# A Quantitative Understanding of Pattern Matching 

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#### Abstract

This paper shows that the recent approach to quantitative typing systems for programming languages can be extended to pattern matching features. Indeed, we define two resource-aware type systems, named $\mathscr{U}$ and $\mathscr{E}$, for a $\lambda$-calculus equipped with pairs for both patterns and terms. Our typing systems borrow some basic ideas from [19], which characterises (head) normalisation in a qualitative way, in the sense that typability and normalisation coincide. But, in contrast to [19], our systems also provide quantitative information about the dynamics of the calculus. Indeed, system $\mathscr{U}$ provides upper bounds for the length of (head) normalisation sequences plus the size of their corresponding normal forms, while system $\mathscr{E}$, which can be seen as a refinement of system $\mathscr{U}$, produces exact bounds for each of them. This is achieved by means of a non-idempotent intersection type system equipped with different technical tools. First of all, we use product types to type pairs instead of the disjoint unions in [19], which turn out to be an essential quantitative tool because they remove the confusion between "being a pair" and "being duplicable". Secondly, typing sequents in system $\mathscr{E}$ are decorated with tuples of integers, which provide quantitative information about normalisation sequences, notably time (cf. length) and space (cf. size). Moreover, the time resource information is remarkably refined, because it discriminates between different kinds of reduction steps performed during evaluation, so that beta, substitution and matching steps are counted separately. Another key tool of system $\mathscr{E}$ is that the type system distinguishes between consuming (contributing to time) and persistent (contributing to space) constructors.


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## 1 Introduction

Pattern matching mechanisms are used in several modern programming languages and proof assistants as they provide an efficient way to process and decompose data. However, the semantics of programming languages usually focus on $\lambda$-calculi -a much more basic formalismthus causing a conceptual gap between theory and practice, simply because some properties of the $\lambda$-calculus do not translate directly to languages with matching primitives. Several

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examples of this mismatch can be cited, e.g. solvability [19], standardisation for pattern calculi [37], and neededness [15]. It is then crucial to study the semantics of programming languages with pattern matching features by means of formal calculi equipped with built-in patterns, referred to as pattern calculi (e.g. [21, 35, 32, 37, 6]).

The notion of $\lambda$-abstraction in pattern-calculi is generalised to functions of the form $\lambda p . t$, where $p$ is a pattern specifying the expected structure of their arguments. For instance, in calculi equipped with pair constructors for both patterns and terms, the term $\lambda\langle x, y\rangle . x$ becomes a valid abstraction, to be only successfully evaluated against pairs, i.e. arguments of the form $\langle u, v\rangle$, and yielding the first projection of this pair, i.e. the first component $u$ of the pair. In this work we focus on such a pattern calculus. This can be seen as a simplified form of algebraic pattern matching, but still powerful enough to reason about the most interesting features of existing syntactical matching mechanisms.

Type information, and in particular the size of arbitrary type derivations in some special type disciplines, has been used as a powerful quantitative tool to reason about time (length of evaluation sequences) and space ( size of normal forms). More precisely, when $t$ evaluates to $t^{\prime}$, then the size of the type derivation of $t^{\prime}$ is smaller than that of $t$, thus the size of type derivations provides an upper bound for the length of normalisation sequences as well as for the size of their corresponding normal forms. This was first done for the (call-by-name) notions of head and leftmost evaluation implemented by two variants of the Krivine's abstract machine (KAM) [22, 23, 44], and it was later appropriately extended to other formalisms, e.g. $[8,20,25,38]$.

Now we discuss some interesting features of the underlying type system that we use in this paper. While (idempotent) intersection types [11] allow terms to be typed with distinct types by means of an intersection operator $\cap$, which verifies not only associativity and commutativity but also idempotency given by $\sigma \cap \sigma=\sigma$, non-idempotent intersection types distinguishes between $\sigma \cap \sigma$ and $\sigma$, thus also discriminating quantitative information in type derivations. For this reason, idempotent (resp. non-idempotent) types are often represented by sets (resp. multisets). For example, the term $\lambda x . \lambda y$.xyy can be typed with $\{\{\sigma\} \rightarrow\{\sigma\} \rightarrow \tau\} \rightarrow\{\sigma\} \rightarrow \tau$ in the first model, while the non-idempotent version becomes $[[\sigma] \rightarrow[\sigma] \rightarrow \tau] \rightarrow[\sigma, \sigma] \rightarrow \tau$. As a consequence, a type derivation for ( $\lambda x$. $\lambda y$.xyy)uv: $\tau$ in the idempotent system only depends on one type derivation for $u:\{\sigma\} \rightarrow\{\sigma\} \rightarrow \tau$ and another one for $v: \sigma$, while for its reduct uvv: $\tau$, two derivations for $v: \sigma$ are needed. In contrast, a type derivation in the non-idempotent system already requires two derivations for $v: \sigma$ to correctly infer $(\lambda x . \lambda y . x y y) u v: \tau$. Therefore, while type derivations may increase after reductions in the former, they decrease in the latter.

Non-idempotent intersection (also called nowadays quantitative) type systems, have been independently introduced in the framework of the $\lambda$-calculus by Gardner [27] and Kfoury [43]. Although widely unnoticed, the quantitative power of such systems turned out to be crucial in several resource aware consumption investigations. It was only after [16] that this quantitative feature was highlighted, and since De Carvalho's thesis in 2007 (see also [22]) its relation with linear logic [28] and quantitative relational models has been deeply explored. As its idempotent counterpart, non-idempotent intersection type systems may characterise different notions of normalisation (such as head, weak and strong) [23, 13, 20] but, instead of using some semantical argument (e.g. reducibility) to prove such a characterisation, simple combinatorial arguments are enough to guarantee normalisation of typable terms.

If instead of upper bounds one wants to obtain exact bounds, then the crucial point is to measure only minimal typing derivations, which give the notion of all and only information for typings (cf. [56] for an abstract definition). Syntactic notions of minimal typings were
supplied for the head evaluation strategy implemented by the KAM [22], then for the maximal evaluation strategy [13] for the $\lambda$-calculus. The technique was further developed in [2] with the introduction of an appropriate notion of tightness capturing minimal typings, thus systematically broadening the definition of exact bounds for different evaluation strategy of the $\lambda$-calculus. In all these works, it is possible to extract from a (minimal) type derivation, both the length of the reduction sequence to normal form as well as the size of this normal form. The tightness technique was also applied for call-by-value [3], call-by-need [4], linear head evaluation [3], as well as for several evaluation strategies in classical logic [42]. Our paper extends these results to a $\lambda$-calculus with pair pattern matching by providing two sound and complete typing systems, named $\mathscr{U}$ and $\mathscr{E}$, that respectively provide upper bounds an exact measures for the length of (head) normalisation sequences, as well as for the size of the corresponding reached normal forms.

## Contributions

The first contribution of this paper is to go beyond the qualitative characterisation of head normalisation for pair pattern calculi [19] by providing a typing system $\mathscr{U}$ being able to compute upper bounds for head evaluation. To achieve this, we have introduced different key tools on the untyped side - the reduction calculus- as well as on the typed side -the type system itself.

On the untyped side, one of the main reasons why the type system in [19] fails to provide upper bounds or exact measures for head normalisation is because commuting conversions are considered as independent rules of the reduction relation associated to the underlying pattern calculus. A typical example is the commuting rule $t[p \backslash v] u \mapsto_{\sigma}(t u)[p \backslash v]$ pushing out an explicit matching from an application when there is no capture of free variables. Indeed, the size of type derivations is not strictly decreasing w.r.t. commuting conversions. We solve this problem by integrating these (structural) commuting conversions into the non-structural operational reduction rules, so that the resulting system, based on explicit matchings, implements reduction at a distance [5]. Thus for example, the operational Beta-rule $(\lambda p . t) u \mapsto t[p \backslash u]$ in $[19]$ becomes here $\mathrm{L} \llbracket \lambda p . t \rrbracket u \mapsto \mathrm{~L} \llbracket t[p \backslash u\rfloor \rrbracket$, which combines the commuting conversion $\mapsto_{\sigma}$ with the (non-structural) Beta-reduction rule into a single rule. Even more interesting cases are presented in Sec. 2. Moreover, our presentation provides a suitable deterministic head evaluation strategy which is complete w.r.t. the (non-deterministic) notion of head-normalisation defined in [19], in the sense that both notions turn out to be equivalent, thus answering one of the open questions in [19].

On the typed side, we adopt standard product types specified by means of a pair type. This stands in contrast to the disjoint unions used in [19], which have an important undesirable consequence, because multisets of types in this model carry two completely different meanings: being a pair (but not necessarily a duplicable pair), or being a duplicable term (but not necessarily a pair). Our product types restore a crucial idea in non-idempotent type theory: multisets of types are only assigned to terms that are going to be duplicated during evaluation.

The new specification of the deterministic reduction system at a distance is now wellbehaved w.r.t. our first type system $\mathscr{U}$ : if $t$ is well typed in $\mathscr{U}$, then the size of its type derivation gives an upper bound to the (deterministic) head-reduction sequence from $t$ to its (head) normal form. Our system $\mathscr{U}$ can then be seen as a form of quantitative (relational) model for the pair pattern calculus (Sec. 4), following the lines of [17, 18, 48].

The second contribution of this paper is to go beyond upper bounds by providing a typing system $\mathscr{E}$ being able to provide exact bounds for head evaluation. This is done by using several key tools.

An important notion used in system $\mathscr{E}$ is the clear distinction between consuming and persistent constructors. This has some intuition coming from the theory of residuals [10], where any symbol occurring in a normal form can be traced back to the original term. A constructor is consuming (resp. persistent) if it is consumed (resp. not consumed) during head-reduction. For instance, in $(\lambda z . z)(\lambda z . z)$ the first abstraction is consuming while the second one is persistent. This dichotomy between consuming/persistent constructors has already been highlighted in [41, 42] for the $\lambda \mu$-calculus, and it is adapted here for the pattern calculus. Indeed, patterns and terms are consumed when the pair constructor is destroyed during the execution of the pattern matching rule. Otherwise, patterns and pairs are persistent, and they do appear in the normal form of the original term. For example, in the term $(\lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z)\langle u, v\rangle$, the pair $\langle u, v\rangle$ is going to be duplicated, but only one of its copies is going to be consumed by the matching operations. The other copy will be persistent and contribute to the normal form of the term.

Another major ingredient of our approach is the use of tight types, inspired by [2], and the corresponding notion of tight (cf. minimal) derivations. This is combined with the introduction of counters in the typing judgements, which are now of the form $\Gamma \vdash^{(b, e, m, f)} t: \sigma$. These counters are used to discriminate between the different sorts of reduction steps performed during evaluation, so that firing beta (b), computing substitution $(e)$ or matching ( $m$ ) steps are exactly and independently counted for each tight type derivation. More precisely, soundness of our system $\mathscr{E}$ guarantees that if a judgement $\Gamma \vdash^{(b, e, m, f)} t: \sigma$ is tightly derivable, then $b$ (resp $e$ and $m$ ) corresponds to the number of beta firing (resp. substitution and matching) rules used to head evaluate the term $t$, while $f$ is exactly the size of the corresponding normal form. Moreover, completeness, given by the reverse implication of the previous statement, also holds.

The following list summarises our contributions:

- A deterministic head-strategy for the pattern calculus which is complete w.r.t. the notion of head-normalisation.
- A sound and complete type system $\mathscr{U}$, which provides upper bounds for the length of head-normalisation sequences plus the size of its corresponding normal forms.
- Refinement of system $\mathscr{U}$ to a sound and complete system $\mathscr{E}$, being able to provide independent exact bounds for both the length of head-normalisation sequences and the size of its corresponding normal forms.


## Other Related Works

Non-idempotent intersection types have been applied to the $\lambda$-calculus for the characterisation of termination with respect to a variety of evaluation strategies, such as call-by-value $[25,3]$, call-by-need $[36,8,4]$ and (linear) head reduction [27, 2]. They have been well-adapted also to some explicit resource calculi $[13,38,39]$, as well as to pattern calculi $[12,19,9]$, proof-nets [24], classical logic [40, 42] and call-by-push-value [26, 29].

Closer to our work, non-idempotent intersection types have been used to characterise strong normalisation in a calculus with fix-point operators and pattern matching on constructors [12]. Similarly, a strong call-by-need strategy for a pattern matching language was defined in [9], and completeness of the strategy was shown by means of non-idempotent intersection types by extending the technique introduced in [36, 8]. In both cases, despite the use of non-idempotent types, the result was qualitative, as no quantitative results were obtained by means of the typing system.

Even closer to our work, [19] studied the solvability property in a pair pattern calculus, the main result being that solvability is equivalent to typability plus inhabitation in a nonidempotent intersection type system. One of the contributions of [19] is a characterisation of
(non-deterministic) head-normalisation by means of typability, which is merely qualitative, as it does not give any upper bound/exact measure for head-evaluation, as discussed in the previous subsection.

More practical type-based approaches (i.e. mostly sound but not necessarily complete) to quantitative analysis are sized-types [49, 54, 7, 45, 46] and (automatic) amortised resource analysis $[33,53,30,34,31,51]$.

Sized Types. This line of work is based on the use of types indexed by sizes, which are essentially ordinals. The approach is based on the fact that the compiler checks if the program is typed with the correct size, so that resource usages of programs can be derived from the sized types informations.

While space cost is determined in [54], upper bounds for both space and time costs are obtained in [49, 7]. In particular, sized types are used in [7] to obtain space bounds, while time bounds are computed by using a kind of clock, achieved by means of ticking monadic transformations, originally introduced in [55] as ticking monads to get time complexity for lazy languages. This is done for a call-by-value functional language with (a fixed set of) inductive datatypes, enriched with index polymorphism: functions can be polymorphic in their size annotations. In this respect, sized types enriched with intersection types have been used in [52] to handle time costs for a call-by-value strategy in a more restricted language. Sized types are also extended in [14] to guarantee termination of general higher-order rewriting. Yet another approach based on (first-order) size indices is given by linear dependent types [45, 46], where time and space bounds are obtained by establishing a relation with linear logic, a key tool used to define quantitative types through non-idempotent intersection types. Completeness depends on an oracle for the first order theory on indices describing the semantical properties of the function symbols, so the approach cannot be fully turned into an automatic tool.

Amortised resource analysis. This line of work is motivated by the fact that the worst-case run time analysis per operation, rather than per algorithm, can lead to very pessimistic complexity bounds. This is then replaced by an approach considering both the costly and less costly operations together over the whole set of operations of the algorithm. Automatisation of amortised resource analysis has been achieved in a series of works [33, 30, 53, 34, 31, 51], including space usage [53] and more general usages [31], all regarding a lazy functional language. The pioneer work in [30] has evolved to more sophisticated tools, leading today to RAML [50], a language applied to an industrial strength compiler [31]. Lazy evaluation is not handled in RAML, however, [51] proposes a practical tool to estimate resource usage for Haskell expressions.
Road-map: Sec. 2 introduces the pattern calculus. Sec. 3 presents the typing system $\mathscr{U}$, together with some of its quantitative properties, and Sec. 4 suggests a relational model for our pattern calculus based on the type system. In Sec. 5, we refine $\mathscr{U}$ to extract exact bounds, which leads to the definition of our second typing system $\mathscr{E}$. The soundness (resp. completeness) proof for $\mathscr{E}$ is given in Sec. 6 (resp. Sec. 7). Conclusions and future work are discussed in Sec. 8. All proofs are presented in the Appendix.

## 2 The Pattern Calculus

In this section we introduce the pattern calculus, an extension of the $\lambda$-calculus where abstraction is extended to pair patterns and terms are extended to pairs. We start by introducing the syntax of the calculus.

Terms and contexts of the pattern calculus are defined by means of the following grammars:

| (Patterns) | $p, q$ | $::=x \mid\langle p, q\rangle$ |
| :--- | :--- | :--- |
| (Terms) | $t, u, v$ | $::=x\|\lambda p . t\|\langle t, u\rangle\|t u\| t[p \backslash u]$ |
| (List Contexts) | L | $::=\square \mid \mathrm{L}[p \backslash u]$ |
| (Contexts) | C | $::=\square\|\lambda p . \mathrm{C}\|\langle\mathrm{C}, t\rangle\|\langle t, \mathrm{C}\rangle\| \mathrm{C} t\|t \mathrm{C}\| \mathrm{C}[p \backslash t] \mid t[p \backslash \mathrm{C}]$ |

where $x, y, z, w \ldots$ range over a countable set of variables, and every pattern $p$ is assumed to be linear, i.e. every variable appears at most once in $p$. The term $x$ is called a variable, $\lambda p . t$ is an abstraction, $\langle t, u\rangle$ is a pair, $t u$ is an application and $t[p \backslash u]$ is a closure, where $[p \backslash u]$ is an explicit matching operator. Special terms are $\mathrm{I}:=\lambda z . z, \Delta:=\lambda z . z z$ and $\Omega:=\Delta \Delta$. As usual we use the abbreviation $\lambda p_{1} \ldots p_{n} . t_{1} \ldots t_{m}$ for $\lambda p_{1}\left(\ldots\left(\lambda p_{n} \cdot\left(\left(t_{1} t_{2}\right) \ldots t_{m}\right)\right) \ldots\right), n \geq 0$, $m \geq 1$.

We write $\operatorname{var}(p)$ to denote the variables in the pattern $p$. Free and bound variables of terms and contexts are defined as expected, in particular $\mathrm{fv}(\lambda p . t):=\mathrm{fv}(t) \backslash \operatorname{var}(p)$, $\mathrm{fv}(t[p \backslash u]):=(\mathrm{fv}(t) \backslash \operatorname{var}(p)) \cup \mathrm{fv}(u)$ and $\operatorname{bv}(\lambda p . t):=\mathrm{bv}(t) \cup \operatorname{var}(p), \operatorname{bv}(t[p \backslash u]):=\mathrm{bv}(t) \cup$ $\operatorname{var}(p) \cup \mathrm{bv}(u)$. We also define the domain of a list context as $\operatorname{dlc}(\square)=\emptyset$ and $\operatorname{dlc}(\mathrm{L}[p \backslash u])=\operatorname{dlc}(\mathrm{L}) \cup \operatorname{var}(p)$. We write $p \# q$ if $\operatorname{var}(p)$ and $\operatorname{var}(q)$ are disjoint. As usual, terms are considered modulo $\alpha$-conversion, so that for example $\lambda\langle x, y\rangle \cdot x z={ }_{\alpha} \lambda\left\langle x^{\prime}, y^{\prime}\right\rangle \cdot x^{\prime} z$ and $x[\langle x, y\rangle \backslash z]={ }_{\alpha} x^{\prime}\left[\left\langle x^{\prime}, y^{\prime}\right\rangle \backslash z\right]$. Given a list context L and a term $t, \mathrm{~L} \llbracket t \rrbracket$ denotes the term obtained by replacing the unique occurrence of $\square$ in L by $t$, possibly allowing the capture of free variables of $t$. We use $t\{x \backslash u\}$ to denote the meta-level substitution operation which replaces all the free occurrences of $x$ in $t$ by the term $u$. As usual, this operation is performed modulo $\alpha$-conversion so that capture of free variables is avoided. We use the predicate abs $(t)$ when $t$ is of the form $L \llbracket \lambda p . u \rrbracket$. The reduction relation $\rightarrow_{\mathrm{p}}$ on terms is given by the closure over all contexts of the following rewriting rules.

$$
\begin{aligned}
& \mathrm{L} \llbracket \lambda p . t \rrbracket u \quad \mapsto \quad \mathrm{~L} \llbracket t[p \backslash u \rrbracket \rrbracket \quad \operatorname{dlc}(\mathrm{~L}) \cap \mathrm{fv}(u)=\emptyset \\
& t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \quad \mapsto \quad \mathrm{L} \llbracket t\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right\rfloor \rrbracket \quad \operatorname{dlc}(\mathrm{L}) \cap \mathrm{fv}(t)=\emptyset \\
& t[x \backslash u] \quad \mapsto \quad t\{x \backslash u\}
\end{aligned}
$$

The reduction relation $\rightarrow_{p}$ defined above is related to that in [19], called $\rightarrow_{\Lambda_{p}}$, in the following sense: $\rightarrow_{\Lambda_{\mathrm{p}}}$ contains two subsystem relations, one to deal with clashes, which are not handled by the reduction system in the present calculus since we consider typable terms only (cf. Lem. 6), and another one containing the following five rules:

$$
\begin{aligned}
& (\lambda p . t) u \quad \mapsto t[p \backslash u] \\
& t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash\left\langle u_{1}, u_{2}\right\rangle\right] \quad \mapsto \quad t\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \\
& t[x \backslash u] \quad \mapsto \quad t\{x \backslash u\} \\
& t[p \backslash v] u \quad \mapsto(t u)[p \backslash v] \quad \mathrm{fv}(u) \cap \operatorname{var}(p)=\emptyset \\
& t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash u[q \backslash v]\right] \quad \mapsto \quad t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash u\right][q \backslash v] \quad \mathrm{fv}(t) \cap \operatorname{var}(q)=\emptyset
\end{aligned}
$$

The two last rules can be seen as commuting conversions, which are integrated in the first (two) rules of our reduction system $\rightarrow_{\mathrm{p}}$ by using the substitution at a distance paradigm [5]. It is worth noticing that $t \rightarrow_{\mathrm{p}} t^{\prime}$ can be simulated by $t \rightarrow_{\Lambda_{\mathrm{p}}}^{+} t^{\prime}$. For instance, $(\lambda p . t)\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] u \rightarrow_{\mathrm{p}} t[p \backslash u]\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]$ can be simulated by:

$$
\begin{aligned}
(\lambda p . t)\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] u \rightarrow_{\Lambda_{\mathrm{p}}}\left((\lambda p . t)\left[p_{1} \backslash u_{1}\right] u\right)\left[p_{2} \backslash u_{2}\right] & \rightarrow_{\Lambda_{\mathrm{p}}}((\lambda p . t) u)\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \\
& \rightarrow_{\Lambda_{\mathrm{p}}} t[p \backslash u]\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]
\end{aligned}
$$

Our formulation of the pattern calculus at a distance, given by the relation $\rightarrow_{p}$, as well as the corresponding head strategy that we present below, are one of the essential untyped tools used in this paper to get quantitative results about head-normalisation (cf. Sec. 3 and Sec. 5).

Although the reduction relation $\rightarrow_{p}$ is non-deterministic, it can easily been shown to be confluent, for example using the same technique in [19]. However, in order to study exact bounds of evaluation, we need to define a deterministic strategy for the pattern calculus, i.e. a subrelation of $\rightarrow_{p}$ that is able to compute the same normal forms. Fig. 1 gives an operational semantics for the pattern calculus, which turns out to be an extension of the well-known notion of head-reduction for $\lambda$-calculus, then also named head-reduction, and denoted by $\rightarrow_{\mathrm{h}}$. In the following inductive definition $t \rightarrow_{\mathrm{h}} u$ means that $t$ head-reduces to $u$, and $t \nrightarrow_{\mathrm{h}}$ means that $t$ is a head normal-form, i.e. there is no $u$ such that $t \rightarrow_{\mathrm{h}} u$.

$$
\begin{aligned}
& \frac{\operatorname{dlc}(\mathrm{L}) \cap \mathrm{fv}(u)=\emptyset}{\mathrm{L} \llbracket \lambda p . t \rrbracket u \rightarrow_{\mathrm{h}} \mathrm{~L} \llbracket t[p \backslash u] \rrbracket}(\mathrm{b}) \quad \frac{t \nrightarrow_{\mathrm{h}}}{t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{h}} \mathrm{~L} \llbracket t\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket}(\mathrm{m}) \quad \frac{t \nrightarrow_{\mathrm{h}}}{t[x \backslash u] \rightarrow_{\mathrm{h}} t\{x \backslash u\}} \text { (e) } \\
& \frac{t \rightarrow_{\mathrm{h}} t^{\prime}}{\lambda p . t \rightarrow_{\mathrm{h}} \lambda p . t^{\prime}} \quad \frac{t \rightarrow_{\mathrm{h}} t^{\prime} \quad \neg \mathrm{abs}(t)}{t u \rightarrow_{\mathrm{h}} t^{\prime} u} \quad \frac{t \rightarrow_{\mathrm{h}} t^{\prime}}{t[p \backslash u] \rightarrow_{\mathrm{h}} t^{\prime}[p \backslash u]} \quad \frac{t \nrightarrow_{\mathrm{h}} \quad p \neq x \quad u \rightarrow_{\mathrm{h}} u^{\prime}}{t[p \backslash u] \rightarrow_{\mathrm{h}} t\left[p \backslash u^{\prime}\right]}
\end{aligned}
$$

Figure 1 The head-reduction strategy for the pattern calculus.

Rule $b$ fires the computation of terms by transforming an application of a function to an argument into a closure term. Decomposition of patterns and terms is performed by means of rule $m$, when a pair pattern is matched against a pair term. Substitution is performed by rule e, i.e. an explicit (simple) matching of the form $[x \backslash u]$ is executed. This form of syntactic pattern matching is very simple, and does not consider any kind of failure result, but is already expressive enough to specify the well-known mechanism of successful matching. Context closure is similar to the call-by-name $\lambda$-calculus case, but not exactly the same. Indeed, head-reduction is performed on the left-hand side of applications and closures whenever possible. Otherwise, arguments of explicit matching operators must be head-reduced in order to unblock these operators, i.e. in order to decompose $[p \backslash u]$ when $p$ is a pair pattern but $u$ is still not a pair. Notice however that when $u$ is already a pair, no head-reduction inside $u$ can take place, thus implementing a lazy strategy for pattern matching. Standardisation of calculi as the one in this paper has been studied in [37].

Given any (one-step) reduction relation $\rightarrow_{\mathcal{R}}$, we use $\rightarrow_{\mathcal{R}}^{*}$, or more precisely $\rightarrow_{\mathcal{R}}^{k}(k \geq 0)$ to denote the reflexive-transitive closure of $\rightarrow_{\mathcal{R}}$, i.e. the composition of $k \mathcal{R}$-steps. In the case of head-reduction, we may use the alternative notation $\rightarrow_{h}^{(b, e, m)}$ to emphasize the number of reduction steps in a given reduction sequence, i.e. if $\rho: t \rightarrow_{\mathrm{h}}^{(b, e, m)} u$, then there are exactly $b$ b-steps, $e$ e-steps and $m$ m-steps in the reduction sequence $\rho$. We will often use the notation $\rightarrow_{\mathrm{b}}$ to explicitly refer to a b-step (resp. $\rightarrow_{\mathrm{e}}$ and $\rightarrow_{\mathrm{m}}$ for e and m steps). The reduction relation $\rightarrow_{\mathrm{h}}$ is in fact a function:

- Proposition 1. The relation $\rightarrow_{h}$ is deterministic.
- Example 2. Let us consider the combinators $\mathrm{I}:=\lambda z . z$ and $\mathrm{K}:=\lambda x_{1} \cdot \lambda y_{1} \cdot x_{1}$. Then we have $(\lambda\langle x, y\rangle . x(\mathrm{I} y))[z \backslash \mathrm{I}](\mathrm{I}\langle\mathrm{K}, w\rangle) \rightarrow_{\mathrm{h}}^{(4,6,1)} \lambda y_{1} \cdot w$ :

$$
\begin{array}{llll} 
& (\lambda\langle x, y\rangle \cdot x(\mathrm{I} y))[z \backslash \mathrm{I}](\mathrm{I}\langle\mathrm{~K}, w\rangle) & \rightarrow_{\mathrm{b}} & (x(\mathrm{I} y))[\langle x, y\rangle \backslash \mathrm{I}\langle\mathrm{~K}, w\rangle][z \backslash \mathrm{I}] \\
\rightarrow_{\mathrm{b}} & (x(\mathrm{I} y))[\langle x, y\rangle \backslash z[z \backslash\langle\mathrm{~K}, w\rangle]][z \backslash \mathrm{I}] & \rightarrow_{\mathrm{e}} & (x(\mathrm{I} y))[\langle x, y\rangle \backslash\langle\mathrm{K}, w\rangle][z \backslash \mathrm{I}] \\
\rightarrow_{\mathrm{m}} & (x(\mathrm{I} y))[x \backslash \mathrm{~K}][y \backslash w][z \backslash \mathrm{I}] & \rightarrow_{\mathrm{e}} & (\mathrm{~K}(\mathrm{I} y))[y \backslash w][z \backslash \mathrm{I}] \\
\rightarrow_{\mathrm{b}} & \left(\lambda y_{1} \cdot x_{1}\right)\left[x_{1} \backslash \mathrm{I} y\right][y \backslash w][z \backslash \mathrm{I}] & \rightarrow_{\mathrm{e}} & \left(\lambda y_{1} \cdot \mathrm{I} y\right)[y \backslash w][z \backslash \mathrm{I}] \\
\rightarrow_{\mathrm{b}} & \left(\lambda y_{1} \cdot z[z \backslash y]\right)[y \backslash w][z \backslash \mathrm{I}] & \rightarrow_{\mathrm{e}} & \left(\lambda y_{1} \cdot y\right)[y \backslash w][z \backslash \mathrm{I}] \\
\rightarrow_{\mathrm{e}} & \left(\lambda y_{1} \cdot w\right)[z \backslash \mathrm{I}] & \rightarrow_{\mathrm{e}} & \lambda y_{1} \cdot w
\end{array}
$$

Head normal-forms may contain ill-formed terms called (head) clashes not representing a desired result for a computation, i.e. (head) terms not syntactically well-formed. For example, a pair applied to another term $\left\langle u_{1}, u_{2}\right\rangle v$, or a matching between a pair pattern and a function $t\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \lambda p . u\right]$ are considered to be (head) normal clashes. Formally, a term is said to be a (head) clash if it is generated by the following grammar:

```
(Head Clash) U :}:=c=c|\lambdap.U|\textrm{U}t|\textrm{U}[p\t]|t[\langle\mp@subsup{p}{1}{},\mp@subsup{p}{2}{}\rangle\\textrm{U}
(Clash) c ::= L\llbracket{u\mp@subsup{u}{1}{},\mp@subsup{u}{2}{}\rangle\rrbracketv|t[\langle\mp@subsup{p}{1}{},\mp@subsup{p}{2}{}\rangle\L\llbracket\lambdap.u\rrbracket]
```

Then, a term $t$ is said to be (head) clash-free if $t$ does not head-reduce to a (head) clash, i.e. if there is no $u \in \mathrm{U}$ such that $t \rightarrow_{\mathrm{h}}^{*} u$. Remark in particular that every pair is (head) clash-free. A rewriting system raising a warning (i.e. a failure) when detecting a (head) clash has been defined in [19], allowing to restrict the attention to a smaller set of terms, called canonical terms, that are intended to be the (head) clash-free terms that are not reducible by the relation $\rightarrow_{\mathrm{h}}$. Canonical terms can be characterised inductively as follows:

```
(canonical forms) }\mathcal{M}::=\lambdap.\mathcal{M}|\langlet,t\rangle|\mathcal{M}[\langle\mp@subsup{p}{1}{},\mp@subsup{p}{2}{}\rangle\\mathcal{N}]|\mathcal{N
(pure canonical forms) \mathcal{N}\quad::= x |\mathcal{N}t|\mathcal{N}[\langle\mp@subsup{p}{1}{},\mp@subsup{p}{2}{}\rangle\\mathcal{N}]
```

In summary, canonical terms and irreducible terms are related as follows:

- Proposition 3. $t \in \mathcal{M}$ if and only if $t$ is (head) clash-free and $t \nrightarrow_{h}$.

Size of canonical terms is given by: $|x|:=0,|\langle t, u\rangle|:=1,|\mathcal{N} t|:=|\mathcal{N}|+1,|\lambda p . \mathcal{M}|:=$ $|\mathcal{M}|+1$, and $\left|\mathcal{M}\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathcal{N}\right]\right|:=|\mathcal{M}|+|\mathcal{N}|+1$. As an example, the terms $\lambda\langle x, y\rangle .\langle x, \mathrm{I}\rangle$ and $\lambda x . y(\langle x, z\rangle \mathrm{I})$ are canonical forms of size 2 while $x \Omega$ and $z[\langle z, w\rangle \backslash x \Omega]$ are pure canonical terms of size 1 and 2 respectively. The term $\langle x, \mathrm{I}\rangle w$ is none of them, and the term $\mathrm{I} x$ can head-reduce to the canonical term $x$.

Finally, we define a term $t$ to be head-normalisable if there exists a canonical form $u \in \mathcal{M}$ such that $t \rightarrow_{\mathrm{p}}^{*} u$. Moreover, $t$ is said to be head-terminating if there exists a canonical form $u \in \mathcal{M}$ and an integer $k \geq 0$ such that $t \rightarrow_{\mathrm{h}}^{k} u$. The relation between the non-deterministic reduction relation $\rightarrow_{p}$ and the deterministic strategy $\rightarrow_{h}$ will be established later, but we can already say that, while $t$ head-terminating immediately implies $t$ head-normalisable, the completeness of the head-strategy w.r.t. head-normalisation is not trivial (Thm. 7).

## 3 The $\mathscr{U}$ Typing System

In this section we introduce our first typing system $\mathscr{U}$ for the pattern calculus. We start by defining the sets of types and multiset types, given by means of the following grammars:

```
(Product Types) P : := < (\mathcal{A},\mp@subsup{\mathcal{A}}{2}{})
(Types) }\sigma\quad::= \bullet|\mathcal{P}|\mathcal{A}->
(Multiset Types) \mathcal{A }::=[\mp@subsup{\sigma}{k}{}\mp@subsup{]}{k\inK}{}
```

where $\bullet$ is an atomic type, $K$ is a (possibly empty) finite set of indexes, and a multiset type is an unordered list of (not necessarily different) elements, where [] denotes the empty multiset. We write $|\mathcal{A}|$ to denote the number of elements of the multiset $\mathcal{A}$. For example $[\bullet,[] \rightarrow \bullet, \bullet]$ is a multiset type of 3 elements, representing the intersection type $(\bullet \cap([] \rightarrow \bullet)) \cap \bullet$, where $\cap$ is an associative, commutative and non-idempotent intersection type constructor. We write $\sqcup$ to denote multiset union. Multiset types are used to specify how programs consume terms: intuitively, the empty multiset is assigned to terms that are erased during (head) reduction, while duplicable terms are necessarily typed with non-empty multisets. As usual the arrow type is right-associative.

A product type, representing the type of a pair, is defined as the product of two (possibly empty) multisets of types. This formulation of product types turns out to be a key tool in our quantitative framework, and constitutes an essential difference with the product types proposed in [19], which are modeled by disjoint unions, so that any pair $\langle t, u\rangle$ of typed terms $t$ and $u$ has necessarily at least two types, one of the form $\times_{1}(\sigma)$ where $\sigma$ is the type of $t$, and one of the form $\times_{2}(\tau)$, where $\tau$ is the type of $u$. Indeed, in op. cit., multiset types carry two completely different meanings: being a pair (but not necessarily a pair to be duplicated), or being a duplicable term (but not necessarily a pair). Our specification of products can then be interpreted as the use of the exponential isomorphism $!(A \diamond B) \equiv!A \otimes!B$ of multiplicative exponential linear logic [28].

A typing context $\Gamma$ is a map from variables to multiset types, such that only finitely many variables are not mapped to the empty multiset []. We write dom $(\Gamma)$ to denote the domain of $\Gamma$, which is the set $\{x \mid \Gamma(x) \neq[]\}$. We may write $\Gamma \# \Delta$ if and only if $\operatorname{dom}(\Gamma)$ and $\operatorname{dom}(\Delta)$ are disjoint. Given typing contexts $\left\{\Gamma_{i}\right\}_{i \in I}$ we write $\wedge_{i \in I} \Gamma_{i}$ for the context that maps $x$ to $\sqcup_{i \in I} \Gamma_{i}(x)$. One particular case is $\Gamma \wedge \Delta$. We sometimes write $\Gamma ; \Delta$ instead of $\Gamma \wedge \Delta$, when $\Gamma \# \Delta$, and we do not distinguish $\Gamma ; x:[]$ from $\Gamma$. The typing context $\left.\Gamma\right|_{p}$ is such that $\left.\Gamma\right|_{p}(x)=\Gamma(x)$, if $x \in \operatorname{var}(p)$ and [] otherwise. The typing context $\Gamma \backslash \mathcal{V}$ is defined by $(\Gamma \backslash \mathcal{V})(x)=\Gamma(x)$ if $x \notin \mathcal{V}$ and [] otherwise. Finally, $\Gamma \subseteq \Delta$ means that $\operatorname{dom}(\Gamma) \subseteq \operatorname{dom}(\Delta)$ and $\Gamma(x) \sqsubseteq \Delta(x)$ for every $x \in \operatorname{dom}(\Gamma)$, where $\sqsubseteq$ denotes multiset inclusion.

The type assignment system $\mathscr{U}$ is given in Fig. 2 and can be seen as a natural extension of Gardner's system [27] to explicit matching operators, pairs and product types. It assigns types (resp. multiset types) to terms, using an auxiliary (sub)system that assigns multiset types to patterns. We use $\Phi \triangleright \Gamma \vdash t: \sigma$ (resp. $\Phi \triangleright \Gamma \vdash t: \mathcal{A}$ ) to denote term type derivations ending with the sequent $\Gamma \vdash t: \sigma$ (resp. $\Gamma \vdash t: \mathcal{A}$ ), and $\Pi \triangleright \Gamma \Vdash p: \mathcal{A}$ to denote pattern type derivations ending with the sequent $\Gamma \Vdash p: \mathcal{A}$. The size of a derivation $\Phi$, denoted by sz $(\Phi)$, is the number of all the typing rules used in $\Phi$ except many ${ }^{1}$ (this is particularly appropriate in the proof of the substitution lemma).

Note that when assigning types (multiset types) to terms, we only allow the introduction of multiset types on the right through the many rule.

Most of the rules for terms are straightforward. Rule match is used to type the explicit matching operator $t[p \backslash u]$ and can be seen as a combination of rules app and abs. Rule pat ${ }_{v}$ is used when the pattern is a variable $x$. Its multiset type is the type declared for $x$ in the typing context. Rule pat ${ }_{\times}$is used when the pattern has a product type, which means that the pattern will be matched with a pair. The condition $p \# q$ ensures linearity of patterns. Note that any pair term can be typed, in particular, with $\times([],[])$.

The system enjoys the key property of relevance:

[^0]\[

$$
\begin{array}{cc}
\frac{\Gamma \Vdash p: \mathcal{A} \quad \Delta \Vdash q: \mathcal{B} \quad p \# q}{x: \mathcal{A} \Vdash x: \mathcal{A}}\left(\text { pat }_{\mathrm{v}}\right) & \left.\frac{\Gamma \Vdash p \mathrm{pat}}{\times}\right) \\
\frac{\Gamma \wedge \Delta \Vdash\langle p, q\rangle:[\times(\mathcal{A}, \mathcal{B})]}{x:[\sigma] \vdash x: \sigma}(\mathrm{ax}) & \frac{\left(\Gamma_{k} \vdash t: \sigma_{k}\right)_{k \in K}}{\wedge_{k \in K} \Gamma_{k} \vdash t:\left[\sigma_{k}\right]_{k \in K}}(\text { many }) \\
\frac{\Gamma \vdash t:\left.\sigma \quad \Gamma\right|_{p} \Vdash p: \mathcal{A}}{\Gamma \backslash \operatorname{var}(p) \vdash \lambda p \cdot t: \mathcal{A} \rightarrow \sigma}(\mathrm{abs}) & \frac{\Gamma \vdash t: \mathcal{A} \rightarrow \sigma \quad \Delta \vdash u: \mathcal{A}}{\Gamma \wedge \Delta \vdash t u: \sigma}(\mathrm{app}) \\
\frac{\Gamma \vdash t: \mathcal{A} \quad \Delta \vdash u: \mathcal{B}}{\Gamma \wedge \Delta \vdash\langle t, u\rangle: \times(\mathcal{A}, \mathcal{B})}(\text { pair }) & \frac{\Gamma \vdash t:\left.\sigma \quad \Gamma\right|_{p} \Vdash p: \mathcal{A} \quad \Delta \vdash u: \mathcal{A}}{(\Gamma \backslash \operatorname{var}(p)) \wedge \Delta \vdash t[p \backslash u]: \sigma}(\text { match })
\end{array}
$$
\]

Figure 2 Typing System $\mathscr{U}$.

- Lemma 4 (Relevance). Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then, $\operatorname{dom}(\Gamma) \subseteq \mathrm{fv}(t)$.

Proof. By induction on $\Phi$ (cf. App. A).

Moreover, typing is stable by reduction and expansion, and the size of derivations is decreasing (resp. strictly decreasing) for $\rightarrow_{\mathrm{p}}$ reduction (resp. $\rightarrow_{\mathrm{h}}$ reduction).

- Lemma 5. Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then,

1. (Upper Subject Reduction). $t \rightarrow_{p} t^{\prime}$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ s.t. $\mathrm{sz}(\Phi) \geq \mathrm{sz}\left(\Phi^{\prime}\right)$, and $t \rightarrow_{h} t^{\prime}$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ s.t. $\mathrm{sz}(\Phi)>\mathrm{sz}\left(\Phi^{\prime}\right)$.
2. (Upper Subject Expansion). $t^{\prime} \rightarrow_{p} t$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ such that $\mathrm{sz}\left(\Phi^{\prime}\right) \geq \mathbf{s z}(\Phi)$ and $t^{\prime} \rightarrow_{h} t$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ such that $\mathbf{s z}\left(\Phi^{\prime}\right)>\mathbf{s z}(\Phi)$.

Proof. By induction on $\Phi$, item 1 (resp. item 2) uses a substitution (resp. anti-substitution) lemma (see Lem. 22 and Lem. 23 in App. A for details).

Typed terms are (head) clash-free, i.e. they cannot head reduce to a clash.

- Lemma 6 (Clash-Free). Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then $t$ is (head) clash-free.

Proof. By induction on $\Phi$ (cf. App. A).

Although the system in [19] already characterises head-normalisation in the pattern calculus, it does not provide upper bounds for the length of the head strategy. This is mainly due to the fact that the reduction system in [19] does not always decrease the measure of the typed terms, even when reduction is performed in the so-called typed occurrences. We can recover this situation, as witnessed by the following soundness and completeness result:

- Theorem 7 (Characterisation of Head-Normalisation and Upper Bounds). Let $t$ be a term in the pattern calculus. Then (1) $t$ is typable in system $\mathscr{U}$ iff (2) $t$ is head-normalisable iff (3) $t$ is head-terminating. Moreover, if $\Phi \triangleright \Gamma \vdash t: \sigma$, then the head-strategy terminates on $t$ in at most sz ( $\Phi$ ) steps.

Proof. The statement $(1) \Rightarrow(3)$ holds by upper subject reduction (Lem. 5.1) for $\rightarrow_{h}$. The statement $(3) \Rightarrow(2)$ is straightforward since $\rightarrow_{\mathrm{h}}$ is included in $\rightarrow_{\mathrm{p}}$. Finally, the statement $(2) \Rightarrow(1)$ holds by the fact that canonical terms are typable (easy), and by using upper subject expansion for $\rightarrow_{\mathrm{p}}$ (Lem. 5.2).

The previous upper bound result is especially possible thanks to the upper subject reduction property, stating in particular that reduction $\rightarrow_{\mathrm{h}}$ strictly decreases the size of typing derivations. It is worth noticing that the reduction relation in [19] does not enjoy this property, particularly in the case of the rule $t[p \backslash v] u \rightarrow(t u)[p \backslash v]$, which is a permuting conversion rule, (slightly) changing the structure of the type derivation, but not its size.

## 4 Towards a Relational Model for the Pattern Calculus

Denotational and operational semantics have tended to abstract quantitative information (e.g. time and space) as computational resource consumption. Since the invention of Girard's linear logic [28], where formulas are interpreted as resources, quantitative interpretation of programs, such as relational models [17, 18, 22], have been naturally defined and studied by following the simple idea that multisets are used to record the number of times a resource is consumed. Thus, relational models for the $\lambda$-calculus use multisets to keep track of how many times a resource is used during a computation.

In this brief section we emphasize a semantical result that is implicit in the previous section. Since relational models are often presented by means of typing systems [48, 47], our system $\mathscr{U}$ suggests a quantitative model for our pair pattern calculus in the following way. Indeed, consider a term $t$ such that $\mathrm{fv}(t) \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$, in which case we say that the list $\vec{x}=\left(x_{1}, \ldots, x_{n}\right)$ is suitable for $t$. Then, given $\vec{x}=\left(x_{1}, \ldots, x_{n}\right)$ suitable for $t$, define the interpretation of a term $t$ for $\vec{x}$ as

$$
\llbracket t \rrbracket_{\vec{x}}=\left\{\left(\left(\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}\right), \sigma\right) \mid \text { there exists } \Phi \triangleright x_{1}: \mathcal{A}_{1}, \ldots, x_{n}: \mathcal{A}_{n} \vdash t: \sigma\right\}
$$

A straightforward corollary of upper subject reduction and expansion properties (Lem. 5.1 and Lem. 5.2, respectively) is that $t={ }_{\mathrm{p}} u$ implies $\llbracket t \rrbracket_{\vec{x}}=\llbracket u \rrbracket_{\vec{x}}$, where $={ }_{\mathrm{p}}$ is the equational theory generated by the reduction relation $\rightarrow_{\mathrm{p}}$. Thus, p-equivalent programs have the same meaning.

## 5 The $\mathscr{E}$ Typing System

In this section we introduce our second typing system $\mathscr{E}$ for the pattern calculus, which is obtained by refining the System $\mathscr{U}$ presented in Sec. 3.

| (Product Types) | $\mathcal{P}$ | $::=\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)$ |
| :--- | :--- | :--- |
| (Tight Types) | t | $::=\bullet_{\mathcal{N}} \mid \bullet \mathcal{M}$ |
| (Types) | $\sigma$ | $::=\mathrm{t}\|\mathcal{P}\| \mathcal{A} \rightarrow \sigma$ |
| (Multiset Types) | $\mathcal{A}$ | $::=\left[\sigma_{k}\right]_{k \in K}$ |

Types in $t$, which can be seen as a refinement of the base type $\bullet$ of System $\mathscr{U}$, denote the so-called tight types. The constant $\bullet \mathcal{M}$ denotes the type of any term head reducing to a canonical form, while $\bullet_{\mathcal{N}}$ denotes the type of any term head reducing to a pure canonical form. We write $\operatorname{tight}(\sigma)$, if $\sigma$ is of the form $\bullet_{\mathcal{M}}$ or $\bullet_{\mathcal{N}}$ (we use $\bullet$ to denote either form). We extend this notion to multisets of types and typing contexts as expected, that is, tight $\left(\left[\sigma_{i}\right]_{i \in I}\right)$ if $\operatorname{tight}\left(\sigma_{i}\right)$ for all $i \in I$, and $\operatorname{tight}(\Gamma)$ if $\operatorname{tight}(\Gamma(x))$, for all $x \in \operatorname{dom}(\Gamma)$.

The crucial idea behind the grammar of types is to distinguish between consuming constructors typed with standard types, and persistent constructors typed with tight types, as hinted in the introduction. A constructor is consuming (resp. persistent) if it is consumed (resp. not consumed) during head-reduction. Indeed, the pair constructor is consumed (on the pattern side as well as on the term side) during the execution of the pattern matching rule m . Otherwise, patterns and pairs are persistent, and they do appear in the normal form of the original term. This dichotomy between consuming and persistent constructors, inspired from [41, 42], is reflected in the typing system by using different typing rules to type them, notably for the abstraction, the application, the pair terms and the pair patterns.

The type assignment system $\mathscr{E}$, given in Fig. 3, is based on sequents for terms (resp. patterns) with counters having the form $\Gamma \vdash^{(b, e, m, f)} t: \sigma$ or $\Gamma \vdash^{(b, e, m, f)} t: \mathcal{A}$ (resp. $\Gamma \Vdash^{(e, m, f)} p: \mathcal{A}$ ). Intuitively, if $\Gamma \vdash^{(b, e, m, f)} t: \sigma$ is "tightly" derivable (defined below), then $t \rightarrow_{\mathrm{h}}^{(b, e, m)} v$, where $b$ is the number of b-steps, $e$ the number of e-steps, $m$ the number of m -steps and $f$ is the size of the head normal-form $v$. Similarly, the derivability of $\Gamma \Vdash^{(e, m, f)} p: \mathcal{A}$ means that the pattern $p$ generates $e$ substitution e-steps, $m$ matching m-steps and $f$ symbols contributing to the normal form.

We write $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$ (resp. $\left.\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \mathcal{A}\right)$ to denote term type derivations ending with the sequent $\Gamma \vdash^{(b, e, m, f)} t: \sigma$ (resp. $\Gamma \vdash^{(b, e, m, f)} t: \mathcal{A}$ ), and $\Pi \triangleright \Gamma \Vdash^{(e, m, f)} p: \mathcal{A}$ to denote pattern type derivations ending with the sequent $\Gamma \Vdash^{(e, m, f)}$ $p: \mathcal{A}$. Often in examples, we will use the notation $\Phi^{(b, e, m, f)}$ (resp. $\left.\Pi^{(e, m, f)}\right)$ to refer to a term derivation (resp. pattern derivation) ending with a sequent annotated with indexes $(b, e, m, f)$ (resp. $(e, m, f)$ ).

As mentioned in the introduction, exact bounds can only be extractable from minimal derivations. In our framework this notion is implemented by means of tightness [2]. We say that a derivation $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$ (resp. $\left.\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \mathcal{A}\right)$ is tight, denoted by $\operatorname{tight}(\Phi)$, if and only if $\operatorname{tight}(\Gamma)$ and $\operatorname{tight}(\sigma)(\operatorname{resp} . \operatorname{tight}(\mathcal{A}))$. The size of derivations is defined as in System $\mathscr{U}$.

We now give some intuition behind the typing rules in Fig. 3, by addressing in particular the consuming/persistent paradigm.

- Rule ax: Since $x$ is itself a head normal-form, it will not generate any b, e or m steps, and its size is 0 .
- Rule abs: Used to type abstractions $\lambda$ p.t to be applied (i.e. consumed), therefore it has a functional type $\mathcal{A} \rightarrow \sigma$. Final indexes of the abstraction are obtained from the ones of the body and the pattern, and 1 is added to the first index since the abstraction will be consumed by a b-reduction step.
- Rule absp: Used to type abstractions $\lambda p . t$ that are not going to be applied/consumed (they are persistent). Only the last index (size of the normal form) is incremented by one since the abstraction remains in the normal form (the abstraction is persistent). Note that both the body $t$ and the variables in $p$ should be typed with a tight type.
- Rule app: Types applications $t u$ where $t$ will eventually become an abstraction, and thus the application constructor will be consumed. Indexes for $t u$ are exactly the sum of the indexes for $t$ and $u$. Note that we do not need to increment the counter for b steps, since this was already taken into account in the abs rule.
- Rule app ${ }_{p}$ : Types applications $t u$ where $t$ is neutral, therefore will never become an abstraction, and the application constructor becomes persistent. Indexes are the ones for $t$, adding one to the (normal term) size to count for the (persistent) application.
- Rule pair: Types pairs consumed during some matching step. We add the indexes for the two components of the pair without incrementing the number of $m$ steps, since it is incremented when typing a consuming abstraction, with rule abs.

$$
\begin{aligned}
& \overline{x: \mathcal{A} \Vdash^{(1,0,0)} x: \mathcal{A}}\left(\text { pat }_{\mathrm{v}}\right) \\
& \frac{\Gamma \Vdash^{\left(e_{p}, m_{p}, n_{p}\right)} p: \mathcal{A} \quad \Delta \Vdash^{\left(e_{q}, m_{q}, n_{q}\right)} q: \mathcal{B} \quad p \# q}{\Gamma \wedge \Delta \Vdash^{\left(e_{p}+e_{q}, 1+m_{p}+m_{q}, n_{p}+n_{q}\right)}\langle p, q\rangle:[\times(\mathcal{A}, \mathcal{B})]}\left(\text { pat }_{\times}\right) \\
& \frac{\operatorname{dom}(\Gamma) \subseteq \operatorname{var}(\langle p, q\rangle) \quad \operatorname{tight}(\Gamma)}{\Gamma \Vdash^{(0,0,1)}\langle p, q\rangle:[\bullet \mathcal{N}]}\left(\operatorname{pat}_{\mathrm{p}}\right) \\
& \overline{x:[\sigma] \vdash^{(0,0,0,0)} x: \sigma}(\mathrm{ax}) \\
& \frac{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t:\left.\sigma \quad \Gamma\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}}{\Gamma \ \operatorname{var}(p) \vdash^{\left(b_{t}+1, e_{t}+e_{p}, m_{t}+m_{p}, f_{t}+f_{p}\right)} \lambda p . t: \mathcal{A} \rightarrow \sigma}(\mathrm{abs}) \\
& \frac{\Gamma \vdash^{(b, e, m, f)} t: \mathrm{t} \quad \operatorname{tight}\left(\left.\Gamma\right|_{p}\right)}{\Gamma \rrbracket \operatorname{var}(p) \vdash^{(b, e, m, f+1)} \lambda p . t: \bullet \mathcal{M}}\left(\mathrm{abs}_{\mathrm{p}}\right) \\
& \frac{\left(\Gamma_{k} \vdash^{\left(b_{k}, e_{k}, m_{k}, f_{k}\right)} t: \sigma_{k}\right)_{k \in K}}{\wedge_{k \in K} \Gamma_{k} \vdash^{\left(+{ }_{k \in K} b_{k},+_{k \in K} e_{k},+_{k \in K} m_{k},+_{k \in K} f_{k}\right)} t:\left[\sigma_{k}\right]_{k \in K}} \text { (many) } \\
& \frac{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \mathcal{A} \rightarrow \sigma \quad \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}}{\Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t u: \sigma}(\mathrm{app}) \\
& \frac{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \bullet \mathcal{N}}{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}+1\right)} t u: \bullet_{\mathcal{N}}}\left(\operatorname{app}_{\mathrm{p}}\right) \\
& \frac{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \mathcal{A} \quad \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{B}}{\Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+b_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)}\langle t, u\rangle: \times(\mathcal{A}, \mathcal{B})} \text { (pair) } \\
& \overline{\vdash^{(0,0,0,1)}\langle t, u\rangle: \bullet \mathcal{M}}\left(\text { pair }_{\mathrm{p}}\right) \\
& \frac{\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t:\left.\sigma \quad \Gamma\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A} \quad \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}}{(\Gamma \backslash \operatorname{var}(p)) \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}+e_{p}, m_{t}+m_{u}+m_{p}, f_{t}+f_{u}+f_{p}\right)} t[p \backslash u]: \sigma} \text { (match) }
\end{aligned}
$$

Figure 3 Typing System $\mathscr{E}$.

- Rule pair $\mathrm{p}_{\mathrm{p}}$ : Used to type pairs that are not consumed in a matching step (they are persistent), therefore appear in the head normal-form. Since the pair is already a head normal-form its indexes are zero except for the size, which counts the pair itself.
- Rule match: Note that we do not need separate cases for consuming and persistent explicit matchings, since in both cases typable occurrences of $u$ represent potential head reduction steps for $u$, which need to be taken into account in the final counter of the term.
- Rule pat ${ }_{\mathrm{v}}$ : Typed variables always generate one e and zero m steps, even when erased.
- Rule pat $\times$ : Used when the pattern has a product type, which means that the pattern will be matched with a pair. We add the counters for the two components of the pair and increment the counter for the $m$ steps.
- Rule pat ${ }_{p}$ : Used when the pattern has a tight type, which means that it will not be matched with a pair and therefore will be blocked (it is persistent). This kind of pairs generate zero e and m steps, and will contribute with one blocked pattern to the size of the normal form.

The system also enjoys the relevance and clash-free properties, easily proved by induction:

- Lemma 8 (Relevance). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$. Then, $\operatorname{dom}(\Gamma) \subseteq \mathrm{fv}(t)$.
- Lemma 9 (Clash-Free). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$. Then, $t$ is (head) clash-free.

We now discuss two examples.

- Example 10. Let us consider $t_{0}=(\lambda\langle x, y\rangle .(\lambda\langle w, z\rangle . w y z) x)\langle\langle K, a\rangle, b\rangle$, with the following head-reduction sequence:

$$
\begin{array}{llll} 
& (\lambda\langle x, y\rangle .(\lambda\langle w, z\rangle \cdot w y z) x)\langle\langle\mathrm{K}, a\rangle, b\rangle & \rightarrow_{\mathrm{b}} & ((\lambda\langle w, z\rangle \cdot w y z) x)[\langle x, y\rangle \backslash\langle\langle\mathrm{K}, a\rangle, b\rangle] \\
\rightarrow_{\mathrm{b}} & (w y z)[\langle w, z\rangle \backslash x][\langle x, y\rangle \backslash\langle\langle\mathrm{K}, a\rangle, b\rangle] & \rightarrow_{\mathrm{m}} & (w y z)[\langle w, z\rangle \backslash x][x \backslash\langle\mathrm{~K}, a\rangle][y \backslash b] \\
\rightarrow_{\mathrm{e}} & (w y z)[\langle w, z\rangle \backslash\langle\mathrm{K}, a\rangle][y \backslash b] & \rightarrow_{\mathrm{m}} & (w y z)[w \backslash \mathrm{~K}][z \backslash a][y \backslash b] \\
\rightarrow_{\mathrm{e}} & (\mathrm{~K} y z)[z \backslash a][y \backslash b] & \rightarrow_{\mathrm{b}} & \left(\left(\lambda y_{1} \cdot x_{1}\right)\left[x_{1} \backslash y\right] z\right)[z \backslash a][y \backslash b] \\
\rightarrow_{\mathrm{b}} & x_{1}\left[y_{1} \backslash z\right]\left[x_{1} \backslash y\right][z \backslash a][y \backslash b] & \rightarrow_{\mathrm{e}} & x_{1}\left[x_{1} \backslash y\right][z \backslash a][y \backslash b] \\
\rightarrow_{\mathrm{e}} & y[z \backslash a][y \backslash b] & \rightarrow_{\mathrm{e}} & y[y \backslash b]
\end{array}
$$

$$
\rightarrow_{\mathrm{e}} \quad b
$$

Note that, there are two matching steps in the head-reduction sequence, but the second step is only created after the substitution of $x$ by $\langle\mathrm{K}, a\rangle$. Our method allows us to extract this information from the typing derivations because of the corresponding types for $\langle x, y\rangle$ and $\langle w, z\rangle$. Indeed, both patterns are typed with a product type (cf. the forthcoming tight typing derivations), and therefore the corresponding pairs are consumed and not persistent.

Since $t_{0}=(\lambda\langle x, y\rangle .(\lambda\langle w, z\rangle . w y z) x)\langle\langle\mathrm{K}, a\rangle, b\rangle \rightarrow_{\mathrm{h}}^{(4,6,2)} b$, the term $t_{0}$ should be tightly typable with counter $(4,6,2,0)$, where 0 is the size of $b$. In the construction of such tight derivation we proceed by pieces. Let $T_{\mathrm{K}}=\left[\bullet_{\mathcal{N}}\right] \rightarrow[] \rightarrow \bullet_{\mathcal{N}}$. We first construct the following pattern derivation for $\langle w, z\rangle$ :

$$
\Pi_{\langle w, z\rangle} \triangleright \frac{w:\left[T_{\mathrm{K}}\right] \Vdash^{(1,0,0)} w:\left[T_{\mathrm{K}}\right] \quad \Vdash^{(1,0,0)} z:[]}{w:\left[T_{\mathrm{K}}\right] \Vdash^{(2,1,0)}\langle w, z\rangle:\left[\times\left(\left[T_{\mathrm{K}}\right],[]\right)\right]}
$$

In the following $T_{\langle w, z\rangle}=\left[\times\left(\left[T_{\mathrm{K}}\right],[]\right)\right]$. We construct a similar pattern derivation for $\langle x, y\rangle$ :

$$
\Pi_{\langle x, y\rangle} \triangleright \frac{x: T_{\langle w, z\rangle} \Vdash^{(1,0,0)} x: T_{\langle w, z\rangle} \quad y:\left[\bullet_{\mathcal{N}}\right] \Vdash^{(1,0,0)} y:\left[\bullet_{\mathcal{N}}\right]}{x: T_{\langle w, z\rangle} ; y:[\bullet \mathcal{N}] \Vdash^{(2,1,0)}\langle x, y\rangle:\left[\times\left(T_{\langle w, z\rangle},\left[\bullet_{\mathcal{N}}\right]\right)\right]}
$$

In the rest of the example $T_{\langle x, y\rangle}=\left[\times\left(T_{\langle w, z\rangle},\left[\bullet_{\mathcal{N}}\right]\right)\right]$. We build a type derivation for $\lambda\langle x, y\rangle .(\lambda\langle w, z\rangle . w y z) x$, where $\Gamma_{w}=w:\left[T_{\mathrm{K}}\right], \Gamma_{y}=y:\left[\bullet_{\mathcal{N}}\right], \Gamma=\Gamma_{w} ; \Gamma_{y}$, and $\Gamma_{x}=x: T_{\langle w, z\rangle}$. Furthermore, in this example and throughout the paper, we will use $(\overline{0})$ to denote the tuple $(0,0,0,0)$.

$$
\begin{aligned}
& \frac{\frac{\Gamma_{w} \vdash^{(\overline{0})} w: T_{\mathrm{K}} \quad \Gamma_{y} \vdash^{(\overline{0})} y:[\bullet \mathcal{N}]}{\Gamma \vdash^{(\overline{0})} w y:[] \rightarrow \bullet_{\mathcal{N}}} \vdash^{(\overline{0})} z:[]}{\Gamma \vdash^{(\overline{0})} w y z: \bullet \mathcal{N}} \Pi_{\langle w, z\rangle}^{(2,1,0)} \\
& \frac{\frac{\Gamma_{y} \vdash^{(1,2,1,0)} \lambda\langle w, z\rangle \cdot w y z: T_{\langle w, z\rangle} \rightarrow \bullet_{\mathcal{N}}}{\Gamma_{y} ; \Gamma_{x} \vdash^{(1,2,1,0)}(\lambda\langle w, z\rangle . w y z) x: \bullet_{\mathcal{N}}}}{\vdash^{(2,4,2,0)} \lambda\langle x, y\rangle \cdot(\lambda\langle w, z\rangle . w y z) x: T_{\langle x, y\rangle} \rightarrow \bullet_{\mathcal{N}}} x: T_{\langle w, z\rangle} \\
& \Phi_{1} \triangleright \frac{\vdash_{\langle x, y\rangle}^{(2,1,0)}}{} \\
& \Phi_{\mathrm{K}} \triangleright \frac{\frac{x_{1}:\left[\bullet_{\mathcal{N}}\right] \vdash^{(\overline{0})} x_{1}: \bullet_{\mathcal{N}} \quad \vdash^{(1,0,0)} y_{1}:[]}{x_{1}:\left[\bullet_{\mathcal{N}}\right] \vdash^{(1,1,0,0)} \lambda y_{1} \cdot x_{1}:[] \rightarrow \bullet_{\mathcal{N}}}}{\vdash^{(2,2,0,0)} \mathrm{K}: T_{\mathrm{K}}} x_{1}:\left[\bullet_{\mathcal{N}}\right] \vdash^{(1,0,0)} x_{1}:\left[\bullet_{\mathcal{N}}\right]
\end{aligned}
$$

From $\Phi_{1}$ and $\Phi_{\mathrm{K}}$ we build the final tight derivation for $t_{0}=(\lambda\langle x, y\rangle .(\lambda\langle w, z\rangle . w y z) x)\langle\langle\mathrm{K}, a\rangle, b\rangle$ :

Therefore, $\Phi^{(4,6,2,0)}$ gives the expected exact bounds. It is worth noticing that the pair $\langle\langle K, a\rangle, b\rangle$ is typed here with a singleton multiset, while it would be typable with a multiset having at least two elements in the typing system proposed in [19], even if the term is not going to be duplicated.

Example 11. We now consider the term $t_{1}=(\lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z)\langle u, v\rangle$, having the following head-reduction sequence to head normal-form:

$$
\begin{array}{llll} 
& (\lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z)\langle u, v\rangle & \rightarrow_{\mathrm{b}} & ((\lambda\langle x, y\rangle . \mathrm{I}) z z)[z \backslash\langle u, v\rangle] \\
\rightarrow_{\mathrm{b}} & (\mathrm{I}[\langle x, y\rangle \backslash z] z)[z \backslash\langle u, v\rangle] & \rightarrow_{\mathrm{b}} & w[w \backslash z][\langle x, y\rangle \backslash z][z \backslash\langle u, v\rangle] \\
\rightarrow_{\mathrm{e}} & z[\langle x, y\rangle \backslash z][z \backslash\langle u, v\rangle] & \rightarrow_{\mathrm{e}} & \langle u, v\rangle[\langle x, y\rangle \backslash\langle u, v\rangle] \\
\rightarrow_{\mathrm{m}} & \langle u, v\rangle[x \backslash u][y \backslash v] & \rightarrow_{\mathrm{e}} & \langle u, v\rangle[y \backslash v] \\
\rightarrow_{\mathrm{e}} & \langle u, v\rangle & &
\end{array}
$$

We have 3 b-steps, 4 e-steps, and 1 m -step to the normal form $\langle u, v\rangle$ of size 1 . Note that the pair $\langle u, v\rangle$ is copied twice during the reduction, but only one of the copies is consumed by a matching. The copy of the pair that is not consumed will persist in the term, therefore it will be typed with $\bullet_{\mathcal{M}}$. The other copy will be consumed in a matching step, however its components are not going to be used, therefore we will type it with $[o]$, where $o$ denotes $\times([],[])$.

Since $t_{1}=(\lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z)\langle u, v\rangle \rightarrow_{\mathrm{h}}^{(3,4,1)}\langle u, v\rangle$, we need to derive a tight derivation for $t_{1}$ decorated with counter (3, 4, 1, 1). We first consider the following derivation:

$$
\Phi_{1} \triangleright \frac{\frac{w:[\bullet \mathcal{M}] \vdash^{(\overline{0})} w: \bullet \mathcal{M} \quad w:[\bullet \mathcal{M}] \Vdash^{(1,0,0)} w:[\bullet \mathcal{M}]}{\vdash^{(1,1,0,0)} \mathrm{I}:[\bullet \mathcal{M}] \rightarrow \bullet \mathcal{M}} \quad \frac{\Vdash^{(1,0,0)} x:[] \Vdash^{(1,0,0)} y:[]}{\Vdash^{(2,0,0)}\langle x, y\rangle:[o]}}{\vdash^{\left({ }^{(2,3,1,0)} \lambda\langle x, y\rangle . \mathrm{I}:[o] \rightarrow[\bullet \mathcal{M}] \rightarrow \bullet \mathcal{M}\right.}}
$$

From $\Phi_{1}$ we obtain the following derivation $\Phi_{2}$, where $\mathcal{A}_{0}=[o, \bullet \mathcal{M}]$ :

$$
\triangleright \frac{\frac{\Phi_{1}^{(2,3,1,0)}}{\frac{z:[o] \vdash^{(\overline{0})} z:[o]}{z:[o] \vdash^{(2,3,1,0)}(\lambda\langle x, y\rangle . \mathrm{I}) z:[\bullet \mathcal{M}] \rightarrow \bullet \mathcal{M}} \quad z:[\bullet \mathcal{M}] \vdash^{(\overline{0})} z:[\bullet \mathcal{M}]}}{z: \mathcal{A}_{0} \vdash^{(2,3,1,0)}(\lambda\langle x, y\rangle . \mathrm{I}) z z: \bullet \mathcal{M}} \underset{\vdash^{(3,4,1,0)} \lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z: \mathcal{A}_{0} \rightarrow \bullet \mathcal{M}}{z: \mathcal{A}_{0} \Vdash^{(1,0,0)} z: \mathcal{A}_{0}}
$$

Using $\Phi_{2}$ we obtain the final tight derivation, and its expected counter:

$$
\frac{\frac{\vdots}{\Phi_{2}^{(3,4,1,0)}} \frac{\frac{\vdash^{(\overline{0})}\langle u, v\rangle: o}{\vdash^{(0,0,0,1)}\langle u, v\rangle: \bullet \mathcal{M}}}{\vdash^{(0,0,0,1)}\langle u, v\rangle: \mathcal{A}_{0}}}{\vdash^{(3,4,1,1)}(\lambda z .(\lambda\langle x, y\rangle . \mathrm{I}) z z)\langle u, v\rangle: \bullet \mathcal{M}}
$$

## 6 Soundness of System $\mathscr{E}$

This section studies the implication "tight typability implies head-normalisable". The two key properties used to show this implication are minimal counters for canonical forms (Lem. 13) and the exact subject reduction property (Lem. 15). Indeed, Lem. 13 guarantees that a tight derivation for a canonical form $t$ holds the right counter of the form ( $0,0,0,|t|$ ). Lem. 15 gives in fact an (exact) weighted subject reduction property, weighted because head-reduction strictly decreases the counters of typed terms, and exact because only one counter is decreased by 1 for each head-reduction step. Subject reduction is based on a substitution property (Lem. 14). We start with a key auxiliary lemma.

- Lemma 12 (Tight Spreading). Let $t \in \mathcal{N}$. Let $\Phi \triangleright \Gamma^{(b, e, m, f)} t: \sigma$ be a typing derivation such that tight $(\Gamma)$. Then $\sigma$ is tight and the last rule of $\Phi$ does not belong to $\left\{a p p, a b s, \mathrm{abs}_{\mathrm{p}}\right.$, pair, pair $\left.\mathrm{p}_{\mathrm{p}}\right\}$.

Proof. By induction on $t \in \mathcal{N}$, taking into account the fact that $t$ is not an abstraction nor a pair (cf. App. B).

- Lemma 13 (Canonical Forms and Minimal Counters). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$ be a tight derivation. Then $t \in \mathcal{M}$ if and only if $b=e=m=0$.

Proof. The left-to-right implication is by induction on the definition of the set $\mathcal{M}$, using the tight spreading property (Lem. 12) for the cases of application and explicit matching. The right-to-left implication is by induction on $\Phi$ and also uses Lem. 12 (cf. App. B).
$\triangleright$ Lemma 14 (Substitution for System $\mathscr{E}$ ). If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma$, and $\Phi_{u} \triangleright$ $\Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t\{x \backslash u\}: \sigma$.

Proof. By induction on $\Phi_{t}$ (cf. App. B).

- Lemma 15 (Exact Subject Reduction). If $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$, and $t \rightarrow_{h} t^{\prime}$ is an $s$-step, with $s \in\{b, e, m\}$, then $\Phi^{\prime} \triangleright \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} t^{\prime}: \sigma$, where
- $s=b$ implies $b^{\prime}=b-1, e^{\prime}=e, m^{\prime}=m$.
- $s=e$ implies $b^{\prime}=b, e^{\prime}=e-1, m^{\prime}=m$.
- $s=m$ implies $b^{\prime}=b, e^{\prime}=e, m^{\prime}=m-1$.

Proof. By induction on $\rightarrow_{h}$, using the substitution property (Lem. 14) (cf. App. B).

The exact subject reduction property provides a simple argument to obtain the implication "tightly typable implies head-normalisable": if $t$ is tightly typable, and reduction decreases the counters, then head-reduction necessarily terminates. But the soundness implication is in fact more precise than that. Indeed:

- Theorem 16 (Soundness). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$ be a tight derivation. Then there exists $u \in \mathcal{M}$ and a head reduction sequence $\rho$ such that $\rho: t \rightarrow_{h}^{(b, e, m)} u$ and $|u|=f$.

Proof. By induction on $b+e+m$.
If $b+e+m=0$ (i.e. $b=e=m=0$ ), then canonical forms and minimal counters property (Lem. 13) gives $t \in \mathcal{M}$, so that $t \not \not_{\mathrm{h}}$ holds by Prop. 3. We let $u:=t$ and thus $t \rightarrow_{\mathrm{h}}^{(0,0,0)} t$. It is easy to show that tight derivations $\Phi \triangleright \Gamma \vdash^{(0,0,0, f)} t: \sigma$ for terms in $\mathcal{M}$ verify $|t|=f$.

If $b+e+m>0$, we know by Lem. 13 that $t \notin \mathcal{M}$, and we know by the clash-free property (Lem. 9) that $t$ is (head) clash-free. Then, $t$ turns to be head-reducible by Prop. 3, i.e. there exists $t^{\prime}$ such that $t \rightarrow_{\mathrm{h}} t^{\prime}$. By the exact subject reduction property (Lem. 15) there is a derivation $\Phi^{\prime} \triangleright \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} t^{\prime}: \sigma$ such that $b^{\prime}+e^{\prime}+m^{\prime}+1=b+e+m$. The i.h. applied to $\Phi^{\prime}$ then gives $t^{\prime} \rightarrow_{\mathrm{h}}^{\left(b^{\prime}, e^{\prime}, m^{\prime}\right)} u$ and $|u|=f$. We conclude with the sequence $t \rightarrow_{\mathrm{h}} t^{\prime} \rightarrow_{\mathrm{h}}^{\left(b^{\prime}, e^{\prime}, m^{\prime}\right)} u$, with the counters as expected.

## 7 Completeness for System $\mathscr{E}$

In this section we study the reverse implication "head-normalisable implies tight typability". In this case the key properties are the existence of tight derivations for canonical forms (Lem. 17) and the subject expansion property (Lem. 19). As in the previous section these properties are (exact) weighted in the sense that Lem. 17 guarantees that a canonical form $t$ has a tight derivation with the right counter, and Lem. 19 shows that each step of headexpansion strictly increases exactly one of the counters of tightly typed terms. Subject expansion relies on an anti-substitution property (Lem. 18).

- Lemma 17 (Canonical Forms and Tight Derivations). Let $t \in \mathcal{M}$. There exists a tight derivation $\Phi \triangleright \Gamma \vdash^{(0,0,0,|t|)} t: \mathrm{t}$.

Proof. We generalise the property to the two following statements:

- If $t \in \mathcal{N}$, then there exists a tight derivation $\Phi \triangleright \Gamma \vdash^{(0,0,0,|t|)} t: \bullet \mathcal{N}$.
- If $t \in \mathcal{M}$, then there exists a tight derivation $\Phi \triangleright \Gamma \vdash^{(0,0,0,|t|)} t: \mathrm{t}$.

The proof then proceeds by induction on $\mathcal{N}, \mathcal{M}$, using relevance (Lem. 8).

- Lemma 18 (Anti-Substitution for System $\mathscr{E}$ ). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t\{x \backslash u\}: \sigma$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, integers $b_{t}, b_{u}, e_{t}, e_{u}, m_{t}, m_{u}, f_{t}, f_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}, b=b_{t}+b_{u}, e=e_{t}+e_{u}$, $m=m_{t}+m_{u}, f=f_{t}+f_{u}$, and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.

Proof. By induction on $\Phi$ (cf. App. C).

- Lemma 19 (Exact Subject Expansion). If $\Phi^{\prime} \triangleright \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f^{\prime}\right)} t^{\prime}: \sigma$, and $t \rightarrow_{h} t^{\prime}$ is an s-step, with $s \in\{b, e, m\}$, then $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$, where
- $s=b$ implies $b=b^{\prime}+1, e^{\prime}=e, m^{\prime}=m$.
- $s=e$ implies $b^{\prime}=b, e=e^{\prime}+1, m^{\prime}=m$.
- $s=m$ implies $b^{\prime}=b, e^{\prime}=e, m=m^{\prime}+1$.

Proof. By induction on $\rightarrow_{\mathrm{h}}$, using the anti-substitution property (Lem. 18) (cf. App. C).

The previous lemma provides a simple argument to obtain the implication "headnormalisable implies tightly typable", which can in fact be stated in a more precise way:

- Theorem 20 (Completeness). Let $t$ be a head-normalising term such that $t \rightarrow_{h}^{(b, e, m)} u$, $u \in \mathcal{M}$. Then there exists a tight derivation $\Phi \triangleright \Gamma \vdash^{(b, e, m,|u|)} t: \mathrm{t}$.

Proof. By induction on $b+e+m$.

- If $(b+e+m)=0$ then $t=u \in \mathcal{M}$, therefore $\Gamma \vdash^{(0,0,0,|t|)} t$ : t , by the canonical forms and tight derivations property (Lem. 17).
- If $(b+e+m)>0$, then $t \rightarrow_{\mathrm{h}} t^{\prime} \rightarrow_{\mathrm{h}}^{\left(b^{\prime}, e^{\prime}, m^{\prime}\right)} u$, where $b^{\prime}+e^{\prime}+m^{\prime}+1=b+e+m$. By the i.h. $\Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime},|u|\right)} t^{\prime}$ : t. Then from the exact subject expansion property (Lem. 19), it follows that $\Gamma \vdash^{(b, e, m,|u|)} t$ : t .

In summary, soundness and completeness do not only establish an equivalence between tight typability and head-normalisation, but they provide a much refined equivalence property stated as follows:

- Corollary 21. Given a term $t$, the following statements are equivalent
- There is a tight derivation $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t$ : t.
- There exists a canonical form $u \in \mathcal{M}$ such that $t \rightarrow{ }_{h}^{(b, e, m)} u$ and $|u|=f$.


## 8 Conclusion

This paper provides a quantitative insight of pattern matching by using type systems to study some of its dynamical properties. Indeed, our typing system $\mathscr{U}$ (resp. $\mathscr{E}$ ) provides upper bounds (resp. exact measures) about time and space properties related to (dynamic) computation. More precisely, the tuple of integers in the conclusion of a tight $\mathscr{E}$-derivation for a term $t$ provides the exact length of the head-normalisation sequence of $t$ and the size of its normal form. Moreover, the length of the normalisation sequence is discriminated according to different kind of steps performed to evaluate $t$.

Future work includes generalisations to more powerful notions of (dynamic) patterns, and to other reduction strategies for pattern calculi, as well to programs with recursive schemes. Inhabitation for our typing system is conjectured to be decidable, as the one in [19], but this still needs to be formally proved, in which case the result "solvability = typing+ inhabitation" in opt. cit. would be restated in a simpler framework. The quest of a general notion of model for pattern calculi also remains open, particularly for dynamic pattern calculi [32, 6].

Last, but not least, time cost analysis of a language with constructors and pattern matching is studied in [1], where it is shown that evaluation matching rules other than $\beta$-reduction may be negligible, depending on the reduction strategy and the specific notion of value. We expect the type-based quantitative technical tools we provide in this paper to be helpful in such a kind of quantitative analysis.

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## A The $\mathscr{U}$ Typing System

- Lemma 4 (Relevance). Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then, $\operatorname{dom}(\Gamma) \subseteq \operatorname{fv}(t)$.

Proof. Let $\Phi \triangleright \Gamma \vdash t: \sigma$. By straightforward induction on $\Phi$. Note that $\Gamma=(\Gamma \ \operatorname{var}(p)) ;\left.\Gamma\right|_{p}$ in both (abs) and (match) rules.

- Lemma 22 (Substitution for System $\mathscr{U}$ ). If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash t: \sigma$, and $\Phi_{u} \triangleright \Delta \vdash u: \mathcal{A}$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash t\{x \backslash u\}: \sigma$ such that $\mathbf{s z}\left(\Phi_{t\{x \backslash u\}}\right)=\mathbf{s z}\left(\Phi_{t}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|$.

Proof. We generalise the statement as follows: Let $\Phi_{u} \triangleright \Delta \vdash u: \mathcal{A}$.

- If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash t: \sigma$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash t\{x \backslash u\}: \sigma$.
- If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash t: \mathcal{B}$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash t\{x \backslash u\}: \mathcal{B}$.

In both cases sz $\left(\Phi_{t\{x \backslash u\}}\right)=\mathbf{s z}\left(\Phi_{t}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|$.
The proof then follows by induction on $\Phi_{t}$.

- If $\Phi_{t}$ is (ax), then we consider two cases:
$=t=x$ : then $\Phi_{x} \triangleright x:[\sigma] \vdash x: \sigma$ and $\Phi_{u} \triangleright \Delta \vdash u:[\sigma]$, which is a consequence of $\Delta \vdash u: \sigma$. Then $x\{x \backslash u\}=u$, and we trivially obtain $\Phi_{t\{x \backslash u\}} \triangleright \Delta \vdash u: \sigma$. We have $\mathbf{s z}\left(\Phi_{t\{x \backslash u\}}\right)=1+\mathbf{s z}\left(\Phi_{u}\right)-1$ as expected.
$=t=y$ : then $\Phi_{y} \triangleright y:[\sigma] ; x:[] \vdash y: \sigma$ and $\Phi_{u} \triangleright \emptyset \vdash u:$ [] by the (many) rule. Then $y\{x \backslash u\}=y$, and we trivially obtain $\Phi_{t\{x \backslash u\}} \triangleright y:[\sigma] \vdash y: \sigma$. We have $\mathrm{sz}\left(\Phi_{t\{x \backslash u\}}\right)=1+0-0$ as expected.
- If $\Phi_{t}$ ends with (many), then it has premises of the form $\left(\Phi_{t}^{i} \triangleright \Gamma^{i} ; x: \mathcal{A}^{i} \vdash t: \sigma^{i}\right)_{i \in I}$, where $\Gamma=\wedge_{i \in I} \Gamma^{i}, \mathcal{A}=\wedge_{i \in I} \mathcal{A}^{i}$ and $\mathcal{B}=\left[\sigma^{i}\right]_{i \in I}$. The derivation $\Phi_{u}$ can also be decomposed into subderivations $\left(\Phi_{u}^{i} \triangleright \Delta^{i} \vdash u: \mathcal{A}^{i}\right)_{i \in I}$ where $\Delta=\wedge_{i \in I} \Delta^{i}$. The i.h. gives the derivations $\left(\Phi_{t\{x \backslash u\}}^{i} \triangleright \Gamma^{i} \wedge \Delta^{i} \vdash t\{x \backslash u\}: \sigma^{i}\right)_{i \in I}$. Then we apply rule (many) to get $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash t\{x \backslash u\}: \mathcal{B}$. The statement about sz (_) works as expected by the i.h.
- If $\Phi_{t}$ ends with (abs), so that $t=\lambda p . t^{\prime}$ then, without loss of generality, one can always assume that $(\operatorname{fv}(u) \cup\{x\}) \cap \operatorname{var}(p)=\emptyset$. The result will follow easily by induction and relevance of the typing system. The statement about sz (_) works as expected by the i.h.
- If $\Phi_{t}$ ends with (app), so that $t=t^{\prime} u^{\prime}$, then $\Phi_{t^{\prime} u^{\prime}}$ is of the form

$$
\frac{\Phi_{t^{\prime}} \triangleright \Gamma_{t^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \vdash t^{\prime}: \mathcal{B} \rightarrow \sigma \quad \Phi_{u^{\prime}} \triangleright \Gamma_{u^{\prime}} ; x: \mathcal{A}_{u^{\prime}} \vdash u^{\prime}: \mathcal{B}}{\Gamma_{t^{\prime}} \wedge \Gamma_{u^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}} \vdash t^{\prime} u^{\prime}: \sigma}
$$

Also, $\Phi_{u} \triangleright \Delta \vdash u: \mathcal{A}$ is a consequence of $\left(\Delta_{k} \vdash u: \sigma_{k}\right)_{k \in K}$, with $\mathcal{A}=\left[\sigma_{k}\right]_{k \in K}$ and $\Delta=\wedge_{k \in K} \Delta_{k}$. Note that $\mathcal{A}=\mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}}=\left[\sigma_{i}\right]_{i \in K_{t^{\prime}}} \wedge\left[\sigma_{i}\right]_{i \in K_{u^{\prime}}}$, with $K=K_{t^{\prime}} \uplus K_{u^{\prime}}$, from which one can obtain both $\Delta_{t^{\prime}} \vdash u: \mathcal{A}_{t^{\prime}}$ and $\Delta_{u^{\prime}} \vdash u: \mathcal{A}_{u^{\prime}}$, through the (many) rule. By
the i.h. we then have $\Gamma_{t^{\prime}} \wedge \Delta_{t^{\prime}} \vdash t^{\prime}\{x \backslash u\}: \mathcal{B} \rightarrow \sigma$ and $\Gamma_{u^{\prime}} \wedge \Delta_{u^{\prime}} \vdash u^{\prime}\{x \backslash u\}: \mathcal{B}$. Finally, $\Gamma_{t^{\prime}} \wedge \Gamma_{u^{\prime}} \wedge \Delta_{t^{\prime}} \wedge \Delta_{u^{\prime}} \vdash\left(t^{\prime}\{x \backslash u\}\right)\left(u^{\prime}\{x \backslash u\}\right): \sigma$ by the app rule. The statement about sz (_) works as expected by the i.h.

- If $\Phi_{t}$ ends with (pair) or (pair ${ }_{\mathrm{p}}$ ), so that $t=\left\langle t^{\prime}, u^{\prime}\right\rangle$, then the result is obtained by induction following the same reasoning used in rule app. The statement about sz (_) works as expected by the i.h.
- If $\Phi_{t}$ ends with (match), so that $t=t^{\prime}\left[p \backslash u^{\prime}\right]$, then the proof is similar to the application case since $t^{\prime}\left[p \backslash u^{\prime}\right]\{x \backslash u\}=\left(t^{\prime}\{x \backslash u\}\right)\left[p \backslash u^{\prime}\{x \backslash u\}\right]$ and we can assume that $(\operatorname{fv}(u) \cup\{x\}) \cap$ $\operatorname{var}(p)=\emptyset$. The statement about sz (_) works as expected by the i.h.
- Lemma 23 (Anti-Substitution for System $\mathscr{U}$ ). Let $\Phi \triangleright \Gamma \vdash t\{x \backslash u\}: \sigma$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash t: \sigma$, $\Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$ and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$. Moreover, sz $(\Phi)=\mathbf{s z}\left(\Phi_{t}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|$.

Proof. As in the case of the substitution lemma, the proof follows by generalising the property for the two cases where the type derivation $\Phi$ assigns a type or a multiset type:

- Let $\Phi \triangleright \Gamma \vdash t\{x \backslash u\}: \sigma$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash t: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$ and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.
- Let $\Phi \triangleright \Gamma \vdash t\{x \backslash u\}: \mathcal{B}$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash t: \mathcal{B}, \Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$ and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.
In both cases sz $(\Phi)=\mathbf{s z}\left(\Phi_{t}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|$ holds.
We will reason by induction on $\Phi$ and cases analysis on $t$. For all the rules (except many), we will have the trivial case $t\{x \backslash u\}$, where $t=x$, in which case $t\{x \backslash u\}=u$, for which we have a derivation $\Phi \triangleright \Gamma \vdash u: \sigma$. Therefore $\Phi_{t} \triangleright x:[\sigma] \vdash x: \sigma$ and $\Phi_{u} \triangleright \Gamma \vdash u:[\sigma]$ is obtained from $\Phi$ using the (many) rule. We conclude since $\mathbf{s z}(\Phi)=1+\mathrm{sz}\left(\Phi_{u}\right)-1$. We now reason on the different cases assuming that $t \neq x$.
- If $\Phi$ is (ax) then $\Phi \triangleright y:[\sigma] \vdash y: \sigma$ and, since $t \neq x, t=y \neq x$. Then we take $\mathcal{A}=[]$, $\Phi_{t} \triangleright y:[\sigma] ; x:[] \vdash y: \sigma$, and $\Phi_{u} \triangleright \emptyset \vdash u:[]$ from rule (many). We conclude since $\mathbf{s z}(\Phi)=1+0-0$.
- If $\Phi$ ends with (many), then $\Phi \triangleright \wedge_{k \in K} \Gamma_{k} \vdash t\{x \backslash u\}:\left[\sigma_{k}\right]_{k \in K}$ follows from the derivation $\Phi^{k} \triangleright \Gamma_{k} \vdash t\{x \backslash u\}: \sigma_{k}$, for each $k \in K$. By the i.h. there exist $\Phi_{t}^{k}, \Phi_{u}^{k}$, contexts $\Gamma_{t}^{k}, \Gamma_{u}^{k}$ and multitype $\mathcal{A}_{k}$, such that $\Phi_{t}^{k} \triangleright \Gamma_{t}^{k} ; x: \mathcal{A}_{k} \vdash t: \sigma_{k}, \Phi_{u}^{k} \triangleright \Gamma_{u}^{k} \vdash u: \mathcal{A}_{k}, \Gamma_{k}=\Gamma_{t}^{k} \wedge \Gamma_{u}^{k}$. Taking $\mathcal{A}=\wedge_{k \in K} \mathcal{A}_{k}$ and using rule many we get $\wedge_{k \in K} \Gamma_{t}^{k} ; x: \mathcal{A} \vdash t:\left[\sigma_{k}\right]_{k \in K}$. From the premises of $\Phi_{u}^{k}$ for $k \in K$, applying the many rule, we get $\wedge_{k \in K} \Gamma_{u}^{k} \vdash u: \mathcal{A}$. Note that $\Gamma=\wedge_{k \in K} \Gamma_{k}=\left(\wedge_{k \in K} \Gamma_{t}^{k}\right) \wedge\left(\wedge_{k \in K} \Gamma_{u}^{k}\right)$. The statement about sz (_) works as expected by the i.h.
- If $\Phi$ ends with (abs), then $t=\lambda p . t^{\prime}$, therefore $\Phi \triangleright \Gamma \backslash \operatorname{var}(p) \vdash \lambda p .\left(t^{\prime}\{x \backslash u\}\right): \mathcal{B} \rightarrow \sigma$ follows from $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}\{x \backslash u\}: \sigma$ and $\left.\Pi_{p} \triangleright \Gamma\right|_{p} \Vdash p: \mathcal{B}$. Note that, one can always assume that $\operatorname{var}(p) \cap \mathrm{fv}(u)=\emptyset$ and $x \notin \operatorname{var}(p)$. By the i.h., $\Phi_{t^{\prime}} \triangleright \Gamma_{t^{\prime}} ; x: \mathcal{A} \vdash t^{\prime}: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$, with $\Gamma=\Gamma_{t^{\prime}} \wedge \Gamma_{u}$. Then using abs we get $\Phi_{t} \triangleright \Gamma_{t^{\prime}} \backslash \operatorname{var}(p) ; x: \mathcal{A} \vdash \lambda p . t^{\prime}: \mathcal{B} \rightarrow \sigma$. Note that $\left(\Gamma_{t^{\prime}} ; x: \mathcal{A}\right) \backslash \operatorname{var}(p)=\Gamma_{t^{\prime}} \backslash \operatorname{var}(p) ; x: \mathcal{A}$ and $\Gamma \backslash \operatorname{var}(p)=\left(\Gamma_{t^{\prime}} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u}$. The statement about sz (_) works as expected by the i.h.
- The remaining cases for (app), (pair) and (match) also hold by the i.h. and do not present any special difficulty.
- Lemma 5. Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then,

1. (Upper Subject Reduction). $t \rightarrow_{\mathrm{p}} t^{\prime}$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma \mathrm{s} . \mathrm{t} . \mathrm{sz}(\Phi) \geq \mathrm{sz}\left(\Phi^{\prime}\right)$, and $t \rightarrow_{\mathrm{h}} t^{\prime}$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ s.t. $\mathrm{sz}(\Phi)>\mathrm{sz}\left(\Phi^{\prime}\right)$.
2. (Upper Subject Expansion). $t^{\prime} \rightarrow_{\mathrm{p}} t$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ such that $\mathrm{sz}\left(\Phi^{\prime}\right) \geq \mathrm{sz}(\Phi)$ and $t^{\prime} \rightarrow_{\mathrm{h}} t$ implies there is $\Phi^{\prime} \triangleright \Gamma \vdash t^{\prime}: \sigma$ such that $\mathrm{sz}\left(\Phi^{\prime}\right)>\mathrm{sz}(\Phi)$.

## Proof. Let $\Phi \triangleright \Gamma \vdash t: \sigma$.

1. By induction on $\rightarrow_{\mathrm{p}}$ (resp. $\rightarrow_{\mathrm{h}}$ ) and the substitution property (Lem. 22). The first three cases represent the base cases for both reductions, where the size relation is strict. $=t=\mathrm{L} \llbracket \lambda p . v \rrbracket u \rightarrow_{\mathrm{p} / \mathrm{h}} \mathrm{L} \llbracket v\left[p \backslash u \rrbracket \rrbracket=t^{\prime}\right.$. The proof is by induction on the list L . We only show the case of the empty list as the other one is straightforward. The typing derivation $\Phi$ is necessarily of the form

$$
\frac{\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash p: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \vdash \lambda p . v: \mathcal{A} \rightarrow \sigma} \quad \Gamma_{u} \vdash u: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash(\lambda p . v) u: \sigma}
$$

We then construct the following derivation $\Phi^{\prime}$ :

$$
\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash p: \mathcal{A} \quad \Gamma_{u} \vdash u: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash v[p \backslash u]: \sigma}
$$

Moreover, sz $(\Phi)=\mathbf{s z}\left(\Phi^{\prime}\right)+1$.
$=t=v[x \backslash u] \rightarrow_{\mathrm{p}} v\{x \backslash u\}=t^{\prime}$. Then $\Phi$ has two term premises $\Phi_{v} \triangleright \Gamma_{v} ; x: \mathcal{A} \vdash v: \sigma$, $\Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$, and one pattern premise $\Pi_{x} \triangleright x: \mathcal{A} \Vdash x: \mathcal{A}$, where $\Gamma=\Gamma_{v} \wedge \Gamma_{u}$ and $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Phi_{u}\right)+\mathbf{s z}\left(\Pi_{x}\right)+1$. Lem. 22 then gives a derivation $\Phi^{\prime}$ ending with $\Gamma_{v} \wedge \Gamma_{u} \vdash v\{x \backslash u\}: \sigma$, where $|\mathcal{A}| \geq 0$ and $\mathbf{s z}\left(\Pi_{x}\right)=1$ imply

$$
\mathbf{s z}\left(\Phi^{\prime}\right)=\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|<\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Phi_{u}\right)+\mathbf{s z}\left(\Pi_{x}\right)<\mathbf{s z}(\Phi)
$$

When $t=v[x \backslash u] \rightarrow_{\mathrm{h}} v\{x \backslash u\}=t^{\prime}$, where $v \not_{\mathrm{h}}^{\mathrm{h}}$, the same results hold.
$=t=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t^{\prime}$. Let us write $p=\left\langle p_{1}, p_{2}\right\rangle$ and $u=\left\langle u_{1}, u_{2}\right\rangle$. The typing derivation $\Phi$ is necessarily of the form

$$
\frac{\Phi_{v} \triangleright \Gamma_{v} \vdash v:\left.\sigma \quad \Pi_{p} \triangleright \Gamma_{v}\right|_{p} \vdash\left\langle p_{1}, p_{2}\right\rangle: \mathcal{A} \quad \Phi_{u} \triangleright \Gamma_{u} \vdash \mathrm{~L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket: \mathcal{A}}{\Gamma_{v} \backslash \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right) \wedge \Gamma_{u} \vdash v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash\left\lfloor\llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right]: \sigma\right.}
$$

Moreover, $\mathcal{A}=\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]$ and $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{u}\right)+1$.
Then $\Pi_{p}$ is of the form:

$$
\frac{\left.\Pi_{p_{1}} \triangleright \Gamma_{v}\right|_{p_{1}} \Vdash p_{1}:\left.\mathcal{A}_{1} \quad \Pi_{p_{2}} \triangleright \Gamma_{v}\right|_{p_{2}} \Vdash p_{2}: \mathcal{A}_{2} \quad p_{1} \# p_{2}}{\left.\Gamma_{v}\right|_{p} \Vdash\left\langle p_{1}, p_{2}\right\rangle:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

and $\mathbf{s z}\left(\Pi_{p}\right)=\mathbf{s z}\left(\Pi_{p_{1}}\right)+\mathbf{s z}\left(\Pi_{p_{2}}\right)+1$
The proof is then by induction on the list L .
$=$ For $\mathrm{L}=\square$ we have $\Phi_{u}$ of the form:

$$
\frac{\frac{\Phi_{u_{1}} \triangleright \Gamma_{u_{1}} \vdash u_{1}: \mathcal{A}_{1} \quad \Phi_{u_{2}} \triangleright \Gamma_{u_{2}} \vdash u_{2}: \mathcal{A}_{2}}{\Gamma_{u} \vdash\left\langle u_{1}, u_{2}\right\rangle: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}}{\Gamma_{u} \vdash\left\langle u_{1}, u_{2}\right\rangle: \mathcal{A}}
$$

where $\Gamma_{u}=\Gamma_{u_{1}} \wedge \Gamma_{u_{2}}$ and $\mathbf{s z}\left(\Phi_{u}\right)=\mathbf{s z}\left(\Phi_{u_{1}}\right)+\mathbf{s z}\left(\Phi_{u_{2}}\right)+1$. We first construct the following derivation:

$$
\frac{\Phi_{v} \triangleright \Gamma_{v} \vdash v:\left.\sigma \quad \Pi_{p_{1}} \triangleright \Gamma_{v}\right|_{p_{1}} \vdash p_{1}: \mathcal{A}_{1} \quad \Phi_{u_{1}} \triangleright \Gamma_{u_{1}} \vdash u_{1}: \mathcal{A}_{1}}{\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right) \wedge \Gamma_{u_{1}} \vdash v\left[p_{1} \backslash u_{1}\right]: \sigma}
$$

By using Lem. 4 and $\alpha$-conversion, we construct a derivation $\Phi^{\prime}$ with conclusion $\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right) \backslash \operatorname{var}\left(p_{2}\right) \wedge \Gamma_{u_{1}} \wedge \Gamma_{u_{2}} \vdash v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]: \sigma$. Note that $p_{1} \# p_{2}$ implies $\Gamma_{v} \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)=\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)$. Thus, we finally obtain $\mathbf{s z}\left(\Phi^{\prime}\right)=$ $\mathrm{sz}\left(\Phi_{v}\right)+\mathrm{sz}\left(\Pi_{p}\right)+\mathrm{sz}\left(\Phi_{u}\right)<\mathbf{s z}(\Phi)$.
= Let $\mathrm{L}=\mathrm{L}^{\prime}[q \backslash s]$. Then $\Phi_{u}$ is necessarily of the following form:

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \Delta_{u} \vdash \mathrm{~L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket: \times\left.\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \quad \Pi_{q} \triangleright \Delta_{u}\right|_{q} \Vdash q: \mathcal{B} \quad \Phi_{s} \triangleright \Delta_{s} \vdash s: \mathcal{B}}{\Gamma_{u} \vdash \mathrm{~L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket[q \backslash s]: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}
$$

where $\Gamma_{u}=\Delta_{u} \ \backslash \operatorname{var}(q) \wedge \Delta_{s}$.
We will apply the i.h. on the reduction step $v\left[p \backslash \mathrm{~L}^{\prime} \llbracket u \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket$, in particular we type the left-hand side term with the following derivation $\Psi_{1}$ :

$$
\frac{\Phi_{v} \quad \Pi_{p} \quad \frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \Delta_{u} \vdash \mathrm{~L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Delta_{u} \vdash \mathrm{~L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket: \mathcal{A}}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Delta_{u} \vdash v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right]: \sigma}
$$

The i.h. gives a derivation $\Psi_{2} \triangleright \Gamma_{v} \backslash \operatorname{var}(p) \wedge \Delta_{u} \vdash \mathrm{~L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket: \sigma$ verifying $\mathbf{s z}\left(\Psi_{2}\right)<\mathbf{s z}\left(\Psi_{1}\right)$. Let $\Lambda=\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Delta_{u}$. We conclude with the following derivation $\Phi^{\prime}$ :

$$
\frac{\left.\Psi_{2} \quad \Pi_{q} \triangleright \Delta_{u}\right|_{q} \Vdash q: \mathcal{B} \quad \Phi_{s} \triangleright \Delta_{s} \vdash s: \mathcal{B}}{\Lambda \ \operatorname{var}(q) \wedge \Delta_{s} \vdash \mathrm{~L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket[q \backslash s]: \sigma}
$$

Indeed, we first remark that $\left.\Lambda\right|_{q}=\left.\Delta_{u}\right|_{q}$ holds by relevance and $\alpha$-conversion. Secondly, $\Gamma_{v} \| \operatorname{var}(p) \wedge \Gamma_{u}=\Gamma_{v} \backslash \operatorname{var}(p) \wedge\left(\Delta_{u} \backslash \operatorname{var}(q)\right) \wedge \Delta_{s}=\Lambda \ \operatorname{var}(q) \wedge \Delta_{s}$ also holds by Lem. 4 and $\alpha$-conversion. Last, we have

$$
\begin{array}{rlrl}
\mathrm{sz}\left(\Phi^{\prime}\right)= & \mathrm{sz}\left(\Psi_{2}\right)+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1 & & < \\
& \mathbf{s z}\left(\Psi_{1}\right)+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1 & & = \\
& \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{\mathrm{L}^{\prime}}\right)+1+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1 & = \\
& \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{u}\right)+1 & & =\mathbf{s z}(\Phi)
\end{array}
$$

When $t=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t^{\prime}$, where $v \not_{\mathrm{h}}$, the same results hold.

- Most of the inductive cases are straightforward. We only detail here two interesting cases.
$=t=v[p \backslash u] \rightarrow_{\mathrm{p}} v\left[p \backslash u^{\prime}\right]=t^{\prime}$, where $u \rightarrow_{\mathrm{p}} u^{\prime}$. The proof holds here by the i.h. In particular, when $p=x$ and $x \notin \mathrm{fv}(v)$, then by relevance we have $x$ of type [] as well as $u$ of type []. This means that both $u$ and $u^{\prime}$ are typed by a (many) rule with no premise, and in that case we get $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi^{\prime}\right)$.
$=t=v[p \backslash u] \rightarrow_{\mathrm{h}} v\left[p \backslash u^{\prime}\right]=t^{\prime}$, where $v \nrightarrow_{\mathrm{h}}$ and $p \neq x$ and $u \rightarrow_{\mathrm{h}} u^{\prime}$. By construction there are typing subderivations $\Phi_{v} \triangleright \Gamma_{v} \vdash v: \sigma,\left.\Pi_{p} \triangleright \Gamma_{v}\right|_{p} \Vdash p: \mathcal{A}$ and $\Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$ such that $\Gamma=\Gamma_{v} \| \operatorname{var}(p) \wedge \Gamma_{u}$. Since $p$ is not a variable then $\Pi_{p}$ ends with rule pat $\times$. In which case $\mathcal{A}$ contains exactly one type, let us say $\mathcal{A}=\left[\sigma_{u}\right]$. Then $\Phi_{u}$ has the following form

$$
\frac{\Gamma_{u} \vdash u: \sigma_{u}}{\Phi_{u} \triangleright \Gamma_{u} \vdash u:\left[\sigma_{u}\right]}
$$

The i.h. applied to the premise of $\Phi_{u}$ gives a derivation $\Gamma_{u} \vdash u^{\prime}: \sigma_{u}$ and having the expected size relation. To conclude we build a type derivation $\Phi^{\prime}$ for $v\left[p \backslash u^{\prime}\right]$ having the expected size relation.
2. By induction on $\rightarrow_{\mathrm{p}}$ (resp. $\rightarrow_{\mathrm{h}}$ ) and the anti-substitution property (Lem. 23).

- $t^{\prime}=\mathrm{L} \llbracket \lambda p . v \rrbracket u \rightarrow_{\mathrm{p} / \mathrm{h}} \mathrm{L} \llbracket v[p \backslash u \rrbracket \rrbracket=t$. The proof is by induction on the list L . We consider the case $L=\square$, since the other case follows straightforward by i.h. The typing derivation $\Phi$ is necessarily of the form:

$$
\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \vdash p: \mathcal{A} \quad \Gamma_{u} \vdash u: \mathcal{A}}{\Gamma_{v} \backslash \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash v[p \backslash u]: \sigma}
$$

We then construct the following derivation $\Phi^{\prime}$ :

$$
\frac{\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash p: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \vdash \lambda p . v: \mathcal{A} \rightarrow \sigma} \quad \Gamma_{u} \vdash u: \mathcal{A}}{\Gamma_{v} \ \operatorname{var}(p) \wedge \Gamma_{u} \vdash(\lambda p . v) u: \sigma}
$$

Moreover, sz $\left(\Phi^{\prime}\right)=\mathbf{s z}(\Phi)+1$.
$=t^{\prime}=v[x \backslash u] \rightarrow_{\mathrm{p}} v\{x \backslash u\}=t$. Then by Lem. 23, there exist derivations $\Phi_{v}, \Phi_{u}$, contexts $\Gamma_{v}, \Gamma_{u}$ and a multitype $\mathcal{A}$, such that $\Phi_{v} \triangleright \Gamma_{v} ; x: \mathcal{A} \vdash v: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}, \Gamma=\Gamma_{v} \wedge \Gamma_{u}$, and $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|\mathcal{A}|$. Furthermore, one has $\Pi_{x} \triangleright x: \mathcal{A} \Vdash x: \mathcal{A}$ Then one can construct the following derivation $\Phi^{\prime}$.

$$
\frac{\Gamma_{v} ; x: \mathcal{A} \vdash v: \sigma \quad x: \mathcal{A} \Vdash x: \mathcal{A} \quad \Gamma_{u} \vdash u: \mathcal{A}}{\Gamma_{v} \wedge \Gamma_{u} \vdash v[x \backslash u]: \sigma}
$$

Furthermore, sz $\left(\Phi^{\prime}\right)=\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{x}\right)+\mathbf{s z}\left(\Phi_{u}\right)>\mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Phi_{u}\right)-|A|$, since $|\mathcal{A}| \geq 0$ and $\mathbf{s z}\left(\Pi_{x}\right)=1$. The same result holds for $t=v[x \backslash u] \rightarrow_{\mathrm{h}} v\{x \backslash u\}=t^{\prime}$, where $v \nrightarrow h_{h}$.
$=t^{\prime}=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t$. Let us write $p=\left\langle p_{1}, p_{2}\right\rangle$ and $u=\left\langle u_{1}, u_{2}\right\rangle$. The proof is by induction on the list L.
$=\mathrm{L}=\square$, then the typing derivation $\Phi$ is necessarily of the form:

$$
\frac{\left.\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p_{1}} \vdash p_{1}: \mathcal{A}_{1} \Gamma_{1} \vdash u_{1}: \mathcal{A}_{1}}{\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1} \vdash v\left[p_{1} \backslash u_{1}\right]: \sigma}\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right)\right|_{p_{2}} \Vdash p_{2}: \mathcal{A}_{2} \quad \Gamma_{2} \vdash u_{2}: \mathcal{A}_{2}}{\left(\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)\right) \wedge \Gamma_{2} \vdash v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]: \sigma}
$$

where $\Gamma=\left(\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)\right) \wedge \Gamma_{2}$. Moreover, the following equality holds $\left(\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)\right)=\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \backslash \operatorname{var}\left(p_{2}\right) \wedge \Gamma_{1} \backslash \operatorname{var}\left(p_{2}\right)\right)$, since $\left(\Gamma_{v} \ \operatorname{var}\left(p_{1}\right)\right) \backslash \operatorname{var}\left(p_{2}\right)=\Gamma_{v} \backslash \operatorname{var}(p)$ and $\Gamma_{1} \| \operatorname{var}\left(p_{2}\right)=L_{L .4} \Gamma_{1}$. Similarly, $\left.\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right)\right|_{p_{2}}=\left.L .4\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right)\right|_{p_{2}}$ and, by linearity of patterns, we have $\left.\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right)\right|_{p_{2}}=\left.\Gamma_{v}\right|_{p_{2}}$. Hence, we conclude with the following derivation $\Phi^{\prime}$ :

$$
\frac{\Gamma_{v} \vdash v: \sigma}{} \frac{\left.\Gamma_{v}\right|_{p_{1}} \Vdash p_{1}:\left.\mathcal{A}_{1} \quad \Gamma_{v}\right|_{p_{2}} \Vdash p_{2}: \mathcal{A}_{2}}{\left.\Gamma_{v}\right|_{p} \Vdash p:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]} \quad \frac{\frac{\Gamma_{1} \vdash u_{1}: \mathcal{A}_{1} \quad \Gamma_{2} \vdash u_{2}: \mathcal{A}_{2}}{\Gamma_{1} \wedge \Gamma_{2} \vdash u: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}}{\Gamma_{1} \wedge \Gamma_{2} \vdash u:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

Furthermore,

$$
\begin{array}{rll}
\mathrm{sz}(\Phi)= & \mathrm{sz}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p_{1}}\right)+\mathbf{s z}\left(\Phi_{u_{1}}\right)+1+\mathbf{s z}\left(\Pi_{p_{2}}\right)+\mathbf{s z}\left(\Phi_{u_{2}}\right)+1 & = \\
& \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p_{1}}\right)+\mathbf{s z}\left(\Pi_{p_{2}}\right)+1+\mathbf{s z}\left(\Phi_{u_{1}}\right)+\mathbf{s z}\left(\Phi_{u_{2}}\right)+1 & < \\
& \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{u}\right)+1 & =\mathbf{s z}\left(\Phi^{\prime}\right)
\end{array}
$$

$=$ If $\mathrm{L}=\mathrm{L}^{\prime}[q \backslash s]$, then $t^{\prime}=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime}[q \backslash s] \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L}^{\prime}[q \backslash s] \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=$ $\mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket[q \backslash s]=t$, and $\Phi$ is of the form:

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \Gamma_{\mathrm{L}^{\prime}} \vdash \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right\rceil \rrbracket:\left.\sigma \quad \Pi_{q} \triangleright \Gamma_{\mathrm{L}^{\prime}}\right|_{q} \vdash q: \mathcal{A} \quad \Phi_{s} \triangleright \Gamma_{s} \vdash s: \mathcal{A}}{\Gamma_{\mathrm{L}^{\prime}} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash \mathrm{~L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket[q \backslash s]: \sigma}
$$

From $v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket$ and $\Phi_{\mathrm{L}^{\prime}}$ by the i.h. one gets $\Phi_{\mathrm{L}^{\prime}}^{\prime} \triangleright \Gamma_{\mathrm{L}^{\prime}} \vdash v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right]: \sigma$ with $\mathbf{s z}\left(\Phi_{\mathrm{L}^{\prime}}^{\prime}\right)>\mathbf{s z}\left(\Phi_{\mathrm{L}^{\prime}}\right)$. Furthermore $\Phi_{\mathrm{L}^{\prime}}^{\prime}$ is necessarily of the form:

$$
\frac{\Phi_{v} \triangleright \Gamma_{v} \vdash v:\left.\sigma \quad \Pi_{p} \triangleright \Gamma_{v}\right|_{p} \Vdash p:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right] \quad \frac{\Phi_{u} \triangleright \Gamma_{u} \vdash \mathrm{~L}^{\prime} \llbracket u \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Gamma_{u} \vdash \mathrm{~L}^{\prime} \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash v\left[p \backslash\left\llcorner^{\prime} \llbracket u \rrbracket\right]: \sigma\right.}
$$

Then one can construct the following derivation $\Phi_{u}^{\prime}$ :

$$
\frac{\frac{\Gamma_{u} \vdash \mathrm{~L}^{\prime} \llbracket u \rrbracket: \times\left.\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \quad \Gamma_{u}\right|_{q} \Vdash q: \mathcal{A} \quad \Phi_{s}}{\Gamma_{u} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash \mathrm{~L}^{\prime}[q \backslash s] \llbracket u \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}}{\Gamma_{u} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash \mathrm{~L}^{\prime}[q \backslash s] \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

From which we build $\Phi^{\prime}$ :

$$
\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash p:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right] \quad \Gamma_{u} \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash \mathrm{~L}^{\prime}[q \backslash s] \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash v\left[p \backslash \mathrm{~L}^{\prime}[q \backslash s] \llbracket u \rrbracket\right]: \sigma}
$$

With $\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \| \operatorname{var}(q) \wedge \Gamma_{s}=\left(\Gamma_{v} \| \operatorname{var}(p) \wedge \Gamma_{u}\right) \backslash \operatorname{var}(q) \wedge \Gamma_{s}=\Gamma$. Furthermore

$$
\begin{array}{rlrl}
\mathrm{sz}(\Phi)= & \mathrm{sz}\left(\Phi_{\mathrm{L}^{\prime}}\right)+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1 & & i_{i . h .} \\
& \mathbf{s z}\left(\Phi_{\mathrm{L}^{\prime}}^{\prime}\right)+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1 & & = \\
& & \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{u}\right)+\mathbf{s z}\left(\Pi_{q}\right)+\mathbf{s z}\left(\Phi_{s}\right)+1+1 & \\
& & \mathbf{s z}\left(\Phi_{v}\right)+\mathbf{s z}\left(\Pi_{p}\right)+\mathbf{s z}\left(\Phi_{u}^{\prime}\right)+1 & \\
=\mathbf{s z}\left(\Phi^{\prime}\right)
\end{array}
$$

The same result holds for $t^{\prime}=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t$, where $v \not f_{h}$.
= Most of the inductive cases are straightforward. We only detail here two interesting cases.
= $t^{\prime}=v\left[p \backslash u^{\prime}\right] \rightarrow_{\mathrm{p}} v[p \backslash u]=t$, where $u^{\prime} \rightarrow_{\mathrm{p}} u$. The proof holds by the i.h. In particular, when $p=x$ and $x \notin \mathrm{fv}(v)$, then by relevance we have $x$ of type [] as well as $u$ of type []. This means that $u, u^{\prime}$ are typed by a (many) rule with no premise, and in that case we get $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi^{\prime}\right)$.
$=t^{\prime}=v\left[p \backslash u^{\prime}\right] \rightarrow_{\mathrm{h}} v[p \backslash u]=t$, where $v \not_{\mathrm{h}}$ and $p \neq x$ and $u^{\prime} \rightarrow_{\mathrm{h}} u$. By construction there are subderivations $\Phi_{v} \triangleright \Gamma_{v} \vdash v: \sigma,\left.\Pi_{p} \triangleright \Gamma_{v}\right|_{p} \Vdash p: \mathcal{A}$ and $\Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A}$ for some multiset $\mathcal{A}$ and $\Gamma=\left(\Gamma_{v} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u}$. Since $p$ is not a variable then $\Pi_{p}$ ends with rule (pat ${ }_{\times}$), in which case $\mathcal{A}$ contains only one type, let us say $\mathcal{A}=\left[\sigma_{u}\right]$. Then $\Phi_{u}$ has the following form:

$$
\Phi_{u} \triangleright \frac{\Gamma_{u} \vdash u: \sigma_{u}}{\Gamma_{u} \vdash u^{\prime}:\left[\sigma_{u}\right]}
$$

The i.h. applied to the premise of $\Phi_{u}$ gives a derivation $\Gamma_{u} \vdash u^{\prime}: \sigma_{u}$. Therefore, we construct the following derivation $\Phi^{\prime}$ :

$$
\frac{\Gamma_{v} \vdash v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash p:\left[\sigma_{u}\right] \quad \Gamma_{u} \vdash u^{\prime}: \sigma_{u}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash v\left[p \backslash u^{\prime}\right]: \sigma}
$$

Furthermore,

$$
\begin{aligned}
\mathrm{sz}(\Phi)= & \mathrm{sz}\left(\Phi_{v}\right)+\mathrm{sz}\left(\Pi_{p}\right)+\mathrm{sz}\left(\Phi_{u}\right)+1 \quad<_{i . h .} \\
& \mathrm{sz}\left(\Phi_{v}\right)+\mathrm{sz}\left(\Pi_{p}\right)+\mathrm{sz}\left(\Phi_{u^{\prime}}\right)+1=\mathrm{sz}\left(\Phi^{\prime}\right)
\end{aligned}
$$

- Lemma 6 (Clash-Free). Let $\Phi \triangleright \Gamma \vdash t: \sigma$. Then $t$ is (head) clash-free.

Proof. Let $\Phi \triangleright \Gamma \vdash t: \sigma$. By induction on $\mathrm{sz}(\Phi)$, using the syntax-directed aspect of system $\mathscr{U}$.

- The base case for rule (ax) is trivial.
- The cases for rules (many) and (abs) are straightforward from the i.h.
- The case for (pair) is also straightforward since every pair is (head) clash-free.
- Let us consider the case for (match), where $\Phi$ has the following form:

$$
\frac{\Gamma_{t} \vdash t:\left.\sigma \quad \Gamma_{t}\right|_{p} \Vdash p: \mathcal{A} \quad \Delta \vdash u: \mathcal{A}}{\left(\Gamma_{t} \backslash \operatorname{var}(p)\right) \wedge \Delta \vdash t[p \backslash u]: \sigma}(\text { match })
$$

- If $t \rightarrow_{\mathrm{h}} t^{\prime}$ for some $t^{\prime}$, so that $t[p \backslash u] \rightarrow_{\mathrm{h}} t^{\prime}[p \backslash u]$, then the size of the typing derivation of $t^{\prime}[p \backslash u]$ is smaller than that of $\Phi$ by Upper Subject Reduction. The i.h. then gives $t^{\prime}[p \backslash u]$ (head) clash-free and thus $t[p \backslash u]$ is (head) clash-free.
= If $t \nrightarrow_{\mathrm{h}}$ then there are two cases.
* If $p$ is a variable $x$, then $t[x \backslash u] \rightarrow_{\mathrm{h}} t\{x \backslash u\}$ and by Upper Subject Reduction $t\{x \backslash u\}$ has a type derivation strictly smaller than that of $\Phi$, thus by the i.h. $t\{x \backslash u\}$ is (head) clash-free and so is $t[x \backslash u]$.
* Otherwise $p$ is a pair, so that $\mathcal{A} \neq[]$ (i.e. $\mathcal{A}=\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]$ ).
- If $u \rightarrow_{\mathrm{h}} u^{\prime}$ then $t[p \backslash u] \rightarrow_{\mathrm{h}} t\left[p \backslash u^{\prime}\right]$, so that the size of the typing derivation of $t\left[p \backslash u^{\prime}\right]$ is smaller than that of $\Phi$ by Upper Subject Reduction. The i.h. then gives $t\left[p \backslash u^{\prime}\right]$ (head) clash-free and thus $t[p \backslash u]$ is (head) clash-free.
- If $u\rangle_{\mathrm{h}}$ then $t[p \backslash u]$ is a head-normal form. In order to guarantee that $t[p \backslash u]$ is (head) clash-free note that $u$ cannot be of the form $\mathrm{L} \llbracket \lambda q \cdot v \rrbracket$, which can only be typed with a multiset of functional types.
- Let us consider the case for rule (app), where $\Phi$ has the following form

$$
\frac{\Phi_{u} \triangleright \Gamma_{u} \vdash u: \mathcal{A} \rightarrow \sigma \quad \Phi_{v} \triangleright \Gamma_{v} \vdash v: \mathcal{A}}{\Gamma_{u} \wedge \Gamma_{v} \vdash u v: \sigma}
$$

Note that $u$ cannot be of the form $L \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket$ because it is typed with a functional type, thus it is either $\mathrm{L} \llbracket x \rrbracket$ or $\mathrm{L} \llbracket \lambda p . u^{\prime} \rrbracket$.
If $u$ is $L \llbracket x \rrbracket$, then $u$ is (head) clash-free by the i.h. and thus $u v$ is necessarily (head) clash-free.
If $u$ is $\mathrm{L} \llbracket \lambda p \cdot u^{\prime} \rrbracket$ then $t=\mathrm{L} \llbracket \lambda p \cdot u^{\prime} \rrbracket v \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket u^{\prime}\left[p \backslash v \rrbracket \rrbracket=t^{\prime}\right.$ and the size of the type derivation of $t^{\prime}$ is strictly smaller than the size of $\Phi$ by Upper Subject Reduction. The i.h. gives $t^{\prime}$ (head) clash-free, and thus $t$ is also (head) clash-free.

## B Soundness of System $\mathscr{E}$

- Lemma 12 (Tight Spreading). Let $t \in \mathcal{N}$. Let $\Phi \triangleright \Gamma^{(b, e, m, f)} t: \sigma$ be a typing derivation such that tight $(\Gamma)$. Then $\sigma$ is tight and the last rule of $\Phi$ does not belong to $\left\{a p p, a b s, \mathrm{abs}_{\mathrm{p}}\right.$, pair, pair $\left.\mathrm{p}_{\mathrm{p}}\right\}$.

Proof. First note that, since $t \in \mathcal{N}$, then $t$ is not an abstraction nor a pair, therefore one cannot apply any of the rules $\left\{\right.$ abs, abs $_{p}$, pair, pair $r_{p}$, emptypair\}. We now examine the remaining rules.

- $t=x$. Then $\Phi$ is an axiom $x:[\mathrm{t}] \vdash^{(0,0,0,0)} x: \mathrm{t}$ so the property trivially holds.
- $t=u v$, with $u \in \mathcal{N}$. Then $\Phi$ has a (left) subderivation $\Phi_{u} \triangleright \Gamma_{u} \vdash^{(b, e, m, f)} u: \sigma_{u}$, and since $\Gamma_{u} \subseteq \Gamma$, then $\Gamma_{u}$ is necessarily tight. Therefore, by the i.h., $\sigma_{u}=\bullet_{\mathcal{N}}$, from which follows that $\sigma=\bullet_{\mathcal{N}}$ by applying rule ( $\mathrm{app}_{\mathrm{p}}$ ). Note that one cannot apply rule (app) to type $u v$, since $t$ would have to be an arrow type, which contradicts the i.h.
- $t=u[p \backslash v]$, with $u \in \mathcal{N}, v \in \mathcal{N}$. Then $\Phi$ follows from $\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \sigma$, $\left.\Phi_{p} \triangleright \Gamma_{u}\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}$ and $\Phi_{v} \triangleright \Delta \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \mathcal{A}$, where $\Gamma=\left(\Gamma_{u} \backslash \operatorname{var}(p)\right) \wedge \Delta$. Since $\left(\Gamma_{u} \backslash \operatorname{var}(p)\right) \wedge \Delta$ is tight, then $\Delta$ is tight. By the i.h. on $v$ one gets a derivation $\Delta \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \bullet_{\mathcal{N}}$ so that $\Delta \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left[\bullet_{\mathcal{N}}\right]$ follows from many and $\left.\Gamma_{u}\right|_{p} \Vdash^{(0,0,1)}$ $p:\left[\bullet_{\mathcal{N}}\right]$ necessarily follows from rule ( pat $_{\mathrm{p}}$ ). This implies $\left.\Gamma_{u}\right|_{p}$ is tight, therefore $\Gamma_{u}$ is tight. Since $u \in \mathcal{N}$ the i.h. gives $\sigma \in \mathrm{t}$ as expected.
- Lemma 13 (Canonical Forms and Minimal Counters). Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$ be a tight derivation. Then $t \in \mathcal{M}$ if and only if $b=e=m=0$.

Proof. By induction on $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$, where $\Phi$ is tight (right-to-left implication), and by induction on $t \in \mathcal{M}$ (left-to-right implication). The latter is presented below.

- $t=\lambda p$. $u$, with $u \in \mathcal{M}$. Then $\Phi$ cannot end with rule (abs) because $\sigma$ is tight. The last rule of $\Phi$ is necessarily $\left(\mathrm{abs}_{\mathrm{p}}\right)$. The i.h. then applies and gives $b=e=m=0$ and $f-1=|u|$. We conclude since $f=|u|+1=|t|$.
- $t=\left\langle t_{1}, t_{2}\right\rangle$. Then $\Phi$ necessarily ends with rule ( pair $_{\mathrm{p}}$ ) and the counters are as required.
- $t=u[p \backslash v]$, with $u \in \mathcal{M}, v \in \mathcal{N}$. Then $\Phi$ ends with rule (match), so that $u$ (resp. $v$ ) is typable with some context $\Gamma_{u}\left(\right.$ resp. $\left.\Gamma_{v}\right)$, where $\Gamma=\left(\Gamma_{u} \| \operatorname{var}(p)\right) \wedge \Gamma_{v}$. Let us consider the type $\mathcal{A}$ of $u$ in the premise of rule (match). Since $\Gamma_{v}$ is tight and $v \in \mathcal{N}$, then Lem. 12 guarantees that every type of $v$ in $\mathcal{A}$ is tight, and every counter typing $v$ is of the form $(0,0,0,|v|)$. This same multitype $\mathcal{A}$ types the pattern $p$, so that there are in principle two cases:
= Either $p$ is a variable typable with rule ( $\mathrm{pat}_{\mathrm{v}}$ ), but then $t \notin \mathcal{M}$ since $t$ is still reducible. Contradiction.
- Or $p$ is typable with rule $\left(\right.$ pat $\left._{\mathrm{p}}\right)$, so that its counter is $(0,0,1)$, its type is $\left[\bullet_{\mathcal{N}}\right]$ and its context is $\left.\Gamma_{u}\right|_{p}$ necessarily tight by definition of rule (pat ${ }_{p}$ ).
Since $\Gamma_{u} \backslash \operatorname{var}(p)$ is tight by hypothesis, then the whole context $\Gamma_{u}$ is tight. We can then apply the i.h. to $u$ and obtain counters for $u$ of the form $b_{u}=e_{u}=m_{u}=0$ and $f_{u}=|u|$. On the other side, since the type of $p$ is $[\bullet \mathcal{N}]\left(\right.$ rule pat $\left._{\mathrm{p}}\right)$, there is only one premise to type $v$, which is necessarily of the form $\Delta \vdash^{(0,0,0,|v|)} v: \bullet_{\mathcal{N}}$. We then conclude that the counters typing $u[p \backslash v]$ are $b=e=m=0$ and $f=f_{u}+f_{v}+1=|u|+|v|+1=|t|$, as required.
- $t \in \mathcal{N}$. We have three different cases.
- $t=x$. This case is straightforward.
= $t=u v$, with $u \in \mathcal{N}$. Since $\Phi$ is tight, then $\Gamma$ is tight and we can apply Lem. 12. Then $\Phi$ necessarily ends with rule ( $\mathrm{app}_{\mathrm{p}}$ ). The i.h. then applies to the premise typing $u$, thus giving counters $b=e=m=0$ and $f-1=|u|$. We conclude since $f=|u|+1=|t|$.
- $t=u[p \backslash v]$, with $u \in \mathcal{N}, v \in \mathcal{N}$. This case is similar to the third case.

Lemma 14 (Substitution for System $\mathscr{E}$ ). If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma$, and $\Phi_{u} \triangleright$ $\Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t\{x \backslash u\}: \sigma$.

Proof. We generalise the statement as follows: Let $\Phi_{u} \triangleright \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$.

- If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma$, then there exists
$\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t\{x \backslash u\}: \sigma$.
- If $\Phi_{t} \triangleright \Gamma ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \mathcal{B}$, then there exists
$\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t\{x \backslash u\}: \mathcal{B}$.
The proof then follows by induction on $\Phi_{t}$.
- If $\Phi_{t}$ is (ax), then we consider two cases:
$=t=x$ : then $\Phi_{x} \triangleright x:\left[\sigma^{\prime}\right] \vdash^{(0,0,0,0)} x: \sigma^{\prime}$ and $\Phi_{u} \triangleright \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u:\left[\sigma^{\prime}\right]$, which is a consequence of $\Delta \vdash\left(b_{u}, e_{u}, m_{u}, f_{u}\right) u: \sigma^{\prime}$. Then $x\{x \backslash u\}=u$, and we trivially obtain $\Phi_{t\{x \backslash u\}} \triangleright \Delta \vdash^{\left(0+b_{u}, 0+e_{u}, 0+m_{u}, 0+f_{u}\right)} u: \sigma^{\prime}$.
$=t=y$ : then $\Phi_{y} \triangleright y:[\sigma] ; x:[] \vdash^{(0,0,0,0)} y: \sigma$ and $\Phi_{u} \triangleright \emptyset \vdash^{(0,0,0,0)} u:[]$. Then $y\{x \backslash u\}=y$, and we trivially obtain $\Phi_{t\{x \backslash u\}} \triangleright y:[\sigma] \vdash \vdash^{(0+0,0+0,0+0,0+0)} y: \sigma$.
- If $\Phi_{t}$ ends with (many), then it has premises $\left(\Phi_{t}^{i} \triangleright \Gamma^{i} ; x: \mathcal{A}^{i} \vdash\left(b_{t}^{i}, e_{t}^{i}, m_{t}^{i}, f_{t}^{i}\right) t: \sigma^{i}\right)_{i \in I}$, where $\Gamma=\wedge_{i \in I} \Gamma^{i}, \mathcal{A}=\wedge_{i \in I} \mathcal{A}^{i}, b_{t}=+_{i \in I} b_{t}^{i}, e_{t}=+_{i \in I} e_{t}^{i}, m_{t}=+_{i \in I} m_{t}^{i}, f_{t}=+_{i \in I} f_{t}^{i}$ and $\mathcal{B}=\left[\sigma^{i}\right]_{i \in I}$. The derivation $\Phi_{u}$ can also be decomposed into several subderivations $\left(\Phi_{u}^{i} \triangleright \Delta^{i} \vdash\left(b_{u}^{i}, e_{u}^{i}, m_{u}^{i}, f_{u}^{i}\right) u: \mathcal{A}^{i}\right)_{i \in I}$, where $b_{u}=+_{i \in I} b_{u}^{i}, e_{u}=+_{i \in I} e_{u}^{i}, m_{u}=+_{i \in I} m_{u}^{i}$, $f_{u}=+_{i \in I} f_{u}^{i}, \Delta=\wedge_{i \in I} \Delta^{i}$. We can apply the i.h. and we thus obtain derivations $\left(\Phi_{t\{x \backslash u\}}^{i} \triangleright \Gamma^{i} \wedge \Delta^{i} \vdash\left(b_{t}^{i}+b_{u}^{i}, e_{t}^{i}+e_{u}^{i}, m_{t}^{i}+m_{u}^{i}, f_{t}^{i}+f_{u}^{i}\right) t\{x \backslash u\}: \sigma^{i}\right)_{i \in I}$. Then we apply rule (many) to get $\Phi_{t\{x \backslash u\}} \triangleright \Gamma \wedge \Delta \vdash \vdash^{\left(b_{t}+b_{u}, e_{t}+e_{u}, m_{t}+m_{u}, f_{t}+f_{u}\right)} t\{x \backslash u\}: \mathcal{B}$.
- If $\Phi_{t}$ ends with (app), so that $t=t^{\prime} u^{\prime}$, then

$$
\Phi_{t^{\prime} u^{\prime}} \triangleright \Gamma_{t^{\prime}} \wedge \Gamma_{u^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}} \vdash^{\left(b_{t^{\prime}}+b_{u^{\prime}}, e_{t^{\prime}}+e_{u^{\prime}}, m_{t^{\prime}}+m_{u^{\prime}}, f_{t^{\prime}}+f_{u^{\prime}}\right)} t^{\prime} u^{\prime}: \sigma
$$

which follows from the two term premises $\Gamma_{t^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \vdash^{\left(b_{t^{\prime}}, e_{t^{\prime}}, m_{t^{\prime}}, f_{t^{\prime}}\right)} t^{\prime}: \mathcal{B} \rightarrow \sigma$ and $\Gamma_{u^{\prime}} ; x: \mathcal{A}_{u^{\prime}} \vdash^{\left(b_{u^{\prime}}, e_{u^{\prime}}, m_{u^{\prime}}, f_{u^{\prime}}\right)} u^{\prime}: \mathcal{B}$. Also, $\Phi_{u} \triangleright \Delta \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$ is a consequence of $\left(\Delta_{k} \vdash^{\left(b_{u}^{k}, e_{u}^{k}, m_{u}^{k}, f_{u}^{k}\right)} u: \sigma_{k}\right)_{k \in K}$, with $\mathcal{A}=\left[\sigma_{k}\right]_{k \in K}, \Delta=\wedge_{k \in K} \Delta_{k}$ and $b_{u}=+_{k \in K} b_{u}^{k}$, $e_{u}=+_{k \in K} e_{u}^{k}, m_{u}=+_{k \in K} m_{u}^{k}$ and $f_{u}=+_{k \in K} f_{u}^{k}$. Note on the other hand that $\mathcal{A}=\mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}}=\left[\sigma_{i}\right]_{i \in K_{t^{\prime}}} \wedge\left[\sigma_{i}\right]_{i \in K_{u^{\prime}}}$, with $K=K_{t^{\prime}} \uplus K_{u^{\prime}}$, from which one can obtain (using the many rule):
$=\Delta_{t^{\prime}} \vdash^{\left(B_{t^{\prime}}, E_{t^{\prime}}, M_{t^{\prime}}, F_{t^{\prime}}\right)} u: \mathcal{A}_{t^{\prime}}$

- $\Delta_{u^{\prime}} \vdash^{\left(B_{u^{\prime}}, E_{u^{\prime}}, M_{u^{\prime}}, F_{u^{\prime}}\right)} u: \mathcal{A}_{u^{\prime}}$
where $b_{u}=B_{t^{\prime}}+B_{u^{\prime}}=\left(+_{i \in K_{t^{\prime}}} b_{u}^{i}\right)+\left(+_{i \in K_{u^{\prime}}}, b_{u}^{i}\right), e_{u}=E_{t^{\prime}}+E_{u^{\prime}}=\left(+_{i \in K_{t^{\prime}}} e_{u}^{i}\right)+\left(+_{i \in K_{u^{\prime}}} e_{u}^{i}\right)$, $m_{u}=M_{t^{\prime}}+M_{u^{\prime}}=\left(+_{i \in K_{t^{\prime}}} m_{u}^{i}\right)+\left(+_{i \in K_{u^{\prime}}} m_{u}^{i}\right), f_{u}=F_{t^{\prime}}+F_{u^{\prime}}=\left(+_{i \in K_{t^{\prime}}} f_{u}^{i}\right)+\left(+_{i \in K_{u^{\prime}}} f_{u}^{i}\right)$. By the i.h. we have:

$$
\begin{aligned}
& \Gamma_{t^{\prime}} \wedge \Delta_{t^{\prime}} \vdash^{\left(b_{t^{\prime}}+B_{t^{\prime}}, e_{t^{\prime}}+E_{t^{\prime}}, m_{t^{\prime}}+M_{t^{\prime}}, f_{t^{\prime}}+F_{t^{\prime}}\right)} t^{\prime}\{x \backslash u\}: \mathcal{B} \rightarrow \sigma \\
& \Gamma_{u^{\prime}} \wedge \Delta_{u^{\prime}} \vdash^{\left(b_{u^{\prime}}+B_{u^{\prime}}, e_{u^{\prime}}+E_{u^{\prime}}, m_{u^{\prime}}+M_{u^{\prime}}, f_{u^{\prime}}+F_{u^{\prime}}\right)} u^{\prime}\{x \backslash u\}: \mathcal{B}
\end{aligned}
$$

Finally, applying the app rule we obtain:

$$
\Gamma_{t^{\prime}} \wedge \Gamma_{u^{\prime}} \wedge \Delta_{t^{\prime}} \wedge \Delta_{u^{\prime}} \vdash^{(b, e, m, f)}\left(t^{\prime}\{x \backslash u\}\right)\left(u^{\prime}\{x \backslash u\}\right): \sigma
$$

with $b=b_{t^{\prime}}+b_{u^{\prime}}+b_{u}, e=e_{t^{\prime}}+e_{u^{\prime}}+e_{u}, m=m_{t^{\prime}}+m_{u^{\prime}}+m_{u}$ and $f=f_{t^{\prime}}+f_{u^{\prime}}+f_{u}$, as expected.

- If $\Phi_{t}$ ends with (abs), $\left(\mathrm{abs}_{\mathrm{p}}\right),\left(\mathrm{app}_{\mathrm{p}}\right)$ or (match) the result follows from the i.h. by assuming $\alpha$-conversion whenever necessary.
- If $\Phi_{t}$ ends with (pair) or (pair ${ }_{\mathrm{p}}$ ), so that $t=\left\langle t^{\prime}, u^{\prime}\right\rangle$, then we have two cases. The case for pair ${ }_{\mathrm{p}}$, follows from $\Phi_{t}$ being of the form $x:[] \vdash^{(0,0,0,1)}\left\langle t^{\prime}, u^{\prime}\right\rangle: \bullet \mathcal{M}$, which implies $\Phi_{u} \triangleright \vdash^{(0,0,0,0)} u:[]$. Therefore $\emptyset \vdash^{(0+0,0+0,0+0,1+0)}\left\langle t^{\prime}\{x \backslash u\}, u^{\prime}\{x \backslash u\}\right\rangle: \bullet \mathcal{M}$ holds. The case for pair follows by induction following the same reasoning used in rule app.
- Lemma 15 (Exact Subject Reduction). If $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$, and $t \rightarrow_{h} t^{\prime}$ is an s-step, with $s \in\{b, e, m\}$, then $\Phi^{\prime} \triangleright \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} t^{\prime}: \sigma$, where
- $s=b$ implies $b^{\prime}=b-1, e^{\prime}=e, m^{\prime}=m$.
- $s=e$ implies $b^{\prime}=b, e^{\prime}=e-1, m^{\prime}=m$.
- $s=m$ implies $b^{\prime}=b, e^{\prime}=e, m^{\prime}=m-1$.

Proof. By induction on $\rightarrow_{\mathrm{h}}$.

- $t=\mathrm{L} \llbracket \lambda p . v \rrbracket u \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket v\left[p \backslash u \rrbracket \rrbracket=t^{\prime}\right.$. The proof is by induction on the list context L . We only show the case of the empty list as the other one is straightforward. The typing derivation $\Phi$ is necessarily of the form

$$
\frac{\frac{\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash \vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \vdash^{\left(b_{v}+1, e_{v}+e_{p}, m_{v}+m_{p}, f_{v}+f_{p}\right)} \lambda p . v: \mathcal{A} \rightarrow \sigma} \quad \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash^{\left(b_{v}+1+b_{u}, e_{v}+e_{p}+e_{u}, m_{v}+m_{p}+m_{u}, f_{v}+f_{p}+f_{u}\right)}(\lambda p . v) u: \sigma}
$$

We then construct the following derivation $\Phi^{\prime}$ :

$$
\frac{\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A} \quad \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}}{\Gamma_{v} \ \operatorname{var}(p) \wedge \Gamma_{u} \vdash^{\left(b_{v}+b_{u}, e_{v}+e_{p}+e_{u}, m_{v}+m_{u}+m_{p}, f_{v}+f_{p}+f_{u}\right)} v[p \backslash u]: \sigma}
$$

The counters are as expected because the first one has decremented by 1.

- $t=v[x \backslash u] \rightarrow_{\mathrm{h}} v\{x \backslash u\}=t^{\prime}$, where $v \nrightarrow_{\mathrm{h}}$. Then $\Phi$ has two premises $\Gamma_{v} ; x: \mathcal{A} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)}$ $v: \sigma$ and $\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$, where $\Gamma=\Gamma_{v} \wedge \Gamma_{u}, b=b_{v}+b_{u}, e=e_{v}+e_{u}+1$, $m=m_{v}+m_{u}+0$, and $f=f_{v}+f_{u}+0$.
Lem. 14 then gives a derivation ending with $\Gamma_{v} \wedge \Gamma_{u} \vdash^{\left(b_{v}+b_{u}, e_{v}+e_{u}, m_{v}+m_{u}, f_{v}+f_{u}\right)} v\{x \backslash u\}: \sigma$. The context, type, and counters are as expected.
- $t=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t^{\prime}$, where $v \ngtr_{\mathrm{h}}$.

Let $p=\left\langle p_{1}, p_{2}\right\rangle$ and $u=\left\langle u_{1}, u_{2}\right\rangle$. The typing derivation $\Phi$ is necessarily of the form

$$
\frac{\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p} \vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A} \quad \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} \mathrm{L} \llbracket u \rrbracket: \mathcal{A}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash^{\left(b_{v}+b_{u}, e_{v}+e_{u}+e_{p}, m_{v}+m_{u}+m_{p}, f_{v}+f_{u}+f_{p}\right)} v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash L \llbracket u \rrbracket\right]: \sigma}
$$

Then $\mathcal{A}=\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]$, for some multitypes $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$, and so the pattern $\left\langle p_{1}, p_{2}\right\rangle$ is typable as follows:

$$
\frac{\left.\Gamma_{v}\right|_{p_{1}} \Vdash^{\left(e_{1}, m_{1}, f_{1}\right)} p_{1}:\left.\mathcal{A}_{1} \quad \Gamma_{v}\right|_{p_{2}} \Vdash^{\left(e_{2}, m_{2}, f_{2}\right)} p_{2}: \mathcal{A}_{2}}{\left.\Gamma_{v}\right|_{p} \Vdash^{\left(e_{1}+e_{2}, 1+m_{1}+m_{2}, f_{1}+f_{2}\right)}\left\langle p_{1}, p_{2}\right\rangle:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

where $e_{p}=e_{1}+e_{2}, m_{p}=1+m_{1}+m_{2}$ and $f_{p}=f_{1}+f_{2}$. The proof then follows by induction on list L :

- For $\mathrm{L}=\square$ we have term $u$ typable as follows:

$$
\frac{\Gamma_{1} \vdash^{\left(b_{1}^{\prime}, e_{1}^{\prime}, m_{1}^{\prime}, f_{1}^{\prime}\right)} u_{1}: \mathcal{A}_{1} \quad \Gamma_{2} \vdash^{\left(b_{2}^{\prime}, e_{2}^{\prime}, m_{2}^{\prime}, f_{2}^{\prime}\right)} u_{2}: \mathcal{A}_{2}}{\frac{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)}\left\langle u_{1}, u_{2}\right\rangle: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)}\left\langle u_{1}, u_{2}\right\rangle: \mathcal{A}}}
$$

where $\Gamma_{u}=\Gamma_{1} \wedge \Gamma_{2}$ and $\left(b_{u}, e_{u}, m_{u}, f_{u}\right)=\left(b_{1}^{\prime}+b_{2}^{\prime}, e_{1}^{\prime}+e_{2}^{\prime}, m_{1}^{\prime}+m_{2}^{\prime}, f_{1}^{\prime}+f_{2}^{\prime}\right)$.
We first construct the following derivation:

$$
\frac{\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p_{1}} \Vdash^{\left(e_{1}, m_{1}, f_{1}\right)} p_{1}: \mathcal{A}_{1} \quad \Gamma_{1} \vdash^{\left(b_{1}^{\prime}, e_{1}^{\prime}, m_{1}^{\prime}, f_{1}^{\prime}\right)} u_{1}: \mathcal{A}_{1}}{\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right) \wedge \Gamma_{1} \vdash^{\left(b_{v}+b_{1}^{\prime}, e_{v}+e_{1}^{\prime}+e_{1}, m_{v}+m_{1}^{\prime}+m_{1}, f_{v}+f_{1}^{\prime}+f_{1}\right)} v\left[p_{1} \backslash u_{1}\right]: \sigma}
$$

By using relevance and $\alpha$-conversion to assume freshness of bound variables, we can construct a derivation with conclusion

$$
\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right) \backslash \operatorname{var}\left(p_{2}\right) \wedge \Gamma_{u} \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]: \sigma
$$

where $\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)=\left(b_{v}+b_{u}, e_{v}+e_{u}+e_{1}+e_{2}, m_{v}+m_{u}+m_{1}+m_{2}, f_{v}+f_{u}+f_{1}+f_{2}\right)$.
In order to conclude we remark the following facts:

* $\Gamma_{v} \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)=\Gamma_{v} \backslash \backslash \operatorname{var}\left(p_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)$
* $b_{v}+b_{u}=b_{v}+b_{1}^{\prime}+b_{2}^{\prime}$
* $e_{v}+e_{u}+e_{p}=e_{v}+e_{1}^{\prime}+e_{2}^{\prime}+e_{1}+e_{2}$
* $m_{v}+m_{u}+m_{p}=m_{v}+m_{1}^{\prime}+m_{2}^{\prime}+1+m_{1}+m_{2}$
* $f_{v}+f_{u}+f_{p}=f_{v}+f_{1}^{\prime}+f_{2}^{\prime}+f_{1}+f_{2}$

Then the context, type and counters are as expected.
= Let $\mathrm{L}=\mathrm{L}^{\prime}[q \backslash s]$. Then $\Phi_{u}$ is necessarily of the following form:

$$
\frac{\Delta_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket: \times\left.\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \quad \Delta_{u}\right|_{q} \Vdash^{\left(e_{q}, m_{q}, f_{q}\right)} q: \mathcal{B} \quad \Delta_{s} \vdash^{\left(b_{s}, e_{s}, m_{s}, f_{s}\right)} s: \mathcal{B}}{\Gamma_{u} \vdash^{\left(b_{u}^{\prime}+b_{s}, e_{u}^{\prime}+e_{s}+e_{q}, m_{u}^{\prime}+m_{s}+m_{q}, f_{u}^{\prime}+f_{s}+f_{q}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket[q \backslash s]: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}
$$

where $\Gamma_{u}=\Delta_{u} \backslash \operatorname{var}(q) \wedge \Delta_{s}, b_{u}=b_{u}^{\prime}+b_{s}, e_{u}=e_{u}^{\prime}+e_{s}+e_{q}, m_{u}=m_{u}^{\prime}+m_{s}+m_{q}$ and $f_{u}=f_{u}^{\prime}+f_{s}+f_{q}$.
We will apply the i.h. on the reduction step $v\left[p \backslash \mathrm{~L}^{\prime} \llbracket u \rrbracket\right] \rightarrow_{\mathrm{p}} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket$, in particular we type the left-hand side term with the following derivation $\Psi_{1}$ :

$$
\frac{\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A} \quad \frac{\Delta_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Delta_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket: \mathcal{A}}}{\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Delta_{u} \vdash^{\left(b_{v}+b_{u}^{\prime}, e_{v}+e_{u}^{\prime}+e_{p}, m_{v}+m_{u}^{\prime}+m_{p}, f_{v}+f_{u}^{\prime}+f_{p}\right)} v\left[p \backslash \mathrm{~L}^{\prime} \llbracket u \rrbracket: \sigma\right.}
$$

where $b_{1}=b_{v}+b_{u}^{\prime}, e_{1}=e_{v}+e_{u}^{\prime}+e_{p}, m_{1}=m_{v}+m_{u}^{\prime}+m_{p}$ and $f_{1}=f_{v}+f_{u}^{\prime}+f_{p}$. The i.h. gives a derivation $\Psi_{2} \triangleright \Gamma_{v} \backslash \operatorname{var}(p) \wedge \Delta_{u} \vdash^{\left(b_{2}, e_{2}, m_{2}, f_{2}\right)} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket: \sigma$ where $b_{2}=b_{1}, e_{2}=e_{1}, m_{2}=m_{1}-1$ and $f_{2}=f_{1}$. Let $\Lambda=\Gamma_{v} \| \operatorname{var}(p) \wedge \Delta_{u}$. We conclude with the following derivation $\Phi^{\prime}$ :

$$
\frac{\left.\Psi_{2} \quad \Delta_{u}\right|_{q} \Vdash^{\left(e_{q}, m_{q}, f_{q}\right)} q: \mathcal{B} \quad \Delta_{s} \vdash^{\left(b_{s}, e_{s}, m_{s}, f_{s}\right)} s: \mathcal{B}}{\Lambda \backslash \operatorname{var}(q) \wedge \Delta_{s} \vdash^{\left(b_{2}+b_{s}, e_{2}+e_{s}+e_{q}, m_{2}+m_{s}+m_{q}, f_{2}+f_{s}+f_{q}\right)} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket[q \backslash s]: \sigma}
$$

Indeed, we first remark that $\left.\Lambda\right|_{q}=\left.\Delta_{u}\right|_{q}$ holds by relevance and $\alpha$-conversion. Secondly, $\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u}=\Gamma_{v} \| \operatorname{var}(p) \wedge\left(\Delta_{u} \| \operatorname{var}(q)\right) \wedge \Delta_{s}=\Lambda \ \operatorname{var}(q) \wedge \Delta_{s}$ also holds by relevance and $\alpha$-conversion. Last, we conclude with the following remarks:

$$
\begin{aligned}
& * b^{\prime}=b_{2}+b_{s}=b_{1}+b_{s}=b_{v}+b_{u}=b \\
& * e^{\prime}=e_{2}+e_{s}+e_{q}=e_{1}+e_{s}+e_{q}=e_{v}+e_{u}+e_{p}=e \\
& * m^{\prime}=m_{2}+m_{s}+m_{q}=m_{1}-1+m_{s}+m_{q}=m_{v}+m_{u}+m_{p}-1=m-1 \\
& * f^{\prime}=f_{2}+f_{s}+f_{q}=f_{1}+f_{s}+f_{q}=f_{v}+f_{u}+f_{p}=f
\end{aligned}
$$

- Most of the inductive cases are straightforward, so we only show the interesting one. Let $t=v[p \backslash u] \rightarrow_{\mathrm{h}} v\left[p \backslash u^{\prime}\right]=t^{\prime}$, where $v \rightarrow_{\mathrm{h}}$ and $p \neq x$ and $u \rightarrow_{\mathrm{h}} u^{\prime}$. By construction there are typing subderivations $\Phi_{v} \triangleright \Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \sigma,\left.\Phi_{p} \triangleright \Gamma_{v}\right|_{p} \vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}$ and $\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$. Since $p$ is not a variable then $\Phi_{p}$ ends with rule pat ${ }_{p}$ or pat ${ }_{\times}$. In both cases $\mathcal{A}$ contains only one type, let us say $\mathcal{A}=\left[\sigma_{u}\right]$. Then $\Phi_{u}$ has the following form

$$
\frac{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \sigma_{u}}{\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u:\left[\sigma_{u}\right]}
$$

The i.h. applied to the premise of $\Phi_{u}$ gives a derivation $\Gamma_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}\right)} u^{\prime}: \sigma_{u}$ and having the expected counters. To conclude we build a type derivation $\Phi^{\prime}$ for $v\left[p \backslash u^{\prime}\right]$ having the expected counters.

## C Completeness for System $\mathscr{E}$

- Lemma 17 (Canonical Forms and Tight Derivations). Let $t \in \mathcal{M}$. There exists a tight derivation $\Phi \triangleright \Gamma \vdash^{(0,0,0,|t|)} t$ : t.

Proof. We generalise the property to the two following statements, proved by structural induction on $t \in \mathcal{N}, t \in \mathcal{M}$, respectively, using relevance (Lem. 8).

- If $t \in \mathcal{N}$, then there exists a tight derivation $\Phi \triangleright \Gamma^{(0,0,0,|t|)} t: \bullet \mathcal{N}$ :
- If $t=x$, then $x:\left[\bullet_{\mathcal{N}}\right] \vdash^{(0,0,0,0)} x: \bullet_{\mathcal{N}}$ by (ax), where $|x|=0$.
- If $t=u v$ where $u \in \mathcal{N}$, then $|t|=|u|+1$ and by i.h. there is a tight derivation $\Phi_{u} \triangleright \Gamma_{u} \vdash^{(0,0,0,|u|)} u: \bullet_{\mathcal{N}}$. Then

$$
\frac{\Gamma_{u} \vdash^{(0,0,0,|u|)} u: \bullet_{\mathcal{N}}}{\Gamma_{u} \vdash^{(0,0,0,|u|+1)} u v: \bullet_{\mathcal{N}}}
$$

The result then holds for $\Gamma:=\Gamma_{u}$.

- If $t=u\left[\left\langle p_{1}, p_{2}\right\rangle \backslash v\right]$ where $u, v \in \mathcal{N}$, then $|t|=|u|+|v|+1$ and by i.h. there are tight derivations $\Phi_{u} \triangleright \Gamma_{u} \vdash^{(0,0,0,|u|)} u: \bullet_{\mathcal{N}}, \Phi_{v} \triangleright \Gamma_{v} \vdash^{(0,0,0,|v|)} v: \bullet \mathcal{N}$. Then, $\Phi_{v}^{\prime} \triangleright \Gamma_{v} \vdash^{(0,0,0,|v|)} v:[\bullet \mathcal{N}]$ and

$$
\frac{\left.\Phi_{u} \Gamma_{u}\right|_{\left\langle p_{1}, p_{2}\right\rangle} \vdash^{(0,0,1)}\left\langle p_{1}, p_{2}\right\rangle:[\bullet \mathcal{N}] \quad \Phi_{v}^{\prime}}{\left(\Gamma_{u} \backslash \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)\right) \wedge \Gamma_{v} \vdash^{(0,0,0,|u|+|v|+1)} u\left[\left\langle p_{1}, p_{2}\right\rangle \backslash v\right]: \bullet \mathcal{N}}
$$

The result then holds for $\Gamma:=\left(\Gamma_{u} \| \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)\right) \wedge \Gamma_{v}$, since by i.h. $\operatorname{tight}\left(\Gamma_{u}\right)$ and tight $\left(\Gamma_{v}\right)$ thus tight $(\Gamma)$.

- If $t \in \mathcal{M}$, then there exists a tight derivation $\Phi \triangleright \Gamma \vdash \vdash^{(0,0,0,|t|)} t: \mathrm{t}$.
- If $t \in \mathcal{N}$ then by the previous item the result holds for $t:=\bullet_{\mathcal{N}}$.
- If $t=\langle u, v\rangle$ then $|t|=1$ and $\vdash^{(0,0,0,1)}\langle u, v\rangle: \bullet_{\mathcal{M}}$ by (pair $\boldsymbol{p}_{\mathrm{p}}$ ). The result then holds for $\Gamma:=\emptyset$.
- If $t=\lambda p$. $u$ where $u \in \mathcal{M}$ then $|t|=|u|+1$ and, by i.h., there is a tight derivation $\Phi_{u} \triangleright \Gamma_{u} \vdash^{(0,0,0,|u|)} u$ : t. Then

$$
\frac{\Phi_{u} \triangleright \Gamma_{u} \vdash^{(0,0,0,|u|)} u: \mathrm{t}}{\Gamma_{u} \backslash \operatorname{var}(p) \vdash^{(0,0,0,|u|+1)} \lambda p . u: \bullet_{\mathcal{M}}}
$$

The result then holds for $\Gamma:=\Gamma_{u} \backslash \operatorname{var}(p)$. Observe that, since by i.h. $\operatorname{tight}\left(\Gamma_{u}\right)$ holds, and $\Gamma_{u} \| \operatorname{var}(p) \subseteq \Gamma_{u}$ then $\operatorname{tight}\left(\Gamma_{u} \| \operatorname{var}(p)\right)$ trivially holds.
= If $t=u\left[\left\langle p_{1}, p_{2}\right\rangle \backslash v\right]$ where $u \in \mathcal{M}$ and $v \in \mathcal{N}$ then $|t|=|u|+|v|+1$. Moreover, by the previous item there is a tight $\Phi_{v} \triangleright \Gamma_{v} \vdash^{(0,0,0,|v|)} v: \bullet_{\mathcal{N}}$ (so that $\left.\operatorname{tight}\left(\Gamma_{v}\right)\right)$ and, by i.h., there is a tight derivation $\Phi_{u} \triangleright \Gamma_{u} \vdash^{(0,0,0,|u|)} u: \mathrm{t}$. Then, $\Phi_{v}^{\prime} \triangleright \Gamma_{v} \vdash^{(0,0,0,|v|)} v:[\bullet \mathcal{N}]$ and

$$
\frac{\left.\Phi_{u}\left(\Gamma_{u}\right)\right|_{\langle p, q\rangle} \Vdash^{(0,0,1)}\langle p, q\rangle:[\bullet \mathcal{N}] \quad \Phi_{v}^{\prime}}{\left(\Gamma_{u} \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)\right) \wedge \Gamma_{v} \vdash^{(0,0,0,|u|+|v|+1)} u\left[\left\langle p_{1}, p_{2}\right\rangle \backslash v\right]: \mathrm{t}}
$$

The result then holds for $\Gamma:=\left(\Gamma_{u} \backslash \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)\right) \wedge \Gamma_{v}$, since $\operatorname{tight}\left(\Gamma_{v}\right)$ as remarked, and by i.h. $\operatorname{tight}\left(\Gamma_{u}\right)$, thus $\operatorname{tight}(\Gamma)$.

- Lemma 18 (Anti-Substitution for System $\mathscr{E}$ ). Let $\Phi \triangleright \Gamma^{(b, e, m, f)} t\{x \backslash u\}: \sigma$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, integers $b_{t}, b_{u}, e_{t}, e_{u}, m_{t}, m_{u}, f_{t}, f_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}, b=b_{t}+b_{u}, e=e_{t}+e_{u}$, $m=m_{t}+m_{u}, f=f_{t}+f_{u}$, and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.

Proof. As in the case of the substitution lemma, the proof follows by generalising the property for the two cases where the type derivation $\Phi$ assigns a type or a multiset type.

- Let $\Phi \triangleright \Gamma^{(b, e, m, f)} t\{x \backslash u\}: \sigma$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, integers $b_{t}, b_{u}, e_{t}, e_{u}$, $m_{t}, m_{u}, f_{t}, f_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \sigma$, $\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}, b=b_{t}+b_{u}, e=e_{t}+e_{u}, m=m_{t}+m_{u}, f=f_{t}+f_{u}$, and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.
- Let $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t\{x \backslash u\}: \mathcal{B}$. Then, there exist derivations $\Phi_{t}, \Phi_{u}$, integers $b_{t}, b_{u}, e_{t}, e_{u}$, $m_{t}, m_{u}, f_{t}, f_{u}$, contexts $\Gamma_{t}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{t} \triangleright \Gamma_{t} ; x: \mathcal{A} \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t: \mathcal{B}$, $\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}, b=b_{t}+b_{u}, e=e_{t}+e_{u}, m=m_{t}+m_{u}, f=f_{t}+f_{u}$, and $\Gamma=\Gamma_{t} \wedge \Gamma_{u}$.
We will reason by induction on $\Phi$. For all the rules (except many), we will have the trivial case $t\{x \backslash u\}$, where $t=x$, in which case $t\{x \backslash u\}=u$, for which we have a derivation $\Phi \triangleright \Gamma \vdash \vdash^{(b, e, m, f)} u: \sigma$. Therefore $\Phi_{t} \triangleright x:[\sigma] \vdash^{(0,0,0,0)} x: \sigma$ and $\Phi_{u} \triangleright \Gamma \vdash^{(b, e, m, f)} u:[\sigma]$ is obtained from $\Phi$ using the (many) rule. The conditions on the counters hold trivially. We now reason on the different cases assuming that $t \neq x$.
- If $\Phi$ is (ax), therefore $\Phi \triangleright y:[\sigma] \vdash^{(0,0,0,0)} y: \sigma$. We only consider the case where $t=y$ and $y \neq x$. Then we take $\mathcal{A}=[], \Phi_{t} \triangleright y:[\sigma] ; x:[] \vdash^{(0,0,0,0)} y: \sigma$ and $\Phi_{u} \triangleright \vdash^{(0,0,0,0)} u:[]$, using rule (many). The conditions on the counters follow trivially.
- If $\Phi$ ends with (many), then $\Phi \triangleright \wedge_{k \in K} \Gamma_{k} \vdash^{\left(+_{k \in K} b_{k},+_{k \in K} e_{k},+_{k \in K} m_{k},+_{k \in K} f_{k}\right)} t\{x \backslash u\}:\left[\sigma_{k}\right]_{k \in K}$ follows from $\Phi^{k} \triangleright \Gamma_{k} \vdash^{\left(b_{k}, e_{k}, m_{k}, f_{k}\right)} t\{x \backslash u\}: \sigma_{k}$, for each $k \in K$. By the i.h. there exist $\Phi_{t}^{k}, \Phi_{u}^{k}, b_{t}^{k}, b_{u}^{k}, e_{t}^{k}, e_{u}^{k}, m_{t}^{k}, m_{u}^{k}, f_{t}^{k}, f_{u}^{k}$, contexts $\Gamma_{t}^{k}, \Gamma_{u}^{k}$ and multitype $\mathcal{A}_{k}$, such that $\Phi_{t}^{k} \triangleright \Gamma_{t}^{k} ; x: \mathcal{A}_{k} \vdash\left(b_{t}^{k}, e_{t}^{k}, m_{t}^{k}, f_{t}^{k}\right) t: \sigma_{k}, \Phi_{u}^{k} \triangleright \Gamma_{u}^{k} \vdash\left(b_{u}^{k}, e_{u}^{k}, m_{u}^{k}, f_{u}^{k}\right) u: \mathcal{A}_{k}, \Gamma_{k}=\Gamma_{t}^{k} \wedge \Gamma_{u}^{k}$, $b_{k}=b_{t}^{k}+b_{u}^{k}, e_{k}=e_{t}^{k}+e_{u}^{k}, m_{k}=m_{t}^{k}+m_{u}^{k}, f_{k}=f_{t}^{k}+f_{u}^{k}$.
Taking $\mathcal{A}=\wedge_{k \in K} \mathcal{A}_{k}$ and using the (many) rule on the derivations $\left(\Phi_{t}^{k}\right)_{k \in K}$ we get now $\wedge_{k \in K} \Gamma_{t}^{k} ; x: \mathcal{A} \vdash\left(+_{k \in K} b_{t}^{k},+_{k \in K} e_{t}^{k},+_{k \in K} m_{t}^{k},+_{k \in K} f_{t}^{k}\right) t:\left[\sigma_{k}\right]_{k \in K}$. From the premises $\left(\Phi_{k}^{k}\right)_{k \in K}$, by applying again the (many) rule, we get $\wedge_{k \in K} \Gamma_{u}^{k} \vdash\left(+_{k \in K} b_{u}^{k},+_{k \in K} e_{u}^{k},+_{k \in K} m_{u}^{k},+_{k \in K} f_{u}^{k}\right) u: \mathcal{A}$.

Note that $\Gamma=\wedge_{k \in K} \Gamma_{k}=\left(\wedge_{k \in K} \Gamma_{t}^{k}\right) \wedge\left(\wedge_{k \in K} \Gamma_{u}^{k}\right), b=+_{k \in K} b_{k}=\left(+_{k \in K} b_{t}^{k}\right)+\left(+_{k \in K} b_{u}^{k}\right)$, $e=+_{k \in K} e_{k}=\left(+_{k \in K} e_{t}^{k}\right)+\left(+_{k \in K} e_{u}^{k}\right), m=+_{k \in K} m_{k}=\left(+_{k \in K} m_{t}^{k}\right)+\left(+_{k \in K} m_{u}^{k}\right)$ and $f=+_{k \in K} f_{k}=\left(+_{k \in K} f_{t}^{k}\right)+\left(+_{k \in K} f_{u}^{k}\right)$, as expected.

- If $\Phi$ ends with (abs), then $t=\lambda p \cdot t^{\prime}$, therefore

$$
\Phi \triangleright \Gamma \backslash \operatorname{var}(p) \vdash^{\left(b_{t}+1, e_{t}+e_{p}, m_{t}+m_{p}, f_{t}+f_{p}\right)} \lambda p .\left(t^{\prime}\{x \backslash u\}\right): \mathcal{B} \rightarrow \sigma
$$

follows from $\Gamma \vdash^{\left(b_{t}, e_{t}, m_{t}, f_{t}\right)} t^{\prime}\{x \backslash u\}: \sigma$ and $\left.\Gamma\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{B}$. On the other side one can always assume that $\operatorname{var}(p) \cap \mathrm{fv}(u)=\emptyset$ and $x \notin \operatorname{var}(p)$. We can then apply the i.h. to obtain $\Phi_{t^{\prime}} \triangleright \Gamma_{t^{\prime}} ; x: \mathcal{A} \vdash^{\left(b_{t}^{\prime}, e_{t}^{\prime}, m_{t}^{\prime}, f_{t}^{\prime}\right)} t^{\prime}: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$, with $\Gamma=\Gamma_{t^{\prime}} \wedge \Gamma_{u}, b_{t}=b_{t^{\prime}}+b_{u}, e_{t}=e_{t^{\prime}}+e_{u}, m_{t}=m_{t^{\prime}}+m_{u}, f_{t}=f_{t^{\prime}}+f_{u}$. Then using rule (abs) we get $\Phi_{t} \triangleright \Gamma_{t^{\prime}} \backslash \operatorname{var}(p) ; x: \mathcal{A} \vdash^{\left(b_{t^{\prime}}+1, e_{t^{\prime}}+e_{p}, m_{t^{\prime}}+m_{p}, f_{t^{\prime}}+f_{p}\right)} \lambda p . t^{\prime}: \mathcal{B} \rightarrow \sigma$. And $\Gamma \ \backslash \operatorname{var}(p)=\left(\Gamma_{t^{\prime}} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u}, b_{t}+1=b_{t}^{\prime}+1+b_{u}, e_{t}=e_{t^{\prime}}+e_{u}, m_{t}=m_{t^{\prime}}+m_{u}$ and $f_{t}=f_{t^{\prime}}+f_{u}$, as expected.

- If $\Phi$ ends with (app) then $t=t^{\prime} u^{\prime}$, and the derivation

$$
\Phi \triangleright \Gamma \wedge \Delta \vdash^{\left(b_{t^{\prime}}+b_{u^{\prime}}, e_{t^{\prime}}+e_{u^{\prime}}, m_{t^{\prime}}+m_{u^{\prime}}, f_{t^{\prime}}+f_{u^{\prime}}\right)} t^{\prime}\{x \backslash u\} u^{\prime}\{x \backslash u\}: \sigma
$$

follows from $\Gamma \vdash^{\left(b_{t^{\prime}}, e_{t^{\prime}}, m_{t^{\prime}}, f_{t^{\prime}}\right)} t^{\prime}\{x \backslash u\}: \mathcal{B} \rightarrow \sigma$ and $\Delta \vdash \vdash^{\left(b_{u^{\prime}}, e_{u^{\prime}}, m_{u^{\prime}}, f_{u^{\prime}}\right)} u^{\prime}\{x \backslash u\}: \mathcal{B}$. By the i.h. there exist $\Phi_{t^{\prime}}, \Phi_{t^{\prime}}^{u}, b_{t^{\prime \prime}}, b_{t^{\prime}}^{u}, e_{t^{\prime \prime}}, e_{t^{\prime}}^{u}, m_{t^{\prime \prime}}, m_{t^{\prime}}^{u}, f_{t^{\prime \prime}}, f_{t^{\prime}}^{u}$, contexts $\Gamma_{t^{\prime}}, \Gamma_{t^{\prime}}^{u}$ and multitype $\mathcal{A}_{t^{\prime}}$, such that

$$
\Phi_{t^{\prime}} \triangleright \Gamma_{t^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \vdash^{\left(b_{t^{\prime \prime}}, e_{t^{\prime \prime}}, m_{t^{\prime \prime}}, f_{t^{\prime \prime}}\right)} t^{\prime}: \mathcal{B} \rightarrow \sigma \quad \Phi_{t^{\prime}}^{u} \triangleright \Gamma_{t^{\prime}}^{u} \vdash^{\left(b_{t^{\prime}}^{u}, e_{t^{\prime}}^{u}, m_{t^{\prime}}^{u}, f_{t^{\prime}}^{u}\right)} u: \mathcal{A}_{t^{\prime}}
$$

where $b_{t^{\prime}}=b_{t^{\prime \prime}}+b_{t^{\prime}}^{u}, e_{t^{\prime}}=e_{t^{\prime \prime}}+e_{t^{\prime}}^{u}, m_{t^{\prime}}=m_{t^{\prime \prime}}+m_{t^{\prime}}^{u}, f_{t^{\prime}}=f_{t^{\prime \prime}}+f_{t^{\prime}}^{u}$ and $\Gamma=\Gamma_{t^{\prime}} \wedge \Gamma_{t^{\prime}}^{u}$. And by the i.h. applied to the second premise of $\Phi$, there exist $\Phi_{u^{\prime}}, \Phi_{u^{\prime}}^{u}, b_{u^{\prime \prime}}, b_{u^{\prime}}^{u}, e_{u^{\prime \prime}}$, $e_{u^{\prime}}^{u}, m_{u^{\prime \prime}}, m_{u^{\prime}}^{u}, f_{u^{\prime \prime}}, f_{u^{\prime}}^{u}$, contexts $\Gamma_{u^{\prime}}, \Gamma_{u^{\prime}}^{u}$ and multitype $\mathcal{A}_{u^{\prime}}$, such that

$$
\Phi_{u^{\prime}} \triangleright \Delta_{u^{\prime}} ; x: \mathcal{A}_{u^{\prime}} \vdash^{\left(b_{u^{\prime \prime}}, e_{u^{\prime \prime}}, m_{u^{\prime \prime}}, f_{u^{\prime \prime}}\right)} u^{\prime}: \mathcal{B} \quad \Phi_{u^{\prime}}^{u} \triangleright \Delta_{u^{\prime}}^{u} \vdash^{\left(b_{u^{\prime}}^{u}, e_{u^{\prime}}^{u}, m_{u^{\prime}}^{u}, f_{u^{\prime}}^{u}\right) u: \mathcal{A}_{u^{\prime}} .}
$$

where $b_{u^{\prime}}=b_{u^{\prime \prime}}+b_{u^{\prime}}^{u}, e_{u^{\prime}}=e_{u^{\prime \prime}}+e_{u^{\prime}}^{u}, m_{u^{\prime}}=m_{u^{\prime \prime}}+m_{u^{\prime}}^{u}, f_{u^{\prime}}=f_{u^{\prime \prime}}+f_{u^{\prime}}^{u}$ and $\Delta=\Delta_{u^{\prime}} \wedge \Delta_{u^{\prime}}^{u}$. Now, taking $\mathcal{A}=\mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}}$, and using the (app) rule, one gets a derivation of the form $\Phi_{t^{\prime} u^{\prime}} \triangleright \Gamma_{t^{\prime}} \wedge \Delta_{u^{\prime}} ; x: \mathcal{A}_{t^{\prime}} \wedge \mathcal{A}_{u^{\prime}} \vdash^{\left(b_{t^{\prime \prime}}+b_{u^{\prime \prime}}, e_{t^{\prime \prime}}+e_{u^{\prime \prime}}, m_{t^{\prime \prime}}+m_{u^{\prime \prime}}, f_{t^{\prime \prime}}+f_{u^{\prime \prime}}\right)} t^{\prime}: \mathcal{B} \rightarrow \sigma$ and applying the (many) rule to the premises of $\Phi_{t^{\prime}}^{u}$ and $\Phi_{u^{\prime}}^{u}$ one gets a derivation of the form $\Phi_{u} \triangleright \Gamma_{t^{\prime}}^{u} \wedge \Delta_{u^{\prime}}^{u} \vdash^{\left(b_{t^{\prime}}^{u}+b_{u^{\prime}}^{\prime}, e_{t^{\prime}}^{u}+e_{u^{\prime}}^{u}, m_{t^{\prime}}^{u}+m_{u^{\prime}}^{u}, f_{t^{\prime}}^{u}+f_{u^{\prime}}^{u}\right) u: \mathcal{A} \text {. Note that } \Gamma \wedge \Delta=}$ $\left(\Gamma_{t^{\prime}} \wedge \Gamma_{t^{\prime}}^{u}\right) \wedge\left(\Delta_{u^{\prime}} \wedge \Delta_{u^{\prime}}^{u}\right)=\left(\Gamma_{t^{\prime}} \wedge \Delta_{u^{\prime}}\right) \wedge\left(\Gamma_{t^{\prime}}^{u} \wedge \Delta_{u^{\prime}}^{u}\right)$ and $b=b_{t^{\prime}}+b_{u^{\prime}}=\left(b_{t^{\prime \prime}}+b_{t^{\prime}}^{u}\right)+\left(b_{u^{\prime \prime}}+b_{u^{\prime}}^{u}\right)=$ $\left(b_{t^{\prime \prime}}+b_{u^{\prime \prime}}\right)+\left(b_{t^{\prime}}^{u}+b_{u^{\prime}}^{u}\right)$ as expected (the same happens for the remaining counters).

- If $\Phi$ ends with $\left(\mathrm{abs}_{\mathrm{p}}\right),\left(\mathrm{app}_{\mathrm{p}}\right)$ or (match) the result follows from the inductive hypothesis, as in the previous cases.
- If $\Phi$ ends with (pair) or (pair ), so that $t=\left\langle t^{\prime}, u^{\prime}\right\rangle$, then we have two cases. The case for pair ${ }_{\mathrm{p}}$, follows from $\Phi$ being of the form $\vdash^{(0,0,0,1)}\left\langle t^{\prime}\{x \backslash u\}, u^{\prime}\{x \backslash u\}\right\rangle: \bullet \mathcal{M}$. We then take $\mathcal{A}=[], \Phi_{\left\langle t^{\prime}, u^{\prime}\right\rangle} \triangleright x:[] \vdash^{(0,0,0,1)}\left\langle t^{\prime}, u^{\prime}\right\rangle: \bullet \mathcal{M}$ and $\Phi_{u} \triangleright \vdash^{(0,0,0,0)} u:[]$ follows trivially from the (many) rule. Then conditions on counters and contexts hold trivially. The case for (pair) follows by induction using the same reasoning as in rule (app).
- Lemma 19 (Exact Subject Expansion). If $\Phi^{\prime} \triangleright \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f^{\prime}\right)} t^{\prime}: \sigma$, and $t \rightarrow_{h} t^{\prime}$ is an s-step, with $s \in\{b, e, m\}$, then $\Phi \triangleright \Gamma \vdash^{(b, e, m, f)} t: \sigma$, where
- $s=b$ implies $b=b^{\prime}+1, e^{\prime}=e, m^{\prime}=m$.
- $s=e$ implies $b^{\prime}=b, e=e^{\prime}+1, m^{\prime}=m$.
- $s=m$ implies $b^{\prime}=b, e^{\prime}=e, m=m^{\prime}+1$.

Proof. By induction on $\rightarrow_{\mathrm{h}}$, using Lem. 18 .

- $t=\mathrm{L} \llbracket \lambda p . v \rrbracket u \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket v\left[p \backslash u \rrbracket \rrbracket=t^{\prime}\right.$. The proof is by induction on the list context L .

If $\mathrm{L}=\square$ then by construction there are derivations $\Phi_{v} \triangleright \Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \sigma$ and $\Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$ for some multitype $\mathcal{A}$, and $\Phi^{\prime}$ is of the form

$$
\frac{\left.\Phi_{v} \quad\left(\Gamma_{v}\right)\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A} \quad \Phi_{u}}{\left(\Gamma_{v} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u} \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} v[p \backslash u]: \sigma}
$$

where $\Gamma=\left(\Gamma_{v} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u}, b^{\prime}=b_{v}+b_{u}, e^{\prime}=e_{v}+e_{u}+e_{p}, m^{\prime}=m_{v}+m_{u}+m_{p}$ and $f=f_{v}+f_{u}+f_{p}$. Then

$$
\Phi \triangleright \frac{\left.\Phi_{v} \quad\left(\Gamma_{v}\right)\right|_{p} \Vdash \vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}}{\left(\Gamma_{v} \backslash \operatorname{var}(p)\right) \wedge \Gamma_{u} \vdash^{\left(\left(b_{v}+1\right)+b_{u},\left(e_{v}+e_{p}\right)+e_{u},\left(m_{v}+m_{p}\right)+m_{u},\left(f_{v}+f_{p}\right)+f_{u}\right)}(\lambda p . v) u: \sigma}
$$

where $b=\left(b_{v}+1\right)+b_{u}=b^{\prime}+1, e=\left(e_{v}+e_{p}\right)+e_{u}=e^{\prime}$ and $m=\left(m_{v}+m_{p}\right)+m_{u}=m^{\prime}$. If $\mathrm{L} \neq \square$ the proof from the i.h. is straightforward.

- $t=v[x \backslash u] \rightarrow_{\mathrm{h}} v\{x \backslash u\}=t^{\prime}$, where $v \nrightarrow_{\mathrm{h}}$. Then by the anti-substitution property (Lem. 18) there exist derivations $\Phi_{v}, \Phi_{u}$, integers $b_{v}, b_{u}, e_{v}, e_{u}, m_{v}, m_{u}, f_{v}, f_{u}$, contexts $\Gamma_{v}, \Gamma_{u}$, and multitype $\mathcal{A}$ such that $\Phi_{v} \triangleright \Gamma_{v} ; x: \mathcal{A} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \sigma, \Phi_{u} \triangleright \Gamma_{u} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \mathcal{A}$, $b^{\prime}=b_{v}+b_{u}, e^{\prime}=e_{v}+e_{u}, m^{\prime}=m_{v}+m_{u}, f=f_{v}+f_{u}$, and $\Gamma=\Gamma_{v} \wedge \Gamma_{u}$. Then,

$$
\Phi \triangleright \frac{\left.\Phi_{v} \quad\left(\Gamma_{v} ; x: \mathcal{A}\right)\right|_{x} \Vdash^{(1,0,0)} x: \mathcal{A} \quad \Phi_{u}}{\Gamma_{v} \wedge \Gamma_{u} \vdash^{\left(b_{v}+b_{u}, e_{v}+e_{u}+1, m_{v}+m_{u}+0, f_{v}+f_{u}+0\right)} v[x \backslash u]: \sigma}
$$

where $b=b_{v}+b_{u}=b^{\prime}, e=e_{v}+e_{u}+1=e^{\prime}+1$ and $m=m_{v}+m_{u}=m^{\prime}$. Note that $\left(\Gamma_{v} ; x: \mathcal{A}\right) \backslash \operatorname{var}(x)=\Gamma_{v}$.

- $t=v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{h}} \mathrm{L} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket=t^{\prime}$, where $v \not \nrightarrow \mathrm{~h}_{\mathrm{h}}$. Let us abbreviate $p=\left\langle p_{1}, p_{2}\right\rangle$ and $u=\left\langle u_{1}, u_{2}\right\rangle$. The proof is by induction on the list L .
$=\mathrm{L}=\square$, then there are $\Phi_{v} \triangleright \Gamma_{v} \vdash\left(b_{v}, e_{v}, m_{v}, f_{v}\right) v: \sigma, \Phi_{1} \triangleright \Gamma_{1} \vdash\left(b_{u}^{1}, e_{u}^{1}, m_{u}^{1}, f_{u}^{1}\right) u_{1}: \mathcal{A}_{1}$ and $\Phi_{2} \triangleright \Gamma_{2} \vdash\left(b_{u}^{2}, e_{u}^{2}, m_{u}^{2}, f_{u}^{2}\right) u_{2}: \mathcal{A}_{2}$ where

$$
\Phi_{v\left[p_{1} \backslash u_{1}\right]} \triangleright \frac{\left.\Phi_{v} \quad \Gamma_{v}\right|_{p_{1}} \Vdash{ }^{\left(e_{1}, m_{1}, f_{1}\right)} p_{1}: \mathcal{A}_{1} \quad \Phi_{1}}{\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1} \vdash^{\left(b_{v}+b_{u}^{1}, e_{v}+e_{u}^{1}+e_{1}, m_{v}+m_{u}^{1}+m_{1}, f_{v}+f_{u}^{1}+f_{1}\right)} v\left[p_{1} \backslash u_{1}\right]: \sigma}
$$

and

$$
\Phi^{\prime} \triangleright \frac{\left.\Phi_{v\left[p_{1} \backslash u_{1}\right]} \quad\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right)\right|_{p_{2}} \Vdash \Vdash^{\left(e_{2}, m_{2}, f_{2}\right)} p_{2}: \mathcal{A}_{2}}{\Gamma \vdash^{\left(b^{\prime}, e^{\prime}, m^{\prime}, f\right)} v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right]: \sigma}
$$

where $b^{\prime}=b_{v}+_{i=1,2} b_{u}^{i}, e^{\prime}=e_{v}+_{i=1,2} e_{u}^{i}+e_{i}, m^{\prime}=m_{v}+_{i=1,2} m_{u}^{i}+m_{i}, f=$ $f_{v}+_{i=1,2} f_{u}^{i}+f_{i}$ and $\Gamma=\left(\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right) \backslash \operatorname{var}\left(p_{2}\right)\right) \wedge \Gamma_{2}$.
Moreover, $\left(\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right) \ \operatorname{var}\left(p_{2}\right)\right)=\left(\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \backslash \operatorname{var}\left(p_{2}\right) \wedge \Gamma_{1} \backslash \operatorname{var}\left(p_{2}\right)\right)$, where $\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right) \ \operatorname{var}\left(p_{2}\right)=\Gamma_{v} \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)$ and $\Gamma_{1} \backslash \operatorname{var}\left(p_{2}\right)=L_{L .8} \Gamma_{1}$. Similarly, $\left.\left(\left(\Gamma_{v} \| \operatorname{var}\left(p_{1}\right)\right) \wedge \Gamma_{1}\right)\right|_{p_{2}}=\left.{ }^{\text {Lem. } 8}\left(\Gamma_{v} \backslash \operatorname{var}\left(p_{1}\right)\right)\right|_{p_{2}}$ and, by linearity of patterns, $\left.\left(\Gamma_{v} \| \operatorname{var}\left(p_{1}\right)\right)\right|_{p_{2}}=\left.\Gamma_{v}\right|_{p_{2}}$. Hence,

$$
\Phi_{\left\langle p_{1}, p_{2}\right\rangle} \triangleright \frac{\left.\Gamma_{v}\right|_{p_{1}} \Vdash \Vdash^{\left(e_{1}, m_{1}, f_{1}\right)} p_{1}:\left.\mathcal{A}_{1} \quad \Gamma_{v}\right|_{p_{2}} \Vdash \Vdash^{\left(e_{2}, m_{2}, f_{2}\right)} p_{2}: \mathcal{A}_{2}}{\left.\left.\Gamma_{v}\right|_{p_{1}} \wedge \Gamma_{v}\right|_{p_{2}} \Vdash^{\left(e_{1}+e_{2}, 1+m_{1}+m_{2}, f_{1}+f_{2}\right)}\left\langle p_{1}, p_{2}\right\rangle:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

where $\left.\left.\Gamma_{v}\right|_{p_{1}} \wedge \Gamma_{v}\right|_{p_{2}}=\left.\Gamma_{v}\right|_{\left\langle p_{1}, p_{2}\right\rangle}$, and $\Phi_{\left\langle u_{1}, u_{2}\right\rangle} \triangleright \Gamma_{1} \wedge \Gamma_{2} \vdash\left(b_{u}^{1}+b_{u}^{2}, e_{u}^{1}+e_{u}^{2}, m_{u}^{1}+m_{u}^{2}, f_{u}^{1}+f_{u}^{2}\right)$ $\left\langle u_{1}, u_{2}\right\rangle:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]$. Therefore,

$$
\Phi \triangleright \frac{\Phi_{v} \quad \Phi_{\left\langle p_{1}, p_{2}\right\rangle} \Phi_{\left\langle u_{1}, u_{2}\right\rangle}}{\left(\Gamma_{v} \backslash \operatorname{var}\left(\left\langle p_{1}, p_{2}\right\rangle\right)\right) \wedge\left(\Gamma_{1} \wedge \Gamma_{2}\right) \vdash^{(b, e, m, f)} v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash\left\langle u_{1}, u_{2}\right\rangle\right]: \sigma}
$$

where $b=b_{v}+{ }_{i=1,2} b_{u}^{i}=b^{\prime}, e^{\prime}=e_{v}+{ }_{i=1,2} e_{u}^{i}+e_{i}=e^{\prime}$ and $m=1+m_{v}+{ }_{i=1,2} m_{u}^{i}+m_{i}=$ $m^{\prime}+1$.
= If $\mathrm{L}=\mathrm{L}^{\prime}[q \backslash s]$, then $\Phi^{\prime}$ is of the form:
$\frac{\Gamma_{\mathrm{L}^{\prime}} \vdash^{\left(b_{l}^{\prime}, e_{l}^{\prime}, m_{l}^{\prime}, f_{l}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket:\left.\sigma \quad \Gamma_{\mathrm{L}^{\prime}}\right|_{q} \Vdash^{\left(e_{q}, m_{q}, f_{q}\right)} q: \mathcal{A} \quad \Gamma_{s} \vdash^{\left(b_{s}, e_{s}, m_{s}, f_{s}\right)} s: \mathcal{A}}{\Gamma_{\mathrm{L}^{\prime}} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash^{\left(b_{l}^{\prime}+b_{s}, e_{l}^{\prime}+e_{s}+e_{q}, m_{l}^{\prime}+m_{s}+m_{q}, f_{l}^{\prime}+f_{s}+f_{q}\right)} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket[q \backslash s]: \sigma}$ where $b^{\prime}=b_{l}^{\prime}+b_{s}, e^{\prime}=e_{l}^{\prime}+e_{s}+e_{q}, m^{\prime}=m_{l}^{\prime}+m_{s}+m_{q}, f^{\prime}=f_{l}^{\prime}+f_{s}+f_{q}$ and $\Gamma=\Gamma_{\mathrm{L}^{\prime}} \backslash \operatorname{var}(q) \wedge \Gamma_{s}$. From $v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right] \rightarrow_{\mathrm{h}} \mathrm{L}^{\prime} \llbracket v\left[p_{1} \backslash u_{1}\right]\left[p_{2} \backslash u_{2}\right] \rrbracket$ and derivation $\Phi_{\mathrm{L}^{\prime}}^{\prime}$ for the leftmost premise by the i.h. one gets $\Phi_{\mathrm{L}^{\prime}} \triangleright \Gamma_{\mathrm{L}^{\prime}} \vdash^{\left(b_{l}, e_{l}, m_{l}, f_{l}\right)}$ $v\left[\left\langle p_{1}, p_{2}\right\rangle \backslash \mathrm{L}^{\prime} \llbracket\left\langle u_{1}, u_{2}\right\rangle \rrbracket\right]: \sigma$ where $b_{l}=b_{l}^{\prime}, e_{l}=e_{l}^{\prime}, m_{l}=m_{l}^{\prime}+1$ and $f_{l}=f_{l}^{\prime}$. Furthermore $\Phi_{\mathrm{L}^{\prime}}$ is necessarily of the form:

$$
\Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v:\left.\sigma \quad \Gamma_{v}\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right] \quad \frac{\Gamma_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Gamma_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}
$$

$$
\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \vdash^{\left(b_{v}+b_{u}^{\prime}, e_{v}+e_{u}^{\prime}+e_{p}, m_{v}+m_{u}^{\prime}+m_{p}, f_{v}+f_{u}^{\prime}+f_{p}\right)} v\left[p \backslash \mathrm{~L}^{\prime} \llbracket u \rrbracket\right]: \sigma
$$

where $b_{l}=b_{v}+b_{u}^{\prime}, e_{l}=e_{v}+e_{u}^{\prime}+e_{p}, m_{l}=m_{v}+m_{u}^{\prime}+m_{p}, f_{l}=f_{v}+f_{u}^{\prime}+f_{p}$ and $\Gamma_{\mathrm{L}^{\prime}}=\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u}$. Note that, by relevance and $\alpha$-conversion we have that $\left.\Gamma_{\mathrm{L}^{\prime}}\right|_{q}=\left.\Gamma_{u}\right|_{q}$. Then one can construct the following derivation $\Phi_{u}$ :

$$
\frac{\Gamma_{u} \vdash^{\left(b_{u}^{\prime}, e_{u}^{\prime}, m_{u}^{\prime}, f_{u}^{\prime}\right)} \mathrm{L}^{\prime} \llbracket u \rrbracket: \times\left.\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \quad \Gamma_{u}\right|_{q} \Vdash^{\left(e_{q}, m_{q}, f_{q}\right)} q: \mathcal{A} \quad \Phi_{s}}{\frac{\Gamma_{u} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash^{\left(b_{u}^{\prime}+b_{s}, e_{u}^{\prime}+e_{s}+e_{q}, m_{u}^{\prime}+m_{s}+m_{q}, f_{u}^{\prime}+f_{s}+f_{q}\right)} \mathrm{L}^{\prime}[q \backslash s] \llbracket u \rrbracket: \times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}{\Gamma_{u} \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash^{\left(b_{u}^{\prime}+b_{s}, e_{u}^{\prime}+e_{s}+e_{q}, m_{u}^{\prime}+m_{s}+m_{q}, f_{u}^{\prime}+f_{s}+f_{q}\right)} \mathrm{L}^{\prime}[q \backslash s] \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}}
$$

where $b_{u}=b_{u}^{\prime}+b_{s}, e_{u}=e_{u}^{\prime}+e_{s}+e_{q}, m_{u}=m_{u}^{\prime}+m_{s}+m_{q}$ and $f_{u}=f_{u}^{\prime}+f_{s}+f_{q}$. From $\Phi_{v} \triangleright \Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \sigma$ and $\left.\Pi_{p} \triangleright \Gamma_{v}\right|_{p} \Vdash\left(e_{p}, m_{p}, f_{p}\right) p:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]$ we build $\Phi$ :

$$
\frac{\Phi_{v} \quad \Pi_{p} \quad \Gamma_{u} \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} \mathrm{L}^{\prime}[q \backslash s] \llbracket u \rrbracket:\left[\times\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)\right]}{\Gamma_{v} \backslash \backslash \operatorname{var}(p) \wedge \Gamma_{u} \backslash \operatorname{var}(q) \wedge \Gamma_{s} \vdash^{\left(b_{v}+b_{u}, e_{v}+e_{u}+e_{p}, m_{v}+m_{u}+m_{p}, f_{v}+f_{u}+f_{p}\right)} v\left[p \backslash \mathrm{~L}^{\prime}[q \backslash s] \llbracket u \rrbracket\right]: \sigma}
$$

With $\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u} \backslash \backslash \operatorname{var}(q) \wedge \Gamma_{s}=\left(\Gamma_{v} \backslash \operatorname{var}(p) \wedge \Gamma_{u}\right) \backslash \operatorname{var}(q) \wedge \Gamma_{s}=\Gamma$. Furthermore, * $b=b_{v}+b_{u}=b_{v}+b_{u}^{\prime}+b_{s}=b_{l}+b_{s}=b^{\prime}$

* $e=e_{v}+e_{u}+e_{p}=e_{v}+e_{u}^{\prime}+e_{s}+e_{q}+e_{p}=e_{l}+e_{s}+e_{q}=e^{\prime}$
* $m=m_{v}+m_{u}+m_{p}=m_{v}+m_{u}^{\prime}+m_{s}+m_{q}+m_{p}=m_{l}+m_{s}+m_{q}=m^{\prime}+1$
* $f=f_{v}+f_{u}+f_{p}=f_{v}+f_{u}^{\prime}+f_{s}+f_{q}+f_{p}=f_{l}+f_{s}+f_{q}=f^{\prime}$
- Most of the inductive cases are straightforward, so we only show the interesting one. Let $t=v[p \backslash u] \rightarrow_{\mathrm{h}} v\left[p \backslash u^{\prime}\right]=t^{\prime}$, where $v \nrightarrow_{\mathrm{h}}$ and $p \neq x$ and $u \rightarrow_{\mathrm{h}} u^{\prime}$. By construction there are subderivations $\Phi_{v} \triangleright \Gamma_{v} \vdash^{\left(b_{v}, e_{v}, m_{v}, f_{v}\right)} v: \sigma,\left.\Gamma_{v}\right|_{p} \Vdash^{\left(e_{p}, m_{p}, f_{p}\right)} p: \mathcal{A}$ and $\Phi_{u^{\prime}} \triangleright$ $\Gamma_{u^{\prime}} \vdash^{\left(b_{u^{\prime}}, e_{u^{\prime}}, m_{u^{\prime}}, f_{u^{\prime}}\right)} u^{\prime}: \mathcal{A}$ for some multiset $\mathcal{A}$ and $\Gamma=\left(\Gamma_{v} \| \operatorname{var}(p)\right) \wedge \Gamma_{u^{\prime}}$. Since $p$ is not a variable then $\Phi_{p}$ ends with rule (pat ${ }_{p}$ ) or (pat ${ }_{\times}$). In both cases $\mathcal{A}$ contains only one type, let us say $\mathcal{A}=\left[\sigma_{u^{\prime}}\right]$. Then $\Phi_{u^{\prime}}$ has the following form

$$
\Phi_{u^{\prime}} \triangleright \frac{\Gamma_{u^{\prime}} \vdash^{\left(b_{u^{\prime}}, e_{u^{\prime}}, m_{u^{\prime}}, f_{u^{\prime}}\right)} u^{\prime}: \sigma_{u^{\prime}}}{\Gamma_{u^{\prime}} \vdash^{\left(b_{u^{\prime}}, e_{u^{\prime}}, m_{u^{\prime}}, f_{u^{\prime}}\right)} u^{\prime}:\left[\sigma_{u^{\prime}}\right]}
$$

The i.h. applied to the premise of $\Phi_{u^{\prime}}$ gives a derivation $\Gamma_{u^{\prime}} \vdash^{\left(b_{u}, e_{u}, m_{u}, f_{u}\right)} u: \sigma_{u^{\prime}}$ and having the expected counters. To conclude we build a type derivation $\Phi^{\prime}$ for $v\left[p \backslash u^{\prime}\right]$ having the expected counters.


[^0]:    1 An equivalent type system can be presented without the many rule, for example [19]. However, the inductive proofs in the current presentation turn to be more elegant.

[^1]:    References
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