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Switching Diffusions: Applications To Ecological Models, And Numerical Methods For Games In Insurance

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SWITCHING DIFFUSIONS: APPLICATIONS TO ECOLOGICAL MODELS, AND NUMERICAL METHODS FOR GAMES IN INSURANCE

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DEDICATION

To my family and teachers

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CHAPTER 1 INTRODUCTION

1.1 Regime-Switching Systems

After World War II, significant contributions have been made in the realm of probability theory and stochastic processes. Beside the investigation of specific stochastic processes like random walks, Lévy processes, mathematicians have developed the field of stochastic calculus that has been applied widely to build more realistic dynamic system models. However, continuous processes given by differential equations and stochastic differential equations alone are inadequate to improve modeling accuracy. A class of dynamic systems called "hybrid systems" containing both continuous dynamics and discrete events has been adapted to treat a wide variety of situations arising in many real-world situations. Motivated by such development, in this dissertation, we analyze properties of multi-scale stochastic processes and design numerical algorithms for hybrid dynamic systems.

In 1982, Engle considered time series models that exhibit time-varying volatility clustering; see [15]. In [17], the authors treated time series models that are subject to random switching. Such models have been much extended to various ARCH (autoregressive conditional heteroskedasticity) and GARCH (generalized autoregressive conditional heteroskedasticity) models [11] with many applications in finance; see related work in [9] and references therein for real options. The main ingredient is that in lieu of a fixed configuration, one considers a model in which the configuration is changing in accordance with an additional randomly switching process. In a wide range of applications, especially those in control and optimization, there is a major demand for using such models that at different configurations, the behaviors of the systems are drastically different. In recent years, much effort has been devoted to the study of various hybrid systems with stochastic disturbances [42]. In view of the literature, this dissertation has been focusing on dynamical systems involving a Markov chain as the randomly switching process: hybrid competitive Lotka-Volterra ecosystems in [12], and non-zero-sum stochastic differential game between two insurance companies with regime-switching in [13].

1.2 Lotka-Volterra Ecosystems

Introduced by Lotka [25] and Volterra [36], the well-known Lotka-Volterra models have been investigated extensively in the literature and used widely in ecological and population dynamics, among others. When two or more species live in close proximity and share the same basic requirements, they usually compete for resources, food, habitat, or territory.

Recent effort on the so-called hybrid systems has much enlarged the applicability of Lotka-Volterra systems. One class of such hybrid systems uses a continuous-time Markov chain to model environmental changes and other random factors not represented in the usual stochastic differential equations; see [42] for a comprehensive study of switching diffusions, and see [14, 27, 34, 44, 45] for the stochastic Lotka-Volterra with regime-switching. Random perturbations to the Lotka-Volterra model were considered in the literature; see for example [3, 20] and many references therein, and also [18, 21] for up-to-dated progress on stochastic replicator dynamics. However, most recent works focus on Markov chain with a finite state space. To take into consideration of various factors, it is also natural to consider the Markov chain with a countable state space, which is the effort in the first part of this dissertation.

We focus on the study of hybrid Lotka-Volterra systems with a multiple number of species where the species competing against each other. These systems involve a two-time-scale Markov chain with the help of the study of asymptotic properties of two-time-scale Markov chains [40]. Mathematically, the time scale separation is obtained by introducing a small parameter ε. As $\varepsilon \to 0$, we obtain a limit system. We then show if the limit system has certain properties, then the complex original system also preserves the same property for sufficient small ε.

In contrast to the existing results, our contributions are as follows. (i) We model the

Lotka-Volterra ecosystems using hybrid systems in which continuous states (diffusion) and discrete events (switching) coexist and interact. A distinct feature of the modeling point is that the random discrete events take values in a countably infinite set. (ii) Prior to this work, existence and uniqueness of solution, continuity of sample paths, and stochastic boundedness of regime-switching Lotka-Volterra system with random switching taking values in a countable state space were not available. My dissertation establishes these properties. Although general regime-switching diffusions were considered in [32], the spatial variable x there lives in the whole space $\mathbb{R}^n$, whereas for the Lotka-Volterra systems considered here, $x \in \mathbb{R}^n_+$. It needs to be established that the solution is in $\mathbb{R}^n_+$ as well. (iii) This work provides a substantial reduction of complexity. The two-time scale system is a system involving countably infinitely many equations, whereas the limit system is a single diffusion. Using the limit system as a bridge, we then obtain for example, if the limit system is stochastically bounded, or permanent, or going to extinction, then the much more complex original system with switching also preserves such properties as long as the parameter $\varepsilon > 0$ is small enough. The complexity reduction is achieved by using two-time-scale formulation and perturbed Lyapunov function methods. This line of thinking goes back to the work of [10], which has been much expanded to more general setting in [22]; see also [40].

1.3 Optimal Control in Insurance and Risk Management

Insurers tend to accumulate relatively large amount of cash or cash equivalents through the written insurance portfolio. Investing the surplus in a financial market in order to pay future claims and to avoid financial ruin becomes a natural choice. In terms of financial performance the investment income allows significant pricing flexibility in underwriting to the insurers. The surplus is allowed to be invested in a financial market in continuous time.

On the other hand, reinsurance has been considered as an effective risk management tool for insurance companies to transfer their risk exposure to another commercial institution.

The primary insurer pays the reinsurer a certain portion of the premiums. In return, the reinsurer is obliged to share the risk of large claims with the primary insurer. Proportional reinsurance and excess-of-loss reinsurance are two major types of reinsurance strategies. With proportional reinsurance, the reinsurance company covers a fixed percentage of losses. The fraction of risk shared by the reinsurance company is determined when the reinsurance contract is sold. The other type of reinsurance policy is nonproportional reinsurance. The most common nonproportional reinsurance policy is excess-of-loss reinsurance, where the primary insurance carrier (called cedent) will pay all of the claims up to a predetermined amount (termed retention level).

The optimal risk controls for an insurance corporation has been studied extensively since the classical collective risk model was introduced in [26]. The insurance companies can reduce or eliminate the risk of loss by involving in a reinsurance program and reinvesting in the stock market. Recently, the extension of optimal investment and reinsurance problem (Nash equilibrium) in the context of stochastic differential games including zero-sum games and non-zero-sum games has been developed rapidly; see the existence of the Nash equilibrium of non-zero-sum stochastic differential game with N players over an infinite time horizon in [6]. The existence of the Nash equilibrium of a non-zero-sum stochastic differential game between two insurance companies in [43]. [24] studied a zero-sum stochastic differential reinsurance and investment game between two competing insurance companies under VaR constraints for the purpose of risk management. [8] investigated a class of non-zero sum stochastic differential game between two insurers by using the objectives of relative performance and obtained explicit solutions for optimal reinsurance and investment strategies.

Furthermore, people have realized that stochastic hybrid models have advantages to capture discrete movements (such as random environment, market trends, interest rates, business cycles, etc.) in the insurance market. The hybrid systems enable the consideration of the coexistence of continuous dynamics and discrete events in the systems. To reflect the hybrid feature, one of the recent trends is to use a finite state Markov process to describe the transitions among different regimes. The Markov-modulated switching systems are known as regime-switching systems. The formulation of regime-switching models is a more general and versatile framework to describe the complicated financial markets and their inherent uncertainty and randomness. Because the control strategies are affected by the asset prices on the stock market and economic trends change quickly, Markovian regime-switching processes were introduced widely to capture movements of random environment. A comprehensive study of switching diffusions with "state-dependent" switching is in [42]. [8] provided closed-form Nash equilibria for a mixed regime-switching Cramér-Lundberg diffusion approximation process; see also related works [9] and [7] for regime-switching models of real options and real options with competition.

In this work, we are concerned with an insurance market including two insurance companies. The two competing insurance companies adopt optimal investment and reinsurance strategies to manage the insurance portfolios. The surplus process of each insurance company is subject to the randomness of the market. Following the work of [8], the randomness of the market is modelled by a continuous-time finite-state Markov chain and an independent market-index process. Nevertheless, we model the surplus process as a regime-switching jump-diffusion process, in lieu of a mixed regime-switching Cramér-Lundberg diffusion approximation process. This allows us to work with both proportional and excess-of-loss reinsurance policies. Equilibrium strategies are given by solutions of a system of Hamilton-Jacobi-Isaacs (HJI) equations for the value functions of various players, derived from the principle of dynamic programming. Owing to the inclusion of the random switching environment and jump processes, the system of HJI equations becomes more complicated and closed-form solutions are virtually impossible to obtain. Starting by assuming the existence and uniqueness of the Nash equilibrium, we adopt the Markov chain approximation method (MCAM) developed in [23] to deal with a system of HJI (HJI) equations arising from the associated game problems. One of the advantageous is that no regularity of the system of equations is needed since we are using a probabilistic approach. The convergence of the approximation sequence to the jump process and the convergence of the value function will be established. In the actual computation, we will use our approximation schemes for constant absolute risk aversion (CARA) insurers.

1.4 Outline of the Dissertation

The remainder of the dissertation is arranged as follows. In Chapter 2, we first propose the hybrid competitive Lotka-Volterra ecosystems with countable switching states. Section 2.2 studies existence, uniqueness, and continuity of solutions of the competitive Lotka-Volterra systems associated with a continuous-time Markov chains with a countable state space. We then introduce the Lotka-Volterra systems with two-time scales with the use of a singularly perturbed Markov chain and illustrate the properties or their solutions in Section 2.3. We further provide the permanence and extinction of the systems with two-time-scale Markov chains through their limit systems in 2.3.5 and 2.3.6.

Chapter 3 focuses on a class of non-zero-sum investment and reinsurance games for regime-switching jump-diffusion models. A generalized formulation for surplus processes and the associated game problem are presented in Section 3.1. We design the numerical algorithm based on MCAM in Section 3.2. A discrete approximating Markov chain is constructed and is proved to be locally consistent with the original processes. Section 3.3 deals with the convergence of the approximation process and the value functions. Numerical examples are reported in Section 3.4 to illustrate the performance of the method.

Finally, in Chapter 4, we provide further discussions. We summarize the central theme of the dissertation, provide further remarks, and present some directions for future work.

Before proceeding further, it is important to mention that we may use the same notation index with different meaning in different chapters. However, we will use K as a generic constant throughout the dissertation, whose value may change for different appearances.

CHAPTER 2 HYBRID COMPETITIVE LOTKA-VOLTERRA ECOSYSTEMS: COUNTABLE SWITCHING STATES AND TWO-TIME-SCALE MODELS

2.1 Formulation

We model the random environments (e.g., different seasons, changes in nutrition and food resources, and other random factors) in the ecological system by a continuous-time Markov chain $\alpha(t)$ with a countable state space $\mathbb{Z}_+ = \{1, 2, \dots\}$ and a generator $Q = (q_{\alpha\beta})$ satisfying $q_{\alpha\beta} \geq 0$ for $\alpha \in \mathbb{Z}_+$ and $\beta \neq \alpha$, and $\sum_{\beta=1}^{\infty} q_{\alpha\beta} = 0$ for each $\alpha \in \mathbb{Z}_+$. A stochastic Lotka-Volterra system in random environments can be described by the following stochastic differential equation (in the Stratonovich sense) with regime switching

$$dx_i(t) = x_i(t)\left\{\left[b_i(\alpha(t)) - \sum_{j=1}^{n} a_{ij}(\alpha(t))x_j(t)\right]dt + \sigma_i(\alpha(t)) \circ dw_i(t)\right\},\ i = 1, \dots, n,$$

where $w(\cdot) = (w_1(\cdot), \dots, w_n(\cdot))'$ is an n-dimensional standard Brownian motion, $b(\alpha) = (b_1(\alpha), \dots, b_n(\alpha))'$, $A(\alpha) = (a_{ij}(\alpha))$, and $\Sigma(\alpha) = \operatorname{diag}(\sigma_1(\alpha), \dots, \sigma_n(\alpha))$ with $\alpha \in \mathcal{M}$ represent intrinsic growth rates, the community matrices, and noise intensities in different external environments, respectively. It is well known that the above stochastic differential equation in the Stratonovich sense is equivalent to the system in the Itô sense

$$dx_i(t) = x_i(t)\left\{\left[r_i(\alpha(t)) - \sum_{j=1}^{n} a_{ij}(\alpha(t))x_j(t)\right]dt + \sigma_i(\alpha(t))dw_i(t)\right\},\ i = 1, \dots, n, \tag{2.1}$$

where $r_i(\alpha) := b_i(\alpha) + \frac{1}{2}\sigma_i^2(\alpha)$ for each $i = 1, 2, \dots, n$. In ecology and biology, one prefers to start the formulation of stochastic Lotka-Volterra systems using calculus in the Stratonovich sense because each term has its clear ecological meaning. However, for the analysis, the Itô calculus should be used. Assume throughout the chapter that the Markov chain $\alpha(\cdot)$ and the Brownian motion $w(\cdot)$ are independent. Without loss of generality, we also assume that the initial conditions $x(0)$ and $\alpha(0)$ are non-random.

Note that $(x(t), \alpha(t))$ is a Markov process, whose generator $\mathcal{L}$ is given as follows (see [42, Chapter 2] and also [44, 45] for a definition of the generator of a Markov process). For any

$V : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ with $V(\cdot, \alpha)$ being twice continuously differentiable with respect to the variable x for each $\alpha \in \mathbb{Z}_+$, we define

$$\begin{aligned}\mathcal{L}V(x,\alpha) := & \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x,\alpha) x_i \Big(r_i(\alpha) - \sum_{j=1}^{n} a_{ij}(\alpha) x_j \Big) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} V(x,\alpha) x_i^2 \sigma_i^2(\alpha) \\ & + \sum_{\beta \in \mathbb{Z}_+, \beta \neq \alpha} q_{\alpha\beta} [V(x,\beta) - V(x,\alpha)].\end{aligned} \tag{2.2}$$

Comparing to [44, 45], the Markov chain takes values in a countably infinite set.

2.2 Existence, Uniqueness, and Continuity of Solutions

Before getting to the two-time-scale systems, we first examine systems without the time scale separation. Existence, uniqueness, and continuity of solutions of the regime-switching Lotka-Volterra systems when $\mathbb{Z}_+$ is countably infinite are not available. So we present these results first in what follows. Denote

$$\begin{aligned}&\Xi(x,\alpha) := \mathrm{diag}(x_1, \ldots, x_n)[r(\alpha) - A(\alpha)x], \\ &\xi_i(x,\alpha) = x_i(r_i(\alpha) - \sum_{j=1}^{n} a_{ij}(\alpha) x_j), \\ &s_i(x,\alpha) = x_i \sigma_i(\alpha) \\ &S(x,\alpha) = \mathrm{diag}(s_i(x,\alpha)).\end{aligned} \tag{2.3}$$

By a competitive system, we mean that all values in the community matrix $A(\alpha)$ are non-negative ($a_{ij}(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}_+$ and $i, j = 1, 2, \ldots, n$). It is reasonable to assume that the competitions among the same species are strictly positive. Therefore, we assume the following condition holds.

(A1) For each $\alpha \in \mathbb{Z}_+ = \{1, 2, \ldots\}$, $a_{ii}(\alpha) > 0$ and $a_{ij}(\alpha) \geq 0$ for $i, j = 1, 2, \ldots, n$ and $j \neq i$.

In [44], the existence and uniqueness for the switching diffusion model was obtained when the state space of the switching is finite. However, the state space of the Markov chain in our study is countable but not finite. If $\alpha(t-) := \lim_{s \to t^-} \alpha(s) = \alpha$, then it can switch to β at t

with intensity $q_{\alpha\beta}$. Denote for each α, $q_\alpha = \sum_{\beta\in\mathbb{Z}_+,\beta\neq\alpha} q_{\alpha\beta}$. Note that $\alpha(t)$ may be written as the solution to a stochastic differential equation with respect to a Poisson random measure. To be more precisely, let $\mathfrak{p}(dt, dz)$ be a Poisson random measure with intensity $dt \times \mathfrak{m}(dz)$ and $\mathfrak{m}$ be the Lebesgue measure on $\mathbb{R}$ such that $\mathfrak{p}(\cdot,\cdot)$ is independent of the Brownian motion $w(t)$. Using this fact, for each $\alpha \in \mathbb{Z}$, we can construct disjoint sets $\{\Delta_{\alpha\beta}, \beta \neq \alpha\}$ on the real line as follows

$$\begin{aligned}
&\Delta_{12} = [0, q_{12}),\\
&\Delta_{13} = [q_{12}, q_{12} + q_{13}),\\
&\dots\\
&\Delta_{21} = [q_1, q_1 + q_{21}),\\
&\Delta_{23} = [q_1 + q_{21}, q_1 + q_{21} + q_{23}),\\
&\dots
\end{aligned}$$

Define $h : \mathbb{Z}_+ \times \mathbb{R} \mapsto \mathbb{R}$ by $h(\alpha, z) = \displaystyle\sum_{\beta\in\mathbb{Z}_+,\beta\neq\alpha}^{\infty} (\beta - \alpha)\mathbb{1}_{\{z\in\Delta_{\alpha\beta}\}}$, where $\mathbb{1}_{\{z\in\Delta_{\alpha\beta}\}} = 1$ if $z \in \Delta_{\alpha\beta}$ and $\mathbb{1}_{\{z\in\Delta_{\alpha\beta}\}} = 0$, is the indicator function. The process $\alpha(t)$ can be defined as a solution to

$$d\alpha(t) = \int_{\mathbb{R}} h(\alpha(t-), z)\mathfrak{p}(dt, dz),$$

where $\mathfrak{p}(dt, dz)$ is a Poisson measure with intensity $dt\times\mathfrak{m}(dz)$ and $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{R}$. We assume the following condition holds.

(A2) The Markov chain having generator Q is strongly exponentially ergodic (see [1]) in that there exist a $K > 0$ and a $\lambda_0 > 0$ such that

$$\sum_{\beta=1}^{\infty} |p_{\alpha\beta}(t) - \nu_\beta| \leq K \exp(-\lambda_0 t) \tag{2.4}$$

for any positive integer α and $t > 0$, where ν is the stationary distribution associated with the generator Q and $p_{\alpha\beta}(t) = P(\alpha(t) = \beta | \alpha(0) = \alpha)$. Moreover,

$$M = \sup_{\alpha} q_{\alpha} = \sup_{\alpha} \sum_{\beta \neq \alpha} q_{\alpha\beta} < \infty. \tag{2.5}$$

To proceed, we obtain the global solution in $\mathbb{R}^n_+$ for the system. Then we establish the positivity of solution $x(t)$, finite moments, and continuity. One of the main tools is to use an appropriate Lyapunov functions.

Theorem 2.1. *Assume* (A1) *and* (A2)*. Then for any initial data* $x(0) = x_0 \in \mathbb{R}^n_+$ *and* $\alpha(0) = \alpha \in \mathbb{Z}_+$*, there is a unique solution* $x(t) = (x_1(t), \dots, x_n(t))'$ *to* (2.1) *on* $t \geq 0$*, and the solution will remain in* $\mathbb{R}^n_+$ *almost surely, i.e.,* $x(t) \in \mathbb{R}^n_+$ *a.s. for any* $t \geq 0$.

Proof. The proof consists of two parts. In the first part, we show that there is a unique global solution, and in the second part, we show the solution lives in $\mathbb{R}^n_+$.

Step 1: For any $\iota \in \mathbb{Z}_+$, in view of [30, Theorem 2.1] there is a unique strong solution for the following diffusion

$$dx(t) = \Xi(x(t), \iota)dt + S(x(t), \iota)dw(t), \; x(0) = x_0 \in \mathbb{R}^n_+. \tag{2.6}$$

The rest of the proof of this part is similar to that of [32, Theorem 3.1], so we will be brief. For any stopping time τ and an $\mathcal{F}_\tau$-measurable $\mathbb{R}^n$-valued random variable $x(\rho)$, there exists a strong solution to (2.6) in $[\rho, \infty)$; see [31, Remark 3.10]. We proceed to construct the solution with any initial data $(x_0, i_0) \in \mathbb{R}^n_+ \times \mathbb{Z}_+$ by the interlacing procedure [2, Chapter 5]. Denote by $\tilde{x}^{(0)}(t), t \geq 0$ the solution to

$$d\tilde{x}^{(0)}(t) = \Xi(\tilde{x}^{(0)}(t), i_0)dt + S(\tilde{x}^{(0)}(t), i_0)dw(t), \; \tilde{x}^{(0)}(0) = x_0.$$

Set $\rho_1 = \inf\{t > 0 : \int_0^t \int_{\mathbb{R}} h(i_0, z)\mathfrak{p}(ds, dz) \neq 0\}$, $i_1 = i_0 + \int_0^{\rho_1} \int_{\mathbb{R}} h(i_0, z)\mathfrak{p}(ds, dz)$, and let $\tilde{x}^{(1)}(t), t \leq \rho_1$ be the solution to

$$d\tilde{x}^{(1)}(t) = \Xi(\tilde{x}^{(0)}(t), i_1)dt + S(\tilde{x}^{(0)}(t), i_1)dw(t),$$

with initial data $\tilde{x}^{(1)}(\rho_1) = \tilde{x}^{(0)}(\rho_1)$. Continuing this procedure, let $\rho_\infty = \lim\limits_{k\to\infty} \rho_k$ and set $x(t) = \tilde{x}^{(k)}(t)$, $\alpha(t) = i_k$, if $\rho_k \le t < \rho_{k+1}$. Then

$$\begin{cases} x(t\wedge\rho_k) = x_0 + \displaystyle\int_0^{t\wedge\rho_k} \Xi(x(s),\alpha(s))ds + S(x(s),\alpha(s))dw(s), \\ \alpha(t\wedge\rho_k) = i_0 + \displaystyle\int_0^{t\wedge\rho_k} \int_{\mathbb{R}} h(\alpha(s-),z)\mathfrak{p}(ds,dz), \end{cases}$$

where $\mathfrak{p}(ds,dz)$ is a Poisson random measure as defined in [42, p. 29] with modification to countable state space; see also [32]. To verify that $x(t)$ is a global solution, we claim that $\rho_\infty = \infty$. In fact, it can be shown as in [32], for any $T > 0$,$P(\rho_k \le T) \le \displaystyle\sum_{l=k}^{\infty} e^{-MT}\frac{(MT)^l}{l!}$. Thus $P(\rho_k \le T) \to 0$ as $k \to \infty$ so $\rho_\infty = \infty$ a.s. The uniqueness of $x(t)$ follows from the uniqueness of $\tilde{x}^{(k)}(t)$ on $[\rho_k, \rho_{k+1})$. Thus, we have shown that there is a unique global solution to $dx(t) = \Xi(x(t),\alpha(t))dt + S(x(t),\alpha(t))dw(t)$ with arbitrary initial data (x_0, i_0).

Step 2: Show the solution $x(t)$ obtained in Step 1 above remains in $\mathbb{R}^n_+$. The proof is similar to [44, Theorem 2.1] although the switching set is now countable. Let $k_0 \in \mathbb{N}$ be sufficiently large such that every component of $x(0)$ is contained in $(\frac{1}{k_0}, k_0)$. For each $k \ge k_0$, define

$$\zeta_k := \inf\left\{ t \in [0,\zeta) : x_i(t) \notin (\frac{1}{k}, k) \text{ for some } i = 1,2,\ldots,n \right\}. \tag{2.7}$$

The sequence $\zeta_k, k = 1,2,\ldots$ is monotone so there is a limit $\zeta_\infty := \lim_{k\to\infty}\zeta_k$ with $\zeta_\infty \le \zeta$. We are to show $\zeta_\infty = \infty$ a.s. For suppose not, there would exist some $T > 0$ and $\varepsilon > 0$ such that $P\{\zeta_\infty \le T\} > \varepsilon$. Therefore, we can find some $k_1 \ge k_0$ such that

$$P\{\zeta_k \le T\} > \varepsilon, \quad \text{for all} \quad k \ge k_1. \tag{2.8}$$

Now, we consider the following Lyapunov function $V(x,\alpha) = V(x)$ independent of α given by $V(x) = \sum_{i=1}^n [x_i^\gamma - 1 - \gamma \log x_i]$ for $x \in \mathbb{R}^n_+$ and $0 < \gamma < 1$. Detailed calculation shows that for all $x \in \mathbb{R}^n_+$, $V(x) \ge 0$ and $\mathcal{L}V(x) \le K < \infty$, where in the above, we used condition

(A1). In view of Itô's Lemma [33], for any $k \geq k_1$,

$$V(x(T \wedge \zeta_k)) - V(x(0)) = \int_0^{T\wedge\zeta_k} \mathcal{L}V(x(s))ds + \sum_{i=1}^n \int_0^{T\wedge\zeta_k} \gamma\sigma_i(\alpha(s))(x_i^\gamma(s) - 1)dw_i(s).$$

By virtue of Dynkin's formula and the bound $\mathcal{L}v(x) \leq K$, $KT + V(x(0)) \geq E[V(x(T\wedge\zeta_k))] \geq E[V(x(\zeta_k))I_{\{\zeta_k \leq T\}}]$. By the definitions of ζ_k and V, we have $V(x(\zeta_k)) \geq (k^\gamma - 1 - \gamma \log k) \wedge (\frac{1}{k^\gamma} - 1 + \gamma \log k)$, and hence, it follows from (2.8) that

$$KT + V(x(0)) \quad \geq [(k^\gamma - 1 - \gamma \log k) \wedge (\tfrac{1}{k^\gamma} - 1 + \gamma \log k)]P\{\zeta \leq T\} \to \infty, \text{ as } k \to \infty.$$

This is a contradiction, so we must have $\lim_{k\to\infty} \zeta_k = \infty$ a.s., so $\zeta = \infty$ a.s. Thus, the solution of (2.6) remains in $\mathbb{R}_+^n$ almost surely. □

We next consider the stochastic boundedness. First, we recall the definition.

Definition 2.2. The solution $x(t)$ of (2.1) is stochastically bounded (or bounded in probability), if for any $\eta > 0$, there is a constant $H = H_\eta$ such that for any $x_0 \in \mathbb{R}_+^n$,

$$\limsup_{t\to\infty} P\{|x(t)| \leq H\} \leq 1 - \eta. \tag{2.9}$$

Theorem 2.3. *Under the conditions of Theorem* 2.1 *and for any* $p > 0$ *satisfying*

$$\sup_{\alpha\in\mathbb{Z}^+} \sum_{i=1}^n \frac{1 + pb_i(\alpha) + \frac{p^2}{2}\sigma_i^2(\alpha)}{a_{ii}(\alpha)} < \infty, \tag{2.10}$$

we have

$$\sup_{t\geq 0} E\left[\sum_{i=1}^n x_i^p(t)\right] \leq K < \infty. \tag{2.11}$$

Proof. Let $k_0 \in \mathbb{N}$ be sufficiently large such that every component of $x(0)$ is contained in the interval $(\frac{1}{k_0}, k_0)$. For each $k \geq k_0$, we define $\tau_k := \inf\{t \in [0,\infty) : x_i(t) \notin (\frac{1}{k}, k)$ for some $i = 1, 2, \ldots, n\}$. Similar to the proof in Step 2 of Theorem 2.1, we can show that $\lim_{k\to\infty} \tau_k = \infty$ a.s.

Consider $V(x) = \sum_{i=1}^{n} x_i^p$. Then it follows that for $x \in \mathbb{R}_+^n$, we have

$$\begin{aligned}\mathcal{L}V(x) &= p \sum_{\iota \in \mathbb{Z}_+} \sum_{i=1}^{n} x_i^p \Big[b_i(\iota) + \frac{p}{2}\sigma_i^2(\iota) - \sum_{j=1}^{n} a_{ij}(\iota) x_j \Big] I_{\{\alpha=\iota\}} \\ &\le p \sum_{\iota \in \mathbb{Z}_+} \sum_{i=1}^{n} x_i^p \Big[b_i(\iota) + \frac{p}{2}\sigma_i^2(\iota) - a_{ii}(\iota) x_i \Big] I_{\{\alpha=\iota\}},\end{aligned} \tag{2.12}$$

where in the last step, we used condition (A1). By applying generalized Itô's Lemma [33] to $e^t V(x(t))$, we have

$$\begin{aligned}& e^{t\wedge\tau_k} \sum_{i=1}^{n} x_i^p(t\wedge\tau_k) - \sum_{i=1}^{n} x_i^p(0) \\ &= \int_0^{t\wedge\tau_k} e^s (V(x(s)) + \mathcal{L}V(x(s)))ds + p \sum_{i=1}^{n} \int_0^{t\wedge\tau_k} e^s x_i^{p-1}(s) \sigma_i(\alpha(s)) dw_i(s),\end{aligned}$$

where τ_k is the stopping time defined at the beginning of the proof. Thus taking expectations on both sides and using the assumption (H1), we obtain from (2.12) that

$$E[e^{t\wedge\tau_k} \sum_{i=1}^{n} x_i^p(t\wedge\tau_k)] - \sum_{i=1}^{n} x_i^p(0) = E \int_0^{t\wedge\tau_k} e^s (V(x(s)) + \mathcal{L}V(x(s)))ds \le E \int_0^{t\wedge\tau_k} e^s K ds. \tag{2.13}$$

By (2.13), we have

$$E[e^{t\wedge\tau_k} \sum_{i=1}^{n} x_i^p(t\wedge\tau_k)] - \sum_{i=1}^{n} x_i^p(0) \le E \int_0^{t\wedge\tau_k} e^s K ds \le K(e^t - 1).$$

Therefore, by virtue of Fatou's Lemma and letting $k \to \infty$, we obtain that

$$E\Big[\sum_{i=1}^{n} x_i^p(t) \Big] \le e^{-t} \sum_{i=1}^{n} x_i^p(0) + K(1 - e^{-t}) \le K < \infty.$$

In view of the exponential dominance above, taking $\sup_{t\ge 0}$, we obtain the desired result. □

By virtue of Tchebychev's inequality, a direct consequence of Theorem 2.3 is that the solution $x(t)$ is stochastically bounded. Next we obtain the sample path continuity.

Theorem 2.4. *The solution $x(t)$ to (2.1) is continuous a.s.*

Proof. For any $0 \le \tilde{t} \le t$, we have

$$x_i(t) - x_i(\tilde{t}) = \int_{\tilde{t}}^{t} \xi_i(x(r), \alpha(r))dr + \int_{\tilde{t}}^{t} s_i(x(r), \alpha(r))dr,$$

and hence

$$|x_i(t) - x_i(\tilde{t})|^4 \le 8\left| \int_{\tilde{t}}^{t} \xi_i(x(r), \alpha(r))dr \right|^4 + 8\left| \int_{\tilde{t}}^{t} s_i(x(r), \alpha(r))dr \right|^4. \tag{2.14}$$

Detailed computations in Theorem 2.3 and Hölder's inequality lead to

$$E\left| \int_{\tilde{t}}^{t} \xi_i(x(r), \alpha(r))dr \right|^4 \le (t-\tilde{t})^3 E \int_{\tilde{t}}^{t} |\xi_i(x(r), \alpha(r))|^4 dr \le K|t-\tilde{t}|^4. \tag{2.15}$$

Meanwhile, we can show that $E\left| \int_{\tilde{t}}^{t} s_i(x(r), \alpha(r))dr \right|^4 \le K|t-\tilde{t}|^2$. Thus

$$E[|x(t) - x(\tilde{t})|^4] \le K|t-\tilde{t}|^2. \tag{2.16}$$

The desired result then follows from the well-known Kolmogorov continuity criterion. □

Remark 2.5. In fact, almost all sample paths of the solutions (2.1) are Hölder continuous with exponent $\gamma < \frac{1}{4}$. That is, except a null set N with probability 0, for all $\omega \in \Omega \backslash N$, there exists a random variable $h(\omega) > 0$ satisfying

$$P\left\{ \omega : \sup_{0 \le s,t<\infty, |t-s|<h(\omega)} \frac{|x(t,\omega) - x(s,\omega)|}{|t-s|^\gamma} \le \frac{2}{1-2^{-\gamma}} \right\} = 1. \tag{2.17}$$

2.3 Two-Time-Scale Models

2.3.1 Two-Time-Scale Markov Chains

Recall that a generator Q or its corresponding Markov chain is said to be irreducible if the system of equations

$$\nu Q = 0, \qquad \sum_{\alpha=1}^{\infty} \nu_\alpha = 1 \tag{2.18}$$

has a unique solution $\nu(\nu_1, \nu_2, \dots)$ satisfying that $\nu_\alpha > 0$ for $\alpha = 1, 2, \dots$ Such a solution is termed a stationary distribution. Throughout the rest of the paper, we assume that the Markov chain has a fast varying part and slowly varying part in that $\alpha(t) = \alpha^\varepsilon(t)$, with generator

$$Q^{\varepsilon} = \frac{Q}{\varepsilon} + Q_0, \tag{2.19}$$

where Q is a generator of a Markov chain that is irreducible and Q_0 is a generator of another Markov chain. We do not have any restrictions on Q_0. For simplicity, we use (2.19) in this chapter. Although Q_0 appears in (2.19), the asymptotic properties are dominated by Q. It is possible to consider more complex models with more structure on Q. For the subsequent study, we need a couple of preliminary results. The proofs of (i) and (ii) in Lemma 2.6 can be found in [40, Theorem 4.5, Theorem 4.48, Lemma 5.1]. Denote $p^{\varepsilon}(t) = (P(\alpha^{\varepsilon}(t) = \alpha) : \alpha = 1, 2, \dots)$, $p^{\varepsilon}_{\alpha\beta}(t, t_0) = P(\alpha^{\varepsilon}(t) = \beta | \alpha^{\varepsilon}(t_0) = \alpha)$, and $P^{\varepsilon}(t, t_0)$ is the transition matrix $(p^{\varepsilon}_{\alpha\beta}(t, t_0))$.

Lemma 2.6. *Assume that for Q given in* (2.19) *satisfies* (A2)*. Then there exists a positive constant κ_0 such that*

(i) *For the probability distribution vector $p^{\varepsilon}(t) \in \mathbb{R}^{1\times\infty}$*

$$p^{\varepsilon}(t) = \nu + O(\varepsilon + e^{-\kappa_0 t/\varepsilon}) \tag{2.20}$$

uniformly in $(0, t)$.

(ii) *For the transition probability matrix $P^{\varepsilon}(t, t_0)$, we have*

$$P^{\varepsilon}(t, t_0) = P_0(t) + O\Big(\varepsilon + e^{-\kappa_0 (t - t_0)/\varepsilon}\Big), \tag{2.21}$$

uniformly in (t_0, t), where $P_0(t) = \mathbb{1}\nu$. with $\mathbb{1} = (1, 1, \dots)'$ being an infinite column vector having all entries 1, and $\nu = (\nu_1, \nu_2, \dots)$ is the row vector of stationary distribution associated with the Markov chain with generator Q.

Theorem 2.7. *Assume* (2.18)*. Then for each $\alpha = 1, 2, \dots,$*

$$E\left[\int_0^{\infty} e^{-t}(I_{\{\alpha^{\varepsilon}(t)=\alpha\}} - \nu_{\alpha})dt\right]^2 = O(\varepsilon). \tag{2.22}$$

Proof. The proof here is similar to [5]. Direct calculations leads to

$$E\left[\int_0^\infty e^{-t}(I_{\{\alpha^\varepsilon(t)=\alpha\}} - \nu_\alpha)dt\right]^2 = \int_0^\infty \int_0^t e^{-t-s}O(\varepsilon + e^{-\kappa_0(t-s)/\varepsilon}dsdt + \int_0^\infty \int_0^s e^{-t-s}O(\varepsilon + e^{-\kappa_0(s-t)/\varepsilon}dsdt.$$

Furthermore, $O(\varepsilon)\int_0^\infty \int_0^t e^{-t-s}dsdt = O(\varepsilon)$. In addition, for some $K > 0$,

$$\int_0^\infty \int_0^t e^{-t-s}O(e^{-\kappa_0(t-s)/\varepsilon})dsdt \le K\int_0^\infty \int_0^t e^{-t(\kappa_0+\varepsilon)/\varepsilon}e^{s(\kappa_0-\varepsilon)/\varepsilon}dsdt \le K\frac{\varepsilon}{2(\kappa_0-\varepsilon)} = O(\varepsilon).$$

Thus, $\displaystyle\int_0^\infty \int_0^t e^{-t-s}O(\varepsilon + e^{-\kappa_0(t-s)/\varepsilon})dsdt = O(\varepsilon)$. Likewise, by symmetry, we also have $\displaystyle\int_0^\infty \int_0^s e^{-t-s}O(\varepsilon + e^{-\kappa_0(s-t)/\varepsilon})dtds = O(\varepsilon)$. The proof is complete. □

Let $x^\varepsilon(t) \in \mathbb{R}^n$ for $t \ge 0$ be given by

$$dx^\varepsilon(t) = \mathrm{diag}(x_1^\varepsilon(t), \dots, x_n^\varepsilon(t))\Big[(r(\alpha^\varepsilon(t)) - A(\alpha^\varepsilon(t))x^\varepsilon(t))\,dt + \Sigma(\alpha^\varepsilon(t))dw(t)\Big], \tag{2.23}$$

with the initial conditions $x(0) = x_0$ and $\alpha^\varepsilon(0) = \alpha_0 \in \mathbb{Z}_+$. Under (A1) and (A2), we can construct the solutions of the two-time-scale stochastic differential equations by using similar method as in Theorem 2.1. The existence and uniqueness of solutions of the stochastic differential equations (2.23) hold; $0 \in \mathbb{R}^n$ is a stationary point for each equation in (2.23).

Remark 2.8. Note that existence and uniqueness of solutions, path continuity, and moment bounds established in 2.2 hold for the two-time scale system (2.23). Our main effort below is to show how we may reduce the computational complexity.

Lemma 2.9. *Under* (A1) *and* (A2), $\{x^\varepsilon(\cdot)\}$ *given by* (2.23) *converges weakly to* $x(\cdot)$ *such that* $x(\cdot)$ *satisfies*

$$dx(t) = \overline{\Xi}(x(t))dt + \overline{\Lambda}(x(t))dw(t), \tag{2.24}$$

where $\overline{\Xi}(x) = \sum_{\alpha=1}^\infty \Xi(x,\alpha)\nu_\alpha$, $\overline{\Lambda}(x)\overline{\Lambda}'(x) = \sum_{\alpha=1}^\infty S(x,\alpha)S'(x,\alpha)\nu_\alpha$.

Weak convergence of $x^\varepsilon(\cdot)$ to $x(\cdot)$ is a basic notion in stochastic processes. A definition can be found in [42, pp.371-376]. For convenience, we denote

$$\begin{aligned}
&\overline{r}_i = \sum_{\alpha=1}^{\infty} r_i(\alpha)\nu_\alpha,\ \overline{b}_i = \sum_{\alpha=1}^{\infty} b_i(\alpha)\nu_\alpha,\ \overline{a}_{ij} = \sum_{\alpha=1}^{\infty} a_{ij}(\alpha)\nu_\alpha,\ \overline{\sigma}_i = \sqrt{\sum_{\alpha=1}^{\infty} \sigma_i^2(\alpha)\nu_\alpha} \ \text{ and} \\
&\overline{\xi}_i(x) = \sum_{\alpha=1}^{\infty} \xi_i(x,\alpha)\nu_\alpha,\ \overline{\lambda}_i(x) = x_i\overline{\sigma}_i = \sqrt{\sum_{\alpha=1}^{\infty} s_i^2(x,\alpha)\nu_\alpha},
\end{aligned} \tag{2.25}$$

where $s_i(x,\alpha), S(x,\alpha)$ and $\Xi(x,\alpha)$ are defined in (2.3). The averaged system can be written component-wise as

$$dx_i(t) = x_i(t)\left\{\left[\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij}x_j(t)\right]dt + \overline{\sigma}_i dw_i(t)\right\}. \tag{2.26}$$

Remark 2.10. Note the following facts.

- The proof of the above lemma is similar to the development in [40, Ch.8].

- The averaged system (2.24) is a Lotka-Volterra diffusion system, whose coefficients are an average with respect to the stationary measure ν. Hence, under (A1) and (A2), we can prove that the averaged system (2.24) has a unique solution that is continuous together with moment bounds. This follows the way of treating nonlinear stochastic differential equations. First, we show that there is a local solution and then extend the solution to a global solution by using stopping time argument; see for example, [30, Theorem 2.1].

In the study of stochastic population systems, we are interested in the permanence and extinction of the population. We shall study this by means of the corresponding limit system. Treating directly stability of dynamic systems containing two-time-scale Markov chains is a complex matter. However, considering this problem using limit system is much simpler. Some earlier work concerning the stability of those systems can be found in [5]. In this study, our goal is to establish the permanence and extinction of (2.23) for sufficiently small ε. Here, from a Lyapunov function $V(x)$ of the averaged system, we construct a perturbed Lyapunov function for the more complex original system containing the fast varying Markov chain.

The method we use is motivated by arguments in [22, pp. 148-149]. The averaged system is a diffusion without switching, whereas in the original system, the switching states belong to a countably infinite set. Using the limit system, we can examine the original system, which is much easier than dealing with the original system directly. As a result, our approach leads to a significant reduction of complexity.

2.3.2 Preliminary Calculations

To proceed, we first present some preliminary calculations using perturbed Lyapunov function for preparation on study of various properties of the complex original system. Let $\mathcal{F}_t^\varepsilon = \sigma\{x^\varepsilon(s), \alpha^\varepsilon(s), s \le t\}$, and E_t^ε be the expectation conditioned on $\mathcal{F}_t^\varepsilon$. For a suitable function $\zeta(t)$, define the operator $\mathcal{L}^\varepsilon$ by

$$\mathcal{L}^\varepsilon \zeta(t) = \lim_{\delta \downarrow 0} \frac{1}{\delta} E_t^\varepsilon [\zeta(t+\delta) - \zeta(t)]. \tag{2.27}$$

The generator is as defined in (2.2) with the switching part given by (2.19). As a result, the generator of the switching diffusion process is ε dependent. Let $V(x)$ be a Lyapunov function associated with the averaged system (2.24) independent of the discrete component. Using (2.2) for $V(x^\varepsilon(t))$ where $x^\varepsilon(t)$ is the solution of system (2.23), we obtain

$$\mathcal{L}^\varepsilon V(x^\varepsilon(t)) = \sum_{i=1}^n V_{x_i}(x^\varepsilon(t)) \xi_i(x^\varepsilon(t), \alpha^\varepsilon(t)) + \frac{1}{2} \sum_{i=1}^n V_{x_i x_i}(x^\varepsilon(t)) s_i^2(x^\varepsilon(t), \alpha^\varepsilon(t)).$$

Define

$$V_1^\varepsilon(x,t) = E_t^\varepsilon \int_t^\infty e^{t-u} \sum_{i=1}^n V_{x_i}(x)[\xi_i(x, \alpha^\varepsilon(u)) - \overline{\xi}_i(x)] du, \tag{2.28}$$

and

$$V_2^\varepsilon(x,t) = E_t^\varepsilon \int_t^\infty e^{t-u} \frac{1}{2} \sum_{i=1}^n V_{x_i x_i}(x)[s_i^2(x, \alpha^\varepsilon(u)) - \overline{\lambda}_i^2(x)] du. \tag{2.29}$$

This implies that with $\alpha^\varepsilon(t) = \ell$,

$$\begin{aligned} V_1^\varepsilon(x,t) &= E_t^\varepsilon \int_t^\infty e^{t-u} \sum_{i=1}^n V_{x_i}(x) \sum_{k=1}^\infty \xi_i(x,k)[I_{\{\alpha^\varepsilon(u)=k\}} - \nu_k]du \\ &= \int_t^\infty e^{t-u} \sum_{i=1}^n V_{x_i}(x) \sum_{k=1}^\infty \xi_i(x,k)[p_{\ell k}(u) - \nu_k]du. \end{aligned}$$

Hence,

$$V_1^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[V(x^\varepsilon(t)) + 1]. \tag{2.30}$$

To proceed, we use a notation

$$G(x,\alpha) \quad = \sum_{i=1}^n \sum_{k=1}^\infty V_{x_i}(x)\xi_i(x,k)\Big[I_{\{\alpha=k\}} - \nu_k\Big]. \tag{2.31}$$

Using (2.27), (2.30), and (2.31),

$$\begin{aligned} &\mathcal{L}^\varepsilon V_1^\varepsilon(x^\varepsilon(t),t) \\ &= \lim_{\delta\downarrow 0} \tfrac{1}{\delta} E_t^\varepsilon[V_1^\varepsilon(x^\varepsilon(t+\delta),t+\delta) - V_1^\varepsilon(x^\varepsilon(t),t)] \\ &= \sum_{k=1}^\infty \sum_{i=1}^n \sum_{j=1}^n [V_{x_i}(x^\varepsilon(t))\xi_i(x^\varepsilon(t),k)]_{x_j} \xi_j(x^\varepsilon(t),\alpha^\varepsilon(t)) E_t^\varepsilon \int_t^\infty e^{t-u}[I_{\{\alpha^\varepsilon(u)=k\}} - \nu_k]du \\ &\quad + O(\varepsilon)(V(x^\varepsilon(t))+1) - \sum_{i=1}^n V_{x_i}(x^\varepsilon(t))[\xi_i(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\xi}_i(x^\varepsilon(t))] \\ &= O(\varepsilon)(V(x^\varepsilon(t))+1) - \sum_{i=1}^n V_{x_i}(x^\varepsilon(t))[\xi_i(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\xi}_i(x^\varepsilon(t))]. \end{aligned} \tag{2.32}$$

Similar to the estimate of $V_1^\varepsilon(x,t)$, it can be verified that

$$\begin{aligned} &V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[V(x^\varepsilon(t))+1], \\ &\mathcal{L}^\varepsilon V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[V(x^\varepsilon(t))+1] - \frac{1}{2}\sum_{i=1}^n V_{x_ix_i}(x^\varepsilon(t))[s_i^2(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\lambda_i}^2(x^\varepsilon(t))]. \end{aligned} \tag{2.33}$$

Define $V^\varepsilon(x^\varepsilon(t),t) = V(x^\varepsilon(t)) + V_1^\varepsilon(x^\varepsilon(t),t) + V_2^\varepsilon(x^\varepsilon(t),t)$ satisfying the following properties:

$$V^\varepsilon(x^\varepsilon(t),t) = V(x^\varepsilon(t)) + O(\varepsilon)(V(x^\varepsilon(t))+1).$$

$$\mathcal{L}^\varepsilon V^\varepsilon(x^\varepsilon(t),\alpha^\varepsilon(t),t) = O(\varepsilon)(V(x^\varepsilon(t))+1) + \mathcal{L}V(x^\varepsilon(t),\overline{\alpha}),$$

where

$$\mathcal{L}V(x^{\varepsilon}(t),\overline{\alpha}) = \sum_{i=1}^{n} V_{x_i}(x^{\varepsilon}(t))x_i^{\varepsilon}(t)\Big(\overline{r}_i - \sum_{j=1}^{n}\overline{a}_{ij}x_j^{\varepsilon}(t)\Big) + \frac{1}{2}\sum_{i=1}^{n} V_{x_ix_i}(x^{\varepsilon}(t))[x_i^{\varepsilon}(t)]^2\overline{\sigma}_i^2.$$

2.3.3 Stochastic Boundedness

First, under suitable conditions, the averaged system is stochastically bounded. This follows from a specialization of the proof of [44, Theorem 3.1] (for the case that the switching set has only one element), which is a refinement of the arguments of moment bounds in [29].

Lemma 2.11. *Assume that* (A1), (A2), *and* (2.18) *are satisfied. Then the following statements hold for the solution $x(t)$ of* (2.24).

(1) *For any $p > 0$,*

$$\sup_{t\geq 0} E\Big[\sum_{i=1}^{n} x_i^p(t)\Big] \leq K < \infty. \tag{2.34}$$

(2) *For any $p > 0$,*

$$\limsup_{t\to\infty} E\Big[|x(t)|^p\Big] \leq K < \infty. \tag{2.35}$$

(3) *The solution of the averaged system* (2.24), *namely, $x(t)$, is stochastically bounded, i.e., for any $\delta > 0$, there is a constant $H = H(\delta)$ such that for any $x_0 \in \mathbb{R}_+^n$, we have*

$$\limsup_{t\to\infty} P\{|x(t)| \leq H\} \geq 1-\delta. \tag{2.36}$$

With the lemma above, we proceed to show that the solution of system (2.1) also has the same boundedness property if ε is small enough. Note that the next theorem should be compared with Theorem 2.3. Different from Theorem 2.3, the condition (2.10) is not needed in the following theorem.

Theorem 2.12. *Assume that* (A1), (A2), *and* (2.18) *are satisfied. Then the following statements hold for the solution $x^{\varepsilon}(t)$ of* (2.23) *for ε sufficiently small.*

(1) *For any* $p > 0$,

$$\sup_{t\geq 0} E\left[\sum_{i=1}^{n}[x_i^{\varepsilon}(t)]^p\right] \leq K < \infty. \tag{2.37}$$

(2) *For any* $p > 0$,

$$\limsup_{t\to\infty} E\left[|x^{\varepsilon}(t)|^p\right] \leq K < \infty. \tag{2.38}$$

(3) *The process* $x^{\varepsilon}(t)$ *is stochastically bounded. That is, for any* $\delta > 0$, *there is a constant* $H = H(\varepsilon, \delta)$ *such that for any* $x_0^{\varepsilon} \in \mathbb{R}_+^n$, *we have*

$$\limsup_{t\to\infty} P\{|x^{\varepsilon}(t)| \leq H\} \geq 1 - \delta. \tag{2.39}$$

Proof. We use perturbed Lyapunov function methods to prove this theorem. Consider

$$V(x) = \sum_{i=1}^{n}[x_i]^p \quad \text{and} \quad \tilde{V}(x) = \sum_{i=1}^{n}\left[[x_i]^{\gamma} - 1 - \gamma \log x_i\right].$$

Similar to (2.28) and (2.29), we define

$$V_1^{\varepsilon}(x,t) = E_t^{\varepsilon}\int_t^{\infty} e^{t-u}\sum_{i=1}^{n} V_{x_i}(x)[\xi_i(x,\alpha^{\varepsilon}(u)) - \overline{\xi}_i(x)]du,$$

$$V_2^{\varepsilon}(x,t) = E_t^{\varepsilon}\int_t^{\infty} e^{t-u}\frac{1}{2}\sum_{i=1}^{n} V_{x_ix_i}(x)[s_i^2(x,\alpha^{\varepsilon}(u)) - \overline{\lambda}_i^2(x)]du,$$

$$\tilde{V}_1^{\varepsilon}(x,t) = E_t^{\varepsilon}\int_t^{\infty} e^{t-u}\sum_{i=1}^{n} \tilde{V}_{x_i}(x)[\xi_i(x,\alpha^{\varepsilon}(u)) - \overline{\xi}_i(x)]du,$$

$$\tilde{V}_2^{\varepsilon}(x,t) = E_t^{\varepsilon}\int_t^{\infty} e^{t-u}\frac{1}{2}\sum_{i=1}^{n} \tilde{V}x_ix_i(x)[s_i^2(x,\alpha^{\varepsilon}(u)) - \overline{\lambda}_i^2(x)]du,$$

which have the following properties:

$$V_1^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[V(x^\varepsilon(t))+1],$$

$$\tilde V_1^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[\tilde V(x^\varepsilon(t))+1],$$

$$V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[V(x^\varepsilon(t))+1],$$

$$\tilde V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)[\tilde V(x^\varepsilon(t))+1],$$

$$\mathcal{L}^\varepsilon V_1^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)(V(x^\varepsilon(t))+1) - \sum_{i=1}^n V_{x_i}(x^\varepsilon(t))[\xi_i(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\xi}_i(x^\varepsilon(t))],$$

$$\mathcal{L}^\varepsilon \tilde V_1^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)(\tilde V(x^\varepsilon(t))+1) - \sum_{i=1}^n \tilde V_{x_i}(x^\varepsilon(t))[\xi_i(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\xi}_i(x^\varepsilon(t))],$$

$$\mathcal{L}^\varepsilon V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)(V(x^\varepsilon(t))+1) - \sum_{i=1}^n V_{x_ix_i}(x^\varepsilon(t))[s_i^2(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\lambda}_i^2(x^\varepsilon(t))],$$

$$\mathcal{L}^\varepsilon \tilde V_2^\varepsilon(x^\varepsilon(t),t) = O(\varepsilon)(\tilde V(x^\varepsilon(t))+1) - \sum_{i=1}^n \tilde V_{x_ix_i}(x^\varepsilon(t))[s_i^2(x^\varepsilon(t),\alpha^\varepsilon(t)) - \overline{\lambda}_i^2(x^\varepsilon(t))].$$

Define

$$V^\varepsilon(x^\varepsilon(t),t) = V(x^\varepsilon(t)) + V_1^\varepsilon(x^\varepsilon(t),t) + V_2^\varepsilon(x^\varepsilon(t),t) \quad \text{and}$$

$$\tilde V^\varepsilon(x^\varepsilon(t),t) = \tilde V(x^\varepsilon(t)) + \tilde V_1^\varepsilon(x^\varepsilon(t),t) + \tilde V_2^\varepsilon(x^\varepsilon(t),t).$$

Then

$$\begin{aligned}\mathcal{L}^\varepsilon \tilde V^\varepsilon(x^\varepsilon(t),t) &\le O(\varepsilon)(\tilde V(x^\varepsilon(t))+1) + \gamma \sum_{i=1}^n \bigg\{ -\overline{a}_{ii}[x_i^\varepsilon(t)]^{\gamma+1} + \Big(\overline{b}_i + \frac{\gamma}{2}\overline{\sigma}_i^2\Big)[x_i^\varepsilon(t)]^\gamma \\ &\qquad + \Big(\sum_{j=1}^n \overline{a}_{ji}\Big)x_i^\varepsilon(t) - \overline{b}_i\bigg\} \\ &\le O(\varepsilon)\tilde V(x^\varepsilon(t)) + K \quad \text{as} \quad \varepsilon \text{ is small enough;}\end{aligned} \tag{2.40}$$

$$\begin{aligned}\mathcal{L}^\varepsilon V^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)(V(x^\varepsilon(t))+1) + p\sum_{i=1}^n [x_i^\varepsilon(t)]^p \Big[\overline{b}_i + \frac{p}{2}\overline{\sigma}_i^2 - \sum_{j=1}^n \overline{a}_{ij}x_j^\varepsilon(t)\Big] \\ &\le O(\varepsilon)(V(x^\varepsilon(t))+1) + p\sum_{i=1}^n [x_i^\varepsilon(t)]^p \Big[\overline{b}_i + \frac{p}{2}\overline{\sigma}_i^2 - \overline{a}_{ii}x_j^\varepsilon(t)\Big],\end{aligned} \tag{2.41}$$

where we used condition (A1).

Let $k_0 \in \mathbb{N}$ be sufficiently large such that every component of $x^\varepsilon(0)$ is contained within the interval $\left(\frac{1}{k_0}, k_0\right)$. For each $k \geq k_0$, we define

$$\tau_k := \inf\left\{t \in [0,\infty) : x_i^\varepsilon(t) \notin (\frac{1}{k}, k) \text{ for some } i = 1, 2, \ldots, n\right\}. \tag{2.42}$$

Clearly, the sequence τ_k, $k = 1, 2, \ldots$ is monotonically increasing. Set $\tau_\infty := \lim_{k\to\infty} \tau_k$. We want to show that $\tau_\infty = \infty$ a.s. If this were false, there would exist some $T > 0$ and $\tilde{\varepsilon} > 0$ such that $P\{\tau_\infty \leq T\} > \tilde{\varepsilon}$. Therefore, we can find some $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} > \tilde{\varepsilon}, \quad \text{for all } k \geq k_1. \tag{2.43}$$

By (2.40), it can be verified that for any $(x, \alpha) \in \mathbb{R}_+^n \times \mathbb{Z}_+$,

$$\mathcal{L}^\varepsilon \tilde{V}^\varepsilon(x^\varepsilon(t), t) \leq O(\varepsilon)\tilde{V}(x^\varepsilon(t)) + K$$

Using the generalized Itô's Lemma and taking the expectation on both sides, for any $k \geq k_1$, we have

$$E^\varepsilon \tilde{V}^\varepsilon(x^\varepsilon(t \wedge \tau_k), t \wedge \tau_k) - \tilde{V}^\varepsilon(x^\varepsilon(0), 0) \leq E^\varepsilon \int_0^{t\wedge\tau_k} O(\varepsilon)\tilde{V}(x^\varepsilon(s))ds + Kt.$$

Thus,

$$(1 + O(\varepsilon))E^\varepsilon \tilde{V}(x^\varepsilon(t \wedge \tau_k)) \leq \tilde{V}^\varepsilon(x^\varepsilon(0), \alpha^\varepsilon(0), 0) + Kt + E^\varepsilon \int_0^{t\wedge\tau_k} O(\varepsilon)\tilde{V}(x^\varepsilon(s))ds.$$

When ε is small enough, applying the generalized Gronwall's inequality, we obtain

$$E^\varepsilon \tilde{V}(x^\varepsilon(t \wedge \tau_k)) \leq \frac{\tilde{V}^\varepsilon(x^\varepsilon(0), \alpha^\varepsilon(0), 0) + Kt}{1 + O(\varepsilon)} \; e^{\dfrac{O(\varepsilon)(t\wedge\tau_k)}{1+O(\varepsilon)}}.$$

Letting $t = T$, we have $E^\varepsilon \tilde{V}(x^\varepsilon(T \wedge \tau_k)) < \infty$. On the other hand,

$$\begin{aligned} E^\varepsilon \tilde{V}(x^\varepsilon(T \wedge \tau_k)) &\geq E^\varepsilon[\tilde{V}(x^\varepsilon(\tau_k))I_{\{\tau_k \leq T\}}] \\ &> \tilde{\varepsilon}[(k^\gamma - 1 - \gamma \log k) \wedge ((1/k)^\gamma - 1 + \gamma \log k)] \to \infty, \end{aligned}$$

as $k \to \infty$. This is a contradiction so we must have $\lim_{k\to\infty} \tau_k = \infty$ a.s.

By applying generalized Itô's Lemma to $e^t V^\varepsilon(x^\varepsilon(t), \alpha^\varepsilon(t), t)$ and taking the expectations of both sides, we have

$$
\begin{aligned}
&(1+O(\varepsilon))\Big\{E^\varepsilon[e^{t\wedge\tau_k}\sum_{i=1}^n[x_i^\varepsilon(t\wedge\tau_k)]^p]-\sum_{i=1}^n[x_i^\varepsilon(0)]^p\Big\}\\
&=E^\varepsilon\int_0^{t\wedge\tau_k}e^s(V^\varepsilon(x^\varepsilon(s),s)+\mathcal{L}^\varepsilon V^\varepsilon(x^\varepsilon(s),s)ds\\
&\le E^\varepsilon\int_0^{t\wedge\tau_k}\Big[pe^s\sum_{i=1}^n[x_i^\varepsilon(s)]^p\Big(\frac{1+O(\varepsilon)}{p}+\overline{b}_i+\frac{p}{2}\overline{\sigma}_i^2-\overline{a}_{ii}x_i^\varepsilon(s)\Big)+O(\varepsilon)e^s\Big]ds\\
&\le E^\varepsilon\int_0^{t\wedge\tau_k}e^sK(\varepsilon)ds.
\end{aligned}
\tag{2.44}
$$

By (2.44), we have

$$E[e^{t\wedge\tau_k}\sum_{i=1}^n[x_i^\varepsilon(t\wedge\tau_k)]^p]-\sum_{i=1}^n[x_i^\varepsilon(0)]^p\le E^\varepsilon\int_0^{t\wedge\tau_k}e^sKds\le K(e^t-1).$$

Therefore, by virtue of Fatou's Lemma and letting $k\to\infty$, we obtain

$$E\big[\sum_{i=1}^n[x_i^\varepsilon(t)]^p\big]\le e^{-t}\sum_{i=1}^n[x_i^\varepsilon(0)]^p+K(1-e^{-t})\le K<\infty.$$

In view of the exponential dominance above, taking $\sup_{t\ge0}$, we obtain the desired result. The next two parts of the theorem can be obtained similar to Section 2.2. □

2.3.4 Stability in Probability

Stability of dynamic systems with switching containing randomly perturbed processes has been done recently; see [5]. In this study, our first goal is to establish the stability of (2.23) with small ε via the stability of the averaged system (2.24). We first recall the definition of stability for stochastic differential equations; see [19].

Definition 2.13. The equilibrium point $x=0$ of the system (2.24) is said to be stable in probability, if for any $\varepsilon>0$ and any $\alpha\in\mathbb{Z}_+$, $\lim\limits_{y\to0}P\{\sup\limits_{t\ge0}|x^{y,\alpha}(t)|>\varepsilon\}=0$, where $x^{y,\alpha}(t)$ denotes the solution of (2.24) with initial data $x(0)=y$ and $\alpha(0)=\alpha$.

Using similar argument as [19], we establish the following lemma.

Lemma 2.14. *Let $D \in \mathbb{R}^n$ be a neighborhood of 0. Suppose that for each $i \in \mathbb{Z}_+$, there exists a non-negative function $V(\cdot, \alpha) : D \mapsto \mathbb{R}$ such that*

(i) *$V(\cdot, \alpha)$ is continuous in D and vanished only at $x = 0$;*

(ii) *$V(\cdot, \alpha)$ is twice continuously differentiable in $D \backslash \{0\}$ and $\mathcal{L}V(x, \alpha) \leq 0$, $\forall x \in D\backslash\{0\}$.*

Then the equilibrium point $x = 0$ is stable in probability.

Theorem 2.15. *Assume that*

$$(\overline{r}_i - \overline{a}_{ii})^2 + 4\overline{a}_{ii}(\overline{b}_i + \overline{\sigma}_i^2) < 0, \text{ for all } i = 1, 2, \ldots, n. \tag{2.45}$$

Then under assumptions (A1), (A2), *and* (2.18), *the equilibrium point $x = 0$ is stable in probability for the averaged system* (2.24).

Proof. We consider the Lyapunov function

$$V(x) = \sum_{i=1}^{n} x_i - \log(x_i + 1). \tag{2.46}$$

It can be seen that $V(x)$ satisfies condition (i) of Lemma 2.14.

For (2.24), we have

$$\mathcal{L}V(x) = \sum_{i=1}^{n} \frac{x_i^2}{x_i + 1}\Big(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j\Big) + \frac{1}{2}\sum_{i=1}^{n} \frac{1}{(x_i+1)^2} x_i^2 \overline{\sigma}_i^2. \tag{2.47}$$

By condition (A1), the property of solutions and the assumption, we have

$$\begin{aligned} \mathcal{L}V(x) &\leq \sum_{i=1}^{n} \frac{x_i^2}{(x_i+1)^2}\Big\{(x_i+1)(\overline{r}_i - \overline{a}_{ii}x_i) + \frac{1}{2}\overline{\sigma}_i^2\Big\} \\ &= \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i)^2}\Big[- \overline{a}_{ii}x_i^2 + (\overline{r}_i - \overline{a}_{ii})x_i + (\overline{b}_i + \overline{\sigma}_i^2)\Big] < 0 \text{ for all } x \neq 0. \end{aligned} \tag{2.48}$$

Thus, by Lemma 2.14, the equilibrium point $x = 0$ of system (2.24) is stable in probability. □

Theorem 2.16. *Under conditions* (A1), (A2), (2.18), *and* (2.45), *the equilibrium point $x = 0$ is stable in probability for* (2.23) *for sufficiently small ε.*

Proof. With $V(x)$ defined by (2.46), $V_1^\varepsilon(x,t)$ defined by (2.28), $V_2^\varepsilon(x,t)$ defined by (2.29) and their corresponding estimates, it is easy to see that

$$V^\varepsilon(t) = V(x) + V_1^\varepsilon(x,t) + V_2^\varepsilon(x,t)$$

satisfies condition (i) in Lemma 2.14. Note that $V(x)$ is an increasing function and when ε is small enough, by Theorem 2.12, the process $x^\varepsilon(t)$ is stochastically bounded. Hence, $V(x^\varepsilon(t))$ is bounded for ε is small enough.

Furthermore,

$$\mathcal{L}^\varepsilon V^\varepsilon(t) = O(\varepsilon)\Big[V(x^\varepsilon(t)) + 1\Big] + \sum_{i=1}^{n} \frac{(x_i^\varepsilon(t))^2}{x_i^\varepsilon(t)+1}\big(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j^\varepsilon(t)\big) + \frac{1}{2}\sum_{i=1}^{n} \frac{(x_i^\varepsilon(t))^2}{(x_i^\varepsilon(t)+1)^2}\overline{\sigma}_i^2. \tag{2.49}$$

By virtue of (2.48), $\mathcal{L}^\varepsilon V^\varepsilon(t) \le 0$ for all $x^\varepsilon(t) \ne 0$ and ε small enough. This verifies the theorem. $\square$

2.3.5 Extinction

In this section, we show if the averaged system (2.24) is extinct, then the more complex switching system (2.23) is also extinct for sufficiently small ε.

Definition 2.17. The population is said to reach the extinction if $\lim_{t\to\infty} |x(t)| = 0$ a.s., i.e., $\lim_{t\to\infty} \sum_{i=1}^n |x_i(t)| = 0$ a.s.

Theorem 2.18. *Assume that*

$$\overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 \le -c, \text{ for all } i = 1, 2, \ldots, n, \tag{2.50}$$

where c is a positive number. Then under assumptions (A1), (A2), *and* (2.18), *the population of the averaged system* (2.24) *will become extinct exponentially a.s. for sufficiently small ε.*

Proof. For each $i = 1, 2 \ldots, n$, consider

$$V_i(x) = \log(x_i). \tag{2.51}$$

where x_i is the ith component of x. Using the definition of the generator, we have

$$\mathcal{L}V_i(x(t)) = \overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j(t) - \frac{1}{2}\overline{\sigma}_i^2.$$

Applying Itô's Lemma, we obtain

$$\begin{aligned} \log(x_i(t)) &= \log(x_i(0)) + \int_0^t \left(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j(s) - \frac{1}{2}\overline{\sigma}_i^2 \right) ds + \int_0^t \overline{\sigma}_i dw_i(s) \\ &\le \log(x_i(0)) + t\left(\overline{r}_i - \tfrac{1}{2}\overline{\sigma}_i^2\right) + \overline{\sigma}_i w_i(t). \end{aligned}$$

$w_i(t)$ is a Brownian motion. Therefore, the strong law of large numbers for martingales implies that $\lim_{t\to\infty} \dfrac{w_i(t)}{t} = 0$ a.s. It follows by

$$\limsup_{t\to\infty} \frac{\log(x_i(t))}{t} \le \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 \le -c \quad a.s.$$

Thus, the sample Lyapunov exponent of the solution is negative, and the population will become extinct exponentially a.s. $\square$

Theorem 2.19. *Under conditions* (A1), (A2), (2.18), *and* (2.50), *the population of the system* (2.23) *will become extinct exponentially for sufficiently small* ε.

Proof. With $V_i(x)$ defined by (2.51), $V_{i,1}^{\varepsilon}(x,t)$ defined by (2.28), $V_{i,2}^{\varepsilon}(x,t)$ defined by (2.29) and their corresponding estimates, $V_i^{\varepsilon}(x) = V_i(x) + V_{i,1}^{\varepsilon}(x,t) + V_{i,2}^{\varepsilon}(x,t)$ satisfies the following properties:

$$\begin{aligned} &V_i^{\varepsilon}(x^{\varepsilon}(t)) = V_i(x^{\varepsilon}(t)) + O(\varepsilon)(V_i(x^{\varepsilon}(t)) + 1) \\ &\mathcal{L}^{\varepsilon}V_i^{\varepsilon}(x^{\varepsilon}(t)) = O(\varepsilon)(V_i(x^{\varepsilon}(t)) + 1) + \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 - \sum_{j=1}^{n} \overline{a}_{ij} x_j^{\varepsilon}(t). \end{aligned}$$

By the generalized Itô Lemma,

$$\begin{aligned}\log\left(x_i^\varepsilon(t)\right) &= \log\left(x_i^\varepsilon(0)\right) + \int_0^t \left[O(\varepsilon)(V_i(x^\varepsilon(s))+1) + \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 - \sum_{j=1}^n \overline{a}_{ij} x_j^\varepsilon(s)\right] ds \\ &\quad + \int_0^t \sigma_i(\alpha^\varepsilon(s)) dw_i(s) \\ &\le \log\left(x_i^\varepsilon(0)\right) + t\left[O(\varepsilon) + \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2\right] + O(\varepsilon)\int_0^t \log\left(x_i^\varepsilon(s)\right) ds \\ &\quad + \int_0^t \sigma_i(\alpha^\varepsilon(s)) dw_i(s).\end{aligned}$$

Denote $M(t) = \int_0^t \sigma_i(\alpha^\varepsilon(s))dw_i(s)$ and $M(t)$ is a martingale. Using the quadratic variation of this martingale, we obtain that $t^{-1}\langle M, M\rangle_t = t^{-1}\int_0^t \sigma_i^2(\alpha^\varepsilon(s))ds$ is bounded a.s. The strong law of large numbers for martingales leads to $\lim_{t\to\infty} M(t)/t = 0$ a.s. (see [28, Theorem 1.3.4]). In addition, $\lim\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \log\left(x_i^\varepsilon(s)\right) ds = \log(\overline{x}_i(t))$ a.s, where $\overline{x}_i(t)$ is the solution of (2.24) (see [40, Chapter 8]).Then $\limsup\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \log\left(x_i^\varepsilon(s)\right) ds = \log(\overline{x}_i(t))$ a.s. Therefore,

$$\begin{aligned}\limsup_{t\to\infty} \frac{\log\left(x_i^\varepsilon(t)\right)}{t} &\le O(\varepsilon) + \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 + \limsup_{t\to\infty}\frac{O(\varepsilon)}{t}\int_0^t \log\left(x_i^\varepsilon(s)\right) ds \\ &\le O(\varepsilon) + \overline{r}_i - \frac{1}{2}\overline{\sigma}_i^2 + O(\varepsilon)\log(\overline{x}_i(t)) \quad \text{a.s.}\end{aligned}$$

When ε is small enough, under condition (2.50), $\limsup\limits_{t\to\infty} \dfrac{\log\left(x_i^\varepsilon(t)\right)}{t} < 0$ a.s. This results in the exponential extinction of the population. □

2.3.6 Stochastic Permanence

We first recall the definition of stochastic permanence.

Definition 2.20. The population system (2.24) is said to be stochastically permanent if for any $\delta \in (0,1)$, there exist positive constants $H = H(\delta)$ and $K = K(\delta)$ such that

$$\liminf_{t\to\infty} P\{|x(t)| \ge H\} \ge 1-\delta, \quad \liminf_{t\to\infty} P\{|x(t)| \le K\} \ge 1-\delta, \tag{2.52}$$

where $x(t)$ is the solution of the population system (2.24) with any initial condition $x(0) \in \mathbb{R}_+^n$.

Lemma 2.21. *Assume that* (A1), (A2), *and* (2.18) *hold. Then population system* (2.24) *is stochastically permanent when* $\overline{b}_i > 0$ *for* $i = 1, 2, \ldots, n$.

Proof. To obtain the stochastic permanence, we need to prove two inequalities in (2.52) and the first part is followed by Theorem 2.12. Before working on the second part, we first set the notation: $\widetilde{r} := \max \overline{r}_i$, $\hat{r} = \min \overline{r}_i$, $\hat{b} = \min \overline{b}_i$, $\widetilde{a} = \max \overline{a}_{ij}$, $\widetilde{\sigma} = \overline{\sigma}_i$. We begin to work with some estimates for the averaged system (2.24), where $x(t)$ is the solution. Let θ be a positive constant such that $\theta\widetilde{\sigma}^2 < 2\hat{b}$, and $\kappa > 0$ satisfying $0 < \dfrac{2\kappa}{\theta} < 2\hat{b} - \theta\widetilde{\sigma}^2$. Consider

$$V(x) = \sum_{i=1}^{n} x_i,\ U(x) = \frac{1}{V(x)}, \quad \text{and} \quad J(x) = e^{\kappa t}\big(1 + U(x)\big)^{\theta}.$$

By applying Itô's Lemma, we have

$$\begin{aligned} dU(x(t)) = & \Big[- U^2(x(t)) \sum_{i=1}^{n} x_i(t)(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j(t)) + U^3(x(t)) \sum_{i=1}^{n} \overline{\sigma}_i^2 x_i^2(t) \Big] dt \\ & - U^2(x(t)) \sum_{i=1}^{n} \overline{\sigma}_i x_i(t) dw_i(t). \end{aligned}$$

Note that

$$\begin{aligned} dJ(x(t)) = \theta e^{\kappa t}(1 + U(x(t)))^{\theta-2} \Big\{ \Big[& \frac{\kappa}{\theta}(1 + U(x(t)))^2 - (1 + U(x(t)))U^2(x(t)) \sum_{i=1}^{n} x_i(t) \\ & \times(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j(t)) + U^3(x(t)) \sum_{i=1}^{n} \overline{\sigma}_i^2 x_i^2(t) + \frac{\theta+1}{2} U^4(x(t)) \sum_{i=1}^{n} \overline{\sigma}_i^2 x_i^2(t) \Big] dt \\ & - (1 + U(x(t)))U^2(x(t)) \sum_{i=1}^{n} \overline{\sigma}_i x_i(t) dw_i(t) \Big\}. \end{aligned} \tag{2.53}$$

Therefore,

$$\begin{aligned} \mathcal{L}J(x) = \theta e^{\kappa t}(1 + U(x))^{\theta-2} \Big[& \frac{\kappa}{\theta}(1 + U(x))^2 - (1 + U(x))U^2(x) \sum_{i=1}^{n} x_i(\overline{r}_i - \sum_{j=1}^{n} \overline{a}_{ij} x_j) \\ & + U^3(x) \sum_{i=1}^{n} \overline{\sigma}_i^2 x_i^2 + \frac{\theta+1}{2} U^4(x) \sum_{i=1}^{n} \overline{\sigma}_i^2 x_i^2 \Big]. \end{aligned} \tag{2.54}$$

We have

$$
\begin{aligned}
&-(1+U(x(t)))U^2(x(t))\sum_{i=1}^n x_i(t)(\overline{r}_i-\sum_{j=1}^n \overline{a}_{ij}x_j(t))\\
&\quad\le -(1+U(x(t)))U^2(x(t))\sum_{i=1}^n x_i(t)(\overline{r}_i-\widetilde{a}\sum_{j=1}^n x_j(t))\\
&\quad= -(1+U(x(t)))U^2(x(t))\Big[\sum_{i=1}^n x_i(t)(\overline{b}_i+\frac{1}{2}\overline{\sigma}_i^2)-\widetilde{a}(\sum_{i=1}^n x_i(t))(\sum_{j=1}^n x_j(t))\Big]\\
&\quad\le -(1+U(x(t)))U^2(x(t))\sum_{i=1}^n x_i(t)\hat{b}-(1+U(x(t)))U^2(x(t))\sum_{i=1}^n \frac{1}{2}x_i(t)\overline{\sigma}_i^2\\
&\qquad\quad +(1+U(x(t)))\widetilde{a}\\
&\quad\le -\hat{b}(1+U(x(t)))U(x(t))-U^3(x(t))\sum_{i=1}^n x_i(t)\frac{\overline{\sigma}_i^2}{2}+(1+U(x(t)))\widetilde{a}\\
&\quad\le -\hat{b}(1+U(x(t)))U(x(t))-\frac{1}{2}U^4(x(t))\sum_{i=1}^n x_i^2(t)\overline{\sigma}_i^2+(1+U(x(t)))\widetilde{a}.
\end{aligned} \tag{2.55}
$$

$$
U^3(x(t))\sum_{i=1}^n \overline{\sigma}_i^2 x_i^2(t)\le U^3(x(t))\widetilde{\sigma}^2\sum_{i=1}^n x_i^2(t)\le U(x(t))\widetilde{\sigma}^2\frac{\sum_{i=1}^n x_i^2(t)}{(\sum_{i=1}^n x_i(t))^2}\le \widetilde{\sigma}^2 U(x(t)). \tag{2.56}
$$

Thus,

$$
\begin{aligned}
\mathcal{L}J(x(t)) &\le \theta e^{\kappa t}(1+U(x(t)))^{\theta-2}\Big[\frac{\kappa}{\theta}(1+U(x(t)))^2-\hat{b}(1+U(x(t)))U(x(t))\\
&\quad -\frac{U^4(x(t))}{2}\sum_{i=1}^n x_i^2(t)\overline{\sigma}_i^2+(1+U(x(t)))\widetilde{a}+\widetilde{\sigma}^2U(x(t))+\frac{\theta+1}{2}U^4(x(t))\sum_{i=1}^n \overline{\sigma}_i^2x_i^2(t)\Big]\\
&\le \theta e^{\kappa t}(1+U(x(t)))^{\theta-2}\Big[\frac{\kappa}{\theta}+\frac{2\kappa}{\theta}U(x(t))+\frac{\kappa}{\theta}U^2(x(t))-\hat{b}U(x(t))-\hat{b}U^2(x(t)))\\
&\quad +\widetilde{a}+\widetilde{a}U(x(t))+\widetilde{\sigma}^2U(x(t))+\frac{\theta}{2}\widetilde{\sigma}^2U^2(x(t))\Big]\le K\theta e^{\kappa t},
\end{aligned} \tag{2.57}
$$

where K is a positive constant depending on κ, θ, and coefficients of the system. (This inequality is resulted from the choice of θ and κ.)

Integrating and taking expectations on both sides of (2.53), we have: $E[J(x(t))]-$

$J(x(0)) \le K \int_0^t \theta e^{\kappa t} ds$, i.e.,

$$E[(1+U(x(t)))^\theta] \le e^{-\kappa t}(1+U(x(0)))^\theta + \frac{K\theta}{\kappa}.$$

Note that for $x \in \mathbb{R}_+^n$, $(\sum_{i=1}^n x_i)^\theta \le n^\theta |x|^\theta$. For any given $\delta \in (0,1)$, choose $H > 0$ such that $\dfrac{H^\theta n^\theta K\theta}{\kappa} \le \delta$. By Tchebychev's inequality, we obtain

$$\begin{aligned} P(|x(t)| < H) &\le P\left(U^\theta(x(t)) > \frac{1}{H^\theta n^\theta}\right) \\ &\le H^\theta n^\theta E[U(x(t))^\theta] \\ &\le H^\theta n^\theta E[(1+U(x(t)))^\theta] \\ &\le H^\theta n^\theta \left[e^{-\kappa t}(1+U(x(0)))^\theta + \tfrac{K\theta}{\kappa}\right]. \end{aligned}$$

This implies that $\limsup_{t\to\infty} P(|x(t)| < H) \le \dfrac{H^\theta n^\theta K\theta}{\kappa} \le \delta$, i.e. $\liminf_{t\to\infty} P(|x(t)| \ge H) \ge 1-\delta$. This completes the proof. □

Theorem 2.22. *Under conditions* (A1)*,* (A2)*, and* (2.18)*, system* (2.23) *is stochastically permanent when* $\overline{b}_i > 0$ *for each* $i = 1, 2, \ldots, n$ *and sufficiently small* ε*.*

Proof. We apply the definition of the generator (3.1) and obtain the following for the perturbed system (2.23), for each α,

$$\mathcal{L}^\varepsilon J(x) = \sum_{i=1}^n J_{x_i}(x)\xi_i(x,\alpha) + \frac{1}{2}\sum_{i=1}^n J_{x_i x_i}(x) s_i^2(x,\alpha).$$

Similar to (2.28) and (2.29), we define

$$J_1^\varepsilon(x,t) = E_t^\varepsilon \int_t^\infty e^{t-u} \sum_{i=1}^n J_{x_i}(x)[\xi_i(x,\alpha^\varepsilon(u)) - \overline{\xi}_i(x)]du,$$

$$J_2^\varepsilon(x,t) = E_t^\varepsilon \int_t^\infty e^{t-u} \frac{1}{2}\sum_{i=1}^n J_{x_i x_i}(x)[s_i^2(x,\alpha^\varepsilon(u)) - \overline{\lambda}_i^2(x)]du.$$

Then

$$
\begin{aligned}
J_1^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)[J(x^\varepsilon(t))+1],\\
J_2^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)[J(x^\varepsilon(t))+1],\\
\mathcal{L}^\varepsilon J_1^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)(J(x^\varepsilon(t),t)+1)\\
&\quad -\sum_{i=1}^n J_{x_i}(x^\varepsilon(t))[\xi_i(x^\varepsilon(t),\alpha^\varepsilon(t))-\overline{\xi}_i(x^\varepsilon(t))],\\
\mathcal{L}^\varepsilon J_2^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)(J(x^\varepsilon(t),t)+1)\\
&\quad -\sum_{i=1}^n J_{x_ix_i}(x^\varepsilon(t))[s_i^2(x^\varepsilon(t),\alpha^\varepsilon(t))-\overline{\lambda}_i^2(x^\varepsilon(t))].
\end{aligned}
$$

Define

$$J^\varepsilon(x,t)=J(x)+J_1^\varepsilon(x,t)+J_2^\varepsilon(x,t).$$

The functions satisfy

$$
\begin{aligned}
J^\varepsilon(x^\varepsilon(t),t) &= J(x^\varepsilon(t))+O(\varepsilon)(J(x^\varepsilon(t))+1)\\
\mathcal{L}^\varepsilon J^\varepsilon(x^\varepsilon(t),t) &= O(\varepsilon)(J(x^\varepsilon(t))+1)+\theta e^{\kappa t}(1+U(x^\varepsilon(t)))^{\theta-2}\Bigg[\frac{\kappa}{\theta}(1+U(x^\varepsilon(t)))^2\\
&\quad -(1+U(x^\varepsilon(t)))U^2(x^\varepsilon(t))\sum_{i=1}^n x_i^\varepsilon(t)(\overline{r}_i-\sum_{j=1}^n\overline{a}_{ij}x_j^\varepsilon(t))\\
&\quad +U^3(x^\varepsilon(t))\sum_{i=1}^n\overline{\sigma}_i^2(x_i^\varepsilon(t))^2+\frac{\theta+1}{2}U^4(x^\varepsilon(t))\sum_{i=1}^n\overline{\sigma}_i^2(x_i^\varepsilon(t))^2\Bigg]\\
&= O(\varepsilon)(J(x^\varepsilon(t))+1)+\theta e^{\kappa t}K.
\end{aligned}
\tag{2.58}
$$

Integrating on both sides of (2.58) and taking expectation, we have

$$E_t^\varepsilon[J^\varepsilon(x^\varepsilon(t),t)]-J^\varepsilon(x^\varepsilon(0),0)\le O(\varepsilon)E_t^\varepsilon[J(x^\varepsilon(t))]+O(\varepsilon)+\frac{\theta K}{\kappa}e^{\kappa t}.$$

Denote $J_0^\varepsilon=J^\varepsilon(x^\varepsilon(0),0)$. Then

$$(1-O(\varepsilon))E_t^\varepsilon[J(x^\varepsilon(t))]\le O(\varepsilon)+J_0^\varepsilon+\frac{\theta K}{\kappa}e^{\kappa t},$$

i.e., when $\varepsilon > 0$ is small enough, $1 - O(\varepsilon) > 0$ and

$$E_t^{\varepsilon}[(1 + U(x^{\varepsilon}(t)))^{\theta}] \leq \frac{O(\varepsilon) + J_0^{\varepsilon}}{1 - O(\varepsilon)} e^{-\kappa t} + \frac{\theta K}{\kappa}.$$

For any given $\delta \in (0, 1)$, choose $H > 0$ such that $\dfrac{H^{\theta} n^{\theta} K \theta}{\kappa} \leq \delta$, by using Tchebychev's inequality, we can obtain the first inequality in (2.52). The second inequality in (2.52) is obtained from Theorem 2.12. This completes the proof. $\square$

CHAPTER 3 NUMERICAL METHODS FOR GAMES IN INSURANCE

3.1 Formulation

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtered probability space, where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual condition and $\mathcal{F}_t = \mathcal{F}_t^{\alpha} \vee \mathcal{F}_t^{W} \vee \mathcal{F}_t^{N}$. We work with a finite horizon $[0, T]$, where $T < \infty$ is a positive real number. The processes $\{\alpha(t)\}_{t\geq 0}$, $\{W(t)\}_{t\geq 0} = \{W_Z(t), W_S(t)\}_{t\geq 0}$ and $\{N(t,\cdot)\}_{t\geq 0} = \{N_1(t,\cdot), N_2(t,\cdot)\}_{t\geq 0}$ are Brownian motions and jump processes, respectively, whose details will be given in the formulation of the next subsection.

3.1.1 Insurance Models

We are considering an insurance market consisting of two competing insurance companies. Each of them adopts optimal investment and reinsurance strategies to manage the insurance portfolios. The surplus process of each insurance company is subject to the random fluctuation of the market. Following the work of [8], the randomness of the market is modelled by a continuous-time finite-state Markov chain and an independent market-index process.

To delineate the random economy environment and other random economic factors, we use a continuous-time Markov chain $\alpha(t)$ taking values in a finite space $\mathcal{M} = \{1, \ldots, m\}$. The states of economy are represented by the Markov chain $\alpha(t)$. Let the continuous-time Markov chain $\alpha(t)$ be generated by $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$. That is,

$$\mathbf{P}\{\alpha(t+\delta) = j | \alpha(t) = i, \alpha(s), s \leq t\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } j \neq i, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } j = i, \end{cases} \tag{3.1}$$

where $q_{ij} \geq 0$ for $i, j = 1, 2, \ldots, m$ with $j \neq i$ and $\sum_{j\in\mathcal{M}} q_{ij} = 0$ for each $i \in \mathcal{M}$.

Furthermore, we are considering the insurance portfolios in a financial market with a market index $Z(t)$, whose price satisfies

$$dZ(t) = \mu_Z(t, Z(t))dt + \sigma_Z(t, Z(t))dW_Z(t), \tag{3.2}$$

where $W_Z(t)$ is a standard Brownian motion. Denote by $\{\mathcal{F}_t^{W_Z}\}$ the filtration generated by the Brownian motion $\{W_Z(t)\}_{t\geq 0}$. We note that $Z(t)$ captures the dynamics of the financial market. The cash flows of the insurance companies such as the premiums of insurance policies, claims, and expenses, are subject to the performance of financial market. Hence, the key parameters of the surplus process are defined as functionals of both the finite-state Markov chain $\alpha(t)$ and market index $Z(t)$.

Following the classical Cramér-Lundberg process, we assume that $\widehat{X}_k(t), k \in \{1, 2\}$, the surplus of insurance company k without investment and reinsurance satisfies

$$\widehat{X}_k(t) = \hat{x}_k + \int_0^t c_k(\alpha(s), Z(s))ds - Y_k(t), \quad t \geq 0, \tag{3.3}$$

where $\widehat{X}_k(0) := \hat{x}_k$ is the initial surplus, $c_k(\alpha(t), Z(t))$ is the rate of premium, and $Y_k(t) = \sum_{i=1}^{N_k(t)} A_i^k$ is a compound Poisson process with the claim size A_i^k with $\{A_i^k : i > 1\}$ being a sequence of positive, independent and identically distributed random variables.

In this work, we consider a Poisson measure in lieu of the traditionally used Poisson process. Suppose $\Theta \subset \mathbb{R}_+$ is a compact set and the function $q_k(\cdot)$ is the magnitude of the claim sizes.

$$N_k(t, H) = \text{ number of claims on } [0, t] \text{ with claim size taking values in } H \subset \Theta, \tag{3.4}$$

counts the number of claims up to time t, which is a Poisson counting process. For $k = 1, 2$, $Y_k(t)$ is a jump process representing claims for each company with arrival rate λ_k. Note that claim frequencies depend on the economy and financial market states. The function $q_k(\alpha(t), Z(t), \rho_k)$ is assumed to be the magnitude of the claim sizes, where ρ_k has distribution $\Pi_k(\cdot)$, and $q_k(i, \cdot, \rho_k)$ is continuous for each ρ_k and each $i \in \mathcal{M}$. At different regimes and financial market states, taking into consideration of random environment, the values of $q_k(i, \cdot, \rho_k)$

could be much different. Then the Poisson measure $N_k(\cdot,\cdot)$ has intensity $\lambda_k dt \times \Pi_k(d\rho_k)$ where $\Pi_k(d\rho_k) = f(\rho_k)d\rho_k$.

Let $\nu_{k,n}$ denote the time of the n-th claim and $\zeta_{k,n} = \nu_{k,n+1} - \nu_{k,n}$. Let $\{\zeta_{k,v}, \rho_k, v \geq n\}$ be independent of $\{X_k(s), \alpha(s), s \leq \nu_{k,n}, \zeta_{k,v}, \rho_k, v \leq n\}$. Then the n-th claim term is $q_k(\alpha(\nu_{k,n}), Z(\nu_{k,n}), \rho_k)$, and the claim amount of Y_k can be written as $Y_k(t) = \sum_{\nu_{k,n} \leq t} q_k(\alpha(\nu_{k,n}), Z(\nu_{k,n}), \rho_k)$.

3.1.2 Reinsurance and Investment

Let $a_k(t)$ be an $\mathcal{F}_t$-progressively measurable process valued in $[0,1]$, an exogenous retention level, which is a control chosen by the insurance company representing the reinsurance policy and $g(a_k)$ is the reinsurance premium rate. Denote by $\mathcal{A}_k := \{a_k(t) : 0 \leq a_k(t) \leq 1, 0 \leq t \leq T\}$, the set of reinsurance strategies of insurer k. Recall that A_i^k is the size of the ith claim. Let $A_i^k(a_k)$ be the fraction of each claim paid by the primary insurance company. Then the aggregation claim amount paid by the primary insurance company is denoted as $Y_k^{a_k}(t)$.

Remark 3.1. Note that both the claim frequencies and severities are depending on the Markov regimes and market index. It is a more general formulation compared with the work in [8], where only the claim frequencies depends on the Markov regimes and market index. Furthermore, with the compound Poisson jumps, the surplus process forms a controlled jump-diffusion regime-switching process. We aim to find optimal reinsurance strategies under the jump-diffusion regime-switching process formulation numerically.

The insurance companies invest in both risk-free assets $S_0(t)$ and risky assets $S(t)$ with prices satisfying

$$\begin{cases} \dfrac{dS_0(t)}{S_0(t)} = r(\alpha(t))dt, \\ \dfrac{dS(t)}{S(t)} = \mu_S(\alpha(t), Z(t))dt + \sigma_S(\alpha(t), Z(t))dW_S(t), \end{cases} \tag{3.5}$$

where $r(\alpha(t))$ and $\mu(\alpha(t), Z(t))$ are the return rates of the risk-free and risky assets, respectively; $\sigma_S(\alpha(t), Z(t))$ is the corresponding volatility; $W_S(t)$ is a standard Brownian motion independent of $W_Z(t)$. For $k = 1, 2$, the investment behavior of the insurer k is modelled as a portfolio process $b_k(t)$, where $b_k(t)$ is invested in the risky asset $S(t)$. Let $\mathcal{B}_k = \{b_k(t) : 0 \le t \le T\}$ denote the set of investment strategies of insurer k.

Combining the reinsurance and investment strategies, the surplus process of the insurance company k, denoted by $\widetilde{X}_k(t)$, follows

$$\begin{cases} d\widetilde{X}_k(t) = \Big\{ r(\alpha(t))\widetilde{X}_k(t) + b_k(t)[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_k(\alpha(t), Z(t)) - g(a_k(t)) \Big\}\, dt \\ \qquad\qquad\quad + b_k(t)\sigma_S(\alpha(t), Z(t))dW_S(t) - dY_k^{a_k}(t), \\ \widetilde{X}_k(0) = \widetilde{x}_k, \end{cases} \tag{3.6}$$

where $Y_k^{a_k}(t) = \displaystyle\int_0^t \int_{\mathbb{R}_+} \tilde{q}_k(q_k, a_k) N_k(dt, d\rho_k)$ and $\tilde{q}_k(q_k, a_k)$ is the magnitude of the claim sizes with respect to the surplus process.

In this work, we model the competition of the two insurance companies with investment and reinsurance schemes in finite time horizon using a game theoretic formulation. The performance of each company is measured by the relative performance of their surpluses against their competitor's. Thus, the competition between the two companies becomes a game problem with two players, each of which can adjust its reinsurance strategies based on the competitor's scheme. Let the relative surplus performance for insurance company k be $X_k(t) := \widetilde{X}_k(t) - \kappa_k \widetilde{X}_l(t)$, where $l = 3 - k$. Hence, $X_k(t)$ is governed by the following dynamic system

$$
\begin{aligned}
&dX_k(t)\\
&= \sum_{i\in\mathcal{M}} I_{\{\alpha(t)=i\}}\Big\{r(i)X_k(t) + (b_k(t)-\kappa_k b_l(t))[\mu_S(i,Z(t)) - r(i)] + c_k(i,Z(t)) - \kappa_k c_l(i,Z(t))\\
&\quad -(g(a_k(t)) - \kappa_k g(a_l(t)))dt + (b_k(t)-\kappa_k b_l(t))\sigma_S(i,Z(t))dW_S(t)\Big\} - dY_k^{a_k}(t) + \kappa_k dY_l^{a_l}(t),
\end{aligned}
\tag{3.7}
$$

where

$$Y_k^{a_k}(t) = \int_0^t \int_{\mathbb{R}_+} \tilde{q}_k(q_k, a_k) N_k(dt, d\rho_k).$$

3.1.3 Proportional Reinsurance

We allow the insurance companies to continuously reinsure a fraction of its claim with the retention level $a_k \in [0,1]$ with $k = 1,2$. Note that a_k is the exogenous retention level, and the control chosen by the insurance company for the reinsurance policy. Then $\tilde{q}(q_k, a_k) = a_k(t)q_k(\alpha(t), Z(t), \rho_k)$. We have

$$Y_k^{a_k}(t) = \sum_{i=1}^{N_k(t)} A_i^k(a_k) = \sum_{i=1}^{N_k(t)} a_k A_i^k.$$

Considering the proportional reinsurance strategies, for $k = 1,2$. The relative surplus process of the insurance company k, under the reinsurance and investment, follows

$$
\begin{cases}
dX_k(t) = \Big\{r(\alpha(t))X_k(t) + (b_k(t)-\kappa_k b_l(t))[\mu_S(\alpha(t),Z(t)) - r(\alpha(t))] + c_k(\alpha(t),Z(t))\\
\quad -\kappa_k c_l(\alpha(t),Z(t)) - [g(a_k(t)) - \kappa_k g(a_l(t))]\Big\}dt + (b_k(t)-\kappa_k b_l(t))\sigma_S(\alpha(t),Z(t))dW_S(t)\\
\quad -a_k(t)\int_{\mathbb{R}_+} q_k(\alpha(t),Z(t),\rho_k)N_t(dt,d\rho_k) + \kappa_k a_l(t)\int_{\mathbb{R}_+} q_l(\alpha(t),Z(t),\rho_l)N_l(dt,d\rho_l)\\
X_k(0) = \widetilde{x}_k - \kappa_k \widetilde{x}_l.
\end{cases}
\tag{3.8}
$$

3.1.4 Excess-of-loss Reinsurance

We allow the insurance companies to continuously reinsure its claim and pay all of the claims up to a pre-given level of amount (termed retention level). We still let a_k, $k = 1, 2$ be the retention level chosen by the insurance company to determine the reinsurance policy. We have that

$$Y_k^{a_k}(t) = \sum_{i=1}^{N_k(t)} A_i^k(a_k) = \sum_{i=1}^{N_k(t)} (A_i^k \wedge a_k).$$

Then $\tilde{q}_k(q_k, a_k) = q_k(\alpha(t), Z(t), \rho_k) \wedge a_k(t)$.

Considering the excess-of-loss reinsurance strategies, for $k = 1, 2$, under the reinsurance control and investment, the relative surplus process of the insurance company k follows

$$\begin{cases} dX_k(t) &= \Big\{ r(\alpha(t))X_k(t) + (b_k(t) - \kappa_k b_l(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_k(\alpha(t), Z(t)) \\ & \quad -\kappa_k c_l(\alpha(t), Z(t)) - [g(a_k(t)) - \kappa_k g(a_l(t))] \Big\} dt + (b_k(t) - \kappa_k b_l(t))\sigma_S(\alpha(t), Z(t)) dW_S(t) \\ & \quad - \int_{\mathbb{R}_+} (q_k(\alpha(t), Z(t), \rho_k) \wedge a_k) N_k(dt, d\rho_k) + \kappa_k \int_{\mathbb{R}_+} (q_l(\alpha(t), Z(t), \rho_l) \wedge a_l) N_l(dt, d\rho_l) \\ X_k(0) &= \widetilde{x}_k - \kappa_k \widetilde{x}_l. \end{cases} \tag{3.9}$$

3.1.5 Control Problem

For $k = 1, 2$, insurer k has a utility function $U_k : \mathbb{R} \to \mathbb{R}$, where U_k is assumed to be increasing, strictly concave, and satisfies the Inada conditions, i.e.,

$$\partial_x U_k(-\infty) = +\infty, \qquad \partial_x U_k(+\infty) = 0.$$

Following the work [16], the insurer k aims to maximize the expected utility of his relative performance at the terminal time T by adopting a pair of investment and reinsurance strategy $u_k = (a_k, b_k) \in \mathcal{A}_k \times \mathcal{B}_k$, denote $\mathcal{U}_k \triangleq \mathcal{A}_k \times \mathcal{B}_k$. For an arbitrary pair of admissible controls

$u = (u_1, u_2) \in \mathcal{U} \triangleq \mathcal{U}_1 \times \mathcal{U}_2$, the objective function is

$$\begin{aligned} J^k(t, x_k, z, i, u) &= \mathbf{E}\left[U_k\left((1-\kappa_k)\widetilde{X}_k(T) + \kappa_k(\widetilde{X}_k(T) - \widetilde{X}_l(T))\right)\right] \\ &= \mathbf{E}\left[U_k\left(\widetilde{X}_k(T) - \kappa_k \widetilde{X}_l(T)\right)\right], \end{aligned} \tag{3.10}$$

for $k \neq l \in \{1, 2\}$. For $k = 1, 2$, κ_k measures the sensitivity of insurer k to the performance of his competitor.

The control $u_k = (a_k, b_k)$ with $k \in \{1, 2\}$ is said to be *admissible* if a_k and b_k satisfy

(i) $a_k(t), b_k(t)$ are nonnegative for any $t \geq 0$,

(ii) Both a_k, b_k are adapted to $\mathcal{F}_t$.

(iii) $J^k(t, x_k, z, i, u) < \infty$ for any admissible pair $u_k = (a_k, b_k)$.

For $k = 1, 2$, let $\mathfrak{B}(\mathcal{U}_k \times [0, \infty))$ be the σ-algebra of Borel subsets of $\mathcal{U}_k \times [0, \infty)$. We use a relaxed control formulation; see [23] for a definition and more discussions. Recall that an *admissible relaxed control* $m_k(\cdot)$ is a measure on $\mathfrak{B}(\mathcal{U}_k \times [0, \infty))$ such that $m_k(\mathcal{U}_k \times [0, t)) = t$ for each $t \geq 0$. With the given probability space, we say that $m_k(\cdot)$ is an admissible relaxed (stochastic) control for $\mathcal{F}$, if $m_k(\cdot, \omega)$ is a deterministic relaxed control with probability one and if $m_k(A \times [0, t])$ is $\mathcal{F}_t$-adapted for all $A \in \mathfrak{B}(\mathcal{U}_k)$.

Given a relaxed control $m_k(\cdot)$ of $u_k(\cdot)$, we define the derivative $m_{t,k}(\cdot)$ such that

$$m_k(K) = \int_{\mathcal{U}_k \times [0,\infty)} I_{\{(u_k, t) \in K\}} m_{t,k}(d\phi_k) dt$$

for all $K \in \mathfrak{B}(\mathcal{U}_k \times [0, \infty))$, and that for each t, $m_{t,k}(\cdot)$ is a measure on $\mathfrak{B}(\mathcal{U}_k)$ satisfying $m_{t,k}(\mathcal{U}_k) = 1$. For example, we can define $m_{t,k}(\cdot)$ in any convenient way for $t = 0$ and as the left-hand derivative for $t > 0$,

$$m_{t,k}(A) = \lim_{\varrho \to 0} \frac{m_k(A \times [t - \varrho, t])}{\varrho}, \qquad \forall A \in \mathfrak{B}(\mathcal{U}_k).$$

Note that $m_k(d\phi_k dt) = m_{t,k}(d\phi_k)dt$. It is natural to define the relaxed control representation $m_k(\cdot)$ of $u_k(\cdot)$ by $m_{t,k}(A) = I_{\{u_k(t) \in A\}}$ $\forall A \in \mathfrak{B}(\mathcal{U}_k)$. Define the relaxed control $m(\cdot) = (m_1(\cdot) \times m_2(\cdot))$ with derivative $m_t(\cdot) = m_{t,1}(\cdot) \times m_{t,2}(\cdot)$. Thus $m(\cdot)$ is a measure on the Borel sets of $\mathcal{U} \times [0, \infty)$.

3.1.6 Nash Equilibrium

A Nash equilibrium $u^* = (u_1^*, u_2^*) \in \mathcal{U}$ is achieved such that

$$\begin{aligned} &\mathbf{E}[U_1(\widetilde{X}_1^{u_1}(T) - \kappa_1 \widetilde{X}_2^{u_2^*}(T))] \le \mathbf{E}[U_1(\widetilde{X}_1^{u_1^*}(T) - \kappa_1 \widetilde{X}_2^{u_2^*}(T))], \\ &\mathbf{E}[U_2(\widetilde{X}_2^{u_2}(T) - \kappa_2 \widetilde{X}_1^{u_1^*}(T))] \le \mathbf{E}[U_2(\widetilde{X}_2^{u_2^*}(T) - \kappa_1 \widetilde{X}_1^{u_1^*}(T))]. \end{aligned} \tag{3.11}$$

For $\alpha(t) = i \in \mathcal{M}$, $Z(t) = z$, and $X_k(t) = x_k$, where $0 \le t \le T$ and $k \ne l \in \{1, 2\}$, the value function of insurance company k follows

$$V^k(t, x_k, z, i) = \sup_{u_k \in \mathcal{H}_k} \mathbf{E}[U_k(\widetilde{X}_k^{u_k}(T) - \kappa_k \widetilde{X}_l^{u_l^*}(T))], \tag{3.12}$$

where $V^k(\cdot, \cdot, \cdot, \cdot)$ is the value function in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{M}$.

To obtain the system of Hamilton-Jacobi-Bellman (HJB) equations, we assume the existence of optimal control. For an arbitrary $u_k \in \mathcal{U}_k$, $\alpha(t) = i \in \mathcal{M}$, $k \ne l \in \{1, 2\}$, and $V^k(\cdot, \cdot, \cdot, i) \in C^2(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{M})$, define an integro-differential operator $\mathcal{L}^{u_k, u_l}$ by

$$\begin{aligned} \mathcal{L}^{u_k, u_l} V^k(t, x_k, z, i) = {} & V_{x_k}^k(t, x_k, z, i) \Big\{ r(i) x_k + (b_k - \kappa_k b_l)[\mu(i, z) - r(i)] + c_k(i, z) - \kappa_k c_l(i, z) \\ & -(g(a_k) - \kappa_k g(a_l)) \Big\} + V_z^k(t, x_k, z, i) m(t, z) \\ & + \frac{1}{2}(b_k - \kappa_k b_l)^2 \sigma_S^2(i, z) V_{x_k x_k}(t, x_k, z, i) + \frac{1}{2} \sigma_Z^2(t, z) V_{zz}(t, x_k, z, i) \\ & -\lambda_k \int_{\mathbb{R}_+} [V^k(t, x_k - \tilde{q}(i, z, \rho_k), z, i) - V^k(t, x_k, z, i)] f(\rho_k) d\rho_k \\ & +\lambda_l \int_{\mathbb{R}_+} [V^k(t, x_k + \kappa_k \tilde{q}(i, z, \rho_l), z, i) - V^k(t, x_k, z, i)] f(\rho_l) d\rho_l \\ & + QV(t, x_k, z, \cdot)(i), \end{aligned} \tag{3.13}$$

where

$$QV(t, x_k, z, \cdot)(i) = \sum_{j \ne i} q_{ij} (V(t, x_k, z, j) - V(t, x_k, z, i)).$$

Formally, for $k \ne l \in \{1, 2\}$, we conclude that V^k satisfies the following system of integro-differential HJI (Hamilton-Jacobi-Isaacs) equations: for each $i \in \mathcal{M}$,

$$\begin{cases} V_t^k(t, x_k, z, i) + \sup_{u_k \in \mathcal{H}_k} \mathcal{L}^{u_k, u_l^*} V^k(t, x_k, z, i) = 0, \\ V^k(T, x_k, z, i) = U_k(x_k). \end{cases} \tag{3.14}$$

Remark 3.2. In view of (3.14), a total of four control variables satisfy a system of HJI equations. Due to the complexity of our formulation, it is very difficult to establish the existence and uniqueness of the Nash equilibrium strategies constructively. It seems that it might be only possible using an abstract setup such as a similar approach in [9] and [7]. In fact, the existence and uniqueness of the solutions of system (3.14) is difficult to obtain for any $T > 0$; see [6]. From an insurance practical point of view, such strategy always exists in a well-posed formulation. Thus in lieu of construction of optimal solutions, our effort in this dissertation is: Assuming the existence and uniqueness of the equilibrium strategy, we focus on solving the problem numerically to obtain an approximation to the strategies $u^* = (u_1^*, u_2^*)$ satisfying (3.14).

3.2 Numerical Algorithm

We begin by constructing a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion process with regime-switching in the absence of jumps with the dynamic system

$$\begin{aligned} dX_k(t) &= \sum_{i \in \mathcal{M}} I_{\{\alpha(t)=i\}} \Big\{ r(i)X_k(t) + (b_k(t) - \kappa_k b_l(t))[\mu_S(i, Z(t)) - r(i)] + c_k(i, Z(t)) - \kappa_k c_l(i, Z(t)) \\ &\qquad - (g(a_k) + \kappa_k g(a_l))dt + (b_k(t) - \kappa_k b_l(t))\sigma_S(i, Z(t))dW_S(t) \Big\}, \\ dZ(t) &= \mu_Z(t, Z(t))dt + \sigma_Z(t, Z(t))dW_Z(t) \\ X_k(0) &= \widetilde{x}_k - \kappa_k \widetilde{x}_l \\ Z(0) &= z_0. \end{aligned} \tag{3.15}$$

Because the value function depends on both the state x and the time variable t, two stepsizes are needed. That is, we need to discretize both the state and time. We use $h > 0$ as the stepsize of the state and $\delta > 0$ as the stepsize for the time. In fact, for any given $T > 0$, we use $N = N(\delta) = \lfloor T/\delta \rfloor$.

Let e_i denote the standard unit vector in the i-th coordinate direction and $\mathbb{R}^3_h$ denote the uniform h-grid on $\mathbb{R}^3$; i.e. $\mathbb{R}^3_h = \{(x_1, x_2, z) : (x_1, x_2, z) = h(k_1 e_1 + k_2 e_2 + k_3 e_3); k_1, k_2, k_3 = 0, \pm 1, \pm 2, \dots\}$. We use $S_h = \mathbb{R}^3_h$, denote $x = (x_1, x_2, z)$ and $y = (y_1, y_2, z^*)$.

We can rewrite the system in the short form as follows

$$dX(t) = \mu(X(t), \alpha(t), u(t))dt + \sigma(X(t), \alpha(t), u(t))dW(t), \tag{3.16}$$

where

$$\begin{aligned}
&\mu(X(t), \alpha(t), u(t)) \\
&= \begin{pmatrix} r(\alpha(t))X_1(t) + (b_1(t) - \kappa_1 b_2(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_1(\alpha(t), Z(t)) - \kappa_1 c_2(\alpha(t), Z(t)) \\ -(g(a_1(t)) + \kappa_1 g(a_2(t))) \\ r(\alpha(t))X_2(t) + (b_2(t) - \kappa_2 b_1(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_2(\alpha(t), Z(t)) - \kappa_2 c_1(\alpha(t), Z(t)) \\ -(g(a_2(t)) + \kappa_2 g(a_1(t))) \\ \mu_Z(t, Z(t)) \end{pmatrix} \\
&\sigma(X(t), \alpha(t), u(t)) \\
&\quad = \operatorname{diag}\Big((b_1(t) - \kappa_1 b_2(t))\sigma_S(\alpha(t), Z(t)), (b_2(t) - \kappa_2 b_1(t))\sigma_S(\alpha(t), Z(t)), \sigma_Z(t, Z(t)) \Big) \\
&W(t) = \Big(W_S(t), W_S(t), W_Z(t) \Big)',
\end{aligned}$$

where A' is the transpose of A. Let $\{(\xi_n^{h,\delta}, \alpha_n^{h,\delta}, n < \infty\}$ be a controlled discrete-time Markov chain on $\mathbb{R}^3_h \times \mathcal{M}$ and denote by $p_D^{h,\delta}\big((x, i), (y, j), n\delta | \phi\big)$ the transition probability from a state (x, i) to another state (y, j), for $\phi \in \mathcal{U}$. We use $u_n^{h,\delta}$ to denote the random variable that is the control action for the chain at discrete time n and $p_D^{h,\delta}$ is so defined that the constructed Markov chain's evolution well approximates the local behavior of the controlled regime-switching diffusion (3.7).

For each $k = 1, 2$, we construct the transition probability $p_{k,D}^{h,\delta}\big((x,i),(y,j),n\delta|\phi\big)$ which is associated with $J^k(t,x,i,\phi) = \mathbf{E}\left[U_k\left(X_k(T)\right)\right]$ satisfying the followings:

$$\begin{cases} J_t^k(t,x,i,\phi) + \mathcal{L}^{u_k,u_l} J^k(t,x,i,\phi) = 0, \\ \\ J^k(T,x,i,\phi) = U_k(x_k). \end{cases} \tag{3.17}$$

To figure out the form of $p_{k,D}^{h,\delta}\big((x,i),(y,j)|\phi\big)$, we define a finite difference approximation to (3.17) as

$$\begin{aligned}
&J_t^k(t,x,i,\phi) \to \frac{J^k(t+\delta,x,i,\phi) - J^k(t,x,i,\phi)}{\delta}, \\
&J_{x_k}^k(t,x,i,\phi) \to \frac{J^k(t,x+he_k,i,\phi) - J^k(t,x,i,\phi)}{h} \\
&\quad \text{if } r(i)x_k + (b_k - \kappa_k b_l)[\mu_S(i,z) - r(i)] + c_k(i,z) - \kappa_k c_l(i,z) - (g(a_k) - \kappa_k g(a_l)) > 0, \\
&J_{x_k}^k(t,x,i,\phi) \to \frac{J^k(t,x,i,\phi) - J^k(t,x-he_k,i,\phi)}{h} \\
&\quad \text{if } r(i)x_k + (b_k - \kappa_k b_l)[\mu_S(i,z) - r(i)] + c_k(i,z) - \kappa_k c_l(i,z) - (g(a_k) - \kappa_k g(a_l)) < 0, \\
&J_{x_k x_k}^k(t,x,i,\phi) \to \frac{J^k(t,x+he_k,i,\phi) - 2J^k(t,x,i,\phi) + J^k(t,x-he_k,i,\phi)}{h^2}, \\
&J_z^k(t,x,i,\phi) \to \frac{J^k(t,x+he_3,i,\phi) - J^k(t,x,i,\phi)}{h} \quad \text{if} \quad \mu_Z(t,z) > 0, \\
&J_z^k(t,x,i,\phi) \to \frac{J^k(t,x,i,\phi) - J^k(t,x-he_3,i,\phi)}{h} \quad \text{if} \quad \mu_Z(t,z) < 0, \\
&J_{zz}^k(t,x,i,\phi) \to \frac{J^k(t,x+he_3,i,\phi) - 2J^k(t,x,i,\phi) + J^k(t,x-he_3,i,\phi)}{h^2}.
\end{aligned}$$

To proceed, define

$$
\begin{aligned}
&p^{h,\delta}_{k,D}\big((x,i),(x\pm he_k,i),n\delta|\phi\big)=\tfrac{\delta}{h}\Big\{r(i)x_k+(b_k-\kappa_k b_l)[\mu_S(i,z)-r(i)]+c_k(i,z)-\kappa_k c_l(i,z)\\
&\quad-(g(a_k)-\kappa_k g(a_l))\Big\}^{\pm}+\tfrac{\delta}{2h^2}(b_k-\kappa_k b_l)^2\sigma^2_S(i,z),\\
&p^{h,\delta}_{k,D}\big((x,i),(x\pm he_3,i),n\delta|\phi\big)=\tfrac{\delta}{h}\mu_Z(t,z)^{\pm}+\tfrac{\delta}{2h^2}\sigma^2_Z(t,z),\\
&p^{h,\delta}_{k,D}\big((x,i),(x,j),n\delta|\phi\big)=q_{ij}\delta,\\
&p^{h,\delta}_{k,D}\big((x,i),(x,i),n\delta|\phi\big)=1+q_{ii}\delta-\tfrac{\delta}{h}\,|r(i)x_k+(b_k-\kappa_k b_l)[\mu_S(i,z)-r(i)]+c_k(i,z)-\kappa_k c_l(i,z)\\
&\quad-(g(a_k)-\kappa_k g(a_l))|-\tfrac{\delta}{h}\mu_Z(t,z)-\tfrac{\delta}{h^2}(b_k-\kappa_k b_l)^2\sigma^2_S(i,z)-\tfrac{\delta}{h^2}\sigma^2_Z(t,z),\\
&p^{h,\delta}_{k,D}\big(\,\cdot\,\big)=0,\quad \text{otherwise,}
\end{aligned}
\tag{3.18}
$$

where $K^+=\max\{K,0\}$ and $K^-=\min\{-K,0\}$. By choosing δ and h appropriately, we can have $p^{h,\delta}_{k,D}\big((x,i),(x,i),n\delta|\phi\big)$ given in (3.18) nonnegative. Thus, $p^{h,\delta}(\cdot|\phi)$ are well-defined transition probability.

Remark 3.3. To guarantee the nonnegativity of the transition probabilities in (3.18), we need to choose the step sizes h and δ satisfying certain condition. For example, similar to [37], we may choose $h^2=O(\delta)$ from a practical point of view.

Next, we need to approximate the Poisson jumps for ensuring the local properties of claims for (3.7). We can rewrite the system in the matrix form as follow

$$
dX(t)=\mu(X(t),\alpha(t),u(t))dt+\sigma(X(t),\alpha(t),u(t))dB(t)+\widetilde{Y}_1(t)e_1+\widetilde{Y}_2(t)e_2, \tag{3.19}
$$

where $\widetilde{Y}_k(t)$ is the jump process w.r.t the surplus process $X_k(t)$, for $k=1,2$.

The relative surplus process $X_k(t)$ is determined by two jump terms with the arriving rate λ_k and λ_l, respectively. Denote by $R_k(t)$ the difference of the two jumps. That is,

$$
R_k(t)=\int_{\mathbb{R}_+}\widetilde{q}_k(q_k,a_k)N_k(dt,d\rho_k)-\kappa_k\int_{\mathbb{R}_+}\widetilde{q}_l(q_l,a_l)N_l(dt,d\rho_l).
$$

Since the difference of the two Poisson processes is again a Poisson process, events in the new process $R_k(t)$ will occur according to a Poisson process with the rate $\lambda = \lambda_k + \lambda_l$; and each event, independently, will be from the first jump process with probability $\lambda_k/(\lambda_k + \lambda_l)$, yielding the generic claim size

$$\widetilde{A}^k = \begin{cases} A^k(a_k), & \text{with probability } \frac{\lambda_k}{\lambda_k+\lambda_l}, \\ \\ -\kappa_k A^l(a_l), & \text{with probability } \frac{\lambda_l}{\lambda_k+\lambda_l}. \end{cases}$$

Suppose that the current state is $\xi_n^{h,\delta} = x$, $\alpha_n^{h,\delta} = i$, and control is $u_n^{h,\delta} = \phi$. The next interpolation interval is determined by (3.18) and $\widetilde{q}_k^h(q_k(i,z,\rho_k), a_k)$ is the nearest value of $\widetilde{q}_k(q_k(i,z,\rho_k), a_k)$ so that $\xi_{n+1}^{h,\delta} \in S_h$. Then $|\widetilde{q}_k^h(q_k(i,z,\rho_k), a_k) - \widetilde{q}_k(q_k(i,z,\rho_k), a_k)| \to 0$ as $h \to 0$, uniformly in x. To present the claim terms, we determine the next case $(\xi_{n+1}^{h,\delta}, \alpha_{n+1}^{h,\delta})$ by noting:

1. No claims occur in $[n\delta, n\delta+\delta)$ with probability $1-\lambda\delta+o(\delta)$, we determine $(\xi_{n+1}^{h,\delta}, \alpha_{n+1}^{h,\delta})$ by transition probability $p_{k,D}^{h,\delta}(\cdot)$ as in (3.18).

2. There is a claim of the relative surplus process $X_k(t)$ in $[n\delta, n\delta+\delta)$ with probability $\lambda\delta + o(\delta)$, we determine $(\xi_{n+1}^{h,\delta}, \alpha_{n+1}^{h,\delta})$ by

$$\xi_{n+1}^{h,\delta} = \xi_n^{h,\delta} - \widetilde{q}_k^h(q_k(i,z,\rho_k), a_k)e_k, \qquad \alpha_{n+1}^{h,\delta} = \alpha_n^{h,\delta}.$$

So, we define

$$p_k^{h,\delta}\big((x,i),(y,j)|\phi\big) = (1-\lambda\delta+o(\delta))p_{k,D}^{h,\delta}\big((x,i),(y,j)|\phi\big)+(\lambda\delta+o(\delta))\Pi_k\{\rho : \widetilde{q}_h(i,z,\rho)e_k = x-y\}. \tag{3.20}$$

Definition 3.4. For a controlled Markov chain $\{(\xi_n^{h,\delta}, \alpha_n^{h,\delta}), n < \infty\}$, the one-step transition probability $p^{h,\delta}\big((x,i),(y,j)|\phi\big)$ is given by

$$p^{h,\delta}\big((x,i),(y,j)|\phi\big) = p_1^{h,\delta}\big((x,i),(y,j)|\phi\big)I_{\{k=1\}} + p_2^{h,\delta}\big((x,i),(y,j)|\phi\big)I_{\{k=2\}}, \tag{3.21}$$

where k is the index of the cost function.

The piecewise constant interpolations $\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot)$, and $u^{h,\delta}(\cdot)$ are defined as

$$\xi^{h,\delta}(t) = \xi_n^{h,\delta}, \quad \alpha^{h,\delta}(t) = \alpha_n^{h,\delta}, \quad u^{h,\delta}(t) = u_n^{h,\delta} \quad \text{for } t \in [n\delta, n\delta + \delta) \tag{3.22}$$

Use $\mathbb{E}_{x,i,n}^{\phi,h,\delta}, \mathrm{Var}_{x,i,n}^{\phi,h,\delta}$, and $\mathbf{P}_{x,i,n}^{\phi,h,\delta}$ to denote the conditional expectation, variance, and marginal probability given $\{\xi_\iota^{h,\delta}, \alpha_\iota^\delta, u_\iota^{h,\delta}, \iota \le n, \xi_n^{h,\delta} = x, \alpha_n^{h,\delta} = i, u_n^{h,\delta} = \phi\}$, respectively. Define the difference $\Delta\xi_n^{h,\delta} = \xi_{n+1}^{h,\delta} - \xi_n^{h,\delta}$.

With the approximation of the Markov chain constructed above, we can obtain an approximation of the utility function as follows:

$$\begin{aligned} J^{k,\delta}(n\delta, x, i, u^{h,\delta}) &= (1 - \lambda\delta + o(\delta)) \sum_{(y,j)} p_k^{h,\delta}\Big((x,i),(y,j)|u^{h,\delta}\Big) J^{k,\delta}(n\delta + \delta, y, j, u^{h,\delta}) \\ &+(\lambda_k\delta + o(\delta)) \int_{\mathbb{R}_+} J^{k,\delta}(n\delta + \delta, x_k - \widetilde{q}_k^h(q_k(i,z,\rho_k), a_k^{h,\delta}), z, i, u^{h,\delta}) \Pi_k(d\rho_k) \\ &+(\lambda_l\delta + o(\delta)) \int_{\mathbb{R}_+} J^{k,\delta}(n\delta + \delta, x_k + \kappa_k \widetilde{q}_l^h(q_l(i,z,\rho_l), a_l^{h,\delta}), z, i, u^{h,\delta}) \Pi_l(d\rho_l). \end{aligned} \tag{3.23}$$

Moreover,

$$V^{k,h,\delta}(n\delta, x, i) = \sup_{u_{k,n}^{h,\delta}} J^{k,h,\delta}(n\delta, x, i, u^{h,\delta}). \tag{3.24}$$

Definition 3.5. The sequence $\{(\xi_n^{h,\delta}, \alpha_n^{h,\delta})\}$ is said to be **locally consistent**, if it satisfies

1. There is a transition probability $p_D^{h,\delta}$ is **locally consistent** in the sense

$$\begin{aligned} &\mathbb{E}_{x,i,n}^{\phi,h,\delta}[\Delta\xi_n^{h,\delta}] = \mu^{h,\delta}(x,i,\phi)\delta + o(\delta), \\ &\mathrm{Var}_{x,i,n}^{\phi,h,\delta}[\Delta\xi_n^{h,\delta}] = \sigma^{h,\delta}(x,i,\phi)\delta + o(\delta). \end{aligned} \tag{3.25}$$

where

$$\mu^{h,\delta}(x,i,\phi) = \begin{pmatrix} r(i)x_1 + (b_1 - \kappa_1 b_2)[\mu_S(i,z) - r(i)] + c_1(i,z) - \kappa_1 c_2(i,z) - g(a_1) + \kappa_1 g(a_2) \\ r(i)x_2 + (b_2 - \kappa_2 b_1)[\mu_S(i,z) - r(i)] + c_2(i,z) - \kappa_2 c_1(i,z) - g(a_2) + \kappa_2 g(a_1) \\ \mu_Z(t,z) \end{pmatrix}$$

$$\sigma^{h,\delta}(x,i,\phi) = \mathrm{diag}\Big((b_1 - \kappa_1 b_2)^2 \sigma_S^2(i,z), (b_2 - \kappa_2 b_1)^2 \sigma_S^2(i,z), \sigma_Z^2(t,z)\Big).$$

2. The one-step transition probability $p^{h,\delta}((x,i),(y,j)|\phi)$ for the chain can be represented in the form:

$$p^{h,\delta}((x,i),(y,j)|\phi) = (1-\lambda\delta+o(\delta))p_D^{h,\delta}((x,i),(y,j)|\phi) + (\lambda\delta+o(\delta))\Pi\{\rho : q_h(i,z,\rho) = x-y\}$$

3.3 Convergence of Numerical Approximation

3.3.1 Representations of Approximation Sequences

To proceed, we first show that the constructed Markov chain is locally consistent. This ensures that our approximation is reasonable in certain sense.

Lemma 3.6. *The Markov chain $\xi_n^{h,\delta}$ with transition probabilities $p^{h,\delta}(\cdot)$ defined in* (3.21) *is locally consistent with the stochastic differential equation in* (3.7).

Proof. Define $p_D^{h,\delta}(\cdot) = p_{1,D}^{h,\delta}(\cdot)I_{\{k=1\}} + p_{2,D}^{h,\delta}(\cdot)I_{\{k=2\}}$. Using (3.18) and (3.21), it is easy to see that

$$\begin{aligned}\mathbb{E}_{x,i,n}^{\phi,h,\delta}[\Delta\xi_n^{h,\delta}] &= \mathbb{E}_{x,i,n}^{\phi,h,\delta}[\Delta\xi_n^{h,\delta}I_{\{k=1\}}] + \mathbb{E}_{x,i,n}^{\phi,h,\delta}[\Delta\xi^{h,\delta}I_{\{k=2\}}] \\ &= \begin{pmatrix} r(i)x_1 + (b_1-\kappa_1 b_2)[\mu_S(i,z)-r(i)] + c_1(i,z) - \kappa_1 c_2(i,z) - g(a_1) + \kappa_1 g(a_2) \\ r(i)x_2 + (b_2-\kappa_2 b_1)[\mu_S(i,z)-r(i)] + c_2(i,z) - \kappa_2 c_1(i,z) - g(a_2) + \kappa_2 g(a_1) \\ \mu_Z(t,z) \end{pmatrix}\delta.\end{aligned}$$

Likewise, we obtain $\mathbb{E}_{x,i,n}^{\phi,h,\delta}\left[\Delta\xi_n^{h,\delta}(\Delta\xi_n^{h,\delta})'\right]$ and $\mathrm{Var}_{x,i,n}^{\phi,h,\delta}[\Delta\xi_n^{h,\delta}]$. □

Let $\xi^{h,\delta}(0) = x, \alpha^{h,\delta}(0) = i$. Define the relaxed control representation $m_k^{h,\delta}(\cdot)$ of $u_k^{h,\delta}(\cdot)$ by using its derivative $m_{t,k}^{h,\delta}(A) = I_{\{u^{h,\delta}(t)\in A\}}$. Let $H_n^{h,\delta}$ denote the event that $\xi_n^{h,\delta}, \alpha_n^{h,\delta}$ is determined by the case of "no claim occurs" and use $T_n^{h,\delta}$ to denote the event of "one claim occurs". Let $I_{H_n^{h,\delta}}$ and $I_{T_n^{h,\delta}}$ be corresponding indicator functions, respectively. If $H_n^{h,\delta}$ happens, $I_{H_n^{h,\delta}} = 1$, $I_{T_n^{h,\delta}} = 0$, otherwise $I_{H_n^{h,\delta}} = 0$, $I_{T_n^{h,\delta}} = 1$. Therefore, $I_{H_n^{h,\delta}} + I_{T_n^{h,\delta}} = 1$ and we can write

$$\begin{aligned} \xi^{h,\delta}(t) &= \xi^{h,\delta}(0) + \sum_{r=0}^{\lfloor t/\delta \rfloor -1} [\Delta\xi_r^{h,\delta} I_{H_r^{h,\delta}} + \Delta\xi_r^{h,\delta} I_{T_r^{h,\delta}}] \\ &= x + \sum_{r=0}^{\lfloor t/\delta \rfloor -1} \mathbb{E}_{x,i,r}^{\phi,h,\delta}[\Delta\xi_r^{h,\delta} I_{H_r^{h,\delta}}] + \sum_{r=0}^{\lfloor t/\delta \rfloor -1} (\Delta\xi_r^{h,\delta} - \mathbb{E}_{x,i,r}^{\phi,h,\delta}\Delta\xi_r^{h,\delta}) I_{H_r^{h,\delta}} + \sum_{r=0}^{\lfloor t/\delta \rfloor -1} \Delta\xi_r^{h,\delta} I_{T_r^{h,\delta}}. \end{aligned} \tag{3.26}$$

Define $\mathcal{F}_n^{h,\delta}$ as the smallest σ-algebra generated by $\{\xi_r^{h,\delta}, \alpha_r^{h,\delta}, m_r^{h,\delta}, H_r^{h,\delta}, r \le n\}$ and $\mathcal{F}_t^{h,\delta}$ as the smallest σ- algebra generated by $\{\xi^{h,\delta}(s), \alpha^{h,\delta}(s), m^{h,\delta}(s), H^{h,\delta}(s), s \le t\}$.

For $k, l = 1, 2$ and $k \neq l$, denote

$$\begin{aligned} M^{h,\delta}(t) &= \sum_{r=0}^{\lfloor t/\delta \rfloor -1} (\Delta\xi_r^{h,\delta} - \mathbb{E}_{x,i,r}^{\phi,h,\delta}\Delta\xi_r^{h,\delta}) I_{H_r^{h,\delta}} \\ Y_k^{h,\delta}(t) &= - \sum_{r=0}^{\lfloor t/\delta \rfloor -1} [\Delta\xi_r^{h,\delta}]' e_k I_{T_r^{h,\delta}} \\ &= \sum_{\nu_{k,r}^{h,\delta} \le \lfloor t/\delta \rfloor} \widetilde{q}_k^h(q_k(\alpha^{h,\delta}(\nu_{k,r}^{h,\delta}), Z^{h,\delta}(\nu_{k,r}^{h,\delta}), \rho_k), a_k^{h,\delta}(\nu_{k,r}^{h,\delta})) \\ &\quad -\kappa_k \sum_{\nu_{l,r}^{h,\delta} \le \lfloor t/\delta \rfloor} \widetilde{q}_l^h(q_l(\alpha^{\delta}(\nu_{l,r}^{h,\delta}), Z^{\delta}(\nu_{l,r}^{h,\delta}), \rho_l), a_l^{h,\delta}(\nu_{l,r}^{h,\delta})) \\ &= \int_0^t \int_{\mathbb{R}_+} \widetilde{q}_k^h(q_k(\alpha^{h,\delta}(s), Z^{h,\delta}(s), \rho_k), a_k^{h,\delta}(s)) N_k^{h,\delta}(ds, d\rho_k) \\ &\quad - \int_0^t \int_{\mathbb{R}_+} \kappa_k \widetilde{q}_l^h(q_l(\alpha^{h,\delta}(s), Z^{h,\delta}(s), \rho_l), a_l^{h,\delta}(s)) N_l^{h,\delta}(ds, d\rho_l). \end{aligned} \tag{3.27}$$

Then $M^{h,\delta}(t)$ is a martingale with respect to $\mathcal{F}_{\lfloor t/\delta \rfloor}^{h,\delta}$. Now, we represent $M^{h,\delta}(t)$ similar to the diffusion term in (3.7). Define $W^{h,\delta}(\cdot)$ as

$$\begin{aligned} W^{h,\delta}(t) &= \sum_{r=0}^{\lfloor t/\delta \rfloor -1} [\sigma^{h,\delta}(x,i,\phi)]^{-1} (\Delta\xi_r^{h,\delta} - \mathbb{E}_{x,i,r}^{\phi,h,\delta}\Delta\xi_r^{h,\delta}) I_{H_r^{h,\delta}} \\ &= \int_0^t [\sigma^{h,\delta}(\xi^{h,\delta}(s), \alpha^{h,\delta}(s), u^{h,\delta}(s))]^{-1} dM^{h,\delta}(s). \end{aligned} \tag{3.28}$$

Remark 3.7. *For simplicity, we assume that there is a positive number $c > 0$ such that $(b_1 - \kappa_1 b_2)^2 \sigma_S^2(i,z)$, $(b_2 - \kappa_2 b_1)^2 \sigma_S^2(i,z)$, and $\sigma_Z^2(t,z) \ge c$.*

The local consistency leads to

$$\begin{aligned}
&\sum_{r=0}^{\lfloor t/\delta\rfloor-1} \mathbb{E}^{\phi,h,\delta}_{x,i,r}[\Delta\xi^{h,\delta}_r I_{H^{h,\delta}_r}] \\
&= \sum_{r=0}^{\lfloor t/\delta\rfloor-1} \begin{pmatrix} r(\alpha^{h,\delta}_r)\xi^{\delta}_r e_1 + (b^{h,\delta}_{1,r}-\kappa_1 b^{h,\delta}_{2,r})[\mu(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-r(\alpha^{h,\delta}_r)] \\ +c_1(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-\kappa_1 c_2(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-g(a^{h,\delta}_{1,r})+\kappa_1 g(a^{h,\delta}_{2,r}) \\ r(\alpha^{h,\delta}_r)\xi^{h,\delta}_r e_2 + (b^{h,\delta}_{2,r}-\kappa_2 b^{h,\delta}_{1,r})[\mu(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-r(\alpha^{h,\delta}_r)] \\ +c_2(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-\kappa_2 c_1(\alpha^{h,\delta}_r,\xi^{h,\delta}_r e_3)-g(a^{h,\delta}_{2,r})+\kappa_2 g(a^{h,\delta}_{1,r}) \\ \mu_Z(r\delta,\xi^{h,\delta}_r e_3) \end{pmatrix} \delta I_{H^{h,\delta}_r} + o(\delta) I_{H^{h,\delta}_r} \\
&= \int_0^t \mu^{h,\delta}(\xi^{h,\delta}(s),\alpha^{h,\delta}(s),u^{\delta}(s))ds + \varepsilon^{h,\delta}(t).
\end{aligned}$$

For each t, $\mathbf{E}[\text{number of } r : \nu^{h,\delta}_{k,r} \le t] = \lambda_k t$ as $h,\delta \to 0$. This implies that we can drop $I_{H^{h,\delta}_r}$ with no effect on the above limit.

As a consequence, we can rewrite (3.26) as following:

$$\begin{aligned}
\xi^{h,\delta}(t) \quad &= x + \int_0^t \int_{\mathcal{U}} \mu^{h,\delta}(\xi^{h,\delta}(s),\alpha^{h,\delta}(s),\phi) m^{h,\delta}_s(d\phi)ds \\
&\quad + \int_0^t \int_{\mathcal{U}} \sigma^{h,\delta}(\xi^{h,\delta}(s),\alpha^{h,\delta}(s),\phi) m^{h,\delta}_s(d\phi) dW^{h,\delta}(s) \\
&\quad + \Big(\int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \widetilde{q}^h_1(q_1(\alpha^{h,\delta}(s),Z^{h,\delta}(s),\rho_1),a_1) m^{h,\delta}_s(d\phi) N^{h,\delta}_1(ds,d\rho_1) \\
&\qquad - \int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \kappa_1 \widetilde{q}^h_2(q_2(\alpha^{h,\delta}(s),Z^{h,\delta}(s),\rho_2),a_2) m^{h,\delta}_s(d\phi) N^{h,\delta}_2(ds,d\rho_2) \Big) e_1 \\
&\quad + \Big(\int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \widetilde{q}^h_2(q_2(\alpha^{h,\delta}(s),Z^{h,\delta}(s),\rho_2),a_2) m^{h,\delta}_s(d\phi) N^{h,\delta}_2(ds,d\rho_2) \\
&\qquad - \int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \kappa_2 \widetilde{q}^h_1(q_1(\alpha^{h,\delta}(s),Z^{h,\delta}(s),\rho_1),a_1) m^{h,\delta}_s(d\phi) N^{h,\delta}_1(ds,d\rho_1) \Big) e_2 + \varepsilon^{h,\delta}(t).
\end{aligned} \tag{3.29}$$

We can also rewrite (3.19) as

$$\begin{aligned} X(t) &= x + \int_0^t \int_{\mathcal{U}} \mu(X(s), \alpha(s), \phi(s)) m_s(d\phi) ds + \int_0^t \int_{\mathcal{U}} \sigma(X(s), \alpha(s), \phi) m_s(d\phi) dW(s) \\ &\quad + \Big(\int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \widetilde{q}_1(q_1(\alpha(s), Z(s), \rho_1), a_1) m_s(d\phi) N_1(ds, d\rho_1) \\ &\quad - \int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \kappa_1 \widetilde{q}_2(q_2(\alpha(s), Z(s), \rho_2), a_2) m_s(d\phi) N_2(ds, d\rho_2) \Big) e_1 \\ &\quad + \Big(\int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \widetilde{q}_2(q_2(\alpha(s), Z(s), \rho_2), a_2) m_s(d\phi) N_2(ds, d\rho_2) \\ &\quad - \int_0^t \int_{\mathbb{R}_+} \int_{\mathcal{U}} \kappa_2 \widetilde{q}_1(q_1(\alpha(s), Z(s), \rho_1), a_1) m_s(d\phi) N_1(ds, d\rho_1) \Big) e_2. \end{aligned} \tag{3.30}$$

3.3.2 Convergence of Approximating Markov Chains

Lemma 3.8. *Using the transition probability* $\{p^{h,\delta}(\cdot)\}$ *defined in* (3.21)*, the interpolated process of the constructed Markov chain* $\{\alpha^{h,\delta}(\cdot)\}$ *converges weakly to* $\alpha(\cdot)$*, the Markov chain with generator* Q*.*

Proof. The proof can be obtained similar to [41, Theorem 3.1]. The details are thus omitted.

□

Theorem 3.9. *Let the approximating chain* $\{\xi_n^{h,\delta}, \alpha_n^{h,\delta}, n < \infty\}$ *constructed with transition probabilities defined in* (3.21) *be locally consistent with* (3.25)*,* $m^{h,\delta}(\cdot)$ *be the relaxed control representation of* $\{u^{h,\delta}, n < \infty\}$*,* $(\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot))$ *be the continuous-time interpolation defined in* (3.22)*. Then* $\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_1^{h,\delta}(\cdot,\cdot), N_2^{h,\delta}(\cdot,\cdot)\}$ *is tight.*

Proof. Note that $\alpha^{h,\delta}(\cdot)$ is tight. It follows that for each $\Delta > 0$, each $t > 0$, and $0 < \widetilde{t} \leq \Delta$, there is a random variable $\gamma^{h,\delta}(\Delta) > 0$ such that

$$\begin{aligned}\mathbf{E}_t|W^{h,\delta}(t+\widetilde{t}) - W^{h,\delta}(t)|^2 &= \sum_{\lfloor t/\delta \rfloor}^{\lfloor (t+\widetilde{t})/\delta \rfloor - 1} \mathbf{E}_t\{[\sigma^{h,\delta}(x,\phi,i)]^{-1}(\Delta\xi_r^{h,\delta} - \mathbb{E}_{x,i,r}^{\phi,h,\delta}\Delta\xi_r^{h,\delta})I_{H_r^{h,\delta}}\}^2 \\ &\le \mathbf{E}_t\gamma^{h,\delta}(\Delta)\end{aligned} \tag{3.31}$$

satisfying $\lim\limits_{\Delta\to 0}\limsup\limits_{h,\delta\to 0}\mathbf{E}\gamma^{h,\delta}(\Delta) = 0$, which yields the tightness of $W^{h,\delta}(\cdot)$. A similar argument leads to the tightness of $M^{h,\delta}(\cdot)$. The sequence $m^{h,\delta}(\cdot)$ is tight because of its compact range space. By virtue of Theorem 9.2.1 in [23], we obtain the tightness of $\{N_k^{h,\delta}(\cdot),\ k = 1,2\}$ since the mean number of claims on any bounded interval $[t, t+t_1]$ is bounded and

$$\lim_{\Delta\to 0}\inf_{r}\mathbf{P}\{\nu_{k,r+1}^{h,\delta} - \nu_{k,r}^{h,\delta} > \Delta|\nu_{k,r}^{h,\delta}\} = 1.$$

This implies the tightness of $\{R_k^{h,\delta}(\cdot),\ k = 1,2\}$. As a consequence, $\xi^{h,\delta}(\cdot)$ is tight and

$$\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_1^{h,\delta}(\cdot,\cdot), N_2^{h,\delta}(\cdot,\cdot)\} \text{ is tight.}$$

□

Because $\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_1^{h,\delta}(\cdot,\cdot), N_2^{h,\delta}(\cdot,\cdot)\}$ is tight, the Prohorov's theorem implies that it is sequentially compact. Thus we can extract a weakly convergent subsequence. Select such a convergent subsequence and still index the sequence by h,δ for notational simplicity. We proceed to characterize the limit process.

Theorem 3.10. *Let $\{\xi(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N_1(\cdot,\cdot), N_2(\cdot,\cdot)\}$ be the limit of weakly convergent subsequence and $\mathcal{F}_t$ be the σ-algebra generated by $\{X(s), \alpha(s), m(s), W(s), N_1(s,\cdot), N_2(s,\cdot), s \le t\}$. Then $W(\cdot)$ is a standard $\mathcal{F}_t$-Brownian motion and $N_1(\cdot,\cdot)$, $N_2(\cdot,\cdot)$ are $\mathcal{F}_t$-Poisson measures, and $m(\cdot)$ is an admissible relaxed control.*

Proof. The proof is divided into several steps.

Step 1: By the Skorohod representation, with a slight abuse of notation, $\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_1^{h,\delta}(\cdot,\cdot), N_2^{h,\delta}(\cdot,\cdot)\}$ converges to $\{\xi(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N_1(\cdot,\cdot), N_2(\cdot,\cdot)\}$ w.p.1, and the convergence is uniform on any compact set.

To proceed, we first verify that $W(\cdot)$ is an $\mathcal{F}_t$-Brownian motion. For any real-valued and continuous function ψ, define

$$(\psi, m)_t = \int_0^t \int_{\mathcal{U}} \psi(\phi, s) m_s(d\phi). \tag{3.32}$$

For any given $f(\cdot) \in \mathcal{C}_0^2(\mathbb{R}^3)$ (C^2 function with compact support), consider an associate operator $\mathcal{L}_w f(w) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{3} \frac{\partial^2}{\partial w_i \partial w_i} f(w)$. Let $t, \widetilde{t} > 0$ be given with $t + \widetilde{t} \le T$, along with arbitrary positive integers κ and $\widetilde{\kappa}$, arbitrary $t_i \le t$ and continuous functions ψ_j with $i \le \kappa$ and $j \le \widetilde{\kappa}$, any bounded and continuous function $h(\cdot)$, and arbitrary $f \in \mathcal{C}_0^2(\mathbb{R}^3)$. Denote $\{\Gamma_i^\kappa, i \le \kappa\}$ as a sequence of nondecreasing partition of $\mathbb{R}_+$ such that $\Pi(\partial\Gamma_i^\kappa) = 0$ for all i, κ, where $\partial\Gamma_i^\kappa$ is the boundary of the set Γ_i^κ. As $\kappa \to \infty$, let the diameter of the sets Γ_i^κ go to zero.

By the tightness of $W^{h,\delta}(\cdot)$, it converges weakly to a limit $W(\cdot)$. Using the weak convergence and the Skorohod representation, standard argument reveals that $W^{h,\delta}(\cdot)$ is an $\mathcal{F}_t^{h,\delta}$-Brownian motion, and as $h, \delta \to 0$,

$$\begin{aligned}
&\mathbf{E}h(\xi^{h,\delta}(t_i), \alpha^{\delta}(t_i), W^{h,\delta}(t_i), (\psi_j, m^{h,\delta})_{t_i} : i \le \kappa, j \le \widetilde{\kappa})\Big(f(W^{h,\delta}(t+\widetilde{t}) - f(W^{h,\delta}(t))\Big)\\
&\qquad \to \mathbf{E}h(\xi(t_i), \alpha(t_i), W(t_i), (\psi_j, m)_{t_i} : i \le \kappa, j \le \widetilde{\kappa})\Big(f(W(t+\widetilde{t}) - f(W(t))\Big),\\
&\mathbf{E}h(\xi^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), W^{h,\delta}(t_i), (\psi_j, m^{h,\delta})_{t_i} : i \le \kappa, j \le \widetilde{\kappa})\Big(\int_t^{t+\widetilde{t}} \mathcal{L}_w f(W^{h,\delta}(s))ds\Big)\\
&\qquad \to \mathbf{E}h(\xi(t_i), \alpha(t_i), W(t_i), (\psi_j, m)_{t_i} : i \le \kappa, j \le \widetilde{\kappa})\Big(\int_t^{t+\widetilde{t}} \mathcal{L}_w f(W(s))ds\Big).
\end{aligned}$$

Thus,

$$\mathbf{E}h(\xi(t_i), \alpha(t_i), W(t_i), (\psi_j, m)_{t_i} : i \le \kappa, j \le \widetilde{\kappa})\Big(f(W(t+\widetilde{t})) - f(W(t)) - \int_t^{t+\widetilde{t}} \mathcal{L}_w f(W(s))ds\Big) = 0.$$

It follows that $f(W(t)) - f(W(0) - \int_0^t \mathcal{L}_w f(W(s))ds$ is a martingale. Moreover, the quadratic variation of the martingale $W(t)$ is tI_3, where I_3 is an 3×3 identity matrix. Thus, $W(\cdot)$ is an $\mathcal{F}_t$-Brownian motion.

Step 2: We proceed to show that $N_k(\cdot, \cdot)$ is an $\mathcal{F}_t$-Poisson measure for each $k = 1, 2$. Let $\theta(\cdot)$ be a continuous function on $\mathbb{R}_+$ and define the process

$$\Theta_k = \int_0^t \int_{\mathbb{R}^+} \theta(\rho) N_k(ds, d\rho).$$

Using similar argument as in the proof of the Brownian motion above, if $f(\cdot) \in \mathcal{C}_0^2(\mathbb{R}^3)$ then

$$\begin{aligned}&\mathbf{E}h(\xi(t_i), \alpha(t_i), W(t_i), (\psi_j, m)_{t_i}, N(t_i, \Gamma_i^{\kappa}), i \le \kappa, j \le \widetilde{\kappa})\\ &\times \left[f(\Theta_k(t + \widetilde{t})) - f(\Theta_k(t)) - \lambda_k \textstyle\int_t^{t+\widetilde{t}} \displaystyle\int_{\mathbb{R}_+} \big[f(\Theta_k(s) + \theta(\rho)) - f(\Theta_k(s)) \big] \Pi(d\rho) ds \right] = 0.\end{aligned}$$

This implies that $N_k(\cdot, \cdot)$ is an $\mathcal{F}_t$-Possion measure for each $k = 1, 2$.

Step 3.1: We will use (3.26) for the rest of the proof. Note that $\mathbf{E}|\varepsilon^{h,\delta}(t)| \to 0$ as $h, \delta \to 0$. Letting $h, \delta \to 0$ and using the Skorohod representation for (3.26), we have

$$\int_0^t \int_{\mathcal{U}} \mu^{h,\delta}(\xi^{h,\delta}(s), \alpha^{h,\delta}(s), \phi) m_s^{h,\delta}(d\phi) ds - \int_0^t \int_{\mathcal{U}} \mu(\xi(s), \alpha(s), \phi) m_s^{h,\delta}(d\phi) ds \to 0,$$

uniformly on any bounded time interval with probability one. On the other hand, the sequence $m^{h,\delta}(\cdot)$ converges in the compact-weak topology, thus, for any continuous and bounded function $\psi(\cdot)$ with compact support,

$$\int_0^t \int_{\mathcal{U}} \psi(\phi, s) m^{h,\delta}(d\phi ds) \to \int_0^t \int_{\mathcal{U}} \psi(\phi, s) m(d\phi ds) \text{ as } h, \delta \to 0.$$

By virtue of the Skorohod representation and the weak convergence, as $h, \delta \to 0$,

$$\int_0^t \int_{\mathcal{U}} \mu^{h,\delta}(\xi^{h,\delta}(s), \alpha^{h,\delta}(s), \phi) m_s^{h,\delta}(d\phi) ds - \int_0^t \int_{\mathcal{U}} \mu(\xi(s), \alpha(s), \phi) m_s(d\phi) ds \to 0, \tag{3.33}$$

uniformly in t with probability one on any bounded interval.

Step 3.2: For any $t^1 \ge t$, $t^2 \ge 0$ with $t^1 + t^2 \le T$, any $\mathcal{C}_0^{1,2}$ function $f(\cdot)$ (functions that have compact support whose first partial derivative w.r.t. the time variable and the second partial derivatives w.r.t. the state variable x are continuous), bounded and continuous function $h(\cdot)$, any positive integers $\kappa, \widetilde{\kappa}$, t_i, and any continuous function ψ_j satisfying $t \le t_i \le t_1$ and $i \le \kappa$, and $j \le \widetilde{\kappa}$, the weak convergence and the Skorohod representation imply that

$$\begin{aligned}
&\mathbf{E}h(\xi^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), M^{h,\delta}(t_i), (\psi_j, m^{h,\delta})_{t_i} : i \le \kappa, j \le \widetilde{\kappa}) \\
&\quad \times [f(t^1 + t^2, M^{h,\delta}(t^1 + t^2)) - f(t^1, M^{h,\delta}(t^1))] \\
&\to \mathbf{E}h(\xi(t_i), \alpha(t_i), M(t_i), (\psi_j, m)_{t_i} : i \le \kappa, j \le \widetilde{\kappa})[f(t^1 + t^2, M(t^1 + t^2) - f(t^1, M(t^1))]
\end{aligned} \tag{3.34}$$

with $h, \delta \to 0$. Choose a sequence $\{n^\delta\}$ such that $n^\delta \to \infty$ but $\Delta^\delta = \delta n^\delta \to 0$, then

$$\begin{aligned}
&\mathbf{E}h(\xi^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), M^{h,\delta}(t_i), (\psi_j, m^{h,\delta})_{t_i} : i \le \kappa, j \le \widetilde{\kappa}) \\
&\quad \times [f(t^1 + t^2, M^{h,\delta}(t^1 + t^2)) - f(t^1, M^{h,\delta}(t^1))] \\
&= \mathbf{E}h(\xi^{h,\delta}(t_i), \alpha^{\delta}(t_i), M^{h,\delta}(t_i), (\psi_j, m^{h,\delta})_{t_i} : i \le \kappa, j \le \widetilde{\kappa}) \\
&\quad \times \Bigg[\sum_{ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta - 1} f(\delta(ln^\delta + n^\delta), M^{h,\delta}(\delta(ln^\delta + n^\delta))) \\
&\quad - f(\delta ln^\delta, M^{h,\delta}(\delta(ln^\delta + n^\delta))) + f(\delta ln^\delta, M^{h,\delta}(\delta(ln^\delta + n^\delta))) - f(\delta ln^\delta, M^\delta(\delta ln^\delta)) \Bigg].
\end{aligned} \tag{3.35}$$

Note that

$$\begin{aligned}
&\sum_{ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} [f(\delta(ln^\delta + n^\delta), M^\delta(\delta(ln^\delta + n^\delta))) - f(\delta ln^\delta, M^{h,\delta}(\delta(ln^\delta + n^\delta)))] \\
&\quad = \sum_{ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \sum_{k = ln^\delta}^{ln^\delta + n^\delta - 1} [f(\delta(k+1), M^{h,\delta}(\delta(ln^\delta + n^\delta))) - f(\delta k, M^{h,\delta}(\delta(ln^\delta + n^\delta)))] \\
&\quad = \sum_{ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \frac{\partial f(\delta ln^\delta, M^\delta(\delta(ln^\delta + n^\delta)))}{\partial s} \Delta^\delta + o(1),
\end{aligned}$$

where $o(1) \to 0$ in mean uniformly in t as $\delta \to 0$. Letting $\delta ln^\delta \to s$ as $\delta \to 0$, then $\delta(ln^\delta + n^\delta) \to s$ since $\Delta^\delta = \delta n^\delta \to 0$ as $\delta \to 0$. Then, by the weak convergence and the

Skorohod representation, the continuity of $h(\cdot)$, and the smoothness of $f(\cdot)$ imply that

$$\begin{aligned}
&\mathbf{E}h(\xi^{h,\delta}(t_i),\alpha^{h,\delta}(t_i),M^{h,\delta}(t_i),(\psi_j,m^{h,\delta})_{t_i}:i\le\kappa,j\le\widetilde{\kappa})\\
&\times\sum_{ln^\delta}^{(t^1+t^2)/\delta}\Big[f(\delta(ln^\delta+n^\delta),M^{h,\delta}(\delta(ln^\delta+n^\delta)))-f(\delta ln^\delta,M^{h,\delta}(\delta(ln^\delta+n^\delta)))\Big]\\
&\to\mathbf{E}h(\xi(t_i),\alpha(t_i),M(t_i),(\psi_j,m)_{t_i}:i\le\kappa,j\le\widetilde{\kappa})\left[\int_{t^1}^{t^1+t^2}\frac{\partial f(s,M(s))}{\partial s}ds\right]\qquad\text{as } h,\delta\to 0.
\end{aligned}\tag{3.36}$$

The last part of (3.35) can be seen as

$$\begin{aligned}
&\sum_{ln^\delta=t^1/\delta}^{(t^1+t^2)/\delta}[f(\delta ln^\delta,M^{h,\delta}(\delta(ln^\delta+n^\delta)))-f(\delta ln^\delta,M^{h,\delta}(\delta ln^\delta))]\\
&=\sum_{ln^\delta=t^1/\delta}^{(t^1+t^2)/\delta}\left\{\sum_{i=1}^{3}\frac{1}{2}f_{M_iM_i}(\delta ln^\delta,M^{h,\delta}(\delta ln^\delta))\sum_{k=ln^\delta}^{ln^\delta+n^\delta-1}[M_i^{h,\delta}(\delta(ln^\delta+n^\delta))-M_i^{h,\delta}(\delta ln^\delta)]^2\right\}\\
&\qquad+\widetilde{\varepsilon}^{h,\delta}(t^1+t^2)-\widetilde{\varepsilon}^{h,\delta}(t^1),
\end{aligned}$$

where $f_{M_iM_i}$ denotes the second partial derivatives, $M_i^{h,\delta}(\cdot)$ is the i-th component of $M^{h,\delta}(\cdot)$ and $\sup_{t\le t^1\le T}\mathbf{E}|\widetilde{\varepsilon}^{h,\delta}(t^1)|\to 0$ as $h,\delta\to 0$.

By (3.27) and the definition of $\sigma^{h,\delta}(\cdot)$, we have

$$\begin{aligned}
&\sum_{ln^\delta=t^1/\delta}^{(t^1+t^2)/\delta}\left\{\sum_{i=1}^{2}f_{M_iM_i}(\delta ln^\delta,M^{h,\delta}(\delta ln^\delta))\sum_{k=ln^\delta}^{ln^\delta+n^\delta-1}[M_i^{h,\delta}(\delta(ln^\delta+n^\delta))-M_i^{h,\delta}(\delta ln^\delta)]^2\right\}\\
&\to\int_{t^1}^{t^1+t^2}\int_{\mathcal{U}}Tr\Big[H_Mf(s,M(s))\sigma(\xi(s),\alpha(s),\phi)[\sigma(\xi(s),\alpha(s),\phi)]'\Big]m_s(d\phi)ds,
\end{aligned}\tag{3.37}$$

where $H_M(f(s,M(s)))$ is the Hessian matrix of $f(\cdot)$ at time s, $Tr(\cdot)$ represents for the trace of a matrix.

Using (3.34)-(3.37), we have

$$\begin{aligned}
&\mathbf{E}h(\xi(t_i),\alpha(t_i),M(t_i),(\psi_j,m)_{t_i}:i\le\kappa,j\le\widetilde{\kappa})\\
&\quad\times\left[f(t^1+t^2,M(t^1+t^2))-f(t^1,M(t^1))-\int_{t^1}^{t^1+t^2}\frac{\partial f(s,M(s))}{\partial s}ds\right.\\
&\qquad\left.-\int_{t^1}^{t^1+t^2}\int_{\mathcal{U}}\frac{1}{2}Tr\Big(H_Mf(s,M(s))\sigma(\xi(s),\alpha(s),\phi)[\sigma(\xi(s),\alpha(s),\phi)]'\Big)m_s(d\phi)ds\right]=0.
\end{aligned}$$

Therefore,

$$\int_0^t \int_{\mathcal{U}} \sigma^{h,\delta}(\xi^{h,\delta}(s), \alpha^{h,\delta}(s), \phi) m_s^{h,\delta}(d\phi) ds \to \int_0^t \int_{\mathcal{U}} \sigma(\xi(s), \alpha(s), \phi) m_s(d\phi) ds, \tag{3.38}$$

uniformly in t with probability one on any bounded interval.

Using the same arguments as in Step 2 and Step 3.2, we obtain the limits for the latter parts of (3.29). As a result, $\xi(\cdot)$ is the solution of (3.30), which means $\xi(s) = X(s)$ w.p.1 and $m(\cdot)$ is an admissible relaxed control. $\square$

3.3.3 Convergence of the Cost and the Value Functions

Note that since $U_k(\cdot)$ satisfies the Inada's conditions. There exist positive real numbers K and k_0 such that $|U_k(X_k)| \le K(1 + |X_k|^{k_0})$. We proceed to prove the following result.

Theorem 3.11. *Suppose that the utility functions $U_k(\cdot)$ has at most polynomial growth. Then the value functions $V^{k,h,\delta}(t,x,i)$ converges to $V^k(t,x,i)$ for $k = 1, 2$, respectively, as $h, \delta \to 0$.*

Proof. By Theorem 3.9, each sequence $\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_1^{h,\delta}(\cdot,\cdot), N_2^{h,\delta}(\cdot,\cdot)\}$ has a weakly convergent subsequence with the limit $\xi^{h,\delta}(\cdot)$ satisfying (3.30). Using the same notation as above and applying the Skorohod representation, the weak convergence, as $h, \delta \to 0$, $J^{k,h,\delta}(t,x,i,m^{h,\delta}) \to J^k(t,x,i,m)$, for $k = 1, 2$. The cost function is given by (3.23). Since $V^k(t,x,i)$ is the maximizing expected utility, for any admissible control $m(\cdot)$, $J^k(t,x,i,m) \le V^k(t,x,i)$, for $k = 1, 2$. Let $\widetilde{m}^{h,\delta}(\cdot)$ be an optimal relaxed control for $\{\xi^{h,\delta}(\cdot)\}$, which implies

$$V^{k,h,\delta}(t,x,i) = J^{k,h,\delta}(t,x,i,\widetilde{m}^{h,\delta}) = \sup_{m^{h,\delta}} J^{k,h,\delta}(t,x,i,m^{h,\delta}).$$

Choose a subsequence $\{\widetilde{h}, \widetilde{\delta}\}$ of $\{h, \delta\}$ such that

$$\limsup_{h,\delta \to 0} V^{k,h,\delta}(t,x,i) = \lim_{\widetilde{h},\widetilde{\delta} \to 0} V^{k,\widetilde{h},\widetilde{\delta}}(t,x,i) = \lim_{\widetilde{h},\widetilde{\delta} \to 0} J^{k,\widetilde{h},\widetilde{\delta}}(t,x,i,\widetilde{m}^{\widetilde{h},\widetilde{\delta}}).$$

Without loss of generality, we may assume that $\{\xi^{\widetilde{h},\widetilde{\delta}}(\cdot), \alpha^{\widetilde{h},\widetilde{\delta}}(\cdot), W^{\widetilde{h},\widetilde{\delta}}(\cdot), \widetilde{m}^{\widetilde{h},\widetilde{\delta}}(\cdot), N_1^{\widetilde{h},\widetilde{\delta}}(\cdot,\cdot), N_2^{\widetilde{h},\widetilde{\delta}}(\cdot,\cdot)\}$

converges weakly to $\{X(\cdot), \alpha(\cdot), W(\cdot), m(\cdot), N_1(\cdot,\cdot), N_2(\cdot,\cdot)\}$, where $m(\cdot)$ is an admissible relaxed control. Then the weak convergence and the Skorohod representation leads to

$$\limsup_{h,\delta\to 0} V^{k,\delta}(t,x,i) = \lim_{\widetilde{h},\widetilde{\delta}\to 0} J^{k,\widetilde{h},\widetilde{\delta}}(t,x,i,\widetilde{m}^{\widetilde{h},\widetilde{\delta}}) = J^k(t,x,i,m) \le V^k(t,x,i).$$

We claim that $\liminf_{\delta} V^{k,\delta}(t,x,i) \ge V^k(t,x,i)$.

Suppose that $\overline{m}(\cdot)$ is an optimal control with Brownian motion $W(\cdot)$ such that $\overline{X}(\cdot)$ is the associated trajectory. By the chattering lemma (see [37] and page 59-60 of [23]), for any given $\eta, \delta_\eta > 0$, there is an $\varepsilon > 0$ and an ordinary control $u^{\eta,\delta_\eta}(\cdot)$ that takes only finite many values, $u^{\eta,\delta_\eta}(\cdot)$ is a constant in $[\iota\varepsilon, \iota\varepsilon+\varepsilon)$, $\overline{m}^{\eta,\delta_\eta}(\cdot)$ is its relaxed control representation, and $J^k(t,x,i,\overline{m}^{\eta,\delta_\eta}) \ge V^k(t,x,i) - \eta$. For each $\eta, \delta_\eta > 0$, and the corresponding $\varepsilon > 0$, consider an optimal control problem with piecewise constant on $[\iota\varepsilon, \iota\varepsilon+\varepsilon)$. We consider the process $\{X^{\eta,\delta_\eta}(\iota\varepsilon), \alpha^{\eta,\delta_\eta}, m^{\eta,\delta_\eta}(\iota\varepsilon), W^{\eta,\delta_\eta}(\iota\varepsilon)\}$. Let $\hat{u}^{\eta,\delta_\eta}(\cdot)$ be the optimal control, $\hat{m}^{\eta,\delta_\eta}(\cdot)$ the relaxed control representation, and $\hat{X}^{\eta,\delta_\eta}(\cdot)$ the associated trajectory. Since $\hat{m}^{\eta,\delta_\eta}(\cdot)$ is the optimal control, $J^k(t,x,i,\hat{m}^{\eta,\delta_\eta}) \ge J^k(t,x,i,\overline{m}^{\eta,\delta_\eta}) \ge V^k(t,x,i) - \eta$. Using the chattering lemma, we can approximate $\hat{m}^{\eta,\delta_\eta}(\cdot)$ by a sequence of $m^{h,\delta}(\cdot)$. Then

$$V^{k,h,\delta}(t,x,i) \ge J^{k,h,\delta}(t,x,i,m^{h,\delta}) \to J^k(t,x,i,\hat{m}^{\eta,\delta_\eta}).$$

Moreover,

$$\liminf_{h,\delta\to 0} V^{k,h,\delta}(t,x,i) \ge \lim_{h,\delta\to 0} J^{k,h,\delta}(t,x,i,m^{h,\delta}) = J^k(t,x,i,\hat{m}^{\eta,\delta_\eta}).$$

Thus, $\liminf_{h,\delta\to 0} V^{k,h,\delta}(t,x,i) \ge V^k(t,x,i) - \eta$. The arbitrariness of η implies that $\liminf_{\delta\to 0} V^{k,\delta}(t,x,i) \ge V^k(t,x,i)$, which completes the proof. $\square$

3.4 Numerical Examples

In this section, we present some numerical results for the case in which both insurance companies are constant absolute risk aversion (CARA) agents, i.e., each agent has an exponential utility function. More precisely, the utility function of each insurer has the form

$$U_k(X_k) = -\frac{1}{\eta_k}\exp(-\eta_k X_k), \quad \text{for } \eta_k > 0,\ k = 1,2. \tag{3.39}$$

Based on the algorithm constructed above, we carry out the computation by valuing iterations in a backward manner by time.

1. Set $t = T - \delta$ and $J^{k,h,\delta}(T, x, i, u^{h,\delta}) = U_k(x_k)$, for each $k = 1, 2$.

2. By (3.23), we obtain

$$\begin{aligned} J^{k,\delta}(t, x, i, u^{h,\delta}) \quad &= (1 - \lambda\delta) \sum_{(y,j)} p_k^{h,\delta}\Big((x, i), (y, j)|u^{h,\delta}\Big) J^{k,\delta}(t + \delta, y, j, u^{h,\delta}) \\ &+\lambda_k\delta \int_{\mathbb{R}_+} J^{k,\delta}(t + \delta, x_k - \widetilde{q}_k^h(q_k(i, z, \rho_k), a_k^{h,\delta}), z, i, u^{h,\delta})\Pi_k(d\rho_k) \\ &+\lambda_l\delta \int_{\mathbb{R}_+} J^{k,\delta}(t + \delta, x_k + \kappa_k\widetilde{q}_l^h(q_l(i, z, \rho_l), a_l^{h,\delta}), z, i, u^{h,\delta})\Pi_l(d\rho_l). \end{aligned}$$

   Find the pair $\{\hat{u}_k^{h,\delta}, k = 1, 2\}$ and record $u_k^{h,\delta}(t) = \hat{u}_k^{h,\delta}$ satisfying that for any $u_k^{h,\delta} \in \mathcal{U}_k$,:

$$J^{1,h,\delta}(t, x, i, u_1^{h,\delta}, \hat{u}_2^{h,\delta}) \le J^{1,h,\delta}(t, x, i, \hat{u}_1^{h,\delta}, \hat{u}_2^{h,\delta})$$

$$J^{2,h,\delta}(t, x, i, \hat{u}_1^{h,\delta}, u_2^{h,\delta}) \le J^{2,h,\delta}(t, x, i, \hat{u}_1^{h,\delta}, \hat{u}_2^{h,\delta}).$$

3. Let $t = t - \delta$ and continue the procedure until $t = 0$. We consider the case in which the discrete event consists of two states, or equivalently, the Markov chain has two states with given claim size distributions. In addition, we assume that the claim size distributions are identical in each regime. By using the value iteration methods, we numerically solve the optimal control problems. The continuous-time Markov chain $\alpha(t)$ representing the discrete event state has the generator $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ and takes values in $\mathcal{M} = \{1, 2\}$.

The parameters of the utility function are $\eta_1 = 17.0$ and $\eta_1 = 21.0$ respectively. The sensitivities are $\kappa_1 = 0.8$ and $\kappa_2 = 0.7$. The claim severity of both players follows exponential distribution $f(y) = \theta_k e^{-\theta_k y}$ with $\theta_1 = 0.3$ and $\theta_2 = 0.2$. To incorporate the difference between claim densities in different regimes, we assume arriving rates of Poisson jump are different. So is the setup for risk-free return, and premium income rate. The detail of setup is as follows in Table 1.

Regime	r	c_1	c_2	λ_1	λ_2
	0.02	0.05	0.02	0.20	0.30
2	0.03	0.10	0.20	0.80	0.70

Table 1: Parameters values

The reinsurance premium rates are computed from expectation premium principle as:

$$g_k\left(a_k\right) = \left(1 + l_k\right)\left(1 - a_k\right)\mathbb{E}\left[A_k\right],$$

where $l_1 = 1.1$ and $l_2 = 1.15$. Further, the volatility and the drift of the financial market index and the risky return rate are modeled respectively by:

$$\mu_S\left(i, Z\right) = 0.2iZ;$$

$$\sigma_S\left(i, Z\right) = 0.4iZ;$$

$$\mu_Z\left(t, Z\right) = 0.4\left(t + 1\right)Z;$$

$$\sigma_Z\left(t, Z\right) = 0.1\left(t + 1\right)Z.$$

State discretization follows $\delta = 0.04$ and $h = 0.2$. For reinsurance, we discretize the reinsurance rate into six levels uniformly located from 0 to 1. The investment amount is free from restrictions, and can vary from -3 to 3 with 0.2 increase. The trend of investment and reinsurance for the varying relative surplus of player one is plotted in Figure 1, and that for the second player is in Figure 2.

From Figure 1, we can observe that both players always hold a low proportion of claim, which is due to the fact that both players are very risk-averse. A big claim will not only reduce their relative surplus but also drive the surplus of their opponent side up. Precisely, in regime one, the proportion held by player two is 0, which is less than 0.2 of player one. This results from the claim arriving rate of the player two is relatively much higher than that of player one. In regime two, considering the high premium income rate and the same expected claim amount, a small proportion of claim is affordable for both players.

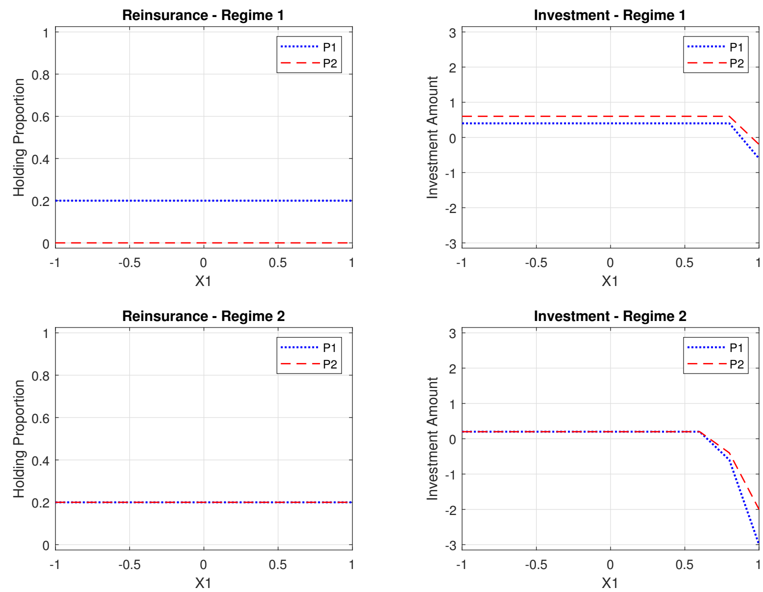

Figure 1: Controls for varying X_1 with $T = 0.08$, $Z = 1.01$, and $X_2 = 0$.

In view of the investment part of Figure 1, player two's investment amount is always no less than that of player one. Since the relative sensitivity of player one to player two is higher, and player two's relative surplus is at a higher relative level initially, player two is willing to accept more risk for higher expected return in order to beat player one. Then, along with growing X_1, since player one's condition is improved, he tends to bear less risk. Meanwhile, player two adopts conservative strategy as well, since he can lower the uncertainty and make use of his advantage that he owns a higher premium income rate. Because a higher market volatility in regime two will introduce more risk, both players choose the investment amount much closer to 0 to lower the uncertainty.

Similar results can be seen from Figure 2. For the reinsurance part, player two initially holds full proportion of a claim, since $X_2 = -1$ leaves player two in a relative bad situation compared to $X_1 = 1$. To change this situation, he chooses to bear lots of risk to reduce the loss from reinsurance premium. This is more obvious for regime 1, where the claim arriving rate is lower. Hence, in more risky scenarios where claim arriving rate is higher, players show

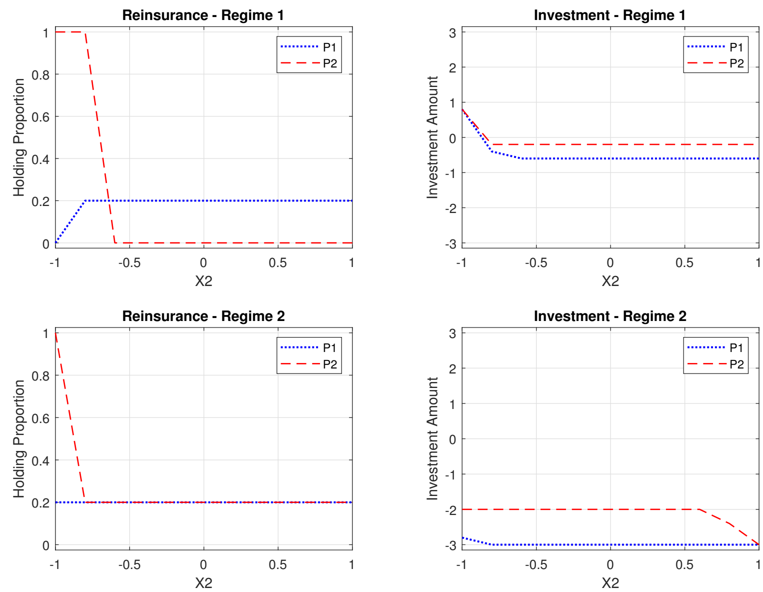

Figure 2: Controls for varying X_2 with $T = 0.08$, $Z = 1.01$, and $X_1 = 1$.

relative risk averse and transfer more risks by reinsurance tools. For the investment part, we can see that player two hold higher positions in risky assets in his portfolio in both regimes, which is consistent with the observations in Figure 1.

CHAPTER 4 CONCLUDING REMARKS AND FUTURE DIRECTIONS

In this dissertation, we have concentrated on properties and numerical solutions for stochastic differential systems with Markovian switching. First, in Chapter 2, we study ecological properties of hybrid competitive Lotka-Volterra models. We formulate the ecosystems as hybrid systems involve both continuous states and discrete events in which the discrete events take values in a countable state space. We demonstrated such properties as existence and uniqueness of solution, stochastic boundedness, sample path continuity for the models.

A main effort is placed on reduction of complexity by introducing a small parameter into the system. This leads to a two-time-scale formulation. Although the two-time-scale system has complex structures, it is shown that there is an associated averaged or reduced system. Using the averaged system, we prove that the original system has similar properties such as extinction and permanence etc. as that of the averaged system for the Lotka-Volterra ecosystems with a two-time-scale Markov chain by perturbed Lyapunov function methods when the ε is small enough.

A number of questions deserve further consideration.

- To begin, instead of the current formulation, we may consider the Markov chain involves both fast and slow motions with more complex structure. For example, two-time-scale Markov chains that are nearly decomposable were considered in [38]. Such setups may be adopted to the ecosystems.

- Other related systems such as mutualism systems can also be formulated and studied. Moreover, one may consider populations suffering sudden environmental shock (e.g., earthquakes, hurricanes, tornadoes, etc.), leading to the consideration regime-switching jump diffusion systems. Designing feedback controls so as to achieve permanence and extinction etc. is another area of future study.

- There is a growing interest to study the associate harvesting problems [35]. To study the harvesting strategies with systems proposed in this paper has not been done to date and is a worthwhile direction.

In Chapter 3, we considered a non-zero-sum stochastic investment and reinsurance game between two insurance companies. Both proportional and non-proportional reinsurance contracts were considered. Although we are able to obtain the systems of HJI equations using dynamic programming principle, solving the problem explicitly is virtually impossible. Based on the assumption that there is a unique Nash equilibrium strategy, we developed a numerical scheme using the Markov chain approximation method (MCAM) to solve the problem. Due to the complexity of the stochastic game formulation, even numerically solving the systems of HJI equations is much more difficult than that of the previous work in stochastic optimization problems. The difficulties arise from the following two aspects. (1) With complex nonlinear state processes, the formulated high-dimension problem adds much difficulties in building approximating Markov chain. (2) The curse of dimensionality makes a significant impact and slow down the computation due to the large numbers of control variables and the dimensions of the HJI systems.

Although the chapter was devoted to a problem arising in risk management and insurance fields, the game problem formulation and the numerical methods developed can be more widely used in various other control and game problems. For our problem, the nature of the Markov chain approximation relies on building a high dimensional lattice of both driving state and control strategy to approximate the value functions under different control scenarios. The optimization on every state follows the same computing rule, leading to the possibility of using parallel acceleration techniques. The first option coming to our mind is to incorporate multi-thread programming techniques into our completed C++ MCAM template library, which enables us to reduce development time by reusing the algorithm architecture of single-thread library. The latest eighth generation Intel CPUs are equipped with six computation cores, which allow maximal twelve threads to run simultaneously. If we parallelize the algorithm using ten threads, we can enhance the time efficiency ten times. However, this is not enough to handle the computational complexity required for our problem. The high dimensionality requires the lattice to be very precise, thus obtaining accurate results relies on generating a large number of nodes. Ten times acceleration seems a big enhancement, but it can only allow us to explore $10^{\frac{1}{7}} \approx 1.39$ times of Z, X_1, X_2, a_1, a_2, b_1, and b_2, which is unable to meet requirements of the computational complexity.

GPU acceleration, e.g., CUDA, is another attractive choice here. Although the frequency of GPU core is much lower than that of CPU, the number of GPU cores is usually hundreds of times of the number of CPU cores, and this makes GPU more suitable for parallel computation. The tenth generation NVidia GPU owns more than two thousand CUDA cores, which make it an easy solution for solving the complexity issue of MCAM algorithm on common stochastic optimization problem, where the maximal or the minimal value on a state is acquired from repeatedly comparing the newly computed value function value against the temporary optimal value so far. However, focusing on MCAM algorithm on our high dimensional game problem, CUDA acceleration is of very limited use. Not like CPU memory, which can be easily more than 64GB, the capacity of GPU memory is usually less than 8GB. The equilibrium strategy is obtained by searching on the value function information stored for different values of the control strategy. As a result, this memory consumption will occur for every GPU thread, which will easily lead the aggregated memory consumed by MCAM algorithm to exceed the GPUs memory capacity. From the above considerations, it appears that using parallel programming techniques to high dimensional game problems needs a lot of more thinking and effort. Finding more efficient way for the numerical solution is our ongoing work. Finally, we note that we have assumed the existence of the Nash equilibrium in this work. From an application point of view, the existence of such equilibrium is reasonable. If no such strategy exists, then the formulation of the physical problem is probably wrong.

Nevertheless, mathematically establishing the existence and uniqueness of the equilibrium strategy is interesting and challenging and deserves in-depth consideration.

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ABSTRACT

SWITCHING DIFFUSIONS: APPLICATIONS TO ECOLOGICAL MODELS, AND NUMERICAL METHODS FOR GAMES IN INSURANCE

by

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August 2019

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Major: Mathematics (Applied)

Degree: Doctor of Philosophy

Recently, a class of dynamic systems called "hybrid systems" containing both continuous dynamics and discrete events has been adapted to treat a wide variety of situations arising in many real-world situations. Motivated by such development, this dissertation is devoted to the study of dynamical systems involving a Markov chain as the randomly switching process: hybrid competitive Lotka-Volterra ecosystems, and non-zero-sum stochastic differential games between two insurance companies with regime-switching.

The first part is concerned with competitive Lotka-Volterra model with Markov switching. A novelty of the contribution is that the Markov chain has a countable state space. Our main objective is to reduce the computational complexity by using two-time-scale systems. Because the existence and uniqueness, as well as continuity of solutions for Lotka-Volterra ecosystems with Markovian switching in which the switching takes place in a countable set are not available, such properties are studied first. The two-time scale feature is highlighted by introducing a small parameter into the generator of the Markov chain. When the small parameter goes to 0, there is a limit system or reduced system. It is established in this work that if the reduced system possesses certain properties such as permanence and extinction, etc., then the complex system also has the same properties when the parameter is sufficiently small. These results are obtained by using the perturbed Lyapunov function methods.

The second part develops an approximation procedure for a class of non-zero-sum stochastic differential investment and reinsurance games between two insurance companies. Both proportional reinsurance and excess-of-loss reinsurance policies are considered. We develop

numerical algorithms to obtain the approximation to the Nash equilibrium by adopting the Markov chain approximation methodology. We establish the convergence of the approximation sequences and the approximation to the value functions. Numerical examples are presented to illustrate the applicability of the algorithms.

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