the density of the fluid and the depth, $p = \rho g h$, the pressure profile is tied to the water height profile. For a single cycle, such as the rise of water level during a storm event, the water level is a parameter for the pressure operator, $\mathbf{P}(\theta) \sim f(h)$:

$$\left|\psi_P\right\rangle = \mathbf{P}\left|\psi_L\right\rangle = p_0\left|0\right\rangle + p_1\left|1\right\rangle \equiv \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \tag{9.29}$$

Similar to the overflow event, as the water level is a function of time, *h(t)*, the profile would take the general form of a bell-shaped curve (Figure 43):

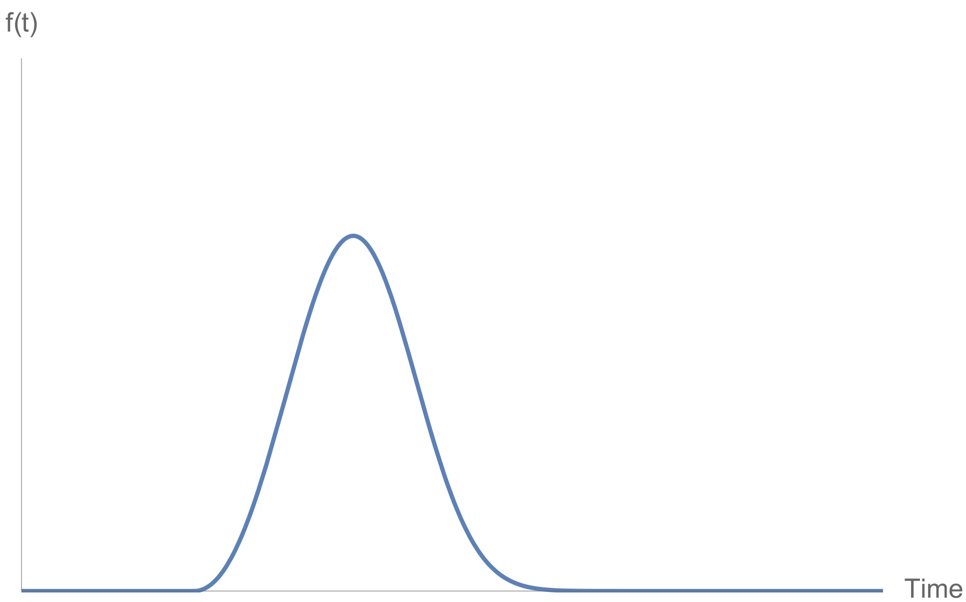

Figure 43: A general profile for the function tracing the change of pressure on the water facing side of the structure.

For Erosion event:

$$\left|\psi_E\right\rangle = \mathbf{E}\left|\psi_L\right\rangle = e_0\left|0\right\rangle + e_1\left|1\right\rangle \equiv \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \tag{9.30}$$

From these state vectors (Equation 9.27, 9.29, 9.30), fragility curves can be constructed by following earlier prescriptions (Chapter 6.1). Figure 44 sketches out samples of curves showing the failure probability as a function of time for three different failure events.

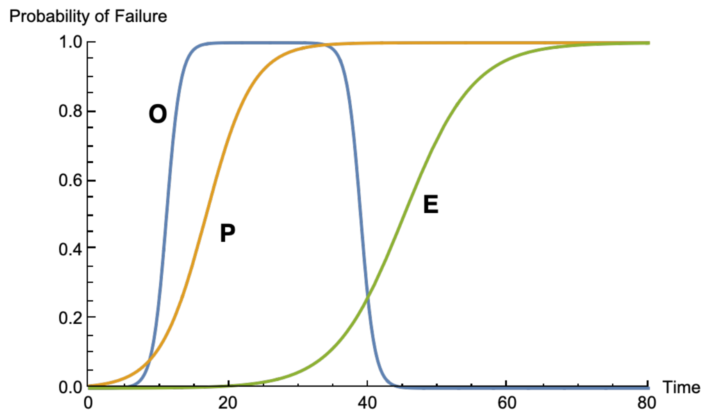


Figure 44: Examples of failure probabilities for different events.

Thus far, we are simply modeling individual events. If we know of the occurrence of an individual event, the above models will provide you with information about the system states. What about the case where we do not know precisely which failure mode can occur? Can we model this type of situations?

The limitations on the availability of information and knowledge led to the use of statistics (data, information, and knowledge) to represent the chance of the concurrent occurrence of the failure modes. Consider the case where vegetative growth on a levee is a possibility, but the precise condition varies according to the environment. While material degradation due to vegetation (**M**) and erosion due to flow (**E**) can result in failures, it is difficult to define precisely their relationships. To determine all the correlations between **M** & **E** can be a challenge. One possible approach is to choose to ignore the details and express the system as a mixed states system in the form of a mixed state density matrix.

If we do not know precisely the occurrences of the failure modes, **M** & **E**, we can model the system as a mixture of both modes, and we have a mixed state system: $\rho_L = p_M |\psi_M\rangle\langle\psi_M| + p_E |\psi_E\rangle\langle\psi_E|$, where $p_M$, $p_E$ are the probabilities for the occurrences of M & E

$$\rho_L = p_M \begin{pmatrix} m_0 m_0 & m_0 m_1 \\ m_1 m_0 & m_1 m_1 \end{pmatrix} + p_E \begin{pmatrix} e_0 e_0 & e_0 e_1 \\ e_1 e_0 & e_1 e_1 \end{pmatrix} \tag{9.31}$$

This formulation can be interpreted as a weighted (statistical) addition of probabilities, in the form of a mixed states density matrix. Here, the difference is that we are weighting the probability of finding the system in certain states that can still be evolving, and not simply a weighted sum of scalar values; we are weighting the states and not the probability. To obtain the probabilities, one performs the usual trace of the density matrix $\langle \mathbf{P} \rangle = tr(\rho_L \mathbf{P})$ for the state or states in question. With the density matrix model for the basic single failure mode formulated, we can now turn to constructing models for systems with multiple failure modes.

9.4.2 Scenario: Concurrent Independent Failure Modes – A & B Separable

For a system that can have both failure modes A and B, the system state vector is constructed from the tensor product of the single failure mode state vectors:

$$|\psi_L\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix} \tag{9.32}$$

$$\left|\psi_L\right\rangle = a_0 b_0 \left|00\right\rangle + a_0 b_1 \left|01\right\rangle + a_1 b_0 \left|10\right\rangle + a_1 b_1 \left|11\right\rangle \quad (9.33)$$

If A and B are independent from each other, then the probability amplitudes are separable. This product state represents a composite system with two totally independent subsystems. The composite probability density can be factorized into two separate components, and the subsystems are not correlated. In this case, operators act on the subsystems independently and separately.

Conventional approach in analyzing the concurrent independent failure modes (this scenario) calls for the use of event trees. For this scenario of a two states system, the model will be a binary event tree (Figure 45). Looking at the state vector $\left|\psi_L\right\rangle = a_0 b_0 \left|00\right\rangle + a_0 b_1 \left|01\right\rangle + a_1 b_0 \left|10\right\rangle + a_1 b_1 \left|11\right\rangle$, the probability amplitudes associated with the basis vectors correspond to the square root of the probabilities at the leaf nodes for the binary event tree. Probabilities obtain from performing the normal projection of the subspace(s) yield the same probabilities found at the leaf nodes. The state vector $\left|\psi_L\right\rangle$ is equivalent to the regular binary event tree. Tensor product of the failure mode state vectors creates an event tree. This is expected as the tensor product enumerates the possible basis states for the combined states for the system AB.

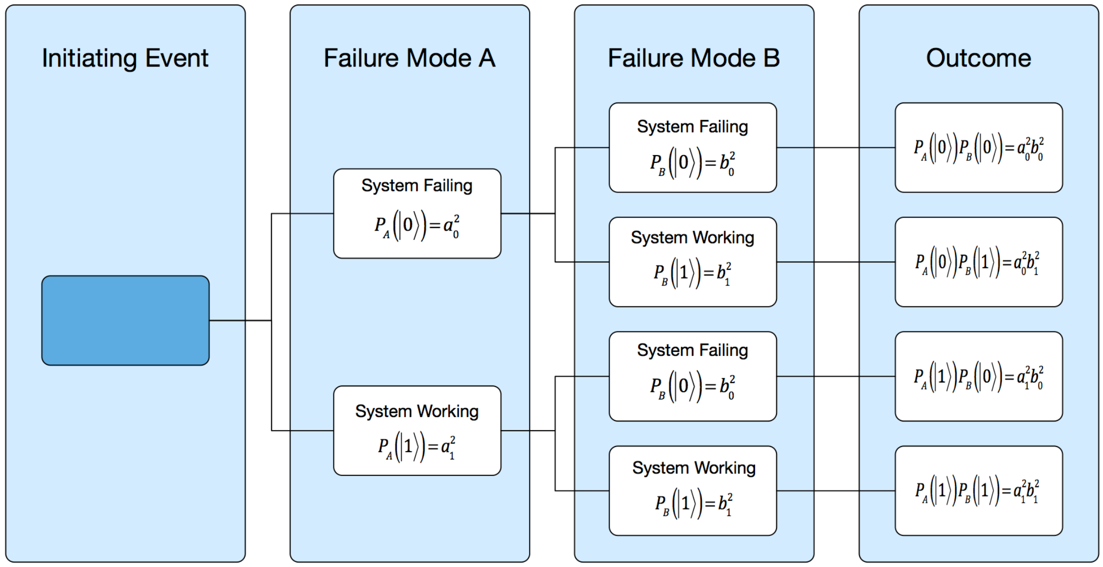


Figure 45: A generic binary event tree.

*Scenario: Concurrent non-mutually exclusive but independent failure modes for a levee.*

The overtopping (O) failure mode and the erosion (E) failure mode can be considered as independent from each other. Erosion is a function of time and overtopping is a function of the height of the levee. A levee can be subjected to both of these events simultaneously. The general state vector for a system that can be subjected to E and O can be constructed from the tensor product:

$$|\psi_L\rangle = |\psi_E\rangle \otimes |\psi_O\rangle = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \otimes \begin{pmatrix} o_0 \\ o_1 \end{pmatrix} = \begin{pmatrix} e_0 o_0 \\ e_0 o_1 \\ e_1 o_0 \\ e_1 o_1 \end{pmatrix} \tag{9.34}$$

$$|\psi_L\rangle = e_0 o_0 |00\rangle + e_0 o_1 |01\rangle + e_1 o_0 |10\rangle + e_1 o_1 |11\rangle \tag{9.35}$$

Since E and O are independent, the probability amplitudes are separable which can be factorized into separate products, and the subsystems are not correlated. The state vector $|\psi_L\rangle$ is equivalent to the regular binary event tree (Figure 46).

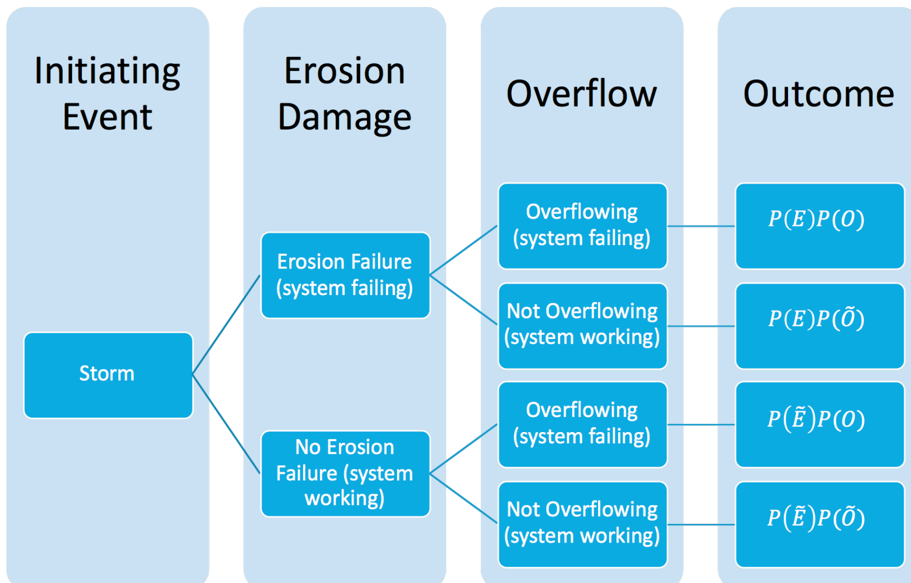

Figure 46: The binary event tree a system with independent E and O failure modes.

If one computes the outcome of the states from the state vector/density matrix, one obtains:

$$P(|00\rangle) = tr(\rho_L|00\rangle\langle 00|) = (e_0 o_0)^2 = P(E)P(O) \tag{9.36}$$

$$P(|01\rangle) = tr(\rho_L|01\rangle\langle 01|) = (e_0 o_1)^2 = P(E)P(\tilde{O}) \tag{9.37}$$

$$P(|10\rangle) = tr(\rho_L|10\rangle\langle 10|) = (e_1 o_0)^2 = P(\tilde{E})P(O) \tag{9.38}$$

$$P(|11\rangle) = tr(\rho_L|11\rangle\langle 11|) = (e_1 o_1)^2 = P(\tilde{E})P(\tilde{O}) \tag{9.39}$$

For this scenario, the key assumption is that the failure modes are separable, which means the failure modes are not correlated. The total probability is, therefore, simply obtained by summing the corresponding states that deemed to be contributing to the failure of the system.

This summation is different than those from other techniques (e.g., event tree) in a number of ways. First, no approximation has been made regarding the magnitudes of the probabilities — no small magnitude approximation. Second, the probabilities are not just failure probabilities associated with a single event isolated to a single component or subsystem. The probabilities at the leaf nodes are probabilities associated with finding the system in a particular state. Contextual information and knowledge are used in the model construction process to build the composite state vectors. Third, the construction of the system model using tensor products ensures that the total probability of all possible events sums to unity, and thus preserving the law of total probability. Lastly, the problem with over counting can be avoided since we are not performing summations of event probabilities. Rather, we are summing the probabilities for finding the system in certain states specified by the initial question asked.

While state vectors contain information that can be used to derive outcomes similar to those of an event tree, they are not equivalent to them. The probabilities derived from the state vectors are by products of the construction of the initial system state model. The model then undergoes various configurations (such as the application of constraints based on information and knowledge) and reconfigurations (change of the system parameters, changes from event occurrences, and changes over time). The changes in the probabilities correspond to the changes in the system. To derive the information applicable to answering questions, such as the probability of finding a system in certain failure states, *project measurements* are performed on the model. The projective measurements (Section 4.3.2) extracts different type of information embedded in the state vector for answers. This circles back to the notion that the question shapes the formulation of the model and determines the type of results to be extracted from the system mode.

9.4.3 Scenario: Concurrent Dependent Failure Modes – A & B Correlated/Entangled

If A and B are not independent, then the failure modes (the subsystems) are correlated in some way and the subsystems are "entangled". With the state vector representation, the probability amplitudes are now represented by joint probability amplitudes in contrast to separable amplitudes. The state vector now takes the more general form:

$$|\psi_L\rangle = k_{00}|00\rangle + k_{01}|01\rangle + k_{10}|10\rangle + k_{11}|11\rangle \tag{9.40}$$

From this perspective, it is no longer relevant to talk about failure mode A and failure mode B as separate entities; rather, it is a composite failure mode, AB, where the composite behaviors dictate the evolutionary behavior of the system states. Event operators now reflect how the composite failure mode AB changes the states. The physics of the system is captured by the operators; the relationships between A and B are embedded within the operators which modify the joint probability amplitudes. The degree of correlation can be represented and coarsely measured by the density matrix properties: $\rho^2 ?= \rho$, $\mathrm{Tr}(\rho^2) ?= 1$ (Section 7.5). Instead of tracking 8 states before when we have separable probability amplitudes, now we can only track 4 states, $k_{00}, k_{01}, k_{10}, k_{11}$. We are losing details about individual component behaviors but retain the ability to make forecast on the composite system. This further support the notion that the concept of operators can be part of the theoretical structure that represent epistemic uncertainty (Section 7.4).

*Scenario: Concurrent not-mutually exclusive and not independent failure modes for a levee*

The Pressure (**P**) and Overtopping (**O**) failure modes are not independent since they are both functions of the height of water levels. The probability amplitudes for the failure modes are not separable:

$$\left|\psi_L\right\rangle = k_{00}\left|00\right\rangle + k_{01}\left|01\right\rangle + k_{10}\left|10\right\rangle + k_{11}\left|11\right\rangle, \text{ with } k_{ij} \sim f(O,P) \sim f(h) \qquad (9.41)$$

In this case, the individual **P** and **O** operators will need to be replaced by a new composite operator, acting on $\left|\psi_L\right\rangle$, which will take on different functional form. The composite operator, in this case, corresponds to the physical model that describe the behavior (change of state) of the system as a whole.

Operators act on the composite state vectors, $\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle$ and not on the sub-component failure modes. We are ignoring the individual component behaviors (e.g. failure mode A, failure mode B) and focus on the composite behaviors. From the mathematical perspectives, the required conservation of total probability helps to constraint the how the operators act on and alter the probability amplitudes for the joint states. The modeling efforts focus on the characterization of the joint states; while one loses the ability to fully differentiate the parts, one gains in efficiency with fewer states to track. In addition, risk assessments can be performed without exact detailed knowledge of the component behaviors. In certain cases, actual observation/measurement of one of the components can specify the other part if the correlations are fully characterized. Decomposition into individual components might not actually add additional meaningful insights and values.

But what is the meaning of the state vector $|\psi_L\rangle$? It is the superposition of four different possible composite states ($|00\rangle, |01\rangle, |10\rangle, |11\rangle$) corresponding to the different possible combinations of failure or not-failure modes. Consider this scenario where a levee is subjected to a high flow volume on one side. At a given time, the levee with two failure modes **P** & **O** is described by $|\psi_P\rangle \otimes |\psi_O\rangle$:

1) The system is working with no failure $|11\rangle$ — e.g. water level below the top of the levee and no seepage (Figure 47). Both failure modes are not occurring. Water is kept on one side of the levee and there is no flooding on the other side.

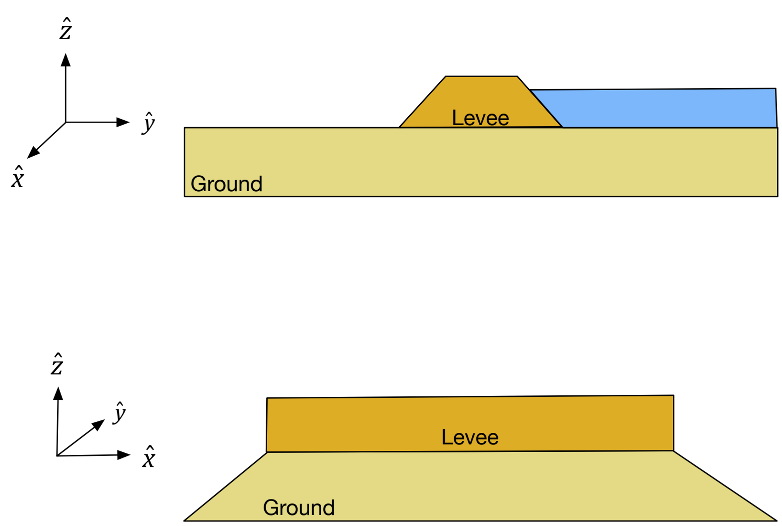


Figure 47: Both failure modes are not occurring.

2) Failing with mode **P**, $|01\rangle$ — e.g. water pressure creates a structural gap with water seepage (Figure 48). Increase in water pressure leads to a higher probability of water seepage for the levee structure. While water is flowing into the other side, the water level is not high enough to overtop the levee and will not lead to the overtopping condition.

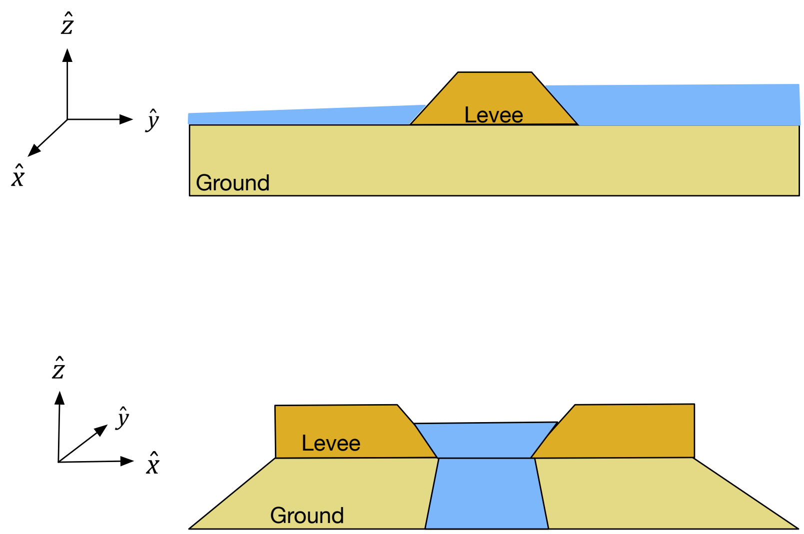


Figure 48: One failure mode, P, is occurring.

3) Failing with mode **O**, $|10\rangle$ — e.g. the levee structure is intact, but water overtopping occurs (Figure 49). Increase in water flow volume leads to a higher probability of water overtopping the levee. However, the structure of the levee is intact and not affected by the building up of water pressure.

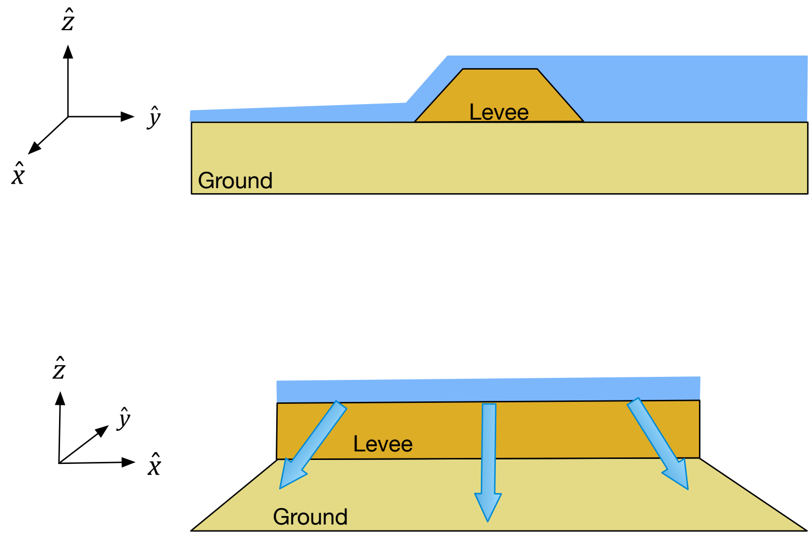

Figure 49: One failure mode, O, is occurring.

4) Failing with both modes **P** & **O**, $|00\rangle$— e.g. both structural failure and overtopping occurs (Figure 50). This is true concurrent failure modes when both modes are actively acting on the system.

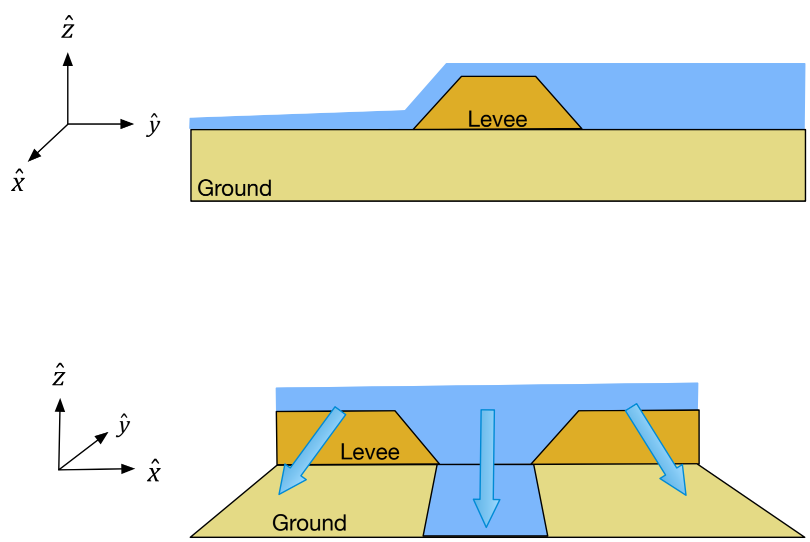

Figure 50: Both failure modes, P & O, are occurring.

The composite system states shifted our perspectives to focus on the modeling of the integrate behaviors, which becomes the basis of handling concurrent failure modes. The composite failure mode (AB) becomes the focus and not the individual modes (A and/or B). This has significant implication on how one approach answering questions about risks.

Previously, the risk question has been redefined as the probability of finding the system in the joint states and not the individual states. In conventional approaches, individual failure modes are considered and evaluated separately (e.g. event tree). In the quantum framework, the behaviors of the joint states take the center stage. Switching from the components to the composite aspect of the systems, the framework points to the possibility that risk assessments can be performed if we have full knowledge of the behaviors of the composite states, even without detailed knowledge of the individual components. In cases where risk assessments on the composite system behaviors are needed, it is the integrated behaviors of the composite system and the composite failure mode that counts.

The quantum approach introduces a different way of thinking: all failure and non-failure modes are considered together as different superposition composite states of a system. The composite states can be in a product state (separable) or entangled state (correlated) (Chapter 5.1). Contextual information and knowledge constrain and configure the system model. The risk analyses focus on tracking the various composite states and their evolutionary behaviors. How much information and knowledge we possess about the system, the components, and their behaviors shape the risk question and determine how the system model configures and evolves over time. In some cases, questions at the component level could be irrelevant and not meaningful (e.g. not independent and/or non mutually exclusive) and one cannot perform the analyses based on the component states. In the case of entangled systems, only the composite states and behaviors are meaningful. In modeling the joint states, the precise behaviors of the components might be ignored. For these scenarios, modeling the joint states behaviors might be sufficient to answer risk questions.

The loss of precision as a result of ignoring the component state behaviors is counterbalanced by a gain of computational efficiency. Instead of tracking 8 states, the composite model only tracks 4 states. To obtain the total failure probability, one calculates the summation of the three failure states $|00\rangle$, $|01\rangle$, $|10\rangle$ probabilities, obtain by extracting the subspace via projections. This is a philosophical change of the mindset on how we formulate the question: instead of focusing on the parts, we focus on modeling the sum of the joint states.

9.4.4 The Calculation of the Total Probability

At this stage, system information and knowledge have been applied to configure the system, and they served as constraints to shape the system model to reflect realistic physical and logical conditions. Additional contextual information and knowledge, such as temporal events and environmental changes, are then applied to bring the system model to the state in time relevant to the question at hand. Changes that happened to the HPS over time are examples of such kinds of contextual information and knowledge. Depending on the nature of these information and knowledge, they are incorporated into the model in a number of ways. Specific events can be modeled as an operator that alter the states of the system (hence changing the probability amplitudes). New states (pure state) or mixtures of states (mixed state) might be incorporated into the model to reflect the physical reality of the situation and condition.

This is an iterative process and the objective is to bring the system model to match the system parameters in the state space and in time, in alignment with the problem statement and questions. For example, a typical risk analysis scenario might ask for finding a levee failing 25 years from initial construction. The levee scenario models developed earlier can be advanced in the time dimension, and projective measurements such as $P(|00\rangle)$ can be performed on the model to extract suitable information to answer the question.

### 9.4.5 The Calculation of the Total Probability

How are the total failure probabilities calculated? The probability of finding a system in the specific states is obtained from the application of the Born Rule (Section 4.1, 4.6) on the projection of those states, which is essentially the sum of the probabilities associated with the failing states.

In the quantum treatment, contextual knowledge and information are captured in the states, in the form of state vectors or density matrix, and the concurrent failure mode probabilities are obtained directly from the joint states

$$|\psi_L\rangle = a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle \tag{9.42}$$

or

$$|\psi_L\rangle = k_{00}|00\rangle + k_{01}|01\rangle + k_{10}|10\rangle + k_{11}|11\rangle \tag{9.43}$$

depending on whether the modes are mutually exclusive, concurrent independent, or concurrent dependent. The density matrix

$$\rho = \sum_i p_i |\psi_L\rangle\langle\psi_L| \tag{9.44}$$

represents mixed states where precise knowledge might not be available.

Keys to this process are the construction of the risk questions, the selection of the risk states, the change of states, the availability and use of knowledge, which collectively go beyond probability distributions and their mapping to represent uncertainties. Fragility curves, binary event trees and the outcome probabilities are all derivative products from the state vectors and density matrices.

The quantum framework tracks the change of system states, which includes both the evolution of the system states and the number of states in the system. The total number of states in a system can change over time, reflecting the actual changes in the system itself. By design, the quantum system is re-normalized when the number of system states changes as a result of the increase or decrease of the number of states from tensor products or projection operations. This ensures that the total probability will not exceed unity.

These two specific characteristics directly address some of the inconsistencies with some conventional techniques in the calculation of the total probabilities. The renormalization of the probabilities to resolve the inconsistency with the computation of the total probability for concurrent failure modes in the case with fragility curves is now a formal structure of the quantum framework. The notion with the potential over-counting of the independent Bernoulli events can be addressed by state vectors where all the different combinations of states are tracked. The process to build composite system model with tensor product further ensures all event states are tracked and accounted for, without double counting.

## *9.5 Insights*

How does the quantum approach compare to the conventional probabilistic approach? The quantum framework provides a rich theoretical structure to model engineering system risk behaviors. The framework provides a structure where solutions from conventional methods can be derived from the quantum framework as products, and new insights can be obtained from the additional features introduced by the framework. However, one must accept the notion that the conventional and quantum approaches are fundamentally different, and side-by-side comparisons between the two might not be as valuable as examining their deviations. The remaining of this chapter focuses on a number of similarities between the two and explore how quantum deviates from the conventional approach.

### 9.5.1 How Risk Problems are Represented

Quantum thinking changes how one models systems since the mathematical framework moves away from conventional approaches. As discussed in earlier sections, the representations between conventional and quantum frameworks are conceptually similar but not the same. The quantum treatment embedded contextual information into the state vectors and the operators. Answers are derived from the system state vectors via extraction processes (such as projections) according to the questions. The formulation of a quantum system model mirrors the question: "What is the probability of finding the system in the working or failing state?" The system is model as a whole and not just a sum of parts.

The quantum model allows the simultaneous representation of all possible system states through superposition. The notion of mutual exclusivity or not becomes a non-issue, since they are all represented in the model as a result of superposition. Independent, dependent, and joint events can be handled by product and entangled states. Information and knowledge constraint and adjust the system states. The exact representation reflects the amount of knowledge we have; we can be as precise as we want if we have the knowledge; or we can choose to ignore some of the knowledge if computational complexity requires of it (at a cost of the introduction of additional uncertainty). Subspace extraction provides the mechanism to select the appropriate and relevant states to represent the system, and the various probabilities can be computed from projecting out the relevant states.

### 9.5.2 How Conditionals are Represented

Conventional probabilistic methodologies model concurrent risk events utilizing the following properties from probability theory:

Complementary:

$$P(\sim A) = 1 - P(A) \tag{9.45}$$

Mutually exclusive:

$$\begin{aligned} &P(A \text{ or } B) = P(A \cup B) = P(A + B) = P(A) + P(B) \\ &P(A \text{ and } B) = P(A \cap B) = P(AB) = 0, P(A \mid B) = 0, P(B \mid A) = 0 \end{aligned} \tag{9.46}$$

Not mutually exclusive:

$$P(A \text{ or } B) = P(A \cup B) = P(A + B) - P(A \cap B) \tag{9.47}$$

Conditional Probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{9.48}$$

Independent:

$$P(A \cap B) = P(A)P(B) \Leftrightarrow P(A \mid B) = P(A) \Leftrightarrow P(B \mid A) = P(B) \tag{9.49}$$

Dependent:

$$P(A \mid B) \neq P(A) \Leftrightarrow P(B \mid A) \neq P(B) \tag{9.50}$$

Joint events, conditionally dependent:

$$P(A \text{ and } B) = P(A \cap B) = P(AB) = P(B \mid A)P(B) \tag{9.51}$$

Joint events, independent:

$$P(A \text{ and } B) = P(A \cap B) = P(AB) = P(A)P(B) \tag{9.52}$$

While concepts analogous to the above are available from quantum probability, their effects are different. For example, in the quantum treatment, mutual exclusivity of the basis state vectors defines the vector space they are in; whereas in conventional probabilities, mutual exclusivity does not define the sample space itself.

With the quantum approach, the question affects the formulation of the problem, and in turn, affects how the system model evolves and change. In the quantum framework, due to how operators and quantum operations can change not only the states but also the system itself, the prior shapes the evolution of the risk questions. This means that the questions we ask will also shape the outcome of the analysis. The right question has to be asked (the right set of risk states need to be identified) in order to get the right answers.

### 9.5.3 How Probabilities are Updated

In conventional probabilistic approaches, probabilities can be updated via Bayes Theorem. The process of updating the probabilities are referred to as Bayesian updating where new data and information are used to update the current probability model. Bayesian technique allows the incorporation of additional information and evidence into the probabilistic models, and hereby conditioning and updating the "prior" with new evidence, represented by a "likelihood function," to derive the "posterior" probability. The posterior probability is obtained from the application of the Bayes' Theorem.

With the quantum framework, both the system and the probabilities can be updated. Technically, the probabilities are not updated directly. The system models with the system states are updated. The system itself can be updated with the additional and removal of system states (pure state) or by adjusting the mixture of states (mixed state). This process is driven by data (new data from physical observations and measurements), information (parameters in models), and knowledge (contextual). These are encapsulated into operators that act on system models, transition system models from states to states. Probabilities derived from the transitioned models are therefore reflecting the updated probabilities.

Furthermore, an additional type of "conditional" comes with the operator-based framework. Operators can be commutative or non-commutative. For commutative operations, the orders of the operation do not matter: $AB - BA = 0$. For non-commutative operations, the orders do: $AB - BA \neq 0$. The commutative property adds dynamic elements to how system can evolve under a variety of conditions.

Some of the conventional processes used to adjust structures, such as the adjustment of event trees, in the quantum framework, it is part of the framework itself. The operators can adjust the trees.

### 9.5.4 How Problems are Analyzed

The quantum thinking changes how one analyzes the problem. The quantum thinking changes how one formulates and analyzes risks associated with complex engineering systems. The issue with computing total probability for concurrent multiple failure modes, regarding the necessity to assume small and independent failure mode probabilities to allow the use of approximation techniques, does not manifest itself in the quantum approach. The process to compute the probabilities in the quantum framework is different than conventional probability. The tracking of the probabilities in the quantum framework is exact, all states are represented and tracked. The calculation of the probabilities does not require approximations. With the quantum framework, the "approximation" in the calculation is more of a reflection on how much knowledge one knows about the system, and not some mathematical constructs designed to address a particular aspect of the problem. With the quantum model, the summation of probabilities does not need to exceed unity since the conservation of total probability is guaranteed by the way of construction.

The problem with the re-normalization of individual failure mode probabilities so that the sum adds to one are also addressed. With the quantum framework, the re-normalization is not about the individual probabilities; *rather, the re-normalization is done as part of the construction of the system model*. The system re-normalization is a recognition of a change of the configuration of the system, a very real physical system. The re-normalization is performed not as a requirement to make the summation to be less than one; rather, it is driven by the reconciliation of the model with the actual physical system. The quantum framework focuses on the state of the system, and probabilities are attributes associated with the system, derived from probability amplitudes.

In terms of the notion of over counting, the argument that uses the independent Bernoulli events to show that the process of enumerating the different probabilities and then summing them will lead to total probabilities greater than 1 is now avoidable. In the construction of the state vectors, the tensor products enumerate and track the changes and perform the counting. The reason why the states will not be double counted is due to the fact that the probabilities extracted from the state vectors do not come from simple addition of probabilities. The probabilities are calculated by projecting the states out from a system that is properly combined and normalized. If the process is properly followed, the states will not be counted more than once!

Turning towards the ambiguity problem, the quantum models can potentially address it by capturing contextual information and knowledge using the additional degrees of freedom resulting from a scalar to a vector framework. The additional degrees of freedom can encapsulate knowledge and information directly in the system models, using the density operator formalism for example. The mixed state density matrix provides different pathways to model systems based on how much information and what type of knowledge one possesses, as well as the relationships between the different system states. In certainly case where the system behaves like an ensemble, some degree of ignorance can be applied to simplify the tracking and computations.

9.5.5 Summary

In this chapter, the problem about finding the total probability for systems with concurrent failure modes was analyzed with the quantum framework. The quantum formulation allows the integration of complex models and physical models in the form of state vectors, which reflect our knowledge about the relationships between the failure modes. Additional information, knowledge, and physical models are expressed in terms of the operators, connecting the physical problem with the change and evolution of failure probabilities.

The quantum framework provides a powerful set of tools to differentiate the characteristics of the system beyond a simple probability framework. Binary event trees can be constructed from the state vectors using tensor products. The quantum framework does not predefine the paths; rather, it maintains that all states are possible as a result of superposition, and the system evolves as events unfold. The probabilities are extracted at the end of the system evolution. The quantum framework will result in total probabilities consistent with the law of probability when framed properly. The quantum calculation relies on projection to extract information; one does not just sum but select what to "sum" over via a purposeful extraction.

The quantum framework is structured such that the different event probabilities, when properly framed and combined, will result in total probabilities consistent with the law of probability. The mathematical operations require the preservation of the law of probability, because it is a requirement to keep track of the states when system evolves, with the system proper normalized to ensure the total number of possible states and the total number of actual available states are the same. The sample space is always the state space. The number of states can change in the quantum models via operations performed on the system state vectors. As a consequence, instead of common cause adjustments where one re-arrange the structure connecting the samples within a sample space, the quantum models simply track the system states, without any re-arrangements. While there are many different individual frameworks that can perform similar analysis, this new quantum approach integrates them under a single framework. Certainly, one can debate and argue the philosophical interpretation regarding special cases and conditions, but the use of the quantum framework provides a solid and coherent foundation where other techniques and mechanisms can build on.

# Chapter 10: Quantum Experimental Model II: Scheduling Risk for Software Development

*If you don't know where you are going, you'll end up someplace else.*

*- Yogi Berra*

In principle, the quantum framework is problem agnostic. Besides modeling physical engineering systems, it can be applied to analyze other classes of problem. This chapter explores the use of the quantum framework to analyze scheduling risks associated with the development of software systems for NASA missions.

Conventional project schedule risk analysis starts with the question: "Given the uncertainties and multiple risks associated with a project, what is the probability that the project will complete within a given cost and duration?" Typically, to answer this question, different probabilistic models are developed for the cost-loaded schedule network scenarios, and statistical simulations are performed on those scenarios to generate duration predictions. The results are then compared to the actual schedule milestone constraints and cost constraints to determine the risk of missing schedule target.

Schedule risk analysis does not and should not be confined to the initial project planning phase. From the project manager's perspective, the key to success rests on how to continuously monitor and gain insights during project execution so that one can ensure the project meets the target schedule. As pointed out earlier, current risk analysis techniques gravitate towards the formulation of uncertainties to model problems. Often, current frameworks and techniques lack comprehensive perspectives on the incorporation of the continuous evolution of uncertainties, the transitions of the "system" from states to states, and the contextual information about the activities into the uncertainty models.

The quantum approach could realign the scheduling risk modeling process to better represent the scheduling lifecycle, reflecting how different software development workflow methodologies and processes dynamically change the scheduling risks. This chapter explores how the quantum approach offers new insights into the analysis of scheduling risks.

## *10.1 Background and Current Approaches*

> *House Spending Bill, Report 116-HR 648*
>
> *James Webb Space Telescope (JWST).—The agreement includes $304,600,000 for JWST. There is profound disappointment with both NASA and its contractors regarding mismanagement, complete lack of careful oversight, and overall poor basic workmanship on JWST, which has undergone two significant reviews because of failures on the part of NASA and its commercial sector partner. NASA and its commercial partners seem to believe that congressional funding for this project and other development efforts is an entitlement, unaffected by failures to stay on schedule or within budget. This attitude ignores the opportunity cost to other NASA activities that must be sacrificed or delayed. The agreement includes a general provision to adjust the cap for JWST to $8,802,700,000, an increase of $802,700,000 above the previous cap. NASA should strictly adhere to this cap or, under this agreement, JWST will have to find cost savings or cancel the mission. NASA and its contractors are expected to implement the recommendations of both the most recent independent review and the previous Casani report and to continue cooperation with JWST's standing review board. The agreement does not adopt the reorganization of JWST into Astrophysics, and the JWST Program Office shall continue the reporting structure adopted after the Casani report and reiterated by the recent Webb Independent Review Board.*
>
> *https://docs.house.gov/billsthisweek/20190121/116-hr648-ExplanatoryStatementDivB.pdf*

In space missions, PRA techniques are routinely used by NASA in assessing mission failure risks associated with hardware and software development programs. The NASA Probabilistic Risk Assessment Procedure Guide for NASA Managers and Practitioners (Stamatelatos and Dezfuli, 2011) contains detailed specifications on how to perform PRAs to analyze different scenarios including Common Cause Failures, Human Reliability Analysis, Software Risk Assessment, Probabilistic Structural Analysis, Launch Abort Models, Physical and Phenomenological Models. For the analyses of software risks, conventional PRA techniques are coupled with Context-Based Software Risk Model (CSRM) technique to further integrate probability and contextual information in the modeling of scenarios.

While PRA techniques are traditionally used in the area of reliability analysis, there are other types of risk problems associated with space mission development that conventional probabilistic techniques can be augmented with alternative techniques to gain insights. A significant number of space missions have not developed and launched on schedule (NRC, 2010), and software development has been identified as a major contribution to schedule delays or cost overruns which become risk drivers (see for example GAO, 2016, 2017, 2018). The standard approach to analyze schedule risks focuses on the identification, quantification, and probabilistic representations of cost and schedule uncertainties. The focus is on the identification of the different types of uncertainties, the occurrence probabilities, and their collective behaviors.

The basic analysis patterns:

1) Collect data about risk events: event and activity descriptions, occurrence probabilities, schedule durations, impacts on the activities and schedules.
2) Model uncertainties with probability distributions and associate the probability distributions to the different events and activities. This step focuses on common cause variations. Aleatory uncertainties are the primary focus, and epistemic uncertainties are included where possible.
3) If only quantitative probabilistic uncertainties (qualitative methodologies are more process driven than computational, see Dezfuli et al., 2011) are considered (step 1 & 2), the event chain probabilities are simulated via Monte Carlo (or other) methods to calculate the probability distributions for the scheduling durations.
4) Additional contextual factors can be incorporated into the model, such as "special-cause variations" (project specific variations, e.g. environmental effects) that might introduce or eliminate uncertainties, and "risk drivers" that add additional weights to the scaling of the probability distributions.
5) Additional activity correlations and branching (for example, in series or parallel) can be introduced to alter the network structure.
6) The schedule network with the final probability distributions is then use as the input model for Monte Carlo (or other statistical) simulations to obtain the distribution for the scheduling durations, which is then used to answer the question of whether the project will complete on target.

For tracking how uncertainties evolve and how risk states change over time, the current approaches often focus on the development of uncertainty models associated with the schedule, and periodically re-run the models during project execution to incorporate new data and contextual information. Bayesian techniques are used to update the probabilities for example, which help to improve the accuracy of the models in reflecting reality. Statistical simulations, such as Monte Carlo simulations, are often used to analyze schedule risks and derive results for complex projects with uncertainties.

Advancements in the field over the past 20 years greatly enhance schedule risk analysis techniques. Simulation techniques have been refined to improve abilities and capabilities to model specific risks and uncertainties, to extend risk models to include multiple activities, in series and in parallel, with different durations and priorities. Advanced techniques, such as coupling Bayesian network with Monte Carlo simulations or system dynamics (Khodakarami et al., 2007; Ordóñez Arízaga, 2007; Fenton & Neil, 2011; Varuttamaseni, 2011) have been developed to better model uncertainties, and it is still an active area of research. The Systems dynamics approach simulates the dynamical behaviors of how the different modeling variables and parameters might change the rate and flow of the network. More data can now be incorporated to provide more comprehensive systematic models to improve fidelity. The use of probabilistic branching structures allows better incorporation and connection of scenarios. Techniques have been developed to allow risk analysis at different levels of granularity and groupings.

For software development scheduling risks, there are certainly many possible reasons as to why a development effort might miss target schedule. There are the internal (controllable) elements, such as development methodology and lifecycles, the external (uncontrollable) elements, such as external dependencies and budget delays, and finally the unknown-unknowns - the unpredictable events. Traditional approaches associate probability density functions (pdf) to duration uncertainties and then introduce these elements into the modeling process. Depending on the problem, these elements might be scaled and organized into different groupings and branching structures such as in the case with multiple related or correlated activities.

Challenges and limitations remain in areas such as the identification and the interpretation of aleatory and epistemic uncertainties, the choice of the probability distribution functions to represent uncertainties, difficulties in separating and isolating concurrent risk impacts within a single event, which might not be feasible (Ordóñez Arízaga, 2007; Hulett, 2017). Statistical simulation techniques also have limitations on tracking states at the event level, since by nature these techniques are based on random statistical simulations; if one cannot fully track state information, prioritize risk mitigations might prove to be difficult since the needed state and contextual information are not available within the model.

These challenges are all very similar to our earlier investigations with engineering systems. Perhaps the quantum approach can also provide new insights? One significant difference is the way the initial line of questioning differs between conventional and quantum approaches. Perhaps the quantum line of questioning might offer new insights on how to manage software engineering scheduling risks?

*10.2 The Problem*

Consider the development of the ground software system for space missions. The software system consists of multiple subsystems and components. To develop the software system, a typical software engineering process starts with decomposing the problem into lower level requirements, and the requirements are then allocated to development teams. After the development teams completed their development at the component level, the components are integrated by the integration and test (I&T) team into the final product. Often the components have different developmental schedules and they can also have dependencies on one another other. The development and integration of all the components into a coherent system within planned schedule present project management challenges.

For space missions, if the software system is not sufficiently integrated beyond a critical operation threshold, the mission cannot proceed to launch. That means the integration process of the software system is connected to scheduling risk, the risk of not be able to complete verification and validation (V&V) on time (the event), which can result in launch delay (consequence), and add cost to the mission. The risk question is therefore: "What is the risk of not having an integrated working system ready by launch?"

The same question can be asked in other ways, such as "what is the risk of the mission failing to launch at the target date?" The consequence of missing the launch schedule would be a launch slip, which carries cost impacts. The probability of the integrated system not ready for launch can be challenging to quantify and model since the probabilities are connected to the development lifecycle methodologies as well as other constantly evolving and changing factors. Parameterization along this line of questioning can be complicated. However, the question can be recast as the probability of finding the system in a certain state since “a system not ready for launch” is one of many possible states the schedule network can be found in. To facilitate further explorations with this notion, a reference scenario drawn from real life space mission development situations is described below.

*The Reference Scenario for the Models* — A typical space mission has three major systems: 1) the payload (satellite, space probe, etc), 2) the launch vehicle (the rocket), and 3) the ground system (the software and hardware that support mission operations). The three systems have to be operational in order for launch to proceed. For this scenario, we shall only focus on the software part of the ground system. The launch go/no-go decision requires answering the question of whether the ground system is ready and operational to support the launch and commissioning of the space mission.

For this scenario, we are interested in assessing the scheduling risks associated with the development of the software system. A standard NASA systems engineering process (NASA, 2017) to ensure that software is working properly is to execute a verification and validation program on the system against requirements and operational scenarios; the verification and validation tests have binary pass or fail states. The progressions of these tests are influenced by various engineering development methodologies. There are different types of modern engineering methodologies in the realization of software products, such as the V-model, Multi-V, iterative, spiral, or agile (Turner, 2007), and these methodologies affect how the schedule network (the system) transition from states to states. As such, the answer to the scheduling risk question must derived from both the states of the system and the process of how to go from one state to another. For this investigation, the conventional V-model (Figure 51) will serve as a reference model for the exploration on how the quantum framework can lead to different insights.

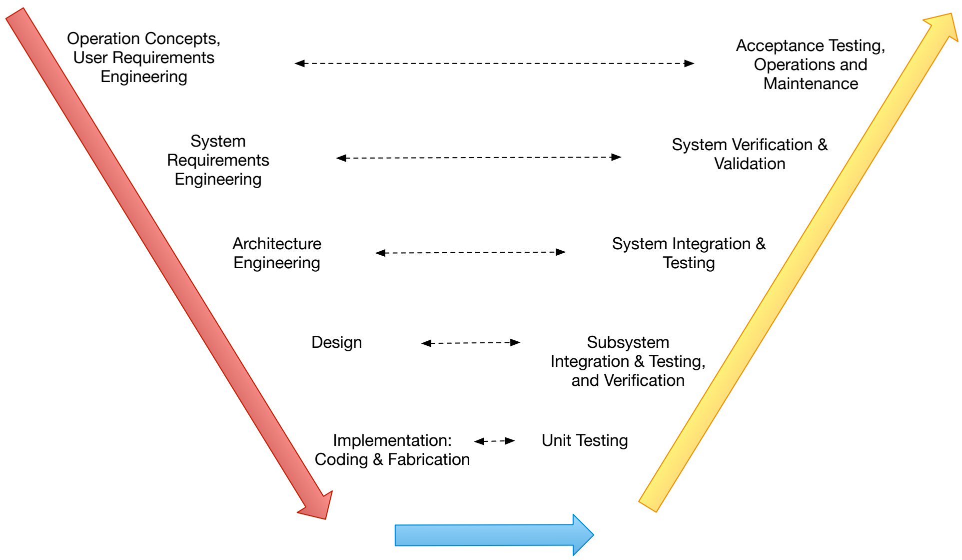


Figure 51. A standard V-model for systems engineering lifecycle process.

The following analysis will consider a simplified and generalized software development scenario derived from the development lifecycle for a subsystem that is part of a current NASA flagship mission ground system. This software subsystem, refer to as W, consists of three components: A, B, and C. These components are developed by three separate teams from three different organizations at different geographical locations. In this scenario, the development of components A and B are in parallels, and the development of component C depends on the completion of A and B. This development process can be represented by the following schedule network diagram (Figure 52).

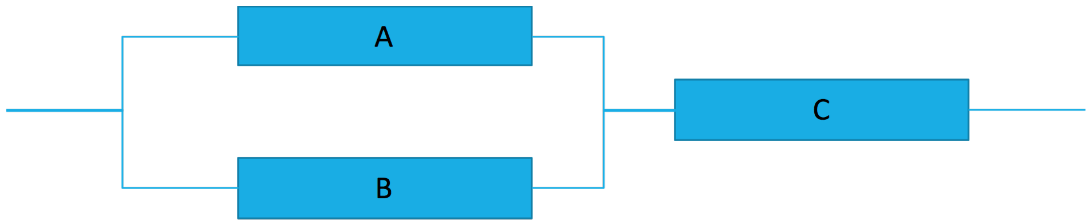

Figure 52: Schedule network for a software subsystem development scenario.

The properties, characteristics, and challenges associated with this scenario:

1) A, B, and C might follow different development approaches, processes, and methodologies.
2) Software development are often iterative in nature. At a given instance, a component can turn from working to not working, and might not follow simple probabilistic distributions.
3) The system requirements can evolve over time. The interfaces between A, B, and C, can change over time.
4) With modern software engineering methodologies, the states of the system can change dynamically throughout the development lifecycle.
5) The organizational and geographic separations between the development teams lead to:
    a. each team might have a different way to conduct their development activities.
    b. processes may or may not be shared between the teams,
    c. probabilistic models could be different between activities, and

d. team deliverables come together only at the established schedule milestones.

6) The development schedule is tightly connected with upstream and downstream dependencies. The interactions can be highly dynamic. Furthermore, statistical data about the scheduling metrics could be limited due to the nature of the work, which are often themselves technological demonstrations and prototypes. Simulation techniques, such as Monte Carlo simulations, might have significant uncertainties associated with the results and outcomes.

7) Information about the rate of development is an important piece of contextual knowledge that is required for strong forecasting and prediction, which can be difficult to build into the statistical simulations.

As mentioned in the previous section, current techniques focus on the modeling of uncertainties with probability distributions and then associated them to activities. In reality, however, the activity durations can be dynamically changed not just due to uncertainties; development methodologies and team dynamics often alter activity durations. Conventional techniques have some limited ways to address this. Conventional schedule risk analysis focuses on modeling the duration uncertainties and simulating the scheduling network to determine the type of risks that can impact the completion of a project. Systems dynamics approach simulates the dynamical behaviors of how the different modeling variables and parameters might affect the interactions of the components, the feedbacks, stock and flow of the component network.

The quantum framework, given its ease of tracking system states, the evolution and transition of states, and the change of probabilities of finding the system in certain states, might offer new perspectives to address this problem. Workflow methodologies and processes can be captured with operators, which dynamically shape the evolution of the schedule network and provide a level of integration where other techniques cannot easily achieve.

### *10.3 The Quantum Model*

The quantum approach takes a different perspective and approach, which starts by redefining the questions. The schedule network can be thought of as a complex system with many different states where each state represents the developmental maturity of a software component, for example. The quantum model represents the various possible states in which the schedule network can be found, and the quantum operators model the behaviors of the system, the change of states, and the dynamic schedule behaviors of the system components.

This perspective reformulates how we approach the scheduling problem and offers a new line of reasoning. The uncertainties are associated not with whether or not a component is developed on schedule (the duration) but the probability of finding the component in a working (completed development) or a non-working (still in development) state. The scheduling risk question is now recast as the probability of finding the system in certain states and the associated consequences.

#### 10.3.1 Basic Construction of the Quantum Model for the Schedule Network

The construction of the quantum scheduling model starts with building the system model from the system state vectors. Scheduling information is built into a system model in the form of evolution operators. In essence, the modeling process focuses on modeling the complex system, the operators evolve the system over time, and the state of the system at different times can be extracted from the model via projection.

The basic construction of the quantum model for the schedule network (Figure 52) follows these steps:

1) Construct the system components state vector: treat each development unit as a component, and let the states be passed or failed (or ready or not ready). The state vectors are the state of the system component at time $t$.
2) Construct the operators. Operators change the components from states to states. One can think of them representing development lifecycles and activities that can change the system states.
3) Construct the density operator for the components. Pure state density operators are used when we know precisely the development lifecycle and duration. Mixed state density operators are used when we do not know for sure the lifecycle and duration parameters. This is similar to the traditional notion of assigning PDF to durations; however, with the density operator, we have additional degrees of freedom to capture different scenarios and the ability to track states over time. The additional capabilities stand in contrast to the conventional approach, where one would associate different PDFs to the

durations, adjust with multipliers and weights, and then perform statistical simulations. The density operator approach incorporates additional information into the model in the sense of uncertainties about lifecycle and duration parameters!

4) Construct the composite scheduling network system (the integration of components). Using the density matrix:

$$\rho_S = \rho_A \otimes \rho_B \otimes \rho_C \tag{10.1}$$

Uncertainty modeling is done comprehensively across step 2-4 via operators.

For this approach, the risk question is reframed as: "What is the probability of finding the system (the components) in a particular sequence of states (can be all working, or some of them might not be if not all the components have to be fully operation at launch, for example) at the target time?" In the quantum framework, the answers can be obtained from the construction of the projection operator to "collapse" the system into the state of interest, thus extracting the probability. The process projects out a targeted state of the system (summarizing the launch state requirements, such as which components are necessary for launch, which are good to have, or which are not necessary), and the measurement **Ri** (Chapter 4.6) performs on the target state gives the risk value.

### 10.3.2 Operators and the Change of States

The schedule network system, represented by the density operator, describes the state of the system. There are a number of activities and events that could alter the system states. Some typical development lifecycle activities are:

- normal development maturity lifecycle as a function of time, typically follows a S-curve profile,

- agile type developmental maturity lifecycle, like a rolling wave, where the maturity of the system is successively approaching completion in cycles,
- spiral type development, which is incremental and similar to agile,
- software fault, failure, and rework that can regress the state of the system and system maturity, which in the traditional schedule network might not be fully captured and represented.

Since the purpose of this analysis is to illustrate how to model the schedule network system with the quantum framework, the rest of this exposition will utilize a simple time (**T**) operator model with a simple S-curve. The choice of modeling with the S-curve stems from the fact that the S-curve behaviors are widely observed in actual software development programs, and the S-curve behaviors have been widely studied in project management and complex systems. In project management, the S-curve is a key element for modeling the probability of potential impacts in cost risk simulations (see PMBOK, 2013, 2017 for details). Figure 53, 54, 55, and 56 are examples of S-curves that are commonly observed.

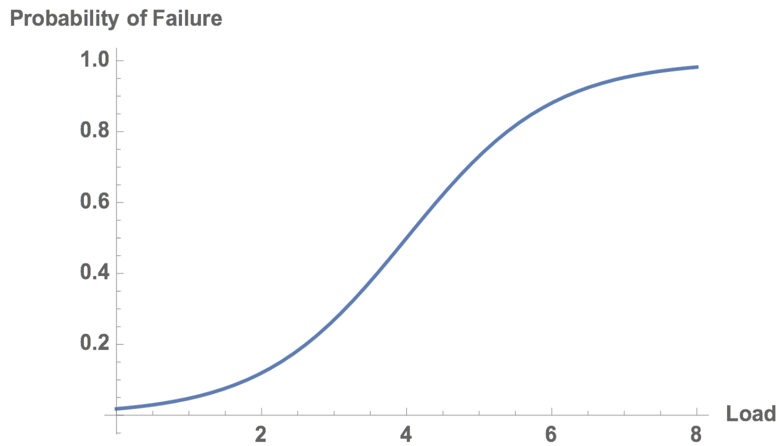


Figure 53: A fragility curve traces a S curve.

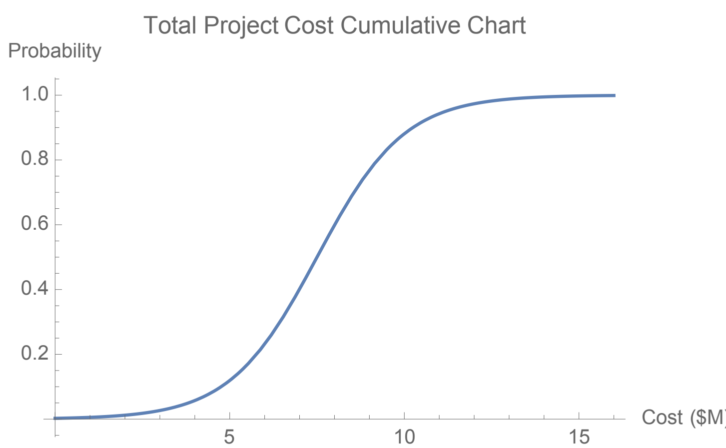


Figure 54: A S-curve for project costing. Adapted from PMBOK 2013 (PMI, 2013).

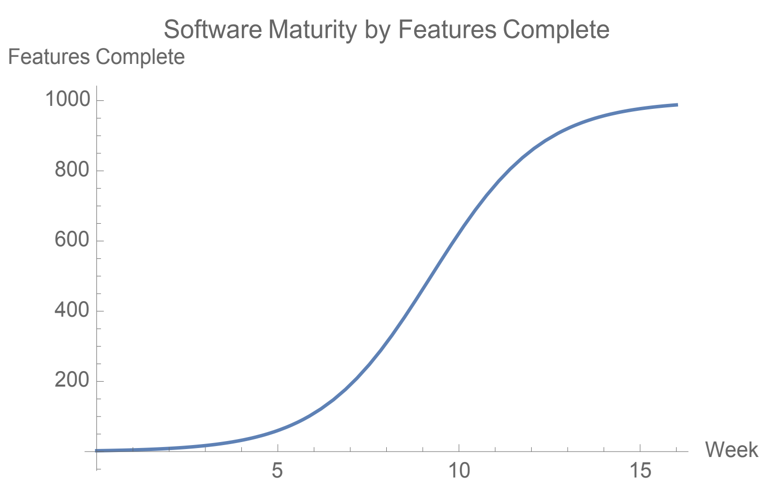


Figure 55: A S-curve representing software maturity. Adapted from Anderson (2004).

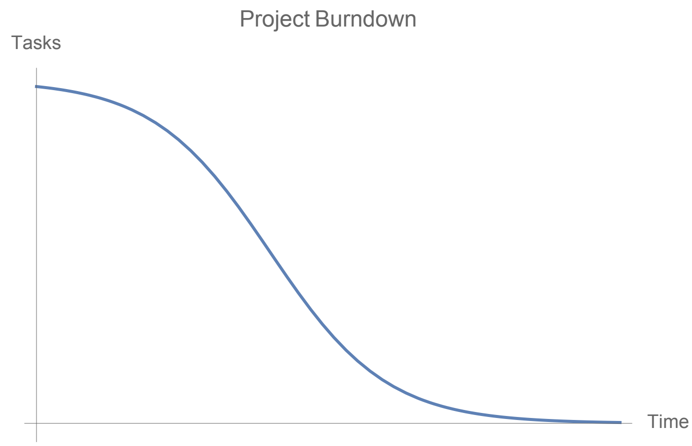

Figure 56: A reverse S-curve for project burndown. Adapted from Sandofsky (2015).

Let the $\mathbf{T}_i$ operator denote general time operations, where *i* is the *i-th* system component. The general **T** operator basically contains the activity rate information, such as the maturity rate of system components, with ties to development workflow methodologies. For this experimental model, we shall consider only the simplest type of time operation, system maturity (M) with a development methodology, which we will denote by the $\mathbf{M}_i$ operator; $\mathbf{M}_i$ is a specific type of the **T** operator. For a system component A, the state of the development will be:

$$\left|A_f\right\rangle = \mathbf{M}_A\left|A_i\right\rangle \tag{10.2}$$

The operator models the time behavior of the development activity maturity, with the interpretation that the successful completion of a development activity will result in a milestone achievement in the form of the delivery of a software component. In this scenario, the $\mathbf{M}_i$ operator traces and reflects the output and outcome resulting from the implementation and execution of a development methodology. Traditional development methodologies such as the waterfall model, can be described by the S curve. Thus, earlier operator models developed with the sigmoid function can be adapted to model the developmental activities associated with a scheduling system component.

The sigmoid curve/sigmoid function, represented by the special case of the logistic function:

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1} \tag{10.3}$$

can be used to model the behavior of the software development maturity process in $\mathbf{M}_i$. In this representation, the x-axis will be time and the y-axis will be a function tracing the maturity of the system, such as the fraction of tasks completed and/or the ratio of (# of tests passed)/(total number of tests). Operators representing different activity workflow processes can be constructed using different types of rate curves. Figure 57 is a composite curve from a composite operator obtained via the tensor product of two operators (representing different S curves for different components) acting on different components:

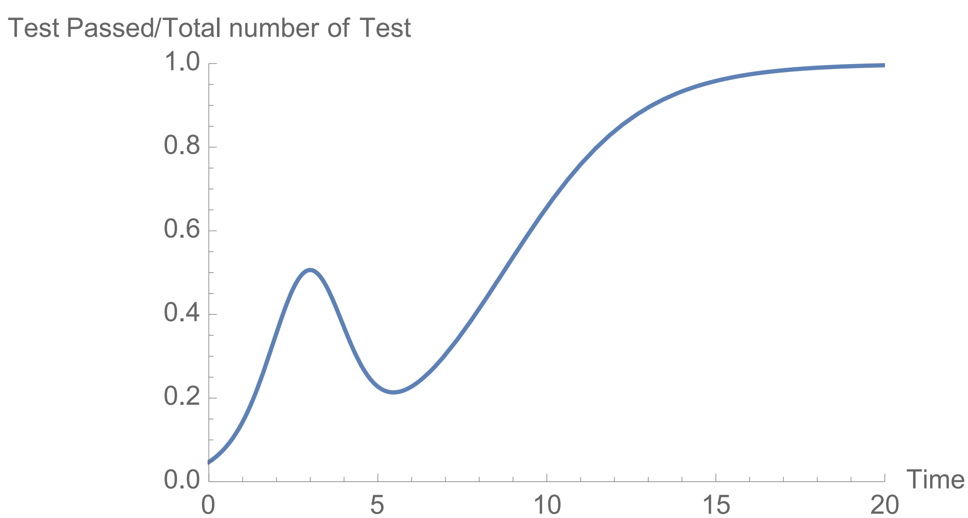


Figure 57: Composite S curves.

From the conceptional viewpoint, this $\mathbf{M}_i$ operator will adjust the probability of finding the system in working or failing state, and the shape of the curve traces the state of the system corresponding to the rate of completion of the scheduled task activities. This $\mathbf{M}_i$ operator correlates to the time behavior of the development methodologies. **M** tracks the completeness of a lifecycle process, such as the development of a software system, by tracking the successful completion of tasks or the verification of requirements. The particular choice of the tracing variables depends on how one defines what constitute an acceptable working state of the system.

A key is the activity operator models the development workflow lifecycle. Essentially, the operator models the behavior of the development methodology and how the methodology affects the schedule as measured by the maturity state of the system. This aspect is not easily captured by PDFs in common schedule risk analysis models, especially in a dynamic fashion; current simulation-based techniques can only capture the effects of how the methodology affects the probability distributions that are used to model the schedule durations. In the quantum framework, the schedule network is actually a by-product of the system model formed out of the tensor products of the system state vectors.

10.3.3 Construction of the Schedule Network Density Operator

To represent the schedule network system, S, we start with the density operator for the individual components A, B, and C, then form the network, the composite system, via tensor product. For this scenario two different methodologies $M_1$ and $M_2$ (e.g. waterfall and agile) can be used for the development processes. The schedule unit (the system component) states are then represented by:

$$\left|A_1\right\rangle = M_1\left|A_0\right\rangle \quad (10.4)$$

$$\left|A_2\right\rangle = M_2\left|A_0\right\rangle \quad (10.5)$$

In an actual development process, precisely which methodology and how they are used can be a form of uncertainty, and in fact hybrid approach where different methodologies are mixed together can be entirely possible. Unless one knows precisely that the development team follows a workflow methodology exactly and meticulously, a mixed state density operator would be appropriate to represent the development state of component A:

$$\rho_A = w_{A1}\left|A_1\right\rangle\left\langle A_1\right| + w_{A2}\left|A_2\right\rangle\left\langle A_2\right| \quad (10.6)$$

where $w_i$ represents the adoption (utilization) probability of methodology *i* in the practice. Similar density operators can be constructed to represent activities B and C. The density operators for components B and C are respectively:

$$\rho_B = w_{B1}\left|B_1\right\rangle\left\langle B_1\right| + w_{B1}\left|B_2\right\rangle\left\langle B_2\right| \quad (10.7)$$

$$\rho_C = w_{C1}\left|C_1\right\rangle\left\langle C_1\right| + w_{C2}\left|C_2\right\rangle\left\langle C_2\right| \quad (10.8)$$

For S, the density operator is given by the tensor product of the component density matrices:

$$\rho_S = \rho_A \otimes \rho_B \otimes \rho_C \quad (10.9)$$

At this point, the density operator $\rho_S$ is the most general and comprehensive representation for the system, containing detailed information about the scheduling network. The computational cost, the information compilation and maintenance cost for this model scale with the complexity of the system.

Consider the following simplified scenario, where the development teams adopt and adhere to a specific methodology, either $M_1$ or $M_2$. In this case, we can model each component in pure state.

$$\left|A\right\rangle = M_1\left|A_0\right\rangle \tag{10.10}$$
$$\left|B\right\rangle = M_2\left|B_0\right\rangle \tag{10.11}$$
$$\left|C\right\rangle = M_1\left|C_0\right\rangle \tag{10.12}$$

The possible states one can find the scheduling system to be in can be formed with tensor product:

$$\left|ABC\right\rangle = \left|A\right\rangle \otimes \left|B\right\rangle \otimes \left|C\right\rangle \tag{10.13}$$

and the possible states are:

$$\left|111\right\rangle, \left|110\right\rangle, \left|101\right\rangle, \left|100\right\rangle, \left|011\right\rangle, \left|010\right\rangle, \left|001\right\rangle, \left|000\right\rangle.$$

With the states defined, we then turn to specifying the questions one can ask of the system that can help us to analyze and understand the schedule risk associated with this development program. For space systems development, it often comes down to: Is the system ready to support launch? But how does that translate into in the context of this quantum model? The system needs to be in working states for some specific combination of $\left|ABC\right\rangle$, as dictated by the launch and commissioning requirements. For example, the requirements can simply be that for a system to be ready for launch support, only components A and C have to be fully developed and verified to be working at launch time, *t*, which correspond to the states $|111\rangle, |110\rangle$.

10.3.4 Answering the Schedule Risk Question: Projection of Particular Events

In traditional simulations, statistical runs enumerate what the system will be under certain assumptions and conditions, like in the process of going down the paths of a tree. At each node a choice is made in keeping the linear pathway from the head node to the leaf node. Most software development workflows do not follow this simplistic linear deterministic transition process. Whereas, the quantum approach aligns with how schedules behave in the real world, especially with modern development workflow like the agile methodologies. The quantum approach tracks the evolution of all the states, with the analysis focusing on what the intended end state one would like to achieve or expect. The probabilities are then extracted from the system via projections.

To determine the schedule risk with the quantum model, the first step is to identify what constitutes the target end state. Certainly, a definitive end state is when all the components are working, which is simply the $|111\rangle$ state for the three components scheduling network system. But in reality, for some space missions it is entirely possible that some of the components do not have to be in a fully working state before launch. In fact, some of the ground software components cannot be fully realized until actual data can be obtained from the spacecraft at the target location and operational environment; in such case, say if component B is not required at launch, then the desirable states would be $|111\rangle$ or $|101\rangle$. Once the desirable end state of the system at launch has been identified, measurement and projection operations can be constructed and performed on the system model.

The probability of the scheduling system at the desirable end state can be obtained by first transition the system from the initial states to the target end states via the application of the different transition operators. Essentially, one is transitioning the system from $t_0$ to the planned completion milestone, say $t_f$. But the quantum approach is different in that all possible states, the superposition states, are evolved; in contrast, conventional approaches nominally follow linear pathways. After the system transitioned into the targeted state at $t_f$, the projection operator is then applied to the system and collapse it to the target state, such as $|111\rangle\langle 111|$, which gives the probability of the program hitting the milestone.

The quantum thinking is about identifying the states in which the system can be found, which is informed and shaped by prior experience. Prior knowledge also influences the formulation of the problem and the construction of the state representations. The following scenarios highlight how the selection of the end states in the query determines the outcome.

*Scenario: All components have to be fully operational at launch. What is the probability of finding the system ready for launch?*

In this scenario, at launch time $t_L$, A, B, and C should and can only be in one state, the $|111\rangle$ state. The state vectors at $t_L$ are given by:

$$\left|A(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|A_0\right\rangle \tag{10.14}$$
$$\left|B(t_L)\right\rangle = \mathbf{M}_2(t_L)\left|B_0\right\rangle \tag{10.15}$$
$$\left|C(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|C_0\right\rangle \tag{10.16}$$

and

$$\left|ABC\right\rangle = \left|A(t_L)\right\rangle \otimes \left|B(t_L)\right\rangle \otimes \left|C(t_L)\right\rangle \tag{10.17}$$

The probability that the system is ready for launch is then:

$$\left\langle ABC | 111 \right\rangle \left\langle 111 | ABC \right\rangle \tag{10.18}$$

■

*Scenario: Only components A and C have to be fully operational at launch. B needs actual flight data at the stationary orbit to complete calibration for operation. What is the probability of finding the system ready for launch?*

In this scenario, there is no requirement for component B to be operational at launch. There is no requirement for a specific state of B at launch, so both working and not working states for B are acceptable. Therefore, at launch time $t_L$, A, B, and C can be in these two states: the $\left|111\right\rangle$ and the $\left|101\right\rangle$ state.

For this scenario, the state vectors at $t_L$ are given by:

$$\left|A(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|A_0\right\rangle \tag{10.19}$$
$$\left|B(t_L)\right\rangle = \mathbf{M}_2(t_L)\left|B_0\right\rangle \tag{10.20}$$
$$\left|C(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|C_0\right\rangle \tag{10.21}$$

and

$$\left|ABC\right\rangle = \left|A(t_L)\right\rangle \otimes \left|B(t_L)\right\rangle \otimes \left|C(t_L)\right\rangle \tag{10.22}$$

The risk question is asking for the probability of finding the system in one of the two required states. The probability that the system is ready for launch is then:

$$\langle ABC|111\rangle\langle 111|ABC\rangle + \langle ABC|101\rangle\langle 101|ABC\rangle \tag{10.23}$$

■

*Scenario: Only components A and C have to be fully operational at launch. B needs actual flight data at the stationary orbit to complete calibration for operation, but B has to be 80% complete. What is the probability of finding the system ready for launch?*

In this scenario, at launch time $t_L$, A, B, and C should and can only be in two states: the $|111\rangle$ and the $|101\rangle$ state. Furthermore, there is a requirement that B has to be found in a maturity state with 80% functional completion. This particular requirement alters how the risk question is being asked, which also affects how we formulate the composite system. With the requirement, instead of considering both the $|111\rangle$ and the $|101\rangle$ state, we now should only look at the $|111\rangle$ state. The end state of the system, the $|111\rangle$ state, is different than the end state from the other two scenarios. For subsystem A and C, the $|111\rangle$ is defined as the system at 100% completion; whereas for subsystem B, the $|111\rangle$ is defined as the system at 80% completion. The maturity operator, $\mathbf{M}_{B2}$, will need to be constructed differently to take into the account of the 80% maturity requirement.

The maturity operator, $\mathbf{M}_{B2}$, can be constructed in a number of ways, depending on how one traces system maturity. For example, one can trace system maturity by mapping to tasks completion, requirements fulfillment, or the number of verified and validated requirements. Regardless of the choice of the tracking parameters, the key is to define the $\left|111\right\rangle$ state with the 80% completion as the target. In essence, we are scaling the $\theta$ function to match to the 80% completion requirement and use that in the construction of the $\mathbf{M}_{B2}$ operator.

The state vectors at $t_L$ are given by:

$$\left|A(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|A_0\right\rangle \tag{10.24}$$

$$\left|B(t_L)\right\rangle = \mathbf{M}_{B2}(t_L)\left|B_0\right\rangle \tag{10.25}$$

$$\left|C(t_L)\right\rangle = \mathbf{M}_1(t_L)\left|C_0\right\rangle \tag{10.26}$$

and

$$\left|ABC\right\rangle = \left|A(t_L)\right\rangle \otimes \left|B(t_L)\right\rangle \otimes \left|C(t_L)\right\rangle \tag{10.27}$$

The probability that the system is ready for launch is then:

$$\left\langle ABC\middle|111\right\rangle\left\langle 111\middle|ABC\right\rangle \tag{10.28}$$

■

In the last scenario examples, an important point is being made. The quantum approach does not necessarily follow a deterministic path, and prior knowledge, experience and expectations can influence the conclusion by affecting how the questions are being casted and how the systems are being constructed and evolved. In conventional probability, conclusions are drawn from data, such as the frequency of occurrence of events. Bayesian thinking adds a new element to this by introducing prior knowledge and expectations into the data via Bayesian updates, and conclusions are drawn from data and prior expectations together. The quantum framework takes this to another level where prior expectations actually may influence the formulation of the questions, directly shaping the structures of the data and the type of answers one might derive from the data, which then conclusions are drawn.

### *10.4 Insights*

The quantum approach offers interesting insights which differ from those of conventional techniques.

*Reformulate the problem and open new line of questioning and reasoning*

Whereas conventional approaches to scheduling risk analysis focuses on asking the probability and confidence of a project completing on target, the quantum reformulation treats the scheduling network as part of the system model and ask the chance of finding the system in a certain state or states. The schedule network is a by-product of the system model. The uncertainties are associated not with whether or not a component is developed on schedule but the probability of finding the component in a working (completed development) or a non-working (still in development) state. The scheduling risk question is now recast as the probability of finding the system in certain states, of which impact information is considered not just as a scaling factor but a deciding one.

As brought up earlier, the selection of the target system state incorporates an impact decision. For the launch decision of a spacecraft, not all ground software components have to be fully functional in order for the issue of a go for launch. Certain hardware in the spacecraft could be failing, but the overall system can still be viable for a "go" for launch. The quantum framework can accommodate these in a straightforward systematic manner (the construction of projectors to extract subspace) when compare to conventional techniques.

The quantum approach allows us to explore a few more lines of questioning and reasoning. Traditional methods answer the question about the schedule. In order to explore different conditions and scenarios, different schedule networks will have to be constructed, and different simulation runs will need to be executed. In contrast, the density matrix represents and captures all possible states of the system, which means all of the scenarios are simultaneously represented in the density matrix, and the system can be queried on multiple scenarios and perspectives at the same time, with the benefits of gaining procedural and computational efficiencies.

Furthermore, the original schedule network is now modeled as a system in a complementary problem space. In this complementary problem space, operators represent processes, which *directly* model development methodologies and lifecycles. Operators change the states of the system, and the operator framework has more dimensions and degrees of freedom when compare to conventional approaches where the methodologies can only be represented *indirectly* by PDFs. Traditional approach focuses on modeling the uncertainty of the activity durations. Quantum approach focuses on modeling how the system (and the uncertainties) changes in time, or how the uncertainties change as the system changes and evolves in time. *This is a key philosophical difference from conventional approaches!*

*New Capabilities*

New risk analysis capabilities are introduced as we adjust to modeling how the system changes in time. This framework supports more intricate modeling of the actual processes, events and activities. The software development methodologies can be directly modeled to evaluate how they can impact the chance of realizing the system on schedule. Furthermore, the quantum framework adds another dimension for representing complex uncertainties with imprecisions. The density matrix representation allows some degree of impreciseness, which makes it possible to derive useful results without specifying precisely the exact type of methodology the development team employs. The model can capture a specific and well-defined workflow, but it also has the flexibility to handle hybrid development workflow methodologies. Before, modeling the individual methodologies can be complicated and are limited to adjusting the PDFs.

The capability to query individual or combination of system states also changes how one formulates the questions. This encourages analysts to expand both breadth and depth in their investigation by probing the system from different angles and ask different kinds of questions, such as uncertainties introduced by human factors and lifecycle processes. The probabilities derive from the density matrix is the starting point; different operations can be performed on the density matrix to alter the scenarios to gain additional insights. Conventional techniques, from Bayesian updates to common cause adjustments can be incorporated into the operators and projectors, adjusting the number of system states, and rate parameters (S-curves for process maturity for example) can be fed directly into the model. Traditional statistical simulation-based solutions now have an additional analytical extension.

A commonly used tool to communicate project completion rate, the burn-down charts and curves, can be derived from the quantum model. In its most basic and simplistic form, burn-down charts and curves are simply the different projections of the state vector over time. More importantly, with operators, different scenarios can be evaluated (e.g. different combinations of states, $|111\rangle$ or $|101\rangle$, for example), very much in the same way we discussed earlier with fragility curves and how to combine probabilities.

One thing not to be missed here, is that the activity operator models the development lifecycle and is essentially modeling the development methodology. How the development methodology affects the schedule has never been fully incorporated in other schedule risk analysis models in this fashion; the most is to capture the effects of the methodology in affecting the probability distributions modeling the schedule duration.

*Getting useful information without full knowledge; flexibility to explore scenarios*

The density matrix formulation captures multiple scenarios within the model, allowing some degree of mixing (hence mixed states), such as the mixed representation of several software development methodologies. The operators, with temporal information embedded in it, modulate the behaviors of uncertainties over time. Different conditions can be applied to change the states captured by the density matrix, and the states can be evaluated at any given time slice.

The formulation does not limit to questions about meeting project schedule milestones, but the risk scenarios can be decomposed and analyzed by components or a collection of components, which provides great flexibility in dealing with complex dynamic systems and processes. The risk question steers the construction of the projection operator to the probability of finding the system and the components in certain states. The projector itself is a specification of the consequence applicable to the risk decision, e.g., in the earlier scenario examples, the selection of which components are required to support launch reflects the incorporation of consequence into the formulation of the risk question itself. The value derived from the projection operation itself will be the choice of what is acceptable risk (or not)!

A different way to think about the density matrix representation is that instead of associating statistical distributions, typically triangular distributions or other pdfs, to activity durations, the quantum analysis focuses on the modeling of the system and its behavior over time. The schedule network is a product, an attribute, a part of the system model. An analogy would be like procedural programming vs. object-oriented programming (OOP). In procedural programming, the connecting steps represent the model; whereas in OOP, the objects and how they interact, together, is the model. In OOP, one queries the object to derive state information based on the attributes, that change over the course of interactions between the object with other objects. The interactions between the objects are not necessarily deterministic; an object's attribute values constantly evolve according to its interactions with others. These interactions alter and change the states of the system of objects. The exact behavior of the system is not defined *a priori*.

*Software verification and validation*

While this Chapter aims at exploring how quantum probability can be applied to model scheduling risk, the approach and interpretation have broader context beyond scheduling. This Chapter begins with the risk question: "What is the risk of not having an integrated working software system ready by launch?" This question typically is answer by the verification and validation (V&V) of the software system. In this context, what type of uncertainties are associate with the V&V processes? How can quantum probability model these uncertainties?

The answers to these questions depend on the context of the problem, and how one interprets the probability framework under that context. Three areas are briefly explored here: 1) requirement verification in system realization, 2) complex event-driven interacting systems, and 3) software system operating in the space environment.

In the realization of software systems, conventional processes focus on verifying the software system against requirements and validating the system against use cases. Requirements come with different definition and specifications, and this discussion elaborates on the verification of system performances over time. In this case, the uncertainties can be associated with system processing performance scalability and stability over time. System performance can degrade over time for a variety of reasons, such as computational hardware failures, new software defects and bugs introduced from software patches, logical fault, or algorithmic induced latencies. Probabilistic models have been developed to assess these scenarios based on test data samples. Quantum probability can potentially be applied to model the intrinsic failures, such as random hardware faults and failures (aleatory), and event-based failure modes, such as performance degradation due to algorithmic latencies (epistemic).

The second area relates to the modeling of complex software system failure probability. One common problem for modern software systems lies with the difficulty to verify all possible event modes, pathways, and scenarios for event-driven software systems. This can be particularly daunting for highly interactive event-driven complex software system where the enumeration of all possible pathways can be a non-polynomial (NP) problem (takes a very long time). Probabilistic models have been developed to represent the frequency of event occurrences based on statistical sampling with smaller scale systems, which provide the means to quantify the uncertainties. Quantum probability can also be evaluated as a probability framework to model these types of problems, with failure probability (aleatory) associates with completeness measures and algorithmic complexities map to epistemic uncertainties.

Lastly, software systems operating in space environment present an interesting case with a different range of failure modes and characteristics. A software system operating in the space environment can be subjected to a number of failure situations. Failures due to software algorithmic errors and out of bound conditions are relatively common. These types of failure modes are epistemic in nature, such as when operation parameters go out of bound, leading to an unforeseen failure mode. Another type of failure, the Single Event Upset (SEU), is more random in nature. SEU events are caused by energetic ionizing particle (such as cosmic rays) striking a microelectronic device, such as the spacecraft computer, causing a bit-flip in memory that can potentially lead to software errors. Quantum probability can potentially be evaluated as a probability framework to model these types of system as well, where the SEUs are represented in the state vectors, the failure modes to be represented by the operators, and discrete probability distribution of failure modes in the density operator.

The above explorations point to a common underlying theme. Quantum probability can be an alternative probability framework for modeling uncertainties in a variety of situations and problems. The interpretation of quantum probability in the context of the problem plays a significant role in determining its applicability and usefulness in the modeling process. For certain problems, conventional Kolmogorov based techniques might be the optimal tools, but quantum probability can also be explored for problems with characteristics that are in alignment with the quantum framework.

*Concluding thoughts*

Conventional techniques place strong focus on the formulation of uncertainties. While accurate representations of uncertainty are important, the quantum framework also captures how uncertainties change and evolve over time via operators that model processes. The quantum approach extends the perspectives to incorporate additional contextual information about the behaviors of the system's states and their transitions. This framework tracks the change of probabilities of finding the states besides tracking the states. Such is a fundamental change on how we view a schedule network. The approach illustrated in this chapter view the schedule network as a system with states and dependencies. Temporal information and knowledge are captured and embedded in the operators to moderate state transitions. The models describe how each component transitions from states to states at different time.

The change of perspectives also enables the incorporation of additional elements such as workflow models into considerations. Workflow models to describe different development methodologies can be directly integrated into the process to "evolve" the probabilities, and not just statistical data from *a priori* or *a posteriori*. This allows additional degree of freedom to test and evaluate various constraints to improve the schedule risk assessment results. The direct incorporation of the "questions" in shaping the analyses aligns and reflects better with the dynamic nature of complex engineering systems, where traditional deterministic flow might not necessarily be true (e.g. event driven workflow).

The new framework also facilitates the notion of making decisions with incomplete information, providing a bridge between pure statistical methods and non-pure probabilistic methods (recall earlier references). The question risk managers might ask: "Can I do a quick risk assessment without knowing everything about the requirements? As a manager, I need to be able to make decisions with incomplete information" can be partially addressed. The quantum formulation opens up new lines of questioning and thinking beyond what conventional techniques provide.

# Chapter 11: Conclusions and Future Directions

*You can't connect the dots looking forward; you can only connect them looking backwards. So you have to trust that the dots will somehow connect in your future.*

*- Steve Jobs, Stanford Commencement Speech 2005*

Exploring different and new approaches to represent uncertainty and risk for complex engineering projects has been an ongoing quest for risk researchers. Challenges with modeling complex engineering problems continue to drive researchers to explore ways to augment probabilistic techniques in the representation of uncertainty. One research avenue is alternative probability theories within the spectrum from pure to imprecise probability theories.

In this dissertation, the theory of quantum probability and the associated tools of quantum mechanics were applied to investigate their usefulness as a risk analysis and assessment tool for engineering. This research investigates the application of the quantum framework to study engineering systems where the tracking of states and contextual knowledge can be challenging. The dissertation lays out the case for why the quantum framework can provide new tools to capture information in engineering risk system models; how quantum tools monitor, change and evolve the system; and how this characterization of information can support risk analysis.

From a theoretical perspective, a case is argued for the quantum framework's coherent and integrated representation of aleatory and certainty types of epistemic uncertainties. These models can capture different aspects of system states, and more importantly, the uncertainties about the system states.

Fundamental aleatory uncertainties about system states are contained within the state vectors in the form of probability amplitudes. Epistemic uncertainties about models and parameters are partially quantified by operators (with the physics models built into them) acting on state vectors. Epistemic uncertainties are also captured by the density operator, interpreted as a statistical ensemble of possibilities. Finally, subjective beliefs and expert opinions are incorporated in the density matrix, using the probabilities as weighing functions.

Compared to conventional frameworks, the quantum framework potentially brings benefits and extends risk modeling capabilities in several directions. The quantum framework provides a new way to encode information in a high capacity format as the density operator. Event information and physics are encoded in operators, which process the higher capacity information in different ways not easily available with conventional techniques. Ignorance or lack of information can be incorporated in the encoding of system states with the density matrix. The framework extends the common data and statistics driven mapping of PDFs to scenarios, to a function and physics-models driven approach; this provides a different treatment for static and dynamic uncertainties.

The quantum approach, besides offering a new set of tools, also introduces a different way to construct questions about risk in engineering systems. Risk questions can evolve as the problem and system condition changes. The changes become new constraints, adjusting the system states. Results consistent with the new system states are derived from the evolved system model as by-products of the construction of quantum risk models themselves.

This research suggests that the quantum framework may have broader applications beyond the study of risk for engineering systems. The use of the quantum approach to model software project schedule risk with the incorporation of workflow models points to the possibility of utilizing this framework as a general theoretical framework for the study of different types of risk problems. The framework can serve as a platform to advance research in a number of areas, such as the framing problem: "the answer and conclusion depends on how the question was asked and how the data were structured, organized, and presented" (Fenton & Neil, 2013), and the lack of a scientific platform to describe uncertainty, knowledge and risk (Aven & Zio, 2014).

*New Knowledge and Limitations*

Quantum probability is a mathematical framework for describing quantum systems in physics. Others have interpreted this mathematical framework in the context of cognitive and decision science. This dissertation presents an original interpretation of the quantum probability framework in the context of analyzing risks for physical macroscopic engineering systems, levees and floodwalls, and risks for software development processes. Quantum probability, as a mathematical framework, does not have any contextual notion about macroscopic engineering systems or software development processes. This work provides an original interpretation, proposing the corresponding mathematical representation of the risk equation in the context of quantum probability — the Risk Operator, and the associating computational algorithms for representing aleatory and epistemic uncertainties in system models, the transition of system states, and the equation for calculating risk values for dynamic systems.

This research work also gained insights in understanding the potential limitations of the proposed approach. First, in order to build the quantum models, sufficient knowledge about the systems, events, and system dynamics have to be available for the construction of the state vectors and operators. This requirement itself is often a significant roadblock in the construction of a viable model. Not all engineering problems can be formulated with the quantum framework.

Second, while quantum models with the combination of two probability frameworks might provide some computational efficiency for certain problems, general quantum models might not scale well in terms of computation complexity with *classical computation.* To model complex interacting composite engineering systems, the dimensionality of the system model can grow exponentially, and the corresponding matrix size can present computational challenges. While there are algorithms and techniques to manage and mitigate the sizing of the models, nonetheless the sizing problem is still a limitation with the current *classical computing platforms*.

The computational scalability limitation can potentially be mitigated with the advance of *quantum computing*. Quantum computers offer a different computational platform with algorithms and paradigms that are capable of performing calculations that classical computers might take astronomical timescale to complete. While the technology is still at its early demonstration stage, recent rapid advancements in the field by companies like IBM, D-Wave, and Google are bringing quantum computing steps closer to the general engineering community. This is the time for us to be ready for quantum, for which this research work hopes to contribute a small step.

*The Tip of the Iceberg*

This dissertation attempts to demonstrate concepts, illustrating the setup of a risk model with quantum probability and the quantum mechanics apparatus. The quantum framework is rich with features, and as a conceptual framework, it has the potential to be used for modeling a wide range of problem.

To fully realize the potential, new research will need to take place beyond this dissertation. At this stage, the conceptual model here has not been tested against a broader range of problems. This work suggests that the framework can reproduce and replicate products from common risk analysis techniques. Furthermore, the framework allows the incorporation of new elements, offering new capabilities to augment conventional techniques. These capabilities will need to be explored to assess the potential of the approach.

Follow on research can move in at least three different directions. One direction is to experiment with the above models and see how well it can make predications for complex modern storm protection systems. The simplistic models developed in this work can be refined and adapted to address new problems and to make predictions that can be compared with new observations and data. New observations and data can in turn be used to refine and enhance the quantum models, and Bayesian concepts of using new data to update a prior can be applied to the probabilities in the density operator and the probability amplitudes.

A second direction is to extend the application of the quantum framework to study problems from other engineering domains, such as agile software engineering workflows and practices. Specifically, common risk analysis approaches have been difficult to apply to agile managed projects and it is possible that quantum approaches may bring new capabilities to analyze the problem. This work has shown that the framework can be adapted to work with other risk problem types, such as scheduling networks. Future work can extend the scope to over a wider range of engineering and project management problems.

Finally, the third direction is to continue to map out the various theoretical quantum constructs and explore how they can be used to perform risk modeling. This dissertation proposed the application of quantum probability as a mathematical framework to represent aleatory and epistemic uncertainties using the density operator. The density operator also has other properties that match with the characteristics of other problem domains. In earlier chapters one came across the case where density matrices derived from different mixed states can be identical, leading to ambiguity. Perhaps quantum structures such as the density operator and matrix can shed new lights on the ambiguity problem. In another front, to account for *unknown-unknown* in analyzing multi-hazard risks, researchers have been exploring different concepts related to the entropy of information, the measurement of disorder such as Shannon Entropy, to quantify the quality and value of information available for use in making risk management decisions. Alternatives such as *Decision Entropy* (Gilbert et al., 2016) have been proposed to characterize the quality of available information. The quantum framework provides potentially another alternative as a concept of entropy can be formulated using the density operator. The von Neumann entropy, *S*,

$$S = -tr(\rho \ln \rho), \tag{11.1}$$

in which $\rho$ is the density operator, extends the classical Gibbs entropy (a classical measurement of disorder in a system) into the quantum realm. This von Neumann entropy opens up an entirely different area for researchers to explore different ways to provide quantitative measures for degree of knowledge and information about the state of a system, analogues to the Shannon Entropy and Fisher's Information Measures.

This is possibly only the tip of the iceberg, with many directions awaiting others to explore!

# Appendices

# Appendix A: General Survey of Current PRA Techniques

This appendix provides a general overview of some current probabilistic based risk assessment methodologies and techniques. Classical probability techniques and methods are generally based on the works of Bernoulli (1738), Laplace (1814), and evolved over the years into the modern form by theoreticians such as Keynes (1921), Jeffreys (1939), Cox (1961), Jaynes (1957, 1968, 2003), and the current modern PRA formalism is usually based on the Kolmogorov probability framework (1933). However, there are several "interpretations" of the concept of probability (Hájek, 2012), and they generally fall into two main classes, the frequentist and the Bayesianism. This section will start with a brief review of the philosophies, and then proceed to review the PRAs techniques.

## *A.1 Frequentist vs. Bayesian*

Mathematically, the frequentist and the Bayesian interpretations of probability are equivalent, but the philosophical interpretations are quite different between the two. In the frequentist interpretation of probability, the configuration space is defined as the set of all possible events, and a particular state of a system follows a well-defined distribution which can be obtained by repeated trials of measurement for the outcome of the system states. In order words, more data from trials will increase precisions until one reaches the fundamental intrinsic variations of the states, and the probability values derived from this process do not subject to external (outside of the system) influences. The Bayesian interpretation approaches form a different perspective, in which the probability represents not the absolute event states but the rational belief of the observer based on the available information at hand at the time. The probability, therefore, can be updated and refined with posterior facts and information using the Bayes' Theorem:

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)} \quad \text{(A.1)}$$

Where the left-hand side, the posterior probability, is calculated from the prior probability, P(A), and the likelihood function from new evidence, P(B|A)/P(B). The probability values derived from the Bayesian thinking can therefore be subjected to external influences. Future interactions of the system with external elements can alter the probabilities.

While practitioners tend to align to one philosophy or the other, it is the author's opinion that our reality is a mixture of both and more. One can interpret that the frequentist's thinking aligns well with aleatory uncertainties, and the Bayesian thinking aligns well with epistemic uncertainties. The subjective aspect of picking and choosing the boundaries firmly aligns with the human factors associated with the decision process. A mechanism unifying these concepts might lead to new and better approaches to represent and manage risks.

Perhaps from another perspective, the choice between the two interpretations is a simple reflection of whether a problem can be sampled within a timescale suitable for frequentists or Bayesian. Bayesian takes chances based on available information, and frequentist waits for solid information to make inferences. For problems with short timescale, in terms of seconds, minutes, days, such as drug testing, then conventional frequentist statistics makes sense, since you have enough time to collect the data and isolate the intrinsic stochastic probabilities. But if the problem has timescales of years and centuries, such as geo-technical applications, or problems with cosmic timescales, then one simply does not have the time to collect enough statistics to make frequentist-based inferences. Bayesian is then a better approach.

### *A.2 Static vs. Dynamics Methodologies*

Besides philosophical interpretations, another way to classify and group risk assessment methodologies is by their capabilities to represent changes in the system and the states over time. In the context of this dissertation, static system means a system S stays in the exact same state at all times, or the state of the system is independent of time, $dS/dt = 0$. Dynamic system is one that can transition from one state to another state over time, or the state of the system is a function of time, $dS/dt \neq 0$. Static methodologies, under this context, are assessment methods applicable to systems that are in a steady state or at a fixed point in time. Dynamics methodologies consider a system's changing states and the state transition processes. Often, the uncertainties of the dynamic system are treated as stochastic Markov processes.

#### A.2.1 Markov Process

A lot of processes in nature are random, and one of those processes are the Markov process. Markov process has 2 elements: a process consists of a sequence of transition stages (over time) and a stochastic (random) element for the transition. A stochastic process is a sequence of events in which their occurrences and the outcomes are probabilistic and random. Example of such include radioactive decay, Brownian motion, random walks, and the financial market (some quantitative financial experts might argue differently). A Markov process is a special kind of stochastic process exhibiting the Markov property:

- The future outcome or state depends only on the present state, and not what preceded (the Markov property); future events (condition probabilities) depend only on current event and are independent of past events. The conditional probabilities are called transition probabilities.
- Future predictions depend solely on the present state. This property is usually referred to as the memoryless property as prior results do not affect the future states.

The Markov process is continuous in time, and a Markov process that evolves in discrete-time element is known as a Markov chain. Markov chain exhibits additional properties:

1) The Markovian property,
2) There are a finite number of possible states or outcome,
3) The probabilities do not change as a function of time (stationary transition probabilities), and
4) There is an initial state or boundary conditions.

Therefore, the state changes can occur at any time for a Markov process, but for a Markov chain, the state space is discrete and their changes are discreet in time.

## *A.3 Static Methodologies*

### A.3.1 Risk Matrices

Risk matrices is traditionally traced to the use of a 5×5 matrix to map the severity (consequence) and the Probability (likelihood), see for example, Table 4.1

and 4.2 of Cox (2009). This methodology is an example of a scenario driven, semi-quantitative approach for risk assessment. Essentially, a 2-dimensional risk profile of a risk event is developed by assigning values, such as 3-points or 5-points high-medium-low value or a distribution function, to the consequence and the likelihood of the impact when the risk event happens (Figure A-1). The matrix element values are simple computation of the product between the consequence and likelihood, following the risk equation: Risk = Probability × Consequence.

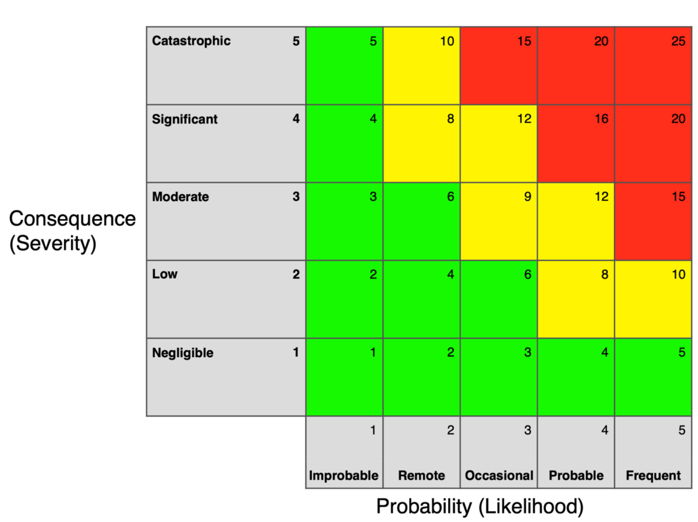


Figure A-1. A Risk Matrix.

To use the matrix, one would simply interpret the values as representation to the significance of the risk event, and the larger the value, the more "risky" the event is. This technique is widely used by many, and most might not realize the potential flaw in the quantitative computational values from the matrix. As point out by Cox (2008, 2009), in certain situations, the computed risk value can provide the wrong indications, such as a mismatch between the qualitative ratings and the quantitative values. Some other limitations of this methodology include: the inability to handle aggregate risks or risks that interact with each other, the inability to reflect uncertainties in the risk parameters, the lack of numerical range and resolution to represent risks, and in the case where probabilities are allocated for the parameters, the joint quantitative probability might not match with the qualitative ratings from the matrix (Cox, 2008, 2009; Dezfuli, 2011). Finally, studies have also demonstrated that the use of risk matrices might not necessarily lead to better-than-random decisions (Cox, 2009).

#### A.3.2 Event Tree or Event Sequence Diagrams

The development of event tree can trace back to the work by Rasmussen (1975) and by Swaminathan & Smidts (1999) to quantitative and probabilistic risk analysis. An event tree is an inductive logical or chronological decomposition of an event into a progressive series of events leading to some subsequent outcomes, consequences, or end states. An event tree is a representation of Sample Space and need not represent a chronology, although it is often used that way in engineering system safety studies (Hartford and Baecher, 2003). The decomposed events, system elements, and steps are represented as a branching tree graph or flowchart, and Boolean logic serves as the connectors or nodes between them. Probabilities for the chance of occurrences of the events can be associated with the nodes. Each event tree represents different scenarios formulated by the risk analysts to describe the various event paths leading to various outcomes. The tree structure provides an excellent way to visualize the different scenarios, and probabilities embedded in the nodes facilitate the computation of risk events. In other words, this tool can serve both as a qualitative description of the scenarios as well as the mathematical apparatus to evaluate the outcome probabilities. The graphical representation of the scenarios also offers another advantage in communicating the risk scenarios to stakeholders in a visual, readable and easy to follow format. However, a key deficiency of this technique stem from the fact that the scenario building process requires the risk analysts to possess very comprehensive knowledge about the system, which might not be feasible for a lot of real-life scenarios, and once the scenarios have been built, changes to the system will require the risk analyst to redevelop the tree, which can be daunting. Subjective uncertainties can also be introduced into the system by the analysts.

#### A.3.3 Decision Tree

The decision tree is a variant of the event tree methodology with a focus on decision uncertainty and decision analysis in operations research and risk analysis. Similar to the event tree, the decision tree is a representation of scenarios of what can happen that lead to a particular outcome. Event states are connected to each other by nodes, consist of 3 different types: decision nodes, chance nodes, and end nodes. Each decision node is a logical branching point with a test condition (or decision), and each branch is a possible outcome of the test condition. A chance node represents stochastic event, and the end node, as the name implies, terminates the tree. The decision tree, therefore, is an abstraction of a decision problem, graphically describing the various possible outcomes, the decision pathways, the uncertainties and trade-offs. While the flowchart representation can serve as great qualitative visual aid to support decision makers to come up with a decision strategy, the decision tree can also be used to compute quantitative expectation values based on the probabilities assigned to or associated with the nodes, resulting in a quantitative measure to determine the best, optimal, or alternative paths.

This technique, however, comes with deficiencies like those of the event trees. The analysts will need to have the knowledge to describe and model the many different combinations of events and decisions. A full characterization of a decision is therefore, not an easy task (Keeney & Raiffa, 1993) since it requires the analysts or decision makers to weight the paths at the decision points and the consequences at the end of the tree. Furthermore, subjective uncertainties and biases can be introduced into the system by the analysts or sometime by the very nature of the decision problem. Lastly, if there are many decision points and many uncertainties, quantitative computations can be very complex and have large errors (see Clemen & Reilly, 1999; Goodwin & Wright, 2004; Keeney & Raiffa, 1993).

#### A.3.4 Fault Tree Analysis

The Fault Tree Analysis (FTA) technique was developed in the early 1960s for the U.S. Air Force Minuteman Launch Control System (Ericson, 1999). The primary user communities nowadays for this technique come from safety engineering and reliability engineering. U.S. Nuclear Regulatory Commission (NRC), NASA and others further developed and refined this methodology over the years. Extensive description of the methodology can be found in Vesely et al. (1981, 2002). In brief, the FTA technique takes an analytical approach to understand the failure states and behaviors of a system by decomposing the top-level fault event (the root node) into a series of lower-level events. The fault tree itself, a hierarchical structured causality model, is the graphical representation of these connected sequences of events, providing a traceable graph of how lower-level fault events connect and combine into the top-level root via Boolean logic. The fault tree can also be generated from a bottoms-up approach where different possible fault paths are identified. These paths are then joined together via Boolean logic, building up to the top event. Quantitative values, such as probabilities, can be applied to the nodes, and quantitative magnitude for the risks can be arrived at by tracing the paths and summing up the probabilities.

While the FTA technique does allow users to perform quantitative risk assessments, the fault tree model itself is not a quantitative model. The FTA models do not provide a comprehensive and exhaustive enumeration of all possible failure modes for the system in question, nor does it exhaustively trace out all possible failure paths and causes. The models reflect heavily on the modelers' knowledge, expertise, and their judgments. These models serve as a convenient starting point for the quantification process, but the fundamental qualitative nature of the models themselves remains. Intrinsic to the FTA approach is its reliance on Boolean logic and the assumption on the binary outcome (success or failure), but the reality of our world certainly contains many shades of grays.

#### A.3.5 Event-Chain Model

The event-chain model (or Chain-of-Events) is another variant of the event tree methodology for several uncertainty modeling. Originally, the technique was developed for the uncertainty analysis for the component failure scenarios for mechanical systems. This technique now finds a wide range of applications such as schedule network analysis, uncertainty modeling for project management, and accident analysis. In schedule network analysis, the event chains can be altered by some events, and the identification and management of these events can help maintain the project schedule. On the other hand, in accident analysis, the root event (an accident, a project) is decomposed into constituting events, sequentially connected by causal factors, forming a chain of events with the project objective or the accident as the last event.

The fundamental assumption in this technique is that the events are directly and causally related, and each event has both a preceding and a following events. The relationships between the events, like the event tree model, are connected by Boolean operations and probabilities can be associated with the events and the operations. The event-chain can have branches and strands, running in parallels and connecting to others at different time or at some common events. Breaking the connections between the events along the chain will therefore prevent the occurrence of the root event (root cause or the initiating event). In accident analysis, for example, the single event or the sequence of events that can trigger or prevent the occurrence of the accident are the "critical" or the "critical chains" of events. Breaking the chain by the elimination of critical events - the failure modes - can therefore prevent the accident. Examples of event-chain based accident prevent techniques included Failure Mode and Effects Analysis (FMEA) and Fault Tree Analysis (FTA) (Section A.3.4 above).

While this technique helps analysts to identify important failure modes and conditions in a system, it does have some limitations. The construction of the event-chain depends heavily on the skills and domain knowledge of the analyst. The selection of what events to include or not in the chain can be subjective and arbitrary. For modern systems, particularly for event-driven system or activities, the interactions between systems and subsystems greatly increase the complexity, making the model building process highly complex, challenging, and sometime impossible to attain completeness. Missing knowledge and information can further complicate the model development process. Subjective considerations, such as human organizational and management factors, can be difficult to integrate into the model. Finally, the strength of this methodology lies with its ability to identify failure modes and conditions, but it does not identify or point to the sources. Most of the time, the analysis will stop at the identification of the failure modes and cannot provide any further insights as to the cause of the problem nor ways to detect and fix the problems beforehand. (Leveson, 2001). Other paradigms were developed to help mitigate the weakness of the event-chain approach, such as applying system dynamics analysis and Monte Carlo simulation techniques to enhance the technique's ability to deal with dynamic complex engineering systems (Dulac, 2007; Leveson, 2011; Fleming 2015).

#### A.3.6 Bayesian Inference

While semi-quantitative risk assessments can be performed with the risk matrix and fault tree analysis techniques, the fundamental qualitative nature of the techniques, a strong dependence on judgments and beliefs would put a limit to how far quantitatively one can reach with these approaches, as probability is applied after the formation of the models, instead of being built into the models. The uncertainties associated with judgments and beliefs are hard to quantify. For example, FTA is built upon Boolean logic, and the resulting tree itself is "deterministic" in that the probabilities are calculated by following the deterministic fault tree paths. But what about the uncertainty about the paths themselves? This is where the distinction between aleatory and epistemic uncertainties becomes crucial to the discussion. In the FTA example, the individual tree nodes are associated with a probability, and these probabilities represent the aleatory uncertainty at the node. The epistemic uncertainty, the uncertainty about the model, is the overall uncertainty associate with the tree itself. As pointed out earlier, the fault tree models are not exhaustive and they are limited by the knowledge possessed by the modeler, which would naturally have varying levels of epistemic uncertainty.

Bayesian inference techniques address this issue by finding a way to combine together a current probability model with new data and information to "update" the existing model. Bayesian techniques, as the name implies, are based on the Bayes' Theorem on conditional probabilities. Instead of taking and using probability from a pure statistical perspective, Bayesian technique allows the incorporation of additional information and evidence into the probabilistic models, and hereby conditioning and updating the "prior" with new evidence, represented by a "likelihood function," to derive the "posterior" probability. The posterior probability is obtained from the application of the Bayes' Theorem,

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)} \tag{A.2}$$

where the left-hand side, the posterior probability, is calculated from the prior probability, P(A), and the likelihood function from new evidence, P(B|A)/P(B). Risk assessment is then performed, not just on the aleatory probabilistic models, but also the uncertainty of the model itself, the epistemic uncertainties, using all available information and knowledge at the time.

The Bayesian inference methodology generally follow these steps after the risk elements have been identified:

- Develop the aleatory uncertainty probability models for the failure states of the risk elements, think of the individual nodes in a fault tree,
- Develop the epistemic uncertainty probability models for the system, think of the fault tree itself, in which the state of knowledge about the uncertainties of the models about the risk elements themselves are quantified, usually by the specification of a "prior" distribution for the model parameters.
- Observe and collect data about the actual behaviors of the real system itself, or a related reference system, on those risk elements and model parameters.
- The data collected is then used to "update" the "prior" distribution, folding in "new" knowledge, to make the "posterior" distribution better reflecting what we observed.
- Verify and validate the updated model to see if it can make better predictions.

This methodology allows the incorporation and updating of the model with newer information, similar to a learning process where the model learns by assimilation. The inferences are made about the "soundness" of the model by inferring from observational data and evidence collected, and not merely by some assumptions of the statistical behaviors of risk elements.

While this represent a significant advancement in the treatment of uncertainties in risk models, the uncertainty models themselves are still rooted in classical probability and they are still bounded by the constraints that limit our ability to accurately represent what we observed in the real world (this will be further elaborated in the next section when we review various perspectives from the literature and active research programs).

### A.3.7 Bayesian Network and Bayesian Belief Network

Bayesian network, also known as Bayesian belief network, is an implementation of the Bayesian inference methodology with a probabilistic graphical model known as the probabilistic directed acyclic graph. Bayesian network is often viewed as union between Bayesian probability with graph theory to incorporate causality and ordering information into the otherwise traditional commutative algebra-based probability theory. The graphical network connections capture cause and effect associated with the system states, their relationships, and changes. The nodes in the model represent system variables and they are causally linked together with conditional probabilities, expressing how one variable can influence another. These approaches relaxed the constrains so that the models are no longer involve a simple causal explanation or over simplified statistical models. Current techniques based on graphs with nodes data lookup, or system dynamics model with simulations. Casual models, such as Bayesian Networks, allow us to go beyond data and explore the more subjective evidence and information.

This technique finds a wide range of modeling applications, which includes artificial intelligence machine learning, medical diagnostic assessment, reliability analysis, organizational decision theory and analysis, and risk analysis (Cowell, 1999; Jensen, 2001; Pearl, 1988; Russell and Norvig, 2003; Ordóñez Arízaga, 2007; Mohaghegh 2010). Bayesian network can also be combined with other techniques, such as system dynamics, to model socio-technical system (Mohaghegh, 2010), or with PRA to model complex engineering system (Dezfuli et al., 2009). Bayesian network, with Bayesian influence methods embedded within, is suitable for modeling problems where limited information is available or the input data has great uncertainties (Bedford and Cooke, 2001). Yet, Bayesian network's ability to deal with limited information is also a potential weakness and liability for the technique. A Bayesian network is built by a modeler, and the goodness of the network in representing the problem depends greatly on the knowledge, experience, and skills possessed by the modeler. Subjective elements and biases can also be introduced into the process. The conditional probabilities assigned to the model are only as good as what is known; the prior data and knowledge that serve as the starting points can have significant impact to the downstream network, the computational complexity and the validity of the results. Finally, this technique relies on the ability to update the probabilities using the Bayes' Theorem. For complex networks, the computational requirements to update and refresh all the node probabilities could be daunting (could be a NP problem itself). Finally, the network can only represent a particular instance in time; any changes to the system would require a model update or even building a new one. This is just another example of the challenges encountered by static methodologies.

## *A.4 Dynamic Methodologies*

### A.4.1 Systems Dynamics

Systems dynamics is a system modeling technique developed by Forrester (1961) to study the non-linear dynamic behaviors of complex interacting industrial systems. System dynamics originates to support the industrial management process

(Forrester, 1961), and is now being used widely for a range of problems, ranging from plant operations, aerospace safety (Sterman, 2000; Leveson, 2006; Dulac, 2007), human decisions, organizational behaviors, social technical risk (Mohaghegh, 2010), and the modeling of decision making process. The fundamental premise of the modeling technique is based upon the notion that a system is formed out of many interacting parts and components, and their interactions are often complex with many different characteristics such as positive and negative feedback loops, time-dependent relationships and interactions forming casual loops. Casual loops modeling the system behaviors are developed and then transformed into stock and flow models, which can then be quantified and expressed in mathematical equations. The mathematical models are then either solved via analytics or simulations.

Unlike the static techniques before, system dynamics is usually simulation based nowadays due to the highly complex and non-linear nature of the modern systems, although analytical methods are available and can be applied to the study of simpler systems. Both analytical and simulation-based techniques can be highly complex and time consuming, and modern system dynamics applications are computer simulation based. This technique, besides the computational demands, also suffers from the same weaknesses as the earlier techniques, namely the soundness of the model depends greatly on the domain knowledge of the modeler.

### A.4.2 Dynamic Event Tree

The dynamic event tree adapts the event tree methodology to model a system's states evolutionary changes in time. A system can transition from one state to another state within a time interval, and those changes could be due to stochastic change of a part within the system (for example, a failure of a part within a pump), or the changes could be deterministic changes due to wear and tear of a part due to usage. These changes, both stochastic and deterministic, can be modeled based on the physics, parametric equations derived from empirical observations, or simulations. The dynamic aspect was achieved by combining the traditional event tree methodology with Markov analysis. If the time behaviors of a system follow the Markov property, which means the stochastic process does not depend on the system's evolution history, then the dynamic event tree for the system becomes very simple and can be discretized. However, for systems where state changes have strong dependency on evolutionary history of the system, the analysis process becomes an enumeration of a collection of event trees, tracking the changes of the trees over multiple time interval windows. Furthermore, any changes of the tree attributes, from the Boolean conditions to the node probabilities, would require a recompilation of the entire collection of event trees for the particular scenario. Finally, limitations for the static event tree also apply to the case of the dynamic event tree, and the limitation compound due to the additional time dimension (see Varuttamaseni, 2011 for a short summary).

#### A.4.3 Dynamic Fault Tree

The Dynamic Fault Tree (DFT) is a product formed out of the combination of the traditional fault tree methodology with Markov analysis. The DFT is an extension of the static fault tree by adding new elements to give the traditional fault tree methodology the capability to handle time evolutionary behaviors and dependencies between states. Specifically, new logic gates, known as the dynamic gates, were introduced into the framework to model the dynamic behaviors and interactions (Shin and Seong 2008), and these new dynamic gates consist of the following gates: the warm and cold spare gates, functional dependency (FDEP) gate, the priority AND (PAND) gate, and the sequence enforcing (SEQ) gate. These gates basically model the state transitions as well as the ordering of the transitions between events, which is not available with the static fault tree methodology. Moreover, the DFT gates can be converted into Markov models, which simplify the modeling of sequential dependency relationships in a dynamic complex system where a state has dependency on the order of occurrence of prior events and states. However, the ability to convert DFTs into Markov models brings not only the strength of the Markov analysis process, but also the challenges as well. As the complexity of the fault tree increases, the number of transitions and states to track and compute can grow rapidly, and greatly increase the computational complexity and the calculations can be very time consuming. To simplify the computational complexity, several different approaches are utilized to manage DFT computations, such as separating the computation into the static and the dynamic parts (divide-and-conquer), combining DFT with Bayesian network modeling, Monte Carlo simulations, and other approximation methods (Vesely et al., 2002; Pourret et al., 2008; Varuttamaseni, 2011; Lindhe et al., 2012).

#### A.4.4 Discrete Event Transition Models

The dynamic event tree methodology tracks the system state evolution over time. Another approach to model state transition is to look for ways to discretize the system states into different bins and track the state transition between the bins. In essence, this approach groups the system parameters together in bins, instead of tracking the state parameters individually, and model the time evolution of the group over time bins. To do so, the parameters are captured within a vector which describe the states of the system. The analysis is then performed on the vector by assigning transition probabilities to the vector based on physics models or Markov properties. This technique is often couple with the use of Monte Carlo methods, and it finds applications in system reliability analysis and failure analysis.

#### A.4.5 Monte Carlo Simulation

When dealing with deterministic problems we often turn to the analytic approach in finding solutions to the problem, but not all problems can be solved analytically. For complex systems and problems with multiple coupled parameters and many degrees of freedom, analytic solutions are simply not possible. Monte Carlo methods are a broad group of computational methods that rely on random sampling and probabilistic techniques to simulate what might resemble and represent the real system and its behavior. This technique finds application in many different fields and disciplines, from finance, the physical sciences, engineering, to project risk management.

The Monte Carlo technique is essentially a computation algorithm that randomly generate outcomes following a prescribed probability distribution and a set of constraints, test the outcomes against and aggregate those meeting some criteria, and by the repeated sampling and aggregation, the properties or the solutions can then be inferred and drawn from the statistics. Monte Carlo is a powerful technique, and it is often couple with other techniques such as systems dynamics to scale up their capabilities to handle large scale complex problems, sometimes as a last resort when the problem is very difficult or near impossible to be solved with other techniques.

There are a few disadvantages associated with Monte Carlo methods. The computational cost for complex problems is high and can be time consuming. Also, the result is only a simulation, an imitation, and an approximation; the validity of the results strongly depends on the simulation logic, and that itself is limited by the knowledge of the analyst. The technique is also not one that can accurately represent time-evolutionary behaviors of a system. Finally, random sampling requires the specification of the sampling space, and this approach will encounter difficulty with the treatment of rare events, incomplete knowledge, and the unknown-unknowns.

#### A.4.6 Dynamic Bayesian Networks

Similar to other dynamic methodologies, Dynamic Bayesian networks (Dagum et al., 1992) attempt to describe dynamic systems by the incorporation of techniques to relate the variables (the nodes) to other nodes separate by time steps. Traditional Bayesian network models a system at a particular instance in time, and it does not capture time-based dependencies between the system states. In Dynamic Bayesian networks, an instance of the static Bayesian network is viewed as a time slice representing the system and the associated states at that time. Different time slices are then connected together to represent the time evolutionary changes of the states, and this is similar to time series analysis (Figure A-2). Often, additional constraints are placed on the model, such as the Markovian condition.

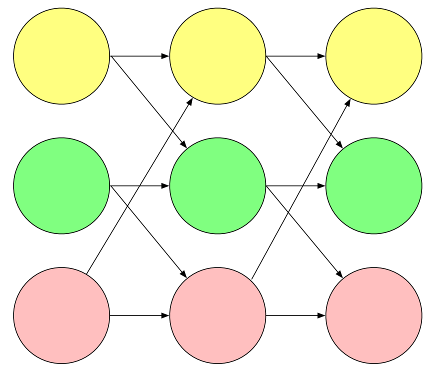


Figure A-2. A graphical example of a Dynamic Bayesian networks.

In this approach, the underlying Bayesian network is assumed to be fixed. This assumption is great when you have a simple system that itself does not change.

## *A.5 Alternative Uncertainty Theories*

Traditional representations of aleatory and epistemic uncertainties have been in the form of probability. Recent research focuses have been on connecting probability with other uncertainty theories, especially in the case of epistemic uncertainties, where researchers are looking to extend classical probability with other theories (Aven, 2010; Aguirre et al., 2014; Flage et al., 2014). Others attempt to develop alternative uncertainty theories by investigating how the state of knowledge in the form of beliefs can be incorporated into the mathematical framework, such as the Dempster-Shafer theory (Dempster, 1967; Shafer, 1976, 2002).

Aguirre et al. (2014) highlights some of the current research directions on alternative uncertainty theories:

- Probability theory: this is to extend classical probability with the inclusion of subjective probabilities, "where a probability measure represents a degree of belief of an agent about the occurrence of an event A." (Aguirre et al., 2014)

- Fuzzy set theory: True or False or either (true or false). This is in essence an extension of the traditional truth table with the inclusion of an additional "either" state (Zadeh, 1965).
- Possibility theory: extending the fuzzy set with the additional condition of normalization. (Zadeh, 1978; Dubois & Prade, 1988, 2001)
- Belief functions theory: the above mentioned Dempster-Shafer theory - theory of evidence (Dempster, 1967; Shafer, 1976). This approach models the degree of belief using mass function, belief function, or plausibility function, and relaxes Kolmogorov's additivity axiom.
- Imprecise probabilities (Walley, 1991): a framework that admits imprecision in probability models and introduces probability bounds.

*A.6 Decision Entropy*

The concept of Decision Entropy was proposed by Gilbert et al. (2016) as a framework that recognizes that there can be different starting points for the initial prior probabilities, and each might lead to different decisions and therefore different risk assessment outcomes. The choice of the initial prior probability depends on the knowledge possessed by the assessor at the time. Given that the prior probability can either be providing informative or non-informative input into the decision process, the choice of the initial prior probability distribution is contextual dependent and relative to how the probabilities will be used for a decision. The theory puts forward that the end usage of the probability for making decisions drives the selection of the starting prior probability.  In other words, the assessment questions determine if a particular prior probability can provide valuable information applicable to the assessment process. This important distinction affects the application of Bayes' theorem, since the basic Bayesian approach does not distinguish the prior probability based on their information context, and thus the initial selection of the prior is often based on what knowledge is available at the time. The choice of the initial prior, therefore, could direct the Bayesian update process down a path that could lead to wrong conclusions because of black swan type events.

Gilbert et al. attempt to development a measure, the decision entropy, by quantifying the informative and non-informative prior probabilities in a mathematical framework which can be used to assess and establish the value of the initial prior probabilities relative to the end decision outcome. The theory postulates that the maximum decision entropy is when the prior probabilities contains the least amount of information; in other words, at maximum decision entropy, the prior probabilities are equal for all possible system states. Probabilities that do not contribute to the eventual decision, the "non-informative priors," represents complete ignorance; whereas informative probabilities serve as input to the decision process and thus affect the management of risk. The non-informative prior sample space becomes important for this consideration because it sets the stage for assessing the conditioning probabilities on any available information (whether subject or objective) and shapes the assessment paths as the probability is being updated. A quantitative measure of the priors can help guide and steer the use of new information and data on how to update the posterior probabilities. This approach can potentially prevent the premature "locking-in" to a specific updating path by a priori assumptions, keeping open the options to consider different alternatives, allowing new information to be properly utilized for updating the posterior probabilities.

The decision entropy theory is a mathematical framework to assess the risk probabilities for systems where the assessor has limited or incomplete knowledge. The framework attempts to look for a quantitative representation of the knowledge about the risk and uncertainty, from one end of the spectrum - zero knowledge - to the other end where we have full knowledge, in the form of an entropy measure. The theory focus on finding the optimal starting point, based on the end question, under limited or incomplete knowledge, using the entropy measure. While this theory present great potentials, it is still under active research and development. The mathematical framework will need to be further developed and refined.

### *A.7 A Closing Note*

The above techniques rely on the use of probability theory to quantify uncertainties. A number of researchers have pointed out that while current tools allow us to sufficiently deal with many risk problems we encounter in different fields, there are still plenty of cases where they fail or fail to adequately represent reality (Cox, 2009, 2012; Samuelson, 2011; Aven & Zio, 2014). The use of probability to represent uncertainties is a foundation element of risk assessment, so a change in the underlying probability theory can have great impact to the efficacy of the techniques. Fundamental research into this area could present game changing results.

# Appendix B: Quantum Probability

*God does not play dice with the universe*

*- Albert Einstein, The Born-Einstein Letters 1916-55.*

When Einstein made the statement, he might not realize how true it is. Indeed, classical probability, where traditionally illustrated with a game of chance with dice, is insufficient to describe the quantum phenomena.

Classical probability is based on traditional Boolean logic and sets. In the real world, physical phenomena cannot be fully accounted for by Boolean logic alone. Quantum probability evolved from the need for us to find alternative ways to describe the new quantum phenomena. For a thorough discussion and treatment of modern quantum theory, the following references can be consulted for in-depth discussions: von Neumann (1955), Dirac (1958), Sakurai (1994), Griffiths (2005), Khrennikov (2010), Busemeyer & Bruza (2012), Susskind & Friedman (2014), and Chang (2015).

## *B.1 Dirac-von Neumann Axioms*

Whereas Kolmogorov probability is based on set theory, Dirac-von Neumann quantum probability is based on the theory of complex vector space. The original set of Dirac-von Neumann axioms were not developed specifically for quantum probability but to serve as the foundation principles for describing quantum mechanical systems. There are two basic formulations: the complex Hilbert space formulation and the C* algebra formulation (or operator algebra formulation). In this dissertation the complex Hilbert space formulation will be used.

*B.2 Mathematical Representations for Hilbert Space, State Space, State Vectors, and their Operations*

The Hilbert space contains the N-dimensional complex vector space with states that define a quantum system. Since the Hilbert space is a vector space, a general vector in it can be expressed in the traditional vector forms, in vector component form or in matrix form. Let **A** be a general vector in a Hilbert space of N dimensions, and $\hat{e}_i$ be the basis vector spanning the vector space, **A** can be expressed in the traditional notation as:

$$\mathbf{A} = \sum_{i=1}^{N} A_i \hat{e}_i \quad \text{(B.1)}$$

In quantum mechanics, an alternative notation, known as the Dirac Bra-Ket notion, is used to represent a vector. The vector **A** is denoted by $|A\rangle$ and is simply called a ket-vector or kets. With this notation, the above vector **A** can be expressed in the ket notation as follow (for N=3):

$$|A\rangle = A_1|e_1\rangle + A_2|e_2\rangle + A_3|e_3\rangle = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \text{(B.2)}$$

Associated with the ket vectors are a number of axioms on their operations.

Axioms for the ket vectors:

1) Addition: $|A\rangle + |B\rangle = |C\rangle$ (B.3)

2) Commutative: $|A\rangle + |B\rangle = |B\rangle + |A\rangle$ (B.4)

3) Associative: $\{|A\rangle + |B\rangle\} + |C\rangle = |A\rangle + \{|B\rangle + |C\rangle\}$ (B.5)

4) The zero vector, or the null ket: $|A\rangle + 0 = |A\rangle$ (B.6)

5) Given any ket, there is a unique ket such that: $|A\rangle + (-|A\rangle) = 0$ (B.7)

6) Linearity:

6a) Linear scalar multiplication, z is complex: $|zA\rangle = z|A\rangle = |B\rangle$ (B.8)

6b) Distributive: $$z\{|A\rangle + |B\rangle\} = z|A\rangle + z|B\rangle, \quad \{z+w\}|A\rangle = z|A\rangle + w|A\rangle \qquad \text{(B.9)}$$

Corresponding to the ket-space is a complex conjugate dual space, call the bra-space. There is a corresponding bra-vector $\langle A|$, which is the dual vector, for every ket-vector.

Axioms for the bra-vectors:

- For the ket-vector $|A\rangle + |B\rangle$, the corresponding bra-vector is $\langle A| + \langle B|$.
- Let z be a complex number. For the ket-vector $z|A\rangle$. the corresponding bra-vector is $\langle A|z^*$, since the bra-vector is in the conjugate dual space.

The bras can also be viewed as operators that operates on ket-vectors with a complex number as the resulting output.

B.2.1 Inner Products

Since the Hilbert space is a complex inner product space, inner products can be formed between a pair of bra-ket vectors: $\langle B|A\rangle$. The inner products take a pair of vectors and map them to a scalar. The inner products follow these axioms:

1) Linearity: $\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle$ (B.10)

2) Conjugate: $\langle B|A\rangle = \langle A|B\rangle^*$ (B.11)

3) Normalized vector: $\langle A|A\rangle = 1$ (B.12)

4) Orthogonal vector: $\langle B|A\rangle = 0$ (B.13)

5) Positive definite: $\langle A|A\rangle \geq 0$ and $\langle A|A\rangle = 0$ iff $|A\rangle = 0$ (B.14)

B.2.2 Operators

We denote and define an operator **M** by its action to a vector, where **M** acts on the vector $|A\rangle$ to give $|B\rangle$:

$$\mathbf{M}|A\rangle = |B\rangle \tag{B.16}$$

Properties:

Linear scalar multiplication:

$$\text{If } \mathbf{M}|A\rangle = |B\rangle \text{ then } \mathbf{M}z|A\rangle = z|B\rangle \tag{B.17}$$

Distributive:

$$\mathbf{M}\{|A\rangle + |B\rangle\} = \mathbf{M}|A\rangle + \mathbf{M}|B\rangle \tag{B.18}$$

The linear operator **M** can be represented in terms of matrix elements. The matrix equation (B.16) can be represented in component form:

$$\sum_j \mathbf{M}|j\rangle \alpha_j = \sum_j \beta_j |j\rangle \tag{B.19}$$

$$\sum_j \langle k|\mathbf{M}|j\rangle \alpha_j = \sum_j \beta_j \langle k|j\rangle \tag{B.20}$$

Adopting an abbreviated form for $\langle k|\mathbf{M}|j\rangle \equiv m_{kj}$, one obtains

$$\sum_j m_{kj} \alpha_j = \beta_k \quad \text{(B.21)}$$

For some particular linear operator, there will be vectors whose directions are the same when they come out as they were when they went in. These special vectors are called eigenvectors. An eigenvector is defined by:

$$\mathbf{M}|\lambda\rangle = \lambda|\lambda\rangle \quad \text{(B.22)}$$

And λ is called the eigenvalue of the eigenvector $|\lambda\rangle$.

The above defines the how a linear operator operates on the ket vector, and a linear operator can operate on the bra vector the same way, but with the adjoint of ***M***, denoted as $\boldsymbol{M}^{\dagger}$ which is obtained from taking the complex conjugate of the transpose of ***M***, defined by:

$$\mathbf{M}^{\dagger} = \left(\mathbf{M}^{*}\right)^{T} \quad \text{(B.23)}$$

### *B.3 Complex Hilbert Space Formulation*

A quantum system is described by a countable infinite dimensional complex vector space known as a Hilbert space, ***H***. In this space, a quantum system is characterized by its states, observables, and expectations (measurements). Associated with these properties are operations describing the behaviors of them. They satisfy the following axioms:

*Axiom 1: States*

The states of a quantum mechanical system, **S**, are composed of normalized unit vectors, $\psi$, of the complex Hilbert space, **H**, also called the space of states. A quantum state is denoted by the normalized state vector $\psi$ or $|\psi\rangle$ satisfying $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$. The state vectors completely specify the properties of a quantum mechanical system.

*Axiom 2: Observables*

The measurable quantities of a quantum system, the observables, are described by self-adjoint linear operators ***A*** in a complex Hilbert space, ***H***. If a measurement ***A*** is performed on a system in a state $|\lambda_i\rangle$, the result of the measurement is $\lambda_i$. $\lambda_i$ is the eigenvalue and $|\lambda_i\rangle$ is the eigenvector. The measurement is denoted by $\mathbf{A}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$.

*Axiom 3: Expectation Value*

Let $|\psi\rangle$, $|\phi\rangle$ denote 2 different quantum states of a system.

(a) The probability of finding $|\psi\rangle$ in state $|\phi\rangle$ is given by the probability *P* where

$$P(|\psi\rangle,|\phi\rangle) = |\langle\psi|\phi\rangle|^2. \qquad \text{(B.24)}$$

This is known as the Born Rule.

(b) The expectation value of an observable (operator) ***A***, denoted by $\langle A\rangle$, for a quantum system in state $|\psi\rangle$ is given by the inner product

$$\langle A\rangle = (\psi, A\psi) = \langle\psi|A|\psi\rangle. \qquad \text{(B.25)}$$

*B.4 What Quantum Probability Brings to the Table*

Quantum probability introduces a number of new properties that are not derivable from classical probability.

*Non-commutativity*

In the classical framework, the order of the occurrence of events A and B do not affect the results from their combination, i.e. “A and B” and “B and A” will arrive at the same result. Non-commutativity simply states that the sequence of events “A and B” might have a different consequence than the sequence of events “B and A”.

*Quantum Superposition*

A quantum state can be formed from the superposition of two or more other distinct quantum states.  Furthermore, a quantum state can be formed from a superposition of infinitely many possible quantum states, and when a measurement is performed on the superposition state, it collapsed into a definite state.

*Quantum Interference*

A quantum state can be a superposition of many distinct states, or a linear combination of many distinct states. When performing a measurement on a quantum state, the expectation value is given by the inner product of the states and the probability is given by the Born Rule. The cross terms as a result of expanding the product are the interference terms, a result of the non-additivity of the probabilities.

*Quantum Entanglement*

Quantum entanglement describes the quantum state of a joint system, such as a pair of particles, where the entangled quantum state describes joint system completely, but the quantum state of each of the particle can be partially or entirely unknown. At maximum entanglement, the state of the joint system is completely known, yet the individual states are complete unknown. Other mixed entangled states are possible.

*Quantum Logic and Computation*

The concept of measurement as defined in quantum mechanics deviates from and is inconsistent with the absolute notions found in classical logic. Quantum logic is a reformulation of propositional logic to arrive at a consistent logical reasoning framework for both classical and quantum systems. Quantum computing systems are systems that perform computations using quantum mechanical properties, such as quantum entanglement and superposition, as well as the use of quantum computation circuits, known as quantum gates, to perform operations. Quantum logic gates can perform operations that classical Boolean circuits cannot.

# Glossary

| | |
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| Aleatory uncertainty | Aleatory uncertainty deals with the inherent, intrinsic random stochastic variations associated with a physical system. |
| Ambiguity | Ambiguity can be defined as the possibility of different interpretations for a result based on the availability of contextual knowledge. |
| Bayesian probability | The probability represents not the absolute event states but the rational belief of the observer based on the available information at hand at the time; the probability, therefore, can be updated and refined with posterior facts and information using the Bayes' Theorem. |
| Belief functions theory | This approach models the degree of belief using mass function, belief function, or plausibility function, and relaxes Kolmogorov's additivity axiom. |
| Born Rule | Probability is given by the square of the probability amplitudes. |
| Common Cause/ Common Mode | Multiple components may be susceptible to a common cause/mode where all the components can fail due to a single failure cause. |
| Common cause | An analysis where common causes are identified and used |

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| analysis | for restructuring logical tree structures. |
| Composite operator | The composite operator is formed from the tensor product of the subsystem operators and operates on the composite system state vectors. |
| Composite system | A system form from 2 or more subsystems; composite systems are formed from component state vectors through the use of tensor product. |
| Deep Uncertainty | Uncertainties due to insufficient data, information, and knowledge about the problem or the system for the development of uncertainty and risk models. |
| Density Operator/Density Matrix | The density operator is a linear combination of the different possible states of the system and is used to construct the mixed state quantum representation. The density operator can be expressed as a matrix with a specified basis, and as such, the term density operator and density matrix are often used interchangeably. |
| Ensemble | An ensemble is a large collection of identical or near identical systems. |
| Entanglement | Quantum entanglement describes the quantum state of a joint system, such as a pair of particles, where the entangled quantum state describes joint system completely, but the quantum state of the individual particle can be partially or |

entirely unknown. At maximum entanglement, the state of the joint system is completely known, yet the individual states are complete unknown. Entanglement is a generalization of the concept of correlation in the quantum sense.

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| Epistemic | Epistemic uncertainty reflects the lack of knowledge and information about some properties and characteristics of a system. The epistemic uncertainty is a quantification of the degree of knowledge or the state-of-knowledge of the fidelity of the models, modeling parameters, and assumptions in representing the reality of the relevant physical processes and the systems' behaviors as defined by the conditions. |
| Events | Events are "points in spacetime" where something happened leading to a condition, an outcome, or a result; within the quantum framework, events are points in a complex inner product vector space known as Hilbert space, a complex vector space contains the complete set of possible outcomes. |
| Event tree | An event tree is an inductive logical and chronological decomposition of an event into a progressive series of events leading to some subsequent outcomes, consequences, |

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| | or end states. The decomposed events, system elements, and steps are represented as a branching tree graph or flowchart, and Boolean logic serves as the connectors or nodes between them. Probabilities for the chance of occurrences of the events can be associated with the nodes. |
| Expectation value | The expectation value of an observable can be thought of as the average or mean value of the measured outcomes of an observable. |
| Fragility curves | Fragility curves plot out the functions describing the conditional probability of system failure over the full range of the parameters, such as loads, where the system is subjected to. |
| Frequentist | Assign event probability based on the frequency of occurrence in a large number of trials. |
| Fuzzy set theory | Fuzzy set theory is in essence an extension of the traditional truth table with the inclusion of an additional “either” state, besides True or False. |
| Hilbert Space | A complex inner product vector space. Objects in Hilbert space are vectors, and events are outcomes which are points in this vector space. |
| Ignorance | The term ignorance is generally referring to the lack of information and knowledge, an absence of awareness of |

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| | missing knowledge, or it can also describe the state where information and knowledge are deliberately discarded or ignored. |
| Imprecise probability | Imprecise probabilities is a framework that admits imprecision in probability models and introduces probability bounds. |
| Inner Products | The inner product space is a vector space where a number of specific structures and operations, such as the notion of length, the angle between vectors, and the inner product between vectors are defined. The inner product can be thought of as a generalization of the Euclidean space scalar product or dot product to multi-dimensional vector space (can be complex and infinite dimension). The inner product operation maps a pair of vectors to a scalar. |
| Knowledge operators | A class of operators that encapsulate historical and contextual knowledge about the system. |
| Kolmogorov probability | Most modern probabilistic methods follow the Kolmogorov formalism, where the Kolmogorov axioms define the probability mathematical framework; Kolmogorov probability can be regarded as a scalar theory with probabilities map to scalar values and functions. |
| Markov process | A Markov process is a special kind of stochastic process |

exhibiting the Markov property: 1) The future outcome or state depends only on the present state, and not what preceded (the Markov property); future events (condition probabilities) depend only on current event and are independent of past events. The conditional probabilities are called transition probabilities. 2) Future predictions depend solely on the present state. This property is usually referred to as the memoryless property as prior results do not affect the future states. The Markov process is continuous in time, and a Markov process that evolves in discrete-time element is known as a Markov chain.

Measurement — The results from the measurement of observables are eigenvalues of the observable operator.

Minimal cut sets — The minimal number of components, when fail in combination, leads to the failure of the system.

Mixed state — Mixed states are simply systems where their states are less than certain and the systems are described by the collection of the quantum superpositions related by probabilistic uncertainty, which can be thought of as probability distributions of the quantum superpositions ensemble. A mixture of different pure states is a mixed state.

Observables — The measurable quantities and properties of a quantum

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| | system are called the observables. They are represented with self-adjoint linear operators. In order words, an observable L, a measurable quantity, is associated with an operator L that acts on the vectors in state space. |
| Operator | The concept of operators was introduced as an operation that takes an input state vector (like the initial state of a system) to another state vector (the system at a different state). The operator can change the state of the system without modifying the system, or the operator can change the state of the system by changing the system itself. |
| Outer Product | The outer product is a product of two vectors forming a linear operator; the Outer product is a special case of the tensor product. |
| Possibility theory | Possibility theory extending the fuzzy set with the additional condition of normalization. |
| Probabilistic Risk Assessment (PRA) | PRA is a class of risk assessment methodologies. Current PRA techniques generally consist of the following three elements: 1) sets of scenarios, which can be physics or probability models to represent events and engineering systems (simple or composite), 2) the frequency of occurrences of the events associated with the scenarios, usually in the form of probabilistic models for events, and |

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| | 3) the consequences associated with the occurrences of the events, which can be numerical values, event triggers, event sequences, or impact statements. PRA techniques based on conventional probabilistic treatments, such as Kolmogorov axiomatic, Frequentist, or Bayesian treatments, have been widely and successfully used to investigate and model many complex engineering problems |
| Probability amplitudes | The complex basis components of a state vector. |
| Product states | For composite systems, individual components are formed by simple products of the probability amplitudes. For composite systems formed from product states they can be factored back into the separate individual subsystems. |
| Projection operator or Projector | Projectors are operators that can select a certain state or a set of states from the system and project them into a subspace, resulting with a state vector with a different set of basis. |
| Projective measurement | The projective measurement is the application of the projector to transition the system from the initial state (before the hurricane hit) to the observed state (after the hurricane hit). |
| Pure state | The state vector describing such a fully known system is called a pure state. A system is in a pure state when |

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| | complete and precise knowledge about it is available and known, such as right after the preparation and construction of a system, or after an observation and measurement. A pure state system is one that we possess full knowledge about the system, and we can continue to track and trace the changes of the system going forward. |
| Quantum (von Neumann) probability | Quantum or von Neumann probability can be regarded as a geometric theory with probabilities map to vectors and operators. |
| Risk | Risk = Probability × Consequence |
| Risk Analysis (Quantum) | Risk analysis (quantum) is the process of identifying corresponding risk states from the density matrix, calculating their chances of occurrence, and associating with them values of consequence. |
| Risk value | Associated with each risk system is an observable called the risk value denoted by the operator Ri. Performing the measurement Ri on a system's basis state yields the scalar risk value representing the significance of the consequence or impact. |
| Risk vector (conventional) | The event probabilities and consequences are represented in the form of vectors. |
| Separable | Separable subsystems refer to those subsystems that do not |

behave in a correlated fashion. The composite system consists of "product states," where the individual components are formed by simple products of the probability amplitudes. The composite system with product states can be factored back into the separate individual subsystems. The ability to decompose the system can be interpreted as having full knowledge of the individual subsystems, their characteristics and behaviors.

State — The first one characterizes the various possible configurations for the system in terms of probabilities, which is call the state of the system.

State Space — A quantum system is described by a state vector, a unit vector in a complex inner product vector space (a Hilbert space) called the space of states (state space). The subspace spanned by all the possible system states forms the state space for the system.

State vector — The collection of all possible events defines the quantum state space and the state of a quantum system is specified by a vector in the system's state space called the state vector. State vectors completely specify the properties of quantum systems; The state vector encapsulates all the information about the probabilities for the occurrence of events, what

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| | can happen, and how systems evolve. |
| Subspace | The vector space contains the complete set of possible outcomes. A particular collection of events forms a subspace in Hilbert space, and many subspaces can be formed within the Hilbert space. |
| Subsystem | Subsystems make up a composite system and subsystems can be similar or different. |
| Superposition | A state vector is expressed as the superposition of the basis states, the orthonormal basis vectors in the system's state space, and it is expressed as the vector sum of the basis vectors. |
| Tensor Product | Tensor product ($\otimes$) is the product of tensors (of which vectors are subsets), and the result is an expanded space formed from combining vector spaces together. |
| Unitary operator/ Unitary transformation | Unitary operators keep the unit vector unchanged in magnitude. In other words, while the system might have changed to a new state, the system itself has not, keeping the same set of basis vectors and unit magnitude for the state vector; events that only change the system states and not the system itself are represented by unitary operators. |
| Verification | The process to ensure that a system is built to specifications. The system is built right. |

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| Validation | The process to ensure that a system is built to solve the problem. The right system is built. |

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