Two Variable Logic with Ultimately Periodic Counting

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Abstract

We consider the extension of FO^2 with quantifiers that state that the number of elements where a formula holds should belong to a given ultimately periodic set. We show that both satisfiability and finite satisfiability of the logic are decidable. We also show that the spectrum of any sentence is definable in Presburger arithmetic. In the process we present several refinements to the "biregular graph method". In this method, decidability issues concerning two-variable logics are reduced to questions about Presburger definability of integer vectors associated with partitioned graphs, where nodes in a partition satisfy certain constraints on their in- and out-degrees.

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1 Introduction

In the search for expressive logics with decidable satisfiability problem, two-variable logic, denoted here as FO^2, is one yardstick. This logic is expressive enough to subsume basic modal logic and many description logics, while satisfiability and finite satisfiability coincide, and both are decidable [23, 15, 9]. However, FO^2 lacks the ability to count. Two-variable logic with counting, C^2, is a decidable extension of FO^2 that adds *counting quantifiers*. In C^2 one can express, for example, $\exists^5 x\ P(x)$ and $\forall x \exists^{\geqslant 5} y\ E(x, y)$ which, respectively, mean that there are exactly 5 elements in unary relation P, and that every element in a graph has at least 5 adjacent edges. Satisfiability and finite satisfiability do not coincide for C^2, but both are decidable [10, 16]. In [16] the problems were shown to be NEXPTIME-complete under a unary encoding of numbers, and this was extended to binary encoding in [18]. However, the numerical capabilities of C^2 are quite limited. For example, one can not express that the number of outgoing edges of each element in the graph is even.

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A natural extension is to combine FO^2 with *Presburger arithmetic* where one is allowed to define collections of tuples of integers from addition and equality using boolean operators and quantifiers. The collections of k-tuples that one can define in this way are the *semi-linear sets*, and the collections of integers (when $k = 1$) definable are the *ultimately periodic sets.* Prior work has considered the addition of *Presburger quantification* to fragments of two-variable logic. For every definable set $\phi(x, y)$ and every ultimately periodic set S, one has a formula $\exists^S y\ \phi(x, y)$ that holds at x when the number of y such that $\phi(x, y)$ is in S. We let $\mathsf{FO}^2_{\mathrm{Pres}}$ denote the logic that adds this construct to FO^2.

On the one hand, the corresponding quantification over general k-tuples (allowing semi-linear rather than ultimately periodic sets) easily leads to undecidability [11, 3]. On the other hand, adding this quantification to modal logic has been shown to preserve decidability [1, 7]. Related *one-variable fragments* in which we have only a unary relational vocabulary and the main quantification is $\exists^S x\ \phi(x)$ are known to be decidable (see, e.g. [2]), and their decidability is the basis for a number of software tools focusing on integration of relational languages with Presburger arithmetic [14]. The decidability of full $\mathsf{FO}^2_{\mathrm{Pres}}$ is, to the best of our knowledge, still open [4]. There are a number of other extensions of C^2 that have been shown decidable; for example it has been shown that one can allow a distinguished equivalence relation [22] or a forest-structured relation [6, 5]. $\mathsf{FO}^2_{\mathrm{Pres}}$ is easily seen to be orthogonal to these other extensions.

In this paper we show that both satisfiability and finite satisfiability of $\mathsf{FO}^2_{\mathrm{Pres}}$ are decidable. Our result makes use of the *biregular graph method* introduced for analyzing C^2 in [13]. The method focuses on the problem of existence of graphs equipped with a partition of vertices based on constraints on the out- and in-degree. Such a partitioned graph can be characterized by the cardinalities of each partition component, and the key step in showing these decidability results is to prove that the set of tuples of integers representing valid sizes of partition components is definable by a formula in Presburger arithmetic. From this "graph constraint Presburger definability" result one can reduce satisfiability in the logic to satisfiability of a Presburger formula, and from there infer decidability using known results on Presburger arithmetic.

The approach is closely-related to the machinery developed by Pratt-Hartmann (the "star types" of [21]) for analyzing the decidability and complexity of C^2, its fragments [19], and its extensions [22, 5]. An advantage of the biregular graph approach is that it is transparent how to extract more information about the shape of witness structures. In particular we can infer that the *spectrum* of any formula is Presburger definable, where the spectrum of a formula ϕ is the set of cardinalities of finite models of ϕ. It is also interesting to note that a more restricted version of our biregular graph method is used to prove the decidability of FO^2 extended with two equivalence relations [12].

Characterising the spectrum for general first order formulas is quite a difficult problem, with ties to major open questions in complexity theory [8]. This work can be seen as a demonstration of the power of the biregular graph method to get new decidability results. We make heavy use of both techniques and results in [13], adapting them to the richer logic. We also require additional inductive arguments to handle the interaction of ordinary counting quantifiers and modulo counting quantification.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

Linear and ultimately periodic sets. A set of the form $\{a + ip \mid i \in \mathbb{N}\}$, for some $a, p \in \mathbb{N}$ is a *linear set*. We will denote such a set by a^{+p}, where a and p are called the *offset* and *period* of the set, respectively. Note that, by definition, $a^{+0} = \{a\}$, which is a linear set. For convenience, we define $\emptyset$ and $\{\infty\}$ (which may be written as ∞^{+p}) to also be linear sets.

An *ultimately periodic set* (*u.p.s.*) S is a finite union of linear sets. Usually we write a u.p.s. $\{c_1\} \cup \cdots \cup \{c_m\} \cup a_1^{+p_1} \cup \cdots \cup a_n^{+p_n}$ as just $\{c_1, \ldots, c_m, a_1^{+p_1}, \ldots, a_n^{+p_n}\}$, and abusing notation, we write $a^{+p} \in S$ for a u.p.s. S if $a + ip \in S$ for every $i \in \mathbb{N}$.

Two-variable logic with ultimately periodic counting quantifiers. An *atomic formula* is either an atom $R(\vec{u})$, where R is a predicate, and $\vec{u}$ is a tuple of variables of appropriate size, or an equality $u = u'$, with u and u' variables, or one of the formulas $\top$ and $\perp$ denoting the True and False values. The logic $\mathsf{FO}^2_{\mathrm{Pres}}$ is a class of first-order formulas using only variables x and y, built up from atomic formulas and equalities using the usual boolean connectives and also *ultimately periodic counting quantification*, which is of the form $\exists^S x\ \phi$ where S is a u.p.s. One special case is where S is a singleton $\{a\}$ with $a \in \mathbb{N}_\infty$, which we write $\exists^a x\ \phi$; in case of $a \in \mathbb{N}$, these are *counting quantifiers*. The semantics of $\mathsf{FO}^2_{\mathrm{Pres}}$ is defined as usual except that, for every $a \in \mathbb{N}$, $\exists^a x\ \phi$ holds when there are *exactly* a number of x's such that ϕ holds, $\exists^\infty\ \phi$ holds when there are infinitely many x's such that ϕ holds, and $\exists^S x\ \phi$ holds when there is some $a \in S$ such that $\exists^a x\ \phi$ holds.

Note that when S is $\{\infty\} \cup 0^{+1} = \mathbb{N}_\infty$, $\exists^S x\ \phi$ is equivalent to $\top$. When S is 0^{+1}, $\exists^S x\ \phi$ semantically means that there are finitely many x such that ϕ holds. We define $\exists^\emptyset x\ \phi$ to be $\perp$ for any formula ϕ. We also note that $\exists^0 x\ \phi$ is equivalent to $\forall x\ \neg\phi$, and $\neg\exists^S x\ \phi$ is equivalent to $\exists^{\mathbb{N}_\infty - S} x\ \phi$.

For example, we can state in $\mathsf{FO}^2_{\mathrm{Pres}}$ that every node in a graph has even degree (i.e., the graph is Eulerian). Clearly $\mathsf{FO}^2_{\mathrm{Pres}}$ extends C^2, the fragment of the logic where only counting quantifiers are used, and FO^2, the fragment where only the classical quantifier $\exists x$ is allowed.

Presburger arithmetic. An *existential Presburger formula* is a formula of the form $\exists x_1 \ldots x_k\ \phi$, where ϕ is a quantifier-free formula over the signature including constants $0, 1$, a binary function symbol $+$, and a binary relation $\leqslant$. Such a formula is a *sentence* if it has no free variables. The notion of a sentence holding in a structure interpreting the function, relations, and constants is defined in the usual way. The structure $\mathcal{N} = (\mathbb{N}, +, \leqslant, 0, 1)$, is defined by interpreting $+, \leqslant, 0, 1$ in the standard way, while the structure $\mathcal{N}_\infty = (\mathbb{N}_\infty, +, \leqslant, 0, 1)$ is the same except that $a + \infty = \infty$ and $a \leqslant \infty$ for each $a \in \mathbb{N}_\infty$.

It is known that the satisfiability of existential Presburger sentences over $\mathcal{N}$ is decidable and belongs to NP [17]. Further, the satisfiability problem for $\mathcal{N}_\infty$ can easily be reduced to that for $\mathcal{N}$. Indeed, we can first guess which variables are mapped to ∞ and then which atoms should be true, then check whether each guessed atomic truth value is consistent with other guesses and determine additional variables which must be infinite based on this choice, and finally restrict to atoms that do not involve variables guessed to be infinite, and check that the conjunction is satisfiable by standard integers.

▶ **Theorem 1.** *The satisfiability problem for existential Presburger sentences over both $\mathcal{N}$ and $\mathcal{N}_\infty$ are both in NP.*

3 Main result

In this section we prove the decidability of $\mathsf{FO}^2_{\mathrm{Pres}}$ satisfiability. Our decision procedure is based on the key notion of regular graphs. Note that whenever we talk about graphs or digraphs (i.e., directed graphs), by default we allow both finite and infinite sets of vertices and edges.

3.1 Regular graphs

In the following we fix an integer $p \geqslant 0$. Let $\mathbb{N}_{\infty,+p}$ denote the set whose elements are either a or a^{+p}, where $a \in \mathbb{N}_\infty$. For integers $t, m \geqslant 1$, let $\mathbb{N}_{\infty,+p}^{t\times m}$ denote the set of matrices with t rows and m columns where each entry is an element from $\mathbb{N}_{\infty,+p}$.

A *t-color bipartite (undirected) graph* is $G = (U, V, E_1, \ldots, E_t)$, where U and V are sets of vertices and $E_1, \ldots, E_t$ are pairwise disjoint sets of edges between U and V. Edges in E_i are called *E_i-edges.* We will write an edge in a bipartite graph as $(u, v) \in U \times V$. For a vertex $u \in U \cup V$, the *E_i-degree* of u is the number of E_i-edges adjacent to u. The degree of u is the sum of the E_i-degrees for $i = 1 \ldots t$. We say that G is *complete*, if $U \times V = \bigcup_{i=1}^t E_i$.

For two matrices $A \in \mathbb{N}_{\infty,+p}^{t\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{t\times n}$, the graph G is a *$A|B$-biregular* graph, if there exist partitions $U_1, \ldots, U_m$ of U and $V_1, \ldots, V_n$ of V such that for every $1 \leqslant i \leqslant t$, for every $1 \leqslant k \leqslant m$, for every $1 \leqslant l \leqslant n$, the E_i-degree of every vertex in U_k is $A_{i,k}$ and the E_i degree of every vertex in V_l is $B_{i,l}$.[1] For each such partition, we say that G *has size* $\bar{M}|\bar{N}$, where $\bar{M} = (|U_1|, \ldots, |U_m|)$ and $\bar{N} = (|V_1|, \ldots, |V_n|)$. The partition $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ is called a *witness partition.* We should remark that some U_i and V_i are allowed to be empty.

The above definition can be easily adapted for the case of directed graphs that are not necessarily bipartite. A t-color *directed graph* (or *digraph*) is $G = (V, E_1, \ldots, E_t)$, where $E_1, \ldots, E_t$ are pairwise disjoint set of directed edges on a set of vertices V. As before, edges in E_i are called E_i-edges. The E_i-indegree and -outdegree of a vertex u, is defined as the number of incoming and outgoing E_i-edges incident to u.

In a t-color digraph G we will assume that (i) there are no self-loops – that is, (v, v) is not an E_i-edge, for every vertex $v \in V$ and every E_i, and (ii) if (u, v) is an E_i-edge, then its inverse (v, u) is not an E_j-edge for any E_j. This will suffice for the digraphs that arise in our decision procedure. We say that a digraph G is *complete*, if for every $u, v \in V$ and $u \neq v$, either (u, v) or (v, u) is an E_i-edge, for some E_i.

We say that G is a *$A|B$-regular* digraph, where $A, B \in \mathbb{N}_{\infty,+p}^{t\times m}$, if there exists a partition $V_1, \ldots, V_m$ of V such that for every $1 \leqslant i \leqslant t$, for every $1 \leqslant k \leqslant m$, the E_i-indegree and -outdegree of every vertex in V_k is $A_{i,k}$ and $B_{i,k}$, respectively. We say that G has size $(|V_1|, \ldots, |V_m|)$, and call $V_1, \ldots, V_m$ a witness partition.

Lemma 2 below will be the main technical tool for our decidability result. Let $\bar{x}$ and $\bar{y}$ be vectors of variables of length m and n, respectively.

▶ **Lemma 2.** *For every $A \in \mathbb{N}_{\infty,+p}^{t\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{t\times n}$, there exists (effectively computable) existential Presburger formula $\textsf{c-bireg}_{A|B}(\bar{x}, \bar{y})$ such that for every $(\bar{M}, \bar{N}) \in \mathbb{N}_\infty^m \times \mathbb{N}_\infty^n$, the following holds: there is complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $\textsf{c-bireg}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$.*

Lemma 3 below is the analog for digraphs.

▶ **Lemma 3.** *For every $A \in \mathbb{N}_{\infty,+p}^{t\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{t\times m}$, there exists (effectively computable) existential Presburger formula $\textsf{c-reg}_{A|B}(\bar{x})$ such that for every $\bar{M} \in \mathbb{N}_\infty^m$, the following holds. There is complete $A|B$-regular digraph with size $\bar{M}$ if and only if $\textsf{c-reg}_{A|B}(\bar{M})$ holds in $\mathcal{N}_\infty$.*

[1] By abuse of notation, when we say that an integer z equals a^{+p}, we mean that $z \in a^{+p}$. Thus, when writing $A_{i,k} = a^{+p}$, we mean that the degree of the vertex is an element in a^{+p}.

Lemmas 2 and 3 can be easily readjusted when we are interested only in finite sizes, i.e., $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$, by requiring the formulas to hold in $\mathcal{N}$, instead of $\mathcal{N}_\infty$. Alternatively, we can also state inside the formulas that none of the variables in $\bar{x}$ and $\bar{y}$ are equal to ∞.

The proofs of these two lemmas are discussed in Section 4.

3.2 Decision procedure

Theorem 4 below is the main result in this paper.

▶ **Theorem 4.** *For every FO^2_{Pres} sentence ϕ, there is an (effectively computable) existential Presburger formula PRES_ϕ such that (i) ϕ has a model iff PRES_ϕ holds in $\mathcal{N}_\infty$ and (ii) ϕ has a finite model iff PRES_ϕ holds in $\mathcal{N}$.*

From the decision procedure for existential Presburger formulas (Theorem 1) mentioned in Section 2, we immediately will obtain the following corollary.

▶ **Corollary 5.** *Both satisfiability and finite satisfiability for FO^2_{Pres} are decidable.*

We will sketch how Theorem 4 is proven, making use of Lemmas 2 and 3. We start by observing that satisfiability (and spectrum analysis) for an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence can be converted effectively into the same questions for a sentence in a variant of Scott normal form:

$$\phi := \forall x \forall y\ \alpha(x,y)\ \wedge\ \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y, \tag{1}$$

where $\alpha(x,y)$ is a quantifier free formula, each $\beta_i(x,y)$ is an atomic formula and each S_i is an u.p.s. The proof, which is fairly standard, will appear in the full version of this paper. By taking the least common multiple, we may assume that all the (non-zero) periods in all S_i are the same.

We recall some standard terminology. A 1-*type* is a maximally consistent set of atomic and negated atomic unary formulas using only variable x. A 1-type can be identified with the quantifier-free formula that is the conjunction of its constituent formulas. Thus, we say that an element a in a structure $\mathcal{A}$ has 1-type π, if π holds on the element a. We denote by A_π the set of elements in $\mathcal{A}$ with 1-type π. Clearly the domain A of a structure $\mathcal{A}$ is partitioned into the sets A_π. Similarly, a 2-*type* is a maximally consistent set of atomic and negated atomic binary formulas using only variables x, y, containing the predicate $x \neq y$. The notion of a pair of elements (a, b) in a structure $\mathcal{A}$ having 2-type E is defined as with 1-types. We denote by $\Pi = \{\pi_1, \pi_2, \dots, \pi_n\}$ and $\mathcal{E} = \{E_1, \dots, E_t, \overleftarrow{E_1}, \dots, \overleftarrow{E_t}\}$ the sets of all 1-types and 2-types, respectively, where $\overleftarrow{E_i}(x,y) = E_i(y,x)$ for each $1 \leqslant i \leqslant t$ – that is, each $\overleftarrow{E_i}$ is the reversal of E_i.

Let $g : \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ be a function. We will use such a function g to describe the "behavior" of the elements in the following sense. Let $\mathcal{A}$ be a structure. We say that an element $a \in A$ *behaves according to* g, if for every $E \in \mathcal{E}$ and for every $\pi \in \Pi$, the number of elements $b \in A_\pi$ such that the 2-type of (a, b) is E belongs to $g(E, \pi)$. We denote by $A_{\pi,g}$ the set of all elements in A_π that behave according to g. The restriction of g on 1-type π is the function $g_\pi : \mathcal{E} \to \mathbb{N}_{\infty,+p}$, where $g_\pi(E) = g(E, \pi)$. We call the function g_π the *behavior* (function) towards 1-type π.

We are, of course, only interested in functions g that are consistent with the sentence ϕ in (1), and we formalize this as follows:

- A 1-type $\pi \in \Pi$ and a function $g : \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ are *incompatible* (w.r.t. $\forall x \forall y\ \alpha(x,y)$), if there is $E \in \mathcal{E}$ and $\pi' \in \Pi$ such that $\pi(x) \wedge E(x,y) \wedge \pi'(y) \models \neg\alpha(x,y)$ and $g(E, \pi') \neq 0$.

- A function $g : \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ is a *good* function (w.r.t. $\bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y$), if for every $\pi \in \Pi$ and for every i the following holds:[2]

$$\sum_{E \models \beta_i(x,y)} \sum_{\pi \in \Pi} g(E,\pi) = a \qquad\qquad \text{for some } a \in S_i.$$

If $\mathcal{A} \models \phi$ then $A_{(\pi,g)} = \emptyset$, whenever π and g are incompatible, and in addition every element in A behaves only according to some good function.

The main idea is to construct the sentence PRES_ϕ that "counts" the cardinality $|A_{(\pi,g)}|$ in every structure $\mathcal{A} \models \phi$, for every π and g. Toward this end, let $\mathcal{G} = \{g_1, g_2, \ldots, g_m\}$ enumerate all good functions. Note that $\mathcal{G}$ can be computed effectively from the sentence ϕ, since it suffices to consider functions $g : \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ with codomain $\{0, \ldots, a, 0^{+p}, \ldots, a^{+p}, \infty\}$, where a is the maximal offset of the (non-∞) elements in $\bigcup_{i=1}^{k} S_i$.

The sentence PRES_ϕ will be of the form

$$\mathsf{PRES}_\phi := \exists \bar{X}\ \mathsf{consistent}_1(\bar{X})\ \wedge\ \mathsf{consistent}_2(\bar{X}) \tag{2}$$

where $\bar{X}$ is a vector of variables $(X_{(\pi_1,g_1)}, X_{(\pi_1,g_2)}, \ldots, X_{(\pi_n,g_m)})$. Intuitively, each $X_{(\pi_i,g_j)}$ represents $|A_{\pi_i,g_j}|$. By the formulas $\mathsf{consistent}_1(\bar{X})$ and $\mathsf{consistent}_2(\bar{X})$, we capture the consistency of the integers $\bar{X}$ with the formulas $\forall x \forall y\ \alpha(x,y)$ and $\bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y$, respectively.

We start by defining the formula $\mathsf{consistent}_1(\bar{X})$. Letting H be the set of all pairs (π, g) where π and g are incompatible, the formula $\mathsf{consistent}_1(\bar{X})$ can be defined as

$$\mathsf{consistent}_1(\bar{X}) := \bigwedge_{(\pi,g) \in H} X_{(\pi,g)} = 0 \tag{3}$$

Towards defining the formula $\mathsf{consistent}_2(\bar{X})$, we introduce some notations. For $\pi \in \Pi$, define the matrices $M_\pi, \overleftarrow{M}_\pi \in \mathbb{N}_{\infty,+p}^{t \times m}$ as follows:

$$M_\pi := \begin{pmatrix} g_1(E_1,\pi) & \cdots & g_m(E_1,\pi) \\ \vdots & \ddots & \vdots \\ g_1(E_t,\pi) & \cdots & g_m(E_t,\pi) \end{pmatrix} \quad \text{and} \quad \overleftarrow{M}_\pi := \begin{pmatrix} g_1(\overleftarrow{E_1},\pi) & \cdots & g_m(\overleftarrow{E_1},\pi) \\ \vdots & \ddots & \vdots \\ g_1(\overleftarrow{E_t},\pi) & \cdots & g_m(\overleftarrow{E_t},\pi) \end{pmatrix}$$

The idea is that M_π captures all possible behavior towards 1-type π, where each column j represents the behavior of g_j towards π. Note that for a structure $\mathcal{A}$ and 1-type π, the restriction of $\mathcal{A}$ on the set A_π can be viewed as a t-color digraph $G = (V, E_1, \ldots, E_t)$. It is sufficient to consider only the 2-types $E_1, \ldots, E_t$, because each E_i determines its reversal $\overleftarrow{E_i}$. Moreover, an element a has an incoming E_i-edge if and only if it has an outgoing $\overleftarrow{E_i}$-edge. Thus, if $\mathcal{A} \models \phi$, the graph G is a complete $M_\pi|\overleftarrow{M}_\pi$-regular digraph.

Now, we explain how to capture the behavior between elements with distinct 1-types. Define matrices $L_\pi, \overleftarrow{L}_\pi \in \mathbb{N}_{\infty,+p}^{2t \times m}$ as follows:

$$L_\pi := \begin{pmatrix} M_\pi \\ \overleftarrow{M}_\pi \end{pmatrix} \quad \text{and} \quad \overleftarrow{L}_\pi := \begin{pmatrix} \overleftarrow{M}_\pi \\ M_\pi \end{pmatrix}$$

That is, in L_π the first t rows come from M_π with the next t rows from $\overleftarrow{M}_\pi$. On the other hand, in $\overleftarrow{L}_\pi$ the first t rows come from $\overleftarrow{M}_\pi$, followed by the t rows from M_π.

[2] Here the operation $+$ on $\mathbb{N}_{\infty,+p}$ is defined to be commutative operation where $a + \infty = a^{+p} + \infty = \infty$ and $a^{+p} + b = a^{+p} + b^{+p} = (a+b)^{+p}$. On integers from $\mathbb{N}$, it is the standard addition operation.

The idea is that for a structure $\mathcal{A}$, the 2-types that are realized between A_π and $A_{\pi'}$ can be viewed as a $2t$-color bipartite graph $G = (A_\pi, A_{\pi'}, E_1, \ldots, E_t, \overleftarrow{E_1}, \ldots, \overleftarrow{E_t})$, where the direction of the edges are ignored. Moreover, a pair (a, b) has 2-type E if and only if (b, a) has 2-type $\overleftarrow{E}$, Thus, if $\mathcal{A} \models \phi$, the graph G is a complete $L_{\pi'} | \overleftarrow{L}_\pi$-biregular graph.

Now we are ready to define the formula $\mathsf{consistent}_2(\bar{X})$. We enumerate all the 1-types $\pi_1, \ldots, \pi_n$ and define $\mathsf{consistent}_2$ as follows:

$$\mathsf{consistent}_2(\bar{X}) := \bigwedge_{1 \leqslant i \leqslant n} \mathsf{c\text{-}reg}_{M_{\pi_i} | \overleftarrow{M}_{\pi_i}}(\bar{X}_{\pi_i}) \ \wedge \bigwedge_{1 \leqslant i < j \leqslant n} \mathsf{c\text{-}bireg}_{L_{\pi_j} | \overleftarrow{L}_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j}). \tag{4}$$

The formula $\mathsf{consistent}_1(\bar{X})$ is Presburger definable by inspection, while $\mathsf{consistent}_2(\bar{X})$ is Presburger definable using Lemmas 2 and 3. The correctness comes directly from the following lemma.

▶ **Lemma 6.** *For every structure $\mathcal{A} \models \phi$, $\mathsf{consistent}_1(\bar{N}) \wedge \mathsf{consistent}_2(\bar{N})$ holds, where $\bar{N} = (|A_{\pi_1,g_1}|, \ldots, |A_{\pi_n,g_m}|)$. Conversely, for every $\bar{N}$ such that $\mathsf{consistent}_1(\bar{N}) \wedge \mathsf{consistent}_2(\bar{N})$ holds, there is $\mathcal{A} \models \phi$ such that $\bar{N} = (|A_{\pi_1,g_1}|, \ldots, |A_{\pi_n,g_m}|)$.*

Proof. Let ϕ be in Scott normal form as in (1). As before, $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ denote the set of all 1-types and $\mathcal{E} = \{E_1, \ldots, E_t, \overleftarrow{E_1}, \ldots, \overleftarrow{E_t}\}$ the set of all 2-types, where $\overleftarrow{E_i}(x, y) = E_i(y, x)$ for each $1 \leqslant i \leqslant t$. Recall that each 2-type E contains the predicate $x \neq y$ and that $\mathcal{G} = \{g_1, \ldots, g_m\}$ is the set of all good functions.

Note that for $\pi, \pi' \in \Pi$ and $E \in \mathcal{E}$, the conjunction $\pi(x) \wedge E(x, y) \wedge \pi'(y)$ corresponds to a boolean assignment of the atomic predicates in $\alpha(x, y)$. Thus, either $\pi(x) \wedge E(x, y) \wedge \pi'(y) \models \alpha(x, y)$ or $\pi(x) \wedge E(x, y) \wedge \pi'(y) \models \neg\alpha(x, y)$. Similarly, $\pi(x) \wedge x = y \models \alpha(x, y)$ or $\pi(x) \wedge x = y \models \neg\alpha(x, y)$.

We first prove the first statement in the lemma. Let $\mathcal{A} \models \phi$. Partition A into $A_{\pi,g}$'s. We will show that $\mathsf{consistent}_1(\bar{X}) \wedge \mathsf{consistent}_2(\bar{X})$ holds when each $X_{\pi,g}$ is assigned with the value $|A_{\pi,g}|$.

Since $\mathcal{A} \models \forall x \forall y\ \alpha(x, y)$, by definition $A_{\pi,g} = \emptyset$, whenever π and g are incompatible. Thus, $\mathsf{consistent}_1(\bar{X})$ holds.

Next, we will show that $\mathsf{consistent}_2(\bar{X})$ holds. Let $\pi \in \Pi$. By definition of A_π, A_π is a complete $M_\pi | \overleftarrow{M}_\pi$-regular digraph $G = (V, E_1, \ldots, E_t)$, with size $(|A_{\pi,g_1}|, \ldots, |A_{\pi,g_m}|)$. Thus, by Lemma 3, $\mathsf{c\text{-}reg}_{M_\pi | \overleftarrow{M}_\pi}(\bar{X}_\pi)$ holds.

For $\pi_i, \pi_j \in \Pi$, where $i < j$, the structure $\mathcal{A}$ restricted to A_{π_i} and A_{π_j} can be viewed as a complete $L_{\pi_j} | \overleftarrow{L}_{\pi_i}$-biregular graph $G = (U, V, E_1, \ldots, E_t, \overleftarrow{E_1}, \ldots, \overleftarrow{E_t})$, where $U = A_{\pi_i}$ and $V = A_{\pi_j}$, and for each $1 \leqslant i \leqslant t$, we have the interpretation denoted (by a slight abuse of notation) as E_i consist of all pairs $(a, b) \in A_{\pi_i} \times A_{\pi_j}$ whose 2-type is E_i, and similarly for $\overleftarrow{E_i}$. By Lemma 2, $\mathsf{c\text{-}bireg}_{L_{\pi_j} | \overleftarrow{L}_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j})$ holds.

Now we prove the second statement. Suppose PRES_ϕ holds. By definition, there exists an assignment to the variables in $\bar{X}$ such that $\mathsf{consistent}_1(\bar{X}) \wedge \mathsf{consistent}_2(\bar{X})$ holds. Abusing notation as we often do in this work, we denote the value assigned to each $X_{\pi,g}$ by the variable $X_{\pi,g}$ itself.

For each (π, g), we have a set $V_{\pi,g}$ with cardinality $X_{\pi,g}$. We denote by $V_\pi = \bigcup_g V_{\pi,g}$. We construct a structure $\mathcal{A}$ that satisfies ϕ as follows.

- The domain is $A = \bigcup_{\pi,g} V_{\pi,g}$.
- For each $\pi \in \Pi$, for each $a \in V_\pi$, the unary atomic formulas on a are defined such that the 1-type of a becomes π.

- For each $\pi \in \Pi$, the binary predicates on $(u,v) \in V_\pi \times V_\pi$ are defined as follows. Since $\mathsf{c\text{-}reg}_{M_\pi|\overleftarrow{M}_\pi}(\bar{X}_\pi)$ holds, there is a complete $M_\pi|\overleftarrow{M}_\pi$-regular digraph $G = (V_\pi, E_1, \ldots, E_t)$ with size $\bar{X}_\pi$. The edges $E_1, \ldots, E_t$ define precisely the 2-types among elements in V_π.
- For each π_i, π_j, where $i < j$, the binary predicates on $(u,v) \in V_{\pi_i} \times V_{\pi_j}$ are defined as follows. Since $\mathsf{c\text{-}bireg}_{L_{\pi_j}|\overleftarrow{L}_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j})$ holds, there is a $L_{\pi'}|\overleftarrow{L}_\pi$-biregular graph $G = (V_{\pi_i}, V_{\pi_j}, E_1, \ldots, E_t, \overleftarrow{E}_1, \ldots, \overleftarrow{E}_t)$ with size $\bar{X}_\pi|\bar{X}_{\pi'}$. The edges $E_1, \ldots, E_t, \overleftarrow{E}_1, \ldots, \overleftarrow{E}_t$ define precisely the 2-types on $(u,v) \in V_{\pi_i} \times V_{\pi_j}$.

We first show that $\mathcal{A} \models \forall x \forall y\ \alpha(x,y)$. Indeed, suppose there exist $u, v \in A$ such that $\pi(u) \wedge E_(u,v) \wedge \pi'(v) \not\models \alpha(u,v)$. By definition, there is g such that $u \in V_{\pi,g}$ and $g(E,\pi') \neq 0$. Thus, $V_{\pi,g} \neq \emptyset$. This also means that π is incompatible with g, which implies that $X_{\pi,g} = 0$ by $\mathsf{consistent}_1(\bar{X})$, thus, contradicts the assumption that $V_{\pi,g} \neq \emptyset$.

Next, we show that $\mathcal{A} \models \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y$. Note that $\mathcal{G} = \{g_1, \ldots, g_m\}$ consists of only good functions. Thus, for every $g \in \mathcal{G}$, for every β_i, the sum $\sum_\pi \sum_{\beta_i(x,y) \in E}\ g(E,\pi)$ is an element in S_i. ◀

4 Proof ideas for Lemmas 2 and 3

We now discuss the proof of the main biregular graph lemmas. Due to space constraints, we deal only with the 1-color case, which gives the flavor of the arguments. The general case, which is much more involved, is deferred to the full version of this paper.

This section is organized as follows. In Subsection 4.1 we will focus on a relaxation of Lemma 2 where the requirement being complete is dropped. This will then be used to prove the complete case in Subsection 4.2. Finally, in Subsection 4.3 we present a brief explanation on how to modify the proof for the biregular graphs to the one for regular digraphs.

4.1 The case of incomplete 1-color biregular graphs

This subsection is devoted to the proof of the following lemma.

▶ **Lemma 7.** *For every $A \in \mathbb{N}_{\infty,+p}^{1\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{1\times n}$, there exists (effectively computable) existential Presburger formula $\mathsf{bireg}_{A|B}(\bar{x}, \bar{y})$ such that for every $(\bar{M}, \bar{N}) \in \mathbb{N}_\infty^m \times \mathbb{N}_\infty^n$ the following holds: there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $\mathsf{bireg}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$.*

The desired formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x}, \bar{y})$ for complete biregular graphs will be defined using the formula $\mathsf{bireg}_{A|B}(\bar{x}, \bar{y})$.

We will use the following notations. The term vectors always refers to row vectors, and we usually use $\bar{a}, \bar{b}, \ldots$ (possibly indexed) to denote them. We write $(\bar{a}, \bar{b})$ to denote the vector $\bar{a}$ concatenated with $\bar{b}$. Obviously 1-row matrices can be viewed as row vectors. For $\bar{a} = (a_1, \ldots, a_k) \in \mathbb{N}_\infty^k$, we write $\bar{a}^{+p}$ to denote the vector $(a_1^{+p}, \ldots, a_k^{+p})$.

Matrix entries of the form a^{+p} are called *periodic* entries. Otherwise, they are called *fixed* entries. By grouping the entries according to whether they are fixed/periodic, we write a 1-row matrix M as $(\bar{a}, \bar{b}^{+p})$, where $\bar{a}$ and $\bar{b}^{+p}$ correspond to the fixed and periodic entries in M. Matrices that contain only fixed (or, periodic) entries are written as $\bar{a}$ (or, $\bar{a}^{+p}$).

To specify $A|B$-biregular graphs, we write $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graphs, where $A = (\bar{a}, \bar{b}^{+p})$ and $B = (\bar{c}, \bar{d}^{+p})$. Similarly, when, say, A contains only fixed entries, it is written as $\bar{a}|(\bar{c}, \bar{d}^{+p})$-biregular. The size of $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graph is written as $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$, where the lengths of $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ are the same as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively. The other cases, when some of $\bar{a}, \bar{b}^{+p}, \bar{c}, \bar{d}^{+p}$ are omitted, are treated in similar manner.

As before, we will write $\bar{x}, \bar{y}$ (possibly indexed) to denote a vector of variables. We write $\bar{1}$ to denote the vector with all components being 1. We use $\cdot$ to denote the standard dot product between two vectors. To avoid being repetitive, when dot products are performed, it is implicit that the vector lengths are the same. In particular, $\bar{x} \cdot \bar{1}$ is the sum of all the components in $\bar{x}$.

We now outline the proof of Lemma 7, focusing only on the case where there is no ∞ degree in the matrices. The case where such a degree exists is similar but simpler. Without loss of generality, we can also assume that none of the fixed entries are zero. For vectors $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ with the same length as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively, we say that $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$ *is big enough for* $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$, if the following holds:

(a) $\bar{M}_0 \cdot \bar{1} + \bar{M}_1 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geqslant 2\delta_{\max}^2 + 3$,

(b) $\bar{M}_1 \cdot \bar{1} \geqslant \delta_{\max}^2 + 1$,

(c) $\bar{N}_1 \cdot \bar{1} \geqslant \delta_{\max}^2 + 1$.

Here $\delta_{\max}$ is $\max(p, \bar{a}, \bar{b}, \bar{c}, \bar{d})$ – that is, the maximal element among p and the components in $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. When $\bar{b}^{+p}$ or $\bar{d}^{+p}$ are missing, the same notion can be defined by dropping condition (b) or (c), respectively. For example, we say that $\bar{M}|\bar{N}$ is big enough for $\bar{a}|\bar{b}$, if $\bar{M} \cdot \bar{1} + \bar{N} \cdot \bar{1} \geqslant 2\delta_{\max}^2 + 3$, where $\delta_{\max} = \max(\bar{a}, \bar{b})$. Similarly, $(\bar{M}_0, \bar{M}_1)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}^{+p})|\bar{c}$, if $\bar{M}_0 \cdot \bar{1} + \bar{M}_1 \cdot \bar{1} + \bar{N} \cdot \bar{1} \geqslant 2\delta_{\max}^2 + 3$, and $\bar{M}_1 \cdot \bar{1} \geqslant \delta_{\max}^2 + 1$, where $\delta_{\max} = \max(p, \bar{a}, \bar{b}, \bar{c})$.

The proof idea is as follows. We first construct a formula that deals with big enough sizes. Then, we construct a formula for each of the cases when one of the conditions (a), (b) or (c) is violated. The interesting case will be when condition (b) is violated. This means that the number of vertices with degrees from $\bar{b}^{+p}$ is fixed, and they can be "encoded" inside the Presburger formula.

We start with the big enough case. When there are only fixed entries, we will use the following lemma.

▶ **Lemma 8.** *For $\bar{M}|\bar{N}$ big enough for $\bar{a}|\bar{b}$, there is a $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $\bar{M} \cdot \bar{a} = \bar{N} \cdot \bar{b}$.*

Proof. Note that if we have a biregular graph with the desired outdegrees on the left, then the total number of edges must be $\bar{M} \cdot \bar{a}$, and similarly the total number of edges considering the requirement for vertices on the right, we see that the total number of edges must be $\bar{N} \cdot \bar{b}$. Thus this condition is always a necessary one, regardless of whether $\bar{M}|\bar{N}$ is big enough.

When both $\bar{M}$ and $\bar{N}$ do not contain ∞, [13, Lemma 7.2] shows that when $\bar{M}|\bar{N}$ is big enough for $\bar{a}|\bar{b}$, the converse holds: $\bar{M} \cdot \bar{a} = \bar{N} \cdot \bar{b}$ implies that there is a $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$. We briefly mention the proof idea there, which we will also see later (e.g., in the proof of Lemma 9). There is a preliminary construction that handles the requirement on vertices on one side in isolation, leaving the vertices on the right with outdegree 1. A follow-up construction merges vertices on the right in order to ensure the necessary number of incoming edges on the right. In doing so we exploit the "big enough" property in order to avoid merging two nodes on the right with a common adjacent edge on the left.

We will now prove that the condition is also sufficient when either $\bar{M}$ or $\bar{N}$ contains ∞. So assume $\bar{M} \cdot \bar{a} = \bar{N} \cdot \bar{b}$, and thus both $\bar{M}, \bar{N}$ contain ∞.

We construct an $\bar{a}|\bar{b}$-biregular graph $G = (U, V, E)$ with size $\bar{M}|\bar{N}$ as follows. Let $\bar{a} = (a_1, \ldots, a_m)$ and $\bar{b} = (b_1, \ldots, b_n)$. Let $\bar{M} = (M_1, \ldots, M_m)$ and $\bar{N} = (N_1, \ldots, N_n)$. We pick pairwise disjoint sets $U_1, \ldots, U_m$, where each $|U_i| = M_i$ and $V_1, \ldots, V_n$, where $|V_i| = N_i$. We set $U = \bigcup_i U_i$ and $V = \bigcup_i V_i$.

The edges are constructed as follows. For each $i \leqslant i \leqslant m$, when $|U_i|$ is finite, we make each vertex $u \in U_i$ have degree a_i, as follows. For each $1 \leqslant j \leqslant t$, we pick a_i "new" vertices from some infinite set V_l – that is, vertices that are not adjacent to any edge, and connect them to u. Likewise, for each vertex $v \in V_i$ when $|V_i|$ is finite. After performing this, every vertex in finite U_i and V_i has degree a_i and b_i, respectively, and every vertex in infinite sets U_i and V_i has degree at most 1.

Finally, we iterate the following process. For every infinite U_i, if $u \in U_i$ has degree other than a_i, we change the degree to a_i by picking "new" vertices from some infinite set V_l, and connect them to u by an appropriate number of edges. Likewise, we can make each vertex v in infinite V_i to have degree b_i. Note that in any iteration, for every infinite set U_i, the degree of a vertex $u \in U_i$ is either a_i, 1, or 0. Likewise, in any iteration, for every infinite set V_i, the degree of a vertex $v \in V_i$ is either b_i, 1, or 0. Since there is an infinite supply of vertices, there are always new vertices that can be picked in any iteration. ◀

Now we move to the case where the entries are still big enough, but some of the entries are periodic on one side. Then we consider the following formula $\Psi_{(\bar{a},\bar{b}^{+p})|\bar{c}}(\bar{x}_0, \bar{x}_1, \bar{y})$:

$$\exists z \quad (z \neq \infty) \ \wedge \ \big(\bar{a} \cdot \bar{x}_0 + \bar{b} \cdot \bar{x}_1 + pz \ = \ \bar{c} \cdot \bar{y}\big). \tag{5}$$

Note that if $G = (U, V, E)$ is a $(\bar{a}, \bar{b}^{+p})|\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|\bar{N}$, then the number of edges $|E|$ should equal the sum of the degrees of the vertices in U, which is $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + zp$, for some integer $z \geqslant 0$. Since this quantity must equal the sum of the degrees of the vertices in V, which is $\bar{c} \cdot \bar{N}$, we again conclude that this formula is a necessary condition – regardless of whether the entries are big enough. We again show the converse.

▶ **Lemma 9.** *For $(\bar{M}_0, \bar{M}_1)|\bar{N}$ big enough for $(\bar{a}, \bar{b}^{+p})|\bar{c}$ the following holds. There is a $(\bar{a}, \bar{b}^{+p})|\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|\bar{N}$ if and only if $\Psi_{(\bar{a},\bar{b}^{+p})|\bar{c}}(\bar{M}_0, \bar{M}_1, \bar{N})$ holds.*

Proof. Assume that $\Psi_{(\bar{a},\bar{b}^{+p})|\bar{c}}(\bar{M}_0, \bar{M}_1, \bar{N})$ holds. As before, abusing notation, we denote the value assigned to variable z by z itself. Suppose $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + pz = \bar{N} \cdot \bar{c}$. Since $(\bar{M}_0, \bar{M}_1)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}^{+p})|\bar{c}$, it follows immediately that $(\bar{M}_0, \bar{M}_1, z)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}, p)|\bar{c}$. Applying Lemma 8, there is a $(\bar{a}, \bar{b}, p)|\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1, z)|\bar{N}$. That is, we have a graph that satisfies our requirements, but there is an additional partition class Z on the left of size z where the number of adjacent vertices is p, rather than being $\bar{b}^{+p}$ as we require. Let $G = (U, V, E)$ be such a graph, and let $U = U_0 \cup U_1 \cup Z$, where U_0, U_1, and Z are the sets of vertices whose degrees are from $\bar{a}$, $\bar{b}$, and from p. Note that $|U_0| = \bar{M}_0 \cdot \bar{1}$, $|U_1| = \bar{M}_1 \cdot \bar{1}$ and $|Z| = z$.

We will construct a $(\bar{a}, \bar{b}^{+p})|\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|\bar{N}$. The idea is to merge the vertices in Z with vertices in U_1. Let $z_0 \in Z$. The number of vertices in U_1 reachable from z_0 in distance 2 is at most $\delta^2_{\max}$. Since $(\bar{M}_0, \bar{M}_1)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}^{+p})|\bar{c}$, we have $|U_1| = \bar{M}_1 \cdot \bar{1} \geqslant \delta^2_{\max} + 1$. Thus, there is a vertex $u \in U_1$ not reachable in distance 2. We merge z_0 and u into one vertex. Since the degree of z_0 is p, such merging increases the degree of u by p, which does not break our requirement. We perform such merging for every vertex in Z. ◀

Finally, we turn to the big enough case where there are periodic entries on both sides. There we will deal with the following formula $\Psi_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$:

$$\exists z_1 \exists z_2 \ (z_1 \neq \infty) \ \wedge \ (z_2 \neq \infty) \ \wedge \big(\bar{a} \cdot \bar{x}_0 + \bar{b} \cdot \bar{x}_1 + pz_1 = \bar{c} \cdot \bar{y}_0 + \bar{d} \cdot \bar{y}_1 + pz_2\big). \tag{6}$$

▶ **Lemma 10.** *For $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$ big enough for $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$ the following holds: there exists a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$ if and only if $\Psi_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1)$ holds.*

Proof. As before, the "only if" part is straightforward, so we focus on the "if" part. Suppose $\Psi_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1)$ holds. Thus, $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + pz_1 = \bar{c} \cdot \bar{N}_0 + \bar{d} \cdot \bar{N}_1 + pz_2$. If $z_1 \geqslant z_2$, then the equation can be rewritten as $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + p(z_1 - z_2) = \bar{c} \cdot \bar{N}_0 + \bar{d} \cdot \bar{N}_1$. By Lemma 9, there is a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$, which of course, is also $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular. The case when $z_2 \geqslant z_1$ is symmetric. ◀

The previous lemmas give formulas that capture the existence of 1-color biregular graphs for big enough sizes. We now turn to sizes that are not big enough – that is, when one of the conditions (a), (b) or (c) is violated. When condition (a) is violated, we have restricted the total size of the graph, and thus we can write a formula that simply enumerate all possible valid sizes. We will consider the case when condition (b) is violated, with the case where condition (c) is violated being symmetric.

If (b) is violated we can fix the value of $\bar{M}_1 \cdot \bar{1}$ as some r, and it suffices to find a formula that works for this r. The idea is that a fixed number of vertices in a graph can be "encoded" as formulas. For $\bar{a} = (a_1, \ldots, a_k)$, $\bar{b} = (b_1, \ldots, b_l)$, $\bar{c} = (c_1, \ldots, c_m)$ and $\bar{d} = (d_1, \ldots, d_n)$, and for integer $r \geqslant 0$, define the formula $\Phi^r_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$ as follows:

1. when $r = 0$, let
$$\Phi^r_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1) := \ \bar{x}_1 \cdot \bar{1} = 0 \ \wedge \ \Psi_{\bar{a}|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{y}_0, \bar{y}_1),$$
where $\Psi_{\bar{a}|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{y}_0, \bar{y}_1)$ is as defined in equation (5);
2. when $r \geqslant 1$, let $\bar{x}_1 = (x_{1,1}, \ldots, x_{1,l})$ and
$$\Phi^r_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1) \ :=$$
$$\exists s \exists \bar{z}_0 \exists \bar{z}_1 \exists \bar{z}_2 \exists \bar{z}_3 \bigvee_{i=1}^{l} \left(\begin{array}{l} \quad (x_{1,i} \neq 0) \ \wedge \ (b_i + ps \ = \ \bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1}) \ \wedge \ (s \neq \infty) \\ \wedge \ (\bar{z}_0 + \bar{z}_1 = \bar{y}_0) \ \wedge \ (\bar{z}_2 + \bar{z}_3 = \bar{y}_1) \\ \wedge \ \Phi^{r-1}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{c}-\bar{1},\bar{d}^{+p},(\bar{d}-\bar{1})^{+p})}(\bar{x}_0, \bar{x}_1 - \mathbf{e}_i, \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3) \end{array} \right),$$
where $\mathbf{e}_i$ denotes the unit vector (with length k) where the i-th component is 1, and the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{y}_0$, and the lengths of $\bar{z}_2$ and $\bar{z}_3$ are the same as $\bar{y}_1$.

The motivation for these formulas will be explained in the proof of the following lemma.

▶ **Lemma 11.** *For every $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, every integer $r \geqslant 0$ and every $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ such that*
1. $\bar{M}_0 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geqslant 2\delta^2_{\max} + 3$,
2. $\bar{N}_1 \cdot \bar{1} \geqslant \delta^2_{\max} + 1$,
3. $\bar{M}_1 \cdot \bar{1} = r$,

where $\delta_{\max} = \max(p, \bar{a}, \bar{c}, \bar{d})$, the following holds: there is a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$ if and only if $\Phi^r_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1)$ holds.

Proof. The proof is by induction on r. The base case $r = 0$ follows from Lemma 9, so we focus on the induction step.

We begin with the "only if" direction, which provides the intuition for these formulas. Suppose $G = (U, V, E)$ is a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular with size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$. We let $U = U_{0,1} \cup \cdots \cup U_{0,k} \cup U_{1,1} \cup \cdots \cup U_{1,l}$, where $\bar{M}_0 = (|U_{0,1}|, \ldots, |U_{0,k}|)$ and $\bar{M}_1 = (|U_{1,1}|, \ldots, |U_{1,l}|)$. Likewise, we let $V = V_{0,1} \cup \cdots \cup V_{0,m} \cup V_{1,1} \cup \cdots \cup V_{1,n}$, where $\bar{N}_0 = (|V_{0,1}|, \ldots, |V_{0,m}|)$ and $\bar{N}_1 = (|V_{1,1}|, \ldots, |V_{1,n}|)$.

Since we are not in the base case, we can assume $\bar{M}_1 \cdot \bar{1} = \sum_{i=1}^{l} |U_{1,i}| = r \neq 0$. Thus we can fix some i with $1 \leqslant i \leqslant l$ such that $U_{1,i} \neq \emptyset$, and fix also some $u \in U_{1,i}$. Based on this u, we define, for each $1 \leqslant j \leqslant m$, $Z_{0,j}$ to be the set of vertices in $V_{0,j}$ adjacent to u. For each $1 \leqslant j \leqslant n$ we let $Z_{1,j}$ be the set of vertices in $V_{1,j}$ adjacent to u. Figure 1 illustrates the situation.

If we omit the vertex u and all its adjacent edges, we have the following:

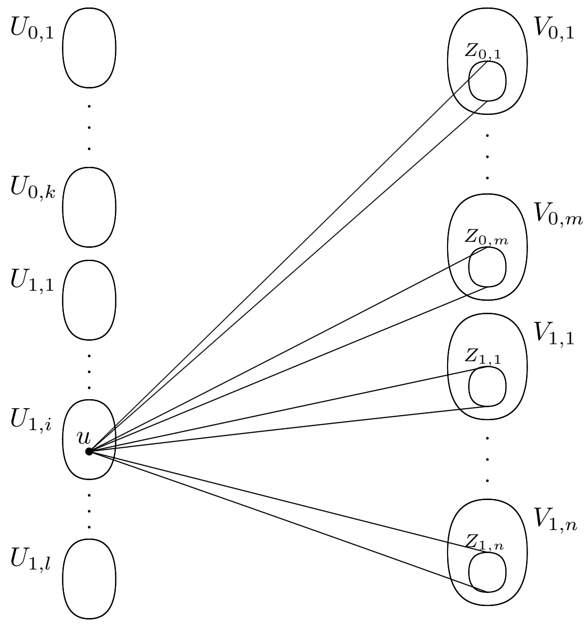

Figure 1 Inductive construction for the "not big enough" case.

1. for every $1 \leqslant j \leqslant m$, every vertex in $Z_{0,j}$ has degree $c_j - 1$,
2. for every $1 \leqslant j \leqslant n$, every vertex in $Z_{1,j}$ has degree $(d_j - 1)^{+p}$.

Thus, we have a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{c} - \bar{1}, \bar{d}^{+p}, (\bar{d} - \bar{1})^{+p})$-biregular graph with size $(\bar{M}_0, \bar{M}_1 - \mathbf{e}_i)|(\bar{K}_{0,0}, \bar{K}_{0,1}, \bar{K}_{1,0}, \bar{K}_{1,1})$, where

$$\bar{K}_0 = (|V_{0,1}| - |Z_{0,1}|, \ldots, |V_{0,m}| - |Z_{0,m}|), \qquad \bar{K}_1 = (|Z_{0,1}|, \ldots, |Z_{0,m}|),$$
$$\bar{K}_2 = (|V_{1,1}| - |Z_{1,1}|, \ldots, |V_{1,n}| - |Z_{1,n}|), \qquad \bar{K}_3 = (|Z_{1,1}|, \ldots, |Z_{1,n}|).$$

We can check that the sizes allow us to apply the induction hypothesis to this graph, keeping in mind that the sizes on the left have now decreased by one. We conclude that $\Phi^{r-1}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{c}-\bar{1},\bar{d}^{+p},(\bar{d}-\bar{1})^{+p})}(\bar{M}_0, \bar{M}_1 - \mathbf{e}_i)|(\bar{K}_{0,0}, \bar{K}_{0,1}, \bar{K}_{1,0}, \bar{K}_{1,1})$ holds. Moreover, since $u \in U_{1,i}$, and hence the degree of u is b_i^{+p}, we have $\bar{K}_1 \cdot \bar{1} + \bar{K}_3 \cdot \bar{1} = b_i + ps$, for some integer $s \geqslant 0$. Note also that $\bar{K}_0 + \bar{K}_1 = \bar{N}_0$ and $\bar{K}_2 + \bar{K}_3 = \bar{N}_1$. Thus, $\Phi^{r}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$ holds where the variables $\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3$ are assigned with $\bar{K}_0, \bar{K}_1, \bar{K}_2, \bar{K}_3$, respectively.

For the "if" direction, suppose $\Phi^{r}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1)$ holds. Then we can fix some $s, \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3$, and i such that (a) $x_{1,i} \neq 0$, (b) $b_i + ps = \bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1}$, (c) $\bar{z}_0 + \bar{z}_1 = \bar{N}_0$, (d) $\bar{z}_2 + \bar{z}_3 = \bar{N}_1$, and (e) $\Phi^{r-1}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{c}-\bar{1},\bar{d}^{+p},(\bar{d}-\bar{1})^{+p})}(\bar{M}_0, \bar{M}_1 - \mathbf{e}_i, \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ holds.

We prove from this that a biregular graph of the appropriate size exists. Note that the hypothesis requires that $\bar{M}_0 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geqslant 2\delta^2_{\max} + 3$, where $\delta_{\max}$ is as defined in the statement of the lemma. Since $\max(p, \bar{a}, \bar{b}, \bar{c}, \bar{c} - \bar{1}, \bar{d}, \bar{d} - \bar{1}) = \delta_{\max}$, the equalities in (c) and (d) imply that $\bar{M}_0 \cdot \bar{1} + \bar{z}_0 \cdot \bar{1} + \bar{z}_1 \cdot \bar{1} + \bar{z}_2 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1}$ is bigger than $2\delta^2_{\max} + 3$.

Note that $(\bar{M}_1 - \mathbf{e}_i) \cdot \bar{1} = r - 1$. Thus we can apply the induction hypothesis and obtain a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{c} - \bar{1}, \bar{d}^{+p}, (\bar{d} - \bar{1})^{+p})$-biregular graph $G = (U, V, E)$ with size $(\bar{M}_0, \bar{M}_1 - \mathbf{e}_i)|(\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$. Let $V = V_0 \cup V_1 \cup V_2 \cup V_3$ be the partition of V, where

$$V_0 = V_{0,1} \cup \cdots \cup V_{0,m}, \qquad V_1 = V_{1,1} \cup \cdots \cup V_{1,m},$$
$$V_2 = V_{2,1} \cup \cdots \cup V_{2,n}, \qquad V_3 = V_{3,1} \cup \cdots \cup V_{3,n},$$

and such that

1. for every $1 \leqslant i \leqslant m$, the degree of vertices in $V_{0,j}$ and $V_{1,j}$ are c_j and $c_j - 1$, respectively;
2. for every $1 \leqslant i \leqslant n$, the degree of vertices in $V_{2,j}$ and $V_{3,j}$ are d_j^{+p} and $(d_j - 1)^{+p}$, respectively.

Note also that $\bar{z}_0 = (|V_{0,1}|, \ldots, |V_{0,m}|)$, $\bar{z}_1 = (|V_{1,1}|, \ldots, |V_{1,m}|)$, $\bar{z}_2 = (|V_{2,1}|, \ldots, |V_{2,m}|)$, and $\bar{z}_3 = (|V_{3,1}|, \ldots, |V_{3,m}|)$.

Let u be a fresh vertex. We can construct a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graph $G' = (U \cup \{u\}, V, E')$, by connecting the vertex u with every vertex in $V_1 \cup V_3$. Note that the formula states that $\bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1} = b_i + ps$, which equals to $|V_1| + |V_3|$, thus, the degree of u is $b_i + ps$, which satisfies our requirement for a vertex to be in U_i. Since prior to the connection, the degrees of $V_{1,j}$ and $V_{3,j}$ are $c_j - 1$ and $(d_j - 1)^{+p}$, after connecting u with each vertex in $V_1 \cup V_3$, their degrees become c_j and d_j^{+p}. That is, the right side vertices now have the desired degrees, i.e., G' is $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular. Moreover, $\bar{z}_0 + \bar{z}_1 = \bar{N}_0$ and $\bar{z}_2 + \bar{z}_3 = \bar{N}_1$. Thus, the resulting graph G' has size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$. ◀

The formula $\mathsf{bireg}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$ characterizing the sizes of $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graphs can be defined by combining all the cases described above.

4.2 Proof of Lemma 2 for 1-color graphs (the complete case)

We now turn to bootstrapping the biregular case to add the completeness requirement imposed in Lemma 2. Let $\bar{a} = (a_1, \ldots, a_k)$, $\bar{b} = (b_1, \ldots, b_l)$, $\bar{c} = (c_1, \ldots, c_m)$ and $\bar{d} = (d_1, \ldots, d_n)$. Let $\bar{x}_0 = (x_{0,1}, \ldots, x_{0,k})$, $\bar{x}_1 = (x_{1,1}, \ldots, x_{1,l})$, $\bar{y}_0 = (y_{0,1}, \ldots, y_{0,m})$, and $\bar{y}_1 = (y_{1,1}, \ldots, y_{1,n})$.

The formula $\mathsf{c\text{-}bireg}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$ for the sizes of complete $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graphs is the conjunction of $\mathsf{bireg}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$ such that

1. for every $1 \leqslant i \leqslant k$, if $x_{0,i} \neq 0$, then $\bar{y}_0 \cdot \bar{1} + \bar{y}_1 \cdot \bar{1} = a_i$;
2. for every $1 \leqslant i \leqslant l$, if $x_{1,i} \neq 0$, then $\bar{y}_0 \cdot \bar{1} + \bar{y}_1 \cdot \bar{1} = b_i + pz_i$, for some z_i;
3. for every $1 \leqslant i \leqslant m$, if $y_{0,i} \neq 0$, then $\bar{x}_0 \cdot \bar{1} + \bar{x}_1 \cdot \bar{1} = c_i$;
4. for every $1 \leqslant i \leqslant n$, if $y_{1,i} \neq 0$, then $\bar{x}_0 \cdot \bar{1} + \bar{x}_1 \cdot \bar{1} = d_i + pz_i$, for some z_i.

To understand these additional conditions, consider a complete biregular graph meeting the cardinality specification. The completeness criterion for 1-color graphs implies that each element on the left is connected to every element on the right. Thus if the size of a partition required to have fixed outdegree a_i is non-empty, we must have that a_i is exactly the cardinality of the number of elements on the right. This is what is captured in the first item. If we have non-empty size for a partition whose outdegree is constrained to be b_i plus a multiple of p, then the total number of elements on the right must be b_i plus a multiple of p. This is what the second item specifies. Considering elements on the left motivates the third and fourth item. Thus we see that these conditions are necessary.

Suppose $\mathsf{c\text{-}bireg}_{(\bar{a},\bar{b}^{+p})|(\bar{c},\bar{d}^{+p})}(\bar{M}_0, \bar{M}_1, \bar{M}_0, \bar{M}_1)$ holds. Then, there is a $(\bar{a}, \bar{b}^{+p})|(\bar{c}, \bar{d}^{+p})$-biregular graph $G = (U, V, E)$ with size $(\bar{M}_0, \bar{M}_1)|(\bar{N}_0, \bar{N}_1)$, which are not necessarily complete. Note that $\bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1}$ is precisely the number of vertices in V. The first item states that the existence of a vertex u with degree a_i implies u is adjacent to every vertex in V. Now, suppose there is a vertex $u \in U$ with degree b_i^{+p}. If u is not adjacent to every vertex in V, then we can add additional edges so that u is adjacent to every vertex in V. The second item states that $|V| = b_i^{+p}$. Thus, adding such edges is legal, since the degree of u stays b_i^{+p}. We can make vertices in V adjacent to every vertex in U using the same argument.

4.3 The proof for regular digraphs

Recall that in the prior argument we consider only digraphs without any self-loop. Thus, a digraph can be viewed as a bipartite graph by splitting every vertex u into two vertices, where one is adjacent to all the incoming edges, and the other to all the outgoing edges. Thus, $A|B$-regular digraphs with size $\bar{M}$ can be characterized as $A|B$-biregular graphs with size $\bar{M}|\bar{M}$. For more details, see [13, Section 8].

5 Extensions and applications

A *type/behavior profile* for a model M is the vector of cardinalities of the sets $A_{\pi,g}$ computed in M, where π ranges of 1-types and g over behavior functions (for a fixed ϕ). Recall that in the proof Theorem 4 we actually showed, in Lemma 6, that we can obtain existential Presburger formulas which define exactly the vectors of integers that arise as the type/behavior profiles of models of ϕ. The domain of the model can be broken up as a disjoint union of sets $A_{\pi,g}$, and thus its cardinality is a sum of numbers in this vector. We can thus add one additional integer variable x_{total} in PRES_ϕ, which will be free, with an additional equation stating that x_{total} is the sum of all $X_{\pi,g}$'s. This allows us to conclude definability of the spectrum.

▶ **Theorem 12.** *From an FO^2_{Pres} sentence ϕ, we can effectively construct a Presburger formula $\psi(n)$ such that $\mathcal{N} \models \psi(n)$ exactly when n is the size of a finite structure that satisfies ϕ, and similarly a formulas $\psi_\infty(n)$ such that $\mathcal{N}_\infty \models \psi_\infty(n)$ exactly when n is the size of a finite or countably infinite model of ϕ.*

We say that ϕ has *NP data complexity of (finite) satisfiability* if there is a non-deterministic algorithm that takes as input a set of ground atoms A and determines whether $\phi \wedge \bigwedge A$ is satisfiable, running in time polynomial in the size of A. Pratt-Hartmann [20] showed that C^2 formulas have NP data complexity of both satisfiability and finite satisfiability. Following the general approach to data complexity from [20], while plugging in our Presburger characterization of $\mathsf{FO}^2_{\mathrm{Pres}}$, we can show that the same data complexity bound holds for $\mathsf{FO}^2_{\mathrm{Pres}}$.

▶ **Theorem 13.** *FO^2_{Pres} formulas have NP data complexity of satisfiability and finite satisfiability.*

Proof. We give only the proof for finite satisfiability. We will follow closely the approach used for C^2 in Section 4 of [20], and the terminology we use below comes from that work.

Given a set of facts D, our algorithm guesses a set of facts (including equalities) on elements of D, giving us a finite set of facts D^+ extending D, but with the same domain as D. We check that our guess is consistent with the universal part α and such that equality satisfies the usual transitivity and congruence rules.

Now consider 1-types and 2-types with an additional predicate Observable. Based on this extended language, we consider good functions as before, and define the formulas $\mathsf{consistent}_1$ and $\mathsf{consistent}_2$ based on them. 1-types with that contain the predicate Observable will be referred to as observable 1-types. The restriction of a behavior function to observable 1-types will be called an *observable behavior*. Given a structure M, an observable one-type π, and an observable behavior function g_0, we let M_{π,g_0} be the elements of M having 1-type π and observable behavior g_0, and we analogously let D_{π,g_0} be the elements of D whose 1-type and behavior in D^+ match π and g_0.

We declare that all elements in A are in the predicate Observable. Add to the formulas $\mathsf{consistent}_1$ and $\mathsf{consistent}_2$ additional conjuncts stating that for each observable 1-type π and for each observable behavior function g_0, the total sum of the number of elements with 1-type π and a behavior function g extending g_0 (i.e., the cardinality of M_{π,g_0}) is the same as $|D_{\pi,g_0}|$. with the cardinality being counted modulo equalities of D^+.

At this point our algorithm returns true exactly when the sentence obtained by existentially quantifying this extended set of conjuncts is satisfiable in the integers. The solving procedure is certainly in NP. In fact, since the number of variables is fixed, with only the constants varying, it is in PTIME [17].

We argue for correctness, focusing on the proof that when the algorithm returns true we have the desired model. Assuming the constraints above are satisfied, we get a graph, and from the graph we get a model M. M will clearly satisfy ϕ, but its domain does not contain the domain of D. Letting O be the elements of M satisfying Observable, we know, from the additional constraints imposed, that the cardinality of O matches the cardinality of the domain of D modulo the equalities in D^+, and for each observable 1-type π_o and observable behavior g_0, $|M_{\pi,g_0}| = |D_{\pi,g_0}$.

Fix an isomorphism λ taking each M_{π,g_0} to (equality classes of) D_{π,g_0}. Create M' by redefining M on O by connecting pairs (o_1, o_2) via E exactly when $\lambda(o_1), \lambda(o_2)$ ise connected via E in D^+. We can thus identify O with D^+ modulo equalities in M'.

Clearly M' now satisfies D. To see that M' satisfies ϕ, we simply note that since all of the observable behaviors are unchanged in moving from an element e in M to the corresponding element $\lambda(e)$ in M', and every such e modified has an observable type, it follows that the behavior of every element in M is unchanged in moving from M to M'. Since the 1-types are also unchanged, M' satisfies ϕ. ◀

Note that the data complexity result here is best possible, since even for FO^2 the data complexity can be NP-hard [20].

6 Conclusion

We have shown that we can extend the powerful language two-variable logic with counting to include ultimately periodic counting quantifiers without sacrificing decidability, and without losing the effective definability of the spectrum of formulas within Presburger arithmetic. We believe that by refining our proof we can obtain a 2NEXPTIME bound on complexity. However the only lower bound we know of is NEXPTIME, inherited from FO^2. We leave the analysis of the exact complexity for future work.

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