On the Approximation Ratio of the k-Opt and Lin-Kernighan Algorithm for Metric and Graph TSP

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Abstract

The k-Opt and Lin-Kernighan algorithm are two of the most important local search approaches for the Metric TSP. Both start with an arbitrary tour and make local improvements in each step to get a shorter tour. We show that for any fixed $k \geq 3$ the approximation ratio of the k-Opt algorithm for Metric TSP is $O(\sqrt[k]{n})$. Assuming the Erdős girth conjecture, we prove a matching lower bound of $\Omega(\sqrt[k]{n})$. Unconditionally, we obtain matching bounds for $k = 3, 4, 6$ and a lower bound of $\Omega(n^{\frac{2}{3k-3}})$. Our most general bounds depend on the values of a function from extremal graph theory and are tight up to a factor logarithmic in the number of vertices unconditionally. Moreover, all the upper bounds also apply to a parameterized version of the Lin-Kernighan algorithm with appropriate parameter. We also show that the approximation ratio of k-Opt for Graph TSP is $\Omega\left(\frac{\log(n)}{\log\log(n)}\right)$ and $O\left(\left(\frac{\log(n)}{\log\log(n)}\right)^{\log_2(9)+\epsilon}\right)$ for all $\epsilon > 0$.

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1 Introduction

The traveling salesman problem (TSP) is probably the best-known problem in discrete optimization. An instance consists of the pairwise distances of n vertices and the task is to find a shortest Hamiltonian cycle, i.e. a tour visiting every vertex exactly once. The problem is known to be NP-hard [12]. A special case of the TSP is the Metric TSP. Here the distances satisfy the triangle inequality. This TSP variant is still NP-hard [15].

Since the problem is NP-hard, a polynomial-time algorithm is not expected to exist. In order to speed up the calculation of a good tour in practice, several approximation algorithms are considered. The approximation ratio is one way to compare approximation algorithms. It is the maximal ratio, taken over all instances, of the output of the algorithm divided by the optimum solution. The best currently known approximation algorithm in terms of approximation ratio for Metric TSP was independently developed by Christofides and Serdjukov [6, 24] with an approximation ratio of $\frac{3}{2}$. However, in practice other algorithms are usually easier to implement and have better performance and runtime [3, 14, 22]. One natural approach is the k-Opt algorithm which is based on local search. It starts with an arbitrary tour and replaces at most k edges by new edges such that the resulting tour is

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shorter. It stops if the procedure cannot be applied anymore. For the 2-Opt algorithm Plesník showed that there are infinitely many instances with approximation ratio $\sqrt{\frac{n}{8}}$, where n is the number of vertices [21]. Chandra, Karloff and Tovey showed that the approximation ratio of 2-Opt is at most $4\sqrt{n}$ [5]. Levin and Yovel observed that the same proof yields an upper bound of $\sqrt{8n}$ [19]. Recently, Hougardy, Zaiser and Zhong closed the gap and proved that the approximation ratio of the 2-Opt algorithm is at most $\sqrt{\frac{n}{2}}$ and that this bound is tight [13]. For general $k > 2$ Chandra, Karloff and Tovey gave a lower bound of $\frac{1}{4}\sqrt[2k]{n}$ [5], no non-trivial upper bound is known so far. In the case where the instances can be embedded into the normed space $\mathbb{R}^d$ the approximation ratio of 2-Opt is between $\Omega(\frac{\log(n)}{\log\log(n)})$ and $O(\log(n))$ [5].

Beyond the worst-case analysis there are also results about the average case behavior of the algorithm. For example the smoothed analysis of the 2-Opt algorithm by Englert, Röglin and Vöcking [7]. In their model each vertex of the TSP instance is a random variable distributed in the d dimensional unit cube by a given probability density function $f_i : [0,1]^d \to [0,\phi]$ bounded from above by a constant $1 \le \phi < \infty$ and the distances are given by the L_p norm. They show that in this case the expected approximation ratio is bounded by $O(\sqrt[d]{\phi})$ for all p. In the model where any instance is given in $[0,1]^d$ and perturbed by a Gaussian noise with standard deviation σ the approximation ratio was improved to $O(\log(\frac{1}{\sigma}))$ by Künnemann and Manthey [17].

One of the best practical heuristics by Lin and Kernighan is based on k-Opt [20]. The Lin-Kernighan algorithm, like the k-Opt algorithm, modifies the tour locally to obtain a new tour. Instead of replacing arbitrary k edges with new edges, which results in a high runtime for large k, it searches for specific changes: Changes where the edges to be added and deleted are alternating in a closed walk, a so called closed alternating walk. Since the Lin-Kernighan algorithm uses a super set of the modification rules of the 2-Opt algorithm, the same upper bound as for 2-Opt also applies. Apart from this, no other upper bound was known.

A special case of the Metric TSP is the Graph TSP. In this case an undirected unweighted graph is given and the distance between two vertices is the distance between them in the graph. Apart from the upper bounds of the Metric TSP, which also apply to the special case, only a lower bound of $2(1-\frac{1}{n})$ on the approximation ratio of the k-Opt algorithm is known so far: Rosenkrantz, Stearns and Lewis describe a Metric TSP instance with this ratio that is also a Graph TSP instance [23].

New results. For fixed $k \ge 3$, we show that the approximation ratio of the k-Opt algorithm is related to the extremal graph theoretic problem of maximizing the number of edges in a graph with fixed number of vertices and no short cycles. Let $\mathrm{ex}(n, 2k)$ be the largest number of edges in a graph with n vertices and girth at least $2k$, i.e. it contains no cycles with less than $2k$ edges. For instances with n vertices we show for Metric TSP that:

▶ **Theorem 1.** *If* $\mathrm{ex}(n, 2k) \in O(n^c)$ *for some* $c > 1$*, the approximation ratio of* k*-Opt is* $O(n^{1-\frac{1}{c}})$ *for all fixed* k*.*

▶ **Theorem 2.** *If* $\mathrm{ex}(n, 2k) \in \Omega(n^c)$ *for some* $c > 1$*, the approximation ratio of* k*-Opt is* $\Omega(n^{1-\frac{1}{c}})$ *for all fixed* k*.*

Using known upper bounds on $\mathrm{ex}(n, 2k)$ in [1] we can conclude:

▶ **Corollary 3.** *The approximation ratio of* k*-Opt is in* $O(\sqrt[k]{n})$ *for all fixed* k*.*

If we further assume the Erdős girth conjecture [10], i.e. $\mathrm{ex}(n, 2k) \in \Theta(n^{1+\frac{1}{k-1}})$, we have:

▶ **Corollary 4.** *Assuming the Erdős girth conjecture, the approximation ratio of k-Opt is in $\Omega(\sqrt[k]{n})$ for all fixed k.*

Using known lower bounds on $\mathrm{ex}(n, 2k)$ from [8, 9, 4, 2, 25, 26, 18] we obtain:

▶ **Corollary 5.** *The approximation ratio of k-Opt is in $\Omega(\sqrt[k]{n})$ for $k = 3, 4, 6$ and in $\Omega(n^{\frac{2}{3k-4+\epsilon}})$ for all fixed k where $\epsilon = 0$ if k is even and $\epsilon = 1$ if k is odd.*

Comparing our upper and lower bounds we obtain:

▶ **Theorem 6.** *Our most general upper bound depending on $\mathrm{ex}(n, 2k)$ is tight up to a factor of $O(\log(n))$.*

The upper bounds can be carried over to a parameterized version of the Lin-Kernighan algorithm:

▶ **Theorem 7.** *The same upper bounds from Theorem 1 and 3 hold for a parameterized version of the Lin-Kernighan algorithm with appropriate parameter.*

Although the Lin-Kernighan algorithm only considers special changes, namely changes by augmenting a closed alternating walk, we are able to show the same upper bound as for the general k-Opt algorithm. For the original version of Lin-Kernighan we get an improved upper bound of $O(\sqrt[3]{n})$. Our results solve two of the four open questions in [5], namely:

- Can the upper bounds given in [5] be generalized to the k-Opt algorithm, i.e. for increasing k the performance guarantee improves?
- Can we show better upper bounds for the Lin-Kernighan algorithm than the upper bound obtained from the 2-Opt algorithm?

We also bound the approximation ratio of the k-Opt algorithm for Graph TSP.

▶ **Theorem 8.** *The approximation ratio of k-Opt with $k \geq 2$ for Graph TSP is $\Omega\left(\frac{\log(n)}{\log\log(n)}\right)$.*

▶ **Theorem 9.** *The approximation ratio of 2-Opt for Graph TSP is $O\left(\left(\frac{\log(n)}{\log\log(n)}\right)^{\log_2(9)+\epsilon}\right)$ for all $\epsilon > 0$.*

Note that the same upper bound also applies to the k-Opt algorithm and the Lin-Kernighan algorithm since they produce 2-optimal tours. Hence, up to a constant factor of at most $\log_2(9)$ in the exponent the k-Opt algorithm does not achieve asymptotically better performance than the 2-Opt algorithm in contrast to the metric case.

Outline of the paper. We start with the basic definitions we need for the analysis in the preliminaries. Then, an outline of the analysis roughly describes the main ideas for the lower and upper bounds on the approximation ratio for Metric and Graph TSP. In the main part of the paper we will only focus on the upper bound on the approximation ratio of the k-Opt algorithm for Metric TSP. Note that the same analysis can be carried over to the Lin-Kernighan algorithm by showing that the k-moves we consider can be performed by augmenting appropriate alternating cycles. For more details on this and the analysis of the other bounds we refer to the full version of the paper.

1.1 Preliminaries

1.1.1 TSP

An instance of Metric TSP is given by a complete weighted graph (K_n, c) where the costs are non-negative and satisfy the triangle inequality: $c(\{x,z\}) + c(\{z,y\}) \geq c(\{x,y\})$ for all $x, y, z \in V(K_n)$. A *cycle* is a closed walk that visits every vertex at most once. A *tour* is a cycle that visits every vertex exactly once. For a tour T, let the *length* of the tour be defined as $c(T) := \sum_{e \in T} c(e)$. The task is to find a tour of minimal length. We fix an *orientation* of the tour, i.e. we consider the edges of the tour as directed edges such that the tour is a directed cycle. From now on, let n denote the number of vertices of the instance.

Graph TSP is a special case of the Metric TSP. Each instance arises from an unweighted, undirected connected graph G. To construct a TSP instance (K_n, c), we set $V(K_n) = V(G)$. The cost $c(\{u,v\})$ of the edge connecting any two vertices $u, v \in V(G)$ is given by the length of the shortest u-v-path in G.

An algorithm A for the traveling salesman problem has *approximation ratio* $\alpha(n) \geq 1$ if for every TSP instance with n vertices it finds a tour that is at most $\alpha(n)$ times as long as a shortest tour and this ratio is achieved by an instance for every n. Note that we require here the sharpness of the approximation ratio deviating from the standard definition in the literature to express the approximation ratio in terms of the Landau symbols. Nevertheless, the results also hold for the standard definition with more complicated notation.

1.1.2 k-Opt and Lin-Kernighan Algorithm

A *k-move* replaces at most k edges of a given tour by other edges to obtain a new tour. It is called *improving* if the resulting tour is shorter than the original one. A tour is called *k-optimal* if there is no improving k-move.

For the 2-Opt algorithm recall the following well known fact: Given a tour T with a fixed orientation, it stays connected if we replace two edges of T by the edge connecting their heads and the edge connecting their tails, i.e. if we replace edges $(a,b), (c,d) \in T$ by (a,c) and (b,d).

An *alternating walk* of a tour T is a walk starting with an edge in T where exactly one of two consecutive edges is in T. An edge of the alternating walk is called *tour edge* if it is contained in T, otherwise it is called *non-tour edge*. A *closed alternating walk* and *alternating cycle* are alternating walks whose edges form a closed walk and cycle, respectively.

We consider a parameterized version of the Lin-Kernighan algorithm described in Section 21.3 of [16] for the analysis. In this version two parameters p_1 and p_2 specify the depth the algorithm is searching for an improvement. Since this extended abstract will focus on the k-Opt algorithm, we do not describe the Lin-Kernighan algorithm here and refer to the Section 21.3 of [16] or the full version of the paper.

1.2 Girth and Ex

The *girth* of a graph is the length of the shortest cycle contained in the graph if it contains a cycle and infinity otherwise. Let $\mathrm{ex}(n, 2k)$ be the maximum number of edges in a graph with n vertices and girth at least $2k$. Moreover, define $\mathrm{ex}^{-1}(m, 2k)$ as the minimal number of vertices of a graph with m edges and girth at least $2k$.

2 Outline of the Analysis

In this section we give an outline of the analysis for the lower and upper bounds of k-Opt for the Metric TSP and Graph TSP.

2.1 Outline of Lower Bound for Metric TSP

In this subsection we sketch the lower bound of k-Opt for the Metric TSP given by Theorem 2. We use the following lemma from [5]:

▶ **Theorem 10** (Lemma 3.6 in [5]). *Suppose there exists a Eulerian unweighted graph $G_{k,n,m}$ with n vertices and m edges, having girth at least $2k$. Then, there is a Metric TSP instance with m vertices and a k-optimal tour T such that $\frac{c(T)}{c(T^*)} \geq \frac{m}{2n}$, where T^* is the optimal tour of the instance.*

For the previous lower bound the theorem was applied to regular Eulerian graphs with high girth. Instead, we show that for every graph that there is a Eulerian subgraph with similar edge vertex ratio and apply the theorem to the Eulerian subgraphs of dense graphs with high girth to get the new bound.

2.2 Outline of Upper Bound for Metric TSP

In this subsection we briefly summarize the ideas for the analysis of the upper bound for the Metric TSP given by Theorem 1. For a fixed k assume that an instance is given with a k-optimal tour T. We fix an orientation of T and assume w.l.o.g. that the length of the optimal tour is 1. To bound the approximation ratio it is enough to bound the length of T. Our general strategy is to construct an auxiliary graph depending on T and bound its girth. More precisely, we show that if this graph has a short cycle this would imply the existence of an improving k-move contradicting the k-optimality of T. Moreover, the auxiliary graph contains many long edges of T so the bound on its girth also bounds the number of long edges in the tour and hence the approximation ratio.

Let the graph G consist of the vertices of the instance and the edges of T, i.e. $G := (V(K_n), T)$. We first divide the edges of T in length classes such that the lth length class consists of the edges with length between c^{l+1} and c^l for some constant $c < 1$, we call these edges *l-long*. For each $l \in \mathbb{N}_0$ we want get an upper bound on the number of l-long edges that depends on the number of vertices.

If we performed the complete analysis on G, we would get a bad bound on the number of l-long edges since G contains too many vertices. To strengthen the result we first construct an auxiliary graph containing all l-long edges for some fixed l but fewer vertices and bound the number of l-long edges in that graph: We partition $V(G)$ into classes with help of the optimal tour such that in each class any two vertices have small distance to each other. We contract the vertices in each class to one vertex and delete self loops to get the multigraph G_1^l. We can partition $V(G)$ in such a way that G_1^l contains all the l-long edges. Note we did not delete parallel edges in G_1^l and hence every edge in G_1^l has a unique preimage in G.

Unfortunately, we cannot directly bound the girth of G_1^l since the existence of a short cycle would not necessarily imply an improving k-move for T. For that we need a property of the cycles in the graph: The common vertex of consecutive edges in any cycle has to be head of both or tail of both edges according to the orientation of T. Therefore, we construct the auxiliary graph G_2^l from G_1^l as follows: We start with G_2^l as a copy of G_1^l and color the vertices of G_2^l red and blue. We only consider l-long edges in G_2^l from a red vertex to a blue vertex according to the orientation of T and delete all other edges. We can show that the coloring can be done in such a way that at least $\frac{1}{4}$ of the l-long edges remain in G_2^l.

We claim that the underlying undirected graph of G_2^l has girth at least $2k$. Note that by construction the graph is bipartite and hence all cycles have even length. Assume that there is a cycle C with $2h < 2k$ edges. We call the preimage of the edges of C in G the *C-edges*. Our aim is to construct a tour T' with the assistance of C that arises from T by an improving k-move.

For every common vertex w of two consecutive edges e_1, e_2 of C in G_2^l we consider the preimage e_1^{-1}, e_2^{-1} of e_1, e_2 in G. Then there have to be endpoints $u \in e_1^{-1}$ and $v \in e_2^{-1}$ such that the images of u and v after the contraction in G_2^l are both w. We will call the edge $\{u, v\}$ a *short edge*. In fact since both endpoints of a short edge are mapped to the same vertex in G_1^l after the contraction and we contracted vertices which have small distance to each other, they are indeed short. Furthermore, we can show that the total length of all the short edges is shorter than that of any single C-edge. The number of the short edges is equal to the number of C-edges which is $2h$. Now, observe that the cycle C defines an alternating cycle in G in a natural way: Let the preimages of C in G be the tour edges and the short edges be the non-tour edges.

To construct a new tour T' from T we start by augmenting the alternating cycle. Afterwards, the tour may split into at most $2h$ connected components. A key property is that the coloring of the vertices in G_2^l ensures that every connected component contains at least two short edges. Since there are $2h$ short edges, we know that after the augmentation we actually get at most h connected components. To reconnect and retain the degree condition we add twice a set L of at most $h-1$ different C-edges, i.e. in total at most $2h-2$ edges. In the end we shortcut to the new tour T' in a particular way without decreasing $|T \cap T'|$.

Note that the original tour T contains $2h$ C-edges, thus T' contains at least 2 fewer C-edges than T. The additional short edges T' contains are cheap, therefore T' is cheaper than T. Moreover, T' arises from T by replacing at most $2h - |L|$ C-edges since we deleted the C-edges and added twice the set L consisting of C-edges. Therefore, we know that T' arises from T by a $2h - |L| \leq 2h$-move. By the k-optimality of T, we have $2h > k$ or $2h \geq k+1$. This already gives us a lower bound of $k+1$ for the girth of the graph G_2^l as C contains $2h$ edges.

In the next step we use the previous result to show that there is actually a cheaper tour T' that arises by an $h+1$-move. This implies that $h+1 > k$ or $2h \geq 2k$, i.e. the girth of G_2^l is at least $2k$. As we have seen above the number of edges we have to replace to obtain T' from T depends on $|L|$, the number of C-edges T' contains. Therefore, we modify T' iteratively such that the number of C-edges in T' increases by 1 after every iteration while still maintaining the property that T' is cheaper than T. We stop when the number of C-edges in T' is $h-1$ as then T' would arise from T by a $2h-(h-1) = h+1$-move.

To achieve this we start with the constructed tour T' and iteratively perform 2-moves that are not necessarily improving but add one more C-edge to T'. In every iteration we consider C-edges e not in the current tour T'. We can show that there is an edge in $T' \backslash T$ incident to each of the endpoints of e. Let the two edges be f_1 and f_2. We want to replace f_1 and f_2 in T' by e_1 and the edge connecting the endpoints of f_1 and f_2 not incident to e. To ensure the connectivity after the 2-move we need to find edges e such that the corresponding edges f_1, f_2 fulfill the following condition: Either both heads or both tails of f_1 and f_2 have to be endpoints of e. It turns out that we can find such edges e in enough iterations to construct T' with the desired properties.

In the end we notice that a lower bound on the girth of G_2^l gives us an upper bound on the number of edges in G_2^l by previous results on extremal graph theory. This implies an upper bound on the number of l-long edges as G_2^l contains at least $\frac{1}{4}$ of the l-long edges in T. That gives us an upper bound on the length of T and thus also an upper bound on the approximation ratio as we assumed that the optimal tour has length 1.

2.3 Outline of Lower Bound for Graph TSP

For the lower bound of Graph TSP given by Theorem 8 let an integer f be given. We construct an instance with approximation ratio $\Theta(f)$ and $(c_1 f)^{c_2 f}$ vertices for some constants $c_1, c_2 > 0$. Thus, the approximation ratio is $\Omega\left(\frac{\log n}{\log\log n}\right)$.

The construction starts with a dense $2f$-regular Eulerian graph G with high girth. Let $W = (v_0, v_1, \ldots, v_{|E(G)|-1})$ be a Eulerian walk of G. Traverse through G according to W starting at v_0 and mark every fth vertex both in G and in W. Whenever we would mark an already marked vertex v in G, we add a new copy v' of v adjacent exactly to the neighbors of v and mark v' instead. Moreover, we replace this occurrence of v in W by v' and mark v'. Let G' be the graph containing G and all the copies of the vertices we made. After the traversal of W, we mark for every unmarked vertex in G' one occurrence of it in W. Since we only need the property that every vertex of G' is marked in W it does not matter which occurrence we mark. The tour T consists of the edges connecting consecutive marked vertices in W.

The proof that T is k-optimal uses the same basic idea as the lower bound of the approximation ratio in the metric case in [5]: If there was an improving k-move, it has to contain an alternating cycle with negative cost. By construction, the length of every edge in T is bounded by f. Thus, such an alternating cycle would imply the existence of a short cycle in G' that can be transformed to a short cycle in G contradicting its high girth.

The constructed instance has approximation ratio $\Theta(f)$ since on the one hand, almost every edge in T has length $\Theta(f)$ leading to a total length of approximately $f|V(G')|$. On the other hand, the length of the optimal tour can be bounded by twice the length of the minimum spanning tree which is at most $2|V(G')|$ in the case of Graph TSP.

2.4 Outline of Upper Bound for Graph TSP

This subsection comprises a sketch of the proof of Theorem 9. Assume that an instance of Graph TSP (K_n, c) is given where c arises from the unweighted graph G. Let a 2-optimal tour T be given for the instance and fix an orientation.

First, note that every edge with length l corresponds to shortest paths with l edges in G between the endpoints of the edges. Now, if the corresponding shortest paths of two edges share a common directed edge, we see that there is an improving 2-move contradicting the assumed 2-optimality of T (Figure 1). Hence, the directed edges of the corresponding shortest paths are disjoint. Note that the optimal tour contains n edges and hence has length at least n. Thus, if the approximation ratio is high, we must have many edges in the union of the shortest paths corresponding to the edges in T and hence also in G. The main challenge now is to exploit this fact in a good way since a simple bound of $n(n-1)$ on the number of directed edges in G would only give an upper bound of $O(n)$ on the approximation ratio, which is worse than the upper bound of $O(\sqrt{n})$ for Metric TSP.

To get a better result we use the same idea from the analysis of the upper bound for Metric TSP: We contract vertices and get a graph with fewer vertices and many edges. Instead of contracting once, we iteratively partition the vertices into sets and contract each set to a single vertex to get a new graph. (We note that we actually just contract the vertices and construct the edges of the new graph in a slightly different way. But let us assume for simplicity that the edges of the new graph are images of the contraction of edges in the old graph.) Starting with G in every iteration we ideally want to partition the vertices of the current graph into sets, contract each set to a vertex and delete self loops such that:

1. The number of vertices decreases much faster than the number of edges.
2. The subgraphs induced by the sets we contract have small diameter.

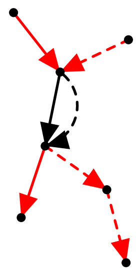

Figure 1 The solid and dashed edges are shortest paths that correspond to two edges in T. If they share a directed edge, there exists an improving 2-move replacing these two edges. The cost of the new edges is bounded by the number of the red edges which is less than the total cost of the two original edges.

The first condition ensures that we get a better bound after every iteration. The second condition builds the connection between the approximation ratio and the number of edges in the contracted graph: It ensures that if the shortest paths corresponding to two edges of T share a directed edge in the contracted graph, then they are also not far away in G, so there is an improving 2-move replacing these two edges. This means that a high approximation ratio would imply a high number of edges in the contracted graph.

Unfortunately, it is not easy to ensure both conditions at the same time even if we know that the graph has many edges as the edges are not equally distributed in the graph. It might happen that there are many vertices with very low degree. If we contract them while still ensuring that the subgraphs have small diameter, the number of vertices cannot decrease fast enough. Therefore, we consider a subset of vertices we call *active* vertices and only require that the number of active vertices decreases fast. If an active vertex has low degree, we will not contract it and consider it as inactive in future iterations. Initially, all vertices are active and we use the following theorem from [11] to find a good partition of the active vertices:

▶ **Theorem 11** (Theorem 6 in [11]). *Given $\epsilon > 0$ every graph G on n vertices can be edge partitioned $E = E_0 \cup E_1 \cup \cdots \cup E_l$ such that $|E_0| \leq \epsilon n^2, l \leq 16\epsilon^{-1}$, and for $1 \leq i \leq l$ the diameter of E_i is at most 4.*

In every iteration we apply the theorem to the subgraph induced by the currently active vertices. The vertices only incident to edges in E_0 become inactive after this iteration. For each of the sets $E_1, \ldots, E_l$ we contract the vertices incident to an edge in the set to a single vertex. These are the active vertices in the next iteration. By choosing ϵ appropriately, we can ensure that the number of vertices decreases significantly and the number of vertices that become inactive in every iteration is small.

After a fixed number of iterations, we have at least one edge and one active vertex remaining. Since the number of active vertices decreased much faster than the edges, we can conclude that G only contains few edges compared to the number of vertices. This implies a bound on the approximation ratio.

3 Upper Bound for Metric TSP

In this section we give an upper bound on the approximation ratio of the k-Opt algorithm.

Fix a $k > 2$ and assume that a worst-case instance with n vertices is given. Let T be a k-optimal tour of this instance. We fix an orientation of the optimal tour and T. Moreover, let w.l.o.g. the length of the optimal tour be 1. We divide the edges of T into length classes.

▶ **Definition 12.** *An edge e is l*-long *if* $(\frac{4k-5}{4k-4})^{l+1} < c(e) \leq (\frac{4k-5}{4k-4})^{l}$.

Note that the shortest path between every pair of vertices has length at most $\frac{1}{2}$ since the optimal tour has length 1. Thus, by the triangle inequality every edge with positive length in T has length at most $\frac{1}{2}$ and is l-long for exactly one l. For every l we want to bound the number of l-long edges. Let us consider from now on a fixed l. In the following we define auxiliary graphs we need for the analysis and show some useful properties of them.

▶ **Definition 13.** *We view the optimal tour as a circle with circumference 1. Let the vertices of the instance lie on that circle in the order of the oriented tour where the arc distance of two consecutive vertices is the length of the edge between them. Divide the optimal tour circle into* $4(k-1)\lceil(\frac{4k-4}{4k-5})^{l}\rceil$ *consecutive arcs of length* $\frac{1}{4(k-1)\lceil(\frac{4k-4}{4k-5})^{l}\rceil}$. *Two vertices are called* near *to each other if they lie on the same arc.*

▶ **Definition 14.** *Let the directed graph $G := (V(K_n), T)$ consist of the vertices of the instance and the oriented edges of T (an example is shown in Figure 2, the colors of the edges will be explained later). The directed multigraph G_1^l arises from G by contracting all vertices near to each other to a vertex and deleting self-loops (Figure 3).*

Note that G_1^l may contain parallel edges. Moreover, edges between vertices which are near to each other are not l-long and hence G_1^l contains all l-long edges.

▶ **Lemma 15.** *There exists a coloring of the vertices of G_1^l with two colors such that at least $\frac{1}{4}$ of the l-long edges in G_1^l go from a red vertex to a blue vertex according to the fixed orientation of T.*

Proof Sketch. The proof is similar to the standard proof that a maximal cut of a graph contains at least $\frac{1}{2}$ of the edges (see for example Theorem 5.3 in [27]). ◀

▶ **Definition 16.** *We obtain the directed multigraph G_2^l by coloring the vertices of G_1^l red and blue according to Lemma 15 and deleting all edges that are not l-long edges from a red vertex to a blue vertex according to the fixed orientation of T (Figure 4, the colors of the edges will be explained later).*

Now, we claim that the underlying undirected graph of G_2^l has girth at least $2k$. In particular, it is a simple graph. Assume the contrary, then there has to be a cycle C with $2h < 2k$ edges since G_2^l is bipartite by construction. We call the preimage of the edges of C in G the *C-edges*. Note that the preimages are unique since we do not delete parallel edges after the contraction.

▶ **Definition 17.** *Let the* connecting paths *be the connected components of $(V(K_n), T \backslash C)$, i.e. the paths in T between consecutive heads and tails of C-edges (the red edges in Figure 2 and 4). Define* head *and* tail *of a path p as the head of the last edge and the tail of the first edge of p according to the orientation of T, respectively. The head and tail of a connecting path are also called the* endpoints *of the connecting path.*

▶ **Definition 18.** *For any two endpoints v_1, v_2 of C-edges in G which are near to each other we call the edge $\{v_1, v_2\}$ a* short edge.

The definition of near ensures that the short edges are indeed short. In fact the total length of all short edges is smaller than that of any C-edge. The number of short edges is $2h$ which is equal to the number of C-edges.

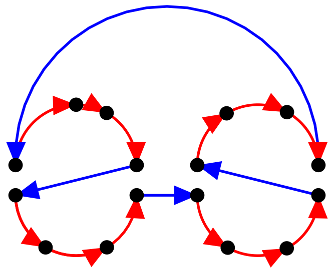

Figure 2 An example instance with a k-optimal tour, i.e. the directed graph G. The blue and red edges are the C-edges and connecting path edges that arise from the chosen cycle in G_2^l in Figure 4, respectively.

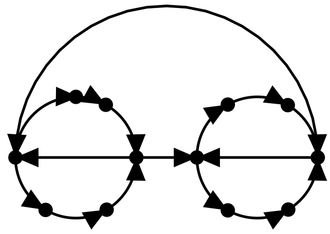

Figure 3 The directed multigraph G_1^l: We contracted vertices that lie near to each other in the optimal tour. Note that the optimal tour is not drawn here, so it is not clear from the figure which vertices to contract.

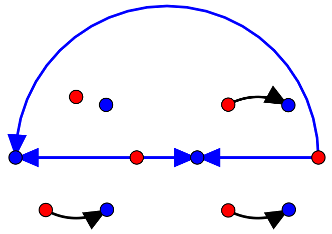

Figure 4 The directed multigraph G_2^l: Coloring the vertices and only considering the l-long edges from red to blue. In this example the upper left edge is not l-long and hence not drawn. The blue edges form the undirected cycle C, the black edges are the remaining edges of the connecting paths corresponding to this cycle.

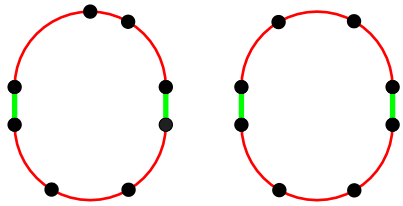

Figure 5 The graph $G_3^{l,C}$: The green edges are the short edges, the red edges are the connecting paths.

▶ **Definition 19.** *We construct the graph $G_3^{l,C}$ as follows: The vertex set of $G_3^{l,C}$ is that of G and the edge set consists of the connecting paths and the short edges (Figure 5).*

▶ **Lemma 20.** *$E(G_3^{l,C})$ is the union of at most h disjoint cycles.*

Proof Sketch. We can show that the degree of every vertex in $G_3^{l,C}$ is two. Moreover, we can see by considering the incident C-edges that the two endpoints of a connecting path are not near to each other since otherwise the common endpoint of the connecting path in G_2^l has to be colored red and blue. Hence, every connected component consists of at least two out of $2h$ connecting paths and we have at most h disjoint cycles. ◀

Now, we show that the existence of C implies that there is an improving k-move contradicting the k-optimality of T.

▶ **Lemma 21.** *There is a tour T' containing the connecting paths and $u-1$ C-edges, where u is the number of connected components of $G_3^{l,C}$.*

Proof Sketch. We construct such a tour T'. Start with a graph G' with the same vertex set and edge set as $G_3^{l,C}$. First, add a set of C-edges to $E(G')$ that makes the graph connected. This is possible since T consists of the C-edges and connecting paths and is connected. We call these C-edges the *fixed C-edges.* Next, add another copy of the fixed C-edges (Figure 6). After shortcutting in a particular way without decreasing $|T \cap T'|$, we get a tour with the desired properties. ◀

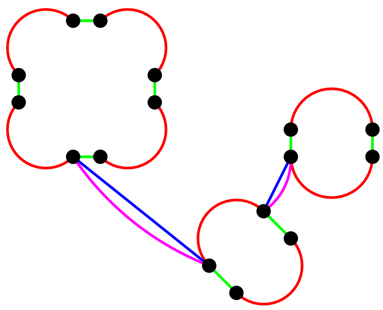

Figure 6 Sketch for Lemma 21. The red curves represent the connecting paths. The green edges are the short edges, the blue edges are the fixed C-edges and the pink edges are the copies of the fixed C-edges. The tour T' results from shortcutting the green and pink edges while leaving the other edges fixed.

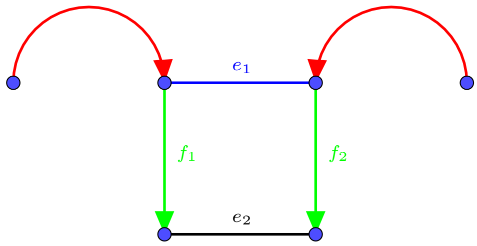

Figure 7 Sketch for Lemma 25. The drawn orientation is that of T'. The red curves represent oppositely oriented connecting paths connected by a C-edge e_1. The green edges f_1 and f_2 are the non-connecting path edges of T' incident to e_1. The edge e_2 connects the other two endpoints of f_1 and f_2 not incident to e_1.

▶ Remark 22. The last lemma already gives us a bound on the girth of G_2^l: By Lemma 20, $G_3^{l,C}$ has at most h connected components. Therefore, we added in the construction above at most $2h-2$ C-edges and some cheap short edges. As T contains $2h$ C-edges, we can show that T' is shorter than T. Moreover, T' arises from T by replacing at most $2h$ C-edges, i.e. by a $2h$-move. If $2h \leq k$, this would contradict the k-optimality of T, hence G_2^l has girth at least $k+1$.

Next, we improve this result and show that G_2^l has girth at least $2k$. We achieve this by starting at T' and iteratively performing 2-moves that are not necessarily improving but include one more C-edge in T'. We stop when the number of C-edges in T' is $h-1$. Then, T' arises from T by a $2h-(h-1)=h+1$ move.

▶ **Definition 23.** *Given a tour T' containing the connecting paths. An* ambivalent 2-move *replaces two non-connecting path edges of T' to obtain a new tour containing at least one more C-edge.*

▶ **Definition 24.** *Fix an orientation of T', we call a connecting path p* wrongly oriented *if the orientation of p in T' is opposite to the orientation in T. Otherwise, it is called* correctly oriented.

▶ **Lemma 25.** *If a tour T' contains a short edge and all connecting paths, then there is an ambivalent 2-move that increases the length of the tour by at most two C-edges.*

Proof Sketch. The coloring of the vertices in G_2^l ensures that every short edge e connects either two heads or two tails of connecting paths. If in addition $e \in T'$, one of them is correctly oriented and the other one is wrongly oriented. Thus, as long as there is a short edge in T', there has to be at least one correctly oriented and one wrongly oriented connecting path. In this case there has to be a C-edge e_1 connecting two oppositely oriented connecting paths since the C-edges connect the connecting paths to the tour T. By definition, every C-edge connects a head and a tail of two connecting paths. If $e_1 \in T'$, the incident connecting paths would be both correctly or both wrongly oriented. Thus, e_1 is not contained in T'. Let e_2, f_1 and f_2 be defined as in Figure 7. Now, we can make a 2-move replacing f_1, f_2 by e_1 and e_2 to obtain a new tour with the additional C-edge e_1. The property that e_1 connects two oppositely oriented connecting paths ensures that the tour stays connected after the 2-move. By the triangle inequality, we have $c(e_2) \le c(f_1) + c(e_1) + c(f_2)$ and thus each of the 2-moves increases the length of the tour by at most two C-edges. ◀

▶ **Lemma 26.** *T is not $h+1$-optimal.*

Proof Sketch. By Lemma 21, we can construct a tour T' using the connecting paths and $u-1$ C-edges where u is the number of connected components of $G_3^{l,C}$. We iteratively perform ambivalent 2-moves to increase the number of C-edges in T'. Since after every such 2-move the number of short edges decreases by at most two, we can perform by Lemma 25 a sufficient number of iterations such that we get a tour with $h-1$ C-edges. There are in total $2h$ C-edges, hence the resulting tour arises by an $2h-(h-1) = h+1$-move from T. As every ambivalent 2-move increases the length of the tour by at most two C-edges, we can show that in the end the resulting T' is still shorter than T. ◀

Since $h < k$, this is a contradiction to the assumption that T is k-optimal. Hence, such a cycle C with less than $2k$ edges cannot exist and this gives us a lower bound of $2k$ on the girth of G_2^l. Next, we conclude an upper bound on the length of T:

▶ **Corollary 27.** *For $l^* := \min_j \{j \mid \sum_{l=0}^{j} 4\,\mathrm{ex}(4(k-1)\lceil (\frac{4k-4}{4k-5})^l \rceil, 2k) \ge n\}$ we have*

$$c(T) \le \sum_{l=0}^{l^*} \frac{4\,\mathrm{ex}(4(k-1)\lceil (\frac{4k-4}{4k-5})^l \rceil, 2k)}{(\frac{4k-4}{4k-5})^l}.$$

Proof Sketch. Let q_l be the number of l-long edges in T. The definition of near ensures that two vertices which are near to each other have shorter distance than the length of any l-long edge. Hence, G_1^l also has q_l l-long edges. Since we have chosen a coloring according to Lemma 15, G_2^l has at least $\frac{1}{4}q_l$ edges. By the k-optimality and Lemma 26, G_2^l has girth at least $2k$ and thus at most $\mathrm{ex}(|V(G_2^l)|, 2k) \le \mathrm{ex}(4(k-1)\lceil (\frac{4k-4}{4k-5})^l \rceil, 2k)$ edges. Therefore, $q_l \le 4\,\mathrm{ex}(4(k-1)\lceil (\frac{4k-4}{4k-5})^l \rceil, 2k)$. This leads to a bound on the length of T. ◀

This is also a bound on the approximation ratio since the length of the optimal tour is 1. With certain assumptions about the growth of $\mathrm{ex}(n, 2k)$, we obtain the main result:

▶ **Theorem 28.** *If $\mathrm{ex}(x, 2k) \in O(x^c)$ for some $c > 1$, the approximation ratio of the k-Opt algorithm is $O(n^{1-\frac{1}{c}})$.*

By a rather technical calculation comparing the upper and lower bound, we get:

▶ **Theorem 29.** *The upper bound from Corollary 27 for the approximation ratio of k-Opt is tight up to a factor of $O(\log(n))$.*

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