Reachability in Two-Dimensional Vector Addition Systems with States: One Test Is for Free

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Abstract

Vector addition system with states is an ubiquitous model of computation with extensive applications in computer science. The reachability problem for vector addition systems is central since many other problems reduce to that question. The problem is decidable and it was recently proved that the dimension of the vector addition system is an important parameter of the complexity. In fixed dimensions larger than two, the complexity is not known (with huge complexity gaps). In dimension two, the reachability problem was shown to be PSPACE-complete by Blondin et al. in 2015. We consider an extension of this model, called 2-TVASS, where the first counter can be tested for zero. This model naturally extends the classical model of one counter automata (OCA). We show that reachability is still solvable in polynomial space for 2-TVASS. As in the work Blondin et al., our approach relies on the existence of small reachability certificates obtained by concatenating polynomially many cycles.

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1 Introduction

Context. Vector addition systems with states (VASS for short) is an ubiquitous model of computation with extensive applications in computer science. This model, equivalent to Petri nets, is defined as a finite state automaton with transitions acting on a set of counters ranging over the nonnegative integers by adding integers. The number of counters is called the dimension and we write d-VASS for a VASS with d counters. The central problem on VASS is the reachability problem since many other problems are reducible to reachability questions. This problem was first proved to be hard for the exponential-space complexity by Lipton [23] in 1976. At that time, the decidability of the problem was open. Three years later [11], the reachability problem for 2-VASS was proved to be decidable by Hopcroft and Pansiot by observing that reachability sets of 2-VASS are *semilinear*. Dimension two is a special case since in contrast reachability sets of 3-VASS are not semilinear in general. A few years later, the reachability problem for VASS was proved to be decidable in any dimension by Mayr [24, 25] thanks to an algorithm simplified later by Kosaraju [12] and Lambert [13]. Recently, the problem was revisited by Leroux [15, 16, 17] by observing that the reachability problem can be decided with a simple algorithm based on semilinear

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inductive invariants. Despite recent improvements on the reachability problem, the exact complexity is still open; the known lower-bound is Tower-hard [4] and the known upper-bound is Ackermannian-easy [18].

When adding to VASS the ability to test counters for zero, the reachability problem becomes undecidable in dimension two via a direct simulation of two-counters Minsky machines [26, Chapter 14]. In dimension one, the class of VASS that we obtain by adding zero-tests are usually called one counter automata (OCA for short). The reachability problem for that class was proved to be NP-complete in [10]. The class of OCA can be naturally extended by introducing the class of d-TVASS (or just TVASS when the dimension d is not fixed) corresponding to a d-VASS extended with zero-tests on the first counter. In that context, a OCA is just a 1-TVASS. The reachability problem is known to be decidable for TVASS in any dimension [28, 2], but the complexity is open, even in dimension two.

In dimension two, the reachability problem for VASS is known to be PSPACE-complete. This result was obtained thanks to a series of results from several authors. The PSPACE lower-bound was proved in [8] and PSPACE membership was obtained as follows (notice that the problem was recently revisited in [5]). First of all, the reachability relation was proved to be semilinear in [19] by observing that it is *flattenable*, meaning that the reachability relation can be captured by a finite set of regular expressions, so called *linear path schemes*, of the form $\alpha_0\beta_1^*\alpha_1\cdots\beta_k^*\alpha_k$ where $\alpha_0,\ldots,\alpha_k$ are paths and $\beta_1,\ldots,\beta_k$ are cycles in the underlying graph of the VASS. It was then proved in [1] that these regular expressions can be exponentially bounded, and k is bounded by a polynomial in the number of states. By introducing a system of inequalities over some variables $n_1,\ldots,n_k$ counting the number of times the cycles $\beta_1,\ldots,\beta_k$ are iterated, an exponential bound on small paths witnessing reachability was derived from a small solution theorem [27, 3]. From such a bound, it follows that the reachability problem is decidable in PSPACE.

Our contribution. In this paper, we are interested in the complexity of the reachability problem for 2-TVASS. We successfully follow the approach used for 2-VASS and outlined above. This approach is not easily lifted to 2-TVASS, because the presence of zero-tests breaks a fundamental property of VASS, namely *monotonicity*. By means of new proof techniques to deal with zero-tests on a single counter, we obtain the following results:

- We show that the reachability relation of a 2-TVASS is flattenable. Our proof does not provide by itself any complexity bound but it is direct and simple, and it provides a description of the reachability relation by *linear path schemes* $\alpha_0\beta_1^*\alpha_1\cdots\beta_k^*\alpha_k$.
- We prove that these linear path schemes can be exponentially bounded, and the number k can be polynomially bounded in the number of states of the 2-TVASS. This bound is obtained via a detour through the class of *weighed one counter automata* (WOCA for short). We believe that our results on WOCA may be of independent interest.
- We derive an exponential bound on paths witnessing reachability thanks to a small solution theorem. From that bound, we deduce that the reachability problem for 2-TVASS is decidable in polynomial space, and so is PSPACE-complete. This is, to our knowledge, one of the few problems on extended VASS whose precise complexity is known.

Related work. TVASS are naturally related to other classical extensions of VASS by observing that a reset is a “weak test”, and a testable counter is a “weak stack”.

By extending d-VASS with resets on the two first counters, we obtain the class of d-RRVASS. It is known that the reachability problem for this class is undecidable if $d \geq 3$ while it is decidable for $d = 2$ [6]. Since a reset can be simulated by a test, the class of

2-TRVASS obtained from 2-VASS by allowing tests on the first counter and resets on the second one, contains the 2-RRVASS. In [9], we proved that the reachability problem for 2-TRVASS is decidable by proving that the reachability relation is effectively semilinear. It worth noticing that this relation is *not* flattenable, and the complexity of the reachability problem for 2-RRVASS and 2-TRVASS is still open. When dealing with the *lossy semantics* (i.e., when counters can be decreased arbitrarily at any step of the execution), tests and resets have exactly the same behavior. In that case, the reachability problem for lossy Minsky machines of arbitrary dimension becomes decidable and the exact complexity is Ackermannian complete [29].

The class of TVASS is also related to the class of pushdown VASS (PVASS for short) obtained by extending VASS with a stack over a finite alphabet. A PVASS can easily simulate any TVASS since a testable counter can be simulated with a stack. The decidability of the reachability problem is open even for 1-PVASS. We proved in [22] that the control-state reachability problem for 1-PVASS is decidable. The complexity is still open.

Due to space constraints, some proofs are missing and some proofs are only sketched. Detailed proofs can be found in the full version of the paper [21].

2 Flattenability of 2-TVASS

Preliminaries. The usual sets of *integers* and *nonnegative integers* are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. For any $a, b \in \mathbb{Z}$, we let $[a, b] \stackrel{\text{def}}{=} \{z \in \mathbb{Z} \mid a \leq z \leq b\}$. A d-dimensional *vector* of integers is a tuple $\mathbf{v} = (v_1, \ldots, v_d)$ in $\mathbb{Z}^d$. Its ith *component* v_i is also written $\mathbf{v}(i)$. We denote by $\|\mathbf{v}\|$ its *infinity norm* $\max\{|v_1|, \ldots, |v_d|\}$. A *word* over some alphabet Σ is a finite sequence $w = a_1 \cdots a_n$ of elements $a_i \in \Sigma$. The *length* of w is $|w| \stackrel{\text{def}}{=} n$. Given two binary relations R and S over some set, we let $R \mathbin{⨾} S \stackrel{\text{def}}{=} \{(x, z) \mid \exists y : x \, R \, y \, S \, z\}$ denote their *relational composition*. The *powers* of a binary relation R are inductively defined by $R^1 \stackrel{\text{def}}{=} R$ and $R^{n+1} \stackrel{\text{def}}{=} R \mathbin{⨾} R^n$.

Vector Addition Systems with States and One Test. A TVASS is a vector addition system with states (VASS) such that the first counter can be tested for zero. Formally, a *d-dimensional TVASS* (shortly called a *d-TVASS*), is a triple $\mathcal{V} = (Q, \Sigma, \Delta)$ where Q is a finite nonempty set of *states*, $\Sigma \subseteq \mathbb{Z}^d \cup \{\mathtt{tst}\}$ is a finite set of *actions*, and $\Delta \subseteq Q \times \Sigma \times Q$ is a finite set of *transitions*. Even though they are not mentioned explicitly, $\mathcal{V}$ implicitly comes with d counters $\mathtt{c}_1, \ldots, \mathtt{c}_d$ whose values range over nonnegative integers. Actions in Σ are either *addition* actions $\mathbf{a} \in \mathbb{Z}^d$ or the *zero-test* action $\mathtt{tst}$. Intuitively, an addition action $\mathbf{a} = (a_1, \ldots, a_d)$ performs the instruction $(\mathtt{c}_1, \ldots, \mathtt{c}_d) \leftarrow (\mathtt{c}_1 + a_1, \ldots, \mathtt{c}_d + a_d)$, provided that all counters remain nonnegative ; the zero-test action $\mathtt{tst}$ tests the first counter for zero and leaves all counters unchanged. We let $A \stackrel{\text{def}}{=} \{(p, \sigma, q) \in \Delta \mid \sigma \in \mathbb{Z}^d\}$ and $T \stackrel{\text{def}}{=} \{(p, \sigma, q) \in \Delta \mid \sigma = \mathtt{tst}\}$ denote the sets of *addition* transitions and *zero-test* transitions, respectively. The notation $\|\Sigma\|$ stands for $\max_{\mathbf{a}} \|\mathbf{a}\|$ where $\mathbf{a}$ ranges over addition actions (or $\{\mathbf{0}\}$ if there are none). A *d-dimensional VASS* (shortly called a *d-VASS*) is a d-TVASS whose set of actions Σ excludes $\mathtt{tst}$, i.e., $\Sigma \subseteq \mathbb{Z}^d$.

We define the operational semantics of a d-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ as follows. A *configuration* of $\mathcal{V}$ is a pair $(q, \mathbf{x})$ where $q \in Q$ is a state and $\mathbf{x} \in \mathbb{N}^d$ is a vector denoting the contents of the counters $\mathtt{c}_1, \ldots, \mathtt{c}_d$. For the sake of readability, configurations $(q, \mathbf{x})$ are written $q(\mathbf{x})$ in the sequel. For each transition $\delta \in \Delta$, we let $\xrightarrow{\delta}$ denote the least binary relation over configurations satisfying the following rules:

$$\frac{\delta = (p, \mathbf{a}, q) \qquad \mathbf{x} \in \mathbb{N}^d \qquad \mathbf{x} + \mathbf{a} \geq \mathbf{0}}{p(\mathbf{x}) \xrightarrow{\delta} q(\mathbf{x} + \mathbf{a})} \qquad\qquad \frac{\delta = (p, \mathtt{tst}, q) \qquad \mathbf{x} \in \mathbb{N}^d \qquad \mathbf{x}(1) = 0}{p(\mathbf{x}) \xrightarrow{\delta} q(\mathbf{x})}$$

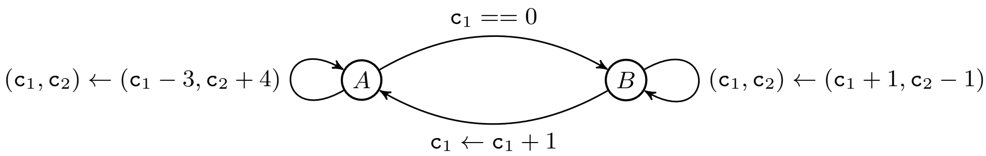

Figure 1 A simple 2-dimensional TVASS with actions written in verbose pseudo-code (to help the reader). Instructions of the form $(\mathtt{c}_1, \mathtt{c}_2) \leftarrow (\mathtt{c}_1 + a_1, \mathtt{c}_2 + a_2)$ stand for addition actions (a_1, a_2). The instruction $\mathtt{c}_1 == 0$ stands for the zero-test action $\mathtt{tst}$.

Given a word $\pi = \delta_1 \cdots \delta_n$ of transitions $\delta_i \in \Delta$, we denote by $\xrightarrow{\pi}$ the binary relation over configurations defined as the relational composition $\xrightarrow{\delta_1} \mathbin{;} \cdots \mathbin{;} \xrightarrow{\delta_n}$. The relation $\xrightarrow{\varepsilon}$ denotes the identity relation on configurations. Given a subset $L \subseteq \Delta^*$, we let $\xrightarrow{L}$ denote the union $\bigcup_{\pi \in L} \xrightarrow{\pi}$. The relation $\xrightarrow{\Delta^*}$, also written $\xrightarrow{*}$, is called the *reachability relation* of $\mathcal{V}$. Observe that $\xrightarrow{*}$ is the reflexive-transitive closure of the *step* relation $\rightarrow \stackrel{\text{def}}{=} \xrightarrow{\Delta}$.

A *run* is a finite, alternating sequence $(q_0(\mathbf{x}_0), \delta_1, q_1(\mathbf{x}_1), \ldots, \delta_n, q_n(\mathbf{x}_n))$ of configurations and transitions, satisfying $q_{i-1}(\mathbf{x}_{i-1}) \xrightarrow{\delta_i} q_i(\mathbf{x}_i)$ for all $i \in [1, n]$. Note that this condition entails that $q_0(\mathbf{x}_0) \xrightarrow{\delta_1 \cdots \delta_n} q_n(\mathbf{x}_n)$. The word $\delta_1 \cdots \delta_n$ is called the *trace* of the run and n is its *length*.

▶ **Example 1.** Consider the 2-TVASS depicted in Figure 1. There are two states, namely A and B, and four transitions, namely:

$$\begin{array}{lcl@{\qquad}lcl} \delta_{AA} & = & (A\,, (-3, 4)\,, A) & \delta_{BB} & = & (B\,, (1, -1)\,, B) \\ \delta_{AB} & = & (A\,, \;\;\mathtt{tst}\;\;, B) & \delta_{BA} & = & (B\,, \;(1, 0)\;\,, A)\,. \end{array}$$

Starting from the configuration $A(3, 5)$, the zero-test transition δ_{AB} cannot be taken as the first counter is not zero. But we can take the loop on A and reach the configuration $A(0, 9)$, which is formally written as the step $A(3, 5) \xrightarrow{\delta_{AA}} A(0, 9)$. We may then move to B via the zero-test transition, take the loop on B four times, and get back to A. This yields the run $\rho = (A(3, 5), \delta_{AA}, A(0, 9), \delta_{AB}, B(0, 9), \delta_{BB}, B(1, 8), \ldots, \delta_{BB}, B(4, 5), \delta_{BA}, A(5, 5))$. The trace of this run is $\pi = \delta_{AA}\delta_{AB}(\delta_{BB})^4\delta_{BA}$, and so we have $A(3, 5) \xrightarrow{\pi} A(5, 5)$.

The run ρ witnesses the fact that $A(3, 5) \xrightarrow{*} A(5, 5)$. In a standard 2-VASS, i.e., without zero-test, the run ρ could be "replayed" from the larger configuration $A(5, 5)$. More precisely, we would have $A(3, 5) \xrightarrow{\pi} A(5, 5) \xrightarrow{\pi} A(7, 5)$. This is not the case in our 2-TVASS. Even though $A(3, 5) \xrightarrow{\pi} A(5, 5)$, it does not hold that $A(5, 5) \xrightarrow{\pi} A(7, 5)$. Indeed, $A(5, 5) \xrightarrow{\delta_{AA}} A(2, 9)$ and the zero-test transition δ_{AB} cannot be taken from $A(2, 9)$.

We cannot replay ρ from the larger configuration $A(5, 5)$. Nonetheless, the configuration $A(7, 5)$ is reachable from $A(3, 5)$ in our 2-TVASS. In fact, it holds that $A(3, 5) \xrightarrow{*} A(3 + 2k, 5)$ for every $k \in \mathbb{N}$. This property will be shown in Example 2. ⌟

Linear Path Schemes and Flattenability. Consider a d-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$. A *path* from a state $p \in Q$ to a state $q \in Q$ is either the empty word ε or a nonempty word $\delta_1 \cdots \delta_n$ of transitions, with $\delta_i = (p_i, \sigma_i, q_i)$, such that $p_0 = p$, $q_n = q$ and $q_{i-1} = p_i$ for all $i \in [1, n]$. Note that for every word $\pi \in \Delta^*$, if the relation $\xrightarrow{\pi}$ is not empty then π is a path. The converse does not hold in general (but it holds if $\pi \in A^*$, i.e., if no zero-test occurs in π). A *cycle* on a state $q \in Q$ is a path from q to q.

A *linear path scheme* from a state $p \in Q$ to a state $q \in Q$ is a regular expression L of the form $L = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ where $\alpha_0 \beta_1 \alpha_1 \cdots \beta_k \alpha_k$ is a path from p to q and each β_i is a cycle. We call $\beta_1, \ldots, \beta_k$ the *cycles* of L. Its *length* is $|L| \stackrel{\text{def}}{=} |\alpha_0 \beta_1 \alpha_1 \cdots \beta_k \alpha_k|$ and its $*$-*length* is $|L|_* \stackrel{\text{def}}{=} k$. We slightly abuse notation and also write L for the language associated to a linear path scheme L.

▶ **Example 2.** We claimed at the end of Example 1 that $A(3,5) \xrightarrow{*} A(3+2k, 5)$ for every $k \in \mathbb{N}$. The case where $k = 0$ is trivial, so let us prove this property assuming that $k > 0$. Consider the linear path scheme

$$L = \delta_{AA} \cdot (\delta_{AB} \delta_{BB} \delta_{BB} \delta_{BA} \delta_{AA})^* \cdot \delta_{AB} \cdot (\delta_{BB})^* \cdot \delta_{BA} .$$

Note in passing that L has length $|L| = 9$ and $*$-length $|L|_* = 2$. To simplify notation, let $\pi \stackrel{\text{def}}{=} \delta_{AB} \delta_{BB} \delta_{BB} \delta_{BA} \delta_{AA}$. Observe that $A(0,x) \xrightarrow{\pi} A(0, x+2)$ for every $x \geq 2$. It follows that $A(3,5) \xrightarrow{\delta_{AA}} A(0,9) \xrightarrow{\pi^{k-1}} A(0, 2k+7) \xrightarrow{\delta_{AB}} B(0, 2k+7) \xrightarrow{(\delta_{BB})^{2k+2}} B(2k+2, 5) \xrightarrow{\delta_{BA}} A(2k+3, 5)$. We have shown that $A(3,5) \xrightarrow{L} A(3+2k, 5)$ for every $k > 0$. ⌟

A binary relation R over configurations is called *flattenable*[1] if there exists a finite set Λ of linear path schemes such that $R \subseteq \bigcup_{L \in \Lambda} \xrightarrow{L}$. It is readily seen that the class of flattenable binary relations is closed under union and relational composition. We say that a d-TVASS $\mathcal{V}$ is *flattenable* when its reachability relation $\xrightarrow{*}$ is flattenable.

Flattenability of TVASS in Dimension Two. We showed sixteen years ago in [19] that 2-VASS are flattenable. Our approach was refined ten years later by Blondin et al. to provide bounds on the resulting linear path schemes and to show that the reachability problem for 2-VASS is solvable in polynomial space [1].

▶ **Theorem 3** ([19, 1])**.** *Every 2-VASS is flattenable. Furthermore, for every configurations $p(\mathbf{x})$ and $q(\mathbf{y})$ of a 2-VASS $\mathcal{V} = (Q, \Sigma, \Delta)$ such that $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$, there exists a linear path scheme L with $|L| \leq (|Q| + \|\Sigma\|)^{O(1)}$ and $|L|_* \leq O(|Q|^2)$ such that $p(\mathbf{x}) \xrightarrow{L} q(\mathbf{y})$.*

The remainder of this section is devoted to the extension of the first part of Theorem 3 to 2-TVASS. The existence of small linear path schemes witnessing flattenability will be shown in Sections 3 and 4, and ensuing complexity results will be presented in Sections 5 and 6.

Consider a 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$. We introduce, for each state $q \in Q$, the binary relation $\updownarrow_q$ over configurations defined by $\updownarrow_q = \{(q(0,x), q(0,y)) \mid q(0,x) \xrightarrow{*} q(0,y)\}$. This relation is called the *vertical loop relation* on q. We let $\updownarrow$ denote the union $\bigcup_{q \in Q} \updownarrow_q$. We first provide a decomposition of $\xrightarrow{*}$ in terms of $\xrightarrow{A^*}$, $\xrightarrow{T}$ and $\updownarrow$. Recall that A and T are the sets of addition transitions and zero-test transitions, respectively.

▶ **Lemma 4.** *It holds that* $\xrightarrow{*} \subseteq \left(\xrightarrow{A^*} \cup \xrightarrow{T} \cup \updownarrow\right)^{2|Q|+1}$.

The binary relations $\xrightarrow{A^*}$ and $\xrightarrow{T}$ are already known to be flattenable. This is a consequence of Theorem 3 for the former, and flattenability is obvious for the latter. As flattenable binary relations are closed under union and relational composition, it remains to show that $\updownarrow$ is flattenable. Vertical loops $q(0,x) \xrightarrow{*} q(0,y)$ either increase the second counter (i.e., $y > x$), or decrease it (i.e., $y < x$), or leave it unchanged (i.e., $y = x$). Let $\uparrow_q$ and $\downarrow_q$ denote the subrelations of $\updownarrow_q$ that correspond to the first and second cases, respectively. We first prove that the relations $\uparrow_q$ are flattenable.

[1] The same notion is often called *flattable* in the literature. It was simply called *flat* in [19].

Fix a state $q \in Q$ and assume that $\uparrow_q$ is not empty (otherwise it is trivially flattenable). We introduce the sequence $(D_x)_{x\in\mathbb{N}}$ of subsets of $\mathbb{N}$ defined by

$$D_x \;=\; \{d \in \mathbb{N} \mid q(0,x) \xrightarrow{*} q(0,x+d)\}$$

We derive from the monotonicity of 2-TVASS with respect to the second counter that $D_0 \subseteq D_1 \subseteq D_2 \cdots$ and that[2] $(D_x + D_x) \subseteq D_x$ for every $x \in \mathbb{N}$. These two properties entail that the sequence $(D_x)_{x\in\mathbb{N}}$ is ultimately stationary. Indeed, suppose by contradiction that there is an infinite increasing subsequence $\{0\} \subsetneq D_{x_0} \subsetneq D_{x_1} \subsetneq D_{x_2} \subsetneq \cdots$. We may extract $c, d_1, d_2, \ldots$ such that $c \in D_{x_0}$, $c > 0$, and $d_i \in D_{x_i} \setminus D_{x_{i-1}}$ for all $i > 0$. By the pigeonhole principle, some congruence class modulo c contains infinitely many d_i. So there exists $0 < i < j$ and $k \in \mathbb{N}$ such that $d_j = d_i + kc$. As d_i and c are both in D_{x_i}, we get from $(D_{x_i} + D_{x_i}) \subseteq D_{x_i}$ that $d_j \in D_{x_i}$, which is impossible since $D_{x_i} \subseteq D_{x_{j-1}}$ and $d_j \notin D_{x_{j-1}}$. We have shown that there exists $t \in \mathbb{N}$ such that $D_x = D_t$ for all $x \geq t$.

Recall that $\uparrow_q$ was assumed to be nonempty. So there exists $h \geq 0$, $m > 0$ and a run from $q(0,h)$ to $q(0,h+m)$. Let β denote the trace of this run. Note that β is a nonempty cycle on q. We derive from $q(0,h) \xrightarrow{\beta} q(0,h+m)$ that $(d+m) \in D_x$ for all $x \in \mathbb{N}$ and $d \in D_x$ with $d \geq h$. It follows that each D_x may be decomposed into $D_x = F_x \cup (B_x + \mathbb{N}m)$ where F_x and B_x are finite subsets of $\mathbb{N}$ such that $b \geq h$ for all $b \in B_x$. For every $d \in D_x$, let $\alpha_{x,d}$ denote the trace of some run from $q(0,x)$ to $q(0,x+d)$. Consider the finite set Λ of linear path schemes defined by

$$\Lambda \;=\; \bigcup_{x \leq t} \Lambda_x \qquad\qquad \text{and} \qquad\qquad \Lambda_x \;=\; \{\alpha_{x,f} \mid f \in F_x\} \;\cup\; \{\alpha_{x,b}\beta^* \mid b \in B_x\}\,.$$

Observe that Λ is finite as it collects the linear path schemes in Λ_x only for $x \leq t$. The linear path schemes in Λ_x with $x > t$ are redundant because of the above-established stabilization property of $(D_x)_{x\in\mathbb{N}}$. We obtain the following lemma by construction.

▶ **Lemma 5.** *It holds that $\uparrow_q \subseteq \bigcup_{L\in\Lambda} \xrightarrow{L}$, hence, the relation $\uparrow_q$ is flattenable.*

Notice that a decreasing vertical loop $q(0,x) \xrightarrow{*} q(0,y)$ with $y < x$ is an increasing vertical loop in the 2-TVASS $\overline{\mathcal{V}}$ obtained from $\mathcal{V}$ by reversing the effect of each transition, i.e., $\overline{\Delta} = \{\overline{\delta} \mid \delta \in \Delta\}$ where $\overline{(p,\mathbf{a},q)} = (q,-\mathbf{a},p)$ and $\overline{(p,\mathtt{tst},q)} = (q,\mathtt{tst},p)$. Put differently, the relation $\downarrow_q$ in $\mathcal{V}$ coincides with the relation $\uparrow_q$ in $\overline{\mathcal{V}}$. By applying Lemma 5 to $\overline{\mathcal{V}}$ and taking the mirror image of the resulting linear path schemes, we get that the relation $\downarrow_q$ in $\mathcal{V}$ is also flattenable. Since $\updownarrow = \bigcup_{q\in Q} \updownarrow_q$ and $\updownarrow_q \subseteq \uparrow_q \cup \downarrow_q \cup \xrightarrow{\varepsilon}$ for every $q \in Q$, we obtain that $\updownarrow$ is flattenable. Together with Lemma 4 and Theorem 3, this concludes the proof of the following theorem.

▶ **Theorem 6.** *Every 2-TVASS is flattenable.*

We have presented in this section a direct and simple proof that the reachability relation of every 2-TVASS is flattenable. This result entails, in particular, that the reachability relation is effectively semilinear for 2-TVASS (which was already known [9]) and computable by cycle acceleration techniques (see, e.g., [20]).

To derive complexity results from flattenability, we need to bound the length of linear path schemes witnessing flattenability. This requires a finer analysis of 2-TVASS runs than what was done for Theorem 6 (the latter will not be used in the remainder of the paper).

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[2] The sum $A + B$ of two subsets $A, B \subseteq \mathbb{Z}$ is defined as $\{a + b \mid a \in A \wedge b \in B\}$.

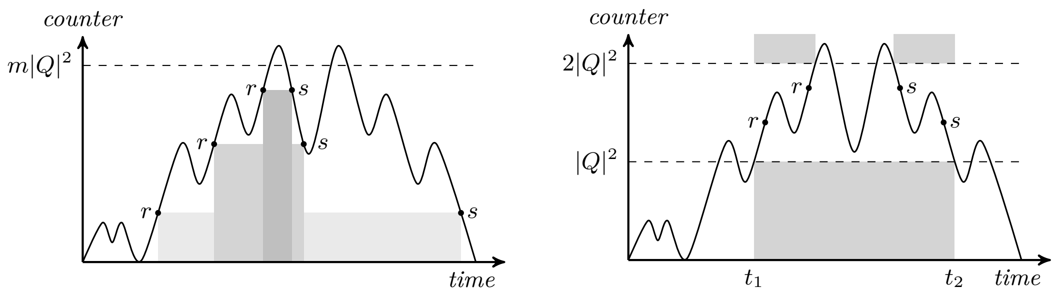

Figure 2 Illustration of the pumping lemmas for one-counter automata (Lemma 7 on the left and Lemma 9 on the right). The curves show the evolution of the counter along a run. Gray areas denote forbidden zones for the counter.

3 A Detour via Weighted One-Counter Automata

We have given in the previous section a simple proof that 2-TVASS are flattenable. This proof provides no bound on the length of the resulting linear path schemes, though. To obtain small linear path schemes witnessing flattenability, we take a detour via weighted one-counter automata. The rationale is that a 2-TVASS behaves like a one-counter automaton equipped with an additional counter (that cannot be tested for zero). When this additional counter is allowed to become negative, actions on it can be seen as weights. We show in this section that, in a weighted one-counter automaton, the reachable weights between two mutually reachable configurations $p(0)$ and $q(0)$ can be obtained via small linear path schemes. This will yield, for 2-TVASS, small linear path schemes for the reachability subrelations $p(0,x) \xrightarrow{*} q(0,y)$ such that x and y are large and $q(0,y) \xrightarrow{*} p(0,z)$ for some z. For simplicity, we consider weighted one-counter automata where addition actions and weights are in $\{-1,0,1\}$

A *weighted one-counter automaton* (shortly called a *WOCA*), is a quadruple $\mathcal{A} = (Q,\Sigma,\Delta,\lambda)$ where (Q,Σ,Δ) is a 1-TVASS such that $\Sigma = \{-1,0,1,\texttt{tst}\}$ and $\lambda : \Delta \to \{-1,0,1\}$ is a *weight* function. All notions defined in Section 2 for 1-TVASS naturally carry over to WOCA. The weight function is extended to words in Δ^* by $\lambda(\delta_1 \cdots \delta_n) = \lambda(\delta_1) + \cdots + \lambda(\delta_n)$. The *weight* of a run is the weight of its trace. For notational convenience, we write $p(x) \xrightarrow[w]{\pi} q(y)$ when $p(x) \xrightarrow{\pi} q(y)$ and $w = \lambda(\pi)$. Similarly, we let $p(x) \xrightarrow[w]{*} q(y)$ stand for the existence of $\pi \in \Delta^*$ such that $p(x) \xrightarrow[w]{\pi} q(y)$.

As mentioned before, this section is devoted to the proof that, for any states p and q in a WOCA, the weights $w \in \mathbb{Z}$ such that $p(0) \xrightarrow[w]{*} q(0) \xrightarrow{*} p(0)$ can be obtained via small linear path schemes (see Theorem 12). We start with two pumping lemmas on runs of one-counter automata. The first one, Lemma 7, can be seen as an iterated version of the pumping lemma for one-counter languages due to Latteux [14]. It is an easy consequence of the classical *hill-cutting* technique for one-counter automata (often attributed to Valiant and Paterson [30]). The second one, Lemma 9, tunes the hill-cutting technique so as to obtain short extracted cycles. This second pumping lemma is crucial to obtain linear path schemes with short cycles. The hill-cutting techniques used in both lemmas are illustrated in Figure 2.

We assume for the remainder of this section that $\mathcal{A} = (Q,\Sigma,\Delta,\lambda)$ is a WOCA and that $p, q \in Q$ are states of $\mathcal{A}$.

▶ **Lemma 7.** *If $p(0) \xrightarrow{\pi} q(0)$ then for every $m > 0$ such that $|\pi| \geq m^2|Q|^3$, there exists a factorization $\pi = \alpha\beta_1 \cdots \beta_m\gamma\theta_m \cdots \theta_1\eta$, with $\beta_i\theta_i \neq \varepsilon$ for all $i \in [1, m]$, verifying*

$$p(0) \xrightarrow{\alpha\beta_1^{n_1}\cdots\beta_m^{n_m}\gamma\theta_m^{n_m}\cdots\theta_1^{n_1}\eta} q(0)$$

for every $n_1, \ldots, n_m \in \mathbb{N}$.

Proof Sketch. If the counter remains below $m|Q|^2$ then some configuration repeats at least $m + 1$ times, and the subruns in-between can be iterated arbitrarily many times. The cycles β_i come from these subruns and the cycles θ_i are empty. Otherwise, the run contains a "high hill" and we extract, for each counter value in $[0, m|Q|^2]$, a pair of configurations with this counter value (see Figure 2 (left)). This extraction proceeds from the inside of the hill towards the outside. Some pair of states (r, s) necessarily occurs $m + 1$ times in this extraction. The subruns between the r configurations provide the cycles β_i and the subruns between the s configurations provide the cycles θ_i. The extraction guarantees that these cycles can be iterated arbitrarily many times. ◀

▶ **Corollary 8.** *If $p(x) \xrightarrow{*} q(y)$ then $p(x) \xrightarrow{\pi} q(y)$ for some $\pi \in \Delta^*$ such that $|\pi| < (|Q|+x+y)^3$.*

▶ **Lemma 9.** *If $p(0) \xrightarrow{\pi} q(0)$ with $|\pi| \geq 2|Q|^3$ then there exists $r, s \in Q$, $x, d \in \mathbb{N}$, and a factorization $\pi = \alpha\beta\gamma\theta\eta$, with $\beta\theta \neq \varepsilon$ and no zero-test transition in γ, such that $x+d \leq 2|Q|^2$, $|\beta\theta| \leq 2|Q|^3$ and verifying*

$$p(0) \xrightarrow{\alpha} r(x) \xrightarrow{\beta^n} r(x + nd) \xrightarrow{\gamma} s(x + nd) \xrightarrow{\theta^n} s(x) \xrightarrow{\eta} q(0)$$

for every $n \in \mathbb{N}$.

Proof Sketch. If the counter remains below $2|Q|^2$ then some configuration repeats at least twice. So there is a subrun of length at most $2|Q|^3$ from some configuration $r(x)$ to the same configuration $r(x)$, and this subrun can be iterated arbitrarily many times. The cycle β comes from this subrun and the cycle θ is empty. Otherwise, the run contains a "high hill" and we extract, for each counter value in $[|Q|^2, 2|Q|^2]$, a pair of configurations with this counter value (see Figure 2 (right)). For the counter value $|Q|^2$, the pair of configurations is extracted from the inside of the hill towards the outside. Let t_1 and t_2 denote the positions of these configurations. For the counter values in $[|Q|^2 + 1, 2|Q|^2]$, this extraction proceeds from the outside of the hill – but limited to $[t_1, t_2]$ – towards the inside. Some pair of states (r, s) necessarily occurs twice in this extraction. The subrun between the two r configurations provides the cycle β and the subrun between the two s configurations provides the cycle θ. The extraction guarantees that these cycles can be iterated arbitrarily many times. By construction, the counter remains below $2|Q|^2$ in the subrun providing the cycle β (except possibly for the last configuration). If $\beta > |Q|^3$ then some configuration repeats at least twice in this subrun and we can proceed as in the first case of the proof. The same reasoning can also be applied to the cycle θ. ◀

We now exploit the two previous pumping lemmas to obtain short runs with appropriate weights. First, we show in Lemma 10 that if there is a run from $p(0)$ to $q(0)$ with positive (resp. negative) weight, then there is a short one. In fact, this lemma will be used in the particular case where $p = q$ to get short cyclic runs with positive (resp. negative) weights. Second, we show in Lemma 11 that, assuming that $p(0)$ and $q(0)$ are mutually reachable, if there is a run from $p(0)$ to $q(0)$ whose weight is in a given congruence class modulo $m > 0$, then there is a short one and the weight difference between the two runs can be "qualitatively compensated" by a cyclic run on $q(0)$.

▶ **Lemma 10.** *If $p(0) \xrightarrow[w]{*} q(0)$ for some $w \neq 0$ then $p(0) \xrightarrow[u]{\pi} q(0)$ for some $u \neq 0$ having the same sign as w and some $\pi \in \Delta^*$ such that $|\pi| \leq 539|Q|^9$.*

Proof. We only consider the case where the weight w is positive. The case where w is negative is symmetric. Assume that the set $\{\pi \in \Delta^* \mid p(0) \xrightarrow{\pi} q(0) \wedge \lambda(\pi) > 0\}$ is not empty, and take a word π of minimal length in that set. If $|\pi| < 2|Q|^3$ then we are done. Otherwise, by Lemma 9, there exists $r, s \in Q$, $x, d \in \mathbb{N}$, and a factorization $\pi = \alpha\beta\gamma\theta\eta$ satisfying the conditions of Lemma 9. Observe that $p(0) \xrightarrow{\alpha\gamma\eta} q(0)$ and $|\alpha\gamma\eta| < |\alpha\beta\gamma\theta\eta|$. We deduce from the minimality of π that $\lambda(\alpha\gamma\eta) \leq 0$. This entails that $\lambda(\beta\theta) = \lambda(\pi) - \lambda(\alpha\gamma\eta) > 0$. By Corollary 8, since $p(0) \xrightarrow{*} r(x)$ and $s(x) \xrightarrow{*} q(0)$, there exists α', η' both of length at most $(|Q| + x)^3$ such that $p(0) \xrightarrow{\alpha'} r(x)$ and $s(x) \xrightarrow{\eta'} q(0)$. Similarly, since $r(x) \xrightarrow{*} s(x)$ via a run with no zero-test, there exists γ' with no zero-test and of length $|\gamma'| \leq (|Q| + 2x)^3$ such that $r(x) \xrightarrow{\gamma'} s(x)$. As $x \leq 2|Q|^2$, we get that $|\alpha'\gamma'\eta'| \leq (5|Q|^2)^3 + 2(3|Q|^2)^3 \leq 179|Q|^6$. Now consider the word π' defined by $\pi' \stackrel{\text{def}}{=} \alpha'\beta^n\gamma'\theta^n\eta'$ where $n = 1 + |\alpha'\gamma'\eta'|$. Note that $p(0) \xrightarrow{\pi'} q(0)$ as γ' contains no zero-test and the factorization $\pi = \alpha\beta\gamma\theta\eta$ satisfies the conditions of Lemma 9. Moreover, $\lambda(\pi') = \lambda(\alpha'\gamma'\eta') + n\lambda(\beta\theta)$, hence, $\lambda(\pi') \geq -|\alpha'\gamma'\eta'| + n > 0$. We deduce from the minimality of π that $|\pi| \leq |\pi'|$. It remains to show that π' is short. By construction, $|\pi'| = |\alpha'\gamma'\eta'| + n|\beta\theta| \leq 179|Q|^6 + 180|Q|^6 \cdot 2|Q|^3$. We obtain that $|\pi'| \leq 539|Q|^9$, which concludes the proof of the lemma. ◀

▶ **Lemma 11.** *If $p(0) \xrightarrow[w]{*} q(0) \xrightarrow{*} p(0)$ then for every $m > 0$, there exists $\pi \in \Delta^*$ with $|\pi| < m^2|Q|^3$ verifying $p(0) \xrightarrow{\pi} q(0)$ and $\lambda(\pi) \equiv w \pmod m$, and such that if $w \neq \lambda(\pi)$ then $q(0) \xrightarrow[v]{*} q(0)$ for some $v \neq 0$ having the same sign as $w - \lambda(\pi)$.*

Proof. Assume that $p(0) \xrightarrow[w]{*} q(0) \xrightarrow{*} p(0)$ and let $m > 0$. Let $C(\pi)$ denote the condition that if $w \neq \lambda(\pi)$ then $q(0) \xrightarrow[v]{*} q(0)$ for some $v \neq 0$ having the same sign as $w - \lambda(\pi)$. Consider the set S of all words $\pi \in \Delta^*$ such that $p(0) \xrightarrow{\pi} q(0)$, $\lambda(\pi) \equiv w \pmod m$ and $C(\pi)$ holds. This set is not empty since $p(0) \xrightarrow[w]{*} q(0)$. Take a word π of minimal length in S and let us prove that $|\pi|$ meets the desired bound. Suppose, by contradiction, that $|\pi| \geq m^2|Q|^3$. By Lemma 7, there exists a factorization $\pi = \alpha\beta_1 \cdots \beta_m\gamma\theta_m \cdots \theta_1\eta$, with $\beta_i\theta_i \neq \varepsilon$ for all $i \in [1, m]$, such that $p(0) \xrightarrow{\alpha\beta_1^{n_1}\cdots\beta_m^{n_m}\gamma\theta_m^{n_m}\cdots\theta_1^{n_1}\eta} q(0)$ for every $n_1, \ldots, n_m \in \mathbb{N}$. Let $u_i \stackrel{\text{def}}{=} \lambda(\beta_1 \cdots \beta_i\theta_i \cdots \theta_1)$ for every $i \in [0, m]$, with the understanding that $u_0 = \lambda(\varepsilon) = 0$, and consider the sequence $u_0, \ldots, u_m$. By the pigeonhole principle, there exists $0 \leq i < j \leq m$ such that u_i and u_j are in the same congruence class modulo m. So $u_j = u_i + u$ for some $u \in \mathbb{Z}m$. This means that $\lambda(\beta_{i+1} \cdots \beta_j\theta_j \cdots \theta_{i+1}) = u$. Now, for each $k \in \mathbb{N}$, let π'_k denote the word obtained from π by taking the cycles $\beta_{i+1}, \ldots, \beta_j$ and $\theta_j, \ldots, \theta_{i+1}$ exactly k times, formally, $\pi'_k \stackrel{\text{def}}{=} \alpha\beta_1 \cdots \beta_i\beta_{i+1}^k \cdots \beta_j^k\beta_{j+1} \cdots \beta_m\gamma\theta_m \cdots \theta_{j+1}\theta_j^k \cdots \theta_{i+1}^k\theta_i \cdots \theta_1\eta$. It is readily seen that $p(0) \xrightarrow{\pi'_k} q(0)$ and $\lambda(\pi'_k) = \lambda(\pi) + (k-1)u$.

Let us prove that $\pi'_0 \in S$. We have already shown that $p(0) \xrightarrow{\pi'_0} q(0)$ and $\lambda(\pi'_0) = \lambda(\pi) - u$, hence, $\lambda(\pi'_0) \equiv w \pmod m$. It remains to show that $C(\pi'_0)$ holds. Let $s, s' \in \{-1, 0, 1\}$ denote the signs of $w - \lambda(\pi)$ and $w - \lambda(\pi'_0)$, respectively. If $s' = 0$ then $C(\pi'_0)$ holds trivially. If $s' \neq 0$ and $s = s'$ then $C(\pi'_0)$ holds because $C(\pi)$ holds. If $s' = 1$ and $s \leq 0$ then $\lambda(\pi'_0) < w \leq \lambda(\pi)$, hence, $u > 0$. It follows from $\lambda(\pi'_k) = \lambda(\pi) + (k-1)u$ that $p(0) \xrightarrow[v]{*} q(0)$ for infinitely many $v > 0$. As $q(0) \xrightarrow{*} p(0)$, we deduce that $q(0) \xrightarrow[v]{*} q(0)$ for some $v > 0$, hence, $C(\pi'_0)$ holds. If $s' = -1$ and $s \geq 0$ then $\lambda(\pi) \leq w < \lambda(\pi'_0)$, hence, $u < 0$. It follows from

$\lambda(\pi'_k) = \lambda(\pi)+(k-1)u$ that $p(0) \xrightarrow[v]{*} q(0)$ for infinitely many $v < 0$. As $q(0) \xrightarrow{*} p(0)$, we deduce that $q(0) \xrightarrow[v]{*} q(0)$ for some $v < 0$, hence, $C(\pi'_0)$ holds. We have shown in all cases that $\pi'_0 \in S$. This contradicts the minimality of π since $|\pi'_0| = |\pi| - |\beta_{i+1} \cdots \beta_j \theta_j \cdots \theta_{i+1}| < |\pi|$. ◀

We are now ready to prove the main result of this section, namely that the reachable weights between two mutually reachable configurations $p(0)$ and $q(0)$ can be obtained via small linear path schemes.

▶ **Theorem 12.** *Let $\mathcal{A} = (Q, \Sigma, \Delta, \lambda)$ be a WOCA. For every states $p, q \in Q$ and weight $w \in \mathbb{Z}$ verifying $p(0) \xrightarrow[w]{*} q(0) \xrightarrow{*} p(0)$, there exists $\alpha, \beta \in \Delta^*$ and $n \in \mathbb{N}$ such that $p(0) \xrightarrow[w]{\alpha\beta^n} q(0)$, $q(0) \xrightarrow{\beta} q(0)$ and $|\alpha\beta| \leq (2|Q|)^{39}$.*

Proof. Assume that $p(0) \xrightarrow[w]{*} q(0) \xrightarrow{*} p(0)$. We start by fixing two short cyclic runs on $q(0)$, one with positive weight and one with negative weight, as follows. By Lemma 10, if $q(0) \xrightarrow[w]{*} q(0)$ for some $w > 0$ then $q(0) \xrightarrow{\beta} q(0)$ for some $\beta \in \Delta^*$ such that $\lambda(\beta) > 0$ and $|\beta| \leq 539|Q|^9$. Let $\beta \stackrel{\text{def}}{=} \varepsilon$ otherwise. Analogously, if $q(0) \xrightarrow[w]{*} q(0)$ for some $w < 0$ then $q(0) \xrightarrow{\theta} q(0)$ for some $\theta \in \Delta^*$ such that $\lambda(\theta) < 0$ and $|\theta| \leq 539|Q|^9$. Let $\theta \stackrel{\text{def}}{=} \varepsilon$ otherwise.

By Lemma 11, for each $m \in \{1, \lambda(\beta), -\lambda(\theta), -\lambda(\beta)\lambda(\theta)\}$ such that $m > 0$, there exists $\alpha_m \in \Delta^*$ with $|\alpha_m| < m^2|Q|^3$ verifying $p(0) \xrightarrow{\alpha_m} q(0)$ and $\lambda(\alpha_m) \equiv w \pmod{m}$, and such that if $w \neq \lambda(\alpha_m)$ then $q(0) \xrightarrow[v]{*} q(0)$ for some $v \neq 0$ having the same sign as $w - \lambda(\alpha_m)$. Let $u_m \stackrel{\text{def}}{=} w - \lambda(\alpha_m)$ and note that $u_m \in \mathbb{Z}m$. Moreover, $u_m > 0$ implies $\beta \neq \varepsilon$ and $u_m < 0$ implies $\theta \neq \varepsilon$. We now consider four cases depending on the emptiness of β and θ.

If $\beta = \theta = \varepsilon$ then we use α_m and u_m for $m \stackrel{\text{def}}{=} 1$. We derive from $\beta = \theta = \varepsilon$ that $u_m = 0$. It follows that $w = \lambda(\alpha_m) + u_m = \lambda(\alpha_m\beta)$.

If $\beta \neq \varepsilon$ and $\theta = \varepsilon$ then we use α_m and u_m for $m \stackrel{\text{def}}{=} \lambda(\beta) > 0$. We derive from $\theta = \varepsilon$ that $u_m \geq 0$. As $u_m \in \mathbb{Z}m$, we get that $u_m = n\lambda(\beta)$ for some $n \in \mathbb{N}$. It follows that $w = \lambda(\alpha_m) + u_m = \lambda(\alpha_m\beta^n)$.

If $\beta = \varepsilon$ and $\theta \neq \varepsilon$ then we use α_m and u_m for $m \stackrel{\text{def}}{=} -\lambda(\theta) > 0$. We derive from $\beta = \varepsilon$ that $u_m \leq 0$. As $u_m \in \mathbb{Z}m$, we get that $u_m = n\lambda(\theta)$ for some $n \in \mathbb{N}$. It follows that $w = \lambda(\alpha_m) + u_m = \lambda(\alpha_m\theta^n)$.

If $\beta \neq \varepsilon$ and $\theta \neq \varepsilon$ then we use α_m and u_m for $m \stackrel{\text{def}}{=} -\lambda(\beta)\lambda(\theta) > 0$. As $u_m \in \mathbb{Z}m$, we get that $u_m = n\lambda(\beta)\lambda(\theta)$ for some $n \in \mathbb{Z}$. If $n \leq 0$ then $w = \lambda(\alpha_m) + u_m = \lambda(\alpha_m\beta^{n\lambda(\theta)})$. If $n \geq 0$ then $w = \lambda(\alpha_m) + u_m = \lambda(\alpha_m\theta^{n\lambda(\beta)})$.

We have shown in each case that $w = \lambda(\alpha_m\gamma^n)$ for some $m \in \{1, \lambda(\beta), -\lambda(\theta), -\lambda(\beta)\lambda(\theta)\}$ with $m > 0$, some $\gamma \in \{\beta, \theta\}$ and some $n \in \mathbb{N}$. Moreover, our choice of β and θ ensures that $m \leq (539|Q|^9)^2$, $q(0) \xrightarrow{\gamma} q(0)$ and $|\gamma| \leq 539|Q|^9$. Recall that $p(0) \xrightarrow{\alpha_m} q(0)$ and $|\alpha_m| < m^2|Q|^3$. It follows that $p(0) \xrightarrow[w]{\alpha_m\gamma^n} q(0)$ and $|\alpha_m| \leq (539|Q|^9)^4|Q|^3$. We obtain that $|\alpha_m\gamma| \leq (2|Q|)^{39}$, which concludes the proof of the theorem. ◀

4 Succinct Flattenability of 2-TVASS

We have shown in Section 2 that 2-TVASS are flattenable. We now prove that flattenability of 2-TVASS can be witnessed by small linear path schemes.

We first introduce a binary relation $\rightsquigarrow$ over the states of a 2-TVASS $\mathcal{V}$, defined by $p \rightsquigarrow q$ if $p(0, x) \xrightarrow{*} q(0, y)$ for some $x, y \in \mathbb{N}$. Notice that this relation is transitive since $p(0, x) \xrightarrow{*} q(0, y)$ and $q(0, x') \xrightarrow{*} r(0, y')$ implies $p(0, x + x') \xrightarrow{*} q(0, y + x') \xrightarrow{*} r(0, y + y')$ by monotonicity. As mentioned in Section 3, a WOCA can be associated to any 2-TVASS by considering actions on the second counter, the one that is not tested for zero, as weights. Under this observation, $p \rightsquigarrow q$ if, and only if, $p(0) \xrightarrow{*} q(0)$ in the associated WOCA. This observation also provides a way to convert Theorem 12 to the following lemma.

▶ **Lemma 13.** *There exists a constant $h \geq 1$ such that, for every 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$, if $p(0,x) \xrightarrow{*} q(0,y)$ with $x, y \geq (|Q| + \|\Sigma\|)^h$ and $q \rightsquigarrow p$, then there exists a linear path scheme L with $|L| \leq (|Q| + \|\Sigma\|)^{O(1)}$ and $|L|_* = 1$ such that $p(0,x) \xrightarrow{L} q(0,y)$.*

Proof. A 2-TVASS cannot be directly translated into a WOCA since some addition transitions $(p, \mathbf{a}, q)$ may satisfy $\|\mathbf{a}\| > 1$. However, by introducing intermediate states and transitions between p and q, we can overcome this problem. It follows that we can assume, without loss of generality, that every addition transition $(p, \mathbf{a}, q)$ satisfies $\|\mathbf{a}\| \leq 1$. Additionally, by introducing for each state p and each addition transition δ an intermediate state, we can assume that if $(p, \mathbf{a}, q)$ and $(p, \mathbf{b}, q)$ are two addition transitions such that $\mathbf{a}(1) = \mathbf{b}(1)$ then $\mathbf{a}(2) = \mathbf{b}(2)$. Thanks to this assumption, we can associate to a 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ a WOCA $(Q, \Sigma', \Delta', \lambda)$ in such a way $(p, (a, b), q)$ is a transition in $\mathcal{V}$ if, and only if, (p, a, q) is a transition in the WOCA weighted by b, and such that $(p, \mathtt{tst}, q)$ is a transition in $\mathcal{V}$ if, and only if, $(p, \mathtt{tst}, q)$ is a transition in the WOCA, and in that case the transition is weighted by zero. Now, let us consider two configurations $p(0,x)$ and $q(0,y)$ with $x, y \geq (2|Q|)^{39}$ such that $p(0,x) \xrightarrow{*} q(0,y)$ and $q \rightsquigarrow p$ in $\mathcal{V}$. Notice that $p(0) \xrightarrow[w]{*} q(0)$ and $q(0) \xrightarrow{*} p(0)$ in the WOCA with $w = y - x$. From Theorem 12, it follows that there exists a path π from p to q, a cycle θ on q, and $n \in \mathbb{N}$ such that $p(0) \xrightarrow[w]{\pi\theta^n} q(0)$ with $|\pi\theta| \leq (2|Q|)^{39}$. Notice that π and θ in the WOCA corresponds to a path α and a cycle β in $\mathcal{V}$, respectively. Since $x, y \geq (2|Q|)^{39} \geq |\alpha\beta|$, observe that $p(0,x) \xrightarrow{\alpha\beta^n} q(0,y)$ since each execution of β can decrease or increase the second counter by a value bounded by $|\beta| \leq (2|Q|)^{39}$. ◀

The previous lemma captures the reachability relation of a 2-TVASS between configurations $p(0,x)$ and $q(0,y)$ with $p \rightsquigarrow q \rightsquigarrow p$ and x, y are large. In order to capture the same relation when x or y are small, the following result will be useful.

▶ **Theorem 14** ([7]). *For every 2-VASS $\mathcal{V} = (Q, \Sigma, \Delta)$, and for every configurations $p(\mathbf{x})$ and $q(\mathbf{y})$ such that $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$ in $\mathcal{V}$, there exists a path π such that $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ and satisfying*

$$|\pi| \leq (|Q| + \|\Sigma\| + \|\mathbf{x}\| + \|\mathbf{y}\|)^{O(1)} .$$

We are now ready to refine Lemma 5 with complexity bounds. Recall that $\updownarrow_q$ is the vertical loop relation on q defined by $\updownarrow_q = \{(q(0,x), q(0,y)) \mid q(0,x) \xrightarrow{*} q(0,y)\}$.

▶ **Lemma 15.** *For every 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ and state $q \in Q$, we have $\updownarrow_q \subseteq \bigcup_{L \in \Lambda} \xrightarrow{L}$ for some finite set Λ of linear path schemes L such that $|L| \leq (|Q| + \|\Sigma\|)^{O(1)}$ and $|L|_* \leq O(|Q|^2)$.*

Proof. Let $h \geq 1$ be the constant of Lemma 13, and let $c \geq 1$ be a constant satisfying $O(1) \leq c$ and $O(|Q|^2) \leq c|Q|^2$ in Lemma 13, Theorem 3, and Theorem 14. Let us consider a 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ and let $N = |Q| + \|\Sigma\|$.

Observe that a run from a configuration $q(0,x)$ to a configuration $q(0,y)$ can be split in such a way:

$$q(0,x) = q_1(0,x_1) \xrightarrow{A^* \cup T} q_2(0,x_2) \cdots \xrightarrow{A^* \cup T} q_k(0,x_k) = q(0,y)$$

Moreover, by removing some parts of such a run, we can assume that the configurations $q_j(0,x_j)$ are pairwise distinct.

Notice that if $x_j < N^h$ for every $j \in [1,k]$, then $k \leq |Q| \cdot N^h \leq N^{h+1}$. By applying Theorem 14, we deduce that for every $j \in [2,k]$, there exists a path α_j such that $q_{j-1}(0, x_{j-1}) \xrightarrow{\alpha_j} q_j(0, x_j)$ with $|\alpha_j| \leq (N + 2N^h)^c$. In particular α defined as $\alpha_2 \cdots \alpha_k$ is a path such that $q(0,x) \xrightarrow{\alpha} q(0,y)$ with $|\alpha| \leq k \cdot (N + 2N^h)^c \leq N^e$ for some constant e. We are done with the linear path scheme $L = \alpha$. So, we can assume that there exists j such that $x_j \geq N^h$. In that case, we introduce j_{min} and j_{max} respectively defined as the minimal and the maximal j satisfying this property.

Let us prove that there exists a linear path scheme L_{min} such that $|L_{\mathsf{min}}| \leq N^e + N^c$ and $|L_{\mathsf{min}}|_* \leq c|Q|^2$ and such that $q(0,x) \xrightarrow{L_{\mathsf{min}}} q_{j_{\mathsf{min}}}(0, x_{j_{\mathsf{min}}})$. Observe that if $j_{\mathsf{min}} = 1$, the proof is immediate with L_{min} reduced to the empty path. If $j_{\mathsf{min}} > 1$, as $x_j < N^h$ for every $1 \leq j < j_{\mathsf{min}}$, we deduce from the previous paragraph that there exists a path α_{min} with a length bounded by N^e such that $q(0,x) \xrightarrow{\alpha_{\mathsf{min}}} q_{j_{\mathsf{min}}-1}(0, x_{j_{\mathsf{min}}-1})$. Recall that $q_{j_{\mathsf{min}}-1}(0, x_{j_{\mathsf{min}}-1}) \xrightarrow{A^* \cup T} q_{j_{\mathsf{min}}}(0, x_{j_{\mathsf{min}}})$. Based on Theorem 3, we deduce that there exists a linear path scheme L_0 such that $q_{j_{\mathsf{min}}-1}(0, x_{j_{\mathsf{min}}-1}) \xrightarrow{L_0} q_{j_{\mathsf{min}}}(0, x_{j_{\mathsf{min}}})$ with $|L_0| \leq N^c$ and $|L_0|_* \leq c|Q|^2$. By considering $L_{\mathsf{min}} = \alpha_{\mathsf{min}} L_0$ we are done. Symmetrically, there exists a linear path scheme L_{max} such that $|L_{\mathsf{max}}| \leq N^e + N^c$ and $|L_{\mathsf{max}}|_* \leq c|Q|^2$ and such that $q_{j_{\mathsf{max}}}(0, x_{j_{\mathsf{max}}}) \xrightarrow{L_{\mathsf{max}}} q(0,y)$.

Note that $q_{j_{\mathsf{max}}} \rightsquigarrow q_k = q_1 \rightsquigarrow q_{j_{\mathsf{min}}}$, hence, $q_{j_{\mathsf{max}}} \rightsquigarrow q_{j_{\mathsf{min}}}$. By applying Lemma 13, we deduce that there exists a linear path scheme L_1 with $|L_1| \leq N^c$ and $|L|_* = 1$ such that $q_{j_{\mathsf{min}}}(0, x_{j_{\mathsf{min}}}) \xrightarrow{L_1} q_{j_{\mathsf{max}}}(0, x_{j_{\mathsf{max}}})$, It follows that the linear path scheme L defined as $L_{\mathsf{min}} L_1 L_{\mathsf{max}}$ satisfies the lemma. ◀

▶ **Corollary 16.** *Every* 2*-TVASS is flattenable. Furthermore, for every configurations* $p(\mathbf{x})$ *and* $q(\mathbf{y})$ *of a* 2*-TVASS* $\mathcal{V} = (Q, \Sigma, \Delta)$ *such that* $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$*, there exists a linear path scheme* L *with* $|L| \leq (|Q| + \|\Sigma\|)^{O(1)}$ *and* $|L|_* \leq O(|Q|^3)$ *such that* $p(\mathbf{x}) \xrightarrow{L} q(\mathbf{y})$*.*

Proof. The proof is a direct corollary of Lemma 4, Theorem 3, and Lemma 15. ◀

▶ **Example 17.** As an illustration of Corollary 16, let us continue Examples 1 and 2 and provide a finite set Λ of "small" linear path schemes such that $A(\mathbf{x}) \xrightarrow{*} B(\mathbf{y})$ if, and only if, $A(\mathbf{x}) \xrightarrow{L} B(\mathbf{y})$ for some $L \in \Lambda$. First, we observe that for every $x, y \in \mathbb{N}$, if $A(0,x) \xrightarrow{*} A(0,y)$ then $x = y$ or the following condition is satisfied:

$$x \geq 2 \ \wedge \ y \geq x + 2 \ \wedge \ (y = x + 3 \Rightarrow x \geq 5) \ \wedge \ (y = x + 5 \Rightarrow x \geq 3)\,.$$

Second, we introduce the paths $\pi = \delta_{AB}\delta_{BB}\delta_{BB}\delta_{BA}\delta_{AA}$ and $\sigma = \delta_{AB}(\delta_{BB})^5\delta_{BA}(\delta_{AA})^2$. It is routinely checked that $A(0,x) \xrightarrow{\pi} A(0,y)$ if, and only if, $x \geq 2$ and $y = x + 2$. Similarly, $A(0,x) \xrightarrow{\sigma} A(0,y)$ if, and only if, $x \geq 5$ and $y = x + 3$. We derive that $A(0,x) \xrightarrow{*} A(0,y)$ if, and only if, $A(0,x) \xrightarrow{\pi^* \cdot \{\varepsilon, \sigma\}} A(0,y)$. We are now done by taking $\Lambda = \{L_1, L_2\}$ where L_1 and L_2 are the linear path schemes defined by $L_1 = (\delta_{AA})^* \cdot \pi^* \cdot \delta_{AB} \cdot (\delta_{BB})^*$ and $L_2 = (\delta_{AA})^* \cdot \pi^* \cdot \sigma\delta_{AB} \cdot (\delta_{BB})^*$. ⌟

5 Linear Path Schemes to Systems of Equations

In this section, we associate to a linear path scheme $L = \alpha_0\beta_1^*\alpha_1 \cdots \beta_k^*\alpha_k$ of a d-TVASS $\mathcal{V}$ from a state p to a state q, and to a vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, a system of linear inequalities $S_{\mathbf{x},L,\mathbf{y}}$ encoding over the variables $(n_1, \ldots, n_k)$ the following constraint:

$$p(\mathbf{x}) \xrightarrow{\alpha_0\beta_1^{n_1}\alpha_1\cdots\beta_k^{n_k}\alpha_k} q(\mathbf{y})$$

Such a system is classical for d-VASS, but for d-TVASS, the presence of zero-test transitions in the linear path scheme L requires some additional work.

Let us first characterize the binary relation $\xrightarrow{\pi}$ thanks to a system of linear inequalities associated to a path π. We introduce the *displacement* $\mathrm{disp}(\delta)$ of a transition δ as the vector in $\mathbb{Z}^d$ defined by $\mathrm{disp}(\delta) \stackrel{\text{def}}{=} \mathbf{a}$ if δ is of the form $(p, \mathbf{a}, q)$ with $\mathbf{a} \in \mathbb{Z}^d$ and $\mathrm{disp}(\delta) \stackrel{\text{def}}{=} \mathbf{0}$ if δ is of the form $(p, \mathtt{tst}, q)$. The *displacement* of a path $\pi = \delta_1 \ldots \delta_n$ is $\mathrm{disp}(\pi) \stackrel{\text{def}}{=} \mathrm{disp}(\delta_1)+\cdots+\mathrm{disp}(\delta_n)$. We also introduce the vector $\mathbf{m}_\pi \in \mathbb{N}^d$ defined component-wise for every $i \in [1, d]$ by $\mathbf{m}_\pi(i) \stackrel{\text{def}}{=} \max_\alpha(-\mathrm{disp}(\alpha)(i))$ where α ranges over the prefixes of π.

A path π from a state p to a state q is said to be *feasible* if $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$. We introduce the partial order $\geq_1$ defined over $\mathbb{N}^d$ by $\mathbf{x} \geq_1 \mathbf{y}$ if $\mathbf{x}(1) = \mathbf{y}(1)$ and $\mathbf{x}(i) \geq \mathbf{y}(i)$ for every $i \in [2, d]$. We let $\succeq_\pi$ denote the partial order over $\mathbb{N}^d$ defined as follows: $\succeq_\pi$ is $\geq_1$ if π contains a zero-test transition, and $\succeq_\pi$ is $\geq$ otherwise.

▶ **Lemma 18.** *Let π be a feasible path from a state p to a state q. For every $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, we have:*

$$p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y}) \iff \mathbf{x} \succeq_\pi \mathbf{m}_\pi \ \wedge\ \mathbf{y} = \mathbf{x} + \mathrm{disp}(\pi)$$

Let us recall that in Section 2 we introduce the d-TVASS $\overline{\mathcal{V}}$ obtained from $\mathcal{V}$ by reversing the effect of each transition, i.e., $\overline{\Delta} = \{\overline{\delta} \mid \delta \in \Delta\}$ where $\overline{(p, \mathbf{a}, q)} = (q, -\mathbf{a}, p)$ and $\overline{(p, T, q)} = (q, T, p)$. Given a path $\pi = \delta_1 \cdots \delta_n$ from p to q in $\mathcal{V}$, we introduce the path $\overline{\pi}$ from q to p in $\overline{\mathcal{V}}$ defined as $\overline{\pi} \stackrel{\text{def}}{=} \overline{\delta_n} \cdots \overline{\delta_1}$. Observe that $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ if, and only if, $q(\mathbf{y}) \xrightarrow{\overline{\pi}} p(\mathbf{x})$.

▶ **Lemma 19.** *We have $\mathbf{m}_{\overline{\pi}} = \mathbf{m}_\pi + \mathrm{disp}(\pi)$.*

Proof. Observe that for any decomposition of π into $\alpha\alpha'$, we have $\mathrm{disp}(\pi) = \mathrm{disp}(\alpha)+\mathrm{disp}(\alpha')$. Hence $-\mathrm{disp}(\alpha) + \mathrm{disp}(\pi) = \mathrm{disp}(\alpha') = -\mathrm{disp}(\overline{\alpha}')$. In particular $\max_\alpha(-\mathrm{disp}(\alpha)(i) + \mathrm{disp}(\pi)(i)) = \max_{\alpha'}(-\mathrm{disp}(\overline{\alpha}')(i))$ for every $i \in [1, d]$ where α ranges over the prefixes of π and α' over the suffixes of π. By observing that $\overline{\alpha}'$ ranges over all the prefixes of $\overline{\pi}$ when α' ranges over the suffixes of π, we get $\mathbf{m}_\pi + \mathrm{disp}(\pi) = \mathbf{m}_{\overline{\pi}}$. ◀

We are now ready to express the relation $\xrightarrow{\beta^n}$ where β is a cycle on a state q and $n \geq 1$ is a positive natural number.

▶ **Lemma 20.** *Let β be a feasible cycle on a state q. For every $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ and $n \in \mathbb{N} \setminus \{0\}$, we have:*

$$q(\mathbf{x}) \xrightarrow{\beta^n} q(\mathbf{y}) \iff \mathbf{x} \succeq_\pi \mathbf{m}_\beta \ \wedge\ \mathbf{y} \succeq_\pi \mathbf{m}_{\overline{\beta}} \ \wedge\ \mathbf{y} = \mathbf{x} + n\,\mathrm{disp}(\beta)$$

A linear path scheme $L = \alpha_0\beta_1^*\alpha_1 \cdots \beta_k^*\alpha_k$ is said to be *feasible* if the paths $\alpha_0, \ldots, \alpha_k$ and the cycles $\beta_1, \ldots, \beta_k$ are feasible. We are now ready to introduce a system of linear inequalities $S_{\mathbf{x},L,\mathbf{y}}$ over the variables $(n_1, \ldots, n_k)$ where $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, and $n_1, \ldots, n_k$ are variables ranging over $\mathbb{N}$ as follows:

$$\bigwedge_{j=0}^{k} \mathbf{x}_j \succeq_{\alpha_j} \mathbf{m}_{\alpha_j} \quad\wedge\quad \bigwedge_{j=1}^{k} \mathbf{y}_{j-1} \succeq_{\beta_j} \mathbf{m}_{\beta_j} \wedge \mathbf{x}_j \succeq_{\beta_j} \mathbf{m}_{\overline{\beta_j}} \quad\wedge\quad \mathbf{y} = \mathbf{y}_k$$

where $\mathbf{x}_0$ is the expression $\mathbf{x}$, and by induction over j, by letting $\mathbf{y}_j$ be the expression $\mathbf{x}_j + \mathrm{disp}(\alpha_j)$ for every $j \in [0, k]$, and $\mathbf{x}_j$ is the expression $\mathbf{y}_{j-1} + n_j\,\mathrm{disp}(\beta_j)$ for every $j \in [1, k]$. From Lemmas 18 and 20, we derive the following corollary.

▶ **Corollary 21.** *Assume that $L = \alpha_0\beta_1^*\alpha_1 \cdots \beta_k^*\alpha_k$ is a feasible linear path scheme from a state p to a state q, and let $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$. If $(n_1, \ldots, n_k)$ is a solution of $S_{\mathbf{x},L,\mathbf{y}}$ then*

$$p(\mathbf{x}) \xrightarrow{\alpha_0\beta_1^{n_1}\alpha_1\cdots\beta_k^{n_k}\alpha_k} q(\mathbf{y})$$

Conversely, a tuple $(n_1, \ldots, n_k)$ with $n_1, \ldots, n_k \geq 1$ that satisfies the previous relation is a solution of $S_{\mathbf{x},L,\mathbf{y}}$.

▶ Remark 22. We can easily extend the definition of $S_{\mathbf{x},L,\mathbf{y}}$ to encode linear path schemes of extended d-TVASS with zero-test actions on any counter.

6 Complexity Results

This section utilizes the results of Sections 4 and 5 to characterize the complexity of the reachability problem in 2-TVASS. We assume that 2-TVASS and their configurations are encoded in binary, and that sizes are defined as expected (up to a polynomial). Under this binary encoding, the reachability problem in 2-TVASS is shown to be PSPACE-complete. Since the reachability problem for 2-VASS (i.e., without zero-test transitions) is already PSPACE-hard [8], we only need to prove PSPACE-membership.

The PSPACE complexity upper-bound is obtained via small solutions of the system of inequalities $S_{\mathbf{x},L,\mathbf{y}}$ associated to a linear path scheme L and a pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^2$. We first recall some results about small solution of systems of equations.

▶ **Theorem 23** ([27])**.** *Let $M = (M_{i,j})$ be a matrix in $\mathbb{Z}^{e\times k}$. Every solution $\mathbf{x} \in \mathbb{N}^k$ of $M\mathbf{x} = \mathbf{0}$ is a finite sum of solutions $\mathbf{y} \in \mathbb{N}^k$ satisfying additionally $\sum_{i=1}^k \mathbf{y}(i) \leq (1+m)^k$ where $m = \max_i \sum_{j=1}^k |M_{i,j}|$.*

We also recall the classical application of the previous theorem to systems of inequalities with constant terms (the vector $\mathbf{b}$ in the following corollary).

▶ **Corollary 24.** *Let $M = (M_{i,j})$ be a matrix in $\mathbb{Z}^{e\times k}$ and let $\mathbf{b} \in \mathbb{Z}^e$. If there exists a solution $\mathbf{x} \in \mathbb{N}^k$ of $M\mathbf{x} \geq \mathbf{b}$ then there exists a solution $\mathbf{y} \in \mathbb{N}^k$ such that $\sum_{i=1}^k \mathbf{y}(i) \leq (2+m)^{k+1+e}$ where $m = \max_i \sum_{j=1}^k |M_{i,j}| + |\mathbf{b}(i)|$.*

Proof. Let $\mathbf{y}$ be the vector in $\mathbb{N}^e$ defined as $\mathbf{y} \stackrel{\text{def}}{=} M\mathbf{x} - \mathbf{b}$ and observe that $(\mathbf{x}, 1, \mathbf{y})$ is a solution of $M\mathbf{x} - t\mathbf{b} - \mathbf{y} = \mathbf{0}$. From Theorem 23 we derive that $(\mathbf{x}, 1, \mathbf{y})$ can be decomposed as a finite sum of "small solutions" $(\mathbf{u}, s, \mathbf{v})$ with $\sum_{j=1}^k \mathbf{u}(j) + s + \sum_{i=1}^e \mathbf{v}(i) \leq (2+m)^{k+1+e}$. Since the sum of those small solutions is 1 on the "s" component, exactly one of them is 1 on that component. This solution $(\mathbf{u}, 1, \mathbf{v})$ provides a vector $\mathbf{u}$ with $\sum_{j=1}^k \mathbf{u}(j) \leq (2+m)^{k+1+e}$ such that $M\mathbf{u} \geq \mathbf{b}$. ◀

We deduce a bound on minimal runs between two configurations.

▶ **Lemma 25.** *For every configurations $p(\mathbf{x})$ and $q(\mathbf{y})$ of a 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ such that $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$, there exists a path π such that $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ and such that:*

$$|\pi| \leq (|Q| + \|\mathbf{x}\| + \|\mathbf{y}\| + \|\Sigma\|)^{O(|Q|^3)}$$

Proof. Let $c \geq 1$ be a constant satisfying Corollary 16, i.e., such that $O(1) \leq c$ and $O(|Q|^3) \leq c|Q|^3$. Consider a 2-TVASS $\mathcal{V}$ and let $p(\mathbf{x})$ and $q(\mathbf{y})$ be two configurations such that $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$. The case where $\Sigma \subseteq \{\mathbf{0}, \mathtt{tst}\}$ is trivial (there is a run of length at most $|Q|$ in that case), so we assume that $\|\Sigma\| \geq 1$ for the remainder of the proof. Let us introduce $N = |Q| + \|\Sigma\|$. Corollary 16 shows that there exists a linear path scheme $L = \alpha_0\beta_1^*\alpha_1 \cdots \beta_k^*\alpha_k$ with $p(\mathbf{x}) \xrightarrow{L} q(\mathbf{y})$ and such that $|L| \leq N^c$ and $k \leq c|Q|^3$. It follows that there exists $n_1, \ldots, n_k \in \mathbb{N}$ such that:

$$p(\mathbf{x}) \xrightarrow{\alpha_0\beta_1^{n_1}\alpha_1\cdots\beta_k^{n_k}\alpha_k} q(\mathbf{y})$$

By removing from L the cycles β_j such that $n_j = 0$, we can assume, without loss of generality, that $n_j \geq 1$ for every $j \in [1, k]$. It follows that L is feasible. From Corollary 21 we deduce that $(n_1, \ldots, n_k)$ is a solution of $S_{\mathbf{x},L,\mathbf{y}}$. From Corollary 24 we deduce that there exist $m_1, \ldots, m_k \in \mathbb{N}$ such that $(m_1, \ldots, m_k)$ satisfies $S_{\mathbf{x},L,\mathbf{y}}$ and such that $m_1 + \cdots + m_k \leq (2+m)^{k+1+e}$ where $m \leq \|\mathbf{x}\|+\|\mathbf{y}\|+\|\Sigma\|\cdot|L|$ and $e \stackrel{\text{def}}{=} 9k+7$. This expression for e comes from the encoding of $\geq_1$ with 3 inequalities. Let us introduce the path $\pi = \alpha_0\beta_1^{m_1}\alpha_1 \cdots \beta_k^{m_k}\alpha_k$ and observe that $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ from Corollary 21. Moreover $|\pi|$ is bounded by:

$$(2+m)^{10k+8}\cdot|L| \leq (2 + \|\mathbf{x}\| + \|\mathbf{y}\| + \|\Sigma\|N^c)^{10c|Q|^3+8}N^c \leq (|Q| + \|\mathbf{x}\| + \|\mathbf{y}\| + \|\Sigma\|)^{O(|Q|^3)}$$

This concludes the proof of the lemma. ◀

We are now ready to characterize the complexity of the reachability problem in 2-TVASS. This decision problem asks, given a 2-TVASS $\mathcal{V} = (Q, \Sigma, \Delta)$ and two configurations $p(\mathbf{x})$ and $q(\mathbf{y})$, whether $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$. By Lemma 25, if $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$ then there exists a run from $p(\mathbf{x})$ to $q(\mathbf{y})$ of length at most exponential in the sizes of $\mathcal{V}$, $p(\mathbf{x})$ and $q(\mathbf{y})$. Notice that configurations along that run have a polynomial size (with respect to the size of the input problem). It follows that a polynomial-space bounded exploration of the reachability set provides a way to decide the reachability problem. We have shown the following theorem.

▶ **Theorem 26.** *The reachability problem for 2-TVASS is PSPACE-complete.*

▶ Remark 27. Other natural problems on 2-TVASS are PSPACE-complete. In the full version of the paper [21, Appendix D], we derive from the succinct flattenability of 2-TVASS that the boundedness problem and the termination problem are both decidable in polynomial space. These results are obtained by providing a polynomial bound on the size of reachable configurations of a bounded 2-TVASS.

7 Conclusion and Perspectives

We have shown in this paper that extending 2-VASS with zero-tests on the first counter is for free, in the sense that the reachability problem remains PSPACE-complete (and so do the boundedness and termination problems). As in the case of 2-VASS, a crucial step in our approach is what we call *succinct flattenability*, i.e., the existence of small linear path schemes witnessing flattenability. Succinct flattenability of 2-VASS was leveraged by Englert et al. in [7] to show that reachability in 2-VASS is NL-complete when the input integers are encoded in unary. The question whether reachability in unary 2-TVASS remains NL-complete is left open. We conjecture that this question can be answered positively, by leveraging our succinct flattenability result for 2-TVASS and by extending [7] with zero-tests on the first counter.

References

1 Michael Blondin, Alain Finkel, Stefan Göller, Christoph Haase, and Pierre McKenzie. Reachability in two-dimensional vector addition systems with states is PSPACE-complete. In *LICS*, pages 32–43. IEEE, 2015.

2 Rémi Bonnet. The reachability problem for vector addition system with one zero-test. In *MFCS*, volume 6907 of *LNCS*, pages 145–157. Springer, 2011.

3 I. Borosh and L. B. Treybig. A sharp bound on positive solutions of linear diophantine equations. *SIAM J. Matrix Analysis Applications*, 13(2):454–458, 1992.

4 Wojciech Czerwiński, Sławomir Lasota, Ranko Lazić, Jérôme Leroux, and Filip Mazowiecki. The reachability problem for Petri nets is not elementary. In *STOC*, pages 24–33. ACM, 2019.

5 Wojciech Czerwiński, Sławomir Lasota, Christof Löding, and Radoslaw Piórkowski. New pumping technique for 2-dimensional VASS. In *MFCS*, volume 138 of *LIPIcs*, pages 62:1–62:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

6 Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset nets between decidability and undecidability. In *ICALP*, volume 1443 of *LNCS*, pages 103–115. Springer, 1998.

7 Matthias Englert, Ranko Lazić, and Patrick Totzke. Reachability in two-dimensional unary vector addition systems with states is NL-complete. In *LICS*, pages 477–484. ACM, 2016.

8 John Fearnley and Marcin Jurdzinski. Reachability in two-clock timed automata is PSPACE-complete. *Inform. Comput.*, 243:26–36, 2015.

9 Alain Finkel, Jérôme Leroux, and Grégoire Sutre. Reachability for two-counter machines with one test and one reset. In *FSTTCS*, volume 122 of *LIPIcs*, pages 31:1–31:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

10 Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell. Reachability in succinct and parametric one-counter automata. In *CONCUR*, volume 5710 of *LNCS*, pages 369–383. Springer, 2009.

11 John Hopcroft and Jean-Jacques Pansiot. On the reachability problem for 5-dimensional vector addition systems. *Theor. Comput. Sci.*, 8(2):135–159, 1979.

12 S. Rao Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *STOC*, pages 267–281. ACM, 1982.

13 Jean-Luc Lambert. A structure to decide reachability in Petri nets. *Theor. Comput. Sci.*, 99(1):79–104, 1992.

14 Michel Latteux. Langages à un compteur. *J. Comput. Syst. Sci.*, 26(1):14–33, 1983.

15 Jérôme Leroux. The general vector addition system reachability problem by Presburger inductive invariants. *Logical Methods in Computer Science*, 6(3), 2010.

16 Jérôme Leroux. Vector addition system reachability problem: a short self-contained proof. In *POPL*, pages 307–316. ACM, 2011.

17 Jérôme Leroux. Vector addition systems reachability problem (A simpler solution). In *Turing-100*, volume 10 of *EPiC Series in Computing*, pages 214–228. EasyChair, 2012.

18 Jérôme Leroux and Sylvain Schmitz. Reachability in vector addition systems is primitive-recursive in fixed dimension. In *LICS*, pages 1–13. IEEE, 2019.

19 Jérôme Leroux and Grégoire Sutre. On flatness for 2-dimensional vector addition systems with states. In *CONCUR*, volume 3170 of *LNCS*, pages 402–416. Springer, 2004.

20 Jérôme Leroux and Grégoire Sutre. Flat counter automata almost everywhere! In *ATVA*, volume 3707 of *LNCS*, pages 489–503. Springer, 2005.

21 Jérôme Leroux and Grégoire Sutre. Reachability in two-dimensional vector addition systems with states: One test is for free, 2020. `arXiv:2007.09096`.

22 Jérôme Leroux, Grégoire Sutre, and Patrick Totzke. On the coverability problem for pushdown vector addition systems in one dimension. In *ICALP (2)*, volume 9135 of *LNCS*, pages 324–336. Springer, 2015.

23 Richard J. Lipton. The reachability problem requires exponential space. Technical Report 62, Yale University, 1976. URL: `http://cpsc.yale.edu/sites/default/files/files/tr63.pdf`.

24 Ernst W. Mayr. An algorithm for the general Petri net reachability problem. In *STOC*, pages 238–246. ACM, 1981.

25 Ernst W. Mayr. An algorithm for the general Petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984.

26 Marvin L. Minsky. *Computation: finite and infinite machines.* Prentice-Hall, Inc., 1967.

27 Loic Pottier. Minimal solutions of linear diophantine systems: Bounds and algorithms. In *RTA*, volume 488 of *LNCS*, pages 162–173. Springer, 1991.

28 Klaus Reinhardt. Reachability in Petri nets with inhibitor arcs. *Electr. Notes Theor. Comput. Sci.*, 223:239–264, 2008.

29 Philippe Schnoebelen. Revisiting Ackermann-hardness for lossy counter machines and reset Petri nets. In *MFCS*, volume 6281 of *LNCS*, pages 616–628. Springer, 2010.

30 Leslie G. Valiant and Mike Paterson. Deterministic one-counter automata. *J. Comput. Syst. Sci.*, 10(3):340–350, 1975.