A Polynomial Kernel for Line Graph Deletion

Eduard Eiben
Department of Computer Science, Royal Holloway University of London, Egham, UK
eduard.eiben@rhul.ac.uk

William Lochet
Department of Informatics, University of Bergen, Bergen, Norway
william.lochet@uib.no

Abstract

The line graph of a graph G is the graph $L(G)$ whose vertex set is the edge set of G and there is an edge between $e, f \in E(G)$ if e and f share an endpoint in G. A graph is called line graph if it is a line graph of some graph. We study the Line-Graph-Edge Deletion problem, which asks whether we can delete at most k edges from the input graph G such that the resulting graph is a line graph. More precisely, we give a polynomial kernel for Line-Graph-Edge Deletion with $\mathcal{O}\left(k^5\right)$ vertices. This answers an open question posed by Falk Hüffner at Workshop on Kernels (WorKer) in 2013.

2012 ACM Subject Classification Theory of computation → Design and analysis of algorithms; Theory of computation → Graph algorithms analysis; Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases Kernelization, line graphs, H-free editing, graph modification problem

Digital Object Identifier 10.4230/LIPIcs.ESA.2020.42

Funding *William Lochet*: Supported by The Bergen Research Foundation (BFS).

1 Introduction

For a family $\mathcal{G}$ of graphs, the general $\mathcal{G}$-Graph Modification problem asks whether we can modify a graph G into a graph in $\mathcal{G}$ by performing at most k simple operations. Typical examples of simple operations well-studied in the literature include vertex deletion, edge deletion, edge addition, or a combination of edge deletion and addition. We call these problems $\mathcal{G}$-Vertex Deletion, $\mathcal{G}$-Edge Deletion, $\mathcal{G}$-Edge Addition, and $\mathcal{G}$-Edge Editing, respectively. By a classical result by Lewis and Yannakakis [20], $\mathcal{G}$-Vertex Deletion is NP-complete for all non-trivial hereditary graph classes. The situation is quite different for the edge modification problems. Earlier efforts for edge deletion problems [13, 24], though having produced fruitful concrete results, shed little light on a systematic answer, and it was noted that such a generalization is difficult to obtain.

$\mathcal{G}$-Graph Modification problems have been extensively investigated for graph classes $\mathcal{G}$ that can be characterized by a finite set of forbidden induced subgraphs. We say that a graph is $\mathcal{H}$-free if it contains none of the graphs in $\mathcal{H}$ as an induced subgraph. For this special case, the $\mathcal{H}$-free Vertex Deletion is well understood. If $\mathcal{H}$ contains a graph on at least two vertices, then all of these problems are NP-complete, but admit a $c^k n^{\mathcal{O}(1)}$ algorithm [4], where c is the size of the largest graph in $\mathcal{H}$ (the algorithms with running time $f(k)n^{\mathcal{O}(1)}$ are called fixed-parameter tractable (FPT) algorithms [7, 11]). On the other hand, the NP-hardness proof of Lewis and Yannakakis [20] excludes algorithms with running time $2^{o(k)}n^{\mathcal{O}(1)}$ under the Exponential Time Hypothesis (ETH) [18]. Finally, as observed by Flum and Grohe [15] a simple application of sunflower lemma [14] gives a *kernel* with $\mathcal{O}(k^c)$ vertices, where c is again the size of the largest graph in $\mathcal{H}$. A kernel is a polynomial time preprocessing algorithm which outputs an equivalent instance of the same problem such that the size of the reduced instance is bounded by some function $f(k)$ that depends only on

28th Annual European Symposium on Algorithms (ESA 2020).
Editors: Fabrizio Grandoni, Grzegorz Herman, and Peter Sanders; Article No. 42; pp. 42:1–42:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

k. We call the function $f(k)$ the size of the kernel. It is well-known that any problem that admits an FPT algorithm admits a kernel. Therefore, for problems with FPT algorithms one is interested in polynomial kernels, i.e., kernels whose size is a polynomial function.

For the edge modification problems, the situation is more complicated. While all of these problems also admit $c^k n^{\mathcal{O}(1)}$ time algorithm, where c is the maximum number of edges in a graph in $\mathcal{H}$ [4], the P vs NP dichotomy is still not known. Only recently Aravind et al. [1] gave the dichotomy for the special case when $\mathcal{H}$ contains precisely one graph H. From the kernelization point of view, the situation is also more difficult. The reason is that deleting or adding an edge to a graph can introduce a new copy of H and this might further propagate. Hence, we cannot use the sunflower lemma to reduce the size of the instance. Cai asked the question whether H-free Edge Deletion admits a polynomial kernel for all graphs H [3]. Kratsch and Wahlström [19] showed that this is probably not the case and gave a graph H on 7 vertices such that H-free Edge Deletion and H-free Edge Editing does not admit a polynomial kernel unless coNP $\subseteq$ NP/poly. Consequently, it was shown that this is not an exception, but rather a rule [5, 16]. Indeed the result by Cai and Cai [5] shows that H-free Edge Deletion, H-free Edge Addition, and H-free-Edge Editing do not admit a polynomial kernel whenever H or its complement is a path or a cycle with at least 4 edges or a 3-connected graph with at least 2 edges missing. Very recently, Marx and Sandeep [21] gave a list of nine graphs, all on 5 vertices such that if H-free-Edge Editing does not admit a kernel for any of these nine graphs under standard complexity assumptions, then H-free-Edge Editing admits a polynomial kernel for $|H| \geq 5$ if and only if H is either empty or complete graph. They also provided a similar characterization for H-free Edge Deletion and H-free Edge Editing. This suggests that actually the H-free edge modification problems with a polynomial kernels are rather rare and only for small graphs H. Recently, Eiben, Lochet, and Saurabh [12] announced a polynomial kernel for the case when H is a paw, which leaves only one last graph on 4 vertices for which the kernelization of H-free edge modification problems remains open, namely $K_{1,3}$ known also as the claw.

The class of claw-free graphs is a very well studied class of graphs with some interesting algorithmic properties. The most prominent example is probably the algorithm of Sbihi [22] for computing the maximum independent set in polynomial time. It also has been extensively studied from a structural point of view, and Chudnosky and Seymour proposed, after a series of papers, a complete characterization of claw-free graphs [6]. Because of such a characterization, it seems reasonable to believe that a polynomial kernel for Claw-free Edge Deletion exists. However, the characterization of Chudnosky and Seymour is quite complex, which makes it hard to use. For this reason, as noted by Cygan et al. [8], trying to show the existence of a polynomial kernel in the cases of sub-classes of claw-free graphs seems like a good first step to try to understand this problem. In this paper, we prove the result for the most famous such class, line graphs.

▶ **Theorem 1.** *Line-Graph Edge Deletion admits a kernel with $\mathcal{O}\left(k^5\right)$ vertices.*

Overview of the Algorithm

As the first step of the kernelization algorithm, we use the characterization of line graphs by forbidden induced subgraphs to find a set S of at most $6k$ vertices such that for every vertex $v \in S$, $G - (S \setminus \{v\})$ is a line graph. This is simply done by a greedy edge-disjoint packing of forbidden induced subgraphs. Having the set S, we use the algorithm by Degiorgi and Simon [9] to find a partition of edges of $G - S$ into cliques such that each vertex is in precisely 2 cliques. Let $\mathcal{C} = \{C_1, \ldots, C_q\}$ be the cliques in the partition. Since $G - (S \setminus \{v\})$

is also a line graph, it is a rather simple consequence of Whitney's isomorphism theorem that the neighborhood of v can be covered by constantly many cliques of $\mathcal{C}$. Furthermore, we will show that if a clique C in $\mathcal{C}$ has more than $k+7$ vertices then the optimal solution does not contain an edge in C. Hence, we can partition the cliques in $\mathcal{C}$ into two groups "large" and "small". Note that if the optimal solution contains an edge in some small clique C, then for this change to be necessary, it has to be propagated from S by modifying small cliques on some clique-path from S to C using only small cliques. We will therefore define the distance of a clique to S, without going into too many details in here, to be basically the length of a shortest clique-path from the clique to S using only small cliques. Since there are only $\mathcal{O}(|S|)$ cliques in immediate neighborhood of S and the number of cliques in the neighborhood of a small clique is bounded by its size, we obtain that there are at most $\mathcal{O}\left(k^d\right)$ cliques at distance at most d. Our main contribution and most technical part of our proof is to show that we can remove the edges covered by cliques at distance at least 5 from G. This is covered in Section 4. Afterwards we end up with an instance with all cliques in $\mathcal{C}$ at distance at least 5 from S being singletons. As discussed above there are only $\mathcal{O}\left(k^4\right)$ cliques at distance at most 4 and because large cliques stay intact in any optimal solution, it suffices to keep $k+7$ vertices in each large clique, which leads to the desired kernel of size $\mathcal{O}\left(k^5\right)$.

2 Preliminaries

We assume familiarity with the basic notations and terminologies in graph theory. We refer the reader to the standard book by Diestel [10] for more information. Given a graph G and a set of edges $F \subseteq E(G)$, we denote by $G - F$ the graph whose set of vertices is $V(G)$ and set of edges is the set $E(G) \setminus F$. Given two vertices $u, v \in V(G)$, we let the *distance* between u and v in G, denoted $\text{dist}_G(u,v)$), be the number of edges on a shortest path from u to v. Furthermore, for $S \subseteq V(G)$ and $u \in V(G)$ we let $\text{dist}_G(u,S) = \min_{v\in S} \text{dist}_G(u,v)$). We omit the subscript G, if the graph is clear from the context.

Parameterized Algorithms and Kernelization. For a detailed illustration of the following facts the reader is referred to [7, 11]. A *parameterized problem* is a language $\Pi \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet; the second component k of instances $(I,k) \in \Sigma^* \times \mathbb{N}$ is called the *parameter*. A parameterized problem Π is *fixed-parameter tractable* if it admits a *fixed-parameter algorithm*, which decides instances (I,k) of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$ for some computable function f.

A *kernelization* for a parameterized problem Π is a polynomial-time algorithm that given any instance (I,k) returns an instance (I',k') such that $(I,k) \in \Pi$ if and only if $(I',k') \in \Pi$ and such that $|I'| + k' \leq f(k)$ for some computable function f. The function f is called the *size* of the kernelization, and we have a polynomial kernelization if $f(k)$ is polynomially bounded in k. It is known that a parameterized problem is fixed-parameter tractable if and only if it is decidable and has a kernelization. However, the kernels implied by this fact are usually of superpolynomial size.

A *reduction rule* is an algorithm that takes as input an instance (I,k) of a parameterized problem Π and outputs an instance (I',k') of the same problem. We say that the reduction rule is *safe* if (I,k) is a *yes*-instance if and only if (I',k') is a *yes*-instance. In order to describe our kernelization algorithm, we present a series of reduction rules.

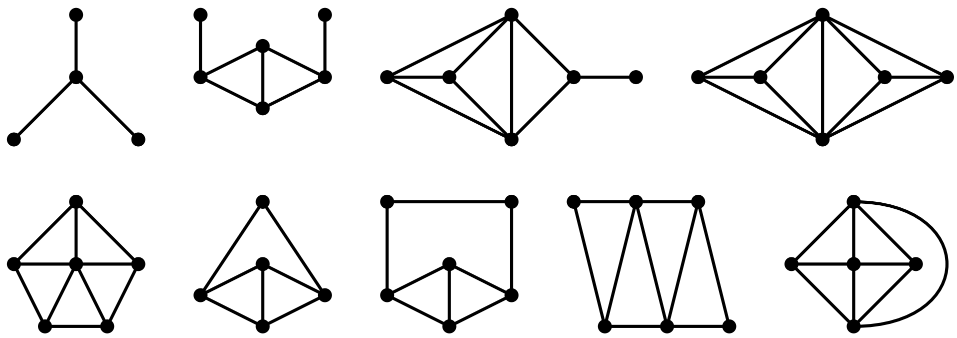


Figure 1 The nine minimal non-line graphs, from characterization of line graphs by forbidden induced subgraphs of Beineke [2]. Note that all of these graphs have at most 6 vertices.

Line graphs. Given a graph G, its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (are incident) in G. It is well known that if the line graphs of two connected graphs G_1 and G_2 are isomorphic then either G_1 and G_2 are K_3 and $K_{1,3}$, respectively, or G_1 and G_2 are isomorphic as well (Whitney's isomorphism theorem [23], see also Theorem 8.3 in [17]). We say that a graph H is a line graph, if there exists a graph G such that $H = L(G)$. Note that in this paper we only consider simple graphs, *i.e.*, the graphs without loops or multiple edges and in particular we also only consider line graphs of simple graphs. Formally, we then study the following parameterized problem:

LINE-GRAPH-EDGE DELETION	
Input:	A graph $G = (V, E)$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a set of edges $F \subseteq E(G)$ such that $G - F$ is a line graph and $\lvert F\rvert \leq k$.

We call a set of edges $F \subseteq V(G)$ such that $G - F$ is a line graph a *solution* for G. A solution F is *optimal*, if there does not exists a solution F' such that $|F'| < |F|$. To obtain our kernel, we will make use of several equivalent characterizations of line graphs.

▶ **Theorem 2** (see, e.g., Theorem 8.4 in [17]). *The following statements are equivalent:*

(1) *G is a line graph.*

(2) *The edges of G can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

(3) *G does not have $K_{1,3}$ as an induced subgraph, and if two odd triangles (triangles with the property that there exists another vertex adjacent to an odd number of triangle vertices) share a common edge, then the subgraph induced by their vertices is K_4.*

(4) *None of nine graphs of Figure 1 is an induced subgraph of G.*

3 Structure of Line Graphs

To obtain our kernel, we heavily rely on different characterizations of line graphs given by Theorem 2. The two main characterizations used throughout the paper are given in points (2) and (4) To ease the presentation of our techniques, we will define a notion of a *clique partition witness* for G, whose existence is implied by the point (2) of Theorem 2. Let G be a line graph, a *clique partition witness* for G is a set $\mathcal{C} = \{C_1, \dots, C_q\}$ be such that:

- $C_i \subseteq V(G)$ for all $i \in [q]$,
- $G[C_i]$ is a complete graph for all $i \in [q]$, that is every C_i is a clique in G,
- $|C_i \cap C_j| \leq 1$ for all $i \neq j \in [q]$,
- every $v \in V(G)$ is in exactly two sets in $\mathcal{C}$, and
- for every edge $uv \in E(G)$ there exists exactly one set $C_i \in \mathcal{C}$ such that $\{u, v\} \subseteq C_i$.

Note that by Theorem 2, G is a line graph if and only if there exists a clique partition witness for G. The following three observations follow directly from the definition of clique partition witness and will be useful throughout the paper.

▶ **Observation 3.** *If $\mathcal{C}$ is clique partition witness for G then every clique in $\mathcal{C}$ is either a singleton, K_2, or a maximal clique in G.*

▶ **Observation 4.** *If $\mathcal{C}$ is clique partition witness for G, then every maximal clique in G of size at least 4 is in $\mathcal{C}$.*

▶ **Observation 5.** *If $\mathcal{C}$ is clique partition witness for G, then any clique of G which is not a sub-clique of some element of $\mathcal{C}$ is a triangle.*

We would like to point out that given a line graph G one can find a clique partition witness for G for example by using an algorithm of Degiorgi and Simon [9] for recognition of line graphs in polynomial time. In the following lemma, we sketch the main procedure of their algorithm together with necessary modifications to actually output a clique partition witness instead of the underlying graph H such that $G = L(H)$, for completeness.

▶ **Lemma 6.** *Given a graph G, there is an algorithm that in time $\mathcal{O}(|E(G)| + |V(G)|)$ decides whether G is a line graph and if so, constructs a clique partition witness for G.*

Proof. The algorithm by Degiorgi and Simon construct the input graph G by adding vertices one at a time, at each step it chooses a vertex to add that is already adjacent to at least one previously-added vertex. That is it construct graphs G_1, G_2, …, $G_n = G$ such that G_i is a connected subgraph of G on i vertices. At each step it maintains a graph H_i such that G_i is a line graph of H_i. In here, we can actually keep a clique partition witness $\mathcal{C}_i$ for G_i such that there is a bijection φ_i between vertices of H_i and clique in $\mathcal{C}_i$ such that $uv \in E(H_i)$ if and only if $|\varphi_i(u) \cap \varphi_i(v)| = 1$.

The algorithm heavily relies on the Whitney's isomorphism theorem that implies that if the underlying graph of G_i has at least 4 vertices, then the underlying graph H_i is unique up to isomorphism. When adding a vertex v to a graph G_i for $i \leq 4$, the algorithm simply brute-forces the possibilities for H_i and $\mathcal{C}_i$.

When adding a vertex v to G_i when $i > 4$, let S be the subgraph of H_i formed by the edges that correspond to the neighbors of v in G_i. Check that S has a vertex cover consisting of one vertex or two non-adjacent vertices, *i.e.*, there are cliques C_1 and C_2 in $\mathcal{C}_i$ with $C_i \cap C_2 = \emptyset$ and $S \subseteq C_1 \cap C_2$. If there are two vertices in the cover, add an edge (corresponding to v) that connects these two vertices in H_i and add v to both C_1 and C_2. If there is only one vertex u in the cover, then add a new vertex to H_i, adjacent to this vertex, add v to the clique $\varphi_i(u)$ in $\mathcal{C}_i$ and add a new clique $\{v\}$ to $\mathcal{C}_i$ to create $\mathcal{C}_{i+1}$. ◀

3.1 Level Structure of Instances

For the rest of the paper, let G be the input graph and let S be a set of at most $6k$ vertices such that for every $v \in S$ the graph $G - (S \setminus \{v\})$ is a line graph. We let $\mathcal{C} = \{C_1, \ldots, C_q\}$ be a clique partition witness for $G - S$. The goal of this subsection is to split the cliques in

$\mathcal{C}$ to levels such that 1) each level contains only bounded number of cliques (that are not singletons) and 2) if we do not remove any edge at level i, then we do not need to remove any edge at level $j > i$. We will later show that we do not need to remove any edges in cliques in level 5. The following lemma is useful to define/bound the number of cliques at the first level, i.e., cliques that interact with S.

▶ **Lemma 7.** *For every vertex $v \in S$ there are at most two cliques $C_1, C_2 \in \mathcal{C}$ such that v is adjacent to all vertices in $C_1 \cup C_2$ and to at most 6 vertices in $V(G) \setminus (S \cup C_1 \cup C_2)$.*

Proof. By the choice of the set S, it follows that $G - (S \setminus \{v\})$ is a line graph. Let $\mathcal{C}'$ be clique partition witness for $G - (S \setminus \{v\})$. By definition, there are at most two cliques C_1' and C_2' in $\mathcal{C}'$ that contains v and all its neighbors. If $|C_i'| \geq 5$, for some $i \in \{1, 2\}$, then by Observation 4, $C_i' \setminus \{v\}$ is a clique in $\mathcal{C}$ and we can set C_i to be $C_i' \setminus \{v\}$. Else $|C_i' \setminus v| \leq 3$ and C_i' contributes to at most 3 neighbors of v in $G - S$. ◀

The following lemma shows that cliques of size at least $k+7$ can serve as kind of separators that will never be changed by a solution of size at most k. Hence, we can remove all cliques separated from S by large cliques. Moreover, it allows us to define the $(i+1)$-st level by only considering the cliques of size at most $k+6$ at level i.

▶ **Lemma 8.** *Let $C \in \mathcal{C}$ such that $|C| \geq k+7$ and let $A \subset E(G)$ be an optimal solution for G. Then $A \cap E(G[C]) = \emptyset$. Moreover, the clique partition witness $\mathcal{C}'$ for $G - A$ contains a clique C' such that $C' \setminus S = C$.*

Proof. Let $\{u, v\} \in A$ such that $\{u, v\} \subseteq C$. Clearly there are at most $k-1$ vertices w in C such that either $\{u, w\} \in A$ or $\{w, v\} \in A$. Let $x \in C$ be such that xv, xu are edges in $G - A$. Similarly, there are at most $k-1$ non-edges to u, v, x in $G - A$, so let $y \in C$ be a vertex such that yu, yv, yx are edges in $G - A$. Repeating the same argument once again, there is $z \in C$ such that zu, zv, zx, zy are edges in $G - A$. However, the subgraph of $G - A$ induced on u, v, x, y, z is K_5 minus an edge, which is one of the forbidden induced subgraphs in the characterization of line graphs.

The moreover part follows from the following argument. Since $|C| \geq k+7 \geq 4$ and, by Observation 4 it follows that the clique partition witness $\mathcal{C}'$ contains a maximal clique $C' \supseteq C$. It remains to show that no vertex in $V(G) \setminus (S \cup C)$ is in C'. Every vertex in $V(G) \setminus S$ is in two cliques C_1, C_2 in $\mathcal{C}$ that cover all its incident edges in $G - S$. If none of these two cliques is C, then C intersect each of these two cliques in at most 1 vertex. It follows that, because $|C| \geq 3$, there is no vertex in $V(G) \setminus (S \cup C)$ adjacent to all vertices of C. ◀

Let us now partition the cliques in $\mathcal{C}$ into two parts $\mathcal{C}_{<k+7}$ and $\mathcal{C}_{\geq k+7}$ such that $\mathcal{C}_{<k+7}$ contains precisely all the cliques in $\mathcal{C}$ with less than $k+7$ vertices and $\mathcal{C}_{\geq k+7}$ contains the remaining cliques. We will refer to the cliques in $\mathcal{C}_{<k+7}$ as *small* cliques and the cliques in $\mathcal{C}_{\geq k+7}$ as *large* cliques. Intuitively, if we are forced to delete some edge in $G - S$, then this change had to be propagated from S only by changes in small cliques.

We are now ready to define the level structure on the cliques in $\mathcal{C}$. We divide the cliques in $\mathcal{C}$ into levels $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_p$, for some $p \in \mathbb{N}$, that intuitively reflects on how far from S the clique $C \in \mathcal{C}$ is if we consider a shortest path using only small cliques. We will define the levels recursively as follows. By Lemma 7 for every vertex $v \in S$ there exists at most two cliques $C_1, C_2 \in \mathcal{C}$ such that v is adjacent to all vertices in $C_1 \cup C_2$ and to at most 6 vertices in $V(G) \setminus (S \cup C_1 \cup C_2)$. Now, for a vertex $v \in S$, let $\mathcal{N}^v$ denote the set of cliques that contains C_1, C_2 and all the cliques in $\mathcal{C}$ that contain at least one of the neighbors of v in $V(G) \setminus (S \cup C_1 \cup C_2)$. We let $\mathcal{L}_1$ be precisely the set $\bigcup_{v \in S} \mathcal{N}^v$. Note that vertices in

$C_1 \cup C_2$ can each appear in one other clique that is not in $\mathcal{N}^v$ and in particular there are cliques that contain a vertex adjacent to a vertex in S and are not in $\mathcal{L}_1$. For $i > 1$, we then let $\mathcal{L}_i$ be the set of cliques C in $\mathcal{C} \setminus (\bigcup_{j\in\{1..i-1\}} \mathcal{L}_j)$ such that there is a small clique C' in the previous level (*i.e.*, $C' \in \mathcal{L}_{i-1} \cap \mathcal{C}_{<k+7}$) such that $C \cap C'$ is not empty.

▶ **Observation 9.** *Let $C \in \mathcal{C}$ and w a vertex in C. If w has a neighbor in S, then either $C \in \mathcal{L}_1 \cup \mathcal{L}_2$ or w is in a large clique.*

Proof. Let $v \in S$ be a neighbor of w. Then $\mathcal{N}^v \subseteq \mathcal{L}_1$ contains a clique C' with $w \in C'$. Clearly C' intersects C in w. Hence either C' is a large clique or by the definition of $\mathcal{L}_2$ the clique C is in $\mathcal{L}_1 \cup \mathcal{L}_2$. ◀

Let $p \in \mathbb{N}$ be such that $\mathcal{L}_p \neq \emptyset$ and $\mathcal{L}_{p+1} = \emptyset$. While the following Reduction Rule is not completely necessary and would be subsumed by Reduction Rule 2, we include it to showcase some of the ideas needed for the proof in a simplified setting.

▶ **Reduction Rule 1.** *Remove all vertices in $V(G) \setminus S$ that are not in a clique in $\bigcup_{i\in[p]} \mathcal{L}_i$.*

Proof of safeness. Let H be the resulting graph and let $\mathcal{C}_H$ be a set of cliques of H obtained from $\mathcal{C}$, by taking all cliques in $\bigcup_{i\in[p]} \mathcal{L}_i$ and for every clique in $C \in (\mathcal{C} \setminus \bigcup_{i\in[q]} \mathcal{L}_i)$, $\mathcal{C}_H$ contains $C \cap V(H)$, if it is nonempty. Since H is an induced subgraph of G and line graphs can be characterized by a set for forbidden induced subgraphs, it follows that for every $A \in E(G)$, if $G - A$ is a line graph, then $H - A$ is a line graph. It remains to show that if there is a set of edges $A \in E(H)$ such that $|A| \leq k$ and $H - A$ is a line graph, then $G - A$ is also a line graph. Let A be such a set of edges of minimum size and let $\mathcal{C}_A$ be a clique partition witness for $H - A$. It suffices to show that for every clique in $C \in (\mathcal{C}_H \setminus \bigcup_{i\in[p]} \mathcal{L}_i)$, it holds that $C \in \mathcal{C}_A$. If this is the case, we get a clique partition witness for $G - A$ by replacing the cliques of $\mathcal{C}_H \setminus \bigcup_{i\in[p]} \mathcal{L}_i$ in $\mathcal{C}_A$ by $\mathcal{C} \setminus \bigcup_{i\in[p]} \mathcal{L}_i$.

Now, $C \in (\mathcal{C}_H \setminus \bigcup_{i\in[p]} \mathcal{L}_i)$ means that all cliques intersecting C are large. Moreover, because all vertices in H are in some clique on some level, by Lemma 8, for each clique $C_1 \in \mathcal{C}_H$ that intersect C there is a clique in $C_1' \in \mathcal{C}_A$ that is the union of C_1 and some vertices in S. Hence, all vertices in C are already in at least one clique in $\mathcal{C}_A \setminus C$ and all the edges incident to exactly one vertex in C are already covered by these cliques. And hence every clique that contains a vertex in C and intersects every other clique in $\mathcal{C}_A$ in at most one vertex has to be a subset of C. Moreover, the cliques in $\mathcal{C}_A$ that are subsets of C have to be vertex disjoint, since every vertex is in at most 2 cliques in $\mathcal{C}_A$. Hence, if C is not in $\mathcal{C}_A$, then some of the edges in C have to be in A, but replacing all the subsets of C in $\mathcal{C}_A$ by C gives a clique partition witness for $H - A'$ for some $A' \subsetneq A$ which contradicts the fact that A is of minimum size. ◀

We will also say that $C \in \mathcal{C}$ is at $\mathcal{L}$-distance d from S, denoted by $\text{dist}^{\mathcal{L}}(C)$, if C is in $\mathcal{L}_d$. We note that $\mathcal{C}$ still contains some cliques that are not in any of $\mathcal{L}_i$'s. We will let $\text{dist}^{\mathcal{L}}(C) = \infty$ for such a clique C. We can now upper bound the number of cliques at $\mathcal{L}$-distance d from S.

▶ **Lemma 10.** *There are at most $14|S|(k+6)^{d-1}$ cliques in $\mathcal{C}$ at level d,* i.e.*, in $\mathcal{L}_d$.*

Proof. By the definition of $\mathcal{L}_1 = \bigcup_{v\in S} \mathcal{N}^v$, where $\mathcal{N}^v$ denote the set of cliques that contains C_1, C_2 and all the cliques in $\mathcal{C}$ that contain at least one of the neighbors of v in $V(G) \setminus (S \cup C_1 \cup C_2)$. By Lemma 7 for every vertex $v \in S$ there exists at most two cliques $C_1, C_2 \in \mathcal{C}$ such that v is adjacent to all vertices in $C_1 \cup C_2$ and to at most 6 vertices in $V(G) \setminus (S \cup C_1 \cup C_2)$. Since every vertex appears in two cliques of $\mathcal{C}$, it follows that $|\mathcal{N}^v| \leq 14$ and consecutively

$\mathcal{L}_1$ contains at most $14|S|$ cliques. Now by the definition of $\mathcal{L}_d$ we know that for any $d \geq 2$ a clique is at level d if and only if it shares a vertex with a small clique at level $d-1$. Since no three cliques in $\mathcal{C}$ can share a vertex the number of cliques at level d is at most the number of vertices in the small cliques at level $d-1$ and the lemma follows by a simple induction on d. ◀

The remainder of the algorithm consists of two steps. First, in Section 4, we show that we can remove all edges from cliques that are at $\mathcal{L}$-distance at least 5 from S. Afterwards, due to Lemma 10, we are left with only $\mathcal{O}\left(k^4\right)$ non-singleton cliques in $\mathcal{C}$. To finish the algorithm in Section 5, for each clique $C \in \mathcal{C}$ that is not a singleton, we mark an arbitrary subset of $k+7$ vertices in C and remove all unmarked vertices from G. It is then rather straightforward consequence of Lemma 8 that this rule is safe and we get an equivalent instance with $\mathcal{O}\left(k^5\right)$ vertices.

4 Bounding the Distance from S

The purpose of this section is to show that it is only necessary to keep the cliques in $\mathcal{C}$ that are at $\mathcal{L}$-distance at most 4 from S (and adding a singleton for vertices covered by exactly one clique at $\mathcal{L}$-distance at most 4). To do so, we need to show that there is always a solution that does not change the cliques at $\mathcal{L}$-distance 5 at all. For this purpose, we first need to understand the interaction of cliques at $\mathcal{L}$-distance 4 from S with the solution. The first step will be to show that there is an optimal solution A with clique partition witness $\mathcal{C}_A$ such that all cliques in $\mathcal{C}_A$ that share an edge with a clique in $\mathcal{C}$ at $\mathcal{L}$-distance at least 4 from S are actually subcliques of a clique in $\mathcal{C}$ (when restricted to $G-S$). It is a simple consequence of Lemma 8 that this is true for any clique that intersect a large clique in an edge. Hence, we can only care about cliques in $\mathcal{C}_A$ that intersect a small clique C in an edge. By Observation 9, no vertex in C has a neighbor in S. It then follows by Observation 5 that any clique in $\mathcal{C}_A$ that intersects C in an edge and is not a subclique of a clique in $\mathcal{C}$ is indeed a triangle. This leads us to the following definition.

▶ **Definition 11** (bad triangle). *Let $A \subseteq E(G)$ be such that $G-A$ is a line graph and let $\mathcal{C}_A$ be a clique partition witness of $G-A$. A triangle $xyz \in \mathcal{C}_A$ is said to be* bad *if it is not a sub-clique of a clique in $\mathcal{C}$, and one of the edges of the triangle, say xy, is an edge contained in a clique of $\mathcal{L}$-distance at least 4 from S.*

▶ **Lemma 12.** *There exists an optimal solution without any bad triangle.*

Proof. Let A be an optimal solution and $\mathcal{C}_A$ the clique partition witness of $G-A$. Suppose xyz is a bad triangle and let C_1, C_2 and C_3 be the elements of $\mathcal{C}$ containing the edges xy, yz and zx respectively. See also Figure 2 for an illustration. Since xyz is a bad triangle, no clique in $\mathcal{C}_A$ is a superset of C_i, $i \in \{1,2,3\}$ and it is a simple consequence of Lemma 8 that C_i is a small clique. By definition of bad triangle, at least one of C_1, C_2, and C_3 is at $\mathcal{L}$-distance at least 4 from S and hence all of these cliques are at $\mathcal{L}$-distance at least 3 from S. Let X (resp. Y, Z) denote the other clique of $\mathcal{C}_A$ containing x (reps. y, z). Let us define $X_1 = X \cap C_1, X_3 = X \cap C_3, Y_1 = Y \cap C_1, Y_2 = Y \cap C_2, Z_3 = Z \cap C_3$ and $Z_2 = Z \cap C_2$.

Let $C'_1 = X_1 \cup Y_1$, $C'_2 = Y_2 \cup Z_2$ and $C'_3 = Z_3 \cup X_3$. Note that C'_i is a sub-clique of C_i for $i \in [3]$. Now for every $i \in [3]$ we will update C'_i as follows. As long as there exists an edge e in C'_i such that e belongs to $K_i \in \mathcal{C}_A$, K_i is a sub-clique of C_i and $K_i \not\subseteq C'_i$, we set $C'_i := C'_i \cup K_i$ (see also Figure 2b). When this process stops, C'_i corresponds to the union of a set of elements of $\mathcal{C}_A$: $K^i_1, \ldots, K^i_{l_i}$ which are sub-cliques of C_i, and C'_i. Moreover, for any edge e of C'_i which is strictly contained in another clique of $\mathcal{C}_A$ (meaning this clique is not e), then this clique has to be a triangle by Observation 5, as the clique of $\mathcal{C}$ containing e is C_i. Let $e^i_1, \dots, e^i_{s_i}$ denote the set of such edges and let $C^i_1, \dots, C^i_{s_i}$ be the triangles of $\mathcal{C}_A$ containing these edges. Note that $|A \cap C'_1| \geq s_1$, as for any edge e^1_j, either x or y has to be non adjacent to each extremity in $G - A$ or the edge would be in two cliques of $\mathcal{C}_A$ (the same statement is also correct for $|A \cap C'_2|$ and $|A \cap C'_3|$). Let A' be the set obtained from A by

- Removing all the edges of $A \cap C'_1$, $A \cap C'_2$ and $A \cap C'_3$.
- Adding one of the two edges of C^i_j different from e^i_j for every $i \in [3]$ and $j \in [s_i]$ (see Figure 2c illustrating the replacement of C^j_i in $\mathcal{C}_A$ by its proper subclique in $\mathcal{C}_{A'}$ implied by this addition of an edge in A'.).

▷ Claim 13. A' is a set of edges not larger than A and such that $G - A'$ is a line graph with fewer bad triangles than $G - A$.

Proof. The fact that $|A'| \leq |A|$ follows from the fact that $|A \cap C'_i| \geq s_i$ for all $i \in [3]$. To see that $G - A'$ is a line graph, let us show that $\mathcal{C}_{A'}$ defined as follows is a clique partition witness for $G - A'$. Let $\mathcal{C}_{A'}$ be the set defined from $\mathcal{C}_A$ by

- Removing C_A, X, Y, Z, every C^i_j for $i \in [3]$, $j \in [s_i]$, every K^i_j for every $i \in [3]$ and $j \in [l_i]$ and every edge which are contained in one of the C'_i.
- Adding C'_i for $i \in [3]$ and for every $i \in [3]$ and $j \in [s_i]$ the edge of C^i_j which has not been removed from A, as well as singletons for vertices belonging to only one clique.

First it is clear that any set added to $\mathcal{C}'_A$ is a clique as A' does not contain any edge in $A \cap C'_1$, $A \cap C'_2$ and $A \cap C'_3$ and these sets are cliques of G.

Now take B and C two cliques of $\mathcal{C}'_A$. If B and C belong to $\mathcal{C}_A$, then clearly their intersection has size at most 1. If one belongs to $\mathcal{C}_A$ and the other is the remaining edge of C^i_j for $i \in [3]$ and $j \in [s_i]$, then it is also clear as it is true for C^i_j. For $i, j \in [3]^2$, C'_i and C'_j also intersect on one vertex, because C_i and C_j do and moreover, the cliques of $\mathcal{C}_A$ intersecting C'_i on two vertices are exactly the C^i_j, so if $B = C'_i$ and $A \in \mathcal{C}_A$, the intersection has also size at most 1, and we covered all the cases for $|C \cap B|$.

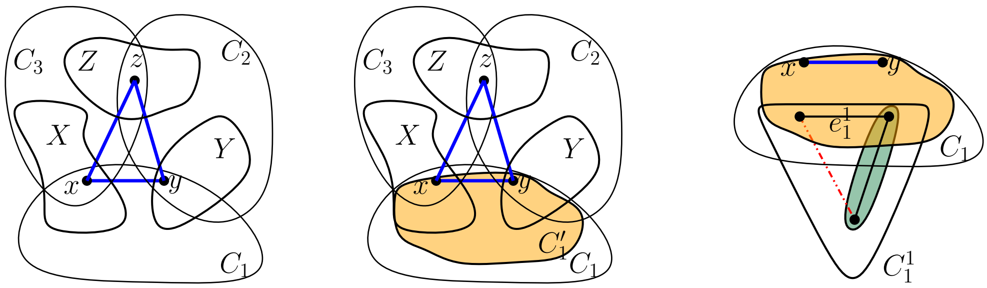
Now for every vertex $x \in V(G)$, if x does not belong to C'_1, C'_2 and C'_3, then it belongs to the same cliques as in $\mathcal{C}_A$ (where the C^i_j have been reduced to an edge and a singleton). For the vertices of C'_1, C'_2 and C'_3 different from x, y, z, we replaced one sub-clique of C_i by another. Finally x belongs to C'_1 and C'_3, y to C'_1 and C'_2 and z to C'_2 and C'_3.

Suppose uv is an edge of $E(G - A')$. If uv belongs to one of the C'_i, then by definition of the C^i_j and because we removed all these triangles, uv only belongs to one clique. For the other edges of $E(G - A')$, the fact that uv belongs to exactly one clique of $\mathcal{C}'_A$ follows from the fact that A' differs on those edges from A only because we added some edges of the C^i_j, and $\mathcal{C}_A$ differs on these vertices only because we changed C^i_j into the remaining edge outside C'_i.

Overall $\mathcal{C}_{A'}$ is indeed a clique partition for $G - A'$. Moreover, to obtain it, we removed at least one bad triangle from $\mathcal{C}_A$ (C_A) without adding one. This ends the proof of the claim. ◁

Finally, we can repeat the process until $\mathcal{C}_{A'}$ is without any bad triangles, which ends the proof of the lemma. ◀

Before we show that indeed all cliques at $\mathcal{L}$-distance at least 5 from S are intact in some optimal solution, we show another auxiliary lemma that is rather simple consequence of Lemma 12, namely that there is a clique partition witness for some optimal solution A such that no two cliques $\mathcal{C}_A$ that intersect the same clique $C \in \mathcal{C}$ at $\mathcal{L}$-distance at least 4 from S in an edge can intersect. This is important later to show that indeed no vertex in a clique $C \in \mathcal{C}$ at $\mathcal{L}$-distance 5 from S will be in two cliques in $\mathcal{C}_A$ that are not subsets of C.



(a) A bad triangle xyz in $\mathcal{C}_A$. C_1, C_2, C_3 are three cliques in $\mathcal{C}$ containing xy, yz, and xz respectively. $X, Y, Z \in \mathcal{C}_A$ are the cliques containing x, y, z other than xyz.

(b) C_1' is the inclusion minimal clique such that $(X \cup Y) \cap C_1 \subseteq C_1' \subseteq C_1$ and for all $K \in \mathcal{C}_A$ if $K \subseteq C_1$ and $|K \cap C_1'| \geq 2$, then $K \subseteq C_1'$. C_2' and C_3' are defined analogously.

(c) C_1^1 intersects C_1' in an edge e_1^1. C_1^1 is replaced by an edge other than e_1^1. This forces to include one edge in C_1^1 to a solution A'. However, this can be seen as replacing an edge between $\{x, y\}$ and endpoints of e_1^1 that is in A.

Figure 2 The treatment of bad triangles. Let $A \subseteq E(G)$ be an optimal solution, $\mathcal{C}_A$ a clique partition witness for A. A bad triangle xyz together with cliques X, Y, Z, as defined in Subfigure 2a are replaced by cliques C_1', C_2', and C_3' defined in Subfigure 2b. Subfigure 2c shows the treatment of cliques in $\mathcal{C}_A$ that intersect C_i' in an edge. By definition of C_i', such clique is not a subclique of C_i and hence a triangle.

▶ **Lemma 14.** *There exists an optimal solution $A \subseteq E(G)$ without any bad triangles and clique partition witness $\mathcal{C}_A$ for $G - A$ such that for every $C \in \mathcal{C}$ of $\mathcal{L}$-distance at least 4 and every $w \in C$, if C_1^w and C_2^w are the two cliques in $\mathcal{C}_A$ containing w, then either $C_1^w \cap C = \{w\}$ or $C_2^w \cap C = \{w\}$.*

Proof. Let $A \subseteq E(G)$ be an optimal solution for G without any bad triangles and clique partition witness $\mathcal{C}_A$ for $G - A$ minimizing the number of pairs (C, w) for which C is at $\mathcal{L}$-distance at least 4, $w \in C$ and the two cliques, denoted C_1^w and C_2^w, in $\mathcal{C}_A$ containing w intersect C in two vertices. Furthermore, it follows from Lemma 8 that C is a small clique, as the clique containing C as a subclique in $\mathcal{C}_A$ would intersect C_1^w in two vertices. Since there are no bad triangles and C is at $\mathcal{L}$-distance at least 4, it follows that $C_1^w \subseteq C$ and $C_2^w \subseteq C$ and in particular $C_1^w \cup C_2^w$ is a clique in G. Indeed, our goal is to replace C_1^w and C_2^w by a clique D such that $C_1^w \cup C_2^w \subseteq D \subseteq C$. We start by setting $D = C_1^w \cup C_2^w$. We will also keep a track of cliques we will remove from $\mathcal{C}_A$. This set will be $\mathcal{D}$ and initialize it as $\mathcal{D} = \{C_1, C_2\}$.

As in the proof of Lemma 12, the only reason why we cannot replace C_1 and C_2 by D and obtain a solution that removes a subset of edges of A is because there exist two vertices $v_1, v_2 \in D$ and a clique $C_{12} \in \mathcal{C}_A$ with $\{v_1, v_2\} \subseteq C_{12}$. Observe that by our assumption there is no bad triangle and $C_{12} \subseteq C$. We let $D = D \cup C_{12}$ and $\mathcal{D} = \mathcal{D} \cup C_{12}$ and repeat until there is no such pair of vertices. Note that every vertex in G is in at most two cliques of $\mathcal{C}_A$. Therefore, this process has to stop after at most $2|C|$ steps.

When there are no two vertices in D that appear together in a different clique, we remove $\mathcal{D}$ from $\mathcal{C}_A$ and replace it by D and $\{v\}$. For every vertex that appears in D, we removed one clique that it appeared in. Hence, every vertex appears in at most 2 cliques and we can always add a singleton to clique partition witness for vertices that are only in one clique. Moreover, no two cliques intersect in two vertices, since D is the only clique we added, and we removed/changed all the cliques that intersected D in at least two vertices. Finally, all edges in $G - A$ remain covered, we only potentially covered some additional edges in D.

Note that this procedure does not introduce any bad triangles or new pair (C', w') for which C' is at $\mathcal{L}$-distance at least 4, $w' \in C'$ and the two cliques in $\mathcal{C}_A$ containing w' intersect C' in two vertices. As it also removes one such pair, we obtain a contradiction with the choice of A. We can therefore deduce that A does not contain such pair (C, w) and the lemma follows. ◀

Finally, we can state the main lemma of this section.

▶ **Lemma 15.** *There exists an optimal solution A for G and a clique partition witness $\mathcal{C}_A$ for $G - A$ such that for every clique $C \in \mathcal{C}$ at $\mathcal{L}$-distance at least 5 it holds that $C \in \mathcal{C}_A$.*

Proof. Let A be an optimal solution without any bad triangles and clique partition witness $\mathcal{C}_A$ for $G - A$ such that for every $C \in \mathcal{C}$ of $\mathcal{L}$-distance at least 4 and every $w \in C$, if C_1^w and C_2^w are the two cliques in $\mathcal{C}_A$ containing w, then either $C_1^w \cap C = \{w\}$ or $C_1^w \cap C = \{w\}$. Note that existence of such a solution is guaranteed by Lemma 14. Moreover let $(A, \mathcal{C}_A)$ be such an optimal solution satisfying properties in Lemma 14 that minimizes the number of cliques $C \in \mathcal{C}$ of $\mathcal{L}$-distance at least 5 such that $C \notin \mathcal{C}_A$. We claim that A satisfies the properties of the lemma.

For a contradiction let $C \in \mathcal{C}$ be a clique at $\mathcal{L}$-distance at least 5 and let $C_1, \dots, C_p$ be the cliques in $\mathcal{C}_A$ that intersects C in at least 2 vertices. Since there is no bad triangle, it follows that $C_i \subseteq C$ for all $i \in [p]$ and by optimality of A, $p = 1$ (else $\bigcup_{i \in [p]} C_i$ is missing at least one edge). We claim that $C = C_1$. Else let $v \in C \setminus C_1$. Note that C is a small clique and hence by Observation 9 v does not have a neighbor in S. In particular all neighbors of v are covered by two cliques in $\mathcal{C}$, one of those cliques is C and let the other clique be C^v. Moreover, Let C_1^v and C_2^v be the two cliques in $\mathcal{C}_A$ containing v. Since $v \in C \setminus C_1$ both C_1^v and C_2^v are subsets of C^v. However, C^v is either a large clique and $\mathcal{C}_A$ contains C^v and the cliques C_1^v and C_2^v are C^v and $\{v\}$ respectively, or C^v is a small clique, in which case C^v is at $\mathcal{L}$-distance at least 4 from S, because it shares a vertex with the clique C at $\mathcal{L}$-distance at least 5 from S. It follows by the choice of A that either $C^v \cap C_1^v = \{v\}$ or $C^v \cap C_2^v = \{v\}$, but then again either C_1^v or C_2^v is the singleton $\{v\}$. However then the clique partition witness $(\mathcal{C}_A \setminus \{C_1, \{v\}\}) \cup \{C_1 \cup \{v\}\}$ defines a better solution. It follows that indeed $C \in \mathcal{C}_A$ for all cliques in $\mathcal{C}$ at $\mathcal{L}$-distance at least 5 in G. ◀

We are now ready to present our main reduction rule. Note that it would seem that we could remove just the vertices that do no appear in a clique at distance at most 4. However, because of the large cliques in at the first four levels, we would be potentially left with many cliques at $\mathcal{L}$-distance infinity that we cannot remove because all of their vertices are in a large clique at $\mathcal{L}$-distance at most 4 from S. While this case could have been dealt with separately, we can actually show a stronger claim, *i.e.,* that we can remove all edges from G that are covered by a clique at $\mathcal{L}$-distance at least 5 from S. Note that in this case we cannot easily claim that if (G, k) is YES-instance then so is the reduced instance and we crucially need the fact that cliques at $\mathcal{L}$-distance at least 5 are kept in clique partition witness of some optimal solution.

▶ **Reduction Rule 2.** *Remove all edges $uv \in E(G)$ such that $\{u, v\} \subseteq C$ for some clique C with $\mathrm{dist}^{\mathcal{L}}(C) \geq 5$. Afterwards remove all isolated vertices from G.*

Let $\mathcal{D}$ be the set of cliques at $\mathcal{L}$-distance at least 5 from S, V_5 the set of vertices that appear in a clique in $\mathcal{D}$ and in a clique in $\mathcal{C} \setminus \mathcal{D}$ and G' be the graph obtained after applying the reduction rule and let $\mathcal{C}' = (\mathcal{C} \setminus \mathcal{D}) \cup \bigcup_{v \in V_5} \{v\}$. Note that $\mathcal{C}'$ is a clique partition witness for $G' - S$ and that $\{v\}$, for $v \in V_5$, is a clique at $\mathcal{L}$-distance at least 5.

Proof of safeness. Let $\mathcal{D}$, V_5, G', $\mathcal{C}'$ be as described above and let A be an optimal solution for G', that is $G' - A$ is a line graph, and let $\mathcal{C}_A$ be clique partition witness for $G' - A$. By Lemma 15, we can assume that $\bigcup_{v \in V_5}\{v\} \subseteq \mathcal{C}_A$. We will show that $(\mathcal{C}_A \setminus \bigcup_{v \in V_5}\{v\}) \cup \mathcal{D}$ is a clique partition witness for $G - A$. Clearly each edge in $G - A$ is either covered by $(\mathcal{C}_A \setminus \bigcup_{v \in V_5}\{v\})$ or by $\mathcal{D}$. It is also easy to see that every vertex is in precisely two cliques. Moreover, two cliques in $\mathcal{D}$ intersect in at most 1 vertex, because $\mathcal{D} \subseteq \mathcal{C}$ and similarly two cliques in $\mathcal{C}_A$ intersect in at most one vertex. Finally, let $D \in \mathcal{D}$ and $C \in (\mathcal{C}_A \setminus \bigcup_{v \in V_5}\{v\})$. Clearly, $D \cap C \subseteq V_5$. Moreover, for $\{u, v\} \subseteq D$, the edge uv is not in G' and hence $\{u, v\} \not\subseteq C$. Hence, $|D \cap C| \leq 1$.

On the other hand, let A be an optimal solution for G and a clique partition witness $\mathcal{C}_A$ for $G - A$ such that for every clique $C \in \mathcal{C}$ at $\mathcal{L}$-distance at least 5 it holds that $C \in \mathcal{C}_A$. Note that the existence of $(A, \mathcal{C}_A)$ is guaranteed by Lemma 15. We claim that $G' - A$ is a line graph. By the choice of $(A, \mathcal{C}_A)$, it follows that $\mathcal{D} \subseteq \mathcal{C}_A$. Moreover, for every edge e that is covered by a clique in $\mathcal{D}$ it holds that $e \notin E(G')$. It follows rather straightforwardly that $\mathcal{C}_A \setminus \mathcal{D} \cup \bigcup_{v \in V_5}\{v\}$ is indeed a clique partition witness for $G' - A$. ◀

5 Finishing the Proof

Suppose now that G, S, and $\mathcal{C}$ correspond to the instance after applying Reduction Rules 1 and 2. Clearly all cliques in $\mathcal{C}$ are either at $\mathcal{L}$-distance at most 4 from S or there are singletons at distance 5 or infinity, depending on whether the singleton intersects a small or a large clique, respectively. It follows from Lemma 10 that there are at most $\mathcal{O}\left(k^4\right)$ cliques at distance at most 4. We let M be any minimal w.r.t. inclusion set of vertices such that for every clique C in $\mathcal{C}$ at $\mathcal{L}$-distance at most 4 it holds that $|M \cap C| \geq \min\{|C|, k + 7\}$. Such a set M can be easily obtained by including arbitrary $\min\{|C|, k + 7\}$ vertices from every clique C at distance at most 4 and then removing the vertices v such that $|(M \setminus \{v\}) \cap C| \geq \min\{|C|, k + 7\}$ for all $C \in \mathcal{C}$ at $\mathcal{L}$-distance at most 4. From this construction it is easy to see that $|M| = \mathcal{O}\left(k^5\right)$.

▶ **Reduction Rule 3.** *Remove all vertices in $V(G) \setminus (S \cup M)$ from G.*

Proof of safeness. Let the clique partition witness $\mathcal{C}'$ for $G - (S \cup M)$ be $\{C \cap M \mid C \in \mathcal{C}, C \cap M \neq \emptyset\}$. Since line graphs are characterized by a finite set of forbidden induced subgraphs, it is easy to see that if $G - A$ is a line graph, for some $A \subseteq E(G)$, then $G[S \cup M] - A = (G - A)[S \cup M]$ is also a line graph. For the other direction, let $A \subseteq E(G)$ be such that $G[S \cup M] - A$ is line graph. We will show that $G - A$ is a line graph. Let $\mathcal{C}_A$ be a clique partition witness for $G[S \cup M] - A$. Now let $\mathcal{C}'_A$ be the set we obtain from $\mathcal{C}_A$ by adding to it all the singleton cliques in $\mathcal{C}$ that do not contain a marked vertex and for every clique $C \in \mathcal{C}_A$ for which there exists $C' \in \mathcal{C}$ with $C \setminus S \subseteq C'$, we replace C by $C' \cup (C \cap S)$.

First let us verify that every vertex in $V(G)$ is in precisely two cliques in $\mathcal{C}'_A$. It is easy to see that this holds for $v \in S \cup M$, because $\mathcal{C}_A$ is a clique partition witness for $G[S \cup M] - A$ and we only added new cliques containing vertices in $V(G) \setminus (M \cup S)$ or extended existing cliques in $\mathcal{C}_A$ by vertices in $V(G) \setminus (M \cup S)$. Now let $v \in V(G) \setminus M$ and let $C_1, C_2 \in \mathcal{C}$ be two cliques that contain v. Because all cliques in $\mathcal{C}$ at $\mathcal{L}$-distance at least 5 are singletons and we keep all vertices of the cliques at $\mathcal{L}$-distance at most 4 of size less than $k + 7$, it follows that C_1 and C_2 either both contain at least $k + 7$ vertices or one of them, say C_2, is a singleton and the other, C_1, contains at least $k + 7$ vertices. If C_2 is a singleton, then $C_2 \in \mathcal{C}'_A$. Else for C_i, $i \in \{1, 2\}$, with $|C_i| \geq k + 7$ there is $C'_i \in \mathcal{C}'$ with $|C'_i| \geq k + 7$ and $C'_i \subseteq C_i$. By Lemma 8, $\mathcal{C}_A$ contains a clique C^A_i such that $C^A_i \setminus S = C'_i \setminus C_i$. By the construction of $\mathcal{C}'_A$ it now follows that $\mathcal{C}'_A$ contains $C_i^A \cup C_i$. From Lemma 7 it follows that if $u \in S$ is adjacent to at least 7 vertices in a clique in $\mathcal{C}$, then it is adjacent to the whole clique. Hence $C_i^A \cup C_i$ indeed induces a complete subgraph of $G - A$. It follows that v is indeed in precisely two cliques in $\mathcal{C}'_A$. Note that above also shows that the sets in $\mathcal{C}'_A$ induce cliques in $G - A$. Furthermore every edge in $G - A$ either has both endpoints in $S \cup M$ and are covered by a clique C in $\mathcal{C}_A$ such that $\mathcal{C}'_A$ contains a superset of C, or they are in the same clique of size at least $k + 7$ in $\mathcal{C}$ that is a subset of a clique in $\mathcal{C}'_A$ as well.

It remains to show that $|C_1 \cap C_2| \leq 1$ for all cliques in $\mathcal{C}'_A$. If $|C_1 \cap C_2| \geq 2$, then at least one of the vertices in $C_1 \cap C_2$ has to be outside $S \cup M$. But then from the above discussion follows that $C_1 \setminus S$ and $C_2 \setminus S$ are in $\mathcal{C}$, $|C_1 \setminus S| \geq k + 7$, $|C_2 \setminus S| \geq k + 7$ and at least $k + 7$ vertices from each of $C_1 \setminus S$ and $C_2 \setminus S$ are in $G[S \cup M]$. Clearly, $C_1 \setminus S$ and $C_2 \setminus S$ intersect in at most one vertex, let us denote it u, and the other vertices in the intersection of C_1 and C_2 are in S. Let v be arbitrary vertex in $C_1 \cap C_2 \cap S$. Note that v is adjacent to at least 7 vertices in both $C_1 \setminus S$ and $C_2 \setminus S$ and by Lemma 7 it is adjacent to all vertices in $(C_1 \cup C_2) \setminus S$. Since $G - (S \setminus \{v\})$ is a line graph, it follows that $G[(C_1 \cup C_2) \setminus (S \setminus \{v\})]$ is a line graph. Every vertex in $C_1 \setminus (S \cup \{u\})$ is in exactly one other clique in $\mathcal{C}$. This clique intersects $C_2 \setminus (S \cup \{u\})$ in at most one vertex. Therefore, there is a pair of vertices $w_1 \in C_1 \setminus (S \cup \{u\})$, $w_2 \in C_2 \setminus (S \cup \{u\})$ such that $w_1 w_2 \notin E(G)$. Now uvw_1 and uvw_2 are two odd triangles (any vertex in $C_i \setminus (S \cup \{u, w_i\})$ is adjacent to three vertices of the triangle uvw_i) that share a common edge, however uvw_1w_2 is not a K_4. Hence, $G[(C_1 \cup C_2) \setminus (S \setminus \{v\})]$ is not a line graph, a contradiction. It follows that if two cliques in $\mathcal{C}$ of size at least $k + 7$ intersect in a vertex in $G - S$, then no vertex in S is adjacent to both cliques and consequently no two cliques in $\mathcal{C}'_A$ intersect in at least two vertices.

It follows that $\mathcal{C}'_A$ is indeed a clique partition witness for $G - A$ and by point (2) in Theorem 2, $G - A$ is indeed a line graph. ◀

We are now ready to prove Theorem 1.

▶ **Theorem 1.** *Line-Graph Edge Deletion admits a kernel with $\mathcal{O}(k^5)$ vertices.*

Proof. We start the algorithm by finding the set S of at most $6k$ vertices such that for every $v \in S$ the graph $G - (S \setminus \{v\})$ is a line graph. This is simply done by greedily finding maximal set of pairwise edge-disjoint forbidden induced subgraphs. Afterwards, we construct a clique partition witness $\mathcal{C}$ for $G - S$ by using the algorithm of Lemma 6. Finally, we apply Reduction Rules 1, 2, and 3 in this order. By the discussion above Reduction Rule 3, after applying all the reduction rules, the resulting instance has $\mathcal{O}(k^5)$ vertices. The correctness of the kernelization algorithm follows from the safeness proofs of the reduction rules. ◀

6 Concluding Remarks

In this paper, we positively answered the open question from WorKer 2013 about kernelization of Line-Graph-Edge Deletion by giving a kernel for the problem with $\mathcal{O}(k^5)$ vertices. Our techniques crucially depend on the structural characterization of the line graphs. We believe that similar techniques could lead also to polynomial kernels for Line-Graph-Edge Addition and Line-Graph-Edge Editing. In particular, a result similar to Lemma 8 still holds when we allow addition of the edges. However, we were not able to bound the distance from S. Main difficulty seems to be the possibility of merging of some small cliques into one in a clique partition witness. It is also worth noting that the line graphs of multigraphs (*i.e.,* graphs that allow multiple edges between the same pair of vertices) have a similar structural characterization with the main difference being that the cliques in a clique partition witness can intersect in more than just one vertex. The kernelization of the edge deletion (as well as addition or editing) to a line graph of a multigraph remains open as well. Finally, the kernelization of Claw-free Edge Deletion as well as of the edge deletion to some of the other natural subclasses of claw-free graphs remain wide open.

References

1 N. R. Aravind, R. B. Sandeep, and Naveen Sivadasan. Dichotomy results on the hardness of H-free edge modification problems. *SIAM J. Discrete Math.*, 31(1):542–561, 2017. doi:10.1137/16M1055797.

2 Lowell W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970. doi:10.1016/S0021-9800(70)80019-9.

3 Hans L Bodlaender, Leizhen Cai, Jianer Chen, Michael R Fellows, Jan Arne Telle, and Dániel Marx. Open problems in parameterized and exact computation-iwpec 2006. *UU-CS*, 2006, 2006.

4 Leizhen Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Inf. Process. Lett.*, 58(4):171–176, 1996. doi:10.1016/0020-0190(96)00050-6.

5 Leizhen Cai and Yufei Cai. Incompressibility of H-free edge modification problems. *Algorithmica*, 71(3):731–757, 2015. doi:10.1007/s00453-014-9937-x.

6 Maria Chudnovsky and Paul Seymour. Claw-free graphs. IV. decomposition theorem. *Journal of Combinatorial Theory, Series B*, 98(5):839–938, 2008. doi:10.1016/j.jctb.2007.06.007.

7 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

8 Marek Cygan, Marcin Pilipczuk, Michał Pilipczuk, Erik Jan van Leeuwen, and Marcin Wrochna. Polynomial kernelization for removing induced claws and diamonds. *Theory Comput. Syst.*, 60(4):615–636, 2017. doi:10.1007/s00224-016-9689-x.

9 Daniele Giorgio Degiorgi and Klaus Simon. A dynamic algorithm for line graph recognition. In Manfred Nagl, editor, *Graph-Theoretic Concepts in Computer Science, 21st International Workshop, WG '95, Aachen, Germany, June 20-22, 1995, Proceedings*, volume 1017 of *Lecture Notes in Computer Science*, pages 37–48. Springer, 1995. doi:10.1007/3-540-60618-1_64.

10 R. Diestel. *Graph Theory, 4th Edition*. Springer, 2012.

11 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

12 Eduard Eiben, William Lochet, and Saket Saurabh. A polynomial kernel for paw-free editing. *CoRR*, abs/1911.03683, 2019. arXiv:1911.03683.

13 Ehab S. El-Mallah and Charles J. Colbourn. The complexity of some edge deletion problems. *IEEE Transactions on Circuits and Systems*, 35(3):354–362, 1988. doi:10.1109/31.1748.

14 Paul Erdős and Richard Rado. Intersection theorems for systems of sets. *Journal of the London Mathematical Society*, 1(1):85–90, 1960.

15 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. doi:10.1007/3-540-29953-X.

16 Sylvain Guillemot, Frédéric Havet, Christophe Paul, and Anthony Perez. On the (non-)existence of polynomial kernels for P_ℓ-free edge modification problems. *Algorithmica*, 65(4):900–926, 2013. doi:10.1007/s00453-012-9619-5.

17 Frank Harary. *Graph theory*. Addison-Wesley series in mathematics. Addison-Wesley Pub. Co., 1969.

18 Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-SAT. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727.

19 Stefan Kratsch and Magnus Wahlström. Two edge modification problems without polynomial kernels. *Discrete Optimization*, 10(3):193–199, 2013. doi:10.1016/j.disopt.2013.02.001.

20 John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *J. Comput. Syst. Sci.*, 20(2):219–230, 1980. doi:10.1016/0022-0000(80)90060-4.

21 Dániel Marx and R. B. Sandeep. Incompressibility of H-free edge modification problems: Towards a dichotomy. *CoRR*, abs/2004.11761, 2020. arXiv:2004.11761.

22 Najiba Sbihi. Algorithme de recherche d'un stable de cardinalite maximum dans un graphe sans etoile. *Discrete Mathematics*, 29(1):53–76, 1980. doi:10.1016/0012-365X(90)90287-R.

23 Hassler Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932. URL: http://www.jstor.org/stable/2371086.

24 Mihalis Yannakakis. Edge-deletion problems. *SIAM J. Comput.*, 10(2):297–309, 1981. doi:10.1137/0210021.