

# Multicriteria Cuts and Size-Constrained $k$ -Cuts in Hypergraphs

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## Abstract

We address counting and optimization variants of multicriteria global min-cut and size-constrained min- $k$ -cut in hypergraphs.

1. For an  $r$ -rank  $n$ -vertex hypergraph endowed with  $t$  hyperedge-cost functions, we show that the number of multiobjective min-cuts is  $O(r2^{tr}n^{3t-1})$ . In particular, this shows that the number of parametric min-cuts in constant rank hypergraphs for a constant number of criteria is strongly polynomial, thus resolving an open question by Aissi, Mahjoub, McCormick, and Queyranne [1]. In addition, we give randomized algorithms to enumerate all multiobjective min-cuts and all pareto-optimal cuts in strongly polynomial-time.
2. We also address node-budgeted multiobjective min-cuts: For an  $n$ -vertex hypergraph endowed with  $t$  vertex-weight functions, we show that the number of node-budgeted multiobjective min-cuts is  $O(r2^r n^{t+2})$ , where  $r$  is the rank of the hypergraph, and the number of node-budgeted  $b$ -multiobjective min-cuts for a fixed budget-vector  $b \in \mathbb{R}_+^t$  is  $O(n^2)$ .
3. We show that min- $k$ -cut in hypergraphs subject to constant lower bounds on part sizes is solvable in polynomial-time for constant  $k$ , thus resolving an open problem posed by Queyranne [11]. Our technique also shows that the number of optimal solutions is polynomial.

All of our results build on the random contraction approach of Karger [12]. Our techniques illustrate the versatility of the random contraction approach to address counting and algorithmic problems concerning multiobjective min-cuts and size-constrained  $k$ -cuts in hypergraphs.

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## 1 Introduction

Cuts and partitioning play a central role in combinatorial optimization and have numerous theoretical as well as practical applications. We consider multicriteria cut problems in hypergraphs. Let  $G = (V, E)$  be an  $n$ -vertex hypergraph and  $c_1, \dots, c_t : E \rightarrow \mathbb{Z}_+$  be  $t$  non-negative hyperedge-cost functions, where  $t$  is a constant. The cost of a subset  $F$  of hyperedges under criterion  $i \in [t]$  is  $c_i(F) := \sum_{e \in F} c_i(e)$ . For a positive integer  $k$ , a subset



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of hyperedges that crosses a  $k$ -partition  $(U_1, \dots, U_k)$  of the vertex set is said to be a  $k$ -cut. We refer to a 2-cut simply as a cut. We recall that the rank of a hypergraph  $G$  is the size of the largest hyperedge in  $G$  (the rank of a graph is 2).

Since we have several criteria, there may not be a single cut that is best for all criteria. In multicriteria optimization, there are three important notions to measure the quality of a cut: (i) parametric min-cuts, (ii) pareto-optimal cuts, and (iii) multiobjective min-cuts. A cut  $F$  is a *parametric min-cut* if there exist positive multipliers  $\mu_1, \dots, \mu_t \in \mathbb{R}_+$  such that  $F$  is a min-cut in the hypergraph  $G$  with hyperedge costs given by  $c_\mu(e) := \sum_{i=1}^t \mu_i c_i(e)$  for all  $e \in E$ . A cut  $F$  *dominates* another cut  $F'$  if  $c_i(F) \leq c_i(F')$  for every  $i \in [t]$  and there exists  $i \in [t]$  such that  $c_i(F) < c_i(F')$ . A cut  $F$  is *pareto-optimal* if it is not dominated by any other cut. For a budget-vector  $b \in \mathbb{R}_+^{t-1}$ , a cut  $F$  is a *b-multiobjective min-cut* if  $c_i(F) \leq b_i$  for every  $i \in [t-1]$  and  $c_t(F)$  is minimum subject to these constraints. A cut  $F$  is a *multiobjective min-cut* if there exists a non-negative budget-vector  $b \in \mathbb{R}_+^{t-1}$  for which  $F$  is a  $b$ -multiobjective min-cut. These three notions satisfy the following relationship with the containment being possibly strict (see the full version of this paper for a proof [4]):

$$\text{Parametric min-cuts} \subseteq \text{Pareto-optimal cuts} \subseteq \text{Multiobjective min-cuts}. \quad (1)$$

There is also a natural notion of min-cuts under node-weighted budget constraints. Let  $w_1, \dots, w_t : V \rightarrow \mathbb{R}_+$  be vertex-weight functions and  $c : E \rightarrow \mathbb{R}_+$  be a hyperedge-cost function. For a budget-vector  $b \in \mathbb{R}_+^t$ , a subset  $F \subseteq E$  of hyperedges is a *node-budgeted b-multiobjective min-cut* if  $F = \delta(U)$  for some subset  $\emptyset \neq U \subsetneq V$  with  $\sum_{u \in U} w_i(u) \leq b_i$  for all  $i \in [t]$  and  $c(F)$  is minimum among all such subsets of  $E$ . A cut  $F$  is a *node-budgeted multiobjective min-cut* if there exists a non-negative budget-vector  $b$  for which  $F$  is a node-budgeted  $b$ -multiobjective min-cut. In this work, we address the following natural questions concerning multiobjective min-cuts and min- $k$ -cuts:

1. Multiobjective min-cuts: Is the number of multiobjective min-cuts at most strongly polynomial?
2. Node-budgeted multiobjective min-cuts: Is the number of node-budgeted multiobjective min-cuts at most strongly polynomial?
3. Size-constrained min- $k$ -cut: For fixed positive integers  $k$  and  $s_1, \dots, s_k$  (all constants), a vertex-weight function  $w : V \rightarrow \mathbb{Z}_+$ , and a hyperedge-cost function  $c : E \rightarrow \mathbb{R}_+$ , can we compute a  $k$ -cut  $F$  with minimum  $c(F)$  subject to the constraint that  $F$  is the set of hyperedges crossing some  $k$ -partition  $(U_1, \dots, U_k)$  of  $V$  where  $\sum_{u \in U_i} w(u) \geq s_i$  for every  $i \in [k]$  in polynomial-time? Is the number of optimal solutions strongly polynomial?

**Previous Work.** For single criterion, a classic result of Dinitz, Karzanov, and Lomonosov [7] shows that the number of min-cuts in an  $n$ -vertex graph is  $O(n^2)$  (also see Karger [12]). The same upper bound was shown to hold for constant-rank hypergraphs by Kogan and Krauthgamer [14] and for arbitrary-rank hypergraphs by Chekuri and Xu [6] and by Ghaffari, Karger, and Panigrahi [8] via completely different techniques. For  $t = 2$  criteria in graphs, Mulmuley [15] showed an  $O(n^{19})$  upper bound on the number of parametric min-cuts. For  $t$  criteria in constant-rank hypergraphs, Aissi, Mahjoub, McCormick, and Queyranne [1] showed that the number of parametric min-cuts is  $\tilde{O}(m^t n^2)$ , where  $m$  is the number of hyperedges, using the fact that the number of approximate min-cuts in constant-rank hypergraphs is polynomial. Karger [13] improved this bound to  $O(n^{t+1})$  by a clever and subtle argument based on his random contraction algorithm; we will describe his argument later. Karger also constructed a graph that exhibited  $\Omega(n^{t/2})$  parametric min-cuts.

Armon and Zwick [3] showed that all pareto-optimal cuts in graphs can be enumerated in pseudo-polynomial time. For  $t = 2$  criteria in constant-rank hypergraphs, Aissi et al [1] showed an upper bound of  $\tilde{O}(n^5)$  on the number of pareto-optimal cuts – this was the first result showing a strongly polynomial upper bound. Aissi et al raised the question of whether the number of pareto-optimal cuts is strongly polynomial for a constant number  $t$  of criteria in constant-rank hypergraphs (or even in graphs). Note that, by containment relationship (1), answering our first question affirmatively would also answer their open question. On a related note, Aissi, Mahjoub, and Ravi [2] designed a random contraction based algorithm to solve the  $b$ -multiobjective min-cut problem in graphs. The correctness analysis of their algorithm also implies that the number of  $b$ -multiobjective min-cuts in graphs for a fixed budget-vector  $b \in \mathbb{R}_+^{t-1}$  is  $O(n^{2t})$ . We emphasize the subtle, but important, distinction between the number of  $b$ -multiobjective min-cuts for a fixed budget-vector  $b$  and the number of multiobjective min-cuts.

Node-budgeted multiobjective min-cut has a rich literature extending nicely to submodular functions. For graphs, Armon and Zwick [3] gave a polynomial-time algorithm to find a minimum valued cut with at most  $b$  vertices in the smaller side. Goemans and Soto [9] addressed the more general problem of minimizing a symmetric submodular function  $f : 2^V \rightarrow \mathbb{R}$  over a downward-closed family  $\mathcal{I}$ . Recall that the hypergraph cut function is symmetric submodular and the family of vertex subsets satisfying node-weighted budget constraints is in fact downward-closed. Goemans and Soto extended Queyranne’s submodular minimization algorithm to enumerate all the  $O(n)$  minimal minimizers in  $\mathcal{I}$  using  $O(n^3)$  oracle calls to the function  $f$  and the family  $\mathcal{I}$ . Their result implies that the number of minimal minimizers is  $O(n)$ , but it is straightforward to see that the total number of minimizers could be exponential. For the special case of node-budgeted multiobjective min-cuts in graphs, Aissi, Mahjoub, and Ravi [2] gave a faster algorithm than that of Goemans and Soto – their algorithm is based on random contraction, runs in  $\tilde{O}(n^2)$ -time, and shows that the number node-budgeted  $b$ -multiobjective min-cuts in graphs for a fixed budget-vector  $b \in \mathbb{R}_+^t$  is  $O(n^2)$ .

For size-constrained min- $k$ -cut, if we allow arbitrary sizes (i.e., arbitrary lower bounds), then the problem becomes NP-hard even for  $k = 2$  as it captures the well-studied min-bisection problem in graphs. If we consider constant sizes but arbitrary  $k$ , then the problem is again NP-hard in graphs [10]. So, our focus is on constant  $k$  and constant sizes. Guinez and Queyranne [11] raised size-constrained min- $k$ -cut with unit vertex-weights as a sub-problem towards resolving the complexity of the submodular  $k$ -partitioning problem. In submodular  $k$ -partitioning, we are given a submodular function  $f : 2^V \rightarrow \mathbb{R}$  (by value oracle) and a fixed constant integer  $k$  (e.g.,  $k = 2, 3, 4, 5, \dots$ ) and the goal is to find a  $k$ -partition  $(U_1, \dots, U_k)$  of the ground set  $V$  so as to minimize  $\sum_{i=1}^k f(U_i)$ . The complexity of even special cases of this problem are open: e.g., if the submodular function  $f$  is the cut function of a given hypergraph, then its complexity is unknown.<sup>1</sup> Guinez and Queyranne showed surprisingly strong non-crossing properties between optimum solutions to size-constrained  $(k - 1)$ -partitioning (constant size lower bounds on the parts) and optimum solutions to  $k$ -partitioning. This motivated them to study the size-constrained min- $k$ -cut problem in hypergraphs for unit vertex-weights as a special case. They showed that size-constrained min- $k$ -cut for unit vertex-weights is solvable in polynomial-time in constant-rank hypergraphs

<sup>1</sup> We note that if the submodular function  $f$  is the cut function of a given hypergraph, then the submodular  $k$ -partition problem is not identical to hypergraph  $k$ -cut as the two objectives are different. However, if the submodular function is the cut function of a given graph, then the submodular  $k$ -partition problem coincides with the graph  $k$ -cut problem which is solvable in polynomial-time.

(with exponential run-time dependence on the rank) and mention the open problem of designing an algorithm for it in arbitrary-rank hypergraphs. The size-constrained min- $k$ -cut problem for unit sizes (i.e., all size lower-bounds  $s_1, \dots, s_k$  are equal to one) is known as the hypergraph  $k$ -cut problem. The hypergraph  $k$ -cut problem was shown to admit a polynomial-time algorithm only recently [5] via a non-uniform random contraction algorithm.

## 1.1 Our Contributions

Our high-level contribution is in showing the versatility of the random contraction technique to address algorithmic and counting problems concerning multiobjective min-cuts and size-constrained min- $k$ -cuts in hypergraphs. All of our results build on the random contraction technique with additional insights.

Our first result is a strongly polynomial upper bound on the number of multiobjective min-cuts in constant-rank hypergraphs.

► **Theorem 1.1.** *The number of multiobjective min-cuts in an  $r$ -rank,  $n$ -vertex hypergraph  $G$  with  $t$  hyperedge-cost functions is  $O(r2^{rt}n^{3t-1})$ .*

We emphasize that our upper bound is over all possible non-negative budget-vectors (in contrast to the number of  $b$ -multiobjective min-cuts for a fixed budget-vector  $b$ ). Theorem 1.1 and Proposition 1 imply that the number of pareto-optimal cuts in constant-rank hypergraphs is  $O(n^{3t-1})$  and hence, is strongly polynomial for constant number of criteria. This answers the main open question posed by Aissi, Mahjoub, McCormick, and Queyranne [1]. We also design randomized polynomial time algorithms to enumerate all multiobjective min-cuts and all pareto-optimal cuts in constant-rank hypergraphs (see Section A.3). Independent of our work, Rico Zenklusen has also shown Theorem 1.1. We learned after submission of this work that his approach leads to *deterministic* polynomial time algorithms to enumerate all multiobjective min-cuts and all pareto-optimal cuts in constant-rank hypergraphs.

Given the upper bound in Theorem 1.1, a discussion on the lower bound is in order. We recall that Karger [13] constructed a graph with  $t$  edge-cost functions that exhibited  $\Omega(n^{t/2})$  parametric min-cuts. This is also a lower bound on the number of pareto-optimal cuts and multiobjective min-cuts by (1). We improve this lower bound for pareto-optimal cuts by constructing a graph with  $t$  edge-cost functions that exhibits  $\Omega(n^t)$  pareto-optimal cuts (see Section A.4). Our instance also exhibits the same lower bound on the number of  $b$ -multiobjective min-cuts for a fixed budget-vector  $b$ .

Our next result is an upper bound on the number of node-budgeted multiobjective min-cuts and node-budgeted  $b$ -multiobjective min-cuts.

► **Theorem 1.2.** *1. The number of node-budgeted multiobjective min-cuts in an  $r$ -rank,  $n$ -vertex hypergraph with  $t$  vertex-weight functions is  $O(r2^r n^{t+2})$ .  
2. For a fixed budget-vector  $b \in \mathbb{R}_+^t$ , the number of node-budgeted  $b$ -multiobjective min-cuts in an  $n$ -vertex hypergraph with  $t$  vertex-weight functions is  $O(n^2)$ .*

We draw the reader's attention to the distinction between the two parts in Theorem 1.2. The first part implies that the number of node-budgeted multiobjective min-cuts is strongly polynomial in constant-rank hypergraphs for constant number of vertex-weight functions. The second part implies that the number of node-budgeted  $b$ -multiobjective min-cuts for any fixed budget-vector  $b \in \mathbb{R}_+^t$  is strongly polynomial in arbitrary-rank hypergraphs for any number  $t$  of vertex-weight functions.

Our final result shows that the size-constrained min- $k$ -cut problem can be solved in polynomial time for constant  $k$  and constant sizes (in arbitrary-rank hypergraphs).

► **Theorem 1.3.** *Let  $k \geq 2$  be a fixed positive integer and let  $1 \leq s_1 \leq s_2 \leq \dots \leq s_k$  be fixed positive integers. Let  $G$  be an  $n$ -vertex hypergraph with hyperedge-cost function  $c : E \rightarrow \mathbb{R}_+$ . Then, there exists a polynomial-time algorithm that takes  $(G, c)$  as input and returns a fixed  $w$ -weighted  $s$ -size-constrained min- $k$ -cut for any choice of vertex-weight function  $w : V \rightarrow \mathbb{Z}_+$  with probability  $\Omega\left(\frac{1}{n^{2\sigma_{k-1}+1}}\right)$ , where  $\sigma_{k-1} := \sum_{i=1}^{k-1} s_i$ .*

Theorem 1.3 resolves an open problem posed by Guinez and Queyranne [11]. A structural consequence of Theorem 1.3 is that the number of size-constrained min- $k$ -cuts (over all possible node-weight functions  $w : V \rightarrow \mathbb{Z}_+$ ) in a given hypergraph is polynomial for constant sizes and constant  $k$ .

We refer the reader to Table 1 for a comparison of known results and our contributions.

■ **Table 1** Text in gray refers to known results while text in black illustrates results from this work. Here,  $t$  denotes the number of criteria (i.e., the number of hyperedge-cost/vertex-weight functions),  $r$  denotes the rank of the hypergraph, and  $n$  denotes the number of vertices.

Problem	Graphs	$r$ -rank Hypergraphs	Hypergraphs
# of Parametric Min-Cuts	$O(n^{t+1})$ [13] $\Omega(n^{t/2})$ [13]	$O(2^r n^{t+1})$ [13]	OPEN
# of Pareto-Optimal Cuts	$\tilde{O}(n^5)$ for $t = 2$ [1] $O(n^{3t-1})$ [Thm 1.1] $\Omega(n^t)$ [Thm A.2]	$O(r2^{rt} n^{3t-1})$ [Thm 1.1]	OPEN
# of $b$ -Multiobjective Min-Cuts	$O(n^{2t})$ [2] $\Omega(n^t)$ [Thm A.2]	$O(r2^{rt} n^{2t})$ [Thm 2.1]	OPEN
# of Multiobjective Min-Cuts	$O(n^{3t-1})$ [Thm 1.1] $\Omega(n^t)$ [Thm A.2]	$O(r2^{rt} n^{3t-1})$ [Thm 1.1]	OPEN
# of Node-Budgeted $b$ -Multiobjective Min-Cuts	$O(n^2)$ [2]	$O(n^2)$ [Thm 1.2]	$O(n^2)$ [Thm 1.2]
# of Node-Budgeted Multiobjective Min-Cuts	$O(n^{t+2})$ [Thm 1.2]	$O(r2^r n^{t+2})$ [Thm 1.2]	OPEN
Node-Weighted $s$ -Size-Constrained $k$ -cut (const. $k$ and const. $s \in \mathbb{Z}^k$ )	Poly-time [11]	Poly-time [11]	Poly-time [Thm 1.3]

## 1.2 Technical Overview

As mentioned earlier, all our results build on the random contraction technique introduced by Karger [12] to solve the global min-cut problem in graphs. Here, a *uniform* random edge of the graph is contracted in each step until the graph has only two nodes; the set of edges between the two nodes is returned as the cut. Karger showed that this algorithm returns a fixed global min-cut with probability  $\Omega(n^{-2})$ . As a consequence, the number of min-cuts in an  $n$ -vertex graph is  $O(n^2)$ . The algorithm extends naturally to  $r$ -rank hypergraphs, however the naive analysis will only show that the algorithm returns a fixed global min-cut with probability  $\Omega(n^{-r})$ . Kogan and Krauthgamer [14] introduced an LP-based analysis thereby showing that the algorithm indeed succeeds with probability  $\Omega(2^{-r} n^{-2})$ . As a consequence, the number of global min-cuts in constant-rank hypergraphs is also  $O(n^2)$ .

In a recent work, Karger observed that uniform random contraction can also be used to bound the number of parametric min-cuts in constant-rank hypergraphs. We describe his argument for graphs since two of our theorems build on it. Suppose we fix the multipliers

$\mu_1, \dots, \mu_t$  in the parametric min-cut problem, then a fixed min  $c_\mu$ -cost cut can be obtained with probability  $\Omega(n^{-2})$  by running the random contraction algorithm with respect to the edge-cost function  $c_\mu$ . Karger suggested an alternative viewpoint of the execution of the algorithm for the edge-cost function  $c_\mu$ . For simplicity, we assume parallel edges instead of costs, i.e.,  $c_i(e) \in \{0, 1\}$  for every edge  $e$  and every criterion  $i \in [t]$ . Let  $E_i$  be the set of edges with non-zero weight in the  $i$ 'th criterion. The execution of the random-contraction algorithm wrt  $c_\mu$  can alternatively be specified as follows: a permutation  $\pi_i$  of the edges in  $E_i$  for each  $i \in [t]$  and an *interleaving* indicating at each step whether the algorithm contracts the next edge from  $\pi_1$  or  $\pi_2$  or  $\dots$  or  $\pi_t$ . Critically, the sequences  $\pi_i$  for every  $i \in [t]$  can be assumed to be uniformly random. Thus, we can move all randomness upfront, namely pick a uniform random permutation  $\pi_i$  for each criterion  $i \in [t]$ . Now, instead of returning one cut, we return the collection of cuts produced by contracting along all possible interleavings. This modified algorithm no longer depends on the specific multipliers  $\mu_1, \dots, \mu_t$  and hence, a parametric min-cut for any fixed choice of multipliers  $\mu_1, \dots, \mu_t$  will be in the output collection with probability at least  $\Omega(n^{-2})$ . It remains to bound the number of interleavings since that determines the number of cuts in the returned collection: the crucial observation here is that the number of interesting interleavings is only  $n^{t-1}$ . This is because interleaved contractions produce the same final graph as performing a certain number of contractions according to  $\pi_1$  (until the number of vertices is  $n_1$  say), then a certain number of contractions based on  $\pi_2$  (until the number of vertices is  $n_2$  say), and so on. So, the order of contractions becomes irrelevant and only the number of vertices  $n_1, \dots, n_t$  are relevant. Overall, this implies that the number of parametric min-cuts is  $O(n^{t+1})$ . We emphasize that this *interleaving argument* relies crucially on the basic random contraction algorithm picking edges to contract according to a *uniform distribution* (allowing the permutations  $\pi_1, \dots, \pi_t$  to be uniform random permutations).

Next, we describe our approach underlying the proof of Theorem 1.1, but for graphs. In order to bound the number of multiobjective min-cuts through the interleaving argument, we first need a random-contraction based algorithm to solve the  $b$ -multiobjective min-cut problem. Indeed, Aissi, Mahjoub, and Ravi [2] designed a random-contraction based algorithm to solve the  $b$ -multiobjective min-cut problem in graphs. Their algorithm proceeds as follows: Let  $U_0 := \emptyset$  and for each  $i \in [t-1]$ , let  $U_i$  be the set of vertices  $u \in V - \cup_{j=1}^{i-1} U_j$  for which  $c_i(\delta(u)) > b_i$  (known as the set of  $i$ -infeasible vertices), and let  $U_t := V - \cup_{j=1}^{t-1} U_j$ . In each step, they pick  $i \in [t]$  with probability proportional to the number of  $i$ -infeasible vertices (i.e.,  $|U_i|$ ) and pick a random edge  $e$  among the ones incident to  $U_i$  with probability proportional to  $c_i(e)$ , contract  $e$ , and repeat. Unfortunately, this algorithm does not have the *uniform distribution* that is crucially necessary to apply Karger's interleaving argument. To introduce uniformity to the distribution, we modify this algorithm in two ways:

1. At each step we deterministically choose the criterion  $i$  corresponding to the largest  $U_i$  (as opposed to picking  $i$  randomly with probability proportional to  $|U_i|$ ).
2. Next, we choose a uniform random edge  $e$  from among all edges in the graph with probability proportional to  $c_i(e)$  (as opposed to picking an edge only from among the edges incident to  $U_i$ ). We contract this chosen edge  $e$ .

These two features bring a *uniform distribution* property to the algorithm, which in turn, allows us to apply the interleaving argument. With these two features, we show that the algorithm returns a fixed  $b$ -multiobjective min-cut for a fixed budget-vector  $b$  with probability  $\Omega(n^{-2t})$ . Armed with the two features, we move all randomness upfront using the interleaving argument. As a consequence, we obtain that the total number of multiobjective min-cuts (irrespective of the choice of budget-vector  $b$ ) is  $O(n^{3t-1})$ . For constant-rank hypergraphs, we

perform an LP-based analysis of our algorithm for  $b$ -multiobjective min-cut (thus, extending Kogan and Krauthgamer’s analysis) to arrive at the same success probability. The interleaving argument for constant-rank hypergraphs proceeds similarly.

We emphasize that the interleaving argument does not extend to arbitrary-rank hypergraphs. This is because, the random contraction based algorithm that we know for arbitrary-rank hypergraphs crucially requires non-uniform contractions (the next hyperedge to contract is chosen from a distribution that depends on the current sizes of all hyperedges), so we cannot assume that the permutations  $\pi_1, \pi_2, \dots, \pi_t$  are uniformly random. Consequently, we do not even know if the number of parametric min-cuts in a hypergraph is at most strongly polynomial. Another interesting open question here is whether the  $b$ -multiobjective min-cut problem in hypergraphs is solvable in polynomial-time even for  $t = 2$  criteria. We have arrived at hypergraph instances (with large rank) for which Aissi, Mahjoub, and Ravi’s approach (as well as our modified approach) will never succeed, even with non-uniform random contractions.

Next, we outline the proof of Theorem 1.2. The approach is to again design a random-contraction algorithm that returns a fixed node-budgeted  $b$ -multiobjective min-cut with probability  $\Omega(n^{-2})$  (in both constant-rank and arbitrary-rank hypergraphs). Such an algorithm would imply the second part of the theorem immediately while the first part would follow if we can apply an interleaving-like argument (i.e., the designed algorithm performs uniform random contractions). Our approach is essentially an extension of the approach by Goemans and Soto who suggested contracting the infeasible vertices together (a vertex  $u$  is infeasible if  $w_i(u) > b_i$  for some  $i \in [t]$ ). Aissi, Mahjoub, and Ravi show that doing this additional step after each random contraction step returns a fixed node-budgeted  $b$ -multiobjective min-cut with probability  $\Omega(n^{-2})$  in graphs. Our main contribution is showing that this additional “contracting infeasible vertices together” step in conjunction with (1) uniform random contractions for constant-rank hypergraphs and (2) non-uniform random contractions for arbitrary hypergraphs succeeds with the required probability. Next, a naive interleaving-like argument can be applied for constant-rank hypergraphs to conclude that the number of node-budgeted multiobjective min-cuts is  $O(n^{t+3})$ . We improve this to  $O(n^{t+2})$  with a more careful argument.

Finally, we outline our approach for Theorem 1.3. Guinez and Queyranne address size-constrained  $k$ -cut in constant-rank hypergraphs for unit vertex-weights by performing uniform random contractions until the number of nodes in the hypergraph is close to  $\sum_{i=1}^k s_i$  at which point they return a uniform random cut. Their success probability has exponential dependence on the rank. The key technical ingredient to bring down the exponential dependence on rank is the use of non-uniform contractions. For the special case of unit sizes and unit vertex-weights (i.e., the hypergraph  $k$ -cut problem), Chandrasekaran, Xu, and Yu [5] introduced an explicit non-uniform distribution that leads to a success probability of  $\Omega(n^{-2(k-1)})$ . Our algorithm extends the non-uniform distribution to arbitrary but constant sizes (as opposed to just unit sizes), yet without depending on vertex-weights. Our analysis takes care of the vertex-weight function through weight tracking, i.e., by declaring the weight of a contracted node to be the sum of the weight of the vertices in the hyperedge being contracted. We note that our algorithm’s success probability when specialized to the case of unit vertex-weights and unit sizes is weaker than the success probability of the algorithm by Chandrasekaran, Xu, and Yu (by a factor of  $n$ ). We leave it as an open question to improve this. On the other hand, our algorithm has the added advantage that it does not even take the vertex-weight function  $w$  as input and yet succeeds in returning a  $w$ -vertex-weighted  $s$ -size-constrained  $k$ -cut for any choice of  $w$  with inverse polynomial probability.

**Organization.** In Section 2, we bound the number of multiobjective min-cuts (and prove Theorem 1.1). Due to page limit, we give efficient algorithms to enumerate all multiobjective min-cuts and all pareto-optimal cuts, and present lower bounds on the number of pareto-optimal cuts and the number of  $b$ -multiobjective min-cuts in the appendix. In Section 3, we give an algorithm to solve size-constrained min- $k$ -cut, thereby proving Theorem 1.3. Due to page limit, we defer the details of node-budgeted multiobjective min-cuts and a proof of Theorem 1.2 to the full version of the paper [4].

### 1.3 Preliminaries

We define the random contraction procedure that is central to all of our algorithms. Let  $G = (V, E)$  be a hypergraph, and let  $U \subseteq V$  be a set of vertices in  $G$ . We define  $G$  contract  $U$ , denoted  $G/U$ , to be a hypergraph on the vertex set  $(V \setminus U) \cup \{u\}$ , where  $u$  is a newly introduced vertex. The hyperedges of  $G/U$  are defined as follows. For each hyperedge  $e \in E$  of  $G$ , such that  $e \not\subseteq U$ ,  $G/U$  has a corresponding hyperedge  $e'$ , where  $e' := e$  if  $e \subseteq V \setminus U$  and  $e' := (e \setminus U) \cup \{u\}$  otherwise. If  $w$  is a vertex-weight function for  $G$ , then we will also use  $w$  as a vertex-weight function for  $G/U$ . We define the weight of the newly introduced vertex  $u$  as  $w(u) := \sum_{v \in U} w(v)$ .

We will need the following lemma that will be used in the analysis of two of our algorithms. We refer the reader to the full version of this paper for the proof.

► **Lemma 1.4.** *Let  $r, \gamma, n$  be positive integers with  $n \geq \gamma \geq r + 1 > 2$ . Let  $f : \mathbb{N} \rightarrow \mathbb{R}_+$  be a positive-valued function defined over natural numbers. Then, the optimum value of the linear program  $(LP_1)$  defined below is  $\min_{2 \leq j \leq r} (1 - \frac{j}{\gamma - r + j}) f(n - j + 1)$ .*

$$\begin{array}{ll} \underset{x_2, \dots, x_r, y_2, \dots, y_r}{\text{minimize}} & \sum_{j=2}^r (x_j - y_j) f(n - j + 1) \end{array} \quad (LP_1)$$

$$\text{subject to} \quad 0 \leq y_j \leq x_j \quad \forall j \in \{2, \dots, r\} \quad (2)$$

$$\sum_{j=2}^r x_j = 1 \quad (3)$$

$$\gamma \sum_{j=2}^r y_j \leq \sum_{j=2}^r j \cdot x_j \quad (4)$$

## 2 Multiobjective Min-Cuts and Pareto-Optimal Cuts

In this section, we give upper and lower bounds on the number of multiobjective min-cuts and pareto-optimal cuts and prove Theorem 1.1.

Let  $G = (V, E)$  be an  $r$ -rank hypergraph and let  $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$  be cost functions on the hyperedges of  $G$ . For a subset  $F$  of hyperedges, we will use  $c_i(F)$  to denote  $\sum_{e \in F} c_i(e)$ . For a subset  $U$  of vertices, we will use  $\bar{U}$  to denote  $V \setminus U$  and  $\delta(U)$  to denote the set of hyperedges that intersect both  $U$  and  $\bar{U}$ . For a vertex  $v$ , we will use  $\delta(v)$  to denote  $\delta(\{v\})$ . A subset  $F$  of hyperedges is a cut if there exists a partition  $(U, \bar{U})$  such that  $F = \delta(U)$ . We refer the reader to Section 1 for the definitions of  $b$ -multiobjective min-cuts, multiobjective min-cuts, and pareto-optimal cuts.

We begin with a randomized algorithm for the  $b$ -multiobjective min-cut problem in Section 2.1. We take an alternative viewpoint of this randomized algorithm in Section 2.2 to prove Theorem 1.1. Since all pareto-optimal cuts are multiobjective min-cuts, Theorem 1.1



also implies that the number of pareto-optimal cuts in an  $r$ -rank  $n$ -vertex hypergraph  $G$  with  $t$  hyperedge-cost functions is  $O(r2^{rt}n^{3t-1})$ . In Section A.3, we give randomized polynomial-time algorithms to enumerate all pareto-optimal cuts and all multiobjective min-cuts. In Section A.4, we show a lower bound of  $\Omega(n^t)$  on the number of pareto-optimal cuts and on the number of  $b$ -multiobjective min-cuts.

## 2.1 Finding $b$ -Multiobjective Min-Cuts

In this section, we design a randomized algorithm for the  $b$ -multiobjective min-cut problem, which is formally defined below. We use Algorithm 1. We summarize its correctness and run-time guarantees in Theorem 2.1.

$b$ -MULTIOBJECTIVE MIN-CUT Given: A hypergraph $G = (V, E)$ with hyperedge-cost functions $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$ and a budget-vector $b \in \mathbb{R}_+^{t-1}$ . Goal: A $b$ -multiobjective min-cut.
---

### Algorithm 1 $b$ -MULTIOBJECTIVE MIN-CUT.

$b$ -MULTIOBJECTIVE-MIN-CUT( $G, r, t, c, b$ ):

**Input:** An  $r$ -rank hypergraph  $G = (V, E)$ , hyperedge-cost functions  $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$  and a budget-vector  $b \in \mathbb{R}_+^{t-1}$ .

If  $|V| \leq rt$ :

$X \leftarrow$  a random subset of  $V$

return  $\delta(X)$

For  $i = 1, \dots, t-1$ :

$U_i \leftarrow \{v \in V \setminus (\bigcup_{j=1}^{i-1} U_j) : c_i(\delta(v)) > b_i\}$

$U_t \leftarrow V \setminus \bigcup_{j=1}^{t-1} U_j$

$i \leftarrow \arg \max_{j \in [t]} |U_j|$

$e \leftarrow$  a random hyperedge chosen according to  $\Pr[e = e'] = \frac{c_i(e')}{c_i(E)}$

$G' \leftarrow G/e$

Return  $b$ -MULTIOBJECTIVE-MIN-CUT( $G', r, t, c, b$ )

► **Theorem 2.1.** *Let  $G = (V, E)$  be an  $r$ -rank  $n$ -vertex hypergraph with hyperedge-cost functions  $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$  and let  $b \in \mathbb{R}_+^{t-1}$  be a budget-vector. Let  $F$  be an arbitrary  $b$ -multiobjective min-cut. Then, Algorithm 1 outputs  $F$  with probability at least  $Q_n$ , where*

$$Q_n := \begin{cases} \frac{1}{2^{rt}} & \text{if } n \leq rt, \text{ and} \\ \frac{2t+1}{2^{rt}(rt+1)} \binom{n-t(r-2)}{2t}^{-1} & \text{if } n > rt. \end{cases}$$

Moreover, the algorithm can be implemented to run in polynomial time.

**Proof.** We note that sets  $U_i$  can be computed in polynomial time, and the algorithm recomputes them at most  $n$  times. Random contraction can also be implemented in polynomial time, and therefore the overall run-time of the algorithm is polynomial.

We now bound the correctness probability by induction on  $n$ . For the base case, we consider  $n \leq rt$ . In this case, the algorithm returns  $\delta(X)$  for a random  $X \subseteq V$ . There are  $2^n$  possible values for  $X$ , and  $F = \delta(X)$  for at least one of them. Thus, the probability that the algorithm returns  $F$  is at least  $\frac{1}{2^n} \geq \frac{1}{2^{rt}} = Q_n$ .

Next, we prove the induction step. Suppose  $n > rt$ . We will need the following claim whose proof appears in the appendix.

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▷ **Claim 2.2.** The algorithm outputs  $F$  with probability at least the optimum value of the following linear program.

$$\begin{aligned}
 & \underset{x_2, \dots, x_r, y_2, \dots, y_r}{\text{minimize}} && \sum_{j=2}^r (x_j - y_j) Q_{n-j+1} && (LP_2) \\
 & \text{subject to} && 0 \leq y_j \leq x_j \quad \forall j \in \{2, \dots, r\} \\
 & && \sum_{j=2}^r x_j = 1 \\
 & && |U_i| \sum_{j=2}^r y_j \leq \sum_{j=2}^r j \cdot x_j
 \end{aligned}$$

Let  $U_i$  be the largest among the sets  $U_1, \dots, U_t$  that the algorithm generates when executed on input  $(G, r, t, c, b)$ . Claim 2.2 tells us that the algorithm outputs  $F$  with probability at least the optimum value of the linear program  $(LP_2)$  from the claim.

The linear program  $(LP_2)$  is exactly the linear program  $(LP_1)$  from Lemma 1.4 with  $\gamma = |U_i|$  and  $f(n) := Q_n$ . To apply Lemma 1.4, we just need to show that  $n \geq |U_i| \geq r + 1$ . We recall that  $U_i$  is the largest of the  $t$  sets that the algorithm constructs. Each of these sets is a subset of  $V$ , so we can conclude that  $|U_i| \leq |V| = n$ . We also know from the construction of the sets  $U_1, \dots, U_t$  that they partition  $V$ . This means that  $\sum_{j=1}^t |U_j| = n$ . Since  $U_i$  is the largest of the sets, we must have  $|U_i| \geq \frac{n}{t}$ . Since  $n > rt$ , this means  $|U_i| \geq \frac{rt+1}{t} > r$ . Thus  $|U_i| > r$ , and since  $r$  and  $|U_i|$  are integers, we conclude that  $|U_i| \geq r + 1$ . Therefore, we can apply Lemma 1.4 with  $\gamma = |U_i|$  to conclude that

$$\Pr[\text{Algorithm returns the cut } F] \geq \min_{2 \leq j \leq r} \left( 1 - \frac{j}{|U_i| - r + j} \right) Q_{n-j+1}.$$

The following claim completes the proof of the theorem. Its proof appears in the appendix. ◀

▷ **Claim 2.3.** For every  $j \in \{2, \dots, r\}$ , we have  $\left( 1 - \frac{j}{|U_i| - r + j} \right) Q_{n-j+1} \geq Q_n$ .

## 2.2 Finding Multiobjective Min-Cuts

In this section, we present Algorithm 2, which does *not* take a budget-vector as input, yet outputs any multiobjective min-cut (for any choice of budget-vector) with inverse polynomial probability. This is accomplished by returning a collection of cuts.

In contrast to Algorithm 1, all of the randomness in Algorithm 2 (except for the selection of a random cut in the base case) occurs upfront through the selection of a permutation of the hyperedges. Theorem 2.4 summarizes the guarantees of Algorithm 2.

► **Theorem 2.4.** *Let  $G$  be an  $r$ -rank,  $n$ -vertex hypergraph with  $t$  hyperedge-cost functions. Then, a fixed multiobjective min-cut  $F$  is in the collection returned by Algorithm 2 with probability  $\frac{\Omega(n^{-2t})}{r^{2rt}}$ . Moreover, the algorithm outputs at most  $n^{t-1}$  cuts.*

**Proof.** We begin by showing the second part of the theorem. The algorithm outputs at most one cut for each choice of  $n_1, \dots, n_{t-1} \in [n]$  (or just one cut if  $|V| \leq rt$ ). Hence, it outputs at most  $n^{t-1}$  cuts.

We now argue that the algorithm retains the same success probability as Algorithm 1, for any fixed budget-vector  $b$ .

---

**Algorithm 2** MULTIOBJECTIVE MIN-CUT.

---

MULTIOBJECTIVE-MIN-CUT( $G, r, t, c_1, \dots, c_t$ ):

**Input:** An  $r$ -rank hypergraph  $G = (V, E)$  and  
hyperedge-cost functions  $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$ .

If  $|V| \leq rt$ :

Pick a random subset  $X$  of  $V$  and return  $\delta(X)$ 

For  $i = 1, \dots, t$ :

 $E_i \leftarrow \{e \in E : c_i(e) > 0\}$ 
 $\pi_i \leftarrow$  a permutation of  $E_i$  generated by repeatedly choosing a not yet  
chosen hyperedge  $e$  with probability proportional to  $c_i(e)$ 
 $R \leftarrow \emptyset$ 

For each sequence  $n_1, \dots, n_t$  with  $n \geq n_1 \geq n_2 \geq \dots \geq n_{t-1} \geq n_t = rt$ :

 $G' \leftarrow G$ 

For  $i = 1, \dots, t$ :

While  $|V(G')| > n_i$ :

 $e \leftarrow$  the first hyperedge from  $\pi_i$  that is still present in  $G'$ 
 $G' \leftarrow G'/e$ 
 $X \leftarrow$  a random subset of  $V(G')$ 

Add  $\delta(X)$  to  $R$  if it is not already present

Return  $R$ 


---

Suppose  $n \leq rt$ . Then both Algorithm 2 and Algorithm 1 return  $\delta(X)$  for a random subset  $X$  of the vertices of  $G$ . Thus, for any cut  $F$ , the two algorithms have the same probability of returning  $F$ . Henceforth, we assume  $n > rt$ .

We will view Algorithm 1 from a different perspective. In that algorithm, whenever we contract a hyperedge  $e$ , we choose, for some  $i \in [t]$ , a hyperedge according to the probability distribution  $\Pr[e = e'] = \frac{c_i(e')}{c_i(E)}$ . In particular, the choice of  $i$  depends on which contractions have been made so far, but the choice of a particular hyperedge, given the choice of  $i$ , does not depend on our previous contractions, except for the fact that we do not contract hyperedges which have already been reduced to singletons. We note that allowing the contraction of singletons would not change the success probability of the algorithm.

Therefore, we could modify Algorithm 1 so that it begins by selecting permutations  $\pi_1, \dots, \pi_t$  of  $E_1, \dots, E_t$  (where  $E_i = \{e \in E : c_i(e) > 0\}$ ) as in Algorithm 2, and then whenever the algorithm asks to contract a random hyperedge with probability proportional to its weight under  $c_i$ , we instead contract the next hyperedge from  $\pi_i$  which is still present in the current hypergraph. This modification does not change at any step the probability that a particular hyperedge is the next contraction of a non-singleton hyperedge, and therefore the success probability of the algorithm remains exactly the same.

Viewing Algorithm 1 in this way, we note that when we reach the base case of  $n \leq rt$ , we will have contracted the first  $m_i$  hyperedges of each  $\pi_i$ , for some  $m_1, \dots, m_t \in \{0, \dots, |E|\}$ . The crucial observation now is that interleaved contractions can be separated. That is, if we know  $m_i$  for every  $i \in [t]$ , the order in which we do the contractions is irrelevant: we will get the same resulting hypergraph if we contract the first  $m_1$  hyperedges from  $\pi_1$ , then contract the first  $m_2$  hyperedges from  $\pi_2$ , and so on up through the first  $m_t$  hyperedges from  $\pi_t$  instead of the interleaved contractions. Let  $n_1$  be the number of vertices in the hypergraph obtained after contracting the first  $m_1$  hyperedges from  $\pi_1$ , subsequently, let  $n_2$  be the number of vertices in the hypergraph obtained after contracting the first  $m_2$  hyperedges from  $\pi_2$  and so on.

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When we view Algorithm 1 in this way, it is only the choice of the values  $n_1, \dots, n_t$  that depends on the budgets, while the choice of the permutation  $\pi_i$  does not depend on the budgets. Algorithm 2 is running exactly the version of Algorithm 1 that we have just described, except that instead of choosing  $n_1, \dots, n_t$  based on the budgets, it simply tries all possible options (which will certainly include whichever  $n_1, \dots, n_t$  Algorithm 1 would use for the given input budget-vector).

Therefore for every fixed choice of budget-vector  $b$ , and every fixed  $b$ -multiobjective min-cut  $F$ , the probability that  $F$  is in the collection  $R$  output by the algorithm is at least as large as the probability that  $F$  is the cut output by Algorithm 1. By Theorem 2.1, this probability is  $\frac{\Omega(n^{-2t})}{r^{2rt}}$ , as desired. ◀

We derive Theorem 1.1 from Theorem 2.4 now.

► **Theorem 1.1.** *The number of multiobjective min-cuts in an  $r$ -rank,  $n$ -vertex hypergraph  $G$  with  $t$  hyperedge-cost functions is  $O(r2^{rt}n^{3t-1})$ .*

**Proof.** Let  $x$  be the number of multiobjective min-cuts in  $G$ . By Theorem 2.4, the expected number of multiobjective min-cuts output by our algorithm MULTIOBJECTIVE MIN-CUT (i.e., Algorithm 2) is  $(x/r2^{rt})\Omega(n^{-2t})$ . Theorem 2.4 also tells us that the algorithm outputs at most  $n^{t-1}$  cuts. Therefore,  $x = r2^{rt} \cdot O(n^{3t-1})$ . ◀

### 3 Size-Constrained Min- $k$ -Cut in Arbitrary-Rank Hypergraphs

In this section, we consider the problem of finding a minimum cost  $k$ -cut subject to constant lower bounds on the weights of the partition classes and prove Theorem 1.3. Throughout this section, we assume that  $k$  is a constant. We focus on the cardinality case (i.e., unit-cost variant) for the sake of simplicity of exposition and mention that our algorithm also extends to arbitrary non-negative hyperedge costs.

We begin by formally defining the terminology. Let  $G = (V, E)$  be a hypergraph. For a partition  $X = (X_1, \dots, X_k)$  of  $V$ , we define  $\delta(X)$  to be the set of hyperedges that intersect at least two parts of  $X$ . For a weight function  $w : V \rightarrow \mathbb{Z}_+$ , we call  $(G, w)$  a *vertex-weighted hypergraph*. We now define our main object of study, namely size-constrained minimum cuts.

► **Definition 3.1.** *Let  $G = (V, E)$  be a hypergraph,  $w : V \rightarrow \mathbb{Z}_+$  be a vertex-weight function,  $k \geq 2$  be an integer, and  $s \in \mathbb{Z}_+^k$  be a size-vector. A  $k$ -partition  $X$  of  $V$  is an  $s$ -size-constrained  $k$ -partition if  $w(X_i) \geq s_i$  for every  $i \in [k]$ . A set of hyperedges  $F \subseteq E$  is an  $s$ -size-constrained  $k$ -cut if  $F = \delta(X)$  for some  $s$ -size-constrained  $k$ -partition  $X$ . An  $s$ -size-constrained  $k$ -cut of minimum cardinality is said to be an  $s$ -size-constrained min- $k$ -cut.*

The following is the central problem of interest in this section.

$s$ -SIZE-CONSTRAINED MIN- $k$ -CUT
Given: A vertex-weighted hypergraph $(G, w)$ , a positive integer $k$ , and a size-vector $s \in \mathbb{Z}_+^k$ .
Goal: An $s$ -size-constrained min- $k$ -cut.

We give a random contraction based algorithm for this problem. Given a hypergraph  $G = (V, E)$  and a size-constraint vector  $s \in \mathbb{Z}_+^k$ , let  $n = |V|$ . We define  $\sigma_j := \sum_{i=1}^j s_i$ , and  $\alpha_e := \frac{\binom{n-|e|}{\sigma_{k-1}}}{\binom{n}{\sigma_{k-1}}}$ . With these definitions, we solve  $s$ -size-constrained min- $k$ -cut using Algorithm 3. We prove Theorem 1.3 using this algorithm.

■ **Algorithm 3**  $s$ -SIZE-CONSTRAINED MIN- $k$ -CUT.

$s$ -SIZE-CONSTRAINED-MIN- $k$ -CUT( $G, k, s$ ):

**Input:** An  $n$ -vertex hypergraph  $G = (V, E)$ , an integer  $k \geq 2$   
and size-constraint vector  $s = (s_1, \dots, s_k) \in \mathbb{Z}_+$

If  $n \leq \max\{2\sigma_{k-1}, \sigma_k\}$

    Pick a random  $k$ -partition  $X$  of  $V$  and return  $\delta(X)$

$S \leftarrow$  a random subset of  $V$  of size  $2\sigma_{k-1}$

$X_i \leftarrow \emptyset$  for  $i \in \{1, \dots, k\}$

For each  $v \in S$ :

    Pick a random integer  $i \in \{1, \dots, k\}$  and add  $v$  to  $X_i$

$X_k \leftarrow X_k \cup (V \setminus S)$

$X \leftarrow (X_1, \dots, X_k)$

If  $X$  is a  $k$ -partition of  $V$ :

$R \leftarrow \delta(X)$

Else:

$R \leftarrow E$

Compute  $\alpha_e := \frac{\binom{n-|e|}{\sigma_{k-1}}}{\binom{n}{\sigma_{k-1}}}$  for every  $e \in E$

If  $\alpha_e = 0$  for every hyperedge  $e \in E(G)$ :

    Return  $R$

$e \leftarrow$  a random hyperedge of  $G$  chosen with probability proportional to  $\alpha_e$

$R' \leftarrow s$ -SIZE-CONSTRAINED-MIN- $k$ -CUT( $G/e, k, s$ )

Return  $R$  with probability  $\frac{1}{n}$  and  $R'$  with probability  $\frac{n-1}{n}$

► **Theorem 1.3.** Let  $k \geq 2$  be a fixed positive integer and let  $1 \leq s_1 \leq s_2 \leq \dots \leq s_k$  be fixed positive integers. Let  $G$  be an  $n$ -vertex hypergraph with hyperedge-cost function  $c : E \rightarrow \mathbb{R}_+$ . Then, there exists a polynomial-time algorithm that takes  $(G, c)$  as input and returns a fixed  $w$ -weighted  $s$ -size-constrained min- $k$ -cut for any choice of vertex-weight function  $w : V \rightarrow \mathbb{Z}_+$  with probability  $\Omega\left(\frac{1}{n^{2\sigma_{k-1}+1}}\right)$ , where  $\sigma_{k-1} := \sum_{i=1}^{k-1} s_i$ .

**Proof.** We consider Algorithm 3. We first analyze its run-time. Each recursive call reduces the number of vertices in the hypergraph. Thus, the algorithm makes at most  $n$  recursive calls. Apart from the recursion steps, the algorithm only selects random partitions and performs contractions, both of which can be implemented to run in polynomial time.

Now we analyze the success probability. Let  $\mathcal{G}_n$  be the set of all vertex-weighted  $n$ -vertex hypergraphs which contain an  $s$ -size-constrained  $k$ -cut. For a vertex-weighted hypergraph  $(G, w)$ , let  $M(G, w)$  be the set of all  $s$ -size-constrained min- $k$ -cuts in  $(G, w)$ . Define

$$q_n := \min_{(G,w) \in \mathcal{G}_n} \min_{F \in M(G,w)} \Pr[\text{Algorithm returns } F \text{ on input } (G, w, k, s)], \text{ and}$$

$$Q_n := \begin{cases} (k^{\max\{2\sigma_{k-1}, \sigma_k\}})^{-1} & \text{if } n \leq \max\{2\sigma_{k-1}, \sigma_k\}, \text{ and} \\ \left(k^{\max\{2\sigma_{k-1}, \sigma_k\}} n \binom{n}{2\sigma_{k-1}}\right)^{-1} & \text{if } n > \max\{2\sigma_{k-1}, \sigma_k\}. \end{cases}$$

We note that  $Q_n = \Omega(k^{-\sigma_{k-1}-\sigma_k} n^{-2\sigma_{k-1}-1})$ , so it suffices to show that  $q_n \geq Q_n$  for all  $n \geq k$  (for smaller  $n$ , there are no  $k$ -cuts).

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We proceed by induction on  $n$ . Let  $F$  be an  $s$ -size-constrained min- $k$ -cut in  $(G, w)$ . Let  $Y$  be an  $s$ -size-constrained  $k$ -partition with  $F = \delta(Y)$ . We assume that  $|Y_1| \leq |Y_2| \leq \dots \leq |Y_k|$ . This assumption is without loss of generality because we can relabel the parts of an  $s$ -size-constrained  $k$ -partition so that they are in increasing order of size and the resulting partition will still be an  $s$ -size-constrained  $k$ -partition (since we have assumed that the size vector  $s$  is in increasing order).

For the base case, suppose  $n \leq \max\{2\sigma_{k-1}, \sigma_k\}$ . In this case, the algorithm returns  $\delta(X)$  where  $X$  is a  $k$ -partition of  $V$  chosen uniformly at random. Since  $F = \delta(Y)$ , the probability that  $F = \delta(X)$  for the  $X$  randomly chosen by the algorithm is at least  $\Pr[X = Y]$ . The number of  $k$ -partitions of  $V$  is at most  $k^n$ , the number of ways to assign each of the  $n$  vertices to one of  $k$  labeled sets. Thus,  $\Pr[X = Y] \geq k^{-n} \geq k^{-\max\{2\sigma_{k-1}, \sigma_k\}} = Q_n$ .

Now we will prove the inductive step. Assume that  $n > \max\{2\sigma_{k-1}, \sigma_k\}$ . By the inductive hypothesis, we have  $q_{n'} \geq Q_{n'}$  for all  $n' \in \{k, \dots, n-1\}$ . We will show that  $q_n \geq Q_n$ .

Suppose  $|Y_k| \geq n - 2\sigma_{k-1}$ . Let  $T$  be an arbitrary subset of  $Y_k$  of size  $n - 2\sigma_{k-1}$ . Consider the set  $S$  chosen by the algorithm. The probability that  $S$  is equal to  $V - T$  is  $\binom{n}{2\sigma_{k-1}}^{-1}$ . Next, consider the sets  $X_i$  created by the algorithm. The probability that  $X_i = Y_i$  for every  $i \in [k]$  conditioned on  $S = V - T$ , is  $k^{-2\sigma_{k-1}}$ . Thus, the probability that the  $k$ -partition  $X$  obtained in the algorithm is identical to  $Y$  is at least  $k^{-2\sigma_{k-1}} \binom{n}{2\sigma_{k-1}}^{-1}$ . Since the last step of the algorithm returns  $R = \delta(X)$  with probability  $1/n$ , it follows that the algorithm returns  $F$  with probability at least  $\left(nk^{\max\{2\sigma_{k-1}, \sigma_k\}} \binom{n}{2\sigma_{k-1}}\right)^{-1} = Q_n$ .

Henceforth, we assume that  $|Y_k| < n - 2\sigma_{k-1}$ . We will call a hyperedge *large* if it contains at least  $n - 2\sigma_{k-1}$  vertices. Since  $Y_k$  is the largest part of the  $k$ -partition  $Y$ , every large hyperedge must be contained in the  $k$ -cut  $F$ . In particular, if  $\alpha_e = 0$  for a hyperedge  $e$ , then  $\binom{n-|e|}{\sigma_{k-1}} = 0$ , which implies that  $n - |e| < \sigma_{k-1}$ , and hence  $e$  is large and consequently,  $e$  cannot be in  $F$ .

Next, suppose that  $\alpha_e = 0$  for every hyperedge  $e$ . Then every hyperedge is a large hyperedge and therefore,  $F = E$ . In this case, the algorithm will return  $R$ . We note that  $R = E$  if  $X$  is not a  $k$ -partition. We lower bound the probability that  $X$  is not a  $k$ -partition now. If all vertices in  $S$  are assigned to  $X_k$ , then  $X$  is not a  $k$ -partition. The probability that all vertices in  $S$  are assigned to  $X_k$  is  $k^{-2\sigma_{k-1}}$ . Thus, the probability that the algorithm returns  $F = R = E$  is at least  $k^{-2\sigma_{k-1}} \geq Q_n$ .

Henceforth, we assume that  $\alpha_e > 0$  for some hyperedge  $e \in E$ . This means that the algorithm will contract some hyperedge and then recurse on the resulting hypergraph. Let  $e'$  be a random variable for the hyperedge chosen to be contracted. Let  $w'$  be the weight function defined on the vertices of  $G/e'$  as follows:  $w'(v) := w(v)$  for each  $v \in V \setminus e'$  and  $w'(v) := \sum_{u \in e'} w(u)$  when  $v$  is the new vertex resulting from the contraction. If  $e' \notin F$ , then  $F$  will be an  $s$ -size-constrained min- $k$ -cut in  $(G/e', w')$ . Therefore, we have that  $\Pr[R' = F \text{ on input } (G, k, s)]$

$$\begin{aligned} &= \sum_{e' \in E \setminus F} \Pr[e' = e] \cdot \Pr[\text{Algorithm returns } F \text{ on input } (G/e', k, s)] \\ &\geq \sum_{e' \in E \setminus F} \frac{\alpha_e}{\sum_{f \in E} \alpha_f} \cdot q_{n-|e|+1}. \end{aligned}$$

Let  $e$  be a hyperedge that is not in the  $k$ -cut  $F$ . Then,  $e$  cannot be a large hyperedge and hence,  $|e| < n - 2\sigma_{k-1}$ . Consequently,  $n - |e| + 1 > 2\sigma_{k-1} + 1 \geq k$ . Therefore, by applying the induction hypothesis, we have  $q_{n-|e|+1} \geq Q_{n-|e|+1}$ . Hence,

$$\Pr[R' = F \text{ on input } (G, k, s)] \geq \frac{1}{\sum_{f \in E} \alpha_f} \sum_{e \in E \setminus F} \alpha_e \cdot Q_{n-|e|+1}.$$

We need the following two claims. We prove Claim 3.2 in the full version of this paper, and we defer the proof of Claim 3.3 to complete the proof of the theorem.

▷ **Claim 3.2.** For every hyperedge  $e \in E \setminus F$ , we have  $\alpha_e Q_{n-|e|+1} \geq \frac{nQ_n}{n-1}$ .

▷ **Claim 3.3.**  $\sum_{f \in E} \alpha_f \geq 1$ .

By Claim 3.2, we have that  $\Pr[R' = F \text{ on input } (G, k, s)]$

$$= \frac{1}{\sum_{f \in E} \alpha_f} \sum_{e \in E \setminus F} \alpha_e \cdot Q_{n-|e|+1} \geq \frac{1}{\sum_{f \in E} \alpha_f} \sum_{e \in E \setminus F} \frac{nQ_n}{n-1} = \frac{|E \setminus F|}{\sum_{f \in E} \alpha_f} \cdot \frac{nQ_n}{n-1}.$$

Thus, Claim 3.3 implies that

$$\Pr[R' = F \text{ on input } (G, k, s)] \geq \frac{nQ_n}{n-1}.$$

Finally, we note that since we have assumed  $n > \max\{2\sigma_{k-1}, \sigma_k\}$  and  $\alpha_e > 0$  for some  $e$ , the probability that the algorithm returns  $R'$  is  $(n-1)/n$ . Thus, we conclude that  $\Pr[\text{Algorithm returns } F] \geq Q_n$ . ◀

**Proof of Claim 3.3.** Let  $Z = (Z_1, \dots, Z_k)$  be a random  $k$ -partition obtained by picking disjoint sets  $Z_1, \dots, Z_{k-1}$  with  $|Z_i| = s_i$  and setting  $Z_k = V \setminus \bigcup_{i=1}^{k-1} Z_i$ . Since  $n > \sigma_k$  and every vertex has weight at least 1, the  $k$ -partition  $Z$  is an  $s$ -size-constrained  $k$ -partition. Therefore,  $|\delta(Z)|$  is an upper bound on  $|F|$ . In particular,  $|F| \leq \mathbb{E}[|\delta(Z)|] = \sum_{e \in E} \Pr(e \in \delta(Z))$ . Negating the inequality and adding  $|E|$  to both sides gives

$$\begin{aligned} |E \setminus F| &\geq \sum_{e \in E} (1 - \Pr(e \in \delta(Z))) = \sum_{e \in E} \Pr(e \notin \delta(Z)) = \sum_{e \in E} \sum_{i=1}^k \Pr(e \subseteq Z_i) \\ &\geq \sum_{e \in E} \frac{\binom{n-|e|}{\sigma_{k-1}}}{\binom{n}{\sigma_{k-1}}} = \sum_{e \in E} \alpha_e. \end{aligned} \quad \triangleleft$$

► **Remark.** Since our algorithm does not even take the vertex-weights as input, it could trivially be extended to handle a version of the problem where we have multiple weight functions on the vertices (as in the previous sections) each with their own minimum sizes. If we have  $t$  vertex-weight functions,  $w_1, \dots, w_t : V \rightarrow \mathbb{Z}_+$  and each function  $w_j$  has an associated list of lower bounds  $s_{j,1}, \dots, s_{j,k}$ , then we can find a min- $k$ -cut satisfying all of these lower-bound constraints with at least inverse polynomial probability by simply running our algorithm with  $s_i = \max_{j \in [t]} s_{j,i}$  for every  $i \in [k]$ .

## 4 Conclusion and Open problems

In this work, we illustrated the versatility of the random contraction technique by addressing multicriteria versions of min-cut and size-constrained min- $k$ -cut problems. There are several interesting open questions in this area. We conclude by stating a few: (1) For the number of pareto-optimal cuts and multiobjective min-cuts, there is still a gap between our lower

bound (which is  $\Omega(n^t)$ ) and our upper bound (which is  $O(n^{3t-1})$ ). Can we improve either of these bounds? We believe that improving our bounds for the number of  $b$ -multiobjective min-cuts for a fixed budget-vector  $b \in \mathbb{R}_+^{t-1}$  would be a first-step towards this goal. (2) We gave a polynomial-time algorithm to solve the  $b$ -multiobjective min-cut problem in constant-rank hypergraphs. How about arbitrary-rank hypergraphs? Is the  $b$ -multiobjective min-cut problem in arbitrary rank hypergraphs (even for  $t = 2$  criteria) solvable in polynomial-time or is it NP-hard?

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## A Appendix

### A.1 Proof of Claim 2.2

Proof of Claim 2.2. Since  $n > rt$ , when the algorithm is executed on  $G$  it will contract a randomly chosen hyperedge and recurse. Let  $e'$  be the random hyperedge chosen by the algorithm. If  $e' \notin F$ , then  $F$  will still be a  $b$ -multiobjective min-cut in  $G/e'$ . We observe that  $G/e'$  is a hypergraph with  $n - |e'| + 1$  vertices and the rank of  $G/e'$  is at most the rank of  $G$ . Therefore, if  $e' \notin F$ , then the algorithm will output  $F$  with probability at least  $Q_{n-|e'|+1}$ .

Let  $i \in [t]$  be the index of the cost function chosen by the algorithm. Let

$$\begin{aligned} E_j &:= \{e \in E : |e| = j\}, \\ x_j &:= \Pr[e' \in E_j] = \frac{c_i(E_j)}{c_i(E)}, \text{ and} \\ y_j &:= \Pr[e' \in E_j \cap F] = \frac{c_i(E_j \cap F)}{c_i(E)}. \end{aligned}$$

We note that  $E_j$  is the set of hyperedges of size  $j$ ,  $x_j$  is the probability of picking a hyperedge of size  $j$  to contract, and  $y_j$  is the probability of picking a hyperedge of size  $j$  from  $F$  to contract. We know that

$$\Pr[\text{Algorithm returns the cut } F] \geq \sum_{j=2}^r (x_j - y_j) Q_{n-j+1}. \quad (5)$$

The values of  $x_j$  and  $y_j$  will depend on the structure of  $G$ . However we can deduce some relationships between them. Since  $0 \leq c_i(E_j \cap F) \leq c_i(E_j)$  for every  $j \in \{2, \dots, r\}$ , we know that

$$0 \leq y_j \leq x_j \text{ for every } j \in \{2, \dots, r\}. \quad (6)$$

Moreover,  $x_j$  is the probability of picking a hyperedge of size  $j$ . Hence,

$$\sum_{j=2}^r x_j = 1. \quad (7)$$

Next, we show that for every  $i \in [t]$  and every  $v \in U_i$ , we have

$$c_i(F) \leq c_i(\delta(v)). \quad (8)$$

If  $i < t$ , then  $c_i(F) \leq b_i < c_i(\delta(v))$  for every  $v \in U_i$ . Let  $i = t$ . We recall that  $F$  is a  $b$ -multiobjective min-cut. Since every cut induced by a single vertex in  $U_t$  satisfies all of the budgets, no such cut can have a better  $c_t$ -cost than  $F$ , so again  $c_i(F) \leq c_i(\delta(v))$  for every  $v \in U_i$ .

From inequality (8), we conclude that

$$c_i(F) \leq \frac{\sum_{v \in U_i} c_i(\delta(v))}{|U_i|} \leq \frac{\sum_{v \in V} c_i(\delta(v))}{|U_i|} = \frac{\sum_{e \in E} |e| c_i(e)}{|U_i|} = \frac{\sum_{j=2}^r j \cdot c_i(E_j)}{|U_i|}.$$

Therefore

$$\sum_{j=2}^r y_j = \Pr[e' \in F] = \frac{c_i(F)}{c_i(E)} \leq \frac{1}{|U_i|} \sum_{j=2}^r j \cdot x_j.$$

Thus, we have that

$$|U_i| \sum_{j=2}^r y_j \leq \sum_{j=2}^r j \cdot x_j. \quad (9)$$

The minimum value of our lower bound in equation (5) over all choices of  $x_j$  and  $y_j$  that satisfy inequalities (6), (7), and (9) is a lower bound on the probability that the algorithm outputs  $F$ .  $\triangleleft$

## A.2 Proof of Claim 2.3

Proof of Claim 2.3. Let  $j \in \{2, \dots, r\}$ . The given inequality is equivalent to  $\frac{Q_{n-j+1}}{Q_n} \geq \frac{|U_i|^{-r+j}}{|U_i|^{-r}}$ . Since  $U_i$  is the largest among  $U_1, \dots, U_t$  which together partition  $V$ , we have  $|U_i| \geq \frac{n}{t}$ . Consequently,  $\frac{|U_i|^{-r+j}}{|U_i|^{-r}} = 1 + \frac{j}{|U_i|^{-r}} \leq 1 + \frac{jt}{n-rt}$ . Therefore, it suffices to prove that  $\frac{Q_{n-j+1}}{Q_n} \geq 1 + \frac{jt}{n-rt}$ . We case on the value of  $n-j+1$ .

1. Suppose that  $n-j+1 > rt$ . Then, we have

$$\frac{Q_{n-j+1}}{Q_n} = \frac{\binom{n-t(r-2)}{2t}}{\binom{n-j+1-t(r-2)}{2t}} = \prod_{\ell=0}^{2t-1} \frac{n-t(r-2)-\ell}{n-j+1-t(r-2)-\ell}. \quad (10)$$

We consider two sub-cases based on the value of  $j$ .

a. Suppose that  $j > 2t$ . Then, we observe that

$$\begin{aligned} \prod_{\ell=0}^{2t-1} \frac{n-t(r-2)-\ell}{n-j+1-t(r-2)-\ell} &\geq \left( \frac{n-t(r-2)}{n-j+1-t(r-2)} \right)^{2t} \\ &= \left( 1 + \frac{j-1}{n-j+1-t(r-2)} \right)^{2t} \\ &\geq 1 + \frac{2t(j-1)}{n-j+1-t(r-2)} \\ &\geq 1 + \frac{jt + (j-2)t}{n-rt} \\ &\geq 1 + \frac{jt}{n-rt}. \end{aligned}$$

We use  $j > 2t$  in the second to last inequality and  $j \geq 2$  in the final inequality.

b. Suppose that  $j \leq 2t$ . Then we can cancel additional terms from the the right hand side of equation (10) to obtain that

$$\begin{aligned} \prod_{\ell=0}^{2t-1} \frac{n-t(r-2)-\ell}{n-j+1-t(r-2)-\ell} &= \prod_{\ell=0}^{j-2} \frac{n-t(r-2)-\ell}{n-tr-\ell} \\ &\geq \left( \frac{n-t(r-2)}{n-tr} \right)^{j-1} \\ &= \left( 1 + \frac{2t}{n-rt} \right)^{j-1} \\ &\geq 1 + \frac{2t(j-1)}{n-rt} \\ &\geq 1 + \frac{jt}{n-rt}. \end{aligned}$$

Thus, in either subcase, our desired inequality holds.

2. Suppose that  $n - j + 1 \leq rt$ . Now the expression for  $Q_{n-j+1}$  is different. Since we still know that  $n \geq rt + 1$  and  $j \leq r$ , we conclude that

$$\begin{aligned} \frac{Q_{n-j+1}}{Q_n} &= \frac{(rt+1)^{\binom{n-t(r-2)}{2t}}}{2t+1} \\ &\geq \frac{(rt+1)^{\binom{rt+1-t(r-2)}{2t}}}{2t+1} \\ &= rt+1 \\ &\geq 1+jt \\ &\geq 1 + \frac{jt}{n-rt}. \end{aligned}$$

Thus, our desired inequality holds in all cases.  $\triangleleft$

### A.3 Enumerating Multiobjective Min-Cuts and Pareto-Optimal Cuts

In this section, we give algorithms to enumerate all multiobjective min-cuts and pareto-optimal cuts in polynomial time.

We first give a polynomial time algorithm to enumerate all multiobjective min-cuts. We execute our algorithm for MULTIOBJECTIVE MIN-CUT (i.e., Algorithm 2) a sufficiently large number of times so that it succeeds with high probability (i.e., with probability at least  $1 - 1/n$ ): In particular, executing it  $r^{2t}2^{rt}O(n^{2t} \log n)$  many times gives us a collection  $\mathcal{C}$  that is a superset of the collection  $\mathcal{C}_{MO}$  of multiobjective min-cuts with high probability. Moreover, the size of the collection  $\mathcal{C}$  is  $r^{2t}2^{rt}O(n^{3t-1} \log n)$ . We can prune  $\mathcal{C}$  to identify  $\mathcal{C}_{MO}$  in polynomial-time as follows: remove every cut  $F \in \mathcal{C}$  for which there exists a cut  $F' \in \mathcal{C}$  with  $c_t(F') < c_t(F)$  and  $c_i(F') \leq c_i(F)$  for every  $1 \leq i \leq t-1$ .

Next, we give a polynomial time algorithm to enumerate pareto-optimal cuts. By containment relation (1), it suffices to identify all pareto-optimal cuts in the collection  $\mathcal{C}$ . For this, we only need a polynomial-time procedure to verify if a given cut  $F$  is pareto-optimal. Algorithm 4 gives such a procedure. It essentially searches for a cut that dominates the given cut  $F$  by running our algorithm for  $b$ -MULTIOBJECTIVE MIN-CUT with  $t$  different budget-vectors.

■ **Algorithm 4** Verify pareto-optimality of a given cut.

---

VERIFY-PARETO-OPTIMALITY( $G, r, t, c_1, \dots, c_t, F$ ):

**Input:** An  $r$ -rank hypergraph  $G = (V, E)$ , cost functions  $c_1, \dots, c_t : E \rightarrow \mathbb{R}_+$ , and a cut  $F$  in  $G$

For  $i = 1, \dots, t$ :

$\vec{c} \leftarrow (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_t, c_i)$

$\vec{b} \leftarrow (c_1(F), \dots, c_{i-1}(F), c_{i+1}(F), \dots, c_t(F))$

For  $j = 1, \dots, r^{2t}2^{rt}O(n^{2t} \log(n))$ :

$F' \leftarrow b$ -MULTIOBJECTIVE-MIN-CUT( $G, r, t, \vec{c}, \vec{b}$ )

If  $F'$  is a  $b$ -multiobjective cut in  $(G, \vec{c})$  and  $c_i(F') < c_i(F)$ :

Return FALSE

Return TRUE

---

► **Theorem A.1.** *Given an  $r$ -rank,  $n$ -vertex hypergraph  $G$  with  $t$  hyperedge-cost functions and a cut  $F$  in  $G$ , if  $F$  is a pareto-optimal cut, then Algorithm 4 returns TRUE, and if  $F$  is*

not a pareto-optimal cut, then the algorithm returns *FALSE* with high probability. Moreover, the algorithm can be implemented to run in polynomial time.

**Proof.** The run-time of the algorithm is polynomial because our algorithm for  $b$ -MULTIOBJECTIVE MIN-CUT (i.e., Algorithm 1) is a polynomial-time algorithm. Algorithm 4 returns false only if it finds a cut that dominates the input cut  $F$ . If  $F$  is pareto-optimal, no such cut will exist, and therefore the algorithm will return true.

Next, suppose the input cut  $F$  is not pareto-optimal. Then  $F$  must be dominated by some other cut  $F'$ . Let  $i \in [t]$  be such that  $c_i(F') < c_i(F)$  (such an  $i$  is guaranteed to exist by the definition of domination). Let  $b$  be a budget-vector of the costs of  $F$  under the cost functions other than  $c_i$  and let  $c'$  be  $c$  with  $c_i$  moved to the end of the vector of cost functions. Then  $F$  will not be a  $b$ -multiobjective min-cut in  $(G, c')$ , since  $F'$  also satisfies  $b$ , but has a lower  $c_i$ -cost.

Therefore, any  $b$ -multiobjective min-cut will dominate  $F$  (since it will also satisfy  $b$  and cannot have higher  $c_i$  cost than  $F'$ ). Thus, if any of our  $r2^{rt}n^{2t} \log(n)$  calls to our algorithm for  $b$ -MULTIOBJECTIVE-MIN-CUT (i.e., Algorithm 1) for this value of  $i$  returns a multiobjective min-cut, then the algorithm will return false. By Theorem 2.1, our algorithm for  $b$ -MULTIOBJECTIVE MIN-CUT returns a  $b$ -multiobjective min-cut with probability  $\frac{1}{r2^{rt}}\Omega(\frac{1}{n^{2t}})$ . Therefore, if we run this algorithm  $r2^{rt}O(n^{2t} \log(n))$  times, a  $b$ -multiobjective min-cut will be returned at least once with high probability, and our algorithm will correctly return false. ◀

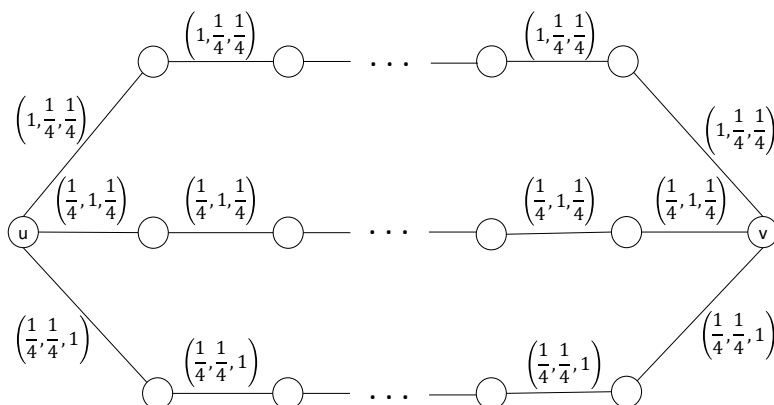
## A.4 Lower Bounds

In this section we discuss lower bounds on the number of distinct pareto-optimal cuts in  $n$ -vertex hypergraphs. Karger gave a family of graphs with  $n^{t/2}$  parametric min-cuts [13]. We recall that every parametric min-cut is a pareto-optimal cut by the containment relation (1). Thus,  $n^{t/2}$  is also a lower bound on the number of pareto-optimal cuts in  $n$ -vertex hypergraphs. To the best of the authors' knowledge, this is the best lower bound on the number of pareto-optimal cuts that is known in the literature. We give an  $\Omega(n^t)$  (for constant  $t$ ) lower bound on the number of pareto-optimal cuts in a graph. Our lower bound construction is different from that of Karger.

► **Theorem A.2.** *For all positive integers  $t$  and  $n$  such that  $n \geq t+2$ , there exists an  $n$ -vertex graph  $G$  with associated edge-cost functions  $c_1, \dots, c_t : E(G) \rightarrow \mathbb{R}_+$  such that  $G$  has at least  $\binom{n-2}{t}^t$  distinct pareto-optimal cuts.*

**Proof.** For fixed  $n$  and  $t$ , construct a graph  $G$  as follows. The graph  $G$  has two special vertices  $u$  and  $v$ . The rest of the vertices are used to form  $t$  distinct paths between  $u$  and  $v$  with each path consisting of at least  $\lfloor \frac{n-2}{t} \rfloor + 1 > \frac{n-2}{t}$  distinct edges. We assign edge costs as follows: If  $e$  is an edge in the  $i$ 'th path, then  $c_i(e) = 1$ , while  $c_j(e) = 1/(t+1)$  for every  $j \in [t] \setminus \{i\}$ . See Figure 1 for an example. We will show that any cut which contains exactly one edge from each path is pareto-optimal. The number of such cuts is at least  $\binom{n-2}{t}^t$ , since each path has at least  $(n-2)/t$  edges, so this will suffice to prove the theorem.

We observe that any cut contains either exactly one edge from each path or at least two edges from some path. Any cut  $F$  which contains exactly one edge from each path will have  $c_i(F) = 2t/(t+1)$  for every  $i \in [t]$ . Any cut  $F'$  that contains at least two edges from some path  $i \in [t]$  will have  $c_i(F') = 2 > 2t/(t+1)$ . Therefore no cut which contains two edges from the same path can dominate a cut which contains exactly one edge from each path. Furthermore, if two different cuts each contain exactly one edge from all paths, then they both have the same cost under every cost function, and thus neither can dominate the other. We conclude that every cut which contains exactly one edge from each path is pareto-optimal. ◀



■ **Figure 1** An illustration of our lower bound construction for  $t = 3$ .

► **Remark A.3.** The lower bound from Theorem A.2 is still significantly smaller than the  $O(n^{3t-1})$  upper bound from Theorem 1.1. We believe that this gap comes from the slack in the analysis of our randomized algorithms.

► **Remark A.4.** We note that the construction in Theorem A.2 also shows that there exists a budget-vector  $b \in \mathbb{R}_+^{t-1}$  such that the number of  $b$ -multiobjective min-cuts is  $\Omega(n^t)$ : consider budget values  $b_i = (2t)/(t+1)$  for every  $i \in [t-1]$ . We emphasize that since not every multiobjective min-cut is pareto-optimal, this lower bound does not imply the one from Theorem A.2. Since distinct pareto-optimal cuts need not be  $b$ -multiobjective min-cuts for the same vector  $b$ , the bound in Theorem A.2 does not immediately imply this bound either.