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Maximum Likelihood Estimations Based on Upper Record Values for Probability Density Function and Cumulative Distribution Function in Exponential Family and Investigating Some of Their Properties

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Cover Page Footnote

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A useful subfamily of the exponential family is considered. The ML estimation based on upper record values are calculated for the parameter, Cumulative Density Function, and Probability Density Function of the subfamily. The relationship between MLE based on record values and a random sample are discussed, along with some properties of these estimators, and its utility is shown for large samples.

Keywords: Exponential family, record values, MLE, asymptotically unbiasedness

Introduction

An exponential family includes a wide range of statistical distributions in two discrete and continuous states, that has a large importance in the distribution theory, and with which it might be possible to integrally investigate many properties of the distribution. Different forms have been presented for this family in the one-parametric state. Generally, a one-variable exponential family is a group of distributions that their probability density function is as follows:

$$\mathrm{f}\left(x;\theta\right)=\exp\left(-\eta\left(\theta\right)\mathrm{T}\left(x\right)-\mathrm{A}\left(\theta\right)\right),$$

where $\mathrm{T}(x)$, $\eta(\theta)$, and $\mathrm{A}(\theta)$ are unknown functions. Equivalently,

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$$f(x;\theta) = h(x)g(\theta)\exp\{-\eta(\theta)T(x)\}$$

in which θ is parameter, T(x) is vector of sufficient statistics. A subfamily from the exponential family is

$$F(x;\theta) = 1 - \exp\{-B(\theta)A(x)\},$$

in which A(x) is a sufficient increasing function and in which

$$B(\theta) > 0; x \in [a,b]; \quad A(a) = 0, A(b) = +\infty; \quad a,b \in \mathbb{R}.$$

It is called the first type exponential family. Many one-parametric continuous distributions can be studied by choosing the above general form, as described in Table 1. Other distributions in this family, may be found, but the reason for choosing the above general family lies in its wideness and comprehensiveness.

Table 1. Instances for the exponential family of the first type

A(x)	B(θ)	Distribution name
A(x) = x	θ	Exponential
A(x) = log(1 + x)	B(θ) = 1 / θ	Lomax
A(x) = x^α	B(θ) = θ	Weibull
A(x) = –ln(x)	B(θ) = θ	Pareto

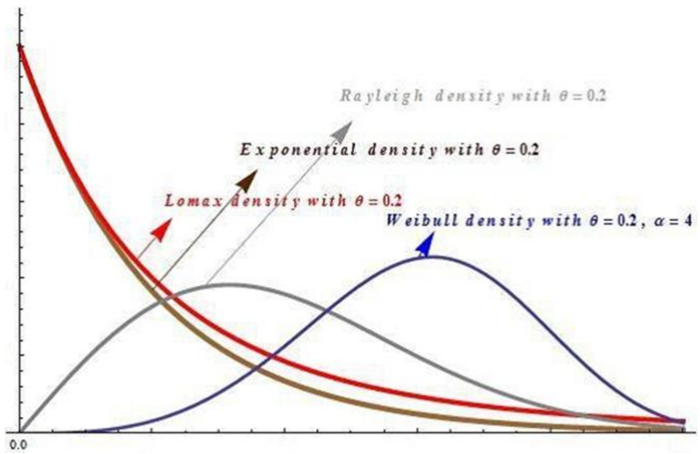

Figure 1. Density functions related to exponential family of the first type

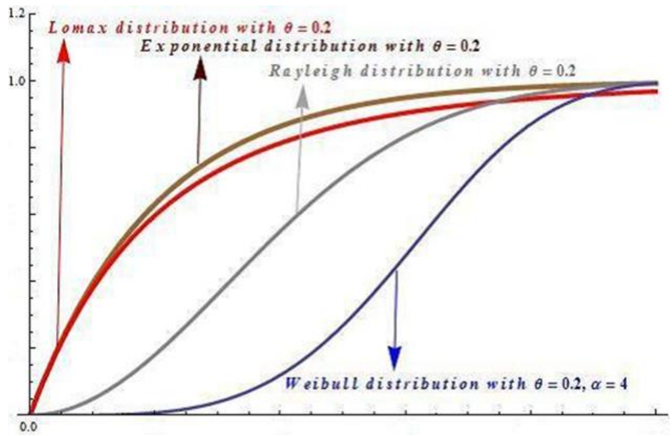

Figure 2. Cumulative distribution functions related to the first type of exponential family

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The Probability density function (PDF) diagram for the distributions in Table 1 and their cumulative distribution function (CDF) diagram are shown in Figures 1 and 2, respectively.

The determination and investigation of the properties of the maximum likelihood estimations based on record values for probability density function and cumulative distribution function in the given exponential family is the goal of this article. The maximum likelihood estimations for the unknown parameters of the exponential family, probability density function, and cumulative distribution function of the exponential family were generally obtained based on a random sample, and also values of upper records. In addition, validity of maximum likelihood estimators based on record values in different states has been presented along with a series of examples. The criterion of Mean Square Error (MSE) was considered in order to investigate and compare them, along with the properties of these estimators and their asymptotic conditions.

Chandler (1952) suggested the concept of records under a mathematical model, but this concept was changed in various fields such as commerce and economics. Shorrock (1972) completed the asymptotic theory of records. Currently, the theory of records is utilized in estimation theory, queuing theory, and etc. Among the works recently done in the field of estimation of distribution parameters based on records, it might be pointed out to those of Ahmadi and Doostparast (2006), Ahmadi et al. (2009), Baklizi (2008), Raqab et al. (2007), and Nasiri and Hosseini (2012). Ahmadi and Balakrishnan (2004) investigated the interval estimations based on record values for quantiles. Nasiri et al. (2013) used a statistic based on record values to estimate Bayesian estimation for exponential distribution.

Maximum Likelihood Estimations Based on Random Sample and Upper Record Values for Unknown Parameter of Exponential Family

Considering an iid random sample of size *n* from the exponential family of the first type, the joint probability density function is

$$\mathrm{f}\left(x_1,\ldots,x_n;\theta\right)=\mathrm{B}^n\left(\theta\right)\prod_{i=1}^{n}\mathrm{A}'\left(x_i\right)\exp\left\{-\mathrm{B}\left(\theta\right)\sum_{i=1}^{n}\mathrm{A}\left(x_i\right)\right\} \qquad (1)$$

where

$$\mathrm{A}'\left(x\right)=\frac{d\,\mathrm{A}\left(x\right)}{dx}$$

where

$$\mathrm{f}\left(x;\theta\right)=\frac{d\,\mathrm{F}\left(x;\theta\right)}{dx}=\mathrm{A}'\left(x\right)\mathrm{B}\left(\theta\right)\exp\left\{-\mathrm{A}\left(x\right)\mathrm{B}\left(\theta\right)\right\}.$$

Hence, the maximum likelihood function is determined as follows considering the joint probability density function ($X_1,\ldots, X_n$) (1).

$$\begin{aligned} L &= \log \mathrm{f}\left(X_1,\ldots,X_n;\theta\right) \\ &= n\log \mathrm{B}\left(\theta\right)+\sum_{i=1}^{n}\log \mathrm{A}'\left(X_i\right)-\mathrm{B}\left(\theta\right)\sum_{i=1}^{n}\mathrm{A}\left(X_i\right) \end{aligned}$$

However, the maximum likelihood estimations are determined from solving the equation

$$\frac{\partial L}{\partial \theta}=0,$$

therefore:

$$\mathrm{B}(\theta)=\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\,.$$

If B(θ) is a one-to-one function from Θ to ℝ,

$$\theta=\mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right). \tag{2}$$

Because the probability density function f(x; θ) = A′(x)B(θ)exp{–A(x)B(θ)} with the properties and conditions described for A and B is a continuous function, it is possible to obtain its first and second order derivatives.

$$\frac{\partial^2 L}{\partial\theta^2}=\frac{n\mathrm{B}''(\theta)\mathrm{B}(\theta)-n\left[\mathrm{B}'(\theta)\right]^2}{\mathrm{B}^2(\theta)}-\mathrm{B}''(\theta)t;\quad t=\sum_{i=1}^{n}\mathrm{A}(x_i) \tag{3}$$

In addition, substituting (2) in (3),

$$\left.\frac{\partial^2 L}{\partial\theta^2}\right|_{\mathrm{B}(\theta)=\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}}=\frac{n\mathrm{B}''(\theta)\left(\frac{n}{t}\right)-n\left[\mathrm{B}'(\theta)\right]^2}{\left(\frac{n}{t}\right)^2}-\mathrm{B}''(\theta)t$$
$$=-\frac{\left[\mathrm{B}'(\theta)t\right]^2}{n}<0$$

Considering the negative sign of the likelihood function,

$$\theta=\mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right)$$

is a likelihood value. With respect to the described issues, ML estimation of the parameter θ based on the random sample ($X_1,\ldots,X_n$) will be

$$\hat{\theta}_{\text{MLE},(X_1,\ldots,X_n)} = \mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n} A(X_i)}\right) \tag{4}$$

A(X) is distributed as exponential distribution by the parameter 1 / B(θ),

$$\mathrm{A}(X) \overset{\text{Distribution}}{=} \exp\left(\frac{1}{\mathrm{B}(\theta)}\right),$$

because

$$\begin{aligned}\mathrm{P}(\mathrm{A}(X) \le t) &= \mathrm{P}(X \le \mathrm{A}^{-1}(t)) \\ &= 1 - \exp\{-\mathrm{B}(\theta)t\}\end{aligned}$$

Because $\underset{i:1,\ldots,n}{X_i}$ is a random sample, $\underset{i:1,\ldots,n}{\mathrm{A}(X_i)}$ is also a random sample and

$$\underset{i:1,\ldots,n}{\mathrm{A}(X_i)} \overset{\text{Distribution}}{=} \exp\left(\frac{1}{\mathrm{B}(\theta)}\right) \tag{5}$$

so

$$T = \sum_{i=1}^{n} \mathrm{A}(X_i) \overset{\text{Distribution}}{=} \text{Gamma}\left(n, \frac{1}{\mathrm{B}(\theta)}\right).$$

Therefore, regarding equations (4) and (5),

$$\hat{\theta}_{\text{MLE},(X_1,\ldots,X_n)} \overset{\text{Distribution}}{=} \mathrm{B}^{-1}\left(\frac{n}{T}\right) \tag{6}$$

in which

$$T \overset{\text{Distribution}}{=} \text{Gamma}\left(n, \frac{1}{\mathrm{B}(\theta)}\right).$$

Considering a random sample of size m which has n values of the upper records ($n < m$), the joint probability density function of the upper records for this sample is determined as follows (Arnold et al., 1998):

$$\begin{aligned} \mathrm{f}\left(r_1,\ldots,r_n;\theta\right) &= \mathrm{f}\left(r_n;\theta\right)\prod_{i=1}^{n-1}\mathrm{h}\left(r_i;\theta\right) \\ &= \mathrm{B}^n\left(\theta\right)\left\{\prod_{i=1}^{n}\mathrm{A}'\left(r_i\right)\right\}\exp\left\{-\mathrm{B}\left(\theta\right)+\mathrm{A}\left(r_n\right)\right\} \end{aligned}$$

The likelihood function based on upper records is

$$\begin{aligned} L_{\text{Records}} &= \log \mathrm{f}\left(r_1,\ldots,r_n;\theta\right) \\ &= n\log \mathrm{B}\left(\theta\right)+\sum_{i=1}^{n}\log \mathrm{A}'\left(r_i\right)-\mathrm{B}\left(\theta\right)\mathrm{A}\left(r_n\right) \end{aligned}$$

The maximum likelihood estimation based on upper record values for parameter θ will be obtained from solving the equation

$$\frac{dL_{\text{Records}}}{d\theta}=0\,;$$

$$\frac{dL_{\text{Records}}}{d\theta}=0 \quad\Rightarrow\quad \frac{n\mathrm{B}'\left(\theta\right)}{\mathrm{B}\left(\theta\right)}-\mathrm{B}'\left(\theta\right)\mathrm{A}\left(r_n\right)=0 \quad\Rightarrow\quad \hat{\theta}=\mathrm{B}^{-1}\left(\frac{n}{\mathrm{A}\left(R_n\right)}\right) \tag{7}$$

$$\frac{d^2L_{\text{Records}}}{d\theta^2}=\frac{n\mathrm{B}''\left(\theta\right)\mathrm{B}\left(\theta\right)-n\left[\mathrm{B}'\left(\theta\right)\right]^2}{\mathrm{B}^2\left(\theta\right)}-\mathrm{B}''\left(\theta\right)l,\quad l=\mathrm{A}\left(r_n\right) \tag{8}$$

Substituting (7) in (8),

$$\left.\frac{d^2 L_{\text{Records}}}{d\theta^2}\right|_{\mathrm{B}(\theta)=\frac{n}{\mathrm{A}(r_n)}} = \frac{n\mathrm{B}''(\theta)\left(\frac{n}{l}\right) - n\left[\mathrm{B}'(\theta)\right]^2}{\left(\frac{n}{l}\right)^2} - \mathrm{B}''(\theta)l$$

$$= -\frac{\left[\mathrm{B}'(\theta)l\right]^2}{n} < 0$$

This implies (7) is a maximum likelihood estimation based on upper record values because it maximizes the likelihood function based on record values, and

$$\hat{\theta}_{\text{MLE},(R_1,\ldots,R_n)} = \mathrm{B}^{-1}\left(\frac{n}{\mathrm{A}(R_n)}\right). \tag{9}$$

Because $\underset{i:1,\ldots,n}{\mathrm{A}(R_i)}$ is distributed as gamma by the parameter (i, 1 / B(θ)), then A(R_n) is distributed as gamma by the parameter (n, 1 / B(θ)) (Arnold et al., 1998).

$$T = \mathrm{A}(R_n) \overset{\text{Distribution}}{=} \text{Gamma}\left(n, \frac{1}{\mathrm{B}(\theta)}\right) \tag{10}$$

Considering (9) and (10) simultaneously leads to

$$\hat{\theta}_{\text{MLE},(R_1,\ldots,R_n)} \overset{\text{Distribution}}{=} \mathrm{B}^{-1}\left(\frac{n}{T}\right) \tag{11}$$

in which

$$T = \mathrm{A}(R_n) \overset{\text{Distribution}}{=} \text{Gamma}\left(n, \frac{1}{\mathrm{B}(\theta)}\right).$$

This may be summarized in the following theorem:

Theorem 1. In the exponential family of the first type, the maximum likelihood estimator based on a random sample of size n and the maximum likelihood estimator based on the first n values of the upper records are identically distributed. In other words, if g is a real function then

$$\text{i} \quad g\left(\hat{\theta}_{\text{MLE},(X_1,\ldots,X_n)}\right) \overset{\text{Distribution}}{=} g\left(\hat{\theta}_{\text{MLE},(R_1,\ldots,R_n)}\right)$$

$$\text{ii} \quad \text{MSE}\left(g\left(\hat{\theta}_{\text{MLE},(X_1,\ldots,X_n)}\right)\right) = \text{MSE}\left(g\left(\hat{\theta}_{\text{MLE},(R_1,\ldots,R_n)}\right)\right)$$

Proof. Regarding (6) and (11), the proof is obvious.

The second part of Theorem 1 can be regarded as the primary reason for using record estimators. In likelihood estimation of any function of θ, a random sample of size n and n values of the upper records have the same errors (if MSE is regarded as the error criterion). Some examples will be presented in which the record estimations are justifiable. As a preface, it is useful to define $\alpha(n)$ as

$$\begin{aligned} \text{MSE}&\left(g\left(\hat{\theta}_{\text{MLE},(X_1,\ldots,X_n)}\right)\right) \\ &= \text{MSE}\left(g\left(\hat{\theta}_{\text{MLE},(R_1,\ldots,R_n)}\right)\right) \\ &= E\left[\left\{g\left(B^{-1}\left(\frac{n}{T}\right)\right) - g(\theta)\right\}^2\right] \\ &= E\left[g^2\left(B^{-1}\left(\frac{n}{T}\right)\right)\right] - 2g(\theta)E\left[g\left(B^{-1}\left(\frac{n}{T}\right)\right)\right] + \left(g(\theta)\right)^2 \\ &= \alpha(n) \end{aligned} \tag{12}$$

Result 1. Considering the first part of Theorem 1 and relation (12), if a number of m upper record values $R_1,\ldots, R_m$ exist in a random sample $X_1,\ldots, X_n$, then the MSE value of ML estimation based on record values in this sample equals with MSE of ML estimation based on a random sample of size m, i.e.

$$\begin{aligned} \text{MSE}\left(\hat{\theta}_{\text{MLE},(R_1,\ldots,R_m)}\right) &= \text{MSE}\left(\hat{\theta}_{\text{MLE based on random sample of size } m,(X_1,\ldots,X_m)}\right) \\ &= \alpha(m) \end{aligned} \tag{13}$$

Result 2. The relation (12) and the result (1) provide a new tool and how to determine if conditions employing the record estimations is better. Two following instances clarify this issue.

Example 1. If A(x) = x and B(θ) = 1 / θ, then $B^{-1}(n / T) = T / n$; it is known that $B^{-1}(n / T) = T / n$ is distributed as gamma by the parameters (n, θ / n),

$$B^{-1}\left(\frac{n}{T}\right)=\frac{T}{n}\overset{\text{Distribution}}{=}\text{Gamma}\left(n,\frac{\theta}{n}\right).$$

Now, considering the relation (12),

$$E\left[\left(B^{-1}\left(\frac{n}{T}\right)\right)^2\right]=\theta^2+\frac{\theta^2}{n},\quad E\left[B^{-1}\left(\frac{n}{T}\right)\right]=\theta.$$

Therefore,

$$MSE=\alpha(n)=\frac{\theta^2}{n}.$$

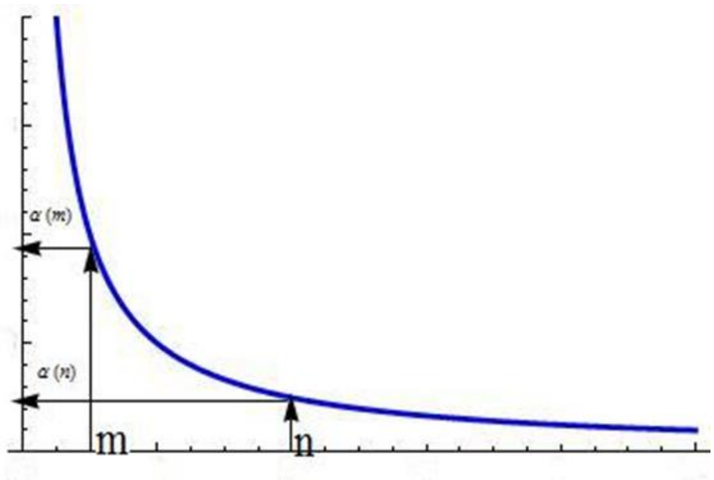

Figure 3. Comparison of maximum likelihood estimation based on random sample and records in an exponential distribution for Example 1

With respect to Figure 3, it is readily concluded that for a random sample of size n, if m records of the upper records are available ($m < n$), then the maximum likelihood estimation based on record values have always more errors.

As a consequence of Example 1, the maximum likelihood estimations based on record values might be inefficient and unsuitable in the first looking. It is shown that this belief is not in general the case and there are circumstances in which estimations based on record values are more efficient (i.e. have less MSEs). The following example clarifies this subject.

Example 2. Assuming $\mathrm{A}(x) = \log(x / k)$ and $\mathrm{B}(\theta) = \theta$, to estimate the parameter $\mathrm{g}(\theta) = k^{\theta}$. As a consequence of the relation (12),

$$\mathrm{B}^{-1}(\theta) = \theta, \quad \mathrm{B}^{-1}\left(\frac{n}{T}\right) = \frac{n}{T}, \quad \mathrm{g}\left(\mathrm{B}^{-1}\left(\frac{n}{T}\right)\right) = k^{\frac{n}{T}}.$$

Thus,

$$k^{\frac{n}{T}} = \exp\left\{\ln\left(k^{\frac{n}{T}}\right)\right\} = \sum_{i=0}^{+\infty} \frac{\left(\ln\left(k^{\frac{n}{T}}\right)\right)^{i}}{i!}.$$

If for example $k = \mathrm{e}$, then

$$k^{\frac{n}{T}} = \exp\left\{\ln\left(k^{\frac{n}{T}}\right)\right\} = \sum_{i=0}^{+\infty} \frac{\left(\ln\left(k^{\frac{n}{T}}\right)\right)^{i}}{i!} = \sum_{i=0}^{+\infty} \frac{\left(\frac{n}{T}\right)^{i}}{i!}.$$

Considering

$$\mathrm{E}\left[\left\{\mathrm{g}\left(\mathrm{B}^{-1}\left(\frac{n}{T}\right)\right)\right\}^{\alpha}\right] = \mathrm{E}\left[\exp\left\{\frac{\alpha n}{T}\right\}\right] = \mathrm{E}\left[\sum_{i=0}^{+\infty} \frac{\left(\frac{\alpha n}{T}\right)^{i}}{i!}\right], \quad \frac{\left(\frac{\alpha n}{T}\right)^{i}}{i!} > 0,$$

each variable $(\alpha n / T)^i / i!$ is greater than zero then, it is possible to interchange the integration and summation:

$$\mathrm{E}\left[\sum_{i=0}^{+\infty}\frac{\left(\frac{\alpha n}{T}\right)^i}{i!}\right]=\sum_{i=0}^{+\infty}\mathrm{E}\left[\frac{\left(\frac{\alpha n}{T}\right)^i}{i!}\right]$$

$$=\sum_{i=0}^{+\infty}\left\{\int_0^{+\infty}\frac{\alpha^i n^i t^{n-1}\theta^n\exp\{-\theta t\}}{t^i i!(n-1)!}dt\right\}$$

$$=\sum_{i=0}^{+\infty}\frac{\alpha^i n^i\theta^n}{i!(n-1)!}\left\{\int_0^{+\infty}t^{n-i-1}\exp(-\theta t)\,dt\right\}$$

n and i are round numbers and that is why

$$\int_0^{+\infty}t^{n-i-1}\exp(-\theta t)\,dt=\frac{\Gamma(n-i)}{\theta^{n-i}}. \tag{14}$$

$$\mathrm{E}\left[\left\{\mathrm{g}\left(\mathrm{B}^{-1}\left(\frac{n}{T}\right)\right)\right\}^{\alpha}\right]=\mathrm{E}\left[\sum_{i=0}^{+\infty}\frac{\left(\frac{\alpha n}{T}\right)^i}{i!}\right]$$

$$=\sum_{i=0}^{+\infty}\frac{\alpha^i n^i\theta^n}{i!(n-1)!}\left\{\int_0^{+\infty}t^{n-i-1}\exp(-\theta t)\,dt\right\}$$

$$=\sum_{i=0}^{+\infty}\frac{\alpha^i n^i\theta^n}{i!(n-1)!}\frac{\Gamma(n-i)}{\theta^{n-i}}$$

Because the Gamma function is defined for positive values, the MSE is

$$\mathrm{MSE}(\mathrm{g}(\theta))=\sum_{i=0}^{n-1}\frac{2^i n^i\theta^n}{i!(n-1)!}\frac{\Gamma(n-i)}{\theta^{n-i}}-2\mathrm{e}^{\theta}\sum_{i=0}^{n-1}\frac{n^i\theta^n}{i!(n-1)!}\frac{\Gamma(n-i)}{\theta^{n-i}}+\mathrm{e}^{2\theta}.$$

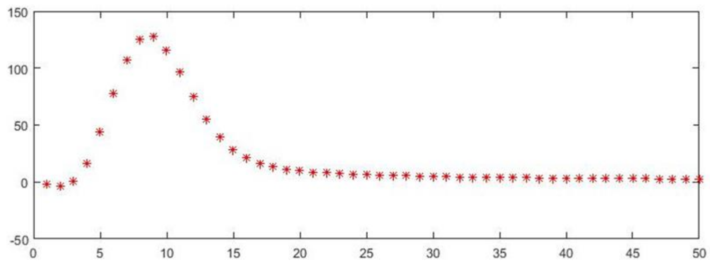

Figure 4. Different values of MSE in terms of *n* when estimating g in Example 2

Consider MSE(g(θ)) for different values of *n* as in Figure 4. If the number of upper records in a sample of size 10 or 11 equals to 5 or 6, then the maximum likelihood estimation based on records does have less amount of error. For another example, it is seen, with respect to Figure 4 that $\alpha(4) < \alpha(5) < \alpha(7) < \alpha(12)$, that is, if a sample of size 12 has 4 values of the upper records, then the ML estimation of g(θ) based on record values does have less amount of error, since it is clearly seen in the figure that $\alpha(4) < \alpha(11)$.

Maximum Likelihood Estimations for the PDF and CDF from the Exponential Family of the First Type Based on Random Sample and Upper Record Values

Suppose a random sample of size *n* from the exponential family of the first type ($X_1,\ldots, X_n$) has *m* values from the upper record ($R_1,\ldots, R_m$). Considering the relation (4) and the point that the maximum likelihood estimations are stable, the ML estimations for f and F are

$$\hat{\theta}_{\mathrm{MLE},(X_1,\ldots,X_n)} = \mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right),$$

$$\begin{aligned}\hat{\mathrm{f}}&_{\mathrm{MLE},(X_1,\ldots,X_n)}\\ &= \mathrm{A}'(x)\mathrm{B}\left(\mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right)\right)\exp\left(-\mathrm{A}(x)\mathrm{B}\left(\mathrm{B}^{-1}\left(\frac{n}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right)\right)\right) \\ &= \frac{n\mathrm{A}'(x)}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\exp\left(-\frac{n\mathrm{A}(x)}{\sum_{i=1}^{n}\mathrm{A}(X_i)}\right)\end{aligned} \tag{15}$$

Similarly, the maximum likelihood estimation for cumulative distribution function based on random sample is determined as

$$\hat{\mathrm{F}}_{\mathrm{MLE},(X_1,\ldots,X_n)} = 1 - \exp\left(-\frac{n\,\mathrm{A}(x)}{\sum_{i=1}^{n} \mathrm{A}(X_i)} \right). \tag{16}$$

In addition, the maximum likelihood estimators $\mathrm{f}_\theta(x)$ and $\mathrm{F}_\theta(x)$, based on the upper record values in this sample (sample of size n), are obtained as follows:

$$\begin{aligned} \hat{\mathrm{f}}_{\mathrm{MLE},(R_1,\ldots,R_m)} &= \frac{m\,\mathrm{A}'(x)}{\mathrm{A}(R_m)} \exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)} \right), \\ \hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)} &= 1 - \exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)} \right) \end{aligned} \tag{17}$$

Theorem 2. If the random sample $(X_1,\ldots, X_n)$ has m values from the upper records $(R_1,\ldots, R_m)$, then the ML estimations of the PDF and CDF based on upper record values and based on a random sample of size m are distributed identically.

Proof. It is easily proven by considering (6), (10), (15), (16), and (17).

Theorem 3. The ML estimations based on random sample and based on the upper record values are biased estimators and

i $\quad \mathrm{E}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] = \sum_{i=0}^{n-2} \left[\frac{\Gamma(n-i-1)}{\Gamma(n)\Gamma(i+1)} (-1)^i \{n\,\mathrm{B}(\theta)\}^{i+1} \mathrm{A}'(x) \right] \{\mathrm{A}(x)\}^i$

ii $\quad \mathrm{E}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] = \sum_{i=0}^{m-2} \left[\frac{\Gamma(m-i-1)}{\Gamma(m)\Gamma(i+1)} (-1)^i \{m\,\mathrm{B}(\theta)\}^{i+1} \mathrm{A}'(x) \right] \{\mathrm{A}(x)\}^i$

iii $\quad \mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] = 1 - \sum_{i=0}^{n-1} \left[\frac{\Gamma(n-i)}{\Gamma(n)\Gamma(i+1)} (-1)^i \{n\,\mathrm{B}(\theta)\}^{i} \right] \{\mathrm{A}(x)\}^i$

iv $\quad \mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] = 1 - \sum_{i=0}^{m-1} \left[\frac{\Gamma(m-i)}{\Gamma(m)\Gamma(i+1)} (-1)^i \{m\,\mathrm{B}(\theta)\}^{i} \right] \{\mathrm{A}(x)\}^i$

Proof. Here, the fourth part is proven; the other parts are proven similarly:

$$\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)} = 1 - \exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right),$$

$$T = \mathrm{A}(R_m) \overset{\text{Distribution}}{=} \mathrm{Gamma}\left(m, \frac{1}{\mathrm{B}(\theta)}\right),$$

$$\mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] = \mathrm{E}\left[1 - \exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right] = 1 - \mathrm{E}\left[\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right],$$

$$\mathrm{E}\left[\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right] = \mathrm{E}\left[\sum_{i=1}^{+\infty} \frac{(-1)^i m^i \mathrm{A}^i(x)}{T^i i!}\right]$$

$$= \mathrm{E}\left[\sum_{\text{even } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)}{T^i i!}\right] - \mathrm{E}\left[\sum_{\text{odd } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)}{T^i i!}\right]$$

Notice the variables $m^i\mathrm{A}^i(x) / T^i i!$ are positive; that is why it is possible to interchange the expectations and summation.

$$\mathrm{E}\left[\sum_{\text{even } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)}{T^i i!}\right] - \mathrm{E}\left[\sum_{\text{odd } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)}{T^i i!}\right] = \sum_{\text{even } is}^{+\infty} \mathrm{E}\left[\frac{m^i \mathrm{A}^i(x)}{T^i i!}\right] - \sum_{\text{odd } is}^{+\infty} \mathrm{E}\left[\frac{m^i \mathrm{A}^i(x)}{T^i i!}\right]$$

$$= \sum_{\text{even } is}^{+\infty} \int_0^{+\infty} \frac{m^i \mathrm{A}^i(x)}{t^i i!} \frac{t^{m-1} \mathrm{B}^m(\theta) \exp(-\mathrm{B}(\theta) t)}{\Gamma(m)} dt$$

$$- \sum_{\text{odd } is}^{+\infty} \int_0^{+\infty} \frac{m^i \mathrm{A}^i(x)}{t^i i!} \frac{t^{m-1} \mathrm{B}^m(\theta) \exp(-\mathrm{B}(\theta) t)}{\Gamma(m)} dt$$

$$= \sum_{\text{even } is}^{+\infty} \frac{m^i \mathrm{A}^i(x) \mathrm{B}^m(\theta)}{i!\Gamma(m)} \int_0^{+\infty} t^{m-i-1} \exp(-\mathrm{B}(\theta)) dt$$

$$- \sum_{\text{odd } is}^{+\infty} \frac{m^i \mathrm{A}^i(x) \mathrm{B}^m(\theta)}{i!\Gamma(m)} \int_0^{+\infty} t^{m-i-1} \exp(-\mathrm{B}(\theta)) dt$$

Using (14) instead of

$$\int_0^{+\infty} t^{m-i-1} \exp(-\mathrm{B}(\theta)) dt,$$

the above expression becomes

$$\sum_{\text{even } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)\mathrm{B}^m(\theta)\Gamma(m-i)}{i!\Gamma(m)\mathrm{B}^{m-i}(\theta)} - \sum_{\text{odd } is}^{+\infty} \frac{m^i \mathrm{A}^i(x)\mathrm{B}^m(\theta)\Gamma(m-i)}{i!\Gamma(m)\mathrm{B}^{m-i}(\theta)}$$
$$= \sum_{i=1}^{+\infty} \frac{\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)} \left(-m\mathrm{A}(x)\mathrm{B}(\theta)\right)^i$$

Finally, having considered the above relations and the fact that the Gamma function is defined for positive values, the expectation is readily considered to be

$$\mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] = 1 - \mathrm{E}\left[\exp\left(-\frac{m\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right]$$
$$= 1 - \sum_{i=0}^{m-1} \frac{\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)} \left(-m\mathrm{A}(x)\mathrm{B}(\theta)\right)^i$$

Theorem 4.

$$\mathrm{MSE}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] = \sum_{i=0}^{n-3} \left[\left(\mathrm{A}'(x)\mathrm{B}(\theta)\right)^2 \frac{\left(-2n\mathrm{A}(x)\mathrm{B}(\theta)\right)^i \Gamma(n-i-2)}{\Gamma(n)\Gamma(i+1)}\right]$$

i $$-2n\mathrm{A}'(x)\left(\mathrm{B}(\theta)\right)^2 \exp\left\{-\mathrm{A}(x)\mathrm{B}(\theta)\right\} \sum_{i=0}^{n-2} \left[\frac{\left(-n\mathrm{A}(x)\mathrm{B}(\theta)\right)^i \Gamma(n-i-1)}{\Gamma(n)\Gamma(i+1)}\right]$$
$$-\left(\mathrm{A}'(x)\mathrm{B}(\theta)\right)^2 \exp\left\{-2\mathrm{A}(x)\mathrm{B}(\theta)\right\}$$

$$\mathrm{MSE}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] = \sum_{i=0}^{m-3} \left[\left(\mathrm{A}'(x)\mathrm{B}(\theta)\right)^2 \frac{\left(-2m\mathrm{A}(x)\mathrm{B}(\theta)\right)^i \Gamma(m-i-2)}{\Gamma(m)\Gamma(i+1)}\right]$$

ii $$-2m\mathrm{A}'(x)\left(\mathrm{B}(\theta)\right)^2 \exp\left\{-\mathrm{A}(x)\mathrm{B}(\theta)\right\} \sum_{i=0}^{m-2} \left[\frac{\left(-m\mathrm{A}(x)\mathrm{B}(\theta)\right)^i \Gamma(m-i-1)}{\Gamma(m)\Gamma(i+1)}\right]$$
$$-\left(\mathrm{A}'(x)\mathrm{B}(\theta)\right)^2 \exp\left\{-2\mathrm{A}(x)\mathrm{B}(\theta)\right\}$$

iii
$$\begin{aligned}\mathrm{MSE}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] &= \sum_{i=0}^{n-1}\left[\frac{\left(-2n\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\Gamma(n-i)}{\Gamma(n)\Gamma(i+1)}\right] \\ &\quad -2\exp\left\{-\mathrm{A}(x)\mathrm{B}(\theta)\right\}\sum_{i=0}^{n-1}\left[\frac{\left(-n\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\Gamma(n-i)}{\Gamma(n)\Gamma(i+1)}\right] \\ &\quad -\exp\left\{-2\,\mathrm{A}(x)\mathrm{B}(\theta)\right\}\end{aligned}$$

iv
$$\begin{aligned}\mathrm{MSE}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right] &= \sum_{i=0}^{m-1}\left[\frac{\left(-2m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\Gamma(m-i)}{\Gamma(m)\Gamma(i+1)}\right] \\ &\quad -2\exp\left\{-\mathrm{A}(x)\mathrm{B}(\theta)\right\}\sum_{i=0}^{m-1}\left[\frac{\left(-m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\Gamma(m-i)}{\Gamma(m)\Gamma(i+1)}\right] \\ &\quad -\exp\left\{-2\,\mathrm{A}(x)\mathrm{B}(\theta)\right\}\end{aligned}$$

Proof. Here the fourth part is proven; the other parts are proved in the same way.

$$\begin{aligned}\mathrm{MSE}\left(\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right) &= \mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}-\mathrm{F}_\theta(x)\right]^2 \\ &= \mathrm{E}\left[\left\{1-\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right\}-\left\{1-\exp\left(-\mathrm{B}(\theta)\mathrm{A}(x)\right)\right\}\right]^2 \\ &= \mathrm{E}\left[\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)-\exp\left(-\mathrm{B}(\theta)\mathrm{A}(x)\right)\right]^2 \\ &= \mathrm{E}\left[\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right] \\ &\qquad -2\exp\left(-\mathrm{B}(\theta)\mathrm{A}(x)\right)\mathrm{E}\left[\exp\left(-\frac{m\,\mathrm{A}(x)}{\mathrm{A}(R_m)}\right)\right] \\ &\qquad +\exp\left(-2\,\mathrm{B}(\theta)\mathrm{A}(x)\right)\end{aligned}$$

From equation (10),

$$T = \mathrm{A}\left(R_n\right) \overset{\text{Distribution}}{=} \text{Gamma}\left(n, \frac{1}{\mathrm{B}(\theta)}\right).$$

Considering above relation,

$$\mathrm{MSE}\left(\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right) = \mathrm{E}\left(\exp\left(\frac{-2m\,\mathrm{A}(x)}{T}\right)\right) - 2\exp\left(-\mathrm{A}(x)\mathrm{B}(\theta)\right)\mathrm{E}\left(\frac{-m\,\mathrm{A}(x)}{T}\right)$$
$$+\exp\left(-2\,\mathrm{A}(x)\mathrm{B}(\theta)\right)$$

In order to obtain the above relation, we should obtain the following general expectation:

$$\begin{aligned}
\mathrm{W}(\alpha) &= \mathrm{E}\left(\exp\left(\frac{-\alpha m\,\mathrm{A}(x)}{T}\right)\right) \\
&= \int_0^{+\infty} \exp\left(\frac{-\alpha m\,\mathrm{A}(x)}{T}\right) \frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)} \\
&= \int_0^{+\infty} \sum_{i=0}^{+\infty} \frac{\left(-\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!} \frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)} \\
&= \int_0^{+\infty} \sum_{\text{even } is}^{+\infty} \frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!} \frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)} \\
&\qquad - \int_0^{+\infty} \sum_{\text{odd } is}^{+\infty} \frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!} \frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)}
\end{aligned}$$

$(\alpha m\mathrm{A}(x))^i / (t^i i!) > 0$, therefore

$$\int_0^{+\infty}\sum_{\text{even } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!}\frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)}dt$$

$$-\int_0^{+\infty}\sum_{\text{odd } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!}\frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)}dt$$

$$=\sum_{\text{even } is}^{+\infty}\int_0^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!}\frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)}dt$$

$$-\sum_{\text{odd } is}^{+\infty}\int_0^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i}{t^i i!}\frac{t^{m-1}\,\mathrm{B}^m(\theta)\exp\left(-\mathrm{B}(\theta)t\right)}{\Gamma(m)}dt$$

$$=\sum_{\text{even } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\int_0^{+\infty}\frac{t^{m-1}\exp\left(-\mathrm{B}(\theta)t\right)}{t^i}dt$$

$$-\sum_{\text{odd } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\int_0^{+\infty}\frac{t^{m-1}\exp\left(-\mathrm{B}(\theta)t\right)}{t^i}dt$$

$$=\sum_{\text{even } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\int_0^{+\infty}t^{m-i-1}\exp\left(-\mathrm{B}(\theta)t\right)dt$$

$$-\sum_{\text{odd } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\int_0^{+\infty}t^{m-i-1}\exp\left(-\mathrm{B}(\theta)t\right)dt$$

$$=\sum_{\text{even } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\frac{\Gamma(m-i)}{\mathrm{B}^{m-i}(\theta)}-\sum_{\text{odd } is}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\frac{\Gamma(m-i)}{\mathrm{B}^{m-i}(\theta)}$$

$$=\sum_{i=0}^{+\infty}\frac{\left(\alpha m\,\mathrm{A}(x)\right)^i\,\mathrm{B}^m(\theta)}{\Gamma(i+1)\Gamma(m)}\frac{\Gamma(m-i)}{\mathrm{B}^{m-i}(\theta)}$$

$$=\sum_{i=0}^{+\infty}\frac{\left(-\alpha m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\,\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)}$$

$$=\sum_{i=0}^{n-1}\frac{\left(-\alpha m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^i\,\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)}$$

Finally,

$$\mathrm{W}(\alpha)=\sum_{i=0}^{m-1}\frac{\left(-\alpha m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^{i}\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)}$$

Now the MSE function is obtained:

$$\begin{aligned}\mathrm{MSE}&=\mathrm{W}(2)-2\exp\left(-\mathrm{A}(x)\mathrm{B}(\theta)\right)\mathrm{W}(1)+\exp\left(-2\,\mathrm{A}(x)\mathrm{B}(\theta)\right)\\&=\sum_{i=0}^{m-1}\frac{\left(-2m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^{i}\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)}\\&\qquad-2\exp\left(-\mathrm{A}(x)\mathrm{B}(\theta)\right)\sum_{i=0}^{m-1}\frac{\left(-2m\,\mathrm{A}(x)\mathrm{B}(\theta)\right)^{i}\Gamma(m-i)}{\Gamma(i+1)\Gamma(m)}\\&\qquad+\exp\left(-2\,\mathrm{A}(x)\mathrm{B}(\theta)\right)\end{aligned}$$

The next theorem clarifies some advantageous aspects of the above likelihood estimators. That is, the above-mentioned estimators are always nearly unbiased in large samples.

Theorem 5. The maximum likelihood estimations based on random sample and on the upper record values for the PDF and CDF in the exponential family of the first type are asymptotically unbiased. In the other words,

i $\quad\lim_{n\to\infty}\mathrm{E}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right]=\mathrm{f}_{\theta}(x)$

ii $\quad\lim_{n\to\infty}\mathrm{E}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right]=\mathrm{f}_{\theta}(x)$

iii $\quad\lim_{n\to\infty}\mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right]=\mathrm{F}_{\theta}(x)$

iv $\quad\lim_{n\to\infty}\mathrm{E}\left[\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right]=\mathrm{F}_{\theta}(x)$

The following lemma is needed in order to prove the above theorem:

Lemma 1.

$$\lim_{n\to\infty}\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)}=1$$

for the reason that

$$\lim_{n\to\infty}\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)}=\lim_{n\to\infty}\frac{n^i}{\prod_{k=1}^{i+1}(n-k)}=\lim_{n\to\infty}\frac{n^i}{n^i}=1.$$

It must be proven

$$\forall\epsilon>0,\exists N\text{ s.t. }\forall n\geq N,\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)}-1<\epsilon.$$

Assuming ϵ is arbitrary, N is chosen as

$$N=\frac{\sqrt[i+1]{\epsilon+1}-i\sqrt[i+1]{\epsilon+1}}{\sqrt[i+1]{\epsilon+1}-1}.$$

$$\begin{aligned}\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)}&=\frac{(n-i-2)!n^{i+1}}{(n-1)!}\\&=\frac{n^{i+1}}{(n-1)(n-2)\ldots(n-i-1)}\\&=\frac{n}{n-1}\underset{(i+1)\text{ times}}{\times\cdots\times}\frac{n}{n-i-1}\leq\frac{n}{n-i-1}\underset{(i+1)\text{ times}}{\times\cdots\times}\frac{n}{n-i-1}\end{aligned}$$

However,

$$\forall n\geq N\text{ and }i\in\{1,2,3,\ldots,N-2\},\frac{n}{n-i-1}\leq\frac{N}{N-i-1}.$$

Thus,

$$\frac{n}{n-1}\underset{(i+1)\text{ times}}{\times\cdots\times}\frac{n}{n-i-1}\leq\frac{N}{N-i-1}\underset{(i+1)\text{ times}}{\times\cdots\times}\frac{N}{N-i-1}$$

and consequently

$$\forall n \geq N, \frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)} < \underbrace{\frac{N}{N-i-1} \times \cdots \times \frac{N}{N-i-1}}_{(i+1)\text{ times}}.$$

By substituting

$$N = \frac{\sqrt[i+1]{\varepsilon+1} - i\sqrt[i+1]{\varepsilon+1}}{\sqrt[i+1]{\varepsilon+1} - 1}$$

in the last relation, the following result is reached:

$$\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)} < \varepsilon + 1.$$

Proof. Only the first part is proven; the other parts have similar proofs.

$$\begin{aligned}\lim_{n\to\infty}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] &= \sum_{i=0}^{\infty}\left[\lim_{n\to\infty}\frac{\Gamma(n-i-1)}{\Gamma(n)\Gamma(i+1)}(-1)^i n^{i+1}\{\mathrm{B}(\theta)\}^{i+1}\mathrm{A}'(x)\{\mathrm{A}(x)\}^i\right] \\ &= \sum_{i=0}^{\infty}\lim_{n\to\infty}\left[\frac{\Gamma(n-i-1)n^{i+1}}{\Gamma(n)}\frac{\{-\mathrm{B}(\theta)\}^i\mathrm{B}(\theta)\mathrm{A}'(x)\{\mathrm{A}(x)\}^i}{\Gamma(i+1)}\right]\end{aligned}$$

By considering Lemma 1,

$$\begin{aligned}\lim_{n\to\infty}\left[\hat{\mathrm{f}}_{\mathrm{MLE},(X_1,\ldots,X_n)}\right] &= \sum_{i=0}^{\infty}\frac{\{-\mathrm{A}(x)\mathrm{B}(\theta)\}^i}{i!}\mathrm{B}(\theta)\mathrm{A}'(x) \\ &= \mathrm{A}'(x)\mathrm{B}(\theta)\exp\{-\mathrm{A}(x)\mathrm{B}(\theta)\}\end{aligned}$$

The above theorem calls attention to convergence in probability and to the following theorem.

Theorem 6. If a random sample ($X_1,\ldots, X_n$) has m values from the upper records ($R_1,\ldots, R_m$), then the maximum likelihood estimations for the CDF and PDF based on upper record values are consistent estimations for the CDF and PDF.

$$\text{i} \quad \hat{f}_{\text{MLE},(R_1,\ldots,R_m)} \xrightarrow{P} f_\theta(x)$$
$$\text{ii} \quad \hat{F}_{\text{MLE},(R_1,\ldots,R_m)} \xrightarrow{P} F_\theta(x)$$

Proof. Part i is proven here; the other part has a similar proof.

$$\hat{f}_{\text{MLE},(R_1,\ldots,R_m)} = \hat{f}_{\text{MLE},R}$$

Based on Markov's theorem,

$$P\left\{\varepsilon \le \left|\hat{f}_{\text{MLE},R} - E\left[\hat{f}_{\text{MLE},R}\right]\right|\right\} \le \frac{\text{Var}\left(\hat{f}_{\text{MLE},R}\right)}{\varepsilon}$$
$$E\left[\left\{\hat{f}_{\text{MLE},R}\right\}^2\right] = \sum_{i=0}^{+\infty} \frac{\Gamma(m-i-2)}{\Gamma(i+1)\Gamma(m)} \left\{A'(x)\right\}^2 \left\{B(\theta)\right\}^2 (-2)^i \left\{n B(\theta)\right\}^i \left\{A(x)\right\}^i$$

and consequently,

$$\begin{aligned}&\text{Var}\left(\hat{f}_{\text{MLE},R}\right)\\ &= \sum_{i=0}^{+\infty} \frac{\Gamma(m-i-2)}{\Gamma(i+1)\Gamma(m)} \left\{A'(x)\right\}^2 \left\{B(\theta)\right\}^2 (-2)^i \left\{m B(\theta)\right\}^i \left\{A(x)\right\}^i - \left\{E\left[\hat{f}_{\text{MLE},R}\right]\right\}^2\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{m\to\infty} \text{Var}\left(\hat{f}_{\text{MLE},R}\right)\\ &= \sum_{i=0}^{+\infty} \lim_{m\to\infty}\left[\frac{m^i\Gamma(m-i-2)}{\Gamma(m)}\right] \frac{\left\{-2B(\theta)\right\}^i \left\{A(x)\right\}^i}{\Gamma(i+1)} \left\{A'(x)\right\}^2 \left\{B(\theta)\right\}^2 - \left\{\hat{f}_\theta(x)\right\}^2\\ &= \left\{A'(x)\right\}^2 \left\{B(\theta)\right\}^2 \exp\left\{-2A(x)B(\theta)\right\} - \left\{\hat{f}_\theta(x)\right\}\\ &= 0\end{aligned}$$

Therefore,

$$\lim_{m\to\infty} P\left\{\left|\hat{f}_{\text{MLE},R} - E\left[\hat{f}_{\text{MLE},R}\right]\right| > \varepsilon\right\} \le 0,$$

so

$$\hat{f}_{\text{MLE},R} \xrightarrow{P} \text{E}\left[\hat{f}_{\text{MLE},R0}\right] \quad \text{or} \quad \hat{f}_{\text{MLE},R} - \text{E}\left[\hat{f}_{\text{MLE},R}\right] \xrightarrow{P} 0 .$$

However, if $\lim_{m\to\infty} a_m = a$ and $X \xrightarrow{P} r$, then (Billingsley, 1995)

$$X + a_m \xrightarrow{P} r + a \tag{18}$$

Considering (18) and respecting the second part of Theorem 5 ($\lim_{m\to\infty} \text{E}[\hat{f}_{\text{MLE},R}] = f_\theta(x)$),

$$\hat{f}_{\text{MLE},R} - \text{E}\left[\hat{f}_{\text{MLE},R0}\right] + \text{E}\left[\hat{f}_{\text{MLE},R}\right] - f \xrightarrow{P} 0$$

or equally,

$$\hat{f}_{\text{MLE},R} \xrightarrow{P} f .$$

The following is essential before expressing the next theorem. If $\{X_n\} \xrightarrow{P} \{X\}$ and D is a continuous function of y $\{\text{D}(y): \mathbb{R} \to \mathbb{R}\}$, then (Billingsley, 1995)

$$\text{D}(\{X_n\}) \xrightarrow{P} \text{D}(\{X\})$$

Theorem 7. Suppose a random sample of size n from the exponential family of the first type is available. Also suppose this sample includes m values from the upper record values. The estimators $\hat{\theta}_{\text{MLE},(R_1,\ldots,R_m)}$ are consistent for θ:

$$\hat{\theta}_{\text{MLE},(R_1,\ldots,R_m)} \xrightarrow{P} \theta .$$

Proof. With respect to the continuity of functions $\text{B}(t)$, $-\log(1-t)$, $\text{F}(t) = 1 - \exp\{\text{A}(t)\text{B}(\theta)\}$ and the continuity theorem of composite functions, it is concluded that the function

$$\mathrm{D}(t) = \mathrm{B}\left(\frac{-\log\left(1 - \mathrm{F}(t)\right)}{\mathrm{A}(x)} \right)$$

is always continuous on $\mathbb{R}$. However, it is known from the second part of Theorem 6 that

$$\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)} \xrightarrow{P} \mathrm{F}_\theta(x)$$

or, equivalently,

$$\mathrm{A}(x)\mathrm{B}\left(\hat{\theta}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right)\exp\left\{-\mathrm{A}(x)\mathrm{B}\left(\hat{\theta}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right)\right\} \xrightarrow{P} \mathrm{F}_\theta(x).$$

With respect to $\mathrm{D}\left(\{X_n\}\right) \xrightarrow{P} \mathrm{D}\left(\{X\}\right)$,

$$\mathrm{D}\left(\hat{\mathrm{F}}_{\mathrm{MLE},(R_1,\ldots,R_m)}\right) \xrightarrow{P} \mathrm{D}\left(\mathrm{F}_\theta(x)\right)$$

which is equivalent to

$$\hat{\theta}_{\mathrm{MLE},(R_1,\ldots,R_m)} \xrightarrow{P} \theta$$

Conclusion

Considering two general subfamilies from the exponential family, it was determined the maximum likelihood estimation for these two families at the first step and general state; then, the maximum likelihood estimations based on upper were calculated for the unknown parameter of these two families. Via various theorems shown to be acceptable in general state, estimations of maximum likelihood and of maximum likelihood based on records were calculated for the cumulative distribution function and for the probability density function of these families. Their corresponding errors were described and studied. Proofs were adduced regarding asymptotical states. These estimations are unbiased estimators. Subsequently, several theorems associated with convergence in probability were expressed for the estimators.

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