Descriptive Complexity on Non-Polish Spaces II

Mathieu Hoyrup
Université de Lorraine, CNRS, Inria, LORIA, Nancy, France
mathieu.hoyrup@inria.fr

Abstract

This article is a study of descriptive complexity of subsets of represented spaces. Two competing measures of descriptive complexity are available. The first one is topological and measures how complex it is to obtain a set from open sets using boolean operations. The second one measures how complex it is to test membership in the set, and we call it symbolic complexity because it measures the complexity of the symbolic representation of the set. While topological and symbolic complexity are equivalent on countably-based spaces, they differ on more general spaces. Our investigation is aimed at explaining this difference and highly suggests that it is related to the well-known mismatch between topological and sequential aspects of topological spaces.

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1 Introduction

This article fits in the line of research extending descriptive set theory, mainly developed on Polish spaces, to other classes of topological spaces relevant to theoretical computer science, such as domains [21], quasi-Polish spaces [2], and represented spaces [13, 4, 1]. We pursue our investigation of descriptive set theory on represented spaces, started in [1].

Theoretical computer science, logic and descriptive set theory closely interact, providing different ways of describing properties, by programs, formulas or boolean operation from basic properties, all intimately related. For instance, a property of real numbers that is decidable in the limit must belong to the class $\undertilde{\mathbf{\Delta}}^0_2$, and every $\undertilde{\mathbf{\Delta}}^0_2$-property is decidable in the limit relative to some oracle.

This correspondence works very well on Polish spaces and more generally countably-based topological spaces. However, little is known for other topological spaces whose points can be represented and processed by a program, and it has been shown in [1] that the correspondence fails, even on natural spaces such as the space of polynomials with real coefficients: there is a property which can be decided with 2 mind-changes, but which is not a difference of two open sets, and is in no level below $\undertilde{\mathbf{\Delta}}^0_2$.

We introduce *symbolic* descriptive complexity, which captures the algorithmic complexity of a set, and compare it to topological descriptive complexity. Our general goal is to understand when and why these two measures of complexity differ, and what topological properties of the underlying space cause this disagreement. Our results suggest that the mismatch between the two measures of complexity reflects the discordance between the sequential and the topological aspects of the space, so that symbolic complexity may be

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interpreted as a measure of *sequential* complexity rather than topological complexity, in the same way as many topological notions have a sequential counterpart (sequential continuity, sequential compactness, sequential closure, etc.).

More precisely, we show that among Hausdorff spaces, the spaces that are not Fréchet-Urysohn exhibit a disagreement between symbolic and topological complexity at the lowest level above the open sets, namely the differences of open sets. This result extends a similar result obtained in [1] for the subclass of coPolish spaces.

We focus on the space of open sets of a Polish space, and relate the disagreement between symbolic and topological complexity to the compactness properties of the Polish space, by dividing Polish spaces into 4 classes, ranging from the locally compact to the non σ-compact spaces, and giving a detailed analysis of descriptive complexity of sets in each case.

Along the way, we develop several tools and techniques that are needed to prove our results and are interesting on their own right. In particular we argue that the classical notion of hardness, which makes sense on countably-based spaces, is too restrictive on other spaces and we solve the problem by introducing the weaker notion of hard* set.

We finally observe that the discordance between topological and sequential aspects is already at the core of the theory of admissibly represented topological spaces. These spaces, also characterized as the T_0 quotients of countably-based spaces, are all sequential and form a subclass of topological spaces which behave particularly well from a categorical perspective: for instance, contrary to general topological spaces, they form a cartesian closed category. More concretely, in this category, the space constructions such as product space or subspaces do not coincide with the ones in the category of topological spaces, but with their sequentializations. Our separation results between symbolic and topological complexity heavily rely on the disagreement between sequential and topological space constructions.

1.1 Summary of the main results

We give a quick overview of the main results, stated informally.

In a represented space $\mathbf{X} = (X, \delta_{\mathbf{X}})$, we introduce the *symbolic* complexity of a set $A \subseteq X$. If Γ is a descriptive complexity class, such as $\utilde{\mathbf{\Sigma}}^0_n$ or $\utilde{\mathbf{D}}_n$ (difference of n open sets), then we define the corresponding symbolic complexity class $[\Gamma]$ as follows:

$$A \in [\Gamma](\mathbf{X}) \iff \delta_{\mathbf{X}}^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}})).$$

In a topological space with an admissible representation, one usually has $\Gamma(\mathbf{X}) \subseteq [\Gamma](\mathbf{X})$ and our goal is to understand when and why the other inclusion does not hold, i.e. when and why the topological and symbolic measures of complexity differ. It is know from [2] that they coincide when $\mathbf{X}$ is a countably-based space.

We first observe that the classical notion of hardness, which is very useful to identify the complexity of a set, is closely related to symbolic rather than topological complexity. We introduce a weaker version, called hard* set and prove:

▶ **Theorem** (Theorem 3.2). *For a Borel subset A of an analytic space $\mathbf{X}$,*

$$\begin{aligned} A \textit{ is } \Gamma\textit{-hard} &\iff A \notin [\check{\Gamma}](\mathbf{X}), \\ A \textit{ is } \Gamma\textit{-hard*} &\iff A \notin \check{\Gamma}(\mathbf{X}). \end{aligned}$$

A topological subspace of a sequential space is not always sequential, so the subspace constructions differ in the categories of topological and sequential spaces. This difference implies a difference between symbolic and topological complexity.

The sequential spaces whose subspaces are sequential are called the Fréchet-Urysohn spaces. The class $\underset{\sim}{\mathbf{D}}_2$ consists of differences of two open sets.

▶ **Theorem** (Theorem 4.1). *If* $\mathbf{X}$ *is admissibly represented, Hausdorff and not Fréchet-Urysohn, then*

$$[\underset{\sim}{\mathbf{D}}_2](\mathbf{X}) \not\subseteq \underset{\sim}{\mathbf{D}}_2(\mathbf{X}).$$

The assumption that the space is Hausdorff is needed. Indeed, spaces of open sets behave better at low complexity levels.

▶ **Theorem** (Theorem 5.1). *If* $\mathbf{X}$ *is admissibly represented then*

$$[\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) = \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X})).$$

However, the proof is not constructive and we show that the corresponding effective classes disagree. The class D_2 consists of differences of two *effective* open sets. Let $\mathcal{N}_1$ be the space of functions $\mathbb{N} \to \mathbb{N}$ having at most 1 non-zero value.

▶ **Theorem** (Theorem 5.3). *One has* $[\mathrm{D}_2](\mathcal{O}(\mathcal{N}_1)) \not\subseteq \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1))$.

Finally, we give a rather detailed study of descriptive complexity on the spaces $\mathcal{O}(\mathbf{X})$ when $\mathbf{X}$ is Polish. More precisely, we connect the relationship between symbolic and topological complexity classes to the compactness properties of $\mathbf{X}$. Some of the proofs heavily rely on the fact that the product topology is not sequential in general, so product space constructions differ in the categories of topological and sequential spaces.

In particular, symbolic and topological complexity differ at higher levels when $\mathbf{X}$ is Polish and not locally compact.

▶ **Theorem** (Theorem 6.5).

- *There exists* $A \in [\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(\mathcal{N}_1))$ *which is* $\underset{\sim}{\mathbf{\Delta}}^0_3$*-complete*.*
- *There exists* $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbb{N} \times \mathcal{N}_1))$ *which is* $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$*-complete*, for each* $k \geq 2$.
- *There exists* $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_2](\mathcal{O}(\mathcal{N}))$ *which is not Borel.*

The paper is organized as follows. In Section 2, after giving the needed background on represented spaces, we introduce symbolic complexity and provide simple tools for its study. In Section 3 we introduce and study the notion of hard* set, used to capture the topological complexity of sets. In Section 4 we prove that Hausdorff spaces that are not Fréchet-Urysohn exhibit a disagreement between symbolic and topological complexity at the lowest level. In Section 5, we study spaces of open sets. In particular, in Section 6 we focus on open subsets of Polish spaces and locate symbolic complexity classes depending on the compactness properties of the Polish space.

We sometimes include the proof in the body of the article, and sometimes only give the intuition. Complete proofs can be found in [9].

2 Symbolic complexity

2.1 Represented spaces

The Baire space is $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, whose elements are either viewed as functions or infinite sequences. To finite sequence of natural numbers $\sigma \in \mathbb{N}^*$, we associate the cylinder $[\sigma]$ which is the set of elements of $\mathcal{N}$ extending σ. The Baire space is then endowed with the topology generated by the cylinders. Every subset of $\mathcal{N}$ is endowed with the subspace topology.

A **represented space** is a pair $\mathbf{X} = (X, \delta_{\mathbf{X}})$ where X is a set and $\delta_{\mathbf{X}} :\subseteq \mathcal{N} \to X$ is a partial surjective function called a **representation**. If $\delta_{\mathbf{X}}(p) = x$, then p is a **name** of x. If $\mathbf{X}, \mathbf{Y}$ are represented spaces then a function $F :\subseteq \mathcal{N} \to \mathcal{N}$ is a **realizer** of $f : \mathbf{X} \to \mathbf{Y}$ if $f \circ \delta_{\mathbf{X}} = \delta_{\mathbf{Y}} \circ F$. f is **computable** if it has a computable realizer. We write $\mathbf{X} \cong \mathbf{Y}$ if there exists a bijection between $\mathbf{X}$ and $\mathbf{Y}$ which is computable in both directions.

A representation δ of a topological space (X, τ) is **admissible** if τ is the final topology of δ and every partial continuous function $f :\subseteq \mathcal{N} \to X$ has a continuous realizer, which is a continuous function $F :\subseteq \mathcal{N} \to \mathcal{N}$ satisfying $f = \delta \circ F$.

If $\mathbf{X}, \mathbf{Y}$ are admissibly represented spaces, then a function $f : \mathbf{X} \to \mathbf{Y}$ is continuous if and only if it has a continuous realizer. In particular, f is continuous if and only if it is computable relative to some oracle.

The topological spaces having an admissible representation are exactly the T_0-spaces that are quotients of countably-based spaces, and are also called QCB_0-spaces. These spaces form a cartesian closed category, with very natural representations for products and function spaces. They enjoy the following remarkable but overlooked properties, as proved by Schröder [16]: if X is a QCB_0-space, then

- X is sequential, separable and has a countable network, i.e. a countable family of subsets such that every open set is a union of them,
- X is first-countable if and only if X is countably-based,
- X is hereditarily Lindelöf,
- When identifying the space of open sets $\mathcal{O}(X)$ with the function space $\mathbb{S}^X$ where $\mathbb{S}$ is the Sierpinski space, the topology on $\mathcal{O}(X)$ is the Scott topology.

As far as topology is concerned, admissibly represented spaces and QCB_0-spaces are the same. However, computability can only be expressed in terms of representations, so we will refer to admissibly represented spaces rather than QCB_0-spaces.

Countably-based T_0-spaces have a particular representation, called standard representation, which is admissible. Once a countable basis indexed by $\mathbb{N}$ has been chosen, say $(B_i)_{i\in\mathbb{N}}$, a name of x is any sequence $p \in \mathcal{N}$ such that $\{i \in \mathbb{N} : \exists n, p(n) = i + 1\} = \{i \in \mathbb{N} : x \in B_i\}$, so that x is described by enumerating its basic neighbourhoods in any order.

2.2 Symbolic complexity

Let Γ be a descriptive complexity class, i.e. a family $\Gamma = \{\Gamma(X)\}$ where X ranges over topological spaces and $\Gamma(X)$ is a collection of subsets of X. The simplest class is $\utilde{\mathbf{\Sigma}}^0_1 = \{\utilde{\mathbf{\Sigma}}^0_1(X)\}$ where $\utilde{\mathbf{\Sigma}}^0_1(X)$ is the collection of open subsets of X. The class $\utilde{\mathbf{\Sigma}}^0_n$ is inductively defined as the class of countable unions of differences of $\utilde{\mathbf{\Sigma}}^0_{n-1}$-sets. The class $\utilde{\mathbf{D}}_n$ is inductively defined as follows: $\utilde{\mathbf{D}}_1 = \utilde{\mathbf{\Sigma}}^0_1$ and $\utilde{\mathbf{D}}_{n+1}$ consists of sets $U \setminus A$ where $U \in \utilde{\mathbf{\Sigma}}^0_1$ and $A \in \utilde{\mathbf{D}}_n$. For any class Γ, the class $\check{\Gamma}$ consists of the complements of sets in Γ.

Let $\mathbf{X} = (X, \delta_{\mathbf{X}})$ be a represented space, which is also a topological space by taking the final topology of $\delta_{\mathbf{X}}$: $U \subseteq \mathbf{X}$ is open iff $\delta_{\mathbf{X}}^{-1}(U)$ is open in $\mathrm{dom}(\delta_{\mathbf{X}})$.

▶ **Definition 2.1.** *Let $\mathbf{X} = (X, \delta_{\mathbf{X}})$ be a represented space. We define the **symbolic** complexity class $[\Gamma](\mathbf{X})$ as follows: for $A \subseteq X$,*

$$A \in [\Gamma](\mathbf{X}) \iff \delta_{\mathbf{X}}^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}})).$$

By definition of the final topology of $\delta_{\mathbf{X}}$, one always has $[\utilde{\mathbf{\Sigma}}^0_1](\mathbf{X}) = \utilde{\mathbf{\Sigma}}^0_1(\mathbf{X})$. A descriptive complexity class Γ is often closed under continuous preimages, and in that case one has $\Gamma(\mathbf{X}) \subseteq [\Gamma](\mathbf{X})$ because $\delta_{\mathbf{X}}$ is continuous. Moreover, for such classes Γ, it is not hard to see that the symbolic complexity class $[\Gamma]$ does not depend on the choice of the admissible representation, hence $\Gamma(\mathbf{X})$ is intrinsic to X as a topological space. However, effective complexity classes are sensitive to the choice of an admissible representation (but not to the choice of a computably admissible one, which we do not discuss here).

Our general goal is to understand symbolic complexity classes, relate them to topological complexity classes and understand for which $\mathbf{X}$ and which Γ we have $[\Gamma](\mathbf{X}) = \Gamma(\mathbf{X})$.

An important result due to de Brecht [2] is that symbolic and topological complexity coincide when $\mathbf{X}$ is a countably-based T_0-space with its standard representation.

▶ **Theorem 2.2** ([2]). *Let $\mathbf{X}$ be a countably-based T_0-space with its standard representation, and let $\alpha, \beta < \omega_1$. One has*

$$[\underset{\sim}{\mathbf{D}}_\alpha(\underset{\sim}{\mathbf{\Sigma}}^0_\beta)](\mathbf{X}) = \underset{\sim}{\mathbf{D}}_\alpha(\underset{\sim}{\mathbf{\Sigma}}^0_\beta)(\mathbf{X}).$$

It was improved in [1] by observing that the equality is uniform and effective, so it also holds for the effective complexity classes $\mathrm{D}_m(\Sigma^0_n)$, $m, n \in \mathbb{N}$.

2.3 Tools

We give a simple way of locating a symbolic complexity class. A **network** in a topological space X is a family $\mathcal{F}$ of subsets of X such that every open set is a union of elements of $\mathcal{F}$ [5]. Every admissibly represented space has a countable network, given by the images of cylinders under the admissible representation.

▶ **Proposition 2.3.** *Let $\mathbf{X}$ be admissibly represented. Assume that $\mathbf{X}$ has a countable network of sets in $\underset{\sim}{\mathbf{\Sigma}}^0_{i+1}(\mathbf{X})$. For all $n \in \mathbb{N}$, one has*

$$[\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X}).$$

Proof. Let $\mathbf{Y}$ be the topological space with underlying set X and whose topology is generated by the countable network of $\mathbf{X}$. $\mathbf{Y}$ is countably-based and inherits the T_0-property of $\mathbf{X}$, let $\delta_{\mathbf{Y}}$ be its standard representation. Therefore, one has $[\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{Y}) = \underset{\sim}{\mathbf{\Sigma}}^0_n(\mathbf{Y})$ by Theorem 2.2.

By definition of a network, every open subset of $\mathbf{X}$ is an open subset of $\mathbf{Y}$. In other words, $\mathrm{id} : \mathbf{Y} \to \mathbf{X}$ is continuous hence continuously realizable, which implies that $[\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq [\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{Y})$. Conversely, every open subset of $\mathbf{Y}$ belongs to $\underset{\sim}{\mathbf{\Sigma}}^0_{i+1}(\mathbf{X})$ which implies, by induction on n, that $\underset{\sim}{\mathbf{\Sigma}}^0_n(\mathbf{Y}) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X})$.

Putting everything together, we obtain $[\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq [\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{Y}) = \underset{\sim}{\mathbf{\Sigma}}^0_n(\mathbf{Y}) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X})$. ◀

We take the following notion from [19]. An admissibly represented space is **quasi-zero-dimensional** if it is the sequentialization of a zero-dimensional space, i.e. if its open sets are the sequentially open sets of a space having a basis of clopen sets. For instance, the spaces $2^{\mathcal{N}}$ and $\mathbb{N}^{\mathcal{N}}$ are not zero-dimensional, as proved by Schröder [18], but they are quasi-zero-dimensional.

▶ **Corollary 2.4.** *Let $\mathbf{X}$ be quasi-zero-dimensional, for instance $\mathbf{X} = 2^{\mathcal{N}}$ or $\mathbb{N}^{\mathcal{N}}$. One has*

$$[\underset{\sim}{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{n+1}(\mathbf{X}).$$

Proof. The images of cylinders under the representation are closed subsets of $\mathbf{X}$ (Proposition 7 in [19]). ◀

A common technique to prove a separation result in a space $\mathbf{Y}$ is to prove it in a simpler space $\mathbf{X}$ and then transfer the result to $\mathbf{Y}$ by including $\mathbf{X}$ into $\mathbf{Y}$.

▶ **Proposition 2.5.** *Let Γ, Γ' be complexity classes that are closed under continuous (resp. computable) preimages.*

Let $\mathbf{X}$ be a continuous (resp. computable) retract of $\mathbf{Y}$. If $[\Gamma](\mathbf{Y}) \subseteq \Gamma'(\mathbf{Y})$, then $[\Gamma](\mathbf{X}) \subseteq \Gamma'(\mathbf{X})$.

Proof. Let $r : \mathbf{Y} \to \mathbf{X}$ and $s : \mathbf{X} \to \mathbf{Y}$ be continuous (resp. computable) functions such that $r \circ s = \mathrm{id}_{\mathbf{X}}$. Let $A \in [\Gamma](\mathbf{X})$ and $B = r^{-1}(A)$. As r is continuous hence continuously realizable (resp. computable), one has $B \in [\Gamma](\mathbf{Y})$ so $B \in \Gamma(\mathbf{Y})$. As s is continuous (resp. computable) and $A = s^{-1}(B)$, we conclude that $A \in \Gamma'(\mathbf{X})$. ◀

If $\mathbf{Y} = (Y, \delta_{\mathbf{Y}})$ is a represented space and $X \subseteq Y$, then $\mathbf{X} := (X, \delta_{\mathbf{X}})$ is a represented space by taking $\delta_{\mathbf{X}}$ as the restriction of $\delta_{\mathbf{Y}}$ to $\delta_{\mathbf{Y}}^{-1}(X)$. Observe that as a topological space, $\mathbf{X}$ is not always a topological subspace of $\mathbf{Y}$, but the sequentialization of the topological subspace [17].

▶ **Proposition 2.6.** *Let Γ be closed under finite intersections and continuous (resp. computable) preimages, and Γ' be closed under continuous (resp. computable) preimages. Let $X \in \Gamma(\mathbf{Y})$. If $[\Gamma](\mathbf{Y}) \subseteq \Gamma'(\mathbf{Y})$, then $[\Gamma](\mathbf{X}) \subseteq \Gamma'(\mathbf{X})$.*

Proof. The representation $\delta_{\mathbf{X}}$ of $\mathbf{X}$ is the restriction of $\delta_{\mathbf{Y}}$ to $\delta_{\mathbf{Y}}^{-1}(X)$.

Let $A \in [\Gamma](\mathbf{X})$. One has $A \in [\Gamma](\mathbf{Y})$. Indeed, $\delta_{\mathbf{Y}}^{-1}(A) = \delta_{\mathbf{X}}^{-1}(A) = S \cap \mathrm{dom}(\delta_{\mathbf{X}})$ for some $S \in \Gamma(\mathcal{N})$, and $\mathrm{dom}(\delta_{\mathbf{X}}) = \delta_{\mathbf{Y}}^{-1}(X) = T \cap \mathrm{dom}(\delta_{\mathbf{Y}})$ for some $T \in \Gamma(\mathcal{N})$. By assumption, $U := S \cap T \in \Gamma(\mathcal{N})$ so $\delta_{\mathbf{Y}}^{-1}(A) = U \cap \mathrm{dom}(\delta_{\mathbf{Y}})$ and $A \in [\Gamma](\mathbf{Y})$.

Threfore, $A \in \Gamma'(\mathbf{Y})$ so by continuity (resp. computability) of the identity from $\mathbf{X}$ to $\mathbf{Y}$, we conclude that $A \in \Gamma'(\mathbf{X})$. ◀

3 Hardness

An important tool to pinpoint the descriptive complexity of a set is provided by the notions of hardness and completeness. If Γ is a descriptive complexity class, then in any topological space X, one can define a set $A \subseteq X$ to be **Γ-hard** if for each $C \in \Gamma(\mathcal{N})$, there is a continuous reduction from C to A, i.e. a continuous function $f : \mathcal{N} \to X$ such that $C = f^{-1}(A)$. Note that the reduction always starts from $\mathcal{N}$. It contrasts with the generalizations of Wadge reducibility between subsets of a topological or represented spaces investigated in [14, 15].

As is well known in descriptive set theory on Polish (and even quasi-Polish) spaces, the hardness of a set is closely related to its complexity: Wadge's Lemma implies that for any class $\Gamma \neq \check{\Gamma}$ of Borel sets and any Borel subset A of a Polish space X,

$$A \text{ is } \Gamma\text{-hard} \iff A \notin \check{\Gamma}(X).$$

However, outside countably-based spaces it turns out that the hardness of a set is related to its symbolic rather than topological complexity, which usually differ as we will see shortly.

Therefore, we need another notion of hardness which reflects the topological complexity of a set.

▶ **Definition 3.1.** *Let (X, τ) be a topological space and Γ a descriptive complexity class. We say that $A \subseteq X$ is Γ-**hard*** if for every countably-based topology $\tau' \subseteq \tau$, A is Γ-hard in (X, τ'). A set is Γ-**complete*** if it belongs to $\Gamma(X)$ and is Γ-hard*.*

Note that when (X, τ) is countably-based, these notions coincide with the standard notions of hardness and completeness.

Now we can state the main result of this section, making clear that hardness is related to symbolic complexity, while hardness* is related to topological complexity. Say that a topological space is analytic if it is a continuous image of $\mathcal{N}$.

▶ **Theorem 3.2.** *Let $\Gamma = \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta)$ where α, β are countable ordinals. For an analytic admissibly represented space X and $A \subseteq X$ Borel,*

$$\begin{aligned} A \text{ is } \Gamma\text{-hard} &\iff A \notin [\check{\Gamma}](X), \\ A \text{ is } \Gamma\text{-hard*} &\iff A \notin \check{\Gamma}(X). \end{aligned}$$

For $\beta = 1$, the assumptions that the space is analytic and that A is Borel can be dropped. The proof assumes $\mathbf{\Sigma}^1_1$-determinacy.

3.1 Hausdorff-Kuratowski Theorem

On Polish and even quasi-Polish spaces, there is no $\mathbf{\Delta}^0_n$-complete set because of the Hausdorff-Kuratowski Theorem. Other spaces may admit $\mathbf{\Delta}^0_n$-complete* sets, and we show that this possibility is again tightly related to the validity of the Hausdorff-Kuratowski Theorem for $\mathbf{\Delta}^0_n$-sets.

▶ **Definition 3.3.** *A topological space X has the **Hausdorff-Kuratowski property at level $\mathbf{\Delta}^0_n$** if*

$$\mathbf{\Delta}^0_n(X) = \bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(\mathbf{\Sigma}^0_{n-1})(X).$$

▶ **Theorem 3.4.** *Let X be an analytic topological space.*

For each $n \geq 2$, X has the Hausdorff-Kuratowski property at level $\mathbf{\Delta}^0_n$ if and only if X has no $\mathbf{\Delta}^0_n$-complete set.*

For $n = 2$, the analyticity assumption can be droppped.

Proof. If the HK property is satisfied, then there is no $\mathbf{\Delta}^0_n$-complete* set. Indeed, such a set A would be in $\mathbf{D}_\alpha(\mathbf{\Sigma}^0_{n-1})$ for some $\alpha < \omega_1$ and some countably-based topology, and $\mathbf{\Delta}^0_n$-hard for that topology, which would imply that $\mathbf{\Delta}^0_n(\mathcal{N}) \subseteq \mathbf{D}_\alpha(\mathbf{\Sigma}^0_{n-1})(\mathcal{N})$, which is known to be false (the difference hierarchies do not collapse on $\mathcal{N}$).

Conversely, if the HK property does not hold, then there exists $A \in \mathbf{\Delta}^0_n(X)$ such that $A \notin \mathbf{D}_\alpha(\mathbf{\Sigma}^0_{n-1})$ for any $\alpha < \omega_1$. If X is analytic or $n = 2$, then A is $\check{\mathbf{D}}_\alpha(\mathbf{\Sigma}^0_{n-1})$-hard* for each $\alpha < \omega_1$ by Theorem 3.2. As a result, A is $\mathbf{\Delta}^0_n$-hard*, hence $\mathbf{\Delta}^0_n$-complete*. ◀

We now give a criterion for the validity of the Hausdorff-Kuratowski property at a given level.

▶ **Theorem 3.5.** *Let (X, τ) be a topological space. If there exists a Polish topology τ' such that $\tau \subseteq \tau' \subseteq \mathbf{\Sigma}^0_n(\tau)$, then (X, τ) has the Hausdorff-Kuratowski property for all levels $\mathbf{\Delta}^0_k$ with $k \geq n + 1$.*

Proof. The proof follows the line of the argument in [10], reducing the case of $\mathbf{\Delta}^0_n$ to $\mathbf{\Delta}^0_2$ by enriching the topology. However, some care is needed because we have to deal with two topologies.

▷ Claim 3.6. For any $k \leq n$ and any countable family $\mathcal{F} \subseteq \mathbf{\Sigma}^0_k(X, \tau)$, there exists a Polish topology $\tau'' \subseteq \mathbf{\Sigma}^0_n(X, \tau)$ containing $\mathcal{F}$.

Proof of the Claim. We prove it by induction on k. For $k = 1$, the result is immediate by taking $\tau'' = \tau'$, as $\mathcal{F}$ is already contained in τ'. Assume the result for $k < n$ and let $\mathcal{F} \subseteq \utilde{\mathbf{\Sigma}}^0_{k+1}(X,\tau)$. There exists a countable family $\mathcal{G} \subseteq \utilde{\mathbf{\Sigma}}^0_k(X,\tau)$ such that each element of $\mathcal{F}$ is a countable union of differences of elements of $\mathcal{G}$. By induction, there is a Polish topology $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$ containing $\mathcal{G}$. Let τ''' be generated by τ'' and the complements of the elements of $\mathcal{G}$. As the latter sets are closed in τ'' which is Polish, τ''' is Polish. Moreover, those sets belong to $\utilde{\mathbf{\Pi}}^0_k(X,\tau) \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$, so $\tau''' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$. Finally, each element of $\mathcal{F}$ is open in τ''', and the claim is proved. ◁

We now prove the theorem. Let $A \in \utilde{\mathbf{\Delta}}^0_{n+1}(X,\tau)$. There exists a countable family $\mathcal{F} \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$ such that A and its complement are countable unions of differences of elements of $\mathcal{F}$. Applying the claim, there exists a Polish topology $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$ containing $\mathcal{F}$. Therefore, $A \in \utilde{\mathbf{\Delta}}^0_2(X,\tau'')$ so applying the Hausdorff-Kuratowski theorem for Polish spaces, one has $A \in \utilde{\mathbf{D}}_\alpha(X,\tau'')$ for some $\alpha < \omega_1$. We conclude by observing that $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X,\tau)$. ◀

We give two simple applications of this result.

The space $\mathbb{R}[X]$ of polynomials with real coefficients is an example of a coPolish space which is not countably-based [3]. A polynomial is represented by giving an upper bound on its degree as well as standard names of its coefficients. On $\mathbb{R}[X]$, hence on $\mathbb{R}[X]^\mathbb{N}$, there is a set in $[\mathrm{D}_\omega]$ which is $\utilde{\mathbf{\Delta}}^0_2$-complete* (Theorem 5.8 in [1]). Theorem 3.5 implies that there is no $\utilde{\mathbf{\Delta}}^0_k$-complete* set for $k \geq 3$.

▶ **Corollary 3.7.** *The space $\mathbb{R}[X]^\mathbb{N}$ has the Hausdorff-Kuratowski property for all levels $\utilde{\mathbf{\Delta}}^0_k$ with $k \geq 3$, therefore that space has no $\utilde{\mathbf{\Delta}}^0_k$-complete* set for $k \geq 3$.*

Proof. For each $n, d \in \mathbb{N}$, the set $C_{n,d} := \{(P_i)_{i\in\mathbb{N}} : \deg(P_i) \leq d\}$ is closed. Enriching the topology on $\mathbb{R}[X]^\mathbb{N}$ with these sets results in a Polish topology contained in $\utilde{\mathbf{\Sigma}}^0_2(\mathbb{R}[X]^\mathbb{N})$ (the space becomes homeomorphic to $\mathbb{R}^\mathbb{N}$). ◀

We will see later that on $\mathcal{O}(\mathcal{N}_1)$, hence on $\mathcal{O}(\mathbb{N}\times\mathcal{N}_1)$, there is a set in $[\mathrm{D}_\omega]$ which is $\utilde{\mathbf{\Delta}}^0_3$-complete* (Theorem 6.5). Theorem 3.5 implies that there is no $\utilde{\mathbf{\Delta}}^0_k$-complete* set for $k \geq 4$.

▶ **Corollary 3.8.** *The space $\mathcal{O}(\mathbb{N}\times\mathcal{N}_1)$ has the Hausdorff-Kuratowski property for all levels $\utilde{\mathbf{\Delta}}^0_k$ with $k \geq 4$, therefore that space has no $\utilde{\mathbf{\Delta}}^0_k$-complete* set for $k \geq 4$.*

Proof. We add the following sets to the topology: for each $(n,f) \in \mathbb{N}\times\mathcal{N}_1$, the closed set $C_{n,f} := \{U : (n,f) \notin U\}$; for each $(n,p) \in \mathbb{N}^2$, the $\utilde{\mathbf{\Pi}}^0_2$-set $P_{n,p} := \{U : \{n\}\times[0^p] \subseteq U\}$, where $[0^p]$ is the set of functions $f \in \mathcal{N}_1$ such that $f(i) = 0$ for all $i < p$. We now show that the resulting topological space is homeomorphic to a closed subset of the Baire space, which implies that it is Polish.

We encode $U \in \mathcal{O}(\mathbb{N}\times\mathcal{N}_1)$ into two sequences g_n, h_n of elements of $\mathcal{N}$, which we can encode into a single element of $\mathcal{N}$. We use a one-to-one enumeration $(f_i)_{i\in\mathbb{N}}$ of the elements of $\mathcal{N}_1$, where f_0 is the null function.

Given U, we define its code $(g_n, h_n)_{n\in\mathbb{N}}$ as follows:

- $g_n(i) = 1$ if $(n, f_i) \in U$ and $g_n(i) = 0$ if $(n, f_i) \notin U$.
- $h_n(0) = 0$ if $(n, f_0) \notin U$, and $h_n(0) = p + 1$ if $(n, f_0) \in U$ and p is minimal such that $\{n\}\times[0^p] \subseteq U$.

It is not hard to see that the function sending U to $(g_n, h_n)_{n\in\mathbb{N}}$ is one-to-one and continuous for the enriched topology, as well as its inverse, and that the subset of $\mathcal{N}$ consisting of the codes of elements of $\mathcal{O}(\mathbb{N}\times\mathcal{N}_1)$ is closed. As a result, the enriched topological space is Polish.

Moreover, the enriched topology is contained in $\utilde{\mathbf{\Sigma}}^0_3(\mathcal{O}(\mathbb{N}\times\mathcal{N}_1))$. ◀

4 Fréchet-Urysohn property

In [1] we have given a characterization of the coPolish spaces on which the symbolic complexity differs from the topological complexity at the level $\boldsymbol{D}_2$: they are exactly the spaces that are not Fréchet-Urysohn.

We can extend part of the argument from coPolish spaces to Hausdorff admissibly represented spaces. We will see later (Theorem 5.1) that the assumption that the space is Hausdorff cannot be dropped.

▶ **Theorem 4.1.** *Let $\mathbf{X}$ be admissibly represented and Hausdorff. If $\mathbf{X}$ is not Fréchet-Urysohn, then*

$$[\boldsymbol{D}_2](\mathbf{X}) \not\subseteq \boldsymbol{D}_2(\mathbf{X}).$$

We use the Arens' space $\mathbf{S_2}$, which is the inductive limit of

$$X_N = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{\frac{1}{n} + \frac{1}{k}X^n : n \leq N, k \in \mathbb{N}\right\}.$$

As in [1], one has $[\boldsymbol{D}_2](\mathbf{S_2}) \not\subseteq \boldsymbol{D}_2(\mathbf{S_2})$ and a witness is the set $A = \{0\} \cup \{\frac{1}{n} + \frac{1}{k}X^n : n, k \in \mathbb{N}\}$. Therefore, Theorem 4.1 is an immediate corollary of the next result together with Proposition 2.6.

▶ **Proposition 4.2.** *Let $\mathbf{X}$ be admissibly represented and Hausdorff. $\mathbf{X}$ is not Fréchet-Urysohn if and only if $\mathbf{X}$ contains a closed copy of $\mathbf{S_2}$.*

▶ Remark 4.3 (Historical remark about Proposition 4.2). Franklin [7] proved that when X is a Hausdorff sequential space, X is Fréchet-Urysohn if and only if it does not contain a set which, endowed with the sequentialization of the subspace topology, is homeomorphic to $\mathbf{S_2}$ (Proposition 7.3 in [7]). It implies that if $\mathbf{X}$ is a Hausdorff admissibly represented space, then $\mathbf{X}$ is not Fréchet-Urysohn if and only if $\mathbf{X}$ does not contain $\mathbf{S_2}$ as a represented subspace.

In [22] and [11] it is proved that when X is a Hausdorff sequential space having a point-countable k-network, X is not Fréchet-Urysohn if and only if it does not contain a *closed* set homeomorphic to $\mathbf{S_2}$ (Theorem 2.12 in [11]). Observe that the subspace topology on a closed subset of a sequential space is always sequential, so there is no need to take the sequentialization of the subspace topology as in Franklin's result. This result implies ours, because admissibly represented spaces are sequential and the images of cylinders under the representation give a countable k-network. However we provide a proof in our setting for self-containedness.

The result was also recently proved in [3] for the subclass of coPolish spaces (Proposition 66 in [3], where $\mathbf{S_2}$ is called $\mathbf{S_{min}}$).

5 Spaces of open sets

As already mentioned, the assumption that the space is Hausdorff is important in Theorem 4.1, because some spaces admitting a rich poset structure behave more smoothly. This phenomenon has been already exploited in many ways in the realm of domain theory. We show that even in the absence of a countable basis, some positive results are still valid.

▶ **Theorem 5.1.** *Let $\mathbf{X}$ be admissibly represented. For every $n \geq 2$,*

$$[\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) = \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X})).$$

We will see that this equality cannot be extended to level ω: if $\mathbf{X}$ is Polish and not locally compact, then $[\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(\mathbf{X}))$ contains a $\underset{\sim}{\mathbf{\Delta}}^0_3$-complete* set (Theorem 6.5).

Informal proof. The strategy is inspired from the one developed in [8] and [20] where a similar result is proved in the context of numbered sets and algebraic domains. We show that the result can be proved in the context of represented spaces, and without assuming a countable basis.

We first isolate a property of Scott open sets: say that a set $A \subseteq \mathcal{O}(\mathbf{X})$ is *approximable* if for every directed set $\Delta \subseteq \mathcal{O}(\mathbf{X})$ such that $\sup \Delta \in A$, Δ intersects A. A set is Scott open, i.e. in $\underset{\sim}{\mathbf{\Sigma}}^0_1(\mathcal{O}(\mathbf{X}))$ if and only if it is upwards closed and approximable.

We prove the following generalization. A set A is in $\underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$ if and only if both A and A^c are approximable and has no $n+1$-chain, i.e. no sequence $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n$ such that $U_i \in A$ iff i is even (it indeed generalizes the case of open sets for $n = 1$).

The last step is to prove that:
- A subset of $\mathcal{O}(\mathbf{X})$ that is not approximable is necessarily $\underset{\sim}{\mathbf{\Pi}}^0_2$-hard,
- A subset of $\mathcal{O}(\mathbf{X})$ having an $n+1$-chain is necessarily $\check{\underset{\sim}{\mathbf{D}}}_n$-hard.

Therefore, if $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_2](\mathcal{O}(\mathbf{X}))$ then A is approximable, and if moreover $A \in [\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X}))$, then A has no $n+1$-chain.

Putting everything together implies that $[\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) \subseteq \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$. ◀

A consequence of the preceding development is a characterization of the class $[\underset{\sim}{\mathbf{\Delta}}^0_2]$ in certain cases.

▶ **Proposition 5.2.** *Let $\mathbf{X}$ be countably-based. The class $[\underset{\sim}{\mathbf{\Delta}}^0_2](\mathcal{O}(\mathbf{X}))$ is precisely the class of approximable and co-approximable sets.*

Proof. We know from the proof of Theorem 5.1 that if $A \in [\underset{\sim}{\mathbf{\Delta}}^0_2](\mathcal{O}(\mathbf{X}))$ then both A and A^c are approximable.

Conversely, assume that $A \subseteq \mathcal{O}(\mathbf{X})$ and its complement are approximable. Observe that if U_i is a growing sequence of open sets with union U, then $\mathbf{1}_A(U_i)$ converges to $\mathbf{1}_A(U)$ as $i \to \infty$, as both A and A^c are approximable. Let $(B_i)_{i\in\mathbb{N}}$ be a countable basis of $\mathbf{X}$, closed under finite intersections and unions. Let $E = \{i \in \mathbb{N} : B_i \in A\}$. From a name of an open set $U \in \mathcal{O}(\mathbf{X})$, one can continuously derive a sequence $(i_n)_{n\in\mathbb{N}}$ such that $B_{i_n} \subseteq B_{i_{n+1}}$ and $\bigcup_n B_{i_n} = U$. Therefore, whether $U \in A$ can be tested with finitely mind changes, by testing whether $i_n \in E$. ◀

5.1 Effectiveness

The proof of Theorem 5.1 is not effective. We show that there is no effective argument by proving that $[\mathrm{D}_2](\mathcal{O}(\mathbf{X})) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathbf{X}))$ for some particular $\mathbf{X}$.

▶ **Theorem 5.3.** *One has*

$$[\mathrm{D}_2](\mathcal{O}(\mathcal{N}_1)) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1)),$$

and a witness can be taken in $\underset{\sim}{\mathbf{D}}_2(\mathcal{O}(\mathcal{N}_1))$.

Such a set is a difference of two open sets, but computationally speaking, its name set is strictly easier to describe than the set itself.

Informal proof. We prove the separation result for the space $\mathbb{N} \times \mathcal{O}(\mathcal{N}_1)$ and observe that this space embeds as a D_2-subset of $\mathcal{O}(\mathcal{N}_1)$, to which the result immediately transfers by Proposition 2.6. The following discussion is rather general, and $\mathcal{O}(\mathcal{N}_1)$ is a possible instance of $\mathbf{X}$.

In [1] we proved that if $\mathbf{X}$ is not countably-based, then for sets in $\undertilde{\mathbf{D}}_2(\mathbf{X})$, one cannot continuously convert $[\undertilde{\mathbf{D}}_2]$-names into $\undertilde{\mathbf{D}}_2$-names. Another way of formulating this result is expressed by the existence of a function $f : \mathbf{Y} \to \undertilde{\mathbf{D}}_2(\mathbf{X})$ for some represented space $\mathbf{Y}$, such that:

- $f : \mathbf{Y} \to [\undertilde{\mathbf{D}}_2](\mathbf{X})$ is continuously realizable,
- $f : \mathbf{Y} \to \undertilde{\mathbf{D}}_2(\mathbf{X})$ is not continuously realizable.

Equivalently, it means that $\{(y, x) : x \in f(y)\}$ belongs to $[\undertilde{\mathbf{D}}_2](\mathbf{Y} \times \mathbf{X})$ but not $\undertilde{\mathbf{D}}_2(\mathbf{Y} \times \mathbf{X})$.

If $f : \mathbf{Y} \to [\undertilde{\mathbf{D}}_2](\mathbf{X})$ is moreover computable and if there is an effective enumeration $(y_i)_{i\in\mathbb{N}}$ of the computable elements of $\mathbf{Y}$, then we can consider the following set:

$$A = \{(i, x) \in \mathbb{N} \times \mathbf{X} : x \in f(y_i)\}.$$

One immediately has $A \in \undertilde{\mathbf{D}}_2(\mathbb{N} \times \mathbf{X})$, $A \in [\mathrm{D}_2](\mathbb{N} \times \mathbf{X})$ and one has to prove that $A \notin \mathrm{D}_2(\mathbb{N} \times \mathbf{X})$, which depends on the details of Y and f.

For $\mathbf{X} = \mathcal{O}(\mathcal{N}_1)$, one can make $\mathbf{Y}$ and $f : \mathbf{Y} \to \undertilde{\mathbf{D}}_2(\mathbf{X})$ very explicit: let $\mathbf{Y} = \overline{\mathbb{N}} \times \overline{\mathbb{N}}$ and

$$\begin{aligned} f(\infty, y) &= \{U \in \mathcal{O}(\mathcal{N}_1) : f_\infty \in U\}, \\ f(n, \infty) &= \emptyset, \\ f(n, p) &= \{U \in \mathcal{O}(\mathcal{N}_1) : f_{n,p} \notin U\}, \end{aligned}$$

where f_∞ is the null function and $f_{n,p} \in \mathcal{N}_1$ is the only function satisfying $f_{n,p}(n) = p + 1$.

One can take the following effective indexing of $\mathbf{Y} = \overline{\mathbb{N}} \times \overline{\mathbb{N}}$: if $\langle .,. \rangle : \mathbb{N}^2 \to \mathbb{N}$ is a computable bijection, and t_i is the halting time of Turing machine number i, then let $y_{\langle i,j \rangle} = (t_i, t_j)$. The set $A \subseteq \mathbb{N} \times \mathcal{O}(\mathcal{N}_1)$ becomes:

$$A = \{(\langle i, j \rangle, U) : f_\infty \in U \text{ and } M_i \text{ does not halt, or } M_i \text{ and } M_j \text{ halt and } f_{t_i,t_j} \notin U\}. \quad \blacktriangleleft$$

▶ **Corollary 5.4.** *If $\mathcal{N}_1$ embeds as a D_2-subset of $\mathbf{X}$, then*

$$[\mathrm{D}_2](\mathcal{O}(\mathbf{X})) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathbf{X})).$$

Proof. $\mathcal{O}(\mathcal{N}_1)$ is a computable retract of $\mathcal{O}(\mathbf{X})$, so the separation result (Theorem 5.3) about $\mathcal{O}(\mathcal{N}_1)$ extends to $\mathcal{O}(\mathbf{X})$ by Proposition 2.5. ◀

We consider the so-called sequential fan $\mathbf{S}(\omega) = \{0\} \cup \{\frac{1}{p}X^n : n, p \in \mathbb{N}\} \subseteq \mathbb{R}[X]$. It is Fréchet-Urysohn but has one point with no countable basis of neighborhoods. The space $\mathbb{N} \times \mathbf{S}(\omega)$ has infinitely many points with no countable basis of neighborhoods.

▶ **Theorem 5.5.** *Let $\mathbf{X} = \mathbb{N} \times \mathbf{S}(\omega)$. One has*

$$[\mathrm{D}_2](\mathbf{X}) \nsubseteq \mathrm{D}_2(\mathbf{X}),$$

and it is witnessed by an open set.

Proof. We follow the same scheme as in the preceding proof. Let

$$A = \{(n, P) : \text{if } M_n \text{ halts then } \deg(P) > t_n\}.$$

First, A is open because it is Σ^0_1 relative the halting set.

We show that $A \in [\mathrm{D}_2](\mathbf{X})$. We are given (n, P), with an upper bound d on $\deg(P)$. We run M_n for d steps. If M_n halts before d steps, then we know the value of t_n so we can test whether $\deg(P) > t_n$ (i.e. we start rejecting (n, P) until this test succeeds, in which case we accept (n, P)). If M_n does not halt before d steps, then we accept (n, P) and eventually reject it if M_n halts (indeed, in that case one has $\deg(P) \leq d \leq t_n$).

We show that $A \notin \mathrm{D}_2(\mathbf{X})$. Observe that for each n, $(n, 0)$ belongs to the closure of A: if M_n does not halt then $(n, 0) \in A$; if M_n halts then $(n, 0)$ is the limit when p grows of $(n, \frac{1}{p}X^{t_n+1}) \in A$.

Assume that $A = U \setminus V$ where U, V are open subsets of $\mathbf{X}$ and U is effectively open. For each n, as $(n, 0)$ belongs to the closure of A then one must have $(n, 0) \notin V$. Therefore, $(n, 0) \in A \iff (n, 0) \in U$. However, $(n, 0) \in A$ iff M_n does not halt, so it cannot be equivalent to $(n, 0) \in U$ which is a c.e. condition. We obtain a contradiction, so $A \notin \mathrm{D}_2(\mathbf{X})$. ◀

Proposition 2.6 immediately implies

▶ **Corollary 5.6.** *If $\mathbb{N} \times \mathbf{S}(\omega)$ computably embeds as a D_2-subset of $\mathbf{X}$, then*

$$[\mathrm{D}_2](\mathbf{X}) \nsubseteq \mathrm{D}_2(\mathbf{X})$$

with a witness in $\underset{\sim}{\mathbf{D}}_2(\mathbf{X})$.

One easily checks that the spaces $\mathbb{R}[X]$, $2^{\mathbb{N}^\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ are instances of this result:

- $\mathbb{N} \times \mathbf{S}(\omega) \subseteq \mathbb{R}[X]$ by identifying (n, P) with $\frac{1}{n} + XP$,
- $\mathbb{N} \times \mathbf{S}(\omega) \subseteq 2^{\mathbb{N}^\mathbb{N}}$ by identifying $(n, 0)$ with $\{f \in \mathcal{N} : f(0) = n\}$ and $(n, \frac{1}{p}X^q)$ with $\{f \in \mathcal{N} : f(0) = n \text{ and } f(q+1) \leq p\}$.

6 Spaces of open subsets of Polish spaces

We now focus on spaces of open subsets of Polish spaces, for which we can establish a rather precise picture of the relationship between symbolic and topological complexity, depending on the compactness properties of the space. We show how the behavior of symbolic complexity classes on $\mathcal{O}(X)$ is closely related to the compactness properties of X.

6.1 The 4 classes

The first observation is that when X is locally compact, for instance $X = \mathbb{R}$, $\mathcal{O}(X)$ is countably-based so it behaves very well in terms of descriptive complexity: symbolic and topological complexity coincide. We split the whole class of Polish spaces into 4 disjoint classes, ranging from the locally compact spaces to the non σ-compact spaces.

Let $X_{\mathrm{nk}} = \{x \in X : x \text{ has no compact neighborhood}\}$, which is a closed subset of X.

▶ **Definition 6.1.** *Let X be a Polish space.*

1. *$X \in$ Class I if $X_{\mathrm{nk}} = \emptyset$, i.e. X is locally compact,*
2. *$X \in$ Class II if $X_{\mathrm{nk}} \neq \emptyset$ is finite,*
3. *$X \in$ Class III if $X_{\mathrm{nk}} \neq \emptyset$ is infinite and X is σ-compact,*
4. *$X \in$ Class IV if X is not σ-compact.*

Observe that the union of Classes I, II, III is the class of σ-compact spaces.

▶ **Example 6.2.** Let us give one example for each class:

1. $\mathbb{R}$ belongs to Class I,
2. $\mathcal{N}_1 = \{f \in \mathcal{N} : f \text{ takes at most one positive value}\}$ belongs to Class II, with one element having no compact neighborhood, namely the zero function f_0,

3. $\mathbb{N} \times \mathcal{N}_1$ belongs to Class III, where the elements with no compact neighborhood are the pairs (n, f_0),
4. $\mathcal{N}$ belongs to Class IV.

Moreover, the three latter spaces are minimal in their respective classes, i.e. embed into every space of their classes.

▶ **Proposition 6.3.** *Let X be Polish.*

- $X \notin$ *Class I* $\iff$ *X contains a closed copy of $\mathcal{N}_1$,*
- $X \notin$ *Classes I or II* $\iff$ *X contains a $\underset{\sim}{\mathbf{D}}_2$ copy of $\mathbb{N} \times \mathcal{N}_1$,*
- $X \notin$ *Classes I, II or III* $\iff$ *X contains a closed copy of $\mathcal{N}$.*

Proof. The backwards implications are easy, because if C is a closed subset, or even a $\underset{\sim}{\mathbf{D}}_2$-subset of $\mathbf{X}$ and $x \in C$ has no compact neighborhood in the subspace C, then x has no compact neighborhood in $\mathbf{X}$.

Assume that $\mathbf{X}$ is not locally compact and let $x_0 \in \mathbf{X}_{\mathrm{nk}}$. We define a double-sequence $x_{i,n}$ by induction on i. Let B_0 be a basic neighborhood of x_0. As $\overline{B_0}$ is not compact, it contains a sequence $x_{0,n}$ with no converging subsequence. In particular, there exists a neighborhood B_1 of x_0 such that $\overline{B_1}$ does not contain any $x_{0,n}$. Again, $\overline{B_1}$ is not compact so it contains a sequence $x_{i,n}$ with no converging subsequence. We continue, making sure that the radius of B_i converges to 0. One easily checks that the set $\{x_0\} \cup \{x_{i,n} : i, n \in \mathbb{N}\}$ is closed and homeomorphic to $\mathcal{N}_1$, by sending x_0 to the zero function, and $x_{i,n}$ to the function f such that $f(i) = n$.

Assume that $\mathbf{X}_{\mathrm{nk}}$ is infinite. It contains a copy D of $\mathbb{N}$ with $D \in \underset{\sim}{\mathbf{D}}_2(\mathbf{X})$. Each point $x \in D$ is contained in a neighrbohood B_x such that $\overline{B_x} \cap \overline{B_y} = \emptyset$ for $x \neq y$. Around each point x of D and inside B_x we can build a closed copy of $\mathcal{N}_1$ as in the previous case. Their union is a copy of $\mathbb{N} \times \mathcal{N}_1$ and belongs to $\underset{\sim}{\mathbf{D}}_2(\mathbf{X})$.

The third statement is a particular case of Hurewicz theorem (Theorem 7.10 in [10]). ◀

6.2 Classification

We now relate the behavior of symbolic complexity on $\mathcal{O}(X)$ to the class of X. We first locate the symbolic complexity classes.

▶ **Theorem 6.4** (Classification – Positive results). *Let X be Polish.*

1. *If $X \in$ Class I, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(X))$ for all k,*
2. *If $X \in$ Class II, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(X))$ for $k \geq 3$,*
3. *If $X \in$ Class III, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+2}(\mathcal{O}(X))$ for $k \geq 2$.*

Informal proof. If $X \in$ Class I, i.e. if X is locally compact, then $\mathcal{O}(\mathbf{X})$ is countably-based [16], so symbolic and topological complexity coincide there (Theorem 2.2).

If $X \in$ Class II, then up to a finite set, X is countably-based, and we show that this finite set do not affect the complexity of sets for levels $k \geq 3$.

If $X \in$ Class III, then X is σ-compact, so its open sets are σ-compact as well. Therefore, for each open set B, the corresponding set $P_B = \{U \in \mathcal{O}(\mathbf{X}) : B \subseteq U\}$ belongs to $\underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{O}(X))$. The countable family $\mathcal{B}$ of finite unions basic open subsets of X induces a countable network $(P_B)_{B \in \mathcal{B}}$ of $\mathcal{O}(\mathbf{X})$ made of $\underset{\sim}{\mathbf{\Pi}}^0_2$-sets. Therefore, we can apply Proposition 2.3. ◀

We then identify gaps between symbolic and topological complexity.

▶ **Theorem 6.5** (Classification – Negative results). *Let X be Polish.*

1. *If $X \notin$ Class I, then $[\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(X))$ contains a $\underset{\sim}{\mathbf{\Delta}}^0_3$-complete* set,*
2. *If $X \notin$ Class II, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X))$ contains a $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$-complete*-set for $k \geq 2$,*
3. *If $X \notin$ Class III, then $[\underset{\sim}{\mathbf{\Sigma}}^0_2](\mathcal{O}(X))$ contains a non-Borel set.*

We observe that two phenomena are possible. For some spaces, the classes $[\underset{\sim}{\mathbf{\Sigma}}^0_k]$ and $\underset{\sim}{\mathbf{\Sigma}}^0_k$ differ for low values of k and then coincide after some rank (if X is in Class II, then they coincide for $k \geq 3$). For other spaces, the classes never coincide (if X is in Class III or IV).

It is open whether $\underset{\sim}{\mathbf{\Sigma}}^0_k(O(X)) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+1}(\mathcal{O}(X))$ when X belongs to Class III. A similar study should be done when X is not Polish.

Informal proof. The difference between symbolic and topological complexity is related to the fact that product spaces usually have two different natural topologies: the product topology (which is the structure obtained as the cartesian product in the category of topological spaces), and its sequentialization (obtained from the cartesian product in the category of QCB_0-spaces, or admissibly represented spaces). These two different topologies obviously induce different topological complexity classes, already at the first level $\underset{\sim}{\mathbf{\Sigma}}^0_1$.

For instance, on $\mathcal{N} \times \mathcal{O}(\mathcal{N})$, the set $\{(f, U) : f \in U\}$ is open but is not Borel for the product topology.

The proof shows how to exploit the difference between the product topology and its sequentialization on the space $\mathcal{N} \times \mathcal{O}(\mathbf{X})$ and turn it into a difference between symbolic and topological complexity on $\mathcal{O}(\mathbf{X})$.

When $\mathbf{X} = \mathbb{N} \times \mathcal{N}_1$, one has $\mathbf{X} \cong \mathbb{N} \times \mathbf{X}$ so $\mathcal{O}(\mathbf{X}) \cong \mathcal{O}(\mathbf{X})^{\mathbb{N}}$. This equality enables one to iterate: if $A_k \in [\Sigma^0_k](\mathcal{O}(\mathbf{X}))$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$-complete*, then the set

$$A_{k+1} = \{(U_i)_{i\in\mathbb{N}} \in \mathcal{O}(\mathbf{X})^{\mathbb{N}} : \exists i, U_i \notin A_k\}$$

belongs to $[\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}](\mathcal{O}(\mathbf{X})^{\mathbb{N}})$ and it $\underset{\sim}{\mathbf{\Sigma}}^0_{k+2}$-complete*. ◀

The fact that $\mathbf{X}$ is Polish is essential in the proofs. A particular property of Polish and quasi-Polish spaces that is used is the following.

▶ **Proposition 6.6.** *If $\mathbf{X}$ and $\mathbf{Y}$ are quasi-Polish, then the topologies on the admissiby represented spaces $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$ and $\mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{Y})$ are the product topologies.*

Proof. As represented spaces, one has $\mathcal{O}(\mathbf{X})^{\mathbb{N}} \cong \mathcal{O}(\mathbb{N} \times \mathbf{X})$ and $\mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{Y}) \cong \mathcal{O}(\mathbf{X} \sqcup \mathbf{Y})$. The topologies on the admissibly represented spaces $\mathcal{O}(\mathbb{N} \times \mathbf{X})$ and $\mathcal{O}(\mathbf{X} \sqcup \mathbf{Y})$ are the Scott topologies.

On the other hand, for any topological spaces X, Y, it is easy to see that the compact-open topology on $\mathcal{O}(\mathbb{N} \times X)$ and $\mathcal{O}(X \sqcup Y)$ is the product topology on $\mathcal{O}(X)^{\mathbb{N}}$ and $\mathcal{O}(X) \times \mathcal{O}(Y)$ respectively, where $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are endowed with the compact-open topology.

When $\mathbf{X}$ and $\mathbf{Y}$ are quasi-Polish, so are $\mathbb{N} \times \mathbf{X}$ and $\mathbf{X} \sqcup \mathbf{Y}$, so $\mathbf{X}$, $\mathbf{Y}$, $\mathbb{N} \times \mathbf{X}$ and $\mathbf{X} \sqcup \mathbf{Y}$ are consonant, i.e. the Scott topology and the compact-open topology coincide on their spaces of open sets [4]. As a result the topology on the represented spaces $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$ and $\mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{Y})$ is the product of the topologies on $\mathcal{O}(\mathbf{X})$ and $\mathcal{O}(\mathbf{Y})$. ◀

Open subsets of the Baire space

We now give the complete proof that the class $[\Sigma_2^0](\mathcal{O}(\mathcal{N}))$ contains a set that is not Borel (Theorem 6.5, item 3.).

Proof. We have seen that two represented spaces $\mathbf{X}$ and $\mathbf{Y}$ naturally induce a third represented space $\mathbf{X} \times \mathbf{Y}$. The topology induced by that representation is not in general the product topology, but its sequentialization.

A simple example is given by $\mathbf{X} = \mathcal{N}$ and $\mathbf{Y} = \mathcal{O}(\mathcal{N})$. The evaluation map $\mathcal{N} \times \mathcal{O}(\mathcal{N}) \to \mathbb{S}$ is continuous (and computable), however it is not continuous w.r.t. the product topology, because $\mathcal{N}$ is not locally compact (see [6] for more details on this topic). In other words the set $\{(f, O) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in O\}$ is not open for the product topology (but it is sequentially open, or open for the topology induced by the representation). It is even worse.

▶ **Proposition 6.7.** *$E = \{(f, O) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in O\}$ is not Borel for the product topology.*

Proof. We prove that for every Borel set A, there exists a dense G_δ-set $G \subseteq \mathcal{N}$ such that for every $f \in G$, $(f, \mathcal{N} \setminus \{f\}) \in A \iff (f, \mathcal{N}) \in A$. It implies the result as it is obviously false for the set E. To prove it, we show that the class of sets satisfying this condition contains the open sets in the product topology and is closed under taking complements and countable unions, which implies that this class contains the Borel sets.

First, consider a basic open set $A = [u] \times \mathcal{U}_K$ where u is a finite sequence of natural numbers, K is a compact subset of $\mathcal{N}$ and $\mathcal{U}_K = \{O \in \mathcal{O}(\mathcal{N}) : K \subseteq O\}$. Define $G = [u]^c \cup [u] \setminus K$, which is a dense open set. For $f \in [u]^c$, no (f, O) belongs to A. For $f \in [u] \setminus K$, both $(f, \mathcal{N} \setminus \{f\})$ and $(f, \mathcal{N})$ belong to A.

If A satisfies the condition with a dense G_δ-set G, then A^c satisfies the condition with the same G. If A_i satisfy the condition with dense G_δ-sets G_i then $\bigcup_i A_i$ satisfies the condition with $G = \bigcap_i G_i$. ◀

We now use the set E to build a set in $[\Sigma_2^0](\mathcal{O}(\mathcal{N}))$ which is not Borel. We show that $\mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N})$, which is the topological space endowed with the product topology, is a $[\Sigma_2^0]$-retract of $\mathcal{O}(\mathcal{N})$. We build:

- A continuous function $s : \mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N}) \to \mathcal{O}(\mathcal{N})$,
- A $[\Sigma_2^0]$-measurable function $r : \mathcal{O}(\mathcal{N}) \to \mathcal{N} \times \mathcal{O}(\mathcal{N})$,
- Such that $r \circ s = \text{id}$.

First, these ingredients enable us to derive the result. Indeed, let E be the set from Proposition 6.7 and $F := r^{-1}(E) \subseteq \mathcal{O}(\mathcal{N})$. As E is open in $\mathcal{N} \times \mathcal{O}(\mathcal{N})$, F is Σ_2^0. However F is not Borel, otherwise $E = s^{-1}(F)$ would be Borel in $\mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N})$.

Let us now build s and r. We identify $\mathcal{O}(\mathcal{N})$ with $\mathcal{O}(\mathcal{N}) \times \mathcal{O}(\mathcal{N})$ and use the fact that the topology on $\mathcal{O}(\mathcal{N}) \times \mathcal{O}(\mathcal{N})$ coincides with the product topology by Proposition 6.6.

▶ **Lemma 6.8.** *$\mathcal{N}$ is a $[\Sigma_2^0]$-retract of $\mathcal{O}(\mathbb{N})$: there exists $r : \mathcal{O}(\mathbb{N}) \to \mathcal{N}$ which is $[\Sigma_2^0]$-measurable, $s : \mathcal{N} \to \mathcal{O}(\mathbb{N})$ which is computable, such that $r \circ s = \text{id}_{\mathcal{N}}$.*

Proof. Let $\langle ., . \rangle : \mathbb{N}^2 \to \mathbb{N}$ be a computable bijection. Let $r(E) = f_E$ be defined by

$$f_E(i) = \begin{cases} \min\{j \in \mathbb{N} : \langle i, j \rangle \in E\} & \text{if that set is non-empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $s(f) = \{\langle i, f(i) \rangle : i \in \mathbb{N}, f(i) \geq 1\}$. One easily checks that r and s satisfy the required conditions. ◀

By Lemma 6.8, $\mathcal{N}$ is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathbb{N})$, which is a computable retract of $\mathcal{O}(\mathcal{N})$, so $\mathcal{N}$ is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathcal{N})$. It is witnessed by two functions $r_0 : \mathcal{O}(\mathcal{N}) \to \mathcal{N}$ and $s_0 : \mathcal{N} \to \mathcal{O}(\mathcal{N})$ such that $r_0 \circ s_0 = \mathrm{id}_{\mathcal{N}}$.

Let us simply pair s_0 and r_0 with the identity on $\mathcal{O}(\mathcal{N})$: let $s(f, O') = (s_0(f), O')$ and $r(O, O') = (r_0(O), O')$. ◀

In particular, that set is not a countable union of differences of open sets, as it should be on Polish or quasi-Polish spaces. More generally, it is not a countable boolean combination of open sets.

In order to overcome the mismatch between the hierarchy inherited from $\mathcal{N}$ via the representation and the class of Borel sets, one may attempt to change the definition of Borel sets. In [12] the Borel sets are redefined as the smallest class containing the open sets and the saturated compact sets, and closed under countable unions and complements. We observe here that this class is too large in the space $\mathcal{O}(\mathcal{N})$. First, if $U \subseteq \mathcal{N}$ is open then the set $\{V \in \mathcal{O}(\mathcal{N}) : U \subseteq V\}$ is compact and saturated in $\mathcal{O}(\mathcal{N})$. From this it is easy to see that the set built above is Borel in this weaker sense. However this notion of Borel sets is too loose, because compact saturated sets do not usually have a Borel pre-image. For instance, the singleton $\{\mathcal{N}\}$ is compact saturated but its pre-image under the representation is a $\boldsymbol{\Pi}^1_1$-complete set, hence is not Borel.

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