

Planar Bichromatic Bottleneck Spanning Trees

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Abstract

Given a set P of n red and blue points in the plane, a *planar bichromatic spanning tree* of P is a geometric spanning tree of P , such that each edge connects between a red and a blue point, and no two edges intersect. In the bottleneck planar bichromatic spanning tree problem, the goal is to find a planar bichromatic spanning tree T , such that the length of the longest edge in T is minimized. In this paper, we show that this problem is NP-hard for points in general position. Our main contribution is a polynomial-time $(8\sqrt{2})$ -approximation algorithm, by showing that any bichromatic spanning tree of bottleneck λ can be converted to a planar bichromatic spanning tree of bottleneck at most $8\sqrt{2}\lambda$.

2012 ACM Subject Classification Theory of computation \rightarrow Computational geometry; Theory of computation \rightarrow Design and analysis of algorithms

Keywords and phrases Approximation Algorithms, Bottleneck Spanning Tree, NP-Hardness

Digital Object Identifier 10.4230/LIPIcs.ESA.2020.1

Funding This work was partially supported by Grant 2016116 from the United States – Israel Binational Science Foundation. Work by J. Mitchell was partially supported by NSF (CCF-1526406, CCF-2007275).

1 Introduction

Let P be a bi-colored set of red and blue points in the plane and let $n = |P|$. A *bichromatic spanning tree* of P is a spanning tree of P whose edges are straight-line segments connecting between points of different colors. A spanning tree is *planar* if its edges do not cross each other. Borgelt et al. [15] studied the problem of computing a minimum-weight planar bichromatic spanning tree, and showed that the problem is NP-hard. Moreover, for points in general position, they gave an $O(\sqrt{n})$ -approximation algorithm, and for points in convex position, they gave an exact cubic-time algorithm. Biniaz et al. [11] studied the problem of computing a maximum-weight planar bichromatic spanning tree and gave a $(1/4)$ -approximation algorithm for the problem.

Algorithmic problems on bichromatic geometric input have appeared in many problems, including, e.g., trees [1, 12, 15], matchings [13, 18], and partitionings [19]. Often the bichromatic input is referred to as “red-blue” input, e.g. in red-blue intersection [4, 23], red-blue separation [9, 14, 17, 20], and red-blue connection problems [5, 10]. For a survey of many geometric problems on bichromatic (red-blue) points, see Kaneko and Kano [21].



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28th Annual European Symposium on Algorithms (ESA 2020).

Editors: Fabrizio Grandoni, Grzegorz Herman, and Peter Sanders; Article No. 1; pp. 1:1–1:16

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Many of the structures studied in computational geometry are planar, including minimum spanning trees, minimum weight matchings, Delaunay/Voronoi diagrams, etc. Therefore, the planarity requirement is quite natural, and indeed many researchers have considered geometric problems dealing with crossing-free configurations in the plane; see, e.g. [2, 3, 6, 7, 8].

In this paper, we study the problem of computing a bottleneck planar bichromatic spanning tree of P , in which we seek a planar bichromatic spanning tree that minimizes the bottleneck, i.e., the length of the longest edge. To the best of our knowledge, this problem has not been considered before.

Our results. In Section 2, we prove that the problem of computing a bottleneck planar bichromatic spanning tree is NP-hard. Our proof is based on a reduction from the planar 3-SAT problem, and is influenced by the proof of Borgelt et al. [15]. As a corollary we obtain that the problem does not admit a PTAS. Next, in Section 3, we present a polynomial-time algorithm for computing a planar bichromatic spanning tree of bottleneck at most $8\sqrt{2}$ times the bottleneck OPT of an optimal such planar bichromatic spanning tree. We first compute a bottleneck bichromatic spanning tree having bottleneck λ that may have crossings (so λ is a lower bound on OPT). Then, we use the length λ to define a partition of the plane into convex cells, and to partition P into subsets according to these cells. Next, we construct planar bichromatic trees for each subset, and we connect these trees to obtain a planar bichromatic spanning tree of P . We show that this tree has a bottleneck at most $8\sqrt{2}\lambda$.

2 Hardness Proof

In this section, we prove the NP-hardness (Theorem 1) of computing a bottleneck planar bichromatic spanning tree; our proof also shows (Corollary 2, below) that there is no approximation factor less than $\sqrt{2}$, unless $P = NP$.

► **Theorem 1.** *Let P be a set of n red and blue points in the plane. Computing a bottleneck planar bichromatic spanning tree of P is NP-hard.*

Proof. We adapt the proof of Borgelt et al. [15], making modifications necessary to address the bottleneck version. For completeness, we explain the ingredients required for the proof.

The proof is based on a reduction from the planar 3-SAT problem. Given a 3-CNF formula F with n variables $X = \{x_1, x_2, \dots, x_n\}$ and m clauses $Y = \{C_1, C_2, \dots, C_m\}$, let $G_F = (V, E)$ be the graph of F , i.e., $V = X \cup Y$ and $E = \{(x_i, C_j) : x_i \text{ or } \tilde{x}_i \text{ appears in } C_j\}$. If G_F is planar, then F is called a planar 3-CNF formula. The planar 3-SAT problem is to determine whether a given planar 3-CNF formula F is satisfiable; the problem is NP-complete [22].

Let F be a planar 3-SAT formula. We construct, in polynomial time, a set P of red and blue points in the plane, such that F is satisfiable if and only if there exists a planar bichromatic spanning tree of P of bottleneck 1. Consider the graph G_F . It is well known that G_F can be embedded in the plane in polynomial time, using polynomial area, i.e., with the $n + m$ nodes of G_F placed at grid points of a regular square grid of size $O(n + m)$ -by- $O(n + m)$ [16].

The construction is based on chains of pairs of red and blue points. We call the pairs in the chain *sites*. The distance between the two points comprising each site is less than 1, and the distance between two points of different colors in consecutive (along the chain) sites is exactly 1; see Figure 1(a). Now, for every two consecutive sites, there are two possible edges to connect them using edges of length at most 1: we either connect (with an edge

of length exactly 1) the blue point of the first site with the red point of the second site, or the other way around. Moreover, the chain is constructed in such a way that if we connect the blue point in the leftmost site to the red point in the next site, this forces the choice of connections in one direction along the chain; see Figure 1(b).

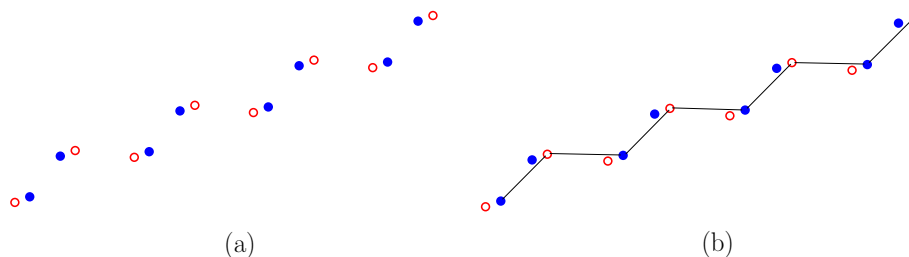


Figure 1 A chain of red-blue sites.

Variables. Each variable $x_i \in X$ is represented by a circular chain of $O(m)$ sites, a special red point r_i , and a red-blue path; see Figure 2. The addition of the red-blue path to the variable gadget of [15] is required, since without it the special red point r_i can be connected to both sides of the chain without increasing the bottleneck, which is not the case in the minimum weight version. The red-blue path forces r_i to be connected exactly to one side.

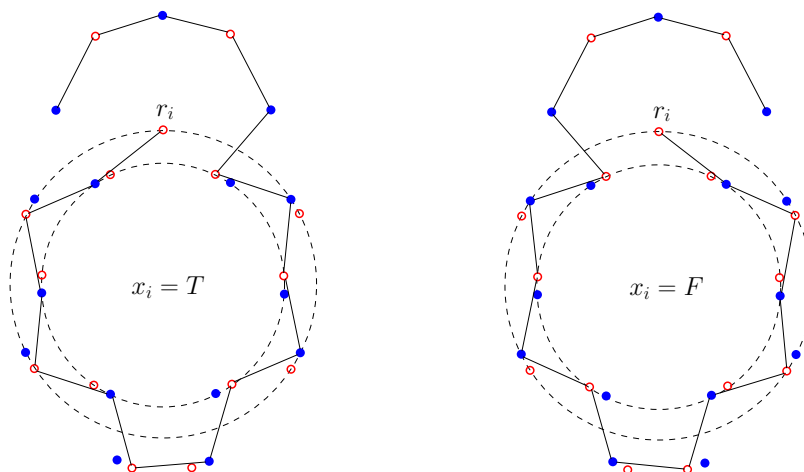
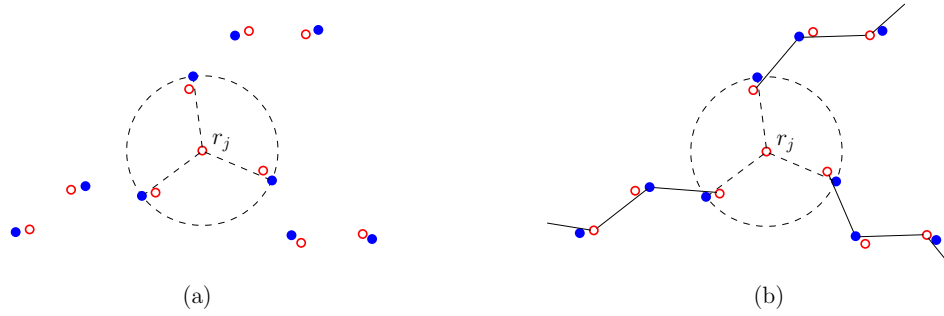


Figure 2 The trees corresponding to the true and the false assignments of x_i .

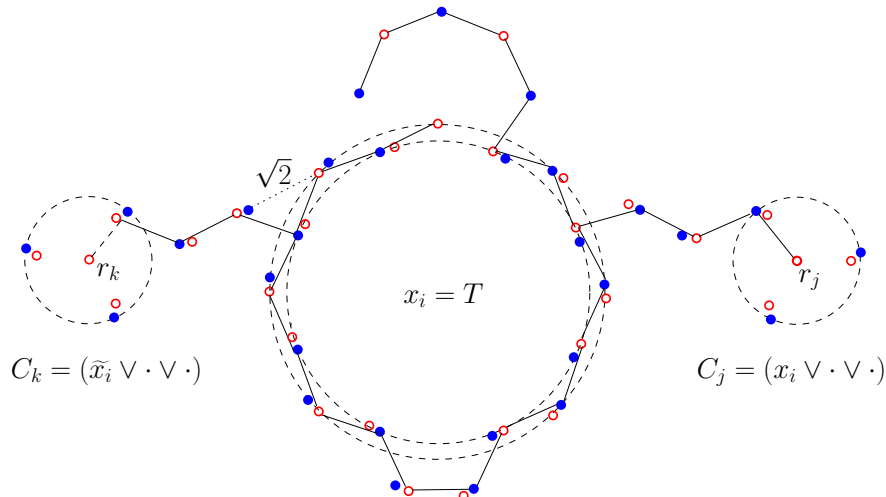
The sites on the circular chain are located on two circles, an inner circle and an outer circle. We locate the points in such a way that the (bichromatic) distances are 1 between consecutive sites on the circular chain, the distance is 1 between r_i and the blue points of its neighboring sites in the chain, and the distance is 1 between the endpoints of the red-blue path and the corresponding points of the chain. Moreover, the points are located such that there are only two possible optimal trees (i.e., planar bichromatic spanning trees of bottleneck 1) of the points, depending on the connection of r_i to the chain. In each of the two trees, r_i is connected to exactly one site of the chain. We associate (arbitrarily) one possibility with the assignment $x_i = T$, and the other with the assignment $x_i = F$; see Figure 2. Thus, the value of x_i will determine the tree of these points, and vice versa. Moreover, if $x_i = T$ (resp., $x_i = F$), then the red points on the right (resp., left) of the inner circle are free to be connected to points outside the gadget, and the red points on the left (resp., right) of the inner circle cannot be connected to points outside the gadget without crossing, and vice versa.

Clauses. Each clause C_j is represented by a single red point r_j and three chains that are connected to the respective variables of C_j ; see Figure 3(a). The distance between r_j and each chain is 1. In any optimal tree, r_j will be connected to at least one of the three chains. However, it cannot be connected to any chain if all the chains are connected to variables that are in the wrong state; see Figure 3(b).



■ **Figure 3** The gadget corresponding to the clause C_j .

We connect between the variables and the clauses such that, in any optimal tree, one of the three chains of the clause has to be connected to a red point on the inner circle of the corresponding variable. Assume that x_i appears unnegated in a clause C_j and negated in a clause C_k , i.e., $C_j = (x_i \vee \cdot \vee \cdot)$ and $C_k = (\tilde{x}_i \vee \cdot \vee \cdot)$. We connect the chain of C_j that is respective to x_i to a site on the right of the inner circle of the gadget x_i , and we connect the chain of C_k that is respective to x_i to a site on the left of the inner circle of the gadget x_i ; see Figure 4. This connection ensures that, if x_i is assigned T , then the red point on the right of the inner circle of x_i is free to be connected to the chain of C_j , and this connection can produce a path through the chain that ends at r_j . On the other hand, if x_i is assigned T , then the red point on the left of the inner circle of x_i cannot be connected to the chain of C_k , which does not allow a connection between the chain and r_k . The same argument holds when x_i is assigned F .



■ **Figure 4** The connection between the variable x_i and the clauses C_j and C_k .

Finally we need to connect all variables to each other by some fixed part of the tree, because the whole construction needs to be a tree and not a forest. These connections can easily be made using red-blue chains having distance at most 1 between every two consecutive points in the path. Also, we need to make sure that the distance between different parts of the construction is large enough to avoid shortcuts.

Overall, the reduction is performed in polynomial (in n, m) time, since the planar graph G_F is drawn on a regular $O(n + m)$ -by- $O(n + m)$ grid, and each of our variable gadgets is of size $O(m)$, resulting in overall polynomial area and a polynomial-size construction that is computed in polynomial (in n, m) time. ◀

Notice that in the reduction we proved that if the 3-SAT formula is not satisfiable, then any planar bichromatic spanning tree of P has an edge of length greater than 1. Actually, we can push the length of this edge to be closer to $\sqrt{2}$. That is, we can draw the connection between each clause and its corresponding variables, such that the distance between each chain of the clause and the corresponding site on the inner circle of the variable is 1, and the distance between each chain of the clause and the sites on the outer circle of the variable is at least $\sqrt{2} - \varepsilon$, for any $0 < \varepsilon < \sqrt{2} - 1$; see Figure 4. This implies that the bottleneck planar bichromatic spanning tree problem cannot be approximated within a factor less than $\sqrt{2}$, unless $P = NP$.

► **Corollary 2.** *The bottleneck planar bichromatic spanning tree problem cannot be approximated within a factor less than $\sqrt{2}$, unless $P = NP$. In particular, there is no PTAS (unless $P = NP$).*

3 Approximation Algorithm

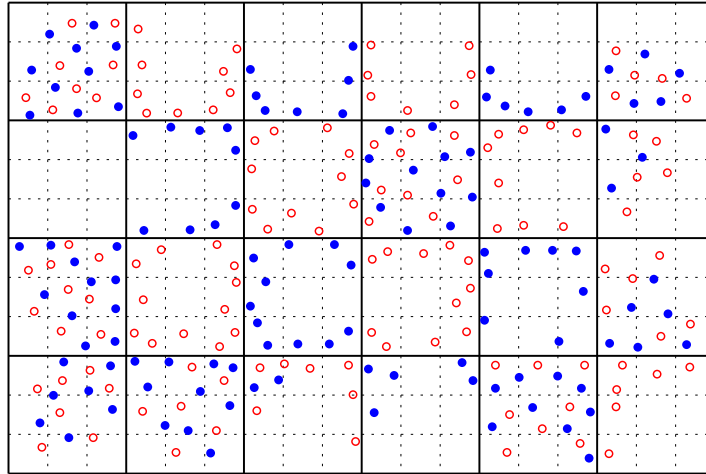
Let P be a set of red and blue points in the plane and let $n = |P|$. Let T be a bichromatic spanning tree of P of minimum bottleneck (T may have crossings and can be computed in $O(n \log n)$ time [12]). Let λ denote the bottleneck of T , i.e., the length of the longest edge in T . Notice that λ is the lower bound for any bichromatic spanning tree of P , in particular for any planar bottleneck bichromatic spanning tree of P . In this section, we show how to compute a planar bichromatic spanning tree of P , such that its bottleneck is at most $8\sqrt{2}\lambda$.

Our algorithm partitions the plane into disjoint cells satisfying the following properties:

1. Each cell is convex and contains points of both colors.
2. In each cell, the distance between any two points is at most $5\sqrt{2}\lambda$.
3. The cells are connected, i.e., if we consider the graph with the cells as its vertices and there is an edge between two cells if they are adjacent (sharing a common boundary), then this graph is connected.
4. We can construct a planar bichromatic spanning tree of the points in each cell and we can connect them without crossings.

Assume, w.l.o.g., that $\lambda = 1$. Consider an axis-parallel grid, with each (square) cell of edge length 3 and all points of P in the interior (not on the boundary) of these cells; see Figure 5. We say that a cell $C_{i,j}$ is bichromatic if it contains points of both colors and we say that $C_{i,j}$ is monochromatic (red or blue) if all of the points in $C_{i,j}$ have the same color, otherwise, we say that $C_{i,j}$ is an empty cell.

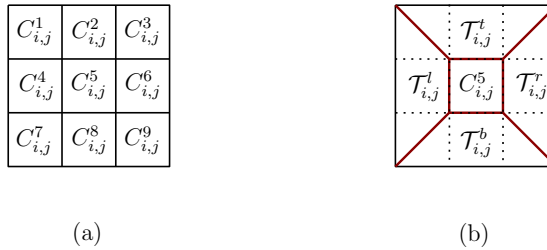
Our algorithm consists of two stages. In Stage 1, we modify the grid cells to satisfy properties (1)-(3), and, in Stage 2, we construct a planar bichromatic spanning tree of the points in each cell and connect between these trees to obtain a planar bichromatic spanning tree of P .



■ **Figure 5** The grid partitioning the points of P .

Stage 1

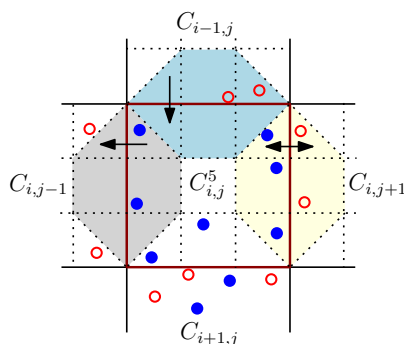
In this stage, we consider the monochromatic cells and we partition and merge portions of them in order to obtain a subdivision in which all cells are convex and bichromatic. Let $C_{i,j}$ be a 3×3 cell of the grid. Since $C_{i,j}$ is a 3×3 cell, $C_{i,j}$ is the union of 9 unit sub-cells, labelled $C_{i,j}^k$, for $k = 1, 2, 3, \dots, 9$, as shown in Figure 6(a). Notice that, since $C_{i,j}$ is a monochromatic cell, the points of $C_{i,j}$ are of distance at most 1 from the boundary of $C_{i,j}$, and therefore, $C_{i,j}^5$ is empty of points of P . The region $C_{i,j} \setminus C_{i,j}^5$ is the union of four trapezoids $\mathcal{T}_{i,j}^l$, $\mathcal{T}_{i,j}^r$, $\mathcal{T}_{i,j}^t$, and $\mathcal{T}_{i,j}^b$, such that $\mathcal{T}_{i,j}^l$ (resp., $\mathcal{T}_{i,j}^r$, $\mathcal{T}_{i,j}^t$, and $\mathcal{T}_{i,j}^b$) is the trapezoid obtained by connecting the left (resp., right, top, and bottom) corners of $C_{i,j}$ by diagonals to the left (resp., right, top, and bottom) corners of $C_{i,j}^5$; see Figure 6(b).



■ **Figure 6** (a) The 9 unit sub-cells of cell $C_{i,j}$. (b) The trapezoids $\mathcal{T}_{i,j}^l$, $\mathcal{T}_{i,j}^r$, $\mathcal{T}_{i,j}^t$, and $\mathcal{T}_{i,j}^b$.

Stage 1.1

In this stage, we introduce a directed graph G in which the vertices are the monochromatic cells and the edges are defined as follows. Let $C_{i,j}$ be a monochromatic cell, and let $\mathcal{N}(C_{i,j}) = \{C_{i,j-1}, C_{i,j+1}, C_{i-1,j}, C_{i+1,j}\}$ be the set of cells that share a grid edge with $C_{i,j}$. Let $C \in \mathcal{N}(C_{i,j})$ be a monochromatic cell and assume, w.l.o.g., that $C = C_{i,j+1}$. There is a directed edge from $C_{i,j}$ to $C_{i,j+1}$ if and only if $C_{i,j}$ and $C_{i,j+1}$ are of different colors and the trapezoid $\mathcal{T}_{i,j}^r$ is not empty of points of P ; see Figure 7.

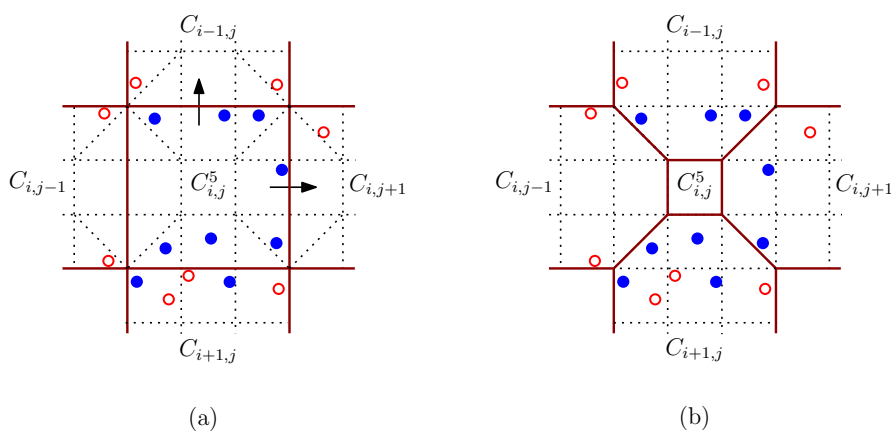


■ **Figure 7** Directed edges between monochromatic cell $C_{i,j}$ and its monochromatic neighbors $(C_{i,j-1}, C_{i-1,j}, C_{i,j+1})$.

Stage 1.2

In this stage, we modify the grid cells by partitioning and merging some of the monochromatic cells with their neighbors, guided by the directed edges introduced in Stage 1.1. Before describing how to modify the grid cells, we describe the following *cell partition procedure* that we will apply in this stage to the empty and some of the monochromatic cells.

Cell partition procedure. For a monochromatic cell $C_{i,j}$, partition $C_{i,j} \setminus C_{i,j}^5$ into trapezoids $\mathcal{T}_{i,j}^l, \mathcal{T}_{i,j}^r, \mathcal{T}_{i,j}^t,$ and $\mathcal{T}_{i,j}^b$, and merge them with the cells $C_{i,j-1}, C_{i,j+1}, C_{i-1,j},$ and $C_{i+1,j}$, respectively; see Figure 8(b).



■ **Figure 8** (a) $d_{in}(C_{i,j}) = 0$ and $d_{out}(C_{i,j}) > 0$. (b) Partitioning and merging $C_{i,j}$ with its neighbors.

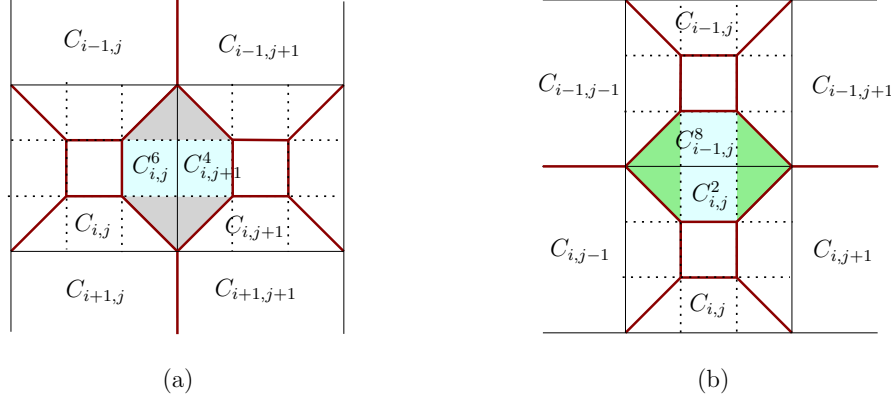
Let $d_{in}(C_{i,j})$ (resp., $d_{out}(C_{i,j})$) denote the in-degree (resp., the out-degree) of the vertex corresponding to the monochromatic cell $C_{i,j}$ in the graph G . We apply the following three steps on the empty and monochromatic cells.

Step 1. We apply this step as long as there exists a cell $C_{i,j}$ with $d_{in}(C_{i,j}) = 0$ and $d_{out}(C_{i,j}) > 0$. For each such cell, we apply the cell partition procedure on $C_{i,j}$ and remove the out-going edges from $C_{i,j}$ and from its neighbors $C_{i,j-1}, C_{i,j+1}, C_{i-1,j},$ and $C_{i+1,j}$; see Figure 8.

Step 2. We apply this step on the monochromatic cells $C_{i,j}$ with $d_{in}(C_{i,j}) > 0$. Consider the grid as an arbitrary white-black chessboard. For each white cell with $d_{in}(C_{i,j}) > 0$, we apply the cell partition procedure on $C_{i,j}$.

Step 3. We apply this step on the empty and the monochromatic cells $C_{i,j}$ with $d_{in}(C_{i,j}) = 0$ and $d_{out}(C_{i,j}) = 0$ (that are not considered in the previous steps). For each such cell, we apply the cell partition procedure on $C_{i,j}$.

We call a cell $C_{i,j}$ a *partitioned cell* if $C_{i,j}$ has been partitioned (using the cell partition procedure), and we call it an *extended cell* otherwise. If we have two adjacent partitioned cells $C_{i,j}$ and $C_{i,j+1}$, then we call the merged area of the two trapezoids $\mathcal{T}_{i,j}^r$ and $\mathcal{T}_{i,j+1}^l$ a *lune*; see Figure 9.



■ **Figure 9** (a) vertical and (b) horizontal lunes.

At the end of this stage, we have three types of non-empty convex cells: original 3×3 cells, extended cells, and lunes. Clearly, each original cell is bichromatic, otherwise, it would have been partitioned or extended in Steps 1–3. Observe that each extended cell $C_{i,j}$ is bichromatic, since $d_{in}(C_{i,j}) > 0$. Observe also that each non-empty lune L is monochromatic. To see this, assume, w.l.o.g., that L was obtained by partitioning the cells $C_{i,j}$ and $C_{i,j+1}$ and merging the trapezoids $\mathcal{T}_{i,j}^r$ and $\mathcal{T}_{i,j+1}^l$. Thus, L cannot be bichromatic, since otherwise, there would be a directed edge from $C_{i,j}$ to $C_{i,j+1}$ and vice versa, which means that one of the cells $C_{i,j}$ and $C_{i,j+1}$ (the black one in the chessboard) is extended in Step 2.

Stage 1.3

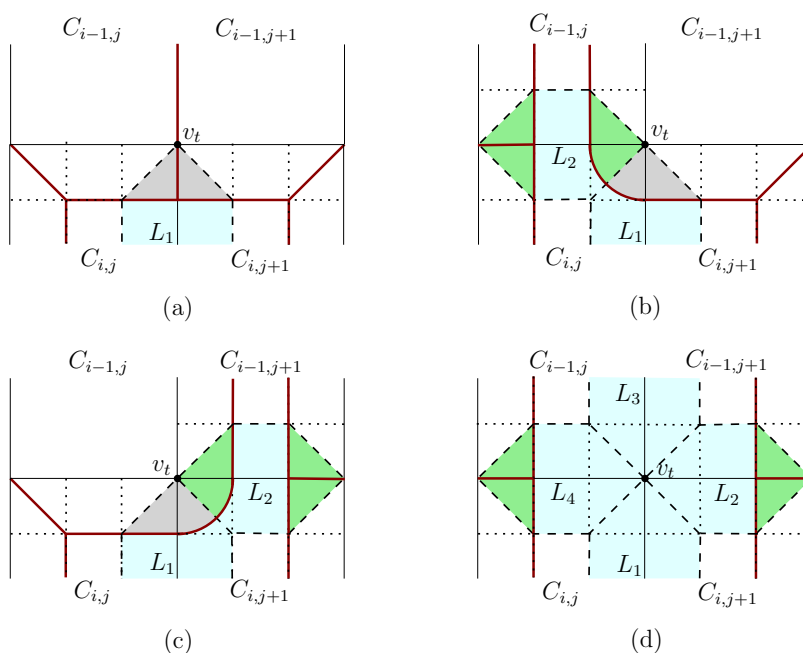
In this stage, we get rid of the lunes, by partitioning each lune into sub-pieces and merging the sub-pieces with adjacent extended cells as follows. Let L_1 be a vertical lune obtained by merging two adjacent trapezoids $\mathcal{T}_{i,j}^r$ and $\mathcal{T}_{i,j+1}^l$; see Figure 9(a). (A horizontal lune will be treated analogously.) As observed above, L_1 is monochromatic (or empty) which means that the subregion $C_{i,j}^6 \cup C_{i,j+1}^4 \subseteq L_1$ is empty of points of P .

We consider the four triangles obtained by removing $C_{i,j}^6$ from $\mathcal{T}_{i,j}^r$ and removing $C_{i,j+1}^4$ from $\mathcal{T}_{i,j+1}^l$, and we merge them with the cells $C_{i-1,j}$, $C_{i-1,j+1}$, $C_{i+1,j}$, and $C_{i+1,j+1}$ according to the following cases. We describe how to merge the top triangles with $C_{i-1,j}$ and $C_{i-1,j+1}$. (Merging the bottom triangles with $C_{i+1,j}$, and $C_{i+1,j+1}$ is done analogously.) Let v_t be the top vertex of L_1 ; see Figure 10.

- If both $C_{i-1,j}$ and $C_{i-1,j+1}$ are extended cells, then we merge the top-left triangle with $C_{i-1,j}$ and the top-right triangle with $C_{i-1,j+1}$; see Figure 10(a).
- If $C_{i-1,j}$ is a partitioned cell and $C_{i-1,j+1}$ is an extended cell, then we have another horizontal lune L_2 between $C_{i-1,j}$ and $C_{i,j}$; see Figure 10(b). Notice that the union of the top-left triangle of L_1 and the right-bottom triangle of L_2 is exactly the sub-cell $C_{i,j}^3$.

Notice also that L_2 is monochromatic and has the same color as L_1 . Thus, the points of P in $C_{i,j}^3$ are of distance 1 from v_t . In this case, we merge the top-right triangle of L_1 and the right-top triangle of L_2 with $C_{i-1,j+1}$. Moreover, we merge the region of $C_{i,j}^3$ intersecting the disk of radius 1 centered at v_t with $C_{i-1,j+1}$; see Figure 10(b).

- If $C_{i-1,j}$ is an extended cell and $C_{i-1,j+1}$ is a partitioned cell, then this case is symmetric to the previous case; see Figure 10(c).
- If both $C_{i-1,j}$ and $C_{i-1,j+1}$ are partitioned cells, then we have four lunes incident to v_t ; see Figure 10(d). Since all of the lunes are monochromatic and have the same color, the triangles of these lunes that are incident to v_t are empty of points of P and, therefore, we remove these triangles from the division.



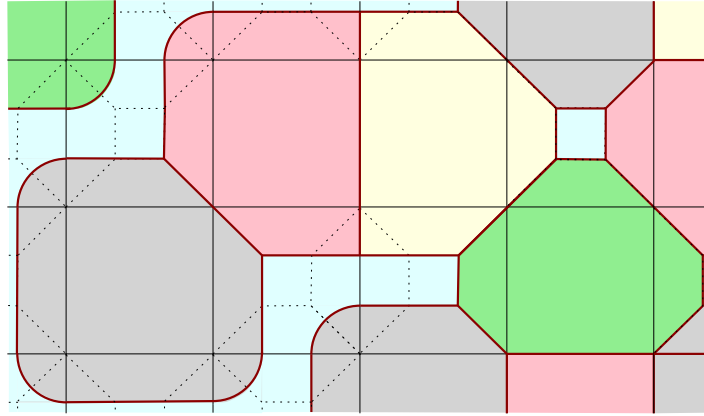
■ **Figure 10** Merging the top triangles of L_1 with the cells $C_{i-1,j}$ and $C_{i-1,j+1}$. The gray and the green regions are part of vertical and horizontal lunes, respectively, and the light blue regions are empty of points of P .

Moreover, in each partitioned cell $C_{i,j}$ such that $i = 1$, $i = n$, $j = 1$, or $j = n$, we treat the trapezoids adjacent to the boundary of the grid as half-lunes and we merge them with their adjacent extended cells as in the lunes case.

Notice that, at the end of this stage, we have two types of non-empty cells: original 3×3 cells and extended cells, and both types are convex and bichromatic cells; see Figure 11. From now on, we refer to both types of these cells as extended cells and denote them by \hat{C} . That is, $\hat{C}_{i,j}$ is either an original 3×3 cell $C_{i,j}$ or an extended cell obtained by merging $C_{i,j}$ with trapezoids from its neighbors.

Stage 2

In this stage, we construct a planar bichromatic spanning tree in each (extended) cell and connect them to each other to obtain, overall, a planar bichromatic spanning tree of P . For each cell $\hat{C}_{i,j}$, we denote by $\hat{P}_{i,j}$ the set of points of P lying in $\hat{C}_{i,j}$. If $C_{i,j}$ has been partitioned, then we set $\hat{P}_{i,j} = \emptyset$.



■ **Figure 11** A subdivision obtained at the end of Stage 1. The light blue regions are empty of points of P .

Stage 2.1

In each cell $\hat{C}_{i,j}$, we construct a planar bichromatic spanning tree $T_{i,j}$ of $\hat{P}_{i,j}$ as follows. We select an arbitrary red point $s \in \hat{P}_{i,j}$ as a center of the tree and connect it to each blue point in the cell to produce a star. We extend the edges of the star to partition the cell into convex cones, possibly except one cone; see Figure 12. If we have a non-convex cone, then we divide it into two convex cones by adding its bisector, as shown in Figure 12(right). Then, we connect all the red points in each cone to one of the blue points on the lines bounding the cone.



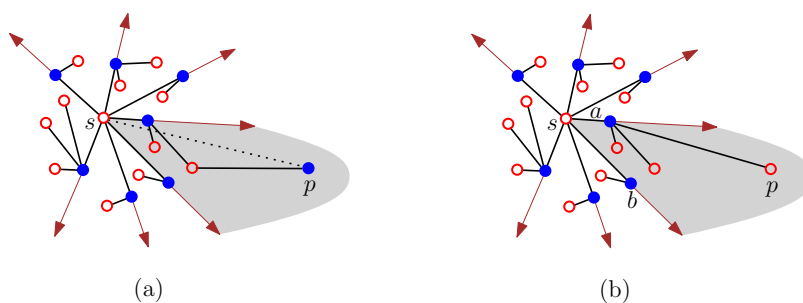
■ **Figure 12** Constructing a planar bichromatic spanning tree in a cell.

► **Lemma 3.** *Let $T_{i,j}$ be a tree constructed in Stage 2.1 in cell $\hat{C}_{i,j}$. Any (red or blue) point p in the plane can be connected to a point of opposite color of $T_{i,j}$ without crossing the edges of $T_{i,j}$.*

Proof. Let s be the center of $T_{i,j}$ and recall that its color is red. Consider the cones produced by the rays between s and the blue points of $T_{i,j}$. Let C be the cone containing p and let a and b be the two blue points defining C . By the way we constructed $T_{i,j}$, all the points in C are red and connected to exactly one of the points a and b , assume, w.l.o.g., a . We distinguish between two cases with respect to the color of p .

Case 1: p is a blue point. If the edge (s, p) does not cross the edges of $T_{i,j}$, then we connect p to s . Otherwise, we connect p to the endpoint of the first edge (from p) crossing (s, p) ; see Figure 13(a).

Case 2: p is a red point. In this case, we connect p to a ; see Figure 13(b). ◀



■ **Figure 13** (a) p and s are of different colors. (b) p and s are of the same color.

Stage 2.2

In this stage, we connect between the trees that are constructed in Stage 2.1 to obtain a planar bichromatic spanning tree of P . Let $\hat{C}_{i,j}$ and $\hat{C}_{k,l}$ be two (extended) cells. We say that $\hat{C}_{k,l}$ is a *side* adjacent (or s-adjacent for short) cell of $\hat{C}_{i,j}$, if one of the following holds:

- $k = i + 1$ and $l = j$, or
- $k = i$ and $l = j + 1$,

and we say that $\hat{C}_{k,l}$ is a *diagonal* adjacent (or d-adjacent for short) cell of $\hat{C}_{i,j}$, if one of the following holds:

- $k = i - 1$, $l = j + 1$, and $C_{i-1,j}$ and $C_{i,j+1}$ have been partitioned, or
- $k = i - 1$, $l = j - 1$, and $C_{i,j-1}$ and $C_{i-1,j}$ have been partitioned.

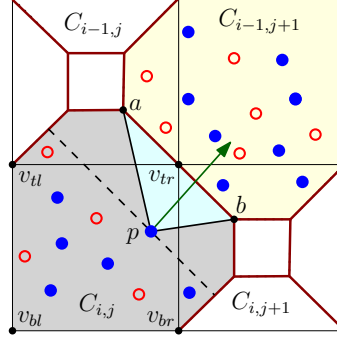
We construct a bichromatic spanning tree T' of P by traversing the cells starting from an arbitrary (non-empty) cell (in breadth first search (BFS) manner). That is, we first initiate a tree T' by an arbitrary tree $T_{i,j}$ that is constructed in a cell $\hat{C}_{i,j}$. Then, we connect T' to all the trees constructed in the cells adjacent to $\hat{C}_{i,j}$, and proceed from these trees. More precisely, in each step, we consider a tree $T_{i,j}$, which is already connected to T' , and we connect T' to all of the trees constructed in the cells adjacent to $\hat{C}_{i,j}$ via $T_{i,j}$ (if they are not connected yet to T'). In the following, we describe how to connect T' to all the trees constructed in the cells adjacent to $\hat{C}_{i,j}$.

Let \hat{C} be a cell adjacent to $\hat{C}_{i,j}$, such that the tree T_C constructed in \hat{C} is not connected yet to T' . Let v_{tr} , v_{tl} , v_{br} , and v_{bl} be the top-right, top-left, bottom-right, and bottom-left vertices of the grid incident to $C_{i,j}$, respectively; see Figure 14. We distinguish between two cases.

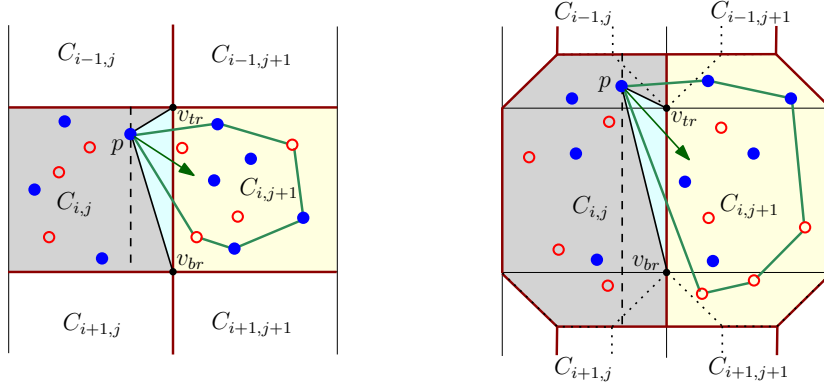
Case 1: \hat{C} is a d-adjacent cell of $\hat{C}_{i,j}$. Assume, w.l.o.g., that $\hat{C} = \hat{C}_{i-1,j+1}$. Then, the boundaries of $\hat{C}_{i,j}$ and $\hat{C}_{i-1,j+1}$ share a common (diagonal) edge \overline{ab} ; see Figure 14. Moreover, the convex hull of $\hat{C}_{i,j} \cup \hat{C}_{i-1,j+1}$ does not contain any point of $P \setminus (\hat{P}_{i,j} \cup \hat{P}_{i-1,j+1})$ (this can be seen clearly in Figure 11). Let $p \in \hat{P}_{i,j}$ be the closest point to the line passing through \overline{ab} , such that no edge of T' crosses the triangle Δpab . By Claim 4, such a point p exists. Then, any edge connecting p to any point of $T_{i-1,j+1}$ does not cross any non-empty cell except $\hat{C}_{i,j}$ and $\hat{C}_{i-1,j+1}$. Therefore, by Lemma 3, we can connect p to $T_{i-1,j+1}$ without crossing any other edge of T' .

Case 2: \hat{C} is an s-adjacent cell of $\hat{C}_{i,j}$. Assume, w.l.o.g., that $\hat{C} = \hat{C}_{i,j+1}$. Let p be the rightmost point in $\hat{P}_{i,j}$, such that no edge of T' crosses the triangle $\Delta pv_{br}v_{tr}$; see Figure 15. By Claim 4, such a point p exists. Let H be the convex hull of $\hat{P}_{i,j+1} \cup \{p\}$. We consider two sub-cases.

Case 2.1: $H \cap (P \setminus (\hat{P}_{i,j} \cup \hat{P}_{i,j+1})) = \emptyset$ (i.e., H does not contain any point of P that is not in $\hat{P}_{i,j} \cup \hat{P}_{i,j+1}$); see Figure 15. Therefore, by Lemma 3, we can connect $T_{i,j+1}$ to $T_{i,j}$ via p , without crossing any other edge of T' .



■ **Figure 14** Illustration of Case 1. We connect $T_{i-1,j+1}$ to $T_{i,j}$ via p .



■ **Figure 15** The convex hull H of $\hat{P}_{i,j+1} \cup \{p\}$ (consisting of green segments) does not contain any point of P that is not in $\hat{P}_{i,j} \cup \hat{P}_{i,j+1}$. We connect $T_{i,j+1}$ to $T_{i,j}$ via p .

Case 2.2: $H \cap (P \setminus (\hat{P}_{i,j} \cup \hat{P}_{i,j+1})) \neq \emptyset$ (i.e., H contains a point of P that is not in $\hat{P}_{i,j} \cup \hat{P}_{i,j+1}$).

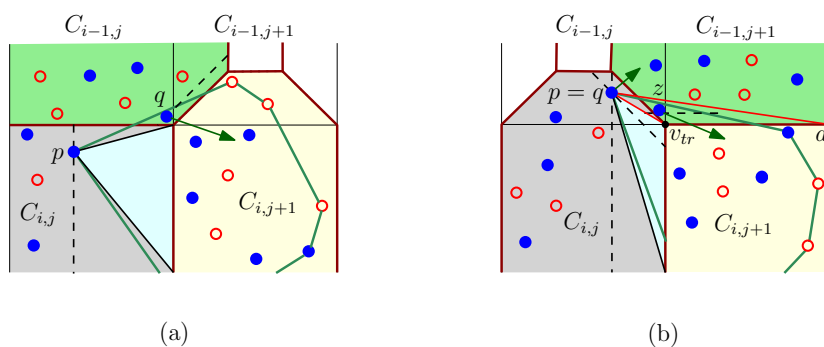
In this case, H contains a point in $\hat{P}_{i-1,j} \cup \hat{P}_{i-1,j+1}$ or in $\hat{P}_{i+1,j} \cup \hat{P}_{i+1,j+1}$. Assume, w.l.o.g., that H contains a point in $\hat{P}_{i-1,j} \cup \hat{P}_{i-1,j+1}$; see Figure 16. Notice that exactly one of the sets $\hat{P}_{i-1,j}$ or $\hat{P}_{i-1,j+1}$ is an empty set, since, in this case, exactly one of the cells $C_{i-1,j}$ or $C_{i-1,j+1}$ has been partitioned. We further distinguish between two cases.

1. $H \cap \hat{P}_{i-1,j} \neq \emptyset$; see Figure 16(a). In this case we first connect $T_{i,j+1}$ to $T_{i-1,j}$ as follows. Let $q \in \hat{P}_{i-1,j}$ be the closest point to the line passing through the boundary edge between $\hat{C}_{i-1,j}$ and $\hat{C}_{i,j+1}$. Then, the convex hull of $\hat{P}_{i,j+1} \cup \{q\}$ does not contain any point of P that is not in $\hat{P}_{i,j+1} \cup \{q\}$. Therefore, by Lemma 3, we can connect $T_{i,j+1}$ to $T_{i-1,j}$ via q , without crossing any other edge of T' .

Moreover, if $T_{i-1,j}$ is not connected yet to T' , then we apply Case 2 on $\hat{C}_{i-1,j}$ to connect $T_{i-1,j}$ to $T_{i,j}$.

2. $H \cap \hat{P}_{i-1,j+1} \neq \emptyset$. If $T_{i-1,j+1}$ is not connected yet to T' , then we first connect $T_{i-1,j+1}$ to $T_{i,j}$ as follows. Let $q \in \hat{P}_{i,j}$ be the closest point to the line passing through the boundary edge between $\hat{C}_{i,j}$ and $\hat{C}_{i-1,j+1}$; see Figure 16(b). Then, the convex hull of $\hat{P}_{i-1,j+1} \cup \{q\}$ does not contain any point of P that is not in $\hat{P}_{i-1,j+1} \cup \{q\}$. Therefore, by Lemma 3, we can connect $T_{i-1,j+1}$ to $T_{i,j}$ via q , without crossing any other edge of T' .

Moreover, we connect $T_{i,j+1}$ to $T_{i-1,j+1}$ as follows. Let a be the bottom-right corner of $C_{i-1,j+1}$. Thus, $\Delta pv_{tr}a \cap \hat{P}_{i-1,j+1} \neq \emptyset$ and $\Delta pv_{tr}a \cap (\hat{P}_{i,j} \setminus \{p\}) = \emptyset$. Let z be the bottommost point in $\Delta pv_{tr}a \cap \hat{P}_{i-1,j+1}$, such that no edge of T' crosses the triangle



■ **Figure 16** (a) H contains points from $\hat{P}_{i-1,j}$. We connect $T_{i,j+1}$ to $T_{i-1,j}$ via q . (b) H contains a point z from $\hat{P}_{i-1,j+1}$. We connect $T_{i-1,j+1}$ to $T_{i,j}$ via q and we connect $T_{i,j+1}$ to $T_{i-1,j+1}$ via z .

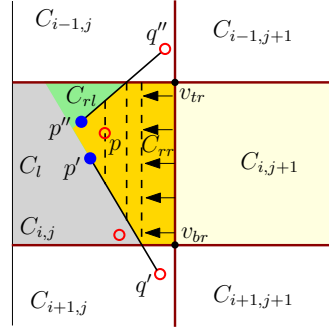
$\Delta zv_{tr}a$; see Figure 16(b). By Claim 4, such a point z exists. Then, the convex hull of $\hat{P}_{i,j+1} \cup \{z\}$ does not contain any point of P that is not in $\hat{P}_{i,j+1} \cup \{z\}$. Therefore, by Lemma 3, we can connect $T_{i,j+1}$ to $T_{i-1,j+1}$ via z , without crossing any other edge of T' . (Notice that the two edges added in this case do not cross each other.)

Correctness Proof

Recall that T is a bichromatic spanning tree of P of minimum bottleneck λ . In this section, we prove that T' is a planar bichromatic spanning tree of P of bottleneck at most $8\sqrt{2}\lambda$. Notice that every point $p \in P$ is contained in a bichromatic cell $\hat{C}_{i,j}$ and hence, it is connected to $T_{i,j}$, the tree constructed in Stage 2.1 in $\hat{C}_{i,j}$. Therefore, to show that T' is a bichromatic spanning tree of P , it is sufficient to show that each tree $T_{i,j}$ is connected to T' .

▷ **Claim 4.** Let T' be the tree constructed at some step during Stage 2.2 and assume that T' is planar. Let $T_{i,j}$ be a tree constructed in $\hat{C}_{i,j}$ and assume that $T_{i,j}$ is already connected to T' . Let \hat{C} be an adjacent cell of $\hat{C}_{i,j}$ that shares an edge \overline{ab} with $\hat{C}_{i,j}$ and let T_C be the tree constructed in \hat{C} , and assume that T_C is not connected yet to T' . Then, there exists a point p in $\hat{C}_{i,j}$, such that no edge of T' crosses the triangle Δpab .

Proof. Assume, w.l.o.g., that $\hat{C} = \hat{C}_{i,j+1}$, $a = v_{tr}$, and $b = v_{br}$; see Figure 17. The following procedure shows the existence of such a point p . We sweep leftwards in $\hat{C}_{i,j}$ with a vertical line l , starting from $\overline{v_{tr}v_{br}}$ until we meet a point, or an edge of T' . If we first meet a point, then this point satisfies the claim. Otherwise, we first meet an edge (p', q') of T' ; see Figure 17. This could only be when exactly one of the endpoints p' or q' is outside $\hat{C}_{i,j}$. Let C_l and C_r be the two sub-cells obtained by partitioning $\hat{C}_{i,j}$ with the line that goes through the points p' and q' . Let C_r be the sub-cell containing v_{tr} and v_{br} . We keep sweeping leftwards only inside C_r . As before, if we first meet a point, then this point satisfies the claim. Otherwise, we meet an edge (p'', q'') of T' before we meet a point. Then, one of the endpoints p'' or q'' is outside $\hat{C}_{i,j}$. Let C_{rl} and C_{rr} be the two sub-cells obtained by partitioning C_r with the line that goes through the points p'' and q'' . Let C_{rr} be the sub-cell containing v_{tr} and v_{br} . We keep sweeping leftwards only inside C_{rr} , until we meet a point, and this point satisfies the claim. Notice that, in the last sweep we meet a point before we meet an edge of T' . This follows from the planarity of T' . ◁



■ **Figure 17** Illustration of the proof of Claim 4.

► **Lemma 5.** *Let $T_{i,j}$ be a tree constructed in $\hat{C}_{i,j}$ in Stage 2.1 and assume that $T_{i,j}$ is already connected to T' . Then, at the end of Stage 2.2, all the trees that are constructed in the cells adjacent to $\hat{C}_{i,j}$ are connected to T' as well.*

Proof. Let \hat{C} be an adjacent cell of $\hat{C}_{i,j}$. We distinguish between two cases.

Case 1: \hat{C} is a d-adjacent cell of $\hat{C}_{i,j}$. Assume, w.l.o.g., that $\hat{C} = \hat{C}_{i-1,j+1}$. Then, in Stage 2.2, Case 1, we connect between $T_{i,j}$ and $T_{i-1,j+1}$.

Case 2: \hat{C} is an s-adjacent cell of $\hat{C}_{i,j}$. Assume, w.l.o.g., that $\hat{C} = \hat{C}_{i,j+1}$. As described in Stage 2.2, we select a point $p \in \hat{C}_{i,j}$ and compute the convex hull H of $\{p\} \cup \hat{P}_{i,j+1}$. Then, we consider two cases. In Case 2.1, when H does not contain any point of $P \setminus (\hat{P}_{i,j} \cup \hat{P}_{i,j+1})$, we connect $T_{i,j}$ directly to $T_{i,j+1}$ (via p). And, in Case 2.2, when H contains a point of $P \setminus (\hat{P}_{i,j} \cup \hat{P}_{i,j+1})$, we connect $T_{i,j+1}$ to $T_{i,j}$ via the tree $T_{i-1,j+1}$, in case that H contains a point of $\hat{P}_{i-1,j+1}$ (or via the tree $T_{i+1,j+1}$, in case that H contains a point of $\hat{P}_{i+1,j+1}$). In the case that H contains a point of $\hat{P}_{i-1,j}$ (symmetrically, H contains a point of $\hat{P}_{i+1,j}$), we connect $T_{i,j+1}$ to $T_{i,j}$ via the tree $T_{i-1,j}$. If $T_{i-1,j}$ is already connected to $T_{i,j}$, then we are done. Otherwise, since $\hat{C}_{i-1,j}$ is an s-adjacent cell of $\hat{C}_{i,j}$, we will try to connect $T_{i-1,j}$ to $T_{i,j}$ in the next iteration in Stage 2.2 (by applying Case 2 once again). In the next iteration, either we connect $T_{i-1,j}$ to $T_{i,j}$ in one of the cases described above or we end up by connecting $T_{i,j-1}$ to $T_{i-1,j}$. In the latter case, if $T_{i,j-1}$ is already connected to $T_{i,j}$, then we are done. otherwise, $T_{i+1,j}$ is already connected to $T_{i,j}$. In this case, we connect $T_{i,j-1}$ to $T_{i,j}$ either directly or via $T_{i+1,j}$, and we are done. ◀

► **Lemma 6.** *Let p and q be two points of P , such that p and q are of different colors, $|pq| \leq \lambda$ and p belongs to T' . Then, q also belongs to T' .*

Proof. Since $|pq| \leq \lambda$, either p and q are in the same cell or they are in adjacent cells. If they are in the same cell $\hat{C}_{i,j}$, then, after Stage 2.1, they are connected in $T_{i,j}$, and the lemma holds. Otherwise, assume, w.l.o.g., that $p \in \hat{C}_{i,j}$ and $q \in \hat{C}$, where \hat{C} is adjacent to $\hat{C}_{i,j}$. Then, after Stage 2.1, p belongs to $T_{i,j}$ and q belongs to $T_{\hat{C}}$, the tree constructed in \hat{C} . Since $T_{i,j}$ is part of T' and \hat{C} is adjacent to $\hat{C}_{i,j}$, by Lemma 5, $T_{i,j}$ is connected to $T_{\hat{C}}$ and therefore q belongs to T' . ◀

► **Lemma 7.** *Let $T_{i,j}$ be a tree constructed in $\hat{C}_{i,j}$. Then, $T_{i,j}$ is connected to T' .*

Proof. Assume by contradiction that $T_{i,j}$ is not connected to T' . Let a be a point from T' and let b be a point from $T_{i,j}$. Since T is a bottleneck bichromatic spanning tree of P , there is a path Π between a and b in T . Let p be the last point (from a) on Π that belongs to T' , i.e., no point of T' appears on the sub-path of Π between p and b . Since b does not belong

to T' , such a point p exists. Let q be the point between p and b on Π that is connected to p . By the selection of p , q does not belong to T' . Since the bottleneck of T is λ , we have $|pq| \leq \lambda$. Therefore, by Lemma 6, p and q are connected in T' , which contradicts that q does not belong to T' . ◀

► **Lemma 8.** T' is planar.

Proof. Each $T_{i,j}$ is planar. We start with $T' = T_{i,j}$, where $T_{i,j}$ is an arbitrary tree constructed in $\hat{C}_{i,j}$, and in each step, we extend T' by connecting it to the trees corresponding to the cells adjacent to the current cell. We connect T' to a “new” tree $T_{i,j}$ by picking a point p in T' , such that the convex hull H of $\{p\} \cup \hat{P}_{i,j}$ is empty of any other points and no edge of T' crosses H . In Claim 4, we showed that such a point p always exists. Thus, connecting p to any point of $T_{i,j}$ will not cross any other edge of T' nor of any other tree. Moreover, in Lemma 3, we show that we can always connect p to $T_{i,j}$ without crossing any of the edges of $T_{i,j}$. Therefore, connecting T' to $T_{i,j}$ does not produce any crossing. ◀

► **Lemma 9.** The bottleneck of T' is at most $8\sqrt{2}\lambda$.

Proof. Consider Figure 11. After Stage 1, each extended cell is contained in a square of size $5\lambda \times 5\lambda$, and hence the bottleneck of each tree constructed in Stage 2.1 is at most $5\sqrt{2}\lambda$. Moreover, every two d-adjacent cells are contained in a square of size $8\lambda \times 8\lambda$ and every two s-adjacent cells are contained in a square of size $8\lambda \times 5\lambda$. Thus, each edge added in Stage 2.2 is of length at most $8\sqrt{2}\lambda$. Therefore, each edge in T' is of length at most $8\sqrt{2}\lambda$. ◀

The algorithm consists of two main stages, and each one of them can be implemented in polynomial time. Therefore, the total running time of the algorithm is polynomial. The following theorem summarizes the result of this section.

► **Theorem 10.** Let P be a set of n red and blue points in the plane. One can compute in polynomial time a planar bichromatic spanning tree of P of bottleneck at most $8\sqrt{2}$ times the bottleneck of an optimal bichromatic spanning tree of P .

References

- 1 M. Abellanas, J. Garcia-Lopez, G. Hernández-Peñalver, M. Noy, and P. A. Ramos. Bipartite embeddings of trees in the plane. *Discr. Appl. Math.*, 93(2-3):141–148, 1999.
- 2 A. K. Abu-Affash, A. Biniiaz, P. Carmi, A. Maheshwari, and M. Smid. Approximating the bottleneck plane perfect matching of a point set. *Comput. Geom.*, 48(9):718–731, 2015.
- 3 A. K. Abu-Affash, P. Carmi, M. J. Katz, and Y. Trabelsi. Bottleneck non-crossing matching in the plane. *Comput. Geom.*, 47(3):447–457, 2014.
- 4 P. K. Agarwal. Partitioning arrangements of lines II: Applications. *Discr. Comput. Geom.*, 5(1):533–573, 1990.
- 5 P. K. Agarwal, H. Edelsbrunner, and O. Schwarzkopf. Euclidean minimum spanning trees and bichromatic closest pairs. *Discr. Comput. Geom.*, 6(1):407–422, 1991.
- 6 O. Aichholzer, S. Cabello, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, F. Hurtado, and D. R. Wood. Edge-removal and non-crossing configurations in geometric graphs. In *EuroCG*, pages 119–122, 2008.
- 7 N. Alon, S. Rajagopalan, and S. Suri. Long non-crossing configurations in the plane. In *SoCG*, pages 257–263, 1993.
- 8 G. Aloupis, J. Cardinal, S. Collette, E. D. Demaine, M. L. Demaine, M. Dulieu, R. Fabila-Monroy, V. Hart, F. Hurtado, S. Langerman, M. Saumell, C. Seara, and P. Taslakian. Matching points with things. In *LATIN, volume 6034 of LNCS*, pages 456–467, 2010.

- 9 S. Arora and K. L. Chang. Approximation schemes for degree-restricted MST and red–blue separation problems. *Algorithmica*, 40(3):189–210, 2004.
- 10 M. J. Atallah and D. Z. Chen. On connecting red and blue rectilinear polygonal obstacles with nonintersecting monotone rectilinear paths. *Int. J. Comput. Geom. Appl.*, 11(4):373–400, 2001.
- 11 A. Biniaz, P. Bose, K. Crosbie, J.-L. De Carufel, D. Eppstein, A. Maheshwari, and M. H. M. Smid. Maximum plane trees in multipartite geometric graphs. *Algorithmica*, 81(4):1512–1534, 2019.
- 12 A. Biniaz, P. Bose, D. Eppstein, A. Maheshwari, P. Morin, and M. H. M. Smid. Spanning trees in multipartite geometric graphs. *Algorithmica*, 80(11):3177–3191, 2018.
- 13 A. Biniaz, A. Maheshwari, and M. Smid. Bottleneck bichromatic plane matching of points. In *CCCG*, 2014.
- 14 J.-D. Boissonnat, J. Czyzowicz, O. Devillers, J. Urrutia, and M. Yvinec. Computing largest circles separating two sets of segments. *Int. J. Comput. Geom. Appl.*, 10(1):41–53, 2000.
- 15 M. G. Borgelt, M. Van Kreveld, M. Löffler, J. Luo, D. Merrick, R. I. Silveira, and M. Vahedi. Planar bichromatic minimum spanning trees. *J. Discrete Algorithms*, 7(4):469–478, 2009.
- 16 H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990. doi:10.1007/BF02122694.
- 17 E. D. Demaine, J. Erickson, F. Hurtado, J. Iacono, S. Langerman, H. Meijer, M. H. Overmars, and S. Whitesides. Separating point sets in polygonal environments. *Int. J. Comput. Geom. Appl.*, 15(4):403–419, 2005.
- 18 A. Dumitrescu and R. Kaye. Matching colored points in the plane: some new results. *Comput. Geom.*, 19(1):69–85, 2001.
- 19 A. Dumitrescu and J. Pach. Partitioning colored point sets into monochromatic parts. *Int. J. Comput. Geom. Appl.*, 12(05):401–412, 2002.
- 20 H. Everett, J.-M. Robert, and M. J. van Kreveld. An optimal algorithm for the ($\leq k$)-levels, with applications to separation and transversal problems. *Int. J. Comput. Geom. Appl.*, 6(3):247–261, 1996.
- 21 A. Kaneko and M. Kano. Discrete geometry on red and blue points in the plane – a survey. *Discr. Comput. Geom.*, 25:551–570, 2003.
- 22 D. Lichtenstein. Planar formulae and their uses. *SIAM J. Comput.*, 11(2):329–343, 1982.
- 23 H. G. Mairson and J. Stolfi. Reporting and counting intersections between two sets of line segments. In *Theoretical Foundations of Computer Graphics and CAD*, volume 40 of *NATO ASI Series*, pages 307–325, 1988.