The Complexity of Approximating the Complex-Valued Potts Model

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Abstract

We study the complexity of approximating the partition function of the q-state Potts model and the closely related Tutte polynomial for complex values of the underlying parameters. Apart from the classical connections with quantum computing and phase transitions in statistical physics, recent work in approximate counting has shown that the behaviour in the complex plane, and more precisely the location of zeros, is strongly connected with the complexity of the approximation problem, even for positive real-valued parameters. Previous work in the complex plane by Goldberg and Guo focused on $q = 2$, which corresponds to the case of the Ising model; for $q > 2$, the behaviour in the complex plane is not as well understood and most work applies only to the real-valued Tutte plane.

Our main result is a complete classification of the complexity of the approximation problems for all non-real values of the parameters, by establishing #P-hardness results that apply even when restricted to planar graphs. Our techniques apply to all $q \geq 2$ and further complement/refine previous results both for the Ising model and the Tutte plane, answering in particular a question raised by Bordewich, Freedman, Lovász and Welsh in the context of quantum computations.

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1 Introduction

The q-state Potts model is a classical model of ferromagnetism in statistical physics [32,37] which generalises the well-known Ising model. On a (multi)graph $G = (V, E)$, configurations of the model are all possible assignments $\sigma : V \to [q]$ where $[q] = \{1, \ldots, q\}$ is a set of q spins with $q \geq 2$. The model is parameterised by y, which corresponds to the temperature of the model and is also known as the *edge interaction*. Each configuration σ is assigned weight $y^{m(\sigma)}$ where $m(\sigma)$ denotes the number of monochromatic edges of G under σ. The partition function of the model is the aggregate weight over all configurations, i.e.,

$$Z_{\mathrm{Potts}}(G; q, y) = \sum_{\sigma\colon V \to [q]} y^{m(\sigma)},$$

When $q = 2$, this model is known as the Ising model, and we sometimes use the notation $Z_{\mathrm{Ising}}(G; y)$ to denote its partition function.

The Ising/Potts models have an extremely useful generalisation to non-integer values of q via the so-called "random-cluster" formulation and the closely related Tutte polynomial. In particular, for numbers q and γ, the Tutte polynomial of a graph G is given by

$$Z_{\mathrm{Tutte}}(G; q, \gamma) = \sum_{A \subseteq E} q^{k(A)} \gamma^{|A|}, \tag{1}$$

where $k(A)$ denotes the number of connected components in the graph (V, A) (isolated vertices do count). When q is an integer with $q \geq 2$, we have $Z_{\mathrm{Potts}}(G; q, y) = Z_{\mathrm{Tutte}}(G; q, y - 1)$, see, for instance, [34]. The Tutte polynomial on planar graphs is particularly relevant in quantum computing since it corresponds to the Jones polynomial of an "alternating link" [37, Chapter 5], and polynomial-time quantum computation can be simulated by additively approximating the Jones polynomial at a suitable value, as we will explain later in more detail, see also [6] for details.

In this paper, we study the complexity of approximating the partition function of the Potts model and the Tutte polynomial on planar graphs as the parameter y ranges in the complex plane. Traditionally, this problem has been mainly considered in the case where y is a positive real, however recent developments have shown that for various models, including the Ising and Potts models, there is a close interplay between the location of zeros of the partition function in the complex plane and the approximability of the problem, even for positive real values of y.

The framework of viewing partition functions as polynomials in the complex plane of the underlying parameters has been well-explored in statistical physics and has recently gained traction in computer science as well in the context of approximate counting. On the positive side, zero-free regions in the complex plane translate into efficient algorithms for approximating the partition function [1, 29] and this scheme has lead to a broad range of new algorithms even for positive real values of the underlying parameters [2, 16–18, 26–28, 30, 31]. On the negative side, the presence of zeros poses a barrier to this approach and, in fact, it has been demonstrated that zeros mark the onset of computational hardness for the approximability of the partition function [4, 5, 11, 15]. These new algorithmic and computational complexity developments stemming from the complex plane mesh with the statistical physics perspective where zeros have long been studied in the context of pinpointing phase transitions, see e.g., [3, 19, 25, 34, 37, 38].

For the problem of exactly computing the partition function of the Potts model, Jaeger, Vertigan and Welsh [20], as a corollary of a more general classification theorem for the Tutte polynomial, established #P-hardness unless (q, y) is one of seven exceptional points, see Section 6.3 of the full version for more details; Vertigan [36] further showed that the same classification applies on planar graphs with the exception of the Ising model ($q = 2$), where the problem is in FP.

For the approximation problem, the only known result that applies for general values y in the complex plane is by Goldberg and Guo [11], which addresses the case $q = 2$; the case $q \geq 3$ is largely open apart from the case when y is real which has been studied extensively even for planar graphs [11–13, 15, 21, 24]. We will review all these results more precisely in the next section, where we also state our main theorems.

1.1 Our main results

In this work, we completely classify the complexity of approximating $Z_{\mathrm{Potts}}(G; q, y)$ for $q \geq 2$ and non-real y, even on planar graphs G; in fact, our results also classify the complexity of the Tutte polynomial on planar graphs for reals $q \geq 2$ and non-real γ. Along the way, we also answer a question for the Jones polynomial raised by Bordewich, Freedman, Lovász, and Welsh [6]. To formally state our results, we define the computational problems we consider. Let K and ρ be real algebraic numbers with $K > 1$ and $\rho > 0$. We investigate the complexity of the following problems for any integer q with $q \geq 2$ and any algebraic number y.[1]

Name: Factor-K-NormPotts(q, y)
Instance: A (multi)graph G.
Output: If $Z_{\mathrm{Potts}}(G; q, y) = 0$, the algorithm may output any rational number. Otherwise, it must output a rational number $\hat{N}$ such that $\hat{N}/K \leq |Z_{\mathrm{Potts}}(G; q, y)| \leq K\hat{N}$.

A well-known fact is that the difficulty of the problem Factor-K-NormPotts(q, γ) does not depend on the constant $K > 1$. This can be proved using standard powering techniques (see [11, Lemma 11] for a proof when $q = 2$). In fact, the complexity of the problem is the same even for $K = 2^{n^{1-\epsilon}}$ for any constant $\epsilon > 0$ where n is the size of the input.

Name: Distance-ρ-ArgPotts(q, y)
Instance: A (multi)graph G.
Output: If $Z_{\mathrm{Potts}}(G; q, y) = 0$, the algorithm may output any rational number. Otherwise, it must output a rational $\hat{A}$ such that, for some $a \in \arg(Z_{\mathrm{Potts}}(G; q, \gamma))$, $|\hat{A} - a| \leq \rho$.

In the special case that q equals 2, we omit the argument q and write Ising instead of Potts in the name of the problem. Similarly, when the input of the problems is restricted to planar graphs, we write PlanarPotts instead of Potts. We also consider the problems Factor-K-NormTutte(q, γ) and Distance-$\pi/3$-ArgTutte(q, γ) for the Tutte polynomial when q, γ are algebraic numbers. Note also that, when q, γ are real, the latter problem is equivalent to finding the sign of the Tutte polynomial, and we sometimes write SignTutte(q, γ) (and, analogously, SignTutte(q, γ)).

Our first and main result is a full resolution of the complexity of approximating $Z_{\mathrm{Potts}}(G; q, y)$ for $q \geq 3$ and non-real y. More precisely, we show the following.

▶ **Theorem 1.** *Let $q \geq 3$ be an integer, $y \in \mathbb{C}\backslash\mathbb{R}$ be an algebraic number, and $K > 1$. Then, Factor-K-NormPlanarPotts(q, y) and Distance-$\pi/3$-ArgPlanarPotts(q, y) are #P-hard, unless $q = 3$ and $y \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly.*

[1] For $z \in \mathbb{C}\backslash\{0\}$, we denote by $|z|$ the norm of z, by $\mathrm{Arg}(z) \in [0, 2\pi)$ the principal argument of z and by $\arg(z)$ the set $\{\mathrm{Arg}(z) + 2\pi j : j \in \mathbb{Z}\}$ of all the arguments of z, so that for any $a \in \arg(z)$ we have $z = |z| \exp(ia)$.

We remark that, for real $y > 0$, the complexity of approximating $Z_{\text{Potts}}(G; q, y)$ on planar graphs is not fully known, though on general graphs the problem is #BIS-hard [13] and NP-hard for $y \in (0, 1)$ [12], for all $q \geq 3$. For real $y < 0$, the problem is NP-hard on general graphs when $y \in (-\infty, 1-q]$ for all $q \geq 3$ ([15])[2] and #P-hard on planar graphs when $y \in (1-q, 0)$ and $q \geq 5$ ([24], see also [14]). Our techniques for proving Theorem 1 allow us to resolve the remaining cases $q = 3, 4$ for $y \in (1-q, 0)$ on planar graphs, as a special case of the following theorem that applies for general $q \geq 3$. This is our second main result.

▶ **Theorem 2.** *Let $q \geq 3$ be an integer, $y \in (-q+1, 0)$ be a real algebraic number, and $K > 1$. Then* Factor-K-NormPlanarPotts(q, y) *and* Distance-$\pi/3$-ArgPlanarPotts(q, y) *are* #P*-hard, unless $(q, y) = (4, -1)$ when both problems can be solved exactly.*

Our third main contribution is a full classification of the range of the parameters where approximating the partition function of the Ising model is #P-hard. Note, on planar graphs G, $Z_{\text{Ising}}(G; y)$ can be computed in polynomial time for all y. For general (non-planar) graphs and non-real y, Goldberg and Guo show #P-hardness on the unit circle ($|y| = 1$) with $y \neq \pm i$, and establish NP-hardness elsewhere. Our next result shows that the NP-hardness results of [11] for non-real y can be elevated to #P-hardness.

▶ **Theorem 3.** *Let $y \in \mathbb{C} \backslash \mathbb{R}$ be an algebraic number, and $K > 1$ be a real. Then,* Factor-K-NormIsing(y) *and* Distance-$\pi/3$-ArgIsing(y) *are* #P*-hard, unless $y = \pm i$ when both problems can be solved exactly.*

For real y, we remark that the problems of approximating $Z_{\text{Ising}}(G; y)$ and determining its sign (when non-trivial) are well-understood:[3] the problem is FPRASable for $y > 1$ and NP-hard for $y \in (0, 1)$ ([21]), #P-hard for $y \in (-1, 0)$ [11, 15], and equivalent to approximating #PerfectMatchings for $y < -1$ [12]. For $y = 0, \pm 1$, $Z_{\text{Ising}}(G; y)$ can be computed exactly.

1.2 Consequences of our techniques for the Tutte/Jones polynomials

While our main results are on the Ising/Potts models, in order to prove them it is convenient to work in the "Tutte world"; this simplifies the proofs and has also the benefit of allowing us to generalise our results to non-integer q. The following result generalises Theorem 1 to non-integer $q > 2$.

▶ **Theorem 4.** *Let $q > 2$ be a real, $\gamma \in \mathbb{C} \backslash \mathbb{R}$ be an algebraic number, and $K > 1$. Then,* Factor-K-NormPlanarTutte(q, γ) *and* Distance-$\pi/3$-ArgPlanarTutte(q, γ) *are* #P*-hard, unless $q = 3$ and $\gamma + 1 \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly.*

Our techniques can further be used to elevate previous NP-hardness results of [12, 15] in the Tutte plane to #P-hardness for planar graphs, and answer a question for the Jones polynomial raised by Bordewich et al. in [6]. A more detailed discussion can be found in Section 4.

[2] Note, for $y \in (-\infty, 1-q) \cup [0, \infty)$, #P-hardness is impossible (assuming NP $\neq$ #P): finding the sign of $Z_{\text{Potts}}(G; q, y)$ is easy, even on non-planar graphs ([15]), and $Z_{\text{Potts}}(G; q, y)$ can be approximated using an NP-oracle. For $y = 1 - q$, the same applies when $q \geq 6$; the cases $q \in \{3, 4, 5\}$ are not fully resolved though [15] shows that $q = 3, 4$ are NP-hard, whereas $q = 5$ should be easy unless Tutte's 5-flow conjecture is false [37, Section 3.5].

[3] Analogously to Footnote 2, for $y \in (-\infty, -1) \cup (0, 1)$ #P-hardness is unlikely since the problem can be approximated with an NP-oracle.

2 Proof outline

In this section we provide some insight on the proofs of our main results. As mentioned earlier, the proofs are performed in the context of the Tutte polynomial.

In previous #P-hardness results [11,15] for the Tutte polynomial, the main technique was to reduce the exact counting #MinimumCardinality (s,t)-Cut problem to the problem of approximating $Z_{\mathrm{Tutte}}(G;q,\gamma)$ using an elaborate binary search based on suitable oracle calls. Key to these oracle calls are gadget constructions which are mainly based on series-parallel graphs which "implement" points (q',γ'); this means that, by pasting the gadgets appropriately onto a graph G, the computation of $Z_{\mathrm{Tutte}}(G;q',\gamma')$ reduces to the computation of $Z_{\mathrm{Tutte}}(G;q,\gamma)$. Much of the work in [11,15], and for us as well, is understanding what values (q',γ') can be implemented starting from (q,γ).

For planar graphs, while the binary-search technique from [11] is still useful, we have to use a different overall reduction scheme since the problem #MinimumCardinality (s,t)-Cut is not #P-hard when the input is restricted to planar graphs [33]. To obtain our #P-hardness results our plan instead is to reduce the problem of exactly evaluating the Tutte polynomial for some appropriately selected parameters q',γ' to the problem of computing its sign and the problem of approximately evaluating it at parameters q,γ; note, this gives us the freedom to use any parameters q',γ' we wish as long as the corresponding exact problem is #P-hard. Then, much of the work consists of understanding what values (q',γ') can be implemented starting from (q,γ), so we focus on that component first.

We first review previous constructions in the literature, known as shifts, and then introduce our refinement of these constructions, which we call polynomial-time approximate shifts, and state our main results about them.

2.1 Shifts in the Tutte plane

Let q be a real number and $\gamma_1,\gamma_2\in\mathbb{C}$. We say that a graph G (q,γ_1)-*implements* (q,γ_2) and that there is a *shift* from (q,γ_1) to (q,γ_2) if there exist vertices s,t in G such that

$$\gamma_2 = q\frac{Z_{st}(G;q,\gamma_1)}{Z_{s|t}(G;q,\gamma_1)},$$

where $Z_{st}(G;q,\gamma_1)$ is the contribution to $Z_{\mathrm{Tutte}}(G;q,\gamma_1)$ from configurations $A\subseteq E$ in which s,t belong to the same connected component in (V,A), while $Z_{s|t}(G;q,\gamma_1)$ is the contribution from all other configurations A (here, E is the edge set of G).

In the following, we will usually encounter shifts in the (x,y)-parametrisation of the Tutte plane, rather than the (q,γ)-parameterisation which was used for convenience here. To translate between these, set $y=\gamma+1$ and $(x-1)(y-1)=q$, see [37, Chapter 3]. We denote by H_q the hyperbola $\{(x,y)\in\mathbb{C}^2:(x-1)(y-1)=q\}$, and we will use both parametrisations as convenient. Section 3.2 of the full version has a more detailed description of shifts that apply to the multivariate Tutte polynomial.

As described earlier, shifts can be used to "move around" the complex plane. If one knows hardness for some $(x_2,y_2)\in H_q$, and there is a shift from $(x_1,y_1)\in H_q$ to (x_2,y_2), then one also obtains hardness for (x_1,y_1), essentially by replacing every edge of the input graph with a distinct copy of the graph G. This approach has been effective when attention is restricted to real parameters [12,14,15], however, when it comes to non-real parameters, the success of this approach has been limited. To illustrate this, in [11], the authors established #P-hardness of the Ising model when $y_2\in(-1,0)$, and used this to obtain #P-hardness for

y_1 on the unit circle by constructing appropriate shifts. However, their shift construction does not extend to general complex numbers, and this kind of result seems unreachable with those techniques.

2.2 Polynomial-time approximate shifts

To obtain our main theorems, we instead need to consider what we call polynomial-time approximate shifts; such a shift from $(x_1, y_1) \in H_q$ to $(x_2, y_2) \in H_q$ is an algorithm that, for any positive integer n, computes in time polynomial in n a graph G_n that (x_1, y_1)-implements $(\hat{x}_2, \hat{y}_2)$ with $|y_2 - \hat{y}_2| \leq 2^{-n}$. In fact, our constructions need to maintain planarity, and we will typically ensure this by either making every G_n a series-parallel graph, in which case we call the algorithm a *polynomial-time approximate series-parallel shift*, or by making every G_n a theta graph, in which case we call the algorithm a *polynomial-time approximate theta shift*.[4]

These generalised shifts allow us to overcome the challenges mentioned above and are key ingredients in our reduction. Our main technical theorem about them is the following.

▶ **Theorem 5.** *Let $q \geq 2$ be a real algebraic number. Let x and y be algebraic numbers such that $(x, y) \in H_q$, $y \in (-1, 0) \cup (\mathbb{C}\backslash\mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (j, j^2), (j^2, j)\}$, where $j = \exp(2\pi i/3)$. Then, for any pair of algebraic numbers $(x', y') \in H_q$ with $y' \in [-1, 1]$ there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y').*

The exceptions $\{(i, -i), (-i, i), (j, j^2), (j^2, j)\}$ are precisely the non-real points of the (x, y) plane where the Tutte polynomial of a graph can be evaluated in polynomial time (see Section 6.3). As we will see, being able to (x, y)-implement approximations of any number in $(-1, 0)$ is essentially the property that makes the approximation problem #P-hard at (x, y).

We remark that the idea of implementing approximations of a given weight or edge interaction has been explored in the literature, though only when all the edge interactions involved are real. We review these results in Section 4 of the full version.

We study the properties of polynomial-time approximate shifts in Section 4 and prove Theorem 5 in Section 5. In the next section, we describe some of the techniques used.

2.2.1 Proof Outline of Theorem 5

Shifts, as defined in Section 2.1, have a transitivity property: if there is a shift from (x_1, y_1) to (x_2, y_2) and from (x_2, y_2) to (x_3, y_3), then there is a shift from (x_1, y_1) to (x_3, y_3), see Section 3.2 of the full version for more details.

The polynomial-time approximate shift given in Theorem 5 is constructed in a similar way. First, we construct a polynomial-time approximate shift from (x, y) to some (x_2, y_2) such that $y_2 \in (-1, 0)$, where x_2 and y_2 depend on x, y. Then, we construct a polynomial-time approximate shift from (x_2, y_2) to (x', y'). Finally, we combine both polynomial-time approximate shifts using an analogue of the transitivity property.

However, when this approach is put into practice, there is a difficulty that causes various technical complications: we only have mild control in our constructions over the intermediate shift (x_2, y_2). In particular, even if the numbers x and y are algebraic, we cannot guarantee that x_2 and y_2 are algebraic, and this causes problems with obtaining the required transitivity

[4] A theta graph consists of two terminals s and t joined by internally disjoint paths [10]. A series-parallel graph with terminals s and t can be obtained from the single-edge graph with edge (s, t) by repeatedly subdividing edges or adding parallel edges [8, Chapter 11].

property. Instead, we have to work with a wider class of numbers, the set $\mathsf{P}_{\mathbb{C}}$ of *polynomial-time computable numbers.* These are numbers that can be approximated efficiently, i.e., for $y \in \mathsf{P}_{\mathbb{C}}$ there is an algorithm that computes $\hat{y}_n \in \mathbb{Q}[i]$ with $|y - \hat{y}_n| \leq 2^{-n}$ in time polynomial in n [23]. We denote by $\mathsf{P}_{\mathbb{R}} = \mathbb{R} \cap \mathsf{P}_{\mathbb{C}}$ the set of polynomial-time computable reals.

Our polynomial-time approximate shifts are constructed in Section 3. The first of these polynomial-time approximate shifts is provided by Lemma 6.

▶ **Lemma 6.** *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in H_q$, $y \in (-1, 0) \cup (\mathbb{C}\backslash\mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (j, j^2), (j^2, j)\}$, where $j = \exp(2\pi i/3)$. Then there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') for some $(x', y') \in H_q$ with $x', y' \in \mathsf{P}_{\mathbb{R}}$ and $y' \in (0, 1)$.*

The construction in Lemma 6 is obtained using a theta graph and trying to get a shift that is very close to the real line. However, we cannot control the point (x', y') that we are approximating, and as mentioned, x', y' might not be algebraic. The proof of Lemma 6 is the most technically demanding new ingredient in our work, and is outlined in Section 3.

Using Lemma 6, we have a series-parallel polynomial-time approximate shift from (x, y) to some $(x', y') \in H_q$ with $x', y' \in \mathsf{P}_{\mathbb{R}}$ and $y' \in (0, 1)$. Next, we have to construct a polynomial-time approximate shift from (x', y') to $(\hat{x}, \hat{y})$, where $(\hat{x}, \hat{y})$ is the point that we want to shift to in Theorem 5. In fact, we actually use a theta shift, which also facilitates establishing the required transitivity property later on. Note that since y' is not necessarily algebraic, we can not directly apply the results that have already appeared in the literature on implementing approximations of edge interactions. In the next lemma, we generalise these results to the setting of polynomial-time computable numbers, where we need to address some further complications that arise from computing with polynomial-time computable numbers instead of algebraic numbers. The proof of the lemma is given in Section 5.5 of the full version.

▶ **Lemma 7.** *Let $q, x, y \in \mathsf{P}_{\mathbb{R}}$ such that $q > 0$, $(x, y) \in H_q$, y is positive and $1 - q/2 < y < 1$. There is a polynomial-time algorithm that takes as an input:*

- *two positive integers k and n, in unary;*
- *a real algebraic number $w \in [y^k, 1]$.*

The algorithm produces a theta graph J that (x, y)-implements $(\hat{x}, \hat{y})$ such that $\left|\hat{y} - w\right| \leq 2^{-n}$. The size of J is at most a polynomial in k and n, independently on w.

Then, we are able to combine the shifts in Lemmas 6 and 7 via a transitivity property for polynomial-time approximate shifts (see Lemma 17), and therefore prove Theorem 5, see Section 5.5 of the full version for the details.

2.3 The reductions

In Section 6.6 of the full version we show how to use a polynomial-time approximate shift from (x_1, y_1) to (x_2, y_2) to reduce the problem of approximating the Tutte polynomial at (x_2, y_2) to the same problem at (x_1, y_1). The following lemma gives such a reduction for the problem of approximating the norm, we also give an analogous result for approximating the argument in the full version (see Lemma 56).

▶ **Lemma 8.** *Let $q \neq 0$, γ_1 and $\gamma_2 \neq 0$ be algebraic numbers, and $K > 1$. For $j \in \{1, 2\}$, let $y_j = \gamma_j + 1$ and $x_j = 1 + q/\gamma_j$. If there is a polynomial-time series-parallel approximate shift from (x_1, y_1) to (x_2, y_2), then we have a reduction from* Factor-K-NormTutte(q, γ_2) *to* Factor-K-NormTutte(q, γ_1). *This reduction also holds for the planar version of the problem.*

In order to prove Lemma 8, we need some lower bounds on the norm of the partition function $Z_{\mathrm{Tutte}}(G;q,\gamma)$. This kind of lower bound plays an important role in several hardness results on the complexity of approximating partition functions [4, 11]. Here, we have to work a bit harder than usual since we have two (algebraic) underlying parameters (in the case of Tutte), and we need to use results in algebraic number theory, see Section 6.1 of the full version for details.

By combining Theorem 5 and Lemma 8 with existing hardness results, we obtain our hardness results for non-real edge interactions in Section 6.8 of the full version. On the way, we collect some hardness on real parameters as well that strengthen previous results in the literature, and part of Section 6 of the full version is devoted to this. The main reason behind these improvements is that previous work on real parameters used reductions from approximately counting minimum cardinality (s,t)-cuts [11, 15], the minimum 3-way cut problem [12], or maximum independent set for planar cubic graphs [14], which are either easy on planar graphs or the parameter regions they cover are considerably smaller or cannot be used to conclude #P-hardness. We instead reduce the exact computation of $Z_{\mathrm{Tutte}}(G;q,\gamma)$ to its approximation, which has the advantage that the problem that we are reducing from is #P-hard for planar graphs [36]. Interestingly, our reduction requires us to apply an algorithm of Kannan, Lenstra and Lovász [22] to reconstruct the minimal polynomial of an algebraic number from an additive approximation of the number. The lower bounds on the partition function $Z_{\mathrm{Tutte}}(G;q,\gamma)$ also play a role in this reduction, the details are given in Section 6.5 of the full version.

3 Polynomial-time approximate shifts with complex weights

We have already outlined in Section 2.2.1 the proof of Theorem 5, the key ingredient in the reductions to obtain our inapproximability results, based on the polynomial-time approximate shifts described in Lemmas 6 and 7. In this section, we give the proof of Lemma 6 which has the "complex part" of the proof and has the new technical ingredients; Lemma 7 follows by generalising results from [14] from algebraic reals to computable numbers.

We will use theta graphs (cf. Footnote 4) to prove Lemma 6. Let J be the theta graph with m internal paths of lengths $l_1,\dots,l_m$ between two vertices s,t. Then J gives a shift, in the sense of Section 2.1, from (q,γ) to (q,γ') where

$$\gamma' = \prod_{j=1}^{m}\left(1+\frac{q}{x^{l_j}-1}\right) - 1 \text{ and } x = 1 + q/\gamma, \tag{4}$$

see Section 3.2 of the full version for details. Note, we will mainly use this in the (x,y)-parametrisation. Also, when $m=1$, we refer to J as an ℓ_1-*stretch*, whereas when $\ell_1 = \ldots = \ell_m$, we refer to J as an m-*thickening*.

We start by giving in Section 3.1 some shifts that we will need, and then show how to use them to obtain the polynomial-time approximate shifts of Lemma 6 in Section 3.2. Let q be a real algebraic number with $q \geq 2$ and let $(x,y) \in H_q$ be a pair of algebraic numbers.

3.1 Some shifts for non-reals

In this section, for (x,y) as in Lemma 6, we will be interested in computing a shift from (x,y) to $(x_1,y_1) \in H_q$ with $x_1 \notin \mathbb{R}$ and $|x_1| > 1$. The following remark will thus be useful.

▶ Remark 23. Let q be a positive real number and let $(x,y) \in H_q$. From $(x-1)(y-1) = q$ it follows that $x = 1 + q/(y-1) = (y+q-1)/(y-1)$, so $|x| \geq 1$ iff $\mathrm{Re}(y) \geq 1 - q/2$, with equality only when $|x| = 1$.

A *root of unity* is a complex number z such that $z^k = 1$ for some positive integer k; there is a polynomial-time algorithm to determine whether an algebraic number z is a root of unity, see [7]. Also, if $z \in \mathbb{C}$ is not a root of unity and $|z| = 1$, then $\{z^j : j \in \mathbb{N}\}$ is dense in the unit circle, see, e.g., [9, Section 1.2].

▶ **Lemma 24.** *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in H_q$ and $\mathrm{Arg}(y) \notin \{0, \pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2\}$. Then we can compute a theta graph J that (x, y)-implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$.*

Proof. We show how to compute n such that $\mathrm{Re}(y^n) > 0$ and $\mathrm{Im}(y^n) > 0$. For such n, we let $y_1 = y^n$ and $x_1 = 1 + q/(y_1 - 1)$, so Remark 23 ensures that $|x_1| > 1$ and $x_1 \notin \mathbb{R}$. Hence, we can return J as the graph with two vertices and n edges joining them. Since y and $|y|$ are algebraic numbers, we can compute the algebraic number $y/|y|$ and detect if $y/|y|$ is a root of unity as explained earlier. There are two cases:

1. $y/|y|$ is not a root of unity. Then we can compute the smallest positive integer n such that $\mathrm{Arg}(y^n) \in [\pi/6, \pi/3]$; such an integer exists because $\{(y/|y|)^j : j \in \mathbb{N}\}$ is dense in the unit circle. Finally, since $\mathrm{Arg}(y^n) \in [\pi/6, \pi/3]$, we have $\mathrm{Re}(y^n) > 0$ and $\mathrm{Im}(y^n) > 0$.
2. $y/|y|$ is a root of unity of order r with $r \geq 5$. Recall that we can compute r by sequentially computing the powers of $y/|y|$ until we obtain 1. Then we have $(y/|y|)^{r+1} = e^{i2\pi/r}$. Note that the real and imaginary parts of $e^{i2\pi/r} = \cos(2\pi/r) + i\sin(2\pi/r)$ are positive. ◀

In the following lemmas, we consider the cases $\mathrm{Arg}(y) \in \{\pi/2, 2\pi/3, 4\pi/3, 3\pi/2\}$, where the exemptions in Lemma 6 arise.

▶ **Lemma 25.** *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in H_q$, $y \neq 0$ and $\mathrm{Arg}(y) \in \{2\pi/3, 4\pi/3\}$. If $q \neq 3$ or $|y| \neq 1$, then we can compute a series-parallel graph J that (x, y)-implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$.*

Proof. Note that $y/|y|$ is a root of unity of order 3. We have $\mathrm{Re}(y) = |y|\cos(2\pi/3) = -|y|/2 < 0$. Let $x = 1 + q/(y - 1)$. We consider three cases.

Case I: $\mathrm{Re}(y) > 1 - q/2$. Then, by Remark 23, $|x| > 1$. We return J as the graph with 2 vertices and one edge joining them.

Case II: $\mathrm{Re}(y) < 1 - q/2$. Then $|x| < 1$. Let $y_n = 1 + q/(x^n - 1)$. An n-stretch gives a shift from (x, y) to (x^n, y_n). Since $x \notin \mathbb{R}$, there are infinitely many values of n such that $y_n \notin \mathbb{R}$. Note that y_n converges to $1 - q \in (-\infty, -1]$, and the distance between $1 - q$ and the set of complex points $\{z \in \mathbb{C} : \mathrm{Arg}(z) \in \{\pi/2, 2\pi/3, 4\pi/3, 3\pi/2\}\}$ is larger than 0. Hence, we can compute n such that $\mathrm{Arg}(y_n) \notin \{0, \pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2\}$. Since $(x^n, y_n) \in H_q$, the result follows from applying Lemma 24 to (x^n, y_n), the transitivity property of shifts and noticing that the obtained graph is series-parallel.

Case III: $\mathrm{Re}(y) = 1 - q/2$. Note that $q > 2$ because for $q = 2$ we would obtain $\mathrm{Re}(y) = 0$; also $|y| \neq 1$, otherwise from $\mathrm{Re}(y) = -|y|/2$, we obtain that $q = 3$, which is excluded. If $|y| < 1$, let $n \geq 1$ be an integer such that $|y|^n < q - 2$ and $\mathrm{Arg}(y^n) = 2\pi/3$. Since $\mathrm{Re}(y^n) = -|y|^n/2$, we have $\mathrm{Re}(y^n) > 1 - q/2$, so $|x_n| > 1$ for $x_n = 1 + q/(x^n - 1)$ and we can return J as the graph with two vertices and n edges joining them. If $|y| > 1$, let $n \geq 1$ be an integer such that $\mathrm{Arg}(y^n) = 2\pi/3$ and $\mathrm{Re}(y^n) = -|y|^n/2 < 1 - q/2$, and apply Case II to (x_n, y^n), where $x_n = 1 + q/(y^n - 1)$, the transitivity property of shifts and noticing that the obtained graph is series-parallel.

This finishes the proof. ◀

▶ **Lemma 26.** *Let q be a real algebraic number with $q \geq 2$. Let y be an algebraic number such that $y \neq 0$ and $\operatorname{Arg}(y) \in \{\pi/2, 3\pi/2\}$.*

If $q > 2$, then we can compute a theta graph J that (x,y)-implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$. If $q = 2$ and $|y| \neq 1$, then we can compute a series-parallel graph J that (x,y)-implements (x_2, y_2) with $y_2 \in (-1, 0)$.

Proof. The hypotheses $y \neq 0$ and $\operatorname{Arg}(y) \in \{\pi/2, 3\pi/2\}$ are equivalent to $y \neq 0$ and $\operatorname{Re}(y) = 0$. Let $x = 1 + q/(y-1)$. If $q > 2$, then $1 - q/2 < 0 = \operatorname{Re}(y)$ and $|x| > 1$ as a consequence of Remark 23, so we return the graph with two vertices and one edge joining them as J. The second claim (case $q = 2$) has been studied in [11, Lemma 3.15], where the graph constructed is a 2-thickening of a k-stretching. ◀

3.2 An approximate shift to (x', y') with $y' \in (0, 1)$

We will use the shifts to complex points (x, y) with $|x| > 1$ in polynomial-time approximate shifts. We start with a technical lemma, which will simplify the proofs, and use this to design the desired polynomial-time approximate shift, cf. Lemma 33. Then, it is just a matter of combining the pieces to conclude the proof of Lemma 6.

▶ **Lemma 32.** *Let $r, c \in (0,1) \cap \mathbb{Q}$, and $\{z_n\}$ be a sequence of algebraic complex numbers satisfying, for every integer $n \geq 1$, $|z_n| < 1$ and*

$$z_n = 1 - f(n) + i g(n) \text{ with } f, g \text{ satisfying } c r^n \leq f(n), g(n) \leq r^n/2.$$

Then there is $w \in (0,1)$ and a bounded sequence of positive integers $\{e_n\}$ such that

$$\left| \prod_{j=1}^{n} z_j^{e_j} - w \right| \leq \left(\frac{\pi}{2} + \frac{\pi}{c(1-r)} \right) r^n \text{ for every integer } n \geq 1.$$

Further, if the representation of the algebraic number z_n can be computed in time polynomial in n, then $w \in \mathsf{P}_{\mathbb{R}}$ and e_n can be computed in polynomial time in n.

Proof Sketch. Write $z_n = \rho_n e^{i\theta_n}$ for some $\rho_n \in (0,1)$ and $\theta_n \in (0, \pi/2)$. Note that $1 - f(n) < \rho_n$. Let $h(n) = 1 - \rho_n$. Using the assumption on $f(n)$, we obtain

$$0 < h(n) < f(n) \leq r^n/2 \quad \text{for integers } n \geq 1. \tag{8}$$

We have $\sin(\theta_n) = \frac{\operatorname{Im}(z_n)}{\rho_n} = \frac{g(n)}{1-h(n)}$, so using that $\sin(x) \leq x \leq \pi \sin(x)/2$ for every $x \in [0, \pi/2]$, we obtain $\frac{g(n)}{1-h(n)} \leq \theta_n \leq \frac{\pi g(n)}{2(1-h(n))}$, and from (8) it follows that

$$g(n) \leq \theta_n \leq \pi g(n) \quad \text{for integers } n \geq 1. \tag{9}$$

Using the assumption $c r^n \leq g(n) \leq r^n/2$, we conclude that, for integers $n \geq 2$,

$$\frac{\theta_{n-1}}{\theta_n} \leq \pi \frac{g(n-1)}{g(n)} \leq \frac{\pi}{2cr}. \tag{10}$$

Let $\tau_0 = 0$. We define τ_n and e_n by induction on n. Let e_n be the largest integer such that $\tau_{n-1} + e_n \theta_n \leq 2\pi$ and let $\tau_n = \tau_{n-1} + e_n \theta_n$. The definition, combined with (9) and (10), yields that (see full version for details)

$$0 \leq e_n \leq \frac{2\pi}{cr} \quad \text{for integers } n \geq 1. \tag{11}$$

The sequence $\{e^{i\tau_n}\}$ converges to 1. In fact, we show that it does so exponentially fast. Note that the derivative of e^{it} has constant norm 1. Therefore, e^{it} is a Lipschitz function with constant 1, that is, $\left|e^{it} - e^{is}\right| \leq |s-t|$ for every $s, t \in \mathbb{R}$. Using (9), it follows that

$$\left|1 - e^{i\tau_n}\right| = \left|e^{i2\pi} - e^{i\tau_n}\right| \leq |2\pi - \tau_n| < \theta_n \leq \pi g(n) \leq \frac{\pi}{2} r^n. \tag{12}$$

Next we study the sequence $\{x_n\}$ defined by $x_n = \prod_{j=1}^n \rho_j^{e_j}$. Since $\rho_j \in (0,1)$, $\{x_n\}$ is decreasing and has a limit $w \in [0,1)$. In the full version, we show that $w > 0$ based on the fact that $e_n \leq 2\pi/(cr)$ and that $h(n)$ is a decreasing geometric series. We next show that $\{x_n\}$ converges exponentially fast to w. Note that $x_n = (1-h(n))^{e_n} x_{n-1}$ and, thus, for $n \geq 2$, using (8), (11) and that $(1-x)^k \geq 1 - kx$ for $x \in (0,1)$ and $k \in \mathbb{Z}_{\geq 0}$,

$$0 \leq x_{n-1} - x_n = x_{n-1}\left(1 - (1-h(n))^{e_n}\right) \leq 1 - (1-h(n))^{e_n} \leq h(n) e_n \leq \tfrac{\pi}{cr} r^n.$$

By telescoping appropriately, we conclude that $|x_n - w| \leq \frac{\pi}{c(1-r)} r^n$ for every integer $n \geq 1$. Using this and (12), we obtain for every positive integer n that

$$\begin{aligned} \left|\prod_{j=1}^n z_j^{e_j} - w\right| &\leq \left|\prod_{j=1}^n z_j^{e_j} - x_n\right| + |x_n - w| = |x_n| \left|\prod_{j=1}^n e^{ie_j\theta_j} - 1\right| + |x_n - w| \\ &\leq \left|\prod_{j=1}^n e^{ie_j\theta_j} - 1\right| + |x_n - w| = \left|e^{i\tau_n} - 1\right| + |x_n - w| \leq \frac{\pi}{2} r^n + \frac{\pi}{c(1-r)} r^n. \end{aligned}$$

Using that e_n is a bounded sequence of positive integers and that z_n can be computed in time polynomial in n, we have that e_n can be computed in time polynomial in n as well. We conclude that w is the limit of a sequence of algebraic numbers that converges exponentially fast and whose n-th element can be computed in time polynomial in n, so $w \in \mathsf{P}_{\mathbb{R}}$. ◀

▶ **Lemma 33.** *Let q be a real algebraic number with $q > 0$. Let x and y be algebraic numbers such that $(x, y) \in H_q$, $y \notin \mathbb{R}$ and $|x| > 1$. Then there is a polynomial-time approximate theta shift from (x, y) to (x', y') for some $(x', y') \in H_q$ with $y' \in (0,1) \cap \mathsf{P}_{\mathbb{R}}$.*

Proof Sketch. Since $y \notin \mathbb{R}$ and $(x-1)(y-1) = q$, we have $x \notin \mathbb{R}$. Let us write $x = Re^{i\theta}$ for some $R > 1$ and $\theta \in (0, 2\pi)$. An m-stretch gives a shift from (x, y) to (x^m, y_m) with $y_m = (x^m + q - 1)/(x^m - 1)$. By plugging $x = Re^{i\theta}$ in the definition of y_m and multiplying by $R^m e^{-im\theta} - 1$ in the numerator and denominator, we obtain

$$y_m = \frac{R^{2m} - q + 1 + (q-2)R^m \cos(m\theta) - iqR^m \sin(m\theta)}{1 + R^{2m} - 2R^m \cos(m\theta)}. \tag{13}$$

If $\theta \in \{\pi/2, 3\pi/2\}$, that is, $x \in i\mathbb{R}$, then for $m \equiv 2 \pmod 4$ such that $1 + R^m > q$, we have $\cos(m\theta) = -1$, $\sin(m\theta) = 0$ and $y_m = \frac{(1+R^m)^2 - q(1+R^m)}{(1+R^m)^2} = \frac{1+R^m-q}{1+R^m} \in (0,1)$, so we can choose $y' = y_m$ and we are done. In the rest of the proof we assume that $\theta \notin \{\pi/2, 3\pi/2\}$.

We are going to apply Lemma 32 to a subsequence of y_m. In particular, we invoke Corollary 21 of the full version in order to find a sequence $\sigma(m)$ of integers, a positive integer k and a positive rational C that satisfy:

- $\sigma(m)$ can be computed in time polynomial in m, while k and C can be computed in constant time from x;
- $\sigma(m) \in [m, m+k-1]$ and $\sin(\sigma(m)\theta)$, $\cos(\sigma(m)\theta) \leq -C$ for every integer $m \geq 1$.

It follows that $\operatorname{Re}\left(x^{\sigma(m)}\right) = \operatorname{Re}\left(R^{\sigma(m)} e^{i\sigma(m)\theta}\right) \leq -CR^{\sigma(m)} \leq -CR^m$. Since $R > 1$, we can compute $m_1 \in \mathbb{Z}_{\geq 0}$ such that for $m \geq m_1$ we have $\operatorname{Re}(x^{\sigma(m)}) < 1 - q/2$ and, thus, $\left|y_{\sigma(m)}\right| < 1$ (recall that $y_m = (x^m + q - 1)/(x^m - 1)$ and Remark 23).

Let a_m, b_m be such that $y_m = 1 - a_m + ib_m$. Then, using the bounds $R^{2\sigma(m)} \le 1 + R^{2\sigma(m)} - 2R^{\sigma(m)}\cos(\sigma(m)\theta) \le 4R^{2\sigma(m)}$, we obtain the bounds

$$\frac{qC}{4}R^{-\sigma(m)} \le a_{\sigma(m)}, b_{\sigma(m)} \le 2qR^{-\sigma(m)} \quad \text{for integers } m \ge 1. \tag{14}$$

Then, let m_2 be an integer such that $m_2 \ge \log_R(4q), m_1$ and c be a rational with $c \in (0, qCR^{-m_2-k-1}/4)$. Then, with $f(m) = a_{\sigma(m+m_2)}$ and $g(m) = b_{\sigma(m+m_2)}$, we obtain from (14) and the inequalities $R^{-m-k+1} \le R^{-\sigma(m)} \le R^{-m}$ that

$$cR^{-m} \le f(m), g(m) \le \frac{1}{2}R^{-m} \quad \text{for integers } m \ge 1. \tag{15}$$

Using that $|y_{\sigma(m)}| < 1$ and (15), we obtain that the sequence $\{z_m\} = \{y_{\sigma(m+m_2)}\}$ satisfies the assumptions of Lemma 32, so applying the lemma yields $y' \in (0,1) \cap \mathsf{P}_{\mathbb{R}}$ and a bounded sequence of positive integers $\{e_m\}$ such that

$$\left|\prod_{j=1}^{m} z_j^{e_j} - y'\right| \le \left(\frac{\pi}{2} + \frac{\pi}{c(1-1/R)}\right)R^{-m}, \tag{16}$$

and e_m can be computed in time polynomial in m, for all integers $m \ge 1$. This gives the following polynomial-time approximate theta shift from (x,y) to (x',y'), where $x' = 1 + q/(y'-1)$. For integer n, pick m to be the smallest integer so that the r.h.s. in (16) is smaller than 2^{-n}, and note that m is linear in n. We return the theta graph J_n that is the parallel composition of the path graphs that are used to implement the weights $y_{\sigma(j+m_2)}$, each one repeated e_j times, for $j \in \{1, \ldots, m\}$. The graph J_n is a shift from (x,y) to $(\hat{x}, \hat{y}) \in H_q$ for $\hat{y} = \prod_{j=1}^{m} z_j^{e_j} = \prod_{j=1}^{m} y_{\sigma(j+m_2)}^{e_j}$, cf. (4). ◀

▶ **Lemma 6.** *Let q be a real algebraic number with $q \ge 2$. Let x and y be algebraic numbers such that $(x,y) \in H_q$, $y \in (-1,0) \cup (\mathbb{C}\backslash\mathbb{R})$ and $(x,y) \notin \{(i,-i), (-i,i), (j,j^2), (j^2,j)\}$, where $j = \exp(2\pi i/3)$. Then there is a polynomial-time approximate series-parallel shift from (x,y) to (x',y') for some $(x',y') \in H_q$ with $x', y' \in \mathsf{P}_{\mathbb{R}}$ and $y' \in (0,1)$.*

Proof. If $y \in (-1,0)$, then a 2-thickening of (x,y) gives the result. Hence, let us assume that $y \notin (-1,0)$ in the rest of the proof. There are two cases to consider.

Case I: $q \ne 2$ or $y \notin i\mathbb{R}$. We apply either Lemma 24, Lemma 25 or Lemma 26, depending on $\operatorname{Arg}(y)$, to find a shift from (x,y) to $(x_1,y_1) \in H_q$ with $y_1 \notin \mathbb{R}$ and $|x_1| > 1$. The graph of this shift is series-parallel. Then we apply Lemma 33 to obtain a polynomial-time approximate theta shift from (x_1,y_1) to some $(x',y') \in H_q$ with $y' \in (0,1) \cap \mathsf{P}_{\mathbb{R}}$. The result follows from the transitivity property of shifts.

Case II: $q = 2$ and $y \in i\mathbb{R}$. Since $y \ne \pm i$, Lemma 26 gives a shift from (x,y) to (x',y') for some $(x',y') \in H_q$ with $y' \in (-1,0)$. A 2-thickening of (x',y') gives the result.

The fact that $x' \in \mathsf{P}_{\mathbb{R}}$ follows from $x' = 1 + q/(y'-1)$ and $y' \in \mathsf{P}_{\mathbb{R}}$. ◀

4 Further consequences of our results

In this final section, we apply our results to the problem of approximating the Jones polynomial of an alternating link, which is connected to the quantum complexity class BQP as explained in [6]. More details can be found in Section 7 of the full version.

We briefly review some relevant facts about links and the Jones polynomial that relate it to the Tutte polynomial on graphs, see [37] for their definitions. Let $V_L(T)$ denote the Jones polynomial of a link L. By a result of Thistlethwaite, when L is an alternating link with associated planar graph $G(L)$, we have $V_L(t) = f_L(t)T(G(L); -t, -t^{-1})$, where $f_L(t)$ is an easily-computable factor that is plus or minus a half integer power of t, and $T(G; x, y)$ is the Tutte polynomial of G in the (x, y)-parametrisation [35, 37]. Moreover, every planar graph is the graph of an alternating link [37, Chapter 2]. Hence, we can translate our results on the complexity of approximating the Tutte polynomial of a planar graph to the complexity of approximating the Jones polynomial of an alternating link, and obtain #P-hardness results for approximating $V_L(t)$. This is done in Corollary 61 of the full version, which shows #P-hardness for all algebraic t with $\mathrm{Re}(t) > 0$ and $t \notin \{1, -e^{2\pi i/3}, -e^{4\pi i/3}\}$.

The case $t = e^{2\pi i/5}$ of Corollary 61 is particularly relevant due to its connection between approximate counting and the quantum complexity class BQP, which was explored by Bordewich, Freedman, Lovász and Welsh in [6], where they posed the question of determining whether $\mathrm{Re}(Z_{\mathrm{Tutte}}(G; q, \gamma)) \geq 0$ or $\mathrm{Re}(Z_{\mathrm{Tutte}}(G; q, \gamma))$ for planar graphs G. The non-planar version has been studied in [11, Section 5], where #P-hardness was shown. Our planar results for the Tutte polynomial allow us to adapt the argument in [11] to answer the question asked in [6], see Corollary 62 of the full version.

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