Topological Influence and Locality in Swap Schelling Games

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Abstract

Residential segregation is a wide-spread phenomenon that can be observed in almost every major city. In these urban areas residents with different racial or socioeconomic background tend to form homogeneous clusters. Schelling's famous agent-based model for residential segregation explains how such clusters can form even if all agents are tolerant, i.e., if they agree to live in mixed neighborhoods. For segregation to occur, all it needs is a slight bias towards agents preferring similar neighbors. Very recently, Schelling's model has been investigated from a game-theoretic point of view with selfish agents that strategically select their residential location. In these games, agents can improve on their current location by performing a location swap with another agent who is willing to swap.

We significantly deepen these investigations by studying the influence of the underlying topology modeling the residential area on the existence of equilibria, the Price of Anarchy and on the dynamic properties of the resulting strategic multi-agent system. Moreover, as a new conceptual contribution, we also consider the influence of locality, i.e., if the location swaps are restricted to swaps of neighboring agents. We give improved almost tight bounds on the Price of Anarchy for arbitrary underlying graphs and we present (almost) tight bounds for regular graphs, paths and cycles. Moreover, we give almost tight bounds for grids, which are commonly used in empirical studies. For grids we also show that locality has a severe impact on the game dynamics.

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1 Introduction

Today's metropolitan areas are populated by a diverse set of residential groups which differ along ethnical, socioeconomic and other traits. A common finding is that cityscapes are not well-mixed, i.e., the different groups of agents tend to separate themselves into largely homogeneous neighborhoods.[1] This phenomenon is well-known as *residential segregation* and is a subject of study in sociology, mathematics and computer science for at least five decades. The most important scientific model addressing residential segregation was proposed by Schelling [31, 32] who simply considered two types of residential agents who are located on a line or on a checkerboard. Each agent is aware of the agents in her neighborhood and is content with her location, if and only if the fraction of neighbors being of her own type is above the tolerance parameter τ, for some $0 < \tau \leq 1$. Discontent agents simply move to another location. Using this basic model Schelling showed that starting from an initially mixed state over time segregated neighborhoods will emerge. While this is to be expected for high τ, Schelling's finding was that this also happens for tolerant agents, i.e., if $\tau \leq \frac{1}{2}$. Thus, only a slight bias towards favoring similar neighbors leads to the emergence of segregation.

Schelling proposed his model as a random process. This has led to an abundance of empirical studies that simulated this process, see, e.g., [20, 13] and the references to chapter 4 in [16]. In these studies, the commonly used underlying topology for modeling the residential area are grid graphs (often toroidal grids where vertices of borders on opposite sides are identified), paths and cycles. A recent line of work [34, 35, 36, 21, 10, 4, 6, 5, 23, 30] rigorously analyzed variants of this random process on paths or grid graphs and it was shown that residential segregation occurs with high probability. However, in reality agents would not move randomly, instead they would move to a location that maximizes their utility.

To address this selfish behavior, a very recent line of work [14, 18, 17, 1] initiated the study of residential segregation from a game-theoretic point of view. The residential area is modeled as a multi-agent system consisting of selfish agents who occupy vertices of an underlying graph and try to maximize their utility, which depends on the agents' types in their immediate neighborhood, by strategically selecting locations. Also strategic segregation in social network formation was considered [2].

This paper sets out to significantly improve and deepen the results on game-theoretic residential segregation for the model investigated in [1] which allows pairs of discontent agents of different type to swap their locations to maximize their utility. This variant of Schelling's model becomes more and more realistic as in many cities the percentage of vacant housing is below 1%. In such settings, location swaps become the only way for agents to improve on their current housing situation. For the model in [1] we consider the influence of the given topology that models the residential area on core game-theoretic questions like the existence of equilibria, the Price of Anarchy and the game dynamics. We thereby focus on popularly studied topologies like grids, paths and cycles. Moreover, we follow-up on a proposal by Schelling [32] to restrict the movement of agents locally and we investigate the influence of this restriction. Such local swaps are realistic since people want to stay close to their working place or important facilities like schools. This also holds when considering dynamics where agents repeatedly perform local moves since these dynamics can be understood as a process which happens over a long timespan and agents adapt to their new neighborhoods over time.

[1] For example, see `https://demographics.virginia.edu/DotMap/`.

1.1 Model, Definitions and Notation

We consider a strategic game played on a given underlying connected, unweighted and undirected graph $G = (V, E)$, with V the set of vertices and E the set of edges. We denote the cardinalities of V and E with n and m, respectively.

For any vertex $v \in V$ we denote the neighborhood of v in G as $N_v = \{u \in V : \{v, u\} \in E\}$ and $\delta_v = |N_v|$ denotes the degree of v in G. Let $\Delta(G) = \max_{v \in V} \delta_v$ and $\delta(G) = \min_{v \in V} \delta_v$ be the maximum and minimum degree of vertices in G, respectively. We call a graph G *α-almost regular* if $\Delta(G) - \delta(G) = \alpha$ and we call α-almost regular graphs *regular* if $\alpha = 0$ and *almost regular* when $\alpha = 1$. Grid graphs will play a prominent role. We will consider *grid graphs with 4-neighbors (4-grids)* which are formed by a two-dimensional lattice with l rows and h columns and every vertex is connected to the vertex on its left, top, right and bottom, respectively, if they exist. In *grid graphs with 8-neighbors (8-grids)*, vertices are additionally also connected to their top-left, top-right, bottem-left and bottom-right vertices, respectively, if they exist.

For a positive integer k, let $[k]$ denote the set $\{1, \ldots, k\}$, moreover, given a graph $G = (V, E)$, let $\mathcal{T}_k(G)$ denote the set of k-tuples of positive integers summing up to $n = |V|$.

A *Swap Schelling Game with k types (k-SSG)* $(G, \mathbf{t})$ is defined by a graph $G = (V, E)$ and a k-tuple $\mathbf{t} = (t_1, \ldots, t_k) \in \mathcal{T}_k(G)$. There are n strategic agents that need to choose vertices in V in such a way that every vertex is occupied by exactly one agent. Every agent belongs to exactly one of the k types and there are t_i agents of type i, for every $i \in [k]$. When $|t_i| = |t_j|$ for each $i, j \in [k]$, we say that the game is *balanced*. For convenience and in all of our illustrations, we associate each agent type $i \in [k]$ with a color. When $k = 2$, we use colors blue and orange and denote by b and $o = n - b$ the number of blue and orange agents, respectively. Additionally, in case of a game with $k = 2$, we will assume that $o \leq b$, i.e., orange is the color of the minority type. For any graph G and any k-dimensional type vector $\mathbf{t} \in \mathcal{T}_k(G)$, let $c : [n] \to [k]$ denote the function which maps any agent $i \in [n]$ to her color $c(i) \in [k]$.

The strategy of an agent is her location on the graph, i.e., a vertex of G. A *feasible strategy profile* $\boldsymbol{\sigma}$ is an n-dimensional vector whose i-th entry corresponds to the strategy of the i-th agent and where all strategies are pairwise disjoint, i.e., $\boldsymbol{\sigma}$ is a permutation of V, and we will treat $\boldsymbol{\sigma}$ as a bijective function mapping agents to vertices, with $\boldsymbol{\sigma}^{-1}$ being its inverse function. Thus, any feasible strategy profile $\boldsymbol{\sigma}$ corresponds to a coloring of G such that for each $i \in [k]$ exactly t_i vertices of G are colored with the i-th color. We say that agent *i occupies vertex v in $\boldsymbol{\sigma}$* if the i-th entry of $\boldsymbol{\sigma}$, denoted as $\boldsymbol{\sigma}(i)$, is v and, equivalently, if $\boldsymbol{\sigma}^{-1}(v) = i$. It will become important to distinguish if two agents i, j occupy neighboring vertices under $\boldsymbol{\sigma}$. For this, we will use the notation $1_{ij}(\boldsymbol{\sigma})$ with $1_{ij}(\boldsymbol{\sigma}) = 1$ if agents i and j occupy neighboring vertices under $\boldsymbol{\sigma}$ and $1_{ij}(\boldsymbol{\sigma}) = 0$ otherwise.

For an agent i and any feasible strategy profile $\boldsymbol{\sigma}$, we denote by $C_i(\boldsymbol{\sigma}) = \{v \in V : c(\boldsymbol{\sigma}^{-1}(v)) = c(i)\}$ the set of vertices of G which are occupied by agents having the same color as agent i. The utility of agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{|N_{\boldsymbol{\sigma}(i)} \cap C_i(\boldsymbol{\sigma})|}{\delta_{\boldsymbol{\sigma}(i)}}$, i.e., as the ratio of the number of agents with the same type which occupy neighboring vertices and the total number of neighboring vertices, and each agent aims at maximizing her utility.

Agents can change their strategies only by swapping vertex occupation with another agent. Consider two strategic agents i and j which occupy vertices $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\sigma}(j)$, respectively. After performing a *swap* both agents exchange their occupied vertex which yields a new feasible strategy profile $\boldsymbol{\sigma}_{ij}$, which is identical to $\boldsymbol{\sigma}$ except that the i-th and the j-th entries are exchanged. Thus, in the induced coloring of G, the coloring corresponding to $\boldsymbol{\sigma}_{ij}$ is identical to the coloring corresponding to $\boldsymbol{\sigma}$ except that the colors of vertices $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\sigma}(j)$ are exchanged. We say that a swap is *local* if the swapping agents occupy neighboring vertices, i.e., if $1_{ij}(\boldsymbol{\sigma}) = 1$.

As agents are strategic and want to maximize their utility, we will only consider *profitable swaps* by agents, i.e., swaps which strictly increase the utility of both agents involved in the swap. It follows that profitable swaps can only occur between agents of different colors. We call a feasible strategy profile $\boldsymbol{\sigma}$ a *swap equilibrium*, or simply, equilibrium, if $\boldsymbol{\sigma}$ does not admit profitable swaps, that is, if for each pair of agents i, j, we have $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ or $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$. We call $\boldsymbol{\sigma}$ a *local swap equilibrium*, or simply local equilibrium, if no profitable local swap exists under $\boldsymbol{\sigma}$. If agents are restricted to performing only local swaps, then we call the corresponding strategic game *Local Swap Schelling Game with k types (local k-SSG)*. Clearly, any swap equilibrium $\boldsymbol{\sigma}$ is also a local swap equilibrium but the converse is not true. Thus the set of local swap equilibria is a superset of the set of swap equilibria.

We measure the quality of a feasible strategy profile $\boldsymbol{\sigma}$ by its *social welfare* $\mathsf{U}(\boldsymbol{\sigma})$, which is the sum over the utilities of all agents, i.e., $\mathsf{U}(\boldsymbol{\sigma}) = \sum_{i=1}^{n} \mathsf{U}_i(\boldsymbol{\sigma})$. For any game $(G, \mathbf{t})$, let $\boldsymbol{\sigma}^*(G, \mathbf{t})$ denote a feasible strategy profile which maximizes the social welfare and let $SE(G, \mathbf{t})$ and $LSE(G, \mathbf{t})$ denote the set of swap equilibria and local swap equilibria for $(G, \mathbf{t})$, respectively. We will study the impact of the agents' selfishness on the obtained social welfare for games played on a given class of underlying graphs $\mathcal{G}$ with k agent types by analyzing the *Price of Anarchy (PoA)* [26], which is defined as $PoA(\mathcal{G}, k) = \max_{G \in \mathcal{G}} \max_{\mathbf{t} \in \mathcal{T}_k(G)} \frac{\mathsf{U}(\boldsymbol{\sigma}^*(G,\mathbf{t}))}{\min_{\boldsymbol{\sigma} \in SE(G,\mathbf{t})} \mathsf{U}(\boldsymbol{\sigma})}$. Analogously, we define the *Local Price of Anarchy (LPoA)*[2] as the same ratio but with respect to local swap equilibria.[3] It follows that, for any $k \geq 2$ and class of graphs $\mathcal{G}$, we have $PoA(\mathcal{G}, k) \leq LPoA(\mathcal{G}, k)$.

We will also investigate the dynamic properties of the (local) k-SSG, i.e., we analyze if the game has the *finite improvement property (FIP)* [29]. In our model, a game possesses the FIP if every sequence of profitable (local) swaps is finite. Since every instance of the (local) k-SSG has a constant minimum improvement per agent, this is equivalent to the existence of an ordinal potential function which guarantees that sequences of profitable (local) swaps will converge to a (local) swap equilibrium of the game. The FIP can be disproved by showing the existence of an *improving response cycle (IRC)*, which is a sequence of feasible strategy profiles $\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^\ell$, with $\boldsymbol{\sigma}^\ell = \boldsymbol{\sigma}^0$, where $\boldsymbol{\sigma}^{q+1}$ is obtained by a profitable swap by two agents in $\boldsymbol{\sigma}^q$, for $q \in [\ell - 1]$. For investigating the FIP, the following function Φ mapping feasible strategy profiles to natural numbers will be important: $\Phi(\boldsymbol{\sigma}) = \left|\left\{\{u, v\} \in E \mid c(\boldsymbol{\sigma}^{-1}(u)) = c(\boldsymbol{\sigma}^{-1}(v))\right\}\right|$. Hence, $\Phi(\boldsymbol{\sigma})$ is the number of edges of G whose endpoints are occupied by agents of the same color under the feasible strategy profile $\boldsymbol{\sigma}$. We will denote such edges as *monochromatic edges* and $\Phi(\boldsymbol{\sigma})$ as the *potential of* $\boldsymbol{\sigma}$. We will see that potential-preserving profitable swaps exist. For analyzing such swaps, we will consider the *extendend potential* $\Psi(\boldsymbol{\sigma})$ which essentially is $\Phi(\boldsymbol{\sigma})$ augmented with a tie-breaker. It is defined as $\Psi(\boldsymbol{\sigma}) = (\Phi(\boldsymbol{\sigma}), n - z(\boldsymbol{\sigma}))$, where $z(\boldsymbol{\sigma})$ is the number of agents having utility 0 under $\boldsymbol{\sigma}$. We compare Ψ for different strategy profiles $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ lexicographically, i.e., on the one hand we have $\Psi(\boldsymbol{\sigma}) > \Psi(\boldsymbol{\sigma}')$ if $\Phi(\boldsymbol{\sigma}) > \Phi(\boldsymbol{\sigma}')$ or $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}')$ and $z(\boldsymbol{\sigma}) < z(\boldsymbol{\sigma}')$. On the other hand we have $\Psi(\boldsymbol{\sigma}) < \Psi(\boldsymbol{\sigma}')$ if $\Phi(\boldsymbol{\sigma}) < \Phi(\boldsymbol{\sigma}')$ or $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}')$ and $z(\boldsymbol{\sigma}) > z(\boldsymbol{\sigma}')$. Note that any profitable swap which increases (decreases) the potential Φ also increases (decreases) the extended potential Ψ.

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[2] In the literature the abreviation LPoA is sometimes also used for the Liquid Price of Anarchy. However, the concepts of the Liquid Price of Anarchy and the Local Price of Anarchy are not related.

[3] We define $PoA(\mathcal{G}, k) = \infty$ or $LPoA(\mathcal{G}, k) = \infty$ if the respective denominator is zero.

1.2 Related Work

We focus on related work on game-theoretic segregation models.

Zhang [35, 36] was the first who introduced a game-theoretic model related to Schelling's original model. There, agents having a noisy single peaked utility function and preferring to be in a balanced neighborhood were employed. Later, Chauhan et al. [14] introduced a game-theoretic model which is much closer to Schelling's formulation. In their model there are two types of agents and the utility of an agent depends on the type ratio in her neighborhood. An agent is content if the fraction of own-type neighbors is above $\tau \in (0, 1]$. Additionally, agents may have a preferred location. To improve their utility, agents can either swap with another agent who is willing to swap (Swap Schelling Game) or jump to an unoccupied vertex (Jump Schelling Game). Their main contribution is an investigation of the convergence properties of many variants of the model. Moreover they provide basic properties of stable placements and their efficiency. Echzell et al. [17] strengthen these results but omitted location preferences. Instead they extended the model to more than two agent types and studied the computational hardness of finding optimal placements.

Elkind et al. [18] investigated a similar model with k types where agents are either strategic or stubborn. Only strategic agents are willing to move and strive for maximizing the fraction of own-type neighbors by jumping to a suitable unoccupied location. This corresponds to the jump version of Chauhan et al. [14] with $\tau = 1$. They show that equilibria are not guaranteed to exist, they analyze the complexity of finding optimal placements and they prove that the PoA can be unbounded. Recently, Agarwal et al. [1] considered swap games in the model of Elkind et al. [18]. They show that on underlying trees equilibria may not exist and that deciding equilibrium existence and the existence of a state with at least a given social welfare is NP-hard. They also establish that the PoA is in $\Theta(n)$ on underlying star graphs if there are at least two agents of each type and between 2.0558 and 4 for balanced games on any graph. Moreover, for $k \geq 3$ the PoA can be unbounded even in balanced games. Additionally, they give a constant lower bound on the Price of Stability and show that it equals 1 on regular graphs. Finally, they introduce a new benchmark for measuring diversity by counting the number of agents having at least one neighbor of different type. In the present paper, we focus on this recent model by Agarwal et al. [1] and extend and improve their PoA results.

Very recently, Kanellopoulos et al. [24] investigated a novel variant of the Jump Schelling Game where the main new aspect is that an agent is included when counting its neighborhood size. This subtle change leads to agents preferring locations with more own-type neighbors.

Hedonic games [15, 9] are related to Schelling games. In particular, Schelling games share a number of properties with fractional hedonic games [7, 27, 3, 12, 28], hedonic diversity games [11] and FEN-hedonic games [22, 19, 25]. However, one of the main differences is that in Schelling games the neighborhoods of coalitions overlap while in hedonic games agents form disjoint coalitions with identical neighborhoods for all agents within the same coalition.

Investigating a local variant of Schelling's model, although proposed by Schelling [32] himself, seems to be a novel approach. To the best of our knowledge, local moves have only been addressed briefly by Vinković and Kirnan [33] in a model which can be understood as a continuous physical analogue of Schelling's model.

1.3 Our Contribution

We follow the model of Agarwal et al. [1], that is, we consider Swap Schelling Games and investigate, on the one hand, the existence of equilibria and the game dynamics and, on the other hand, the quality of the equilibria in terms of the PoA. The novel feature of our analysis is our focus on the influence of the underlying graph and that we also investigate the impact of restricting the agents to performing only local swaps. See Table 1 for a result overview.

While in [1] it was proven that equilibria may fail to exist for arbitrary underlying graphs and in [17] equilibrium existence was shown for regular graphs, we extend and refine these results by investigating almost regular graphs as well as paths, 4-grids and 8-grids. We establish equilibrium existence for all these graph classes and all our results yield polynomial time algorithms for computing an equilibrium. Moreover, we study the PoA in-depth. Since it was shown in [1] that the PoA can be unbounded for $k \geq 3$, we focus on the PoA of the (local) 2-SSG. We give tight or almost tight bounds on the PoA for all mentioned graph classes which in many cases are significant improvements on the $\Theta(n)$ bound proven in [1]. In particular, we also improve the upper bound for balanced games on arbitrary graphs and we give PoA bounds which depend on the minimum and maximum degree in the underlying graph.

Besides analyzing equilibria in the general model of Agarwal et al. [1], we introduce and analyze a local variant of the model, which was already suggested by Schelling [32] but to the best of our knowledge has not yet been explored for Schelling's model. Our results indicate that the local variant has favorable properties. For instance, equilibria are guaranteed to exists on trees in the local version while in [1] it was shown that this is not the case for the general model. Moreover, for many cases we can show that the PoA in the local version deteriorates only slightly compared to the global version.

We refer to [8] for all details which were omitted due to space constraints.

2 Equilibrium Existence and Dynamics

We start by providing a precise characterization which ties equilibria in 2-SSGs with the sum of the utilities experienced by any two agents of different colors.

▶ **Lemma 1.** *A strategy profile $\boldsymbol{\sigma}$ for a 2-SSG is an equilibrium if and only if, for any two agents i and j with $c(i) \neq c(j)$ and $\delta_{\boldsymbol{\sigma}(i)} \leq \delta_{\boldsymbol{\sigma}(j)}$, it holds that $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$.*

Proof. Fix an equilibrium $\boldsymbol{\sigma}$ and consider two agents i and j such that $c(i) \neq c(j)$ and $\delta_{\boldsymbol{\sigma}(i)} \leq \delta_{\boldsymbol{\sigma}(j)}$. Assume w.l.o.g that i is orange and j is blue. Let o_i be the number of orange neighbors of $\boldsymbol{\sigma}(i)$ and b_j be the number of blue neighbors of $\boldsymbol{\sigma}(j)$. It holds that $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{o_i}{\delta_{\boldsymbol{\sigma}(i)}}$, $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{b_j}{\delta_{\boldsymbol{\sigma}(j)}}$ and $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{\delta_{\boldsymbol{\sigma}(j)} - b_j - 1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(j)}}$, $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{\delta_{\boldsymbol{\sigma}(i)} - o_i - 1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$.

As $\boldsymbol{\sigma}$ is an equilibrium, it must be either $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ or $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$. In the first case, we get $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(j)}}$, in the second one, we get $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$. Thus, given that $\delta_{\boldsymbol{\sigma}(i)} \leq \delta_{\boldsymbol{\sigma}(j)}$, in any case we have that $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$.

Now fix a strategy profile $\boldsymbol{\sigma}$ such that, for any two agents i and j with $c(i) \neq c(j)$ and $\delta_{\boldsymbol{\sigma}(i)} \leq \delta_{\boldsymbol{\sigma}(j)}$, it holds that $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$. Assume, by way of contradiction, that $\boldsymbol{\sigma}$ is not an equilibrium. Then, there exist an orange agent i and a blue agent j such that $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ and $\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$. Let o_i be the number of orange neighbors of $\boldsymbol{\sigma}(i)$ and b_j be the number of blue neighbors of $\boldsymbol{\sigma}(j)$. It holds that $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{o_i}{\delta_{\boldsymbol{\sigma}(i)}}$, $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{b_j}{\delta_{\boldsymbol{\sigma}(j)}}$ and $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{\delta_{\boldsymbol{\sigma}(j)} - b_j - 1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(j)}}$, $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{\delta_{\boldsymbol{\sigma}(i)} - o_i - 1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$.

By $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$, we obtain $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(j)}}$. Similarly, by $\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$, we obtain $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\delta_{\boldsymbol{\sigma}(i)}}$. At least one of the two derived inequalities contradicts the assumption on $\boldsymbol{\sigma}$. Thus, $\boldsymbol{\sigma}$ is an equilibrium. ◀

Table 1 Result overview. The "✓" symbol denotes that the respective property holds. Note that a "✓" in the "k-SSG" column implies a "✓" in the local k-SSG column.The "×" symbol denotes that equilibrium existence is not guaranteed and that an IRC exists, respectively. For $k = 2$ we denote by b and o the number of blue and orange agents, respectively and we assume $o \leq b$. If we use α or β in the respective bound, their meaning is defined in the top of the respective column. ϵ is a constant larger zero. We provide bounds for the (L)PoA for several graph classes but due to space constraints for some results we refer to [8]. A large version of the table can be found in [8].

	Equilibrium Existence		Finite Improvement Property		Price of Anarchy			
	k-SSG	local k-SSG	k-SSG	local k-SSG	2-SSG		local 2-SSG	
graph classes						$o = 2\alpha + \beta$		$n = 3\alpha + \beta$
arbitrary	× ([1])		× ([1])		∞ ([1])	$o = 1$	$\left(2n + \frac{8}{n} - 8, 2n - \frac{8}{n}\right)$ (Thm. 12)	$o = \frac{n}{2}$
					≤ 3 (Thm. 9)	$o = \frac{n}{2}$	$\leq 2\left(1 + \frac{\Delta-1}{\delta-1}\right)$ (Thm. 13)	$\delta \geq 2$
					$\leq \frac{no(n-o)-n}{o(o-1)(n-o)}$ (Thm. 9)	otherwise	$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 4(\Delta^2 - \Delta + 1)\right)$ (Thm. 14)	$\Delta \leq n - 2$
regular	✓([17])	✓([17])	✓([17])	✓([17])	$2 + \frac{1}{\alpha}$ (Thm. 17, 18)	$\Delta \in (2\alpha, 2\alpha + 1)$	$2 + \frac{1}{\alpha}$ (Cor. 17, Thm. 18)	$\Delta \in (2\alpha, 2\alpha + 1)$
1-regular	✓(Thm. 2)	✓(Thm. 2)	✓(Thm. 2)	✓(Thm. 2)				
trees	× ([1])	✓(Thm. 5)	× ([1])		$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 4(\Delta^2 - \Delta + 1)\right)$ (Cor. 15 + Thm. 14)	$\Delta \leq n - 2$	$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 4(\Delta^2 - \Delta + 1)\right)$ (Cor. 15 + Thm. 14)	$\Delta \leq n - 2$
cycles	✓([17])	✓([17])	✓([17])	✓([17])	1 (Thm. 19)	$o = 1$	1 (Thm. 20)	$o = 1$
					$\frac{n-2}{b+\beta}$ (Thm. 19)	otherwise	$\frac{n-2}{b-o}$ (Thm. 20)	$o \geq 2$, $b \geq 2o$
							$\frac{n-2}{\alpha+\beta}$ (Thm. 20)	otherwise
paths	✓(Thm. 2)	✓(Thm. 2)	✓(Thm. 2)	✓(Thm. 2)	∞ ([8])	$n = 3$	∞ ([8])	$n = 3$
					$\frac{2n-2}{2n-5}$ ([8])	$n > 3$, $o = 1$	$\frac{2n-2}{2n-5}$ ([8])	$n > 3$, $o = 1$
					$\frac{n-1}{b+1+\beta}$ ([8])	$n > 3$, $o \geq 2$, $\beta \leq 2\alpha + 1$	$\frac{n-1}{b-o-1}$ ([8])	$n > 3$, $o \geq 2$, $b \geq 2o$
					$\frac{n-1}{b+\beta}$ ([8])	otherwise	$\frac{n-1}{\alpha}$ ([8])	otherwise
4-grids	✓(Thm. 3)	✓(Thm. 3)	✓(Thm. 3)	✓(Thm. 3)	$\frac{25}{22}$ (Thm. 21)	$o = 1$	$(3 - \epsilon, 3)$ (Thm. 25)	$2 \times h$ grid, $h \geq 3$
					2 (Thm. 23, 24)	otherwise	$\left(\frac{36}{13} - \epsilon, \frac{36}{13}\right)$ (Thm. 26)	$3 \times h$ grid, $h \geq 3$
							$\left(\frac{5}{2} - \epsilon, \frac{5}{2} + \epsilon\right)$ (Thm. 27)	$l \times h$ grid, $h, l \geq 8 + \frac{20}{\epsilon}$
8-grids	✓(Thm. 8)	✓(Thm. 8)	× (Thm. 7)	✓(Thm. 6)	$\frac{897}{704}$ ([8])	$o = 1$	$\leq \frac{9}{4} + \epsilon$ ([8])	$l \times h$ grid, $h, l \geq 8 + \frac{18}{\epsilon}$
	$k = 2$	$k = 2$			≤ 8 ([8])	otherwise		

By exploiting the potential Φ, Echzell et al. [17] show that, for any $k \geq 2$, k-SSGs played on regular graphs have the FIP and that any sequence of profitable swaps has length at most m. This result can be extended to α-almost regular graphs for some values of α.

▶ **Theorem 2.** *For any $k \geq 2$, k-SSGs played on almost regular graphs has the FIP. Moreover, at most m profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile.*

Theorem 2 cannot be extended beyond almost regular graphs as Agarwal et al. [1] provide a 2-SSG played on a 2-almost regular graph (more precisely, a tree) admitting no equilibria. However, in the next theorem, we show that positive results can be still achieved in games played on 2-almost regular graphs obeying some additional properties.

▶ **Theorem 3.** *Let G be a 2-almost regular graph such that $\Delta(G) \leq 4$ and every vertex of degree δ is adjacent to at most $\delta - 1$ vertices of degree $\Delta(G)$. Then, for any $k \geq 2$, every k-SSG played on G possesses the FIP. Moreover, at most $O(nm)$ profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile.*

As 4-grids meet the conditions required by Theorem 3, we get the following corollary.

▶ **Corollary 4.** *For any $k \geq 2$, every k-SSG played on a 4-grid possesses the FIP. Moreover, at most $O(nm)$ profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile.*

As mentioned before, Agarwal et al. [1] pointed out that 2-SSGs played on trees are not guaranteed to admit equilibria. We show that this is no longer the case in local k-SSGs for any value of $k \geq 2$.

▶ **Theorem 5.** *For any $k \geq 2$, every local k-SSG played on a tree has an equilibrium which can be computed in polynomial time.*

Proof. Root the tree T at a vertex r. We will place the agents color by color, starting with color 1 and ending with color k. Before we place an agent at an inner vertex v all of v's descendants in T have to be occupied. Hence, we place the agents starting from the leaves, and the root r' of every subtree T' is the last vertex in T' which will be occupied. Thus, we ensure that, if the root r' of a subtree T' is occupied by an agent of color $i \in [k]$, T' contains only agents of color $i' \leq i$. Clearly, this construction yields a feasible strategy profile, that we denote by $\boldsymbol{\sigma}$, and can be implemented in polynomial time.

Consider two agents i and j of different colors that occupy two adjacent vertices u and v, respectively. Without loss of generality, we assume that u is the parent of v in T. Since $c(j) < c(i)$, the subtree of T rooted at v contains no vertex of color $c(i)$. As a consequence $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = 0$. Hence σ is a LSE. ◀

Note that, as we move from 4-grids to 8-grids, Corollary 4 does not apply any more. In fact, for 8-grids, we show that the FIP is guaranteed to hold only for local games.

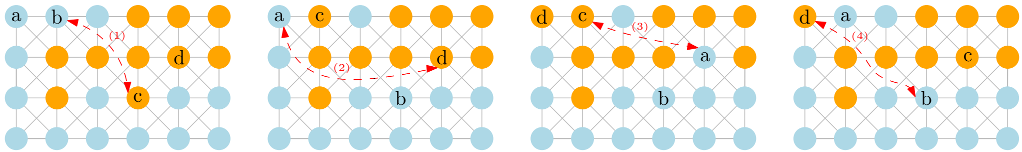
▶ **Theorem 6.** *Any local 2-SSG played on an 8-grid possesses the FIP.*

Proofsketch. It turns out that there are a few local swaps which are improving for both involved agents but which can preserve or decrease Φ. For proving guaranteed convergence we show that after such a Φ-preserving or Φ-decreasing swap a number of swaps must happen before at the same pair of vertices another Φ-preserving or Φ-decreasing swap can occur. This implies that in total the extended potential Ψ increases which then implies the FIP. ◀

Now we will see that compared to the local k-SSG, the k-SSG on 8-grids behaves differently. There the FIP does not hold.

▶ **Theorem 7.** *There cannot exist a potential function for the k-SSG played on an 8-grid, for any $k \geq 2$.*

Proofsketch. We prove the statement by providing an example of an IRC. See Figure 1 for an illustration. ◀



(a) Initial strategy profile **(b)** Strategy profile after the first swap **(c)** Strategy profile after the second swap **(d)** Strategy profile after the third swap

Figure 1 An improving response cycle for the k-SSG played on a 8-grid. The agent types are marked orange and blue.

However, even if convergence to an equilibrium is not guaranteed for $k \geq 2$, they are guaranteed to exist for $k = 2$.

▶ **Theorem 8.** *Every 2-SSG played on an 8-grid has an equilibrium which can be computed in polynomial time.*

Proofsketch. Assume w.l.o.g. that $h \leq l$. We distinguish between two cases. If $o \geq 2h$, then an equilibrium can be obtained by filling the grid with orange agents, starting from the upper-left corner and proceeding sequentially row by row. If $o < 2h$, a more involved construction is needed. We place an orange agent in upper-left corner and proceed essentially along diagonal lines with some careful treatment of the way incomplete diagonals are constructed. ◀

3 Price of Anarchy

In this section, we consider the efficiency of equilibrium assignments and bound the PoA for different classes of underlying graphs. In particular, besides investigating general graphs, we analyze regular graphs, cycles, paths, 4-grids and 8-grids. Agarwal et al. [1] already proved that the PoA for the 2-SSG is in $\Theta(n)$ on underlying star graphs if there are at least two agents of each type and between $\frac{921}{448}$ and 4 for the balanced version, i.e., $o = \frac{n}{2}$. We improve this result by providing an upper bound of 3 which tends to 2 for n going to infinity. Furthermore, the authors of [1] showed that the PoA can be unbounded for $k \geq 3$. Therefore, we concentrate on the (local) 2-SSG for several graph classes.

3.1 General Graphs

Remember that for a 2-SSG game, we assume that o is the less frequent color.

We significantly improve and generalize the results of [1] by providing a general upper bound of $\frac{no(n-o)-n}{o(o-1)(n-o)}$ for the case of $o > 1$. For balanced games, it yields an upper bound of $\frac{2(n+2)}{n}$ which shows that the PoA tends to 2 as the number of vertices increases. Moreover, if $\frac{b}{o} \in \mathcal{O}(1)$, the PoA is constant. With the help of Lemma 1, we can now prove our general upper bound for the 2-SSG.

▶ **Theorem 9.** *The PoA of* 2*-SSGs with* $o > 1$ *is at most* $\frac{no(n-o)-n}{o(o-1)(n-o)}$. *Hence, PoA* $\in \mathcal{O}\left(\frac{b}{o}\right)$.

Proof. Fix a 2-SSG with $o > 1$ orange agents played on a graph G with n vertices. First, we observe that the social welfare of a social optimum is at most $n - 2 + \frac{o-1}{o} + \frac{b-1}{b} = n - \frac{1}{o} - \frac{1}{b}$, as there must be at least one orange vertex that is adjacent to at least one blue vertex, thus getting utility at most $\frac{o-1}{o}$, and at least one blue vertex that is adjacent to at least one orange vertex, thus getting utility at most $\frac{b-1}{b}$.

Given a strategy profile $\boldsymbol{\sigma}'$, a *feasible pair* is a pair of vertices (u, v) such that u and v are occupied by agents of different colors in $\boldsymbol{\sigma}'$ and $\{u, v\} \notin E(G)$, i.e., u and v are not adjacent. Now fix a swap equilibrium $\boldsymbol{\sigma}$ and consider a maximum cardinality matching M of feasible pairs. Clearly $0 \leq |M| \leq o$. Hence, $|M| = o - x$ for some $0 \leq x \leq o$. If $x > 0$, then, there are exactly x orange and at least x blue leftover vertices of V that do not belong to any feasible pair in M. As M has maximum cardinality, each orange leftover vertex has to be adjacent to all leftover blue ones and vice-versa. That is, for each leftover vertex u, we have $\delta_u(G) \geq x$. Let T be a set of pairs of vertices obtained by matching each leftover orange vertex with a leftover blue one. By Lemma 1, it holds for each $(u, v) \in M$, $\mathsf{U}_{\boldsymbol{\sigma}^{-1}(u)}(\boldsymbol{\sigma}) + \mathsf{U}_{\boldsymbol{\sigma}^{-1}(v)}(\boldsymbol{\sigma}) \geq 1$ and for each $(u, v) \in T$, $\mathsf{U}_{\boldsymbol{\sigma}^{-1}(u)}(\boldsymbol{\sigma}) + \mathsf{U}_{\boldsymbol{\sigma}^{-1}(v)}(\boldsymbol{\sigma}) \geq 1 - \frac{1}{x}$. Thus, the social welfare of $\boldsymbol{\sigma}$ is at least $o - x + x(1 - \frac{1}{x}) = o - 1$. ◀

▶ **Corollary 10.** *The PoA of* 2*-SSGs is constant if* $\frac{b}{o}$ *is constant.*

We want to emphasize that in particular for the case where both colors are perfectly balanced, the PoA is constant and tends to 2 which improves the bound by [1]. As for $n = 2$ the 2-SSG is trivial and PoA $= 1$, we get the following corollary.

▶ **Corollary 11.** *The PoA of balanced* 2*-SSGs is at most* $\min\left\{3, \frac{2(n+2)}{n}\right\}$.

We will now show that in contrast to the balanced 2-SSG, the balanced local k-SSG has a much higher LPoA.

▶ **Theorem 12.** *The LPoA of local balanced* 2*-SSGs with* $o > 1$ *is between* $2n + \frac{8}{n} - 8$ *and* $2n - \frac{8}{n}$.

If the underlying graph G does not contain leaf vertices, i.e., all vertices have at least degree 2, we can prove a smaller LPoA. In particular, if the ratio between the maximum and minimum degree of vertices in G is constant, we achieve a constant LPoA.

▶ **Theorem 13.** *The LPoA of local* 2*-SSGs on a graph* G *with minimum degree* $\delta \geq 2$ *and maximum degree* Δ *is at most* $2\left(1 + \frac{\Delta+1}{\delta-1}\right)$.

Proof. Fix a local swap equilibrium $\boldsymbol{\sigma}$ on G with $\delta(G) \geq 2$. Let $\rho := \frac{\delta-1}{2\delta}$ and let o' and b' be the numbers of orange and blue agents that have a utility strictly less than ρ, respectively. Clearly, $o - o'$ and $b - b'$ are the numbers of orange and blue agents that have a utility of at least ρ, respectively. We first prove that $b - b' \geq \frac{\delta o'}{\Delta}$ as well as that $o - o' \geq \frac{\delta b'}{\Delta}$ and show then how these two inequalities imply the theorem statement.

We only prove the first inequality, i.e., $b - b' \geq \frac{\delta o'}{\Delta}$ as the proof of the other inequality is similar. Let i and j, respectively, be a blue agent and an orange agent that occupy two adjacent vertices in G, say $\sigma(i) = u$ and $\sigma(j) = v$, and such that $\mathsf{U}_j(\boldsymbol{\sigma}) < \rho$. By Lemma 1, we have that $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1}{\delta}$, from which we derive $\mathsf{U}_i(\boldsymbol{\sigma}) > 1 - \frac{1}{\delta} - \frac{\delta-1}{2\delta} = \frac{\delta-1}{2\delta} = \rho$.

Let G' be the subgraph of G containing all the non-monochromatic edges, i.e., each edge of G' connects a vertex occupied by an orange agent with a vertex occupied by a blue agent. Clearly, G' is bipartite. Consider the vertex-induced subgraph H of G' in which we have all the o' orange agents having a utility strictly less than ρ on one side and all the $b - b'$

blue agents having a utility of at least ρ on the other side. Since for each vertex v of H occupied by an orange agent, there are at least $(1-\rho)\delta_v \geq \frac{\delta+1}{2}$ vertices adjacent to u that are occupied by blue agents and each such blue agent have a utility of at least ρ, the degree of v in H is at least $\frac{\delta+1}{2}$. Therefore, $|E(H)| \geq \frac{\delta+1}{2}o'$.

Furthermore, since each edge of H is incident to a blue agent that has a utility of at least ρ, the degree in H of every vertex u that is occupied by a blue agent is at most $(1-\rho)\delta_u \leq \frac{\delta+1}{2\delta}\Delta$. Therefore, $|E(H)| \leq \frac{\Delta(\delta+1)}{2\delta}(b-b')$. Merging the two bounds of $|E(H)|$ and simplifying gives $b-b' \geq \frac{\delta}{\Delta}o'$.

Finally, we show how $b-b' \geq \frac{\delta o'}{\Delta}$ and $o-o' \geq \frac{\delta b'}{\Delta}$ imply the theorem statement. The average utility of all the agents in H is at least $\frac{\rho(b-b')}{o'+(b-b')} \geq \frac{\rho\frac{\delta}{\Delta}}{1+\frac{\delta}{\Delta}} = \frac{\delta-1}{2(\delta+\Delta)}$. Similarly, the average utility of the b' blue agents whose utilities are strictly less than ρ and the $o-o'$ orange agents whose utilities are of at least ρ is also at least $\frac{\delta-1}{2(\delta+\Delta)}$. Therefore, the LPoA is at most $\frac{2(\delta+\Delta)}{\delta-1} = 2\left(1+\frac{\Delta+1}{\delta-1}\right)$. ◀

We observe that the *LPoA* on a graph with minimum degree $\delta(G) = 1$ can be unbounded. Consider the star graph with Δ leaves and let $\boldsymbol{\sigma}$ be a strategy profile where the unique orange agent occupies the star center, while all the blue agents occupy the leaves. This is clearly a swap equilibrium of 0 social welfare. Any configuration in which a blue agent occupies the star center has strictly positive social welfare.

However, as the following theorem shows, the LPoA can be upper bounded by a function of Δ if we force $n \geq \Delta + 2$, i.e., we avoid the pathological star graph of $\Delta + 1$ vertices.

▶ **Theorem 14.** *For every $\epsilon > 0$, the LPoA of local 2-SSGs on a graph G with maximum degree $\Delta \leq n-2$ is between $\frac{\Delta(\Delta-1)}{2} - \epsilon$ and $4(\Delta^2 - \Delta + 1)$.*

As shown in the next corollary, the lower bound to the PoA shown in Theorem 14 holds even for the class of trees.

▶ **Corollary 15.** *For every $\epsilon > 0$, the LPoA of the local 2-SSG on a tree G with $\Delta(G) \leq n-2$ is at least $\frac{\Delta(\Delta-1)}{2} - \epsilon$.*

3.2 Regular Graphs

In this section we provide upper and lower bounds to the LPoA for regular graphs, i.e., for graphs where all vertices have the same degree. The key is the following technical lemma.

▶ **Lemma 16.** *Let $\boldsymbol{\sigma}$ be a local swap equilibrium, and let $\Delta = 2\alpha + \beta$, with $\alpha \in \mathbb{N}$ and $\beta \in \{0, 1\}$. Let $X \subseteq V$ be a subset of vertices such that $\delta_v = \Delta$ for every $v \in N_X := \bigcup_{x \in X} N_x$. Finally, let $Z \subseteq N_X$ be the set of vertices occupied by the agents that have a utility strictly larger than $\rho := \frac{\alpha}{2\alpha+1}$. Then, the average utility of the agents that occupy the vertices in $X \cup Z$ is at least ρ.*

▶ **Corollary 17.** *The LPoA of local 2-SSG on a regular graph G with $\Delta(G) = 2\alpha + \beta$, with $\alpha \geq 1$ and $\beta \in \{0, 1\}$ is at most $2 + \frac{1}{\alpha}$.*

Proof. The corollary follows from Lemma 16 by $X = V$. ◀

The matching lower bound is provided in the following.

▶ **Theorem 18.** *The LPoA of local 2-SSG on a regular graph G with $\Delta(G) = 2\alpha + \beta$, with $\alpha \geq 1$ and $\beta \in \{0, 1\}$ is equal to $2 + \frac{1}{\alpha}$.*

Proof. For a fixed degree $\Delta \geq 3$, we define the Δ-regular graph $G(\Delta) := G$ as follows. There are $q := t(\Delta+1)$ gadgets $G^1, \ldots, G^q$. For each $i \in [q]$, gadget G^i is obtained from a complete graph of $\Delta + 1$ vertices, denoted as $v_0^1, \ldots, v_\Delta^i$, by removing edge $\{v_0^i, v_\Delta^i\}$. Observe that, by construction, for any $i \in [q]$, each vertex v_j^i, with $1 \leq j \leq \Delta - 1$, has degree Δ, while vertices v_0^i and v_Δ^i have degree $\Delta - 1$. We obtain G by connecting the q gadgets through edges $\{v_\Delta^i, v_0^{i+1}\}$ for each $i \in [q-1]$ and edge $\{v_\Delta^q, v_0^1\}$. Call these edges *extra-gadget* edges. Thus, G is connected and Δ-regular. Consider now the local 2-SSG played on G in which there are $\lceil \frac{\Delta+1}{2} \rceil q$ blue agents and $\lfloor \frac{\Delta+1}{2} \rfloor q$ orange ones.

On the one hand, the social optimum is at least $n - \frac{4}{\Delta} = q(\Delta+1) - 4\Delta$, as in the strategy profile in which all vertices of the first $\lceil \frac{\Delta+1}{2} \rceil t$ gadgets are colored blue and all vertices of the remaining $\lfloor \frac{\Delta+1}{2} \rfloor t$ gadgets are colored orange there are $n-4$ vertices getting utility 1 and 4 vertices getting utility $\frac{\Delta-1}{\Delta}$.

On the other hand, the strategy profile $\boldsymbol{\sigma}$ in which the first $\lceil \frac{\Delta+1}{2} \rceil$ vertices of each gadget are colored blue and the remaining ones are colored orange is a swap equilibrium. In fact, as extra-gadget edges connect vertices of different colors, every blue vertex is adjacent to $\lceil \frac{\Delta+1}{2} \rceil - 1$ blue ones, while every orange vertex is adjacent to $\lceil \frac{\Delta+1}{2} \rceil$ blue ones. If a blue vertex swaps with an adjacent orange one, it ends up being adjacent to $\lceil \frac{\Delta+1}{2} \rceil - 1$ blue vertices. Thus, no profitable swap exists in $\boldsymbol{\sigma}$.

As the social welfare of $\boldsymbol{\sigma}$ is

$$\begin{aligned} & \frac{q}{\Delta}\left(\left\lceil \frac{\Delta+1}{2} \right\rceil \left(\left\lceil \frac{\Delta+1}{2} \right\rceil - 1\right) + \left\lfloor \frac{\Delta+1}{2} \right\rfloor \left(\left\lfloor \frac{\Delta+1}{2} \right\rfloor - 1\right)\right) \\ = & \begin{cases} \frac{q(\Delta^2-1)}{2\Delta} & \text{if } q \text{ is odd,} \\ \frac{q\Delta}{2} & \text{if } q \text{ is even,} \end{cases} \end{aligned}$$

we get that the LPoA of the game is lower bounded by $\frac{2\Delta(q(\Delta+1)-4\Delta)}{q(\Delta^2-1)}$ when Δ is odd and by $\frac{2(q(\Delta+1)-4\Delta)}{q\Delta}$ when Δ is even. By letting q going to infinity, we get $\frac{2\Delta}{\Delta-1}$ and $\frac{2(\Delta+1)}{\Delta}$, respectively. By using $\Delta = 2\alpha + 1$ in the first case, and $\Delta = 2\alpha$ in the second one, we finally obtain the lower bound of $2 + \frac{1}{\alpha}$. ◀

Next, we provide a full characterization of the (L)PoA of cycles.

▶ **Theorem 19.** *The PoA of 2-SSGs played on cycles with $n \geq 3$ vertices and $o = 2\alpha + \beta$ orange agents, where $\alpha \in \mathbb{N}$, $\beta \in \{0,1\}$, and $b \geq o$, is equal to 1, if $o = 1$; and by $\frac{n-2}{b+\beta}$, otherwise.*

Proofsketch. The social welfare of the social optimum is equal to $n-2$. Let $\boldsymbol{\sigma}$ be a swap equilibrium. Let ℓ be the number of maximal vertex-induced (sub)paths whose vertices are occupied by orange agents only. Clearly, ℓ is also the number of maximal vertex-induced (sub)paths whose vertices are occupied by blue agents only. We claim that $\ell \leq \alpha$ by showing that every agent has a strictly positive utility in $\boldsymbol{\sigma}$ (i.e., each of the 2ℓ maximal paths formed by monochromatic edges contains 2 or more vertices). For the sake of contradiction, assume w.l.o.g that there is an orange agent i such that $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$.

For the matching lower bound, it is enough to consider the strategy profile in which $\ell = \alpha$, i.e., there are $\alpha - 1$ maximal vertex-induced paths occupied by orange (resp. blue) agents only of length 2 each, and one maximal vertex-induced path occupied by orange (resp. blue) agents only of length $2 + \beta$ (resp., $b - 2\alpha + 2$). In this case, the social welfare is equal to $\frac{1}{2}2\alpha + \beta + \frac{\alpha}{2} + (b - 2\alpha) = b + \beta$. ◀

The following theorem provides almost tight upper bounds to the LPoA for cycles.

▶ **Theorem 20.** *The LPoA of local 2-SSGs played on cycles with $n = 3\alpha + \beta$ vertices and b blue agents, where $\alpha \in \mathbb{N}$, $\beta \in \{0, 1, 2\}$, and $b \geq o$, is upper bounded by 1, if $o = 1$; by $\frac{n-2}{b-o}$, if $o \geq 2$ and $b \geq 2o$; and by $\frac{n-2}{\alpha+\beta}$, otherwise, i.e., $o \geq 2$ and $b < 2o$). The upper bounds are tight when (i) $o = 1$ and (ii) $o \geq 2$ and $b \geq 2o$.*

We prove similar results for paths which can be found in [8].

3.3 Grids

We now turn our focus to grid graphs with 4- and 8-neighbors. First, we investigate 2-SSGs in 4-grids and start by characterizing the PoA for the case in which one type has a unique representative.

▶ **Theorem 21.** *The PoA of 2-SSGs played on a 4-grid in which one type has cardinality 1 is equal to $\frac{25}{22}$.*

Clearly, if one type has only one representative, this agent will receive utility zero. However, this is not possible in equilibrium assignments when there are at least two agents of each type.

▶ **Lemma 22.** *In any equilibrium for a 2-SSG played on a 4-grid in which both types have cardinality larger than 1 all agents get positive utility.*

When no agent gets utility zero, the minimum possible utility is $\frac{1}{4}$. Thus, Lemma 22 imply an upper bound of 4 on the PoA. However, a much better result can be shown.

▶ **Theorem 23.** *The PoA of 2-SSGs played on 4-grids is at most 2.*

We now show a matching lower bound.

▶ **Theorem 24.** *The PoA of 2-SSGs played on 4-grids is at least 2, even when both types have the same cardinality.*

Proofsketch. Fix a 2-SSG played on an $n \times n$ grid G, with n being an even number. We define a strategy profile $\boldsymbol{\sigma}$ by giving a coloring rule for any frame of G. There are $\frac{n}{2}$ frames in G that we number from 1 to $\frac{n}{2}$, with frame 1 corresponding to the outer one. Frame i, whose size is $n_i := n - 2(i-1)$, is colored as follows: all vertices in the left column and all vertices in the right column except for the first and the last are of the *basic color* of i, all other vertices take the other color. The basic color of frame i is orange if i is odd and blue otherwise, see Figure 2 for a pictorial example. Observe that every frame evenly splits its vertices between the two colors. We show that $\boldsymbol{\sigma}$ is an equilibrium. ◀

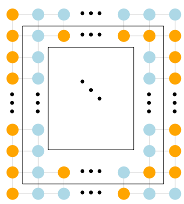


Figure 2 Visualization of the first three frames of G with the coloring induced by the strategy profile defined in the proof of Theorem 24.

We now show matching upper and lower bounds on the LPoA for local 2-SSGs played on grids. By inspecting all the possibilities, the LPoA of local 2-SSGs played on 2×2 grids is 1. Indeed, assuming $b \geq o$, for $o = 1$, all the configurations are isomorphic to each other, while, for $o = 2$, the unique (local) swap equilibrium – up to isomorphisms – is $\left[\begin{smallmatrix} o & b \\ o & b \end{smallmatrix}\right]$.

▶ **Theorem 25.** *The LPoA of local 2-SSGs played on $2 \times h$ 4-grids, with $h \geq 3$ is 3. Furthermore, for every $\epsilon > 0$, there is a value h_0 such that, for every $h \geq h_0$, the PoA of $2 \times h$ 4-grid is at least $3 - \epsilon$.*

▶ **Theorem 26.** *The LPoA of local 2-SSG played on $3 \times h$ 4-grids, with $h \geq 3$ is $\frac{36}{13}$. Furthermore, for every $\epsilon > 0$, there is a value h_0 such that, for every $h \geq h_0$, the PoA of $2 \times h$ 4-grid is at least $\frac{36}{13} - \epsilon$.*

▶ **Theorem 27.** *For every $\epsilon > 0$, the LPoA of local 2-SSG played on $l \times h$ 4-grids, with $\ell, h \geq 8 + \frac{20}{\epsilon}$ is in the interval $\left(\frac{5}{2} - \epsilon, \frac{5}{2} + \epsilon\right]$.*

We prove similar results for 8-grids which can be found in [8].

4 Conclusion and Open Problems

We have shed light on the influence of the underlying graph topology on the existence of equilibria, the game dynamics and the Price of Anarchy in Swap Schelling Games on graphs. Moreover, we have studied the impact of restricting agents to local swaps. We present tight or almost tight bounds for a variety of graph classes.

Clearly, improving on the non-tight bounds is an interesting challenge for future work. Regarding the local Swap Schelling Game, we leave some interesting problems open. Among them is the question whether local swap equilibria are guaranteed to exist for all graph classes and if the local k-SSG always has the finite improvement property. So far, we are not aware of any counter-examples for both questions and extensive agent-based simulations indicate that both equilibrium existence and guaranteed convergence of improving response dynamics may hold. Another interesting line of study is to analyze the Jump Schelling Game with respect to varying underlying graphs and locality.

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