Extending Nearly Complete 1-Planar Drawings in Polynomial Time

Eduard Eiben
Department of Computer Science, Royal Holloway, University of London, Egham, UK
eduard.eiben@rhul.ac.uk

Robert Ganian
Algorithms and Complexity Group, TU Wien, Austria
rganian@ac.tuwien.ac.at

Thekla Hamm
Algorithms and Complexity Group, TU Wien, Austria
thamm@ac.tuwien.ac.at

Fabian Klute
Department of Information and Computing Sciences, Utrecht University, The Netherlands
f.m.klute@uu.nl

Martin Nöllenburg
Algorithms and Complexity Group, TU Wien, Austria
noellenburg@ac.tuwien.ac.at

Abstract

The problem of extending partial geometric graph representations such as plane graphs has received considerable attention in recent years. In particular, given a graph G, a connected subgraph H of G and a drawing $\mathcal{H}$ of H, the extension problem asks whether $\mathcal{H}$ can be extended into a drawing of G while maintaining some desired property of the drawing (e.g., planarity).

In their breakthrough result, Angelini et al. [ACM TALG 2015] showed that the extension problem is polynomial-time solvable when the aim is to preserve planarity. Very recently we considered this problem for partial 1-planar drawings [ICALP 2020], which are drawings in the plane that allow each edge to have at most one crossing. The most important question identified and left open in that work is whether the problem can be solved in polynomial time when H can be obtained from G by deleting a bounded number of vertices and edges. In this work, we answer this question positively by providing a constructive polynomial-time decision algorithm.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Extension problems, 1-planarity

Digital Object Identifier 10.4230/LIPIcs.MFCS.2020.31

Related Version A full version of this article is available at `https://arxiv.org/abs/2007.05346`.

Funding *Robert Ganian*: Robert Ganian acknowledges support by the Austrian Science Fund (FWF, project P31336).
Thekla Hamm: Thekla Hamm acknowledges support by the Austrian Science Fund (FWF, projects P31336 and W1255-N23).

45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020).
Editors: Javier Esparza and Daniel Král'; Article No. 31; pp. 31:1–31:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Planarity is a fundamental concept in graph theory and especially in graph drawing, where planar graphs are exactly those graphs that admit a crossing-free node-link drawing in the plane. It is well known that testing whether a graph is planar can be carried out in polynomial time, and in the positive case one can also construct a plane drawing [16,38]. But what if we are given a more refined question: given a graph G where some subgraph H of G already has a fixed plane drawing $\mathcal{H}$, is it possible to extend $\mathcal{H}$ to a full plane drawing of G? The corresponding problem is an example of so-called drawing extension problems, which are motivated (among others) from network visualization applications: there, important patterns (subgraphs) might be required to have a special layout, or new vertices and edges in a dynamic graph may need to be inserted into an existing (partial) connected drawing which must remain stable in order to preserve the mental map [33].

The problem of extending partial planar drawings was solved thanks to the breakthrough result of Angelini, Di Battista, Frati, Jelinek, Kratochvil, Patrignani and Rutter [2], who provided a linear-time algorithm that answers the above question as well as constructs the desired planar drawing of G (if it exists). Unfortunately, it is often the case that we cannot hope for a plane drawing of G extending $\mathcal{H}$ – either because G itself is not planar, or because the partial drawing $\mathcal{H}$ cannot be extended to a plane one. A natural way to deal with this situation is to relax the restriction from planarity to a more general class of graphs. In our very recent work [20], we investigated the extension problem of partial *1-planar drawings*, one of the most natural and most studied generalizations of planarity [17,29,37]. A graph is 1-planar if it admits a drawing in the plane with at most one crossing per edge. Unlike planar graphs, recognizing 1-planar graphs is NP-complete [22,30], even if the graph is a planar graph plus a single edge [7] – and hence the extension problem of 1-planar drawings is also NP-complete [20].

In spite of this initial observation, we showed that the extension problem for 1-planar drawings is polynomial-time solvable when the *edge deletion distance* between H and G is bounded [20]. However, already in that paper it was pointed out that requiring the edge deletion distance to be bounded is rather restrictive: after all, the deletion of a vertex (including all of its incident edges) from a graph is often considered an atomic operation and yet could have an arbitrarily large impact on the edge deletion measure. That is why that article proposed to measure the distance between H and G in terms of the *edge+vertex deletion distance*, i.e., the minimum number of vertex- and edge-deletion operations required to obtain H from G. Yet – in spite of providing partial results exploring this notion – the existence of a polynomial-time algorithm for extending partial 1-planar drawings of connected graphs with bounded edge+vertex deletion distance was left as a prominent open question. In this paper, we resolve this open question as follows.

▶ **Theorem 1.** *Let κ be a fixed non-negative integer. Given a graph G, a connected subgraph H of G and a 1-planar drawing $\mathcal{H}$ of H such that H can be obtained from G by a sequence of at most κ vertex and edge deletions, it is possible to determine whether $\mathcal{H}$ can be extended to a 1-planar drawing of G in polynomial time, and if so to compute such an extension.*

Proof Techniques. As the first ingredient for our proof, we use the connectedness of H to obtain a bound on the number of edges in $E(G) \setminus E(H)$ which are pairwise crossing. This allows us to perform exhaustive branching to reduce to the case where all that remains is to insert (a possibly large number of) missing edges incident to at most κ vertices (notably those in $V(G) \setminus V(H)$) and where we can assume that these remaining missing edges are pairwise non-crossing. While this step would seem to represent a significant simplification of the problem, it in fact merely exposes its most challenging part. This reduction step is described in Section 3.

Next, in Section 4 we analyze the structure of a hypothetical solution in order to partition each cell[1] containing at least one missing vertex into *base regions*. Intuitively, base regions correspond to a part of a cell which "belongs" to a certain missing vertex, in the sense that edges incident to other missing vertices may only interact with a base region in a limited way (but may still be present). We show that every solution has at most $\mathcal{O}(\kappa^3)$ many base regions, whose boundaries are each determined by the drawing of at most two edges. This allows us to apply a further branching step to identify the boundaries of such base regions.

Third, we show how to subdivide and mark base regions as well as other cells as reserved for drawings of edges that are incident to one of at most two specific added vertices in Section 5. The marked cells can be appropriately grouped together into a bounded number of independent subinstances of a restricted problem, where each such subinstance has the crucial property that it only contains missing edges that are incident to its two assigned vertices and must be routed via the subdivided base regions allocated to the subinstance.

To complete the proof, Section 6 provides an algorithm that can solve the independent subinstances obtained as above. The algorithm expands on the previously developed algorithm for the case of $\kappa = 2$ [20, Section 6] to deal with some added difficulties arising from the fact that the subinstances may geometrically interfere with one another in the plane.

Related Work. The definition of 1-planarity dates back to Ringel (1965) [37] and since then the class of 1-planar graphs has been of considerable interest in graph theory, graph drawing and (geometric) graph algorithms, see the recent annotated bibliography on 1-planarity by Kobourov et al. [29] collecting 143 references. More generally speaking, interest in various classes of beyond-planar graphs (not limited to, but including 1-planar graphs) has steadily been on the rise [17, 24] in the last decade.

Our recent work on the extension problem for 1-planar graphs [20] established the *fixed-parameter tractability* [14, 19] of the problem when parameterized by the edge deletion distance between H and G. The proof of that result heavily relied on the fact that the total number of edge crossings introduced by adding the missing edges was upper-bounded by the number of added edges. In particular, this made it possible to define an auxiliary graph H' of bounded treewidth that captured information about the partial drawing $\mathcal{H}$, whereas the extension problem could then be encoded as a formula in Monadic Second Order Logic over H'. At that point, the problem could be solved by invoking Courcelle's Theorem [13].

The same paper used an extension of this idea to solve the extension problem for the more restrictive IC-planar graphs [1, 5, 31] with respect to the vertex+edge deletion distance – the key distinction here is that while adding κ vertices to an incomplete IC-planar drawing can only create κ new crossings, adding just two vertices to an incomplete 1-planar drawing may require an arbitrarily large number of new crossings. As a final result, the paper provided a polynomial-time algorithm that resolved the special case of adding two vertices into a 1-planar drawing; the core of this algorithm relied on dynamic programming and case analysis. A slightly generalized version of this algorithm is also used as a subroutine in the last part of our proof in this paper.

Other related work also studied extension problems of partial representations (other than drawings) for specific graph classes [3, 4, 6, 8–12, 15, 25–28, 32, 36].

[1] Cells can be viewed as the analogue of faces in 1-planar drawings.

2 Preliminaries

Graphs and Drawings in the Plane. We refer to the standard book by Diestel for basic graph terminology [18]. For a simple graph G, let $V(G)$ be the set of its vertices and $E(G)$ the set of its edges.

A *drawing* $\mathcal{G}$ of G in the plane $\mathbb{R}^2$ is a function that maps each vertex $v \in V(G)$ to a distinct point $\mathcal{G}(v) \in \mathbb{R}^2$ and each edge $e = uv \in E(G)$ to a simple open curve $\mathcal{G}(e) \subset \mathbb{R}^2$ with endpoints $\mathcal{G}(u)$ and $\mathcal{G}(v)$. For ease of notation we often identify a vertex v and its drawing $\mathcal{G}(v)$ as well as an edge e and its drawing $\mathcal{G}(e)$. Throughout the paper we will assume that: (i) no edge passes through a vertex other than its endpoints, (ii) any two edges intersect in at most one point, which is either a common endpoint or a proper *crossing* (i.e., edges cannot touch), and (iii) no three edges cross in a single point. For a drawing $\mathcal{G}$ of G and $e \in E(G)$, we use $\mathcal{G} - e$ to denote the drawing of $G - e$ obtained by removing the drawing of e from $\mathcal{G}$, and for $J \subseteq E(G)$ we define $\mathcal{G} - J$ analogously.

We say that $\mathcal{G}$ is *planar* if no two edges $e_1, e_2 \in E(G)$ cross in $\mathcal{G}$; if the graph G admits a planar drawing, we say that G is planar. A planar drawing $\mathcal{G}$ subdivides the plane into connected regions called *faces*, where exactly one face, the *outer* (or *external*) face is unbounded. The *boundary* of a face is the set of edges and vertices whose drawings delimit the face. Further, $\mathcal{G}$ induces for each vertex $v \in V(G)$ a cyclic order of its neighbors by using the clockwise order of its incident edges. This set of cyclic orders is called a *rotation scheme*. Two planar drawings $\mathcal{G}_1$ and $\mathcal{G}_2$ of the same graph G are *equivalent* if they have the same rotation scheme and the same outer face; equivalence classes of planar drawings are also called *embeddings*. A *plane* graph is a planar graph with a fixed embedding. For a plane graph, its *dual graph* is defined by introducing a vertex for each face, and connecting two faces by an edge, whenever they are adjacent, i.e. share an edge on their boundary.

A drawing $\mathcal{G}$ is *1-planar* if each edge has at most one crossing and a graph G is *1-planar* if it admits a 1-planar drawing. Similarly to planar drawings, 1-planar drawings subdivide the plane into connected regions, which we call *cells* in order to distinguish them from the faces of a planar drawing. The *planarization* $G^\times$ of a 1-planar drawing $\mathcal{G}$ of G is a graph $G^\times$ with $V(G) \subseteq V(G^\times)$ that introduces for each crossing γ of $\mathcal{G}$ a *dummy vertex* $v_\gamma \in V(G^\times)$ and that replaces each pair of crossing edges uv, wx in $E(G)$ by the four *half-edges* $uv_\gamma, vv_\gamma, wv_\gamma, xv_\gamma$ in $E(G^\times)$, where γ is the crossing of uv and wx. In addition all crossing-free edges of $E(G)$ belong to $E(G^\times)$. Obviously, $G^\times$ is planar and the drawing $\mathcal{G}^\times$ of $G^\times$ corresponds to $\mathcal{G}$ with the crossings replaced by the dummy vertices.

Extending 1-Planar Drawings. Given a graph G and a subgraph H of G with a 1-planar drawing $\mathcal{H}$ of H, we say that a drawing $\mathcal{G}$ of G is an *extension* of $\mathcal{H}$ (to the graph G) if the planarization $H^\times$ of $\mathcal{H}$ and the planarization $\mathcal{G}^\times$ of $\mathcal{G}$ restricted to $\mathcal{H}^\times$ have the same embedding. We can now define our problem of interest.

> 1-PLANAR DRAWING EXTENSION
> *Instance:* A graph G, a connected subgraph H of G, and a 1-planar drawing $\mathcal{H}$ of H.
> *Task:* Find a 1-planar extension of $\mathcal{H}$ to G, or correctly identify that there is none.

A brief discussion about the requirement of H being connected is provided in the Concluding Remarks.

A *solution* of an instance $(G, H, \mathcal{H})$ of 1-PLANAR DRAWING EXTENSION is a 1-planar drawing $\mathcal{G}$ of G that is an extension of $\mathcal{H}$. We refer to $V_{\text{add}} := V(G) \setminus V(H)$ as the *added vertices* and to $E_{\text{add}} := E(G) \setminus E(H)$ as the *added edges*. Furthermore, we let

E^H_{add} be the set of added edges whose endpoints are already part of the drawing, i.e., $E^H_{\text{add}} := \{vw \in E_{\text{add}} \mid v, w \in V(H)\}$. It is worth noting that, without loss of generality, we may assume each vertex in V_{add} to be incident to at least one edge in E_{add}.

Since 1-Planar Drawing Extension is NP-complete (and remains NP-complete even if all added edges have at least one endpoint that can be placed freely, i.e., if $E^H_{\text{add}} = \emptyset$ [20]), it is natural to strive for efficient algorithms for the case where $\mathcal{H}$ is nearly a "complete" drawing of G. Deletion distance represents a natural and immediate way of quantifying this notion of completeness. Here, we consider the *vertex+edge deletion distance* κ between H and G, formalized as $\kappa = |V_{\text{add}}| + |E^H_{\text{add}}|$. We note that the vertex+edge deletion distance is in general smaller than the edge deletion distance between H and G.

The remainder of the paper is dedicated to a proof of Theorem 1, which is achieved by developing a polynomial-time algorithm for 1-Planar Drawing Extension when the vertex+edge deletion distance κ between H and an n-vertex graph G is bounded by a fixed constant.

3 Initial Branching

In this section, we introduce the first ingredient for our proof: exhaustive branching over the choice in which cell of $\mathcal{H}$ every vertex in V_{add} will lie in an extension, the drawings of edges in E^H_{add}, and the drawings of remaining added edges which cross another added edge. In order to perform the last step in polynomial time, we obtain a bound on the number of edges in E_{add} which are pairwise crossing. This leaves us with a 1-planar drawing $\mathcal{H}'$ of some graph H' with $H \subseteq H' \subseteq G$ such that $E'_{\text{add}} = E_{\text{add}} \setminus E(H')$ and for every edge $uv \in E'_{\text{add}}$ either $u \in V_{\text{add}}$ or $v \in V_{\text{add}}$ in each branch. We now provide the details of how all of this is done.

First, note that the number of faces of a planarized 1-planar drawing is linearly bounded in the number of vertices of the original graph [34]. Consequently, we can exhaustively branch on the choice of cells of $\mathcal{H}$ containing the drawings of added vertices in $n^{\mathcal{O}(\kappa)}$ steps. Recall that once we have decided into which cell of $\mathcal{H}$ each added vertex is embedded, the exact position of its embedding is irrelevant in terms of extensibility to G [35]. Since κ is a fixed constant, this polynomial-time procedure reduces our initial problem to the problem of finding an extension of $\mathcal{H} + \mathcal{V}$, where $\mathcal{V}$ is an embedding of V_{add} into cells of $\mathcal{H}$, to a 1-planar drawing of G.

In the next step we branch over the placement of some edges in E_{add}. To this end, consider the structure of a 1-planar extension of $\mathcal{H} + \mathcal{V}$ to a drawing of G and observe that the drawing of an added edge $e \in E_{\text{add}}$ might:

(1) cross the drawing of at most one different edge in E_{add},
(2) cross the drawing of at most one edge of H, or
(3) not cross any edge in $E(G)$.

We now show that the number of crossings arising from the first case can be bounded by a function of κ:

▶ **Lemma 2.** *In any extension of $\mathcal{H} + \mathcal{V}$ to a 1-planar drawing of G there are at most $|E^H_{add}| + 3|V_{add}|^2$ crossings between pairs of edges from E_{add}.*

▶ **Remark 3.** The claim of Lemma 2 does not hold if we allow H to be disconnected. Indeed, Figure 1 illustrates how to construct a series of instances with $|V_{\text{add}}| = 2$ that require $\mathcal{O}(|V(H)|)$ pairwise crossings between edges in E_{add} in *every* solution.

Lemma 2 allows us to apply exhaustive branching to determine which edges will be crossed, which cell they will be crossed in, and how the previously placed vertices in V_{add}

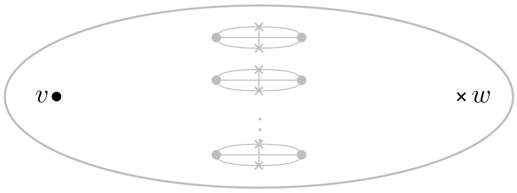

Figure 1 Example for Remark 3. $\mathcal{H}$ is gray, $\mathcal{V} = \{v, w\}$, and E_{add} connects v to vertices in H marked by $\bullet$ and w to vertices in H marked by $\times$.

will be distributed to the new cells created by adding these edges. Moving on, the number of added edges with both endpoints in V_{add} or both endpoints in $V(H)$ is easily seen to be bounded by κ^2. This means that for each such edge e we can once again apply exhaustive branching to determine which of the edges already present in $\mathcal{H}$ at this point, if any, e crosses; if e does not cross another edge of $\mathcal{H}$ then we will also branch to determine which cell e should be drawn in. This, too, requires us to branch on how the vertices in V_{add} will be distributed to the cells created by these newly placed edges. Similarly, we connect all connected components which are not yet connected to H in the partially drawn graph so far by branching on the drawings of at most κ further edges to achieve a drawing of a subgraph of G, each of whose connected components is either connected to H or all of whose edges are drawn. Connected components of the latter type are no longer relevant for extending the respective partial drawing to G since they contain no endpoints of missing edges and can play no role in separating such endpoints, which is why we omit them from all further considerations.

After performing these steps, we are left with a simplified problem which we formally define below.

> UNTANGLED κ-BOUNDED 1-PLANAR DRAWING EXTENSION
> *Instance:* A graph G, a connected subgraph H' of G with $V(H') = V(G)$ and with at most κ marked vertices such that every edge in $E(G) \setminus E(H')$ has precisely one marked endpoint, and a 1-planar drawing $\mathcal{H}'$ of H'.
> *Task:* Find an untangled 1-planar extension of $\mathcal{H}'$ to G, or correctly identify that there is none.

Here, a 1-planar extension of $\mathcal{H}'$ to G is *untangled* if edges in $E(G) \setminus E(H')$ are mutually non-crossing. Note that the marked vertices in the problem statement are precisely the vertices in V_{add}. By applying the abovementioned branching rules, we obtain:

▶ **Corollary 4.** *For every fixed κ, there is a $n^{\mathcal{O}(\kappa^3)}$-time Turing reduction from 1-PLANAR DRAWING EXTENSION restricted to instances of bounded κ to UNTANGLED κ-BOUNDED 1-PLANAR DRAWING EXTENSION.*

The following observation about the obtained instances of UNTANGLED κ-BOUNDED 1-PLANAR DRAWING EXTENSION will be useful later on.

▶ **Observation 5.** *The degree of every vertex in V_{add} in the graph H' lies in $\mathcal{O}(\kappa^2)$.*

4 Base Regions

Let us now consider an instance $(G, H', \mathcal{H}')$ obtained by Corollary 4. To simplify terminology we refer to edges in $E(G) \setminus E(H')$ as *new edges.*

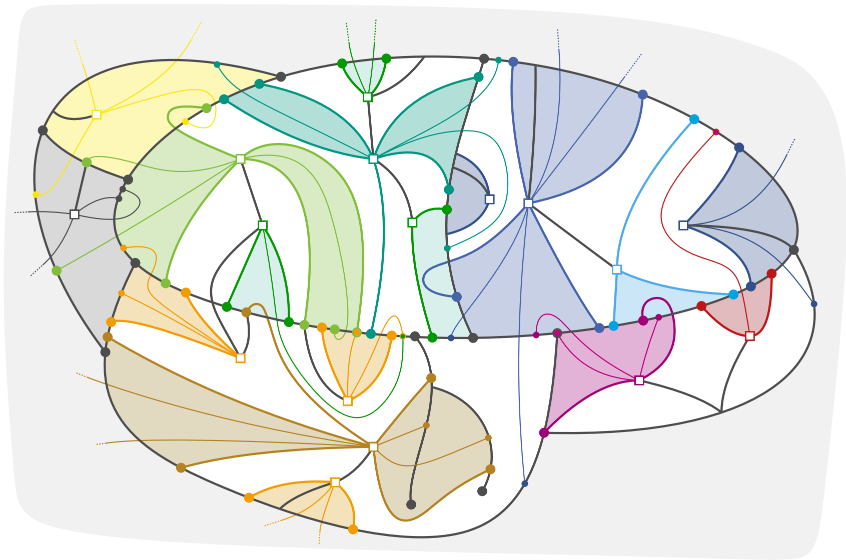

Figure 2 Illustration of Definition 6. Part of $\mathcal{H}'$ is dark-gray, the rest of $\mathcal{H}'$ is indicated by the light-gray background. Vertices in V_{add} are the square marks. Thick colored edges bound the base regions and thin edges are inserted into them. E.g. an edge that intersects a base region of a vertex other than its marked endpoint (see Remark 7) is the orange edge that intersects a green base region.

We will now show that in the planarization of any hypothetical untangled 1-planar extension $\mathcal{G}$ of $\mathcal{H}'$ to G we can identify parts of cells of $\mathcal{H}'$ which only contain parts of drawings of edges in $E(\mathcal{G}^\times) \setminus E(H')$ that are incident to a specific $v \in V_{\text{add}}$. We will call such subsets *base regions* and associate each region to the corresponding $v \in V_{\text{add}}$. Intuitively, base regions of v determine designated areas in which new edges incident to v can start.

▶ **Definition 6.** *A* base region *of some $v \in V_{add}$ in $\mathcal{G}$ is an inclusion maximal connected subset of a cell of $\mathcal{H}'$ containing v, which*
- *does not contain v;*
- *is bounded by parts of $\mathcal{H}'$ and drawings of edges in $E(\mathcal{G}^\times) \setminus E(\mathcal{H}'^\times)$ which are incident to v;*
- *contains the drawing of at least one edge in $\mathcal{G}^\times$ which is incident to v; and*
- *contains no drawing of an edge in $E(G^\times) \setminus E(H'^\times)$ which is incident to some $w \in V_{add} \setminus \{v\}$.*

▶ **Remark 7.** An illustration of Definition 6 is provided in Figure 2; notice that drawings of new edges of G (not $G^\times$) with marked endpoints different from v can still intersect the interior of the base region of v.

Fixing the boundaries of base regions of all vertices in V_{add} to find a hypothetical solution with these base regions determines which edges of H' can be crossed to draw new edges incident to each region-specific vertex. However, base regions do not give explicit structural restrictions on the drawings of new edges beyond the point at which they cross edges of H'.

▶ **Remark 8.** The following basic facts about base regions are easily seen:
1. $v \in V_{\text{add}}$ only has base regions in cells containing v (on their boundary or their interior).
2. Two base regions intersect only in the boundary of a cell of $\mathcal{H}'$.

Our aim for the remainder of this section is to show that we can branch to determine the boundaries of base regions. To this end, it suffices to show that the number of base regions

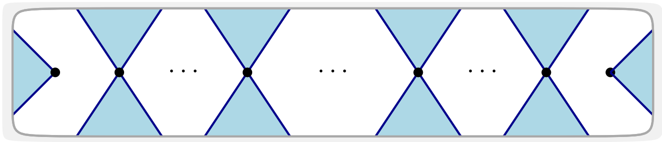

Figure 3 Example for Remark 11. $\mathcal{H}'$ is gray, V_{add} is black and potential base regions are in blue.

is bounded by a function of κ. First, we prove an auxiliary proposition that we then use to show that the number of base regions in each cell of $\mathcal{H}'$ lies in $\mathcal{O}(\kappa)$.

▶ **Proposition 9.** *In every untangled 1-planar extension of $\mathcal{H}'$ to G, each cell of $\mathcal{H}'$ that contains an added vertex contains at least one added vertex that has precisely one base region in that cell.*

▶ **Proposition 10.** *In every untangled 1-planar extension $\mathcal{G}$ of $\mathcal{H}'$ to G, the total number of base regions in every cell of $\mathcal{H}'$ is at most* $\max(1, 2(\kappa - 1))$.

▶ **Remark 11.** The bound in the proof of Proposition 10 is tight; see Figure 3.

In combination with Point 1 of Remark 8 and the degree bound given in Observation 5, we obtain the following.

▶ **Lemma 12.** *The total number of base regions in any untangled 1-planar extension of $\mathcal{H}'$ to G lies in $\mathcal{O}(\kappa^3)$.*

Observe that every base region is bounded by $\mathcal{H}'$ and at most two new edges (since these are each incident to the same vertex in V_{add}, which is not included in the base region and thus cannot be used for connectivity). Hence Lemma 12 allows us to branch on the drawings of the edges that bound base regions in polynomial time, in a similar fashion as the branching carried out in Section 3. In each branch, our aim will be to decide whether the arising 1-planar drawing $\mathcal{H}''$ of H'' (where $H' \subseteq H'' \subseteq G$) can be extended to an untangled 1-planar drawing of G with an additional restriction: notably, in the planarization of such an extension, the newly drawn edges immediately incident to V_{add} are all drawn in $\mathcal{O}(\kappa^3)$ distinguished *base cells* of $\mathcal{H}''$. These base cells are defined as the cells of $\mathcal{H}''$ corresponding to the base regions identified in the given branch.

5 Interactions between Base Cells

Let $\mathcal{H}''$ be an extension of $\mathcal{H}'$ obtained from the previous step described in Section 4. From now on, we refer to edges in $E(G) \setminus E(H'')$ as *new edges*. Recall that we still want to find an untangled 1-planar extension. Additionally, for each $v \in V_{\text{add}}$, we have identified a set $\mathfrak{B}_v$ of base cells, and we require every new edge incident to v to start in one such base cell. Here, we say that a new edge e *starts* in or *exits* through the cell in which, for every arbitrarily small ε, the points on the drawing of e lie at an ε-distance from the unique endpoint of e in V_{add}. We call extensions satisfying this property *based untangled 1-planar*. Furthermore, recall that $\left|\bigcup_{v \in V_{\text{add}}} \mathfrak{B}_v\right| \in \mathcal{O}(\kappa^3)$.

As noted in Remark 7, edges can cross the boundary of base cells (i.e., "cross out" of the base cell they started in) and enter other base cells or cells containing edges from multiple base regions. This makes resolving the remaining 1-planar Extension problem non-obvious. In this and the next section we apply a two-step approach to deal with this issue and complete the proof of our main result: first, we subdivide $\mathcal{H}''$ into parts where only two base cells interact and which can be solved independently (Subsections 5.1 and 5.2), and then we use a dynamic programming algorithm to directly solve each such independent part (Section 6). Note that the newly added edges could subdivide some of the base cells, technically not making them cells in the extended drawing anymore – however, we always use the term "base cell" to refer to the original base cells for all marking procedures.

5.1 Isolating Base Cell Pair Interactions

Our goal here is to somewhat separate interactions of edges starting in many different base cells. To achieve this we aim to reach a state where each cell is "assigned" to at most two base cells, meaning that only edges starting in these base cells can interact in the respective cell.

We begin with an important definition that will be used throughout this subsection.

▶ **Definition 13.** *A cell c of $\mathcal{H}''$ is* accessible *from a base cell $\mathfrak{b} \in \mathfrak{B}_v$ of some $v \in V_{add}$, if an edge from v to a vertex on the boundary of c can be inserted into $\mathcal{H}''$ in a 1-plane way, such that before crossing another edge, it is drawn within $\mathfrak{b}$, and some part of the edge is drawn within c.*

▶ **Observation 14.** *A cell of $\mathcal{H}''$ that is not a neighbor of a base cell $\mathfrak{b}$ in the dual graph of $\mathcal{H}''^{\times}$ is not accessible from $\mathfrak{b}$.*

Our next aim will be to bound the number of cells of $\mathcal{H}''$ that are accessible from three different base cells. To do this, we will use the following lemma which is an immediate consequence of the well-known fact that planar graphs have bounded expansion and Point 2 of Lemma 4.3 of previous work by Gajarský et al. [21].

▶ **Lemma 15** ([21]). *Let $G = (X \cup Y, E)$ be a planar bipartite graph with parts X and Y. Then there are at most $\mathcal{O}(|X|)$ distinct subsets $X' \subseteq X$ such that $X' = N(u)$ for some $u \in Y$.*

Using Observation 14 and Lemma 15, we prove the following.

▶ **Proposition 16.** *There are at most $\mathcal{O}(\kappa^3)$ cells of $\mathcal{H}''$ which are accessible from three different base cells.*

Proposition 16 allows us to employ a more detailed branching procedure on the structure of a hypothetical solution in cells which are accessible from many (notably, at least three) base cells. Our aim is to divide every cell of $\mathcal{H}''$ that is accessible from at least three base cells into parts that delimit interactions of pairs of vertices. This will then allow us to treat the subcells resulting from this division as cells that are accessible from only two added base cells. We note that a hypothetical solution will not induce a unique division of a cell into such parts (in contrast to base regions, which are delimited by edges of $\mathcal{H}$ and hence uniquely determined).

For the remainder of this subsection, let c be a cell of $\mathcal{H}''$ that is accessible from at least three base cells and $\mathcal{G}$ be a based untangled 1-planar extension of $\mathcal{H}''$ to G. We proceed as follows: First, traverse the boundary of the face of $\mathcal{H}''^{\times}$ that corresponds to c, starting from an arbitrary vertex in counterclockwise direction (vertices may appear twice). Let the obtained ordering be given by $v_1, \dots, v_\ell$. Mark each encountered $v \notin V_{\text{add}}$ with each base cell for which v is the endpoint of an edge in $E(\mathcal{G}^{\times})$ which arises from an edge in $E(G) \setminus E(H'')$ that starts in that base cell (see Figure 4 for an example). Note that we mark vertices of a planarized drawing $\mathcal{G}^{\times}$. In particular crossing vertices are also marked. To avoid confusion, we also call attention to the fact that this marking procedure is only defined with respect to a hypothetical solution; keeping that in mind, our next task is to obtain a bound on the number of vertices marked with more than two base cells.

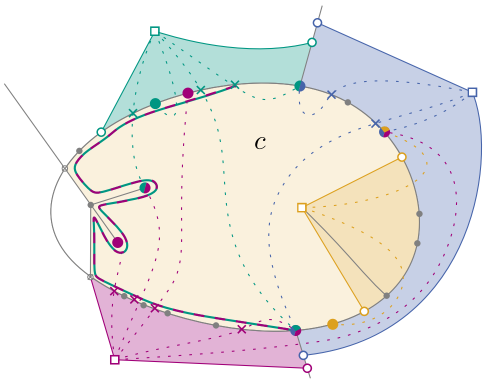

Figure 4 Illustration of the marking of one cell c in $\mathcal{H}''$. Square vertices are in V_{add} and their base cells are marked in the corresponding colors. Filled disks are their neighbors, to which the edges are not yet in $\mathcal{H}''$ on the boundary of c. Crosses and filled disks receive at least one marker. The green-purple curve marks a stretch of the two corresponding base cells. Note that the dashed lines represent a possible 1-planar drawing; not all edges are drawn in c.

▶ **Proposition 17.** *There are at most $\mathcal{O}(\kappa^3)$ vertices of $\mathcal{H}''$ which are marked with at least three different base cells.*

Proposition 17 allows us to branch on the drawings of all missing edges incident to vertices which are marked with three more different base cells, and insert them into $\mathcal{H}''$. Note that this operation could subdivide some cells which are not base cells; whenever that happens, we recompute the accessibility of the new cells, and we observe that the bound given in Proposition 16 still applies. On the other hand, the newly added edges could subdivide some of the base cells, technically not making them cells in the extended drawing anymore – in this case, we still use the term "base cell" to refer to the original base cells for all further marking and labeling procedures.

After adding the above edges in a branch, we can assume that every vertex is marked by at most two base cells. One can divide the boundary of c into connected parts, on which only a specific pair of added vertices appears in the markers.It is then possible to show that the number of such connected parts is bounded, which allows us to finally identify a set of edges that we can later branch on to reach a state where each cell can be "assigned" to at most two designated base cells.

▶ **Lemma 18.** *Let c be a cell of $\mathcal{H}''$. There exist a set $F \subseteq (E(G) \setminus E(H''))$ of at most $\mathcal{O}(\kappa^6)$ new edges such that if $\mathcal{H}''_F$ denotes the restriction of $\mathcal{G}$ to $H + F$, then for every cell $c' \subseteq c$ of $\mathcal{H}''_F$ there exist at most two base cells $\mathfrak{b}_1^{c'}, \mathfrak{b}_2^{c'} \in \bigcup_{v \in V_{add}} \mathfrak{B}_v$ such that all new edges that intersect c' in $\mathcal{G}$ start either in $\mathfrak{b}_1^{c'}$ or in $\mathfrak{b}_2^{c'}$.*

We now recall that there are at most $\mathcal{O}(\kappa^3)$ cells accessible from at least three base cells, and for each such cell c we will branch to determine a set F_c of at most $\mathcal{O}(\kappa^6)$ edges. We will proceed by assuming that the set F_c is precisely the set of edges obtained by applying Lemma 18 on a hypothetical solution $\mathcal{G}$. After accounting for some minor technicalities, this gives rise to a branching factor of $n^{\mathcal{O}(\kappa^9)}$.

As a consequence, we will proceed under the assumption that all cells have already been marked by at most two base cells and every edge that intersects a cell starts in one of the two base cells in the marked set for the cell. However, it is still not possible to cleanly "split" an instance into subinstances that only consist of 2 base cells: the remaining issue is that a

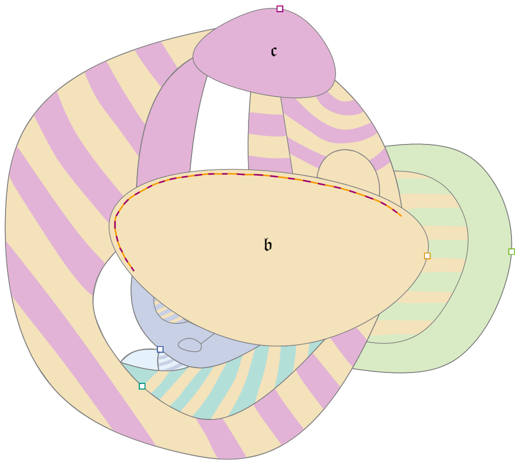

Figure 5 Illustration of interfaces. Square vertices are in V_{add}, and cells are hatched in colors corresponding to a potential accessibility-marking, where every color corresponds to one base cell and a cell is marked as accessible from a base cell if and only if it contains its color. The orange-purple curve marks a $\mathfrak{c}$-interface from the viewpoint of $\mathfrak{b}$.

vertex w on the boundary between cells assigned to different base cells may still be accessed from multiple cells. The number of times such a situation may occur is *not* bounded by a function of κ, and hence a simple branching will not suffice here; the next subsection is dedicated to resolving this obstacle.

5.2 Grouping Interactions

Let $\mathcal{H}'''$ be an extension of $\mathcal{H}''$ obtained from the previous step described in Section 5.1. Recall that we assume that each cell of $\mathcal{H}'''$ is accessible from at most two base cells.

At this point we are roughly in a situation where we could apply our dynamic programming techniques from Section 6, namely having to consider only interactions between at most two vertices at a time. However we cannot apply such techniques for each cell separately as the cells are not independent. More specifically neighbors of vertices in V_{add} on the boundary of multiple cells which are accessible to that added vertex can potentially be reached through any of these cells; which cell is chosen impacts other edges that can be drawn into that cell, hence possibly forcing them to be drawn into other cells.

We can now employ a similar argument as before Lemma 18 to group cells that are consecutive along the boundary of a base cell. For the remainder of this section, let us consider a base cell $\mathfrak{b}$. One can divide the boundary of $\mathfrak{b}$ into connected parts, which only separate $\mathfrak{b}$ from cells marked as accessible from $\mathfrak{b}$ and some specific second base cell or are not marked as accessible from $\mathfrak{b}$. For a base cell $\mathfrak{c}$, we refer to such parts as $\mathfrak{c}$-*interfaces*, and just *interfaces* when we do not want to specify the related base cell. An illustration is provided in Figure 5.

Given a hypothetical solution $\mathcal{G}$, we can design a simple greedy procedure that divides accessible cells of $\mathcal{H}'''$ into interfaces. For the interfaces constructed by this procedure, we can show the following.

▶ **Lemma 19.** *Let $\mathfrak{c}$ be a base cell. After the described procedure, there are at most $\mathcal{O}(\kappa^3)$ $\mathfrak{c}$-interfaces.*

We now formalize the subproblem that captures the case of two added vertices which we will obtain at the end of this section. The subproblem is derived on interfaces for every pair of base cells, and will be solved in the next section.

> 2-SUBDIVIDED BASE CELL ROUTING WITH OCCUPIED CELLS (2-SBCROC)
> *Instance:* A graph S, a connected subgraph T of S with $V(S) = V(T)$ and two vertices x and y and a 1-planar drawing $\mathcal{T}$ of T, with some cells marked as *occupied*, and there is a simple x-*walk* along boundaries of cells of $\mathcal{T}$ that have x on its boundary, and there is a simple y-*walk* along boundaries of different cells of $\mathcal{T}$ that have y on its boundary. Moreover each edge $E(S) \setminus E(T)$ has precisely one endpoint among $\{x, y\}$.
> *Task:* Find an untangled 1-planar extension of $\mathcal{T}$ to S such that no drawing of an edge in $E(S) \setminus E(T)$ intersects the interior of an occupied cell of $\mathcal{T}$, and every edge of T that is crossed by an edge in $E(S) \setminus E(T)$ incident to x lies on the prescribed x-walk, and every edge of T that is crossed by an edge in $E(S) \setminus E(T)$ incident to y lies on the prescribed y-walk.

This is obviously a restriction of 1-PLANAR DRAWING EXTENSION for $(S, T, \mathcal{T})$ and we let the relevant terminology (e.g. *added edges*) carry over.

To arrive at appropriate subinstances, Lemma 19 and the untangledness of the targeted solution ultimately allows us to branch on delimiting edges for interfaces. Neighbors of vertices in V_{add} where incident edges could still be drawn using a choice of two or more interfaces can be carefully separated and handled either by explicitly branching on the missing drawings of all incident edges, or within a special independent subinstance of UNTANGLED κ-BOUNDED 1-PLANAR DRAWING EXTENSION which can be solved using [20, Corollary 17]. In this way we can show:

▶ **Lemma 20.** *For every fixed κ, there is a $n^{\mathcal{O}(\kappa^{28})}$-time Turing reduction from the problem of finding a based untangled 1-planar extension of $\mathcal{H}'''$ conforming to the current branch to 2-SBCROC.*

6 1-Plane Routing of Two Vertices

In this section we give an algorithm to solve 2-SBCROC for an arbitrary instance $(S, T, \mathcal{T})$, which ultimately allows us to prove our main result.

The idea of the algorithm to solve 2-SBCROC is for the most part the same as in [20, Section 6]. In particular our algorithm employs a carefully designed dynamic "delimit-and-sweep" approach to iteratively arrive at situations which can be reduced to a network-flow problem to which standard maximum-flow algorithms can be applied. However several adaptations can, or have to, be made due to the fact that the instances considered here have a slightly different structure (e.g. this algorithm also handles cases, where x and y lie on the boundaries of multiple cells) than the ones considered for the special case considered in [20] and are also more general (e.g. occupied cells have to be taken into acount). In the following, we include a very high-level description of our algorithm for 2-SBCROC.

Our procedure relies on the following considerations: Let λ be a function from the cells of $\mathcal{T}$ to $\{x, y, \emptyset\}$, that maps every occupied cell to $\emptyset$. We say a 1-planar extension of $\mathcal{T}$ to S is λ-*consistent* if whenever the drawing of any edge in $E(S) \setminus E(T)$ which is incident to x intersects the interior of cell c of $\mathcal{T}$ which is not a starting cell of x, then $\lambda(c) = x$, and correspondingly whenever the drawing of any edge in $E(S) \setminus E(T)$ which is incident to y intersects the interior of cell c of $\mathcal{T}$ which is not a starting cell of y, then $\lambda(c) = y$

(i.e., λ restricts the drawings of which added edges may enter which face). Note that a λ-consistent drawing is always untangled. For a given λ it can be shown that we can either find a λ-consistent solution of $(S, T, \mathcal{T})$ for 2-SBCROC or decide that there is none, by constructing an equivalent network flow problem.

▶ **Lemma 21.** *Given λ as above, it is possible to determine whether there exists a λ-consistent solution of $(S, T, \mathcal{T})$ for 2-SBCROC in polynomial time.*

Observe that for a hypothetical 1-planar extension $\mathcal{S}$ of $\mathcal{T}$ to S, λ such that $\mathcal{S}$ is a λ-consistent extension of $\mathcal{T}$ does necessarily exist. This is because some cells of $\mathcal{T}$ might need to be further subdivided, before they can be completely assigned to x or y by any λ. Hence the aim of our dynamic program is to branch on the additional drawings of some edges into $\mathcal{T}$ such that for this extended drawing $\mathcal{T}$ we are able to iteratively extend an assignment λ for which we iteratively extend a λ-consistent extension of $\mathcal{T}$ to an extension $\mathcal{S}$ on S, by applying Lemma 21. For this our dynamic program proceeds along the x-walk and the y-walk simultaneously. Each step of the dynamic program is described by a *record* which, when well-formed encodes a so called *delimiter* D. Intuitively, D is a simple curve from x to y that separates the instance into two subinstances such that

1. the drawing of every added edge in the hypothetical solution of $(S, T, \mathcal{T})$ does not intersect D, i.e. lies completely on one side of D, and
2. we have an assignment λ that maps all cells, the last vertex on the x-walk and the y-walk of which occurs before D on the traversals of the x-walk and the y walk, to $\{x, y, \emptyset\}$.

In this sense D indicates, at which stage of the dynamic program we are; the cells "before" D on the traversals of the x-walk and the y-walk are already sufficiently processed to apply Lemma 21, and we still need to achieve this for the remaining cells. The cases used to define the delimiter are nearly identical to those used in [20, Section 6]. Using the fact that the maximum size of the records is polynomial, and the possibility of transitioning from one record to the next can be checked in polynomial time, we can show:

▶ **Lemma 22.** *2-SBCROC can be solved in $\mathcal{O}(|V(T)|^{12})$.*

We are now ready to formally prove our main theorem.

Sketch of Proof of Theorem 1. By Corollary 4, we can reduce the instance $(G, H, \mathcal{H})$ in $n^{\mathcal{O}(\kappa^3)}$ time into $n^{\mathcal{O}(\kappa^3)}$ many instances of Untangled κ-Bounded 1-Planar Drawing Extension such that $(G, H, \mathcal{H})$ is yes-instance if and only if at least one of these instances is yes-instance. Now let $(G, H', \mathcal{H}')$ be one of these instances. By Lemma 12, there are at most $\mathcal{O}(\kappa^3)$ base regions in any untangled 1-planar extension of $\mathcal{H}'$, and hence we can branch to determine these base regions and their delimiting edges to create the corresponding base cells. We let the resulting graph and its drawing be H'' and $\mathcal{H}''$, respectively.

By Proposition 17, the number of vertices v in $V(G) \setminus V_{\text{add}}$ such that the new edges incident to v start in more than two different base regions in any based untangled 1-planar extension of $\mathcal{H}''$ is bounded by $\mathcal{O}(\kappa^3)$, and hence these can be added to the drawing by branching. Moreover, there are at most $\mathcal{O}(\kappa^3)$ cells that are accessible by three base cells. By Lemma 18, for each such cell c there is a set of at most $\mathcal{O}(\kappa^6)$ new edges such that if we draw these edges, then each subcell of c intersects only new edges that start in two different base cells. By branching on the drawings of these edges for each of $\mathcal{O}(\kappa^3)$ many cells and branching on the assignment of at most two accessible base cells for each newly created cell, we obtain a set of $n^{\mathcal{O}(\kappa^9)}$ new subinstances. In each obtained subinstance, every cell c contains a set of two markers (base cells) $\nu(c)$ and we are only looking for a solution such that if a drawing of a new edge e intersects c, then e starts in one of the base cells in $\nu(c)$.

Finally, the goal is to group the cells marked by the same pair of base cells together so that we can solve the resulting instances separately. This is carried out by applying Lemma 20, which produces a set of subsintances that we solve using Lemma 22. Altogether, this results in a running time of at most $n^{\mathcal{O}(\kappa^{28})}$. ◀

7 Concluding Remarks

In this paper we have presented a constructive polynomial-time algorithm for extending partial 1-planar graph drawings (or report that no such extension exists) in the restricted case that the edge+vertex deletion distance κ between the partial drawing and its extension is bounded. We believe that the most interesting open question in this direction is whether the 1-planar drawing extension problem is fixed-parameter tractable not only w.r.t. the edge deletion distance [20], but also w.r.t. the edge+vertex deletion distance. It would also be worthwhile to obtain an understanding of the complexity of the problem when restricted to instances with natural structural properties, such as having bounded treewidth.

Last but not least, we note that while the requirement on the connectivity of H is well motivated from an application perspective and has also been used in other drawing extension settings [20, 23, 32, 33], the problem of course remains of interest when this requirement is dropped. Our techniques and results do not immediately carry over to the case where H is disconnected, and generalizing the presented results in this direction is an interesting question left for future research.

References

1 Michael Albertson. Chromatic number, independence ratio, and crossing number. *ARS MATHEMATICA CONTEMPORANEA*, 1(1):1–6, 2008. doi:10.26493/1855-3974.10.2d0.

2 Patrizio Angelini, Giuseppe Di Battista, Fabrizio Frati, Vít Jelínek, Jan Kratochvíl, Maurizio Patrignani, and Ignaz Rutter. Testing planarity of partially embedded graphs. *ACM Trans. Algorithms*, 11(4):32:1–32:42, 2015. doi:10.1145/2629341.

3 Alan Arroyo, Martin Derka, and Irene Parada. Extending simple drawings. In Daniel Archambault and Csaba D. Tóth, editors, *27th International Symposium on Graph Drawing and Network Visualization (GD'19)*, volume 11904 of *LNCS*, pages 230–243. Springer, 2019. doi:10.1007/978-3-030-35802-0_18.

4 Alan Arroyo, Fabian Klute, Irene Parada, Raimund Seidel, Birgit Vogtenhuber, and Tilo Wiedera. Extending simple drawings with one edge is hard. *CoRR*, abs/1909.07347, 2019. Accepted for presentation at WG'20. arXiv:1909.07347.

5 Franz J. Brandenburg. Recognizing IC-planar and NIC-planar graphs. *J. Graph Algorithms Appl.*, 22(2):239–271, 2018. doi:10.7155/jgaa.00466.

6 Guido Brückner and Ignaz Rutter. Partial and constrained level planarity. In Philip N. Klein, editor, *Discrete Algorithms (SODA'17)*, pages 2000–2011. SIAM, 2017. doi:10.1137/1.9781611974782.130.

7 Sergio Cabello and Bojan Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013. doi:10.1137/120872310.

8 Erin W. Chambers, David Eppstein, Michael T. Goodrich, and Maarten Löffler. Drawing graphs in the plane with a prescribed outer face and polynomial area. *J. Graph Algorithms Appl.*, 16(2):243–259, 2012. doi:10.7155/jgaa.00257.

9 Timothy M. Chan, Fabrizio Frati, Carsten Gutwenger, Anna Lubiw, Petra Mutzel, and Marcus Schaefer. Drawing partially embedded and simultaneously planar graphs. *J. Graph Algorithms Appl.*, 19(2):681–706, 2015. doi:10.7155/jgaa.00375.

10 Steven Chaplick, Paul Dorbec, Jan Kratochvíl, Mickaël Montassier, and Juraj Stacho. Contact representations of planar graphs: Extending a partial representation is hard. In Dieter Kratsch

and Ioan Todinca, editors, *40th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'14)*, volume 8747 of *LNCS*, pages 139–151. Springer, 2014. doi:10.1007/978-3-319-12340-0_12.

11 Steven Chaplick, Radoslav Fulek, and Pavel Klavík. Extending partial representations of circle graphs. In Stephen K. Wismath and Alexander Wolff, editors, *21st International Symposium on Graph Drawing (GD'13)*, volume 8242 of *LNCS*, pages 131–142. Springer, 2013. doi:10.1007/978-3-319-03841-4_12.

12 Steven Chaplick, Grzegorz Guśpiel, Grzegorz Gutowski, Tomasz Krawczyk, and Giuseppe Liotta. The partial visibility representation extension problem. *Algorithmica*, 80(8):2286–2323, 2018. doi:10.1007/s00453-017-0322-4.

13 Bruno Courcelle. The monadic second-order logic of graphs. I. recognizable sets of finite graphs. *Inf. Comput.*, 85(1):12–75, 1990. doi:10.1016/0890-5401(90)90043-H.

14 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.

15 Giordano Da Lozzo, Giuseppe Di Battista, and Fabrizio Frati. Extending upward planar graph drawings. In Zachary Friggstad, Jörg-Rüdiger Sack, and Mohammad R. Salavatipour, editors, *16th International Symposium on Algorithms and Data Structures (WADS'19)*, volume 11646 of *LNCS*, pages 339–352. Springer, 2019. doi:10.1007/978-3-030-24766-9_25.

16 H. De Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990. doi:10.1007/BF02122694.

17 Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281.

18 Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

19 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013. doi:10.1007/978-1-4471-5559-1.

20 Eduard Eiben, Robert Ganian, Thekla Hamm, Fabian Klute, and Martin Nöllenburg. Extending partial 1-planar drawings. In *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020*, 2020. To appear.

21 Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Sebastian Ordyniak, Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, and Somnath Sikdar. Kernelization using structural parameters on sparse graph classes. *J. Comput. Syst. Sci.*, 84:219–242, 2017. doi:10.1016/j.jcss.2016.09.002.

22 Alexander Grigoriev and Hans L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007. doi:10.1007/s00453-007-0010-x.

23 Seok-Hee Hong and Hiroshi Nagamochi. Convex drawings of graphs with non-convex boundary constraints. *Discrete Applied Mathematics*, 156(12):2368–2380, 2008.

24 Seok-Hee Hong and Takeshi Tokuyama, editors. *Beyond Planar Graphs*. Springer, 2020.

25 Pavel Klavík, Jan Kratochvíl, Tomasz Krawczyk, and Bartosz Walczak. Extending partial representations of function graphs and permutation graphs. In Leah Epstein and Paolo Ferragina, editors, *20th Annual European Symposium on Algorithms (ESA'12)*, volume 7501 of *LNCS*, pages 671–682. Springer, 2012. doi:10.1007/978-3-642-33090-2_58.

26 Pavel Klavík, Jan Kratochvíl, Yota Otachi, Ignaz Rutter, Toshiki Saitoh, Maria Saumell, and Tomás Vyskocil. Extending partial representations of proper and unit interval graphs. *Algorithmica*, 77(4):1071–1104, 2017. doi:10.1007/s00453-016-0133-z.

27 Pavel Klavík, Jan Kratochvíl, Yota Otachi, and Toshiki Saitoh. Extending partial representations of subclasses of chordal graphs. *Theor. Comput. Sci.*, 576:85–101, 2015. doi:10.1016/j.tcs.2015.02.007.

28 Pavel Klavík, Jan Kratochvíl, Yota Otachi, Toshiki Saitoh, and Tomáš Vyskočil. Extending partial representations of interval graphs. *Algorithmica*, 78(3):945–967, 2017. doi:10.1007/s00453-016-0186-z.

29 Stephen G. Kobourov, Giuseppe Liotta, and Fabrizio Montecchiani. An annotated bibliography on 1-planarity. *Computer Science Review*, 25:49–67, 2017. doi:10.1016/j.cosrev.2017.06.002.

30 Vladimir P. Korzhik and Bojan Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *Journal of Graph Theory*, 72(1):30–71, 2013. doi:10.1002/jgt.21630.

31 Giuseppe Liotta and Fabrizio Montecchiani. L-visibility drawings of IC-planar graphs. *Inf. Process. Lett.*, 116(3):217–222, 2016. doi:10.1016/j.ipl.2015.11.011.

32 Tamara Mchedlidze, Martin Nöllenburg, and Ignaz Rutter. Extending convex partial drawings of graphs. *Algorithmica*, 76(1):47–67, 2016. doi:10.1007/s00453-015-0018-6.

33 Kazuo Misue, Peter Eades, Wei Lai, and Kozo Sugiyama. Layout adjustment and the mental map. *J. Visual Languages and Computing*, 6(2):183–210, 1995. doi:10.1006/jvlc.1995.1010.

34 János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.

35 János Pach and Rephael Wenger. Embedding planar graphs at fixed vertex locations. *Graphs and Combinatorics*, 17(4):717–728, 2001. doi:10.1007/PL00007258.

36 Maurizio Patrignani. On extending a partial straight-line drawing. *Int. J. Found. Comput. Sci.*, 17(5):1061–1070, 2006. doi:10.1142/S0129054106004261.

37 Gerhard Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 29(1):107–117, 1965. doi:10.1007/BF02996313.

38 Walter Schnyder. Embedding planar graphs on the grid. In *Symposium on Discrete Algorithms (SODA'90)*, pages 138–148. SIAM, 1990.