A Polynomial Kernel for 3-Leaf Power Deletion

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Abstract

For a non-negative integer ℓ, a graph G is an *ℓ-leaf power of a tree T* if $V(G)$ is equal to the set of leaves of T, and distinct vertices v and w of G are adjacent if and only if the distance between v and w in T is at most ℓ. Given a graph G, 3-Leaf Power Deletion asks whether there is a set $S \subseteq V(G)$ of size at most k such that $G \backslash S$ is a 3-leaf power of some tree T. We provide a polynomial kernel for this problem. More specifically, we present a polynomial-time algorithm for an input instance (G, k) to output an equivalent instance (G', k') such that $k' \leqslant k$ and G' has at most $O(k^{14})$ vertices.

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1 Introduction

Nishimura, Ragde, and Thilikos [31] introduced an ℓ-leaf power of a tree to understand the structure of phylogenetic trees in computational biology. For a non-negative integer ℓ, a graph G is an *ℓ-leaf power of a tree T* if $V(G)$ is equal to the set of leaves of T, and distinct vertices v and w of G are adjacent if and only if the distance between v and w in T is at most ℓ, where the *distance* between vertices x and y in a graph H is the length of a shortest path in H from x to y. We say that G is an *ℓ-leaf power* if G is an ℓ-leaf power of some tree. Note that an ℓ-leaf power could have more than one component. For instance, an ℓ-leaf power of a path of length at least $\ell + 1$ has two components. We remark that a graph is a 2-leaf power if and only if it is a disjoint union of cliques, and is a 3-leaf power if and

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Table 1 The current best known running time of a fixed-parameter algorithm and the current best known upper bound for the number of vertices in a kernel for ℓ-Leaf Power Deletion when ℓ is small. We denote by n the number of vertices and by m the number of edges of an input graph.

ℓ	Running time	Kernel (The number of vertices)	Remark
0	$O(1.2738^k + kn)$ [9]	$2k - \Omega(\log k)$ [27, 29]	Equivalent to Vertex Cover
1	$O(1.2738^k + kn)$ [9]	$2k - \Omega(\log k)$ [27, 29]	Reduced to Vertex Cover
2	$O(2^k k \cdot m\sqrt{n} \log n)$ [23]	$O(k^{5/3})$ [20]	Equivalent to Cluster Deletion
3	$O(37^k \cdot n^7(n+m))$ [3]	$O(k^{14})$ [Theorem 1.1]	-

only if it is a (bull, dart, gem)-free chordal graph [14], where a bull, a dart, and a gem are depicted in Figure 1. There are linear-time algorithms to recognize 4- and 5-leaf powers [6, 8] and a polynomial-time algorithm to recognize 6-leaf powers [16]. For each $\ell \geqslant 7$, there is a linear-time algorithm to recognize ℓ-leaf powers for graphs of bounded degeneracy [18].

We are interested in the following *vertex deletion* problem, which generalizes the corresponding recognition problem.

ℓ-Leaf Power Deletion
Input : A graph G and a non-negative integer k
Parameter : k
Question : Is there a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G \backslash S$ is a ℓ-leaf power?

Vertex deletion problems include some of the best studied NP-hard problems in theoretical computer science, including Vertex Cover and Feedback Vertex Set. In general, the problem asks whether it is possible to delete at most k vertices from an input graph so that the resulting graph belongs to a specified graph class. Lewis and Yannakakis [28] showed that every vertex deletion problem to a non-trivial[1] and hereditary[2] graph class is NP-hard. Since the class of ℓ-leaf powers is non-trivial and hereditary for every non-negative integer ℓ, it follows that ℓ-Leaf Power Deletion is NP-hard.

Vertex deletion problems have been investigated on various graph classes through the parameterized complexity paradigm [12, 15], which measures the performance of algorithms not only with respect to the input size but also with respect to an additional numerical parameter. The notion of vertex deletion allows a highly natural choice of the parameter, specifically the size of the deletion set k. A decidable parameterized problem Π is *fixed-parameter tractable* if it can be solved by an algorithm with running time $f(k) \cdot n^{O(1)}$ where n is input size and $f : \mathbb{N} \to \mathbb{N}$ is a computable function. It is well known that Π is fixed-parameter tractable if and only if it admits a kernel [15]. A *kernel* is basically a polynomial-time preprocessing algorithm that transforms the given instance of the problem into an equivalent instance whose size is bounded above by some function $f(k)$ of the parameter. The function $f(k)$ is usually referred to as the *size* of the kernel. A *polynomial kernel* is then a kernel with size bounded above by some polynomial in k. For a decidable fixed-parameter tractable problem, one of the most natural follow-up questions in parameterized complexity is whether the problem admits a polynomial kernel. The existence of polynomial kernels for vertex deletion problems has been widely investigated; see [21].

[1] A class of graphs $\mathcal{C}$ is *non-trivial* if both $\mathcal{C}$ and the complement of $\mathcal{C}$ contain infinitely many non-isomorphic graphs.
[2] A class of graphs is *hereditary* if it is closed under taking induced subgraphs.

We are going to survey known results of ℓ-Leaf Power Deletion for small values of ℓ; see Table 1. When $\ell = 0$, ℓ-Leaf Power Deletion is identical to Vertex Cover. Currently, the best known fixed-parameter algorithm for Vertex Cover runs in time $O(1.2738^k + k|V(G)|)$, by Chen, Kanj, and Xia [9], and $2k - \Omega(\log k)$ is the best known upper bound for the number of vertices in kernels for Vertex Cover, independently by Lampis [27] and Lokshtanov, Narayanaswamy, Raman, Ramanujan, and Saurabh [29].

When $\ell = 1$, since a graph is a 1-leaf power if and only if it either is isomorphic to K_2, or has no edges, one can easily reduce ℓ-Leaf Power Deletion to Vertex Cover. Thus, 1-Leaf Power Deletion can be solved in time $O(1.2738^k + kn)$ and admits a kernel with $2k - \Omega(\log k)$ vertices.

When $\ell = 2$, ℓ-Leaf Power Deletion was studied under the name of Cluster Deletion. Hüffner, Komusiewicz, Moser, and Niedermeier [23] showed that Cluster Deletion is fixed-parameter tractable by presenting an algorithm with running time $O(2^k k \cdot |E(G)|\sqrt{|V(G)|}\log|V(G)|)$, and Fomin, Le, Lokshtanov, Saurabh, Thomassé, and Zehavi [20] presented a kernel with $O(k^{5/3})$ vertices for Cluster Deletion.

Now, we investigate when $\ell = 3$. Dom, Guo, Hüffner, and Niedermeier [14] already showed that 3-Leaf Power Deletion is fixed-parameter tractable. The algorithm in [17] can be modified to a *single-exponential* fixed-parameter algorithm for 3-Leaf Power Deletion, that is an algorithm with running time $\alpha^k \cdot n^{O(1)}$ for input size n and some constant $\alpha > 1$; see [3]. Here is our main theorem.

▶ **Theorem 1.1.** *3-Leaf Power Deletion admits a kernel with $O(k^{14})$ vertices.*

As another motivation, our result is motivated by vertex deletion problems for chordal graphs and distance-hereditary graphs, which are superclasses of 3-leaf powers. For vertex deletion problems of chordal graphs and distance-hereditary graphs, fixed-parameter algorithms and polynomial kernels have been recently obtained [30, 7, 17, 24, 1, 26].

Roughly speaking, our first step is to find a *"good"* approximate solution, called a *good modulator* of an input graph G, that is a set $S \subseteq V(G)$ of size $O(k^2)$ such that $G \setminus (S \setminus \{v\})$ is a 3-leaf power for every vertex v in S. This technique of computing a good modulator has been used in several kernelization algorithms [24, 25, 26, 2]. To bound the number of components of $G \setminus S$, we introduce two concepts; a *complete split* of a graph G, which is a special type of a clique cut-set of G, and a *blocking pair* for a set $X \subseteq V(G)$, which determine whether $(X, V(G) \setminus X)$ is a complete split of G. A key property, Lemma 4.4, of a blocking pair is that two components of $G \setminus S$ blocked by the same pair in S always contain an obstruction. Through a marking process with pairs in S, we show that if there are many components of $G \setminus S$ blocked by some pairs in S, then we can safely remove all edges inside some of the components. Afterward, we bound the number of isolated vertices of $G \setminus S$ through another marking process, and then design a series of reduction rules to bound the size of the remaining components of $G \setminus S$, which utilize a tree-like structure of 3-leaf powers, introduced by Brandstädt and Le [5].

We organize this paper as follows. In Section 2, we summarize some terminologies in graph theory and introduce 3-leaf powers. In Section 3, we introduce a good modulator of a graph, and then present an algorithm that either confirms that an input instance (G, k) is a no-instance, or constructs a small good modulator of G. In Sections 4 and 5, we design a series of reduction rules that allows us to bound the number of vertices outside of a good modulator of a graph, and prove Theorem 1.1. In Section 6, we conclude this paper with some open problems.

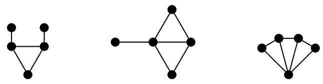

Figure 1 A bull, a dart, and a gem.

2 Preliminaries

In this paper, all graphs are finite and simple. We assume familiarity with the basic notations and terminologies in graph theory and parameterized complexity. We refer the reader to the standard books [12, 13, 15].

For disjoint sets X and Y of vertices of G, we say that X *is complete to* Y if each vertex in X is adjacent to all vertices in Y, and X *is anti-complete to* Y if each vertex in X is non-adjacent to all vertices in Y. By $G\backslash X$ we denote the graph obtained from G by removing all vertices in X and all edges incident with some vertices in X, and $G[X] := G\backslash(V(G)\backslash X)$. For a set T of edges of G, let $G\backslash T$ be a graph obtained from G by removing all edges in T.

A graph G is *trivial* if $|V(G)| \leqslant 1$, and *non-trivial*, otherwise. A graph is *complete* if every pair of distinct vertices is adjacent, and *incomplete*, otherwise. Distinct vertices v and w of G are *twins* in G if $N_G(v)\backslash\{w\} = N_G(w)\backslash\{v\}$. Twins v and w in G are *true* if v and w are adjacent, and *false* if v and w are non-adjacent. A *twin-set* in G is a set of pairwise twins in G. A twin-set is *true* if it is a clique, and *false* if it is an independent set.

For graphs $G_1, \ldots, G_m$, a graph G is $(G_1, \ldots, G_m)$*-free* if G has no induced subgraph isomorphic to one of $G_1, \ldots, G_m$.

It is well known that a parameterized problem Π is fixed-parameter tractable if and only if Π is decidable and admits a kernel; see [15, 19]. An *instance* is an ordered pair (G, k) of a graph G and a non-negative integer k. An instance (G, k) is a *yes-instance* if there is a set $S \subseteq V(G)$ of size at most k such that $G\backslash S$ is a 3-leaf power, and a *no-instance*, otherwise.

The graphs in Figure 1 are called a *bull*, a *dart*, and a *gem*, respectively. A *hole* is an induced cycle of length at least 4. A graph is *chordal* if it has no holes. Dom, Guo, Hüffner, and Niedermeier [14] presented the following characterization of 3-leaf powers.

▶ **Theorem 2.1** (Dom, Guo, Hüffner, and Niedermeier [14, Theorem 1]). *A graph G is a 3-leaf power if and only if G is (bull, dart, gem)-free and chordal.*

We say that a graph H is an *obstruction* if H either is a hole, or is isomorphic to one of the bull, the dart, and the gem. An obstruction H is *small* if $|V(H)| \leqslant 5$. We have the following seven observations about obstructions.

- **(O1)** No obstructions have true twins.
- **(O2)** No small obstructions have an independent set of size at least 4.
- **(O3)** No obstructions have K_4 or $K_{2,3}$ as a subgraph.
- **(O4)** No obstruction H has a cut-vertex v such that $H\backslash v$ has exactly two components H_1 and H_2 with $|V(H_1)| = |V(H_2)|$.
- **(O5)** False twins in an obstruction H have degree 2 in H.
- **(O6)** If a vertex v of an obstruction H has exactly one neighbor w in $V(H)$, then w has degree at least 3 in H.
- **(O7)** A graph H is an obstruction having three distinct vertices of degree 2 in H if and only if H is a hole.

Brandstädt and Le [5] presented a linear-time algorithm to recognize 3-leaf powers, and showed that a graph G is a 3-leaf power if and only if G is obtained from some forest F by substituting each node u of F with a non-empty clique B_u of arbitrary size. We rephrase this characterization by using the following definition.

A *tree-clique decomposition* of a graph G is a pair $(F, \{B_u : u \in V(F)\})$ of a forest F and a family $\{B_u : u \in V(F)\}$ of non-empty subsets of $V(G)$ satisfying the following two conditions.

(1) $\{B_u : u \in V(F)\}$ is a partition of $V(G)$.

(2) Distinct vertices x and y of G are adjacent if and only if F has either a node u such that $\{x, y\} \subseteq B_u$, or an edge vw such that $x \in B_v$ and $y \in B_w$.

We call B_u a *bag of* u for each node u of F. We say that B is a *bag of* G if B is a bag of some node of F. Note that each bag is a clique by (2).

▶ **Theorem 2.2** (Brandstädt and Le [5, Theorem 14])**.** *A graph G is a 3-leaf power if and only if G has a tree-clique decomposition. One can construct a tree-clique decomposition of a 3-leaf power in polynomial time. Moreover, if G is a connected incomplete 3-leaf power, then G has a unique tree-clique decomposition.*

We remark that every connected incomplete 3-leaf power has at least three bags. Brandstädt and Le [5] showed that for a connected incomplete 3-leaf power G, distinct vertices v and w of G are in the same bag of G if and only if v and w are true twins in G. Thus, for such a graph G, B is a bag of G if and only if B is a maximal true twin-set in G.

3 Good modulators

A set S of vertices of a graph G is a *modulator of* G if $G \backslash S$ is a 3-leaf power. A modulator S of a graph G is *good* if $G\backslash(S\backslash\{v\})$ is a 3-leaf power for each vertex v in S. We first collect at most $5k$ vertices by S_1 for vertex-disjoint small obstructions. By using the characterization of graphs without small obstructions [14], when we run the 2-approximation algorithm for Weighted Feedback Vertex Set [4] to $(G\backslash S_1, k)$, we can either confirm that G has no modulator of size at most k, or find a modulator of G having at most $7k$ vertices in time bounded above by a polynomial in $|V(G)| + k$. When we have a modulator S of size at most $7k$, for each vertex $v \in S$, we either find $O(k)$ additional vertices that hit all obstructions containing no vertices in $S\backslash\{v\}$, or decide that v is in every modulator of size at most k. We formalize this in (R1).

▶ **Reduction Rule 1** (R1)**.** *Given an instance (G, k) with $k > 0$, if G has $k + 1$ obstructions $H_1, \ldots, H_{k+1}$ and a vertex v of G such that $V(H_i) \cap V(H_j) = \{v\}$ for every distinct i and j in $\{1, \ldots, k + 1\}$, then replace (G, k) with $(G\backslash v, k - 1)$.*

For small obstructions, we greedily find a maximal packing $\mathcal{P}$ of small obstructions in $G\backslash(S\backslash\{v\})$ that intersect precisely at v in S. If there are more than k such small obstructions in $\mathcal{P}$, we apply (R1) to (G, k). Otherwise, we define $m(v) := |\mathcal{P}|$, and for holes of length at least 6, we apply the result of Jansen and Pilipczuk [24, Lemma 1.3] to $G\backslash((S \cup \bigcup_{H\in\mathcal{P}} V(H))\backslash\{v\})$ and obtain either a set $\mathcal{H}$ of $k - m(v) + 1$ holes that intersect precisely at v in S, or a set S_v of at most $12(k - m(v))$ vertices such that $G\backslash((S_v \cup S \cup \bigcup_{H\in\mathcal{P}} V(H))\backslash\{v\})$ is chordal. By applying this procedure for all vertices v in S, we can obtain the following.

▶ **Lemma 3.1.** *Given an instance (G, k) with $k > 0$, one can find an equivalent instance (G', k') and a good modulator of G' having size at most $84k^2 + 7k$ such that $|V(G')| \leqslant |V(G)|$ and $k' \leqslant k$ in time bounded above by a polynomial in $|V(G)| + k$.*

4 Bounding the number of components outside of a good modulator

Let S be a good modulator of a graph G. We bound the number of components of $G\backslash S$.

4.1 Complete splits and blocking pairs

Cunningham [11] introduced a split of a graph. A *split* of a graph G is a partition (A, B) of $V(G)$ such that $|A| \geqslant 2$, $|B| \geqslant 2$, and $N(A)$ is complete to $N(B)$. We say that a split (A, B) of G is *complete* if $N(A) \cup N(B)$ is a clique. If a graph has a complete split, then obstructions must satisfy some conditions which we prove in the following two lemmas.

▶ **Lemma 4.1.** *Let (A, B) be a complete split of a graph G. If G has a hole H, then $V(H) \cap A = \varnothing$ or $V(H) \cap B = \varnothing$.*

▶ **Lemma 4.2.** *Let (A, B) be a complete split of a graph G. If G has an obstruction H having exactly two vertices in A, then H is isomorphic to the bull.*

Now, we define a blocking pair for a set $X \subseteq V(G)$. A *blocking pair for X* is an unordered pair $\{v, w\}$ of distinct vertices in $N(X)$ such that if v and w are adjacent and $N(v) \cap X = N(w) \cap X$, then $N(v) \cap X$ is not a clique. We say that X is *blocked by* $\{v, w\}$ if $\{v, w\}$ is a blocking pair for X. This definition is motivated by the following lemma that follows rather straightforwardly from the definition of a complete split of a graph.

▶ **Lemma 4.3.** *Let (A, B) be a partition of the vertex set of a graph G such that $|A| \geqslant 2$ and $|B| \geqslant 2$. Then (A, B) is a complete split of G if and only if $N(B)$ is a clique and B has no blocking pairs for A.*

The following lemma shows that if there is a blocking pair $\{v, w\}$ for a set $X \subseteq V(G)$ such that $G[X]$ has two distinct components whose vertex sets are blocked by $\{v, w\}$, then G is not a 3-leaf power.

▶ **Lemma 4.4.** *Let (A, B) be a partition of the vertex set of a graph G such that $|A| \geqslant 2$ and $|B| \geqslant 2$. If $G[A]$ has distinct components C_1 and C_2 such that both $V(C_1)$ and $V(C_2)$ are blocked by $\{v, w\}$ of vertices in B, then $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power.*

4.2 The number of non-trivial components

Let S^+ be the set of vertices v in S such that for each component C of $G \backslash S$, $N_G(v) \cap V(C)$ is a true twin-set in C, and $S^- := S \backslash S^+$. The following proposition shows that $G \backslash S$ has at most $|S^-|$ components having neighbors of S^-.

▶ **Proposition 4.5.** *Let S be a good modulator of a graph G, v be a vertex in S, and C be a component of $G \backslash S$. If $N_G(v) \cap V(C)$ contains distinct vertices w_1 and w_2 that are not true twins in C, then no components of $G \backslash S$ different from C have neighbors of v.*

We present a reduction rule to bound the number of non-trivial components of $G \backslash S$ having no neighbors of S^-. For that, we will use the following definition.

Let X be a set of vertices of a graph Q. For a non-negative integer ℓ, a set $M \subseteq E(Q)$ is an (X, ℓ)*-matching* of Q if each vertex in X is incident with at most ℓ edges in M, and each vertex in $V(Q) \backslash X$ is incident with at most one edge in M.

▶ **Reduction Rule 2** (R2)**.** *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let S^+ be the set of vertices u in S such that for each component C of $G \backslash S$, $N_G(u) \cap V(C)$ is a true twin-set in C, X be the set of 2-element subsets of S^+, and Y be the set of non-trivial components of $G \backslash S$ having no neighbors of $S \backslash S^+$. Let Q be a bipartite graph on $(X \times \{1, 2, 3\}, Y)$ such that the following three statements are true.*

(1) *Elements $(\{v, w\}, 1) \in X \times \{1\}$ and $C \in Y$ are adjacent in Q if and only if $V(C)$ is blocked by $\{v, w\}$.*

(2) *Elements* $(\{v, w\}, 2) \in X \times \{2\}$ *and* $C \in Y$ *are adjacent in* Q *if and only if* C *has a vertex adjacent to both* v *and* w*.*

(3) *Elements* $(\{v, w\}, 3) \in X \times \{3\}$ *and* $C \in Y$ *are adjacent in* Q *if and only if* C *has an edge* xy *such that* x *is adjacent to both* v *and* w*, and* y *is non-adjacent to both* v *and* w*.*

If Q *has a maximal* $(X \times \{1, 2, 3\}, k + 2)$*-matching* M *avoiding some element* U *in* Y*, then replace* (G, k) *with* $(G \backslash E(U), k)$*.*

Proof of Safeness. Let $G' := G \backslash E(U)$. Firstly, we show that if (G, k) is a yes-instance, then so is (G', k). Suppose that G has a modulator S' of size at most k, and $G' \backslash S'$ has an obstruction H. Since $G \backslash S'$ is a 3-leaf power, H has vertices b_1 and b_2 such that $b_1 b_2 \in E(U \backslash S')$. Thus, $|V(U) \backslash S'| \geqslant 2$.

▷ Claim 1. $(V(U) \backslash S', V(G) \backslash (V(U) \cup S'))$ is a split of $G' \backslash S'$.

Proof of Claim 1. We first show that $|V(G) \backslash (V(U) \cup S')| \geqslant 2$. If H is a hole of length 4, then H has at most two vertices of $U \backslash S'$, because $V(U) \backslash S'$ is an independent set of $G' \backslash S'$, and no holes of length 4 have an independent set of size at least 3. Therefore, H has at least two vertices of $G \backslash (V(U) \cup S')$. Thus, we may assume that $|V(H)| \geqslant 5$. By (O2), if H is small, then H has at most three vertices of $U \backslash S'$, and therefore H has at least two vertices of $G \backslash (V(U) \cup S')$. If H is a hole of length at least 6, then H has at most $\lfloor |V(H)|/2 \rfloor$ vertices of $U \backslash S'$, and therefore H has at least $\lceil |V(H)|/2 \rceil \geqslant 2$ vertices of $G \backslash (V(U) \cup S')$.

Therefore, $|V(G) \backslash (V(U) \cup S')| \geqslant 2$. Now, suppose that $(V(U) \backslash S', V(G) \backslash (V(U) \cup S'))$ is not a split of $G' \backslash S'$. Then $G \backslash (V(U) \cup S')$ has vertices v and w such that both v and w have neighbors in $V(U) \backslash S'$, and $N_G(v) \cap (V(U) \backslash S') \neq N_G(w) \cap (V(U) \backslash S')$. Thus, $\{v, w\}$ is a blocking pair for $V(U) \backslash S'$, so for $V(U)$. Then U is adjacent to $(\{v, w\}, 1)$ in Q. Since M is maximal, Y has distinct elements $C_1, \ldots, C_{k+2}$ different from U such that $V(C_i)$ is blocked by $\{v, w\}$ for each $i \in \{1, \ldots, k + 2\}$. Since $|S'| \leqslant k$, two of them, say C_1 and C_2, have no vertices in S'. Then $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power by Lemma 4.4, a contradiction, because it is an induced subgraph of $G \backslash S'$. ◁

Since $V(U) \backslash S'$ is an independent set of $G' \backslash S'$, and H is connected, both b_1 and b_2 have neighbors in $V(G) \backslash (V(U) \cup S')$. Then by Claim 1, b_1 and b_2 are false twins in $G' \backslash S'$. By (O5), both b_1 and b_2 have degree 2 in H. Let z_1 and z_2 be the neighbors of b_1 in $V(H) \cap S$. Then U is adjacent to $(\{z_1, z_2\}, 2)$ in Q. Since M is maximal, Y has distinct elements $C'_1, \ldots, C'_{k+2}$ different from U such that C'_i has a vertex adjacent to both z_1 and z_2 for each $i \in \{1, \ldots, k + 2\}$. Since $|S'| \leqslant k$, two of them, say C'_1 and C'_2, have no vertices in S'. Note that S' has no vertices of H, because H is an induced subgraph of $G' \backslash S'$.

If z_1 and z_2 are non-adjacent, then $G[V(C'_1) \cup V(C'_2) \cup \{z_1, z_2\}]$ has a hole of length 4, a contradiction, because it is an induced subgraph of $G \backslash S'$. Therefore, z_1 and z_2 are adjacent. Since $G[\{b_1, z_1, z_2\}]$ is isomorphic to K_3, H is not a hole, and therefore $|V(H)| = 5$. Let a be a vertex of H different from b_1, b_2, z_1, and z_2. We may assume that a is not in $V(C'_1)$, because otherwise we may swap C'_1 and C'_2. Let c be a vertex of C'_1 adjacent to both z_1 and z_2. Note that $G[\{b_1, b_2, z_1, z_2\}]$ is isomorphic to $K_4 \backslash b_1 b_2$. Since the dart and a hole of length 4 are the only obstructions having false twins, H is isomorphic to the dart. Thus, $N_H(a) = \{z_1\}$ or $N_H(a) = \{z_2\}$. Then $G[\{a, b_1, c, z_1, z_2\}]$ is isomorphic to the gem if c is adjacent to a, and the dart if c is non-adjacent to a, a contradiction, because it is an induced subgraph of $G \backslash S'$. Therefore, if (G, k) is a yes-instance, then so is (G', k).

Secondly, we show that if (G', k) is a yes-instance, then so is (G, k). Suppose that G' has a modulator S' of size at most k, and $G \backslash S'$ has an obstruction H. Since $G' \backslash S'$ is a 3-leaf power, H has an edge of $U \backslash S'$. Thus, $|V(U) \backslash S'| \geqslant 2$. Since S is a good modulator of G, H has at least two vertices in $S \backslash S'$. Then $|V(G) \backslash (V(U) \cup S')| \geqslant 2$, since $S \backslash S' \subseteq V(G) \backslash (V(U) \cup S')$.

▷ Claim 2. $(V(U)\backslash S', V(G)\backslash(V(U) \cup S'))$ is a complete split of $G\backslash S'$.

Since both $U\backslash S'$ and $G\backslash(V(U) \cup S')$ have vertices of H, H is not a hole by Lemma 4.1 and Claim 2, and therefore $|V(H)| = 5$. Let $t_1, \ldots, t_p$ be the vertices of H in $V(U)\backslash S'$, and $s_1, \ldots, s_q$ be the vertices of H in $V(G)\backslash(V(U) \cup S')$. Note that both p and q are at least 2. Since $|V(H)| = 5$, $(p, q) = (3, 2)$ or $(p, q) = (2, 3)$.

If $(p, q) = (3, 2)$, then we may assume that $N_H(s_1) = \{s_2\}$ and $N_H(s_2) = \{s_1, t_1, t_2\}$ by Lemma 4.2 and Claim 2. Since U has no neighbors of $S\backslash S^+$, s_2 is in S^+. Thus, t_1 and t_2 are true twins in $U\backslash S'$, contradicting (O1).

Therefore, $(p, q) = (2, 3)$. By Lemma 4.2 and Claim 2, we may assume that $N_H(t_1) = \{t_2\}$ and $N_H(t_2) = \{t_1, s_1, s_2\}$. Note that s_1 and s_2 are in $S\backslash S'$. Then U is adjacent to $(\{s_1, s_2\}, 3)$ in Q. Since M is maximal, Y has distinct elements $C''_1, \ldots, C''_{k+2}$ different from U such that C''_i has an edge x_iy_i such that x_i is adjacent to both s_1 and s_2, and y_i is non-adjacent to both s_1 and s_2 for each $i \in \{1, \ldots, k+2\}$. Since $|S'| \leqslant k$, two of them, say C''_1 and C''_2, have no vertices in S'. We may assume that s_3 is not in $V(C''_1)$, because otherwise we may swap C''_1 and C''_2. We remark that the bull is the only possible graph to which H is isomorphic. Thus, s_1 and s_2 are adjacent, and s_3 is adjacent to exactly one of s_1 and s_2 in H. Then by considering whether x_1 or y_1 is adjacent to s_3, one can easily show that $G[\{x_1, y_1, s_1, s_2, s_3\}]$ is an obstruction, a contradiction. Therefore, if (G', k) is a yes-instance, then so is (G, k). ◀

▶ **Proposition 4.6.** *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if (R2) is not applicable to (G, k), then $G\backslash S$ has at most $2(k+2)|S|^2$ non-trivial components.*

4.3 The number of isolated vertices

We present a reduction rule to bound the number of isolated vertices of $G\backslash S$. To bound the number, briefly speaking, we take a vertex set $U \subseteq S$ with $|U| \leqslant 4$ and mark at most $k+3$ isolated vertices v of $G\backslash S$ where $U \cup \{v\}$ is possibly a part of some obstruction in G. We prove that after the marking, we can safely remove the remaining isolated vertices from G.

▶ **Reduction Rule 3** (R3). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let $\mathcal{A}$ be the set of ordered pairs (A_1, A_2) of disjoint subsets of S such that $2 \leqslant |A_1| + |A_2| \leqslant 4$, and X be the set of isolated vertices of $G\backslash S$. For each $(A_1, A_2) \in \mathcal{A}$, let X_{A_1,A_2} be a maximal set of vertices v in X such that $N_G(v) \cap (A_1 \cup A_2) = A_1$ and $|X_{A_1,A_2}| \leqslant k+3$. If X has a vertex $u \notin \bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$, then replace (G, k) with $(G\backslash u, k)$.*

▶ **Proposition 4.7.** *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if (R3) is not applicable to (G, k), then $G\backslash S$ has at most $2(k+3)|S|^4/3$ isolated vertices.*

5 Bounding the size of components outside of a good modulator

Let S be a good modulator of a graph G. We first present a reduction rule to bound the size of each complete component of $G\backslash S$, which proceed by a similar marking process as (R3).

▶ **Reduction Rule 4** (R4). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let $\mathcal{A}$ be the set of ordered pairs (A_1, A_2) of disjoint subsets of S such that $2 \leqslant |A_1| + |A_2| \leqslant 4$, and C be a complete component of $G\backslash S$. For each $(A_1, A_2) \in \mathcal{A}$, let X_{A_1,A_2} be a maximal set of vertices v of C such that $N_G(v) \cap (A_1 \cup A_2) = A_1$ and $|X_{A_1,A_2}| \leqslant k+3$. If C has a vertex $u \notin \bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$, then replace (G, k) with $(G\backslash u, k)$.*

▶ **Proposition 5.1.** *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if (R4) is not applicable to (G, k), then every complete component of $G\backslash S$ has at most $2(k+3)|S|^4/3$ vertices.*

In the rest, we present four reduction rules to bound the size of each incomplete component of $G\backslash S$. Firstly, we present a reduction rule to bound the size of a true twin-set in G.

▶ **Reduction Rule 5** (R5). *Given an instance (G, k) with $k > 0$, if G has a true twin-set X such that $|X| \geqslant k + 2$, then replace (G, k) with $(G\backslash v, k)$ for some vertex $v \in X$.*

Later, we will apply (R5) only for true twin-sets in G that are subsets of $V(G)\backslash S$, which one can find in polynomial time by Theorem 2.2.

In the following reduction rules, we start with computing a tree-clique decomposition of $G\backslash S$. We present a reduction rule to remove some bags of $G\backslash S$ which are anti-complete to S.

▶ **Reduction Rule 6** (R6). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let B be a maximal true twin-set in $G\backslash S$. If $G\backslash(S \cup B)$ has a component D having no neighbors of S and $V(D)\backslash N_G(B)$ is non-empty, then replace (G, k) with $(G\backslash(V(D)\backslash N_G(B)), k)$.*

We present two reduction rules to reduce the number of bags of $G\backslash S$. Let C be an incomplete component of $G\backslash S$ with a tree-clique decomposition $(F, \{B_u : u \in V(F)\})$. We use (R7) for bounding the maximum degree of F to $|S| + 2k + 7$, and (R8) for bounding the number of nodes of F having degree 2 in F to $O(|S|)$.

▶ **Reduction Rule 7** (R7). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let B be a maximal true twin-set in $G\backslash S$. If $G\backslash(S \cup B)$ has distinct components $D_1, \ldots, D_{k+4}$ such that $N_G(V(D_1)) = \cdots = N_G(V(D_{k+4}))$, and either $V(D_1) \cup \cdots \cup V(D_{k+4}) \subseteq N_G(B)$, or $\varnothing \neq V(D_i) \cap N_G(B) \neq V(D_i)$ for each $i \in \{1, \ldots, k+4\}$, then replace (G, k) with $(G\backslash V(D_1), k)$.*

To show that (R7) is safe, we will use the following three lemmas. Lemma 5.3 will be useful because it implies that for a good modulator S of G, a subset B of $V(G)\backslash S$ is a true twin-set in $G\backslash S$ if and only if it is a true twin-set in G.

▶ **Lemma 5.2.** *Let P be an induced path of length at least 3 in a graph G. If G has a vertex v adjacent to both ends of P, then $G[V(P) \cup \{v\}]$ is not distance-hereditary.*

▶ **Lemma 5.3.** *Let G be a 3-leaf power having a vertex v such that $G\backslash v$ is connected and incomplete. Then vertices t_1 and t_2 in $V(G)\backslash\{v\}$ are true twins in G if and only if t_1 and t_2 are true twins in $G\backslash v$.*

▶ **Lemma 5.4.** *Let (A, B) be a complete split of a graph G, and S be a non-empty good modulator of G. If G has an obstruction H, and $S \subseteq B\backslash N(A)$, then H has at most one vertex in A.*

Proof of Safeness for (R7). We need to show that if $(G\backslash V(D_1), k)$ is a yes-instance, then so is (G, k). Suppose that $G\backslash V(D_1)$ has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Since $G\backslash(V(D_1) \cup S')$ is a 3-leaf power, H has at least one vertex of D_1. Since S is a good modulator of G, $G\backslash(S\backslash\{v\})$ is a 3-leaf power for each vertex v in S. Thus, if v has a neighbor in a true twin-set X in $G\backslash S$, then $\{v\}$ is complete to X by Lemma 5.3. This means that every true twin-set in $G\backslash S$ is a true twin-set in G as well.

We claim that (a) for each $i \in \{1, \ldots, k+4\}$, $V(D_i) \cap N_G(B)$ is a true twin-set in $G\backslash S$. Suppose that $V(D_i) \cap N_G(B)$ contains two vertices x and y such that x is non-adjacent to

y. Let P be an induced path in D_i from x to y. By Lemma 5.2, the length of P is exactly 2 . Let z be a common neighbor of x and y in $V(P)$. Then $z \in N_G(B)$, because otherwise $V(P)$ with a vertex in B induces a hole of length 4. Then for a vertex v in B, and v' in $V(D_j) \cap N_G(B)$ for some $j \in \{1, \ldots, k+4\} \setminus \{i\}$, $G[\{v, v', x, y, z\}]$ is isomorphic to the dart, contradicting the assumption that S is a modulator of G. Therefore, $V(D_i) \cap N_G(B)$ is a clique. Now, suppose that $G \backslash S$ has a vertex w adjacent to a vertex $t_1 \in V(D_i) \cap N_G(B)$ and non-adjacent to a vertex $t_2 \in V(D_i) \cap N_G(B)$. Note that w is a vertex of $D_i \backslash N_G(B)$. Then for a vertex v in B and a vertex v' of $V(D_j) \cap N_G(B)$ for some $j \in \{1, \ldots, k+4\} \setminus \{i\}$, $G[\{v, v', w, t_1, t_2\}]$ is isomorphic to the bull, a contradiction, and this proves (a).

Suppose that $V(D_1) \cup \cdots \cup V(D_{k+4}) \subseteq N_G(B)$. By (O1), for each $i \in \{1, \ldots, k+4\}$, D_i has at most one vertex of H. By (O2), if H is small, then at most three of $D_1, \ldots, D_{k+4}$ have vertices of H. If H is a hole of length at least 6, then at most two of $D_1, \ldots, D_{k+4}$ have vertices of H, because otherwise H has a vertex of degree at least 3 in H. Since $|S'| \leqslant k$, one of $D_2, \ldots, D_{k+4}$, say D_j, has no vertices in $S' \cup V(H)$. Let s be a vertex of H in D_1 and t be a vertex in D_j. Since $N_G(V(D_1)) = N_G(V(D_j))$, s and t have the same set of neighbors in $V(H)$. Then $G[(V(H) \setminus \{s\}) \cup \{t\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G \backslash (V(D_1) \cup S')$.

Therefore, $\emptyset \neq V(D_i) \cap N_G(B) \neq V(D_i)$ for each $i \in \{1, \ldots, k+4\}$. We claim that (b) $D_i \backslash N_G(B)$ has no neighbors of S. Suppose that $D_i \backslash N_G(B)$ has a neighbor p_i of some vertex v in S. Let $j \in \{1, \ldots, k+4\} \setminus \{i\}$. Since $N_G(V(D_i)) = N_G(V(D_j))$, D_j has a neighbor p_j of v. Since some vertex in B has neighbors in both D_i and D_j, $G \backslash S$ has a path P from p_i to p_j. Note that the length of P is at least 3, because p_i is not in $N_G(B)$. Since v is adjacent to both ends of P, $G[V(P) \cup \{v\}]$ is not distance-hereditary by Lemma 5.2, a contradiction, because it is an induced subgraph of $G \backslash (S \setminus \{v\})$, and this proves (b).

For each $i \in \{1, \ldots, k+4\}$, since $V(D_i) \cap N_G(B)$ is a true twin-set in G, H has at most one vertex in $V(D_i) \cap N_G(B)$ by (O1). Let $D_{i,1}, \ldots, D_{i,m(i)}$ be the components of $D_i \backslash N_G(B)$ for each $i \in \{1, \ldots, k+4\}$. We claim that (c) for each $j \in \{1, \ldots, m(i)\}$, if $|V(D_{i,j})| \geqslant 2$, then $(V(D_{i,j}), V(G) \backslash V(D_{i,j}))$ is a complete split of G. Since $V(D_i) \cap N_G(B)$ is a true twin-set in G, and $D_i \backslash N_G(B)$ has no neighbors of S, it suffices to show that $N_G(N_G(B)) \cap V(D_{i,j})$ is a clique. Suppose that $N_G(N_G(B)) \cap V(D_{i,j})$ contains vertices x and y which are non-adjacent. Let P be an induced path in $D_{i,j}$ from x to y. By Lemma 5.2, the length of P is exactly 2. Let z be a common neighbor of x and y in $V(P)$. Then $z \in N_G(N_G(B))$, because otherwise P with a vertex v in $N_G(B) \cap V(D_i)$ induces a hole of length 4. Then for a vertex v' in B, $G[\{v, v', x, y, z\}]$ is isomorphic to the dart, a contradiction, and this proves (c).

Therefore, each component of $D_i \backslash N_G(B)$ has at most one vertex of H by Lemma 5.4. Each $V(D_i) \cap N_G(B)$ has at most one vertex of H, because $V(D_i) \cap N_G(B)$ is a true twin-set. Therefore, at most one component of $D_i \backslash N_G(B)$ has a vertex of H, because H cannot have false twins of degree at most 1 by (O5). By (O2), if H is small, then at most three of $D_1, \ldots, D_{k+4}$ have vertices of H. If H is a hole of length at least 6, then at most two of $D_1, \ldots, D_{k+4}$ have vertices of H, because otherwise H has a vertex of degree at least 3 in H. Since $|S'| \leqslant k$, one of $D_2, \ldots, D_{k+4}$, say D_i, has no vertices in $S' \cup V(H)$. Note that H has a vertex s_1 in $V(D_1) \cap N_G(B)$, because $D_1 \backslash N_G(B)$ has no neighbors of S, H is connected, and has vertices in both S and $V(D_1)$. Let $t_1 t_2$ be an edge of D_i such that $t_1 \in V(D_i) \cap N_G(B)$ and $t_2 \in V(D_i) \backslash N_G(B)$. Since $N_G(V(D_1)) = N_G(V(D_i))$, and both $V(D_1) \cap N_G(B)$ and $V(D_i) \cap N_G(B)$ are true twin-sets, s_1 and t_1 have the same set of neighbors in $V(H) \backslash V(D_1)$. If H has a vertex s_2 in $V(D_1) \backslash N_G(B)$, then $V(D_1) \cap V(H) = \{s_1, s_2\}$, because both $V(D_1) \cap N_G(B)$ and $V(D_1) \backslash N_G(B)$ have at most one vertex of H. Then $G[(V(H) \setminus \{s_1, s_2\}) \cup \{t_1, t_2\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(V(D_1) \cup S')$. Therefore, H has no vertices in $V(D_1)\backslash N_G(B)$. Then $G[(V(H)\backslash\{s_1\}) \cup \{t_1\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(V(D_1) \cup S')$. ◀

▶ **Reduction Rule 8** (R8). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let $B_1, \dots, B_m$ be pairwise disjoint maximal true twin-sets in $G\backslash S$ for $m \geqslant 6$ such that $N_G(B_i) = B_{i-1} \cup B_{i+1}$ for each $i \in \{2, \dots, m-1\}$. Let ℓ be an integer in $\{3, \dots, m-2\}$ such that $|B_\ell| \leqslant |B_i|$ for each $i \in \{3, \dots, m-2\}$, and G' be a graph obtained from $G\backslash((B_3 \cup \cdots \cup B_{m-2})\backslash B_\ell)$ by making B_ℓ complete to $B_2 \cup B_{m-1}$. Then replace (G, k) with (G', k).*

By applying aforementioned reduction rules exhaustively to an input instance (G, k) with a good modulator S of G, we can bound the size of each incomplete component of $G\backslash S$.

▶ **Proposition 5.5.** *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if none of (R2), (R5), (R6), (R7), and (R8) is applicable to (G, k), then each incomplete component of $G\backslash S$ has at most $(k+1)(k+4)|S|(|S| + 2k + 15)$ vertices.*

To prove Proposition 5.5, we will use the following lemma.

▶ **Lemma 5.6** (Brandstädt and Le [5, Corollary 11]). *Let G be a 3-leaf power. If G has a vertex v of degree at least 1 such that $G\backslash v$ is connected, then $G\backslash v$ has a true twin-set B such that $N_G(v) = B$ or $N_G[v] = N_G[B]$.*

Proof of Proposition 5.5. Let C be an incomplete component of $G\backslash S$ with a tree-clique decomposition $(F, \{B_u : u \in V(F)\})$. Since S is a good modulator of G, $G[V(C) \cup \{v\}]$ is a 3-leaf power for each vertex v in S. Thus, if S has a vertex w having a neighbor in a bag B of C, then $\{w\}$ is complete to B by Lemma 5.3. This means that every bag of C is a true twin-set in G. Since (R5) is not applicable to (G, k), each bag of C contains at most $k+1$ vertices. Therefore, in the remaining of this proof, we are going to bound the number of bags of C. Let X be the set of leaves of F whose bags are anti-complete to S.

▷ Claim 3. If a node u of $F\backslash X$ has degree at most 1 in $F\backslash X$, then $B_u \cap N(S) \neq \varnothing$.

Proof of Claim 3. If $N_F(u) \subseteq X$, then B_u contains a neighbor of S, because otherwise C has no neighbors of S and (R2) is applicable to (G, k). If $N_F(u)\backslash X$ is non-empty, then $N_F(u)\backslash X$ contains exactly one node u_1, because u has degree at most 1 in $F\backslash X$. If B_u contains no neighbors of S, then (R6) is applicable to (G, k) by taking B_{u_1} as B. Therefore, B_u contains a neighbor of S. ◁

▷ Claim 4. The maximum degree of F is at most $|S| + 2k + 7$.

Proof of Claim 4. Suppose that F has a node u of degree at least $|S| + 2k + 8$ in F. For each vertex w in S, if at least two components of $C\backslash B_u$ have neighbors of w, then all components of $C\backslash B_u$ have neighbors of w by Lemma 5.6. Thus, for each vertex w in S, we can choose a component of $C\backslash B_u$, say D, such that either all other components of $C\backslash B_u$ have neighbors of w, or no other components of $C\backslash B_u$ have neighbors of w. Since $C\backslash B_u$ has at least $|S| + 2k + 8$ components, $C\backslash B_u$ has distinct components $D_1, \dots, D_{2k+7}$ different from D such that for each vertex w in S, either all or none of them have neighbors of w. Thus, $N_G(V(D_1)) = \cdots = N_G(V(D_{2k+7}))$. By the pigeonhole principle, $V(D_i) \subseteq N_G(B_u)$ or $\varnothing \neq V(D_i) \cap N_G(B_u) \neq V(D_i)$ is satisfied by at least $k+4$ values of i, contradicting the assumption that (R7) is not applicable to (G, k). ◁

For each vertex v in S, let X_v be the set of nodes of $F \backslash X$ whose bags contain neighbors of v, S_1 be the set of vertices v in S such that X_v contains some leaf of $F \backslash X$, and $S_2 := S \backslash S_1$. Note that by Lemma 5.6, for each vertex v in S, if X_v is non-empty, then $F \backslash X$ has a node, say p, such that $X_v = \{p\}$ or $X_v = N_{F \backslash X}[p]$. Let F' be a tree obtained from $F \backslash X$ by contracting all edges in $F[X_v]$ for each vertex v in S. By Claim 3, F' has at most $|S_1|$ leaves, and therefore it has at most $\max(|S_1| - 2, 0)$ branching nodes. Let Y be the set of nodes of F' which come from X_v for some vertex $v \in S$, and Z be the set of branching nodes of F'. Then $|Y \cup Z| \leqslant |Y| + |Z| \leqslant |S| + \max(|S_1| - 2, 0) \leqslant 2|S|$. Since (R8) is not applicable to (G, k), each component of $F' \backslash (Y \cup Z)$ has at most three nodes. Therefore, $|V(F' \backslash (Y \cup Z))| \leqslant 6|S|$. Then by Claim 4, $|V(F \backslash X)|$ is at most

$$\begin{aligned} |Y|(|S| + 2k + 8) + |Z| + |V(F' \backslash (Y \cup Z))| &\leqslant |S|(|S| + 2k + 8) + |S| + 6|S| \\ &= |S|(|S| + 2k + 15). \end{aligned}$$

Since (R7) is not applicable to (G, k), each node of $F \backslash X$ is adjacent to at most $k + 3$ nodes in X. Thus, $|V(F)| \leqslant (k + 4)|S|(|S| + 2k + 15)$. By (R5), each bag of C has at most $k + 1$ nodes. Therefore, $|V(C)| \leqslant (k + 1)(k + 4)|S|(|S| + 2k + 15)$. ◀

Proof of Theorem 1.1. By Lemma 3.1, we can reduce an input instance to an equivalent instance with a good modulator having at most $O(k^2)$ vertices in polynomial time. Each of (R2), …, (R8) can be applied in polynomial time by Theorem 2.2.

Let (G, k) be the resulting instance and S be a good modulator of G obtained by Lemma 3.1. We are going to show that if none of (R2), …, (R8) are applicable to (G, k), then $|V(G)| = O(k^{14})$. We may assume that $|S| \geqslant k + 1$. By Proposition 4.6, $G \backslash S$ has at most $2(k + 2)|S|^2$ non-trivial components. By Proposition 5.1, each complete component of $G \backslash S$ has at most $2(k + 3)|S|^4/3$ vertices. By Proposition 5.5, each incomplete component of $G \backslash S$ has at most $(k + 1)(k + 4)|S|(|S| + 2k + 15)$ vertices. Therefore, each non-trivial component of $G \backslash S$ has at most $O(k|S|^4)$ vertices. Then the union of all non-trivial components of $G \backslash S$ has at most $2(k + 2)|S|^2 \cdot O(k|S|^4) = O(k^2|S|^6)$ vertices. By Proposition 4.7, $G \backslash S$ has at most $2(k + 3)|S|^4/3$ isolated vertices. Thus, $|V(G)| \leqslant |S| + 2(k + 3)|S|^4/3 + O(k^2|S|^6) = O(k^2|S|^6)$. By Lemma 3.1, $|S| = O(k^2)$, and therefore $|V(G)| = O(k^{14})$. ◀

6 Conclusions

In this paper, we show that 3-Leaf Power Deletion admits a kernel with $O(k^{14})$ vertices. It would be an interesting problem to significantly reduce the size of the kernel.

Gurski and Wanke [22] stated that for every positive integer ℓ, ℓ-leaf powers have bounded clique-width. Rautenbach [32] presented a characterization of 4-leaf powers with no true twins as chordal graphs with ten forbidden induced subgraphs. This can be used to express, in monadic second-order logic, whether a graph is a 4-leaf power and whether there is a vertex set of size at most k whose deletion makes the graph a 4-leaf power. Therefore, by using the algorithm in [10], we deduce that 4-Leaf Power Deletion is fixed-parameter tractable when parameterized by k. It is natural to ask whether 4-Leaf Power Deletion admits a polynomial kernel. For $\ell \geqslant 5$, we do not know whether we can express ℓ-leaf powers in monadic second-order logic. If it is true for some ℓ, then not only ℓ-Leaf Power Deletion is fixed-parameter tractable, but also ℓ-Leaf Power Recognition can be solved in polynomial time, which is still open for $\ell \geqslant 7$.

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