# Quasi-Majority Functional Voting on Expander Graphs

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## Abstract

Consider a distributed graph where each vertex holds one of two distinct opinions. In this paper, we are interested in synchronous *voting processes* where each vertex updates its opinion according to a predefined common local updating rule. For example, each vertex adopts the majority opinion among 1) itself and two randomly picked neighbors in *best-of-two* or 2) three randomly picked neighbors in *best-of-three*. Previous works intensively studied specific rules including best-of-two and best-of-three individually.

In this paper, we generalize and extend previous works of best-of-two and best-of-three on expander graphs by proposing a new model, *quasi-majority functional voting*. This new model contains best-of-two and best-of-three as special cases. We show that, on expander graphs with sufficiently large initial bias, any quasi-majority functional voting reaches consensus within $O(\log n)$ steps with high probability. Moreover, we show that, for any initial opinion configuration, any quasi-majority functional voting on expander graphs with higher expansion (e.g., Erdős-Rényi graph $G(n,p)$ with $p = \Omega(1/\sqrt{n})$) reaches consensus within $O(\log n)$ with high probability. Furthermore, we show that the consensus time is $O(\log n/\log k)$ of best-of-$(2k+1)$ for $k = o(n/\log n)$.

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## 1 Introduction

Consider an undirected graph $G = (V, E)$ where each vertex $v \in V$ initially holds an opinion $\sigma \in \Sigma$ from a finite set $\Sigma$. In *synchronous voting process* (or simply, *voting process*), in each round, every vertex communicates with its neighbors and then all vertices simultaneously update their opinions according to a predefined protocol. The aim of the protocol is to reach a *consensus configuration*, i.e., a configuration where all vertices have the same opinion. Voting process has been extensively studied in several areas including biology, network analysis, physics and distributed computing [10, 32, 30, 22, 26, 2]. For example, in distributed computing, voting process plays an important role in the consensus problem [22, 26].

This paper is concerned with the *consensus time* of voting processes over *binary* opinions $\Sigma = \{0, 1\}$. Then voting processes have state space $2^V$. A state of $2^V$ is called a *configuration*. The *consensus time* is the number of steps needed to reach a consensus configuration. Henceforth, we are concerned with connected and nonbipartite graphs.


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## 1.1 Previous works of specific updating rules

In *pull voting*, in each round, every vertex adopts the opinion of a randomly selected neighbor. This is one of the most basic voting process, which has been well explored in the past [33, 27, 14, 18, 8]. In particular, the expected consensus time of this process has been extensively studied in the literature. For example, Hassin and Peleg [27] showed that the expected consensus time is $O(n^3 \log n)$ for all non-bipartite graphs and all initial opinion configurations, where $n$ is the number of vertices. From the result of Cooper, Elsässer, Ono, and Radzik [14], it is known that on the complete graph $K_n$, the expected consensus time is $O(n)$ for any initial opinion configuration.

In *best-of-two* (a.k.a. *2-Choices*), each vertex $v$ samples two random neighbors (with replacement) and, if both hold the same opinion, $v$ adopts the opinion. Otherwise, $v$ keeps its own opinion. Doerr, Goldberg, Minder, Sauerwald, and Scheideler [21] showed that, on the complete graph $K_n$, the consensus time of best-of-two is $O(\log n)$ with high probability[1] for an arbitrary initial opinion configuration. Since best-of-two is simple and is faster than pull voting on the complete graphs, this model gathers special attention in distributed computing and related area [25, 15, 16, 17, 19, 20, 37]. There is a line of works that study best-of-two on expander graphs [15, 16, 17], which we discuss later.

In *best-of-three* (a.k.a. *3-Majority*), each vertex $v$ randomly selects three random neighbors (with replacement). Then, $v$ updates its opinion to match the majority among the three. It follows directly from Ghaffari and Lengler [25] that, on $K_n$ with any initial opinion configuration, the consensus time of best-of-three is $O(\log n)$ w.h.p. Kang and Rivera [28] considered the consensus time of best-of-three on graphs with large minimum degree starting from a random initial configuration. Shimizu and Shiraga [37] showed that, for any initial configurations, best-of-two and best-of-three reach consensus in $O(\log n)$ steps w.h.p. if the graph is an Erdős-Rényi graph $G(n,p)$[2] of $p = \Omega(1)$.

*Best-of-k* ($k \geq 1$) is a generalization of pull voting, best-of-two and best-of-three. In each round, every vertex $v$ randomly selects $k$ neighbors (with replacement) and then if at least $\lfloor k/2 \rfloor + 1$ of them have the same opinion, the vertex $v$ adopts it. Note that the best-of-1 is equivalent to pull voting. Abdullah and Draief [1] studied a variant of best-of-$k$ ($k \geq 5$ is odd) on a specific class of sparse graphs that includes $n$-vertex random $d$-regular graphs[3] $G_{n,d}$ of $d = o(\sqrt{\log n})$ with a random initial configuration. To the best of our knowledge, best-of-$k$ has not been studied explicitly so far.

In *Majority* (a.k.a. *local majority*), each vertex $v$ updates its opinion to match the majority opinion among the neighbors. This simple model has been extensively studied in previous works [6, 9, 24, 34, 35, 40]. For example, Majority on certain families of graphs including the Erdős-Rényi random graph [6, 40], random regular graphs [24] have been investigated. See [35] for further details.

### Voting process on expander graphs

Expander graph gathers special attention in the context of Markov chains on graphs, yielding a wide range of theoretical applications. A graph $G$ is $\lambda$-*expander* if $\max\{|\lambda_2|, |\lambda_n|\} \leq \lambda$, where $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$ are the eigenvalues of the transition matrix $P$ of the

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[1] In this paper "with high probability" (w.h.p.) means probability at least $1 - n^{-c}$ for a constant $c > 0$.

[2] Recall that the Erdős-Rényi random graph $G(n,p)$ is a graph on $n$ vertices where each of possible $\binom{n}{2}$ vertex pairs forms an edge with probability $p$ independently.

[3] An $n$-vertex random $d$-regular graph $G_{n,d}$ is a graph selected uniformly at random from the set of all labelled $n$-vertex $d$-regular graphs.

simple random walk on $G$. For example, an Erdős-Rényi graph $G(n, p)$ of $p \geq (1+\epsilon)\frac{\log n}{n}$ for an arbitrary constant $\epsilon > 0$ is $O(1/\sqrt{np})$-expander w.h.p. [12]. An $n$-vertex random $d$-regular graph $G_{n,d}$ of $3 \leq d \leq n/2$ is $O(1/\sqrt{d})$-expander w.h.p. [13, 39].

Cooper et al. [14] showed that the expected consensus time of pull voting is $O(n/(1-\lambda))$ on $\lambda$-expander regular graphs for any initial configuration. Compared to pull voting, the study of best-of-two on general graphs seems much harder. Most of the previous works concerning best-of-two on expander graphs put some assumptions on the initial configuration. Let $A$ denote the set of vertices of opinion 0 and $B = V \setminus A$. Cooper, Elsässer, and Radzik [15] showed that, for any regular $\lambda$-expander graph, the consensus time is $O(\log n)$ w.h.p. if $\big||A| - |B|\big| = \Omega(\lambda n)$. This result was improved by Cooper, Elsässer, Radzik, Rivera, and Shiraga [16]. Roughly speaking, they proved that, on $\lambda$-expander graphs, the consensus time is $O(\log n)$ if $|d(A) - d(B)| = \Omega(\lambda^2 d(V))$, where $d(S) = \sum_{v \in S} \deg(v)$ denotes the volume of $S \subseteq V$. To the best of our knowledge, the worst case consensus time of best-of-$k$ on expander graphs has not been studied.

## 1.2 Our model

In this paper, we propose a new class *functional voting* of voting process, which contains many known voting processes as a special case. Let $A \subseteq V$ be the set of vertices of opinion 0 and $A'$ be the set in the next round. Let $B = V \setminus A$ and $B' = V \setminus A'$. For $v \in V$ and $S \subseteq V$, let $N(v) = \{w \in V : \{v, w\} \in E\}$ and $\deg_S(v) = |N(v) \cap S|$.

▶ **Definition 1.1** (Functional voting). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f([0,1]) = [0,1]$ and $f(0) = 0$. A* functional voting with respect to $f$ *is a synchronous voting process defined as*

$$\mathbf{Pr}[v \in A'] = f\left(\frac{\deg_A(v)}{\deg(v)}\right) \quad \text{if } v \in B,$$
$$\mathbf{Pr}[v \in B'] = f\left(\frac{\deg_B(v)}{\deg(v)}\right) \quad \text{if } v \in A.$$

*We call the function $f$ a* betrayal function *and the function*

$$H_f(x) := x\big(1 - f(1-x)\big) + (1-x)f(x)$$

*an* updating function.

Since $f(0) = 0$, consensus configurations are absorbing states. The intuition behind the updating function $H_f$ is that, letting $\alpha = |A|/n$ and $\alpha' = |A'|/n$, on a complete graph $K_n$ (with self-loop), the functional voting with respect to $f$ satisfies $\mathbf{E}[\alpha'] = \frac{|A|}{n}\left(1 - f\left(\frac{|B|}{n}\right)\right) + \frac{|B|}{n} f\left(\frac{|A|}{n}\right) = H_f(\alpha)$.

Functional voting contains many existing models as special cases. For example, pull voting, best-of-two, and best-of-three are functional votings with respect to $x$, $x^2$ and $3x^2 - 2x^3$, respectively. In general, best-of-$k$ is a functional voting with respect to

$$f_k(x) = \sum_{i=\lfloor k/2 \rfloor + 1}^{k} \binom{k}{i} x^i (1-x)^{k-i}. \tag{1}$$

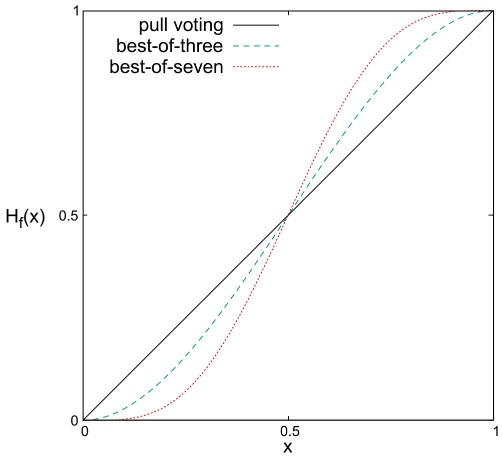


**Figure 1** The updating functions $H_f(x)$ of pull voting (solid line), best-of-three (dashed line) and best-of-seven (dotted line). One can easily observe that best-of-three and best-of-seven are quasi-majority functional voting. Intuitively speaking, quasi-majority functional voting has an updating function $H_f$ with the property so-called "the rich get richer", which coincides with Definition 1.2.

It is straightforward to check that $H_{f_k}(x) = f_k(x)$ if $k$ is odd and $H_{f_k}(x) = f_{k+1}(x)$ if $k$ is even. Majority is a functional voting with respect to

$$f(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x > \frac{1}{2} \end{cases} \tag{2}$$

if a vertex adopts the random opinion when it meets the tie.

## Quasi-majority functional voting

In this paper, we focus on functional voting with respect to $f$ satisfying the following property.

▶ **Definition 1.2** (Quasi-majority). *A function $f$ is* quasi-majority *if $f$ satisfies the following conditions.*

**(i)** *$f$ is $C^2$ (i.e., the derivatives $f'$ and $f''$ exist and they are continuous).*
**(ii)** $0 < f(1/2) < 1$,
**(iii)** *$H_f(x) < x$ whenever $x \in (0, 1/2)$.*
**(iv)** $H'_f(1/2) > 1$,
**(v)** $H'_f(0) < 1$.

*A voting process is a* quasi-majority functional voting *if it is a functional voting with respect to a quasi-majority function $f$.*

Note that $H_f(x)$ is symmetric (i.e., $H_f(1-x) = 1 - H_f(x)$) and thus the condition (iii) implies $H_f(x) > x$ for every $x \in (1/2, 1)$. Intuitively, the conditions (iii) to (v) ensure the drift towards consensus. The conditions (i) and (ii) are due to a technical reasons.

For each constant $k \geq 2$, best-of-$k$ is quasi-majority functional voting but pull voting and Majority are not. Indeed, if $H_{f_k}$ is the updating function of best-of-$k$, then $H'_{f_{2\ell}}(x) = H'_{f_{2\ell+1}}(x) = (2\ell+1)\binom{2\ell}{\ell}x^\ell(1-x)^\ell$. It is straightforward to check that this function satisfies the conditions (iii) to (v) if $\ell \neq 0$ (pull-voting). See Figure 1 for depiction of the updating functions of pull voting, best-of-three and best-of-seven.

## 1.3 Our result

In this paper, we study the consensus time of quasi-majority functional voting on expander graphs[4]. Let $T_{\mathrm{cons}}(A)$ denote the consensus time starting from the initial configuration $A \subseteq V$. For a graph $G = (V, E)$, let $\pi = (\pi(v))_{v \in V}$ denote the *degree distribution* defined as

$$\pi(v) = \frac{\deg(v)}{2|E|}. \tag{3}$$

Note that $\sum_{v \in V} \pi(v) = 1$ holds. We denote by $\|x\|_p := \left(\sum_{v \in V} |x_v|^p\right)^{1/p}$ the $\ell^p$ norm of $x \in \mathbb{R}^V$. For $\pi \in [0,1]^V$ and $A \subseteq V$, let $\pi(A) := \sum_{v \in A} \pi(v)$. Let

$$\delta(A) := \pi(A) - \pi(V \setminus A) = 2\pi(A) - 1$$

denote the *bias* between $A$ and $V \setminus A$.

▶ **Theorem 1.3** (Main theorem). *Consider a quasi-majority functional voting with respect to $f$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$. Then, the following holds:*

**(i)** *Let $C_1 > 0$ be an arbitrary constant and $\varepsilon : \mathbb{N} \to \mathbb{R}$ be an arbitrary function satisfying $\varepsilon(n) \to 0$ as $n \to \infty$. Suppose that $\lambda \le C_1 n^{-1/4}$, $\|\pi\|_2 \le C_1/\sqrt{n}$ and $\|\pi\|_3 \le \varepsilon/\sqrt{n}$. Then, for any $A \subseteq V$, $T_{\mathrm{cons}}(A) = O(\log n)$ w.h.p.*

**(ii)** *Let $C_2$ be a positive constant depending only on $f$. Suppose that $\lambda \le C_2$ and $\|\pi\|_2 \le C_2/\sqrt{\log n}$. Then, for any $A \subseteq V$ satisfying $|\delta(A)| \ge C_2 \max\{\lambda^2, \|\pi\|_2\sqrt{\log n}\}$, $T_{\mathrm{cons}}(A) = O(\log n)$ w.h.p.*

The following result indicates that the consensus time of Theorem 1.3(i) is optimal up to a constant factor.

▶ **Theorem 1.4** (Lower bound). *Under the same assumption of Theorem 1.3(i), $T_{\mathrm{cons}}(A) = \Omega(\log n)$ w.h.p. for some $A \subseteq V$.*

See the full version [38] for the proof of Theorem 1.4.

▶ **Theorem 1.5** (Fast consensus for $H'_f(0) = 0$). *Consider a quasi-majority functional voting with respect to $f$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$. Let $C > 0$ be a constant depending only on $f$. Suppose that $H'_f(0) = 0$, $\lambda \le C$ and $\|\pi\|_2 \le C/\sqrt{\log n}$. Then, for any $A \subseteq V$ satisfying $|\delta(A)| \ge C \max\{\lambda^2, \|\pi\|_2\sqrt{\log n}\}$, it holds w.h.p. that*

$$T_{\mathrm{cons}}(A) = O\left(\log\log n + \log|\delta(A)|^{-1} + \frac{\log n}{\log \lambda^{-1}} + \frac{\log n}{\log(\|\pi\|_2\sqrt{\log n})^{-1}}\right).$$

For example, for each constant $k \ge 2$, best-of-$k$ is quasi-majority with $H'_f(0) = 0$.

▶ Remark 1.6. Roughly speaking, for $p \ge 2$, $\|\pi\|_p$ measures the imbalance of the degrees. For any graphs, $\|\pi\|_p \ge n^{-1+1/p}$ and the equality holds if and only if the graph is regular. For star graphs, we have $\|\pi\|_p \approx 1$.

### Results of best-of-$k$

Our results above do not explore Majority since it is not quasi-majority. A plausible approach is to consider best-of-$k$ for $k = k(n) = \omega(1)$ since each vertex is likely to choose the majority opinion if the number of neighbor sampling increases. Also, note that the betrayal function $f_k$

[4] Throughout the paper, we consider sufficiently large $n = |V|$.

of best-of-$k$ given in (1) converges to that of Majority (i.e., $f_k(x) \to f(x)$ as $k \to \infty$ for each $x \in [0,1]$, where $f$ is the betrayal function (2) of Majority). On the other hand, if $k = O(1)$, there is a tremendous gap between best-of-$k$ and Majority: For any functional voting on the complete graph $K_n$, $T_{\mathrm{cons}}(A) = \Omega(\log n)$ for some $A \subseteq V$ from Theorem 1.4. Majority on $K_n$ reaches the consensus in a single step if $|A| < |V \setminus A| - 1$. This motivates us to consider best-of-$k$ for $k = k(n) \to \infty$ as $n \to \infty$. For simplicity, we focus on best-of-$(2k+1)$ and prove the following result (see the full version [38] for the proof).

▶ **Theorem 1.7.** *Let $k = k(n)$ be such that $k = \omega(1)$ and $k = o(n/\log n)$. Let $C$ be an arbitrary positive constant. Consider best-of-$(2k+1)$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$ such that $\lambda \le Ck^{-1/2}n^{-1/4}$, $\|\pi\|_2 \le Cn^{-1/2}$ and $\|\pi\|_3 \le Ck^{-1/6}n^{-1/2}$. Then, $T_{\mathrm{cons}}(A) = O\left(\frac{\log n}{\log k}\right)$ holds w.h.p. for any $A \subseteq V$.*

## 1.4 Application

Here, we apply our main theorem to specific graphs and derive some useful results.

For any $p \ge (1+\epsilon)\frac{\log n}{n}$ for an arbitrary constant $\epsilon > 0$, $G(n,p)$ is connected and $O(1/\sqrt{np})$-expander w.h.p [12, 23].

▶ **Corollary 1.8.** *Consider a best-of-$k$ on an Erdős-Rényi graph $G(n,p)$ for an arbitrary constant $k \ge 2$. Then, $G(n,p)$ w.h.p. satisfies the following:*

**(i)** *Suppose that $p = \Omega(n^{-1/2})$. Then*
  - **(a)** *for any $A \subseteq V$, $T_{\mathrm{cons}}(A) = O(\log n)$ w.h.p.*
  - **(b)** *for some $A \subseteq V$, $T_{\mathrm{cons}}(A) = \Omega(\log n)$ w.h.p.*

**(ii)** *Suppose that $p \ge (1+\epsilon)\frac{\log n}{n}$ for an arbitrary constant $\epsilon > 0$. Then, for any $A \subseteq V$ satisfying $|\delta(A)| \ge C\max\left\{\frac{1}{np}, \sqrt{\frac{\log n}{n}}\right\}$, $T_{\mathrm{cons}}(A) = O\left(\log\log n + \log|\delta(A)|^{-1} + \frac{\log n}{\log(np)}\right)$ w.h.p., where $C > 0$ is a constant depending only on $f$.*

In Corollary 1.8(i), we stress that the worst-case consensus time on $G(n,p)$ was known for $p = \Omega(1)$ [37]. If $\frac{\log n}{\log(np)} = O(\log\log n)$ (or equivalently, $np = n^{\Omega(1/\log\log n)}$), Corollary 1.8(ii) implies $T_{\mathrm{cons}}(A) = O(\log\log n + \log|\delta(A)|^{-1})$ w.h.p.

▶ **Corollary 1.9.** *Let $k = k(n)$ be such that $k = \omega(1)$ and $k = O(\sqrt{n})$. Consider best-of-$(2k+1)$ on $G(n,p)$ for $p = \Omega(k/\sqrt{n})$. Then, for any $A \subseteq V$, $T_{\mathrm{cons}}(A) = O\left(\frac{\log n}{\log k}\right)$ holds w.h.p.*

From Corollary 1.9, best-of-$n^\epsilon$ on $G(n, n^{-1/2+\epsilon})$ for any constant $\epsilon \in (0, 1/2)$ reaches consensus in $O(1)$ steps. It is known that Majority on $G(n, Cn^{-1/2})$ satisfies $T_{\mathrm{cons}}(A) \le 4$ for large constant $C$ and random $A \subseteq V$ with constant probability [6].

For $3 \le d \le n/2$, $n$-vertex random $d$-regular graph $G_{n,d}$ is connected and $O(1/\sqrt{d})$-expander w.h.p. [13, 39].

▶ **Corollary 1.10.** *Consider a best-of-$k$ on an $n$-vertex random $d$-regular graph $G_{n,d}$ for an arbitrary constant $k \ge 2$. Then, $G_{n,d}$ w.h.p. satisfies the following:*

**(i)** *Suppose that $d = \Omega(n^{1/2})$ and $d \le n/2$. Then,*
  - **(a)** *for any $A \subseteq V$, $T_{\mathrm{cons}}(A) = O(\log n)$ w.h.p.*
  - **(b)** *for some $A \subseteq V$, $T_{\mathrm{cons}}(A) = \Omega(\log n)$ w.h.p.*

**(ii)** *Suppose that $d \ge C$ and $d \le n/2$ for a constant $C > 0$ depending only on $f$. Then, for any $A \subseteq V$ satisfying $|\delta(A)| \ge C\max\left\{\frac{1}{d}, \sqrt{\frac{\log n}{n}}\right\}$, it holds w.h.p. that $T_{\mathrm{cons}}(A) = O\left(\log\log n + \log|\delta(A)|^{-1} + \frac{\log n}{\log d}\right)$.*

▶ **Corollary 1.11.** *Let $k = k(n)$ be such that $k = \omega(1)$ and $k = O(\sqrt{n})$. Consider best-of-$(2k+1)$ on an $n$-vertex random $d$-regular graph $G_{n,d}$ such that $d = \Omega(k\sqrt{n})$ and $d \leq n/2$. Then, for any $A \subseteq V$, $T_{\mathrm{cons}}(A) = O\left(\frac{\log n}{\log k}\right)$ holds w.h.p.*

See the full version [38] for other specific results and examples of quasi-majority functional voting.

## 1.5 Related work

In asynchronous voting process, in each round, a vertex is selected uniformly at random and only the selected vertex updates its opinion. Cooper and Rivera [18] introduced *linear voting model*. In this model, an opinion configuration is represented as a vector $v \in \Sigma^V$ and the vector $v$ updates according to the rule $v \leftarrow Mv$, where $M$ is a random matrix sampled from some probability space. This model captures a wide variety model including asynchronous push/pull voting and synchronous pull voting. Note that best-of-two and best-of-three are not included in linear voting model. Schoenebeck and Yu [36] proposed an asynchronous variant of our functional voting. The authors of [36] proved that, if the function $f$ is symmetric (i.e., $f(1-x) = 1 - f(x)$), smooth and has "majority-like" property (i.e., $f(x) > x$ whenever $1/2 < x < 1$), then the expected consensus time is $O(n \log n)$ w.h.p. on $G(n,p)$ with $p = \Omega(1)$. This perspective has also been investigated in physics (see, e.g., [10]).

Several researchers have studied best-of-two and best-of-three on complete graphs initially involving $k \geq 2$ opinions [5, 4, 7, 25]. For example, the consensus time of best-of-three is $O(k \log n)$ if $k = O(n^{1/3}/\sqrt{\log n})$ [25]. Cooper, Radzik, Rivera, and Shiraga [17] considered best-of-two and best-of-three on regular expander graphs that hold more than two opinions.

Recently, Cruciani, Natale, and Scornavacca [20] studied best-of-two with a random initial configuration on a clustered regular graph. Shimizu and Shiraga [37] obtained phase-transition results of best-of-two and best-of-three on stochastic block models.

# 2 Preliminary and technical result

## 2.1 Formal definition

Let $G = (V, E)$ be an undirected and connected graph. Let $P \in [0,1]^{V \times V}$ be the matrix defined as

$$P(u,v) := \frac{\mathbb{1}_{\{u,v\} \in E}}{\deg(u)} \quad \forall (u,v) \in V \times V \tag{4}$$

where $\mathbb{1}_Z$ denotes the indicator of an event $Z$. For $v \in V$ and $S \subseteq V$, we write $P(v,S) = \sum_{s \in S} P(v,s)$.

Now, let us describe the formal definition of functional voting. For a given $A \subseteq V$, let $(X_v)_{v \in V}$ be independent binary random variables defined as

$$\begin{aligned} \mathbf{Pr}[X_v = 1] &= f\big(P(v,A)\big) \quad \text{if } v \in B, \\ \mathbf{Pr}[X_v = 0] &= f\big(P(v,B)\big) \quad \text{if } v \in A, \end{aligned} \tag{5}$$

where $B = V \setminus A$. For $A \subseteq V$ and $(X_v)$ above, define $A' = \{v \in V : X_v = 1\}$. Note that this definition coincides with Definition 1.1 since $P(v,A) = \frac{\deg_A(v)}{\deg(v)}$. Then, a functional voting is a Markov chain $A_0, A_1, \dots$ where $A_{t+1} = (A_t)'$.

For $A \subseteq V$, let $T_{\mathrm{cons}}(A)$ denote the consensus time of the functional voting starting from the initial configuration $A$. Formally, $T_{\mathrm{cons}}(A)$ is the stopping time defined as

$$T_{\mathrm{cons}}(A) := \min\{t \geq 0 : A_t \in \{\emptyset, V\}, A_0 = A\}.$$

## 2.2 Technical background

Consider best-of-two on a complete graph $K_n$ (with self loop on each vertex) with a current configuration $A \subseteq V$. Let $\alpha = |A|/n$. We have $P(v, A) = \alpha$ for any $v \in V$ and $A \subseteq V$. Then, for any $A \subseteq V$, $\mathbf{E}[\alpha'] = H_f(\alpha) = 3\alpha^2 - 2\alpha^3$. Thus, in each round, $\alpha' = 3\alpha^2 - 2\alpha^3 \pm O(\sqrt{\log n/n})$ holds w.h.p. from the Hoeffding bound. Therefore, the behavior of $\alpha$ can be written as the iteration of applying $H_f$.

The most technical part is the symmetry breaking at $\alpha = 1/2$. Note that $H_f(1/2) = 1/2$ and thus, the argument above does not work in the case of $|\alpha - 1/2| = o(\sqrt{\log n/n})$. To analyze this case, the authors of [21, 11] proved the following technical lemma asserting that $\alpha$ w.h.p. escapes from the area in $O(\log n)$ rounds.

▶ **Lemma 2.1** (Lemma 4.5 of [11] (informal)). *For any constant $C$, it holds w.h.p. that $|\alpha - 1/2| \geq C\sqrt{\log n/n}$ in $O(\log n)$ rounds (the hidden constant factor depends on $C$) if*

**(i)** *For any constant $h$, there is a constant $C_0 > 0$ such that, if $|\alpha - 1/2| = O(\sqrt{\log n/n})$ then $\mathbf{Pr}[|\alpha' - 1/2| > h/\sqrt{n}] > C_0$.*

**(ii)** *If $|\alpha - 1/2| = O(\sqrt{\log n/n})$ and $|\alpha - 1/2| = \Omega(1/\sqrt{n})$, $\mathbf{Pr}[|\alpha' - 1/2| \leq (1+\epsilon)|\alpha - 1/2|] \leq \exp(-\Theta((\alpha - 1/2)^2 n))$ for some constant $\epsilon > 0$.*

Intuitively speaking, the condition (ii) means that the bias $|\alpha' - 1/2|$ is likely to be at least $(1 + \epsilon)|\alpha - 1/2|$ for some constant $\epsilon > 0$. The condition (ii) is easy to check using the Hoeffding bound. The condition (i) means that $\alpha'$ has a fluctuation of size $\Omega(1/\sqrt{n})$ with a constant probability. We can check condition (i) using the Central Limit Theorem (the Berry-Esseen bound). The Central Limit Theorem implies that the normalized random variable $(\alpha' - \mathbf{E}[\alpha'])/\sqrt{\mathbf{Var}[\alpha']}$ converges to the standard normal distribution as $n \to \infty$. In other words, $\alpha'$ has a fluctuation of size $\Theta(\sqrt{\mathbf{Var}[\alpha']})$ with constant probability. Now, to verify the condition (i), we evaluate $\mathbf{Var}[\alpha']$. On $K_n$, it is easy to show that $\mathbf{Var}[\alpha'] = \Theta(1/n)$, which implies the condition (i).

The authors of [16, 17] considered best-of-two on expander graphs. They focused on the behavior of $\pi(A)$ instead of $\alpha$. Roughly speaking, they proved that $\mathbf{E}[\pi(A') - 1/2] \geq (1 + \epsilon)(\pi(A) - 1/2) - O(\lambda^2)$. At the heart of the proof, they showed the following result.

▶ **Lemma 2.2** (Special case of Lemma 3 of [17]). *Consider a $\lambda$-expander graph with degree distribution $\pi$. Then, for any $S \subseteq V$, $\left|\sum_{v\in V} \pi(v)P(v, S)^2 - \pi(S)^2\right| \leq \lambda^2\pi(S)\big(1 - \pi(S)\big)$.*

Then, from the Hoeffding bound, we have $\mathbf{E}[\pi(A') - 1/2] \geq (1 + \epsilon)(\pi(A) - 1/2) - O(\lambda^2 + \|\pi\|_2\sqrt{\log n}))$. Thus, if the initial bias $|\pi(A) - 1/2|$ is $\Omega(\max\{\lambda^2, \sqrt{\log n/n}\})$, we can show that the consensus time is $O(\log n)$.

Unfortunately, we can not apply the same technique to estimate $\mathbf{Var}[\pi(A')]$ on expander graphs, and due to this reason, it seems difficult to estimate the worst-case consensus time on expander graphs. Actually, any previous works put assumptions on the initial bias due to the same reason. It should be noted that Lemma 2.1 is well-known in the literature. For example, Cruciani et al. [20] used Lemma 2.1 from random initial configurations.

The technique of estimating $\mathbf{E}[\pi(A')]$ by Cooper et al. [16, 17] is specialized in best-of-two. Thus, it is not straightforward to prove the estimation of $\mathbf{E}[\pi(A')]$ for voting processes other than best-of-two.

## 2.3 Our technical contribution

For simplicity, in this part, we focus on a quasi-majority functional voting with respect to a *symmetric* function $f$ (i.e., $f(1 - x) = 1 - f(x)$ for every $x \in [0, 1]$) on a $\lambda$-expander graph with degree distribution $\pi$. For example, $f(x) = 3x^2 - 2x^3$ of best-of-three is a symmetric function. Note that $f = H_f$ if $f$ is symmetric. Similar results mentioned in this subsection holds for non-symmetric $f$ (see Lemma 3.5 and 3.6 of the full version [38]). For a $C^2$ function $h : \mathbb{R} \to \mathbb{R}$, let

$$K_1(h) := \max_{x\in[0,1]} |h'(x)|, \quad K_2(h) := \max_{x\in[0,1]} |h''(x)|$$

be constants[5] The following technical result enables us to estimate $\mathbf{E}[\pi(A')]$ and $\mathbf{Var}[\pi(A')]$ of functional voting.

▶ **Lemma 2.3.** *Consider a functional voting with respect to a symmetric $C^2$ function $f$ on a $\lambda$-expander graph with degree distribution $\pi$. Let $g(x) := f(x)(1 - f(x))$. Then, for all $A \subseteq V$,*

$$\left|\mathbf{E}[\pi(A')] - H_f(\pi(A))\right| \leq \frac{K_2(f)}{2}\lambda^2\pi(A)\bigl(1-\pi(A)\bigr),$$
$$\left|\mathbf{Var}[\pi(A')] - \|\pi\|_2^2 g\bigl(\pi(A)\bigr)\right| \leq K_1(g)\lambda\sqrt{\pi(A)\bigl(1-\pi(A)\bigr)}\|\pi\|_3^{3/2}.$$

Note that, if $f$ is symmetric, the corresponding functional voting satisfies that $\mathbf{Pr}[v \in A'] = f(P(v, A))$ for any $v \in V$. Thus we have

$$\mathbf{E}[\pi(A')] = \sum_{v\in V}\pi(v)f\bigl(P(v,A)\bigr), \quad \mathbf{Var}[\pi(A')] = \sum_{v\in V}\pi(v)^2 g\bigl(P(v,A)\bigr).$$

To evaluate $\mathbf{E}[\pi(A')]$ and $\mathbf{Var}[\pi(A')]$ above, we prove the following key lemma that is a generalization of Lemma 2.2 and implies Lemma 2.3.

▶ **Lemma 2.4** (Special case of Lemmas 3.2 and 3.3)**.** *Consider a $\lambda$-expander graph with degree distribution $\pi$. Then, for any $S \subseteq V$ and any $C^2$ function $h : \mathbb{R} \to \mathbb{R}$,*

$$\left|\sum_{v\in V}\pi(v)h\bigl(P(v,S)\bigr) - h\bigl(\pi(S)\bigr)\right| \leq \frac{K_2(h)}{2}\lambda^2\pi(S)\bigl(1-\pi(S)\bigr),$$
$$\left|\sum_{v\in V}\pi(v)^2 h\bigl(P(v,S)\bigr) - \|\pi\|_2^2 h\bigl(\pi(S)\bigr)\right| \leq K_1(h)\lambda\sqrt{\pi(S)\bigl(1-\pi(S)\bigr)}\|\pi\|_3^{3/2}.$$

## Non-symmetric functions

For general $f$, we prove the following.

▶ **Lemma 2.5.** *Consider a functional voting with respect to a $C^2$ function $f$ on a $\lambda$-expander graph. Let $g(x) := f(x)(1 - f(x))$. Then, for all $A \subseteq V$,*

$$\left|\mathbf{E}[\pi(A')] - H_f\bigl(\pi(A)\bigr)\right| \leq K_2(f)\lambda\bigl(|2\pi(A)-1| + \lambda\bigr)\pi(A)\bigl(1-\pi(A)\bigr),$$
$$\left|\mathbf{Var}[\pi(A')] - \|\pi\|_2^2 g\left(\frac{1}{2}\right)\right| \leq K_1(g)\left(\frac{1}{2}\|\pi\|_2^2\,|2\pi(A)-1| + 2\|\pi\|_3^{3/2}\lambda\sqrt{\pi(A)\bigl(1-\pi(A)\bigr)}\right).$$

We refer the proof of Lemma 2.5 to the full-version [38] due to the page limitation.

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[5] For example, for $f(x) = 3x^2 - 2x^3$ of best-of-three, $f''(x) = 6 - 12x$ and $K_2(f) = 6$. It should be noted that we deal with $f$ not depending on $G$ except for best-of-$k$ with $k = \omega(1)$.

## 2.4 Proof sketch of Theorem 1.3

We present proof sketch of Theorem 1.3(i). From the assumption of Theorem 1.3(i) and Lemma 2.3, if $|\pi(A) - 1/2| = o(1)$, we have $\mathbf{Var}[\pi(A')] = \Theta(\|\pi\|_2^2 g(\pi(A))) = \Theta(\|\pi\|_2^2 g(1/2 + o(1))) = \Theta(1/n)$. Moreover, $\mathbf{E}[\pi(A')] = H_f(\pi(A)) \pm O(\pi(A)/\sqrt{n})$ holds for any $A \subseteq V$. Hence, from the Hoeffding bound, $\pi(A') = H_f(\pi(A)) + O(\sqrt{\log n/n})$ holds w.h.p. for any $A \subseteq V$.

- If $|\pi(A) - 1/2| = O(\sqrt{\log n/n})$, we use Lemma 2.1 to obtain an $O(\log n)$ round symmetry breaking. In this phase, since $|\pi(A) - 1/2| = o(1)$, $\mathbf{Var}[\pi(A') - 1/2] = \Theta(1/n)$. Then, from the Berry-Esseen bound, we can check the condition (i). To check the condition (ii), we invoke the condition $H'_f(1/2) > 1$ of the quasi-majority function. From Taylor's theorem and the assumption of Lemma 2.1(ii) ($\pi(A) - 1/2 = \Omega(1/\sqrt{n})$), $\mathbf{E}[\pi(A') - 1/2] = H_f(\pi(A)) - H_f(1/2) - O(1/\sqrt{n}) \approx (1+\epsilon_1)(\pi(A) - 1/2)$ for some positive constant $\epsilon_1 > 0$. Note that $H_f(1/2) = 1/2$.
- If $C_1\sqrt{\log n/n} \le |\pi(A) - 1/2| \le C_2$ for sufficiently large constant $C_1$ and some constant $C_2 > 0$, we use the Hoeffding bound and then obtain $\pi(A') - 1/2 \approx (1 + \epsilon_1)(\pi(A) - 1/2) - O(\sqrt{\log n/n}) \ge (1 + (\epsilon_1/2))(\pi(A) - 1/2)$ w.h.p. Hence, $O(\log n)$ rounds suffice to yield a constant bias. (Note that this argument holds when $|\pi(A) - 1/2| \le C_2$ due to the remainder term of Taylor's theorem.)
- If $C_3 \le \pi(A) < 1/2$, it is straightforward to see that $\pi(A') = H_f(\pi(A)) + O(\sqrt{\log n/n}) \le \pi(A) - \epsilon_2$ w.h.p. for some constant $\epsilon_2 > 0$. Note that we invoke the property that $H_f(x) < x$ whenever $0 < x < 1/2$.
- If $\pi(A) \le C_3$ for sufficiently small constant $C_3$, we use the Markov inequality to show $\pi(A_t) = O(n^{-3})$ w.h.p. for some $t = O(\log n)$. Since $\pi(A) \ge 1/n^2$ whenever $A \ne \emptyset$, this implies that the consensus time is $O(\log n)$ w.h.p. Note that, since $H'_f(0) < 1$, we have $\mathbf{E}[\pi(A')] \le H_f(\pi(A)) + O(\pi(A)/\sqrt{n}) \approx H'_f(0)\pi(A) + O(\pi(A)/\sqrt{n}) \le (1 - \epsilon_3)\pi(A)$ for some constant $\epsilon_3 > 0$.

In the proof of Theorem 1.7, we modify Lemma 2.1 and apply the same argument.

# 3 Reversible Markov chains and Proof of Lemma 2.4

In this section, we prove Lemma 2.4 by showing Lemmas 3.2 and 3.3, which are generalizations of Lemma 2.4 in terms of *reversible Markov chain*. This enables us to evaluate $\mathbf{E}[\pi(A')]$ and $\mathbf{Var}[\pi(A')]$ for functional voting with respect to a $C^2$ function $f$ (see the full version [38] for functional voting with respect to non-symmetric $f$).

## 3.1 Technical tools for reversible Markov chains

To begin with, we briefly summarize the notation of Markov chain, which we will use in this section[6]. Let $V$ be a set of size $n$. A *transition matrix* $P$ over $V$ is a matrix $P \in [0,1]^{V\times V}$ satisfying $\sum_{v\in V} P(u,v) = 1$ for any $u \in V$. Let $\pi \in [0,1]^V$ denote the *stationary distribution* of $P$, i.e., a probability distribution satisfying $\pi P = \pi$. A transition matrix $P$ is *reversible* if $\pi(u)P(u,v) = \pi(v)P(v,u)$ for any $u, v \in V$. It is easy to check that the matrix (4) is

[6] For further detailed arguments about reversible Markov chains, see e.g., [29].

a reversible transition matrix and its stationary distribution is (3). Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $P$. If $P$ is reversible, it is known that $\lambda_i \in \mathbb{R}$ for all $i$. Let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ be the second largest eigenvalue in absolute value[7].

For a function $h : \mathbb{R} \to \mathbb{R}$ and subsets $S, T \subseteq V$, consider the quantity $Q_h(S,T)$ defined as

$$Q_h(S,T) := \sum_{v \in S} \pi(v) h\big(P(v,T)\big). \tag{6}$$

The special case of $h(x) = x$, that is, $Q(S,T) := \sum_{v \in S} \pi(v) P(v,T)$, is well known as *edge measure* [29] or *ergodic flow* [3, 31]. Note that, for any reversible $P$ and subsets $S, T \subseteq V$, $Q(S,T) = Q(T,S)$ holds. The following result is well known as a version of the *expander mixing lemma*.

▶ **Lemma 3.1** (See, e.g., p.163 of [29]). *Suppose $P$ is reversible. Then, for any $S, T \subseteq V$,*

$$|Q(S,T) - \pi(S)\pi(T)| \leq \lambda \sqrt{\pi(S)\pi(T)\big(1-\pi(S)\big)\big(1-\pi(T)\big)}.$$

We show the following lemma which gives a useful estimation of $Q_h(S,T)$.

▶ **Lemma 3.2.** *Suppose $P$ is reversible. Then, for any $S, T \subseteq V$ and any $C^2$ function $h : \mathbb{R} \to \mathbb{R}$,*

$$\Big|Q_h(S,T) - \pi(S)h\big(\pi(T)\big) - h'\big(\pi(T)\big)\big(Q(S,T) - \pi(S)\pi(T)\big)\Big| \leq \frac{K_2(h)}{2}\lambda^2 \pi(T)\big(1-\pi(T)\big).$$

**Proof of Lemma 3.2.** From Taylor's theorem, it holds for any $x, y \in [0,1]$ that

$$|h(x) - h(y) - h'(y)(x-y)| \leq \frac{K_2(h)}{2}(x-y)^2.$$

Hence

$$\begin{aligned}
&\Big|Q_h(S,T) - \pi(S)h\big(\pi(T)\big) - h'\big(\pi(T)\big)\big(Q(S,T) - \pi(S)\pi(T)\big)\Big| \\
&= \left|\sum_{v \in S} \pi(v)\Big(h\big(P(v,T)\big) - h\big(\pi(T)\big) - h'\big(\pi(T)\big)\big(P(v,T) - \pi(T)\big)\Big)\right| \\
&\leq \sum_{v \in S} \pi(v)\Big|h\big(P(v,T)\big) - h\big(\pi(T)\big) - h'\big(\pi(T)\big)\big(P(v,T) - \pi(T)\big)\Big| \\
&\leq \sum_{v \in S} \pi(v)\frac{K_2(h)}{2}\big(P(v,T) - \pi(T)\big)^2 \leq \frac{K_2(h)}{2}\sum_{v \in V} \pi(v)\big(P(v,T) - \pi(T)\big)^2 \\
&\leq \frac{K_2(h)}{2}\lambda^2 \pi(T)\big(1-\pi(T)\big).
\end{aligned}$$

The last inequality follows from Corollary A.2 of the full version [38]. ◀

Next, consider

$$R_h(S,T) := \sum_{v \in S} \pi(v)^2 h\big(P(v,T)\big) \tag{7}$$

for a function $h : \mathbb{R} \to \mathbb{R}$ and $S, T \subseteq V$. For notational convenience, for $S \subseteq V$, let $\pi_2(S) := \sum_{v \in S} \pi(v)^2$. We show the following lemma that evaluates $R_h(S,T)$.

[7] If $P$ is *ergodic*, i.e., for any $u, v \in V$, there exists a $t > 0$ such that $P^t(u,v) > 0$ and $\mathrm{GCD}\{t > 0 : P^t(x,x) > 0\} = 1$, $1 > \lambda_2$ and $\lambda_n > -1$. For example, the transition matrix of the simple random walk on a connected and non-bipartite graph is ergodic.

▶ **Lemma 3.3.** *Suppose that $P$ is reversible. Then, for any $S, T \subseteq V$ and any $C^2$ function $h : \mathbb{R} \to \mathbb{R}$,*

$$\left|R_h(S,T) - \pi_2(S)h\bigl(\pi(T)\bigr)\right| \le K_1(h)\|\pi\|_3^{3/2}\lambda\sqrt{\pi(T)\bigl(1-\pi(T)\bigr)}.$$

**Proof.** We first observe that

$$\left|h(x) - h(y)\right| \le K_1(h)|x-y| \tag{8}$$

holds for any $x, y \in [0,1]$ from Taylor's theorem. Hence,

$$\begin{aligned}
&\left|R_h(S,T) - \pi_2(S)h\bigl(\pi(T)\bigr)\right| \\
&= \left|\sum_{v\in S}\pi(v)^2\Bigl(h\bigl(P(v,T)\bigr) - h\bigl(\pi(T)\bigr)\Bigr)\right| \le \sum_{v\in S}\pi(v)^2\left|h\bigl(P(v,T)\bigr) - h\bigl(\pi(T)\bigr)\right| \\
&\le \sum_{v\in S}\pi(v)^2 K_1(h)\left|P(v,T) - \pi(T)\right| \le K_1(h)\sum_{v\in V}\pi(v)^2\left|P(v,T) - \pi(T)\right|.
\end{aligned}$$

Then, applying the Cauchy-Schwarz inequality and Corollary A.2 of the full version [38],

$$\begin{aligned}
\sum_{v\in V}\pi(v)^2\left|P(v,T) - \pi(T)\right| &\le \sqrt{\left(\sum_{v\in V}\pi(v)^3\right)\left(\sum_{v\in V}\pi(v)\bigl(P(v,T) - \pi(T)\bigr)^2\right)} \\
&\le \|\pi\|_3^{3/2}\lambda\sqrt{\pi(T)\bigl(1-\pi(T)\bigr)}
\end{aligned}$$

and we obtain the claim. ◀

▶ Remark 3.4. The results of this paper can be extended to voting processes where the sampling probability is determined by a reversible transition matrix $P$. This includes voting processes on edge-weighted graphs $G = (V, E, w)$, where $w : E \to \mathbb{R}$ denotes an edge weight function. Consider the transition matrix $P$ defined as follows: $P(u,v) = w(\{u,v\})/\sum_{x:\{u,x\}\in E} w(\{u,x\})$ for $\{u,v\} \in E$ and $P(u,v) = 0$ for $\{u,v\} \notin E$. A *weighted* functional voting with respect to $f$ is determined by $\mathbf{Pr}[v \in A' | v \in B] = f(P(v,B))$ and $\mathbf{Pr}[v \in B' | v \in A] = f(P(v,A))$. For simplicity, in this paper, we do not explore the weighted variant and focus on the usual setting where $P$ is the matrix (4) and its stationary distribution $\pi$ is (3).

## 3.2 Proof of Lemma 2.4

For the first inequality, by substituting $V$ to $S$ of Lemma 3.2, we obtain $\Bigl|Q_h(V,T) - h\bigl(\pi(T)\bigr)\Bigr| \le \frac{K_2(h)}{2}\lambda^2\pi(T)\bigl(1-\pi(T)\bigr)$. Note that $Q(V,T) = Q(T,V) = \pi(T)$ from the reversibility of $P$. Similarly, we obtain the second inequality by substituting $V$ to $S$ of Lemma 3.3. ◀

# 4 Proofs of Theorems 1.3 and 1.5

Consider a quasi-majority functional voting with respect to $f$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$. Let $A_0, A_1, \ldots,$ be the sequence given by the functional voting with initial configuration $A_0 \subseteq V$. Theorems 1.3 and 1.5 follow from the following lemma.

▶ **Lemma 4.1.** *Consider a quasi-majority functional voting with respect to $f$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$. Let $\epsilon_h(f) := H'_f(1/2) - 1$, $\epsilon_c(f) := 1 - H'_f(0)$ and $K(f) := \max\{K_2(f), K_2(H_f)\}$ be three positive constants depending only on $f$. Then, the following holds:*

**(I)** *Let $C_1 > 0$ be an arbitrary constant and $\varepsilon : \mathbb{N} \to \mathbb{R}$ be an arbitrary function satisfying $\varepsilon(n) \to 0$ as $n \to \infty$. Suppose that $\lambda \leq C_1 n^{-1/4}$, $\|\pi\|_2 \leq C_1/\sqrt{n}$ and $\|\pi\|_3 \leq \varepsilon/\sqrt{n}$. Then, for any $A_0 \subseteq V$ such that $|\delta(A_0)| \leq c_1 \log n/\sqrt{n}$ for an arbitrary constant $c_1 > 0$, $|\delta(A_t)| \geq c_1 \log n/\sqrt{n}$ within $t = O(\log n)$ steps w.h.p.*

**(II)** *Suppose that $\lambda \leq \frac{\epsilon_h(f)}{2K(f)}$. Then, for any $A_0 \subseteq V$ s.t. $\frac{2\max\{K(f),8\}}{\epsilon_h(f)} \max\{\lambda^2, \|\pi\|_2\sqrt{\log n}\} \leq |\delta(A_0)| \leq \frac{\epsilon_h(f)}{K(f)}$, $|\delta(A_t)| \geq \frac{\epsilon_h(f)}{K(f)}$ within $t = O(\log|\delta(A_0)|^{-1})$ steps w.h.p.*

**(III)** *Let $c_2, c_3$ be two arbitrary constants satisfying $0 < c_2 < c_3 < 1/2$ and $\epsilon(f) := \min_{x \in [c_2,c_3]}\big(x - H_f(x)\big)$ be a positive constant depending $f, c_2, c_3$. Suppose that $\lambda \leq \frac{\epsilon(f)}{2K(f)}$ and $\|\pi\|_2 \leq \frac{\epsilon(f)}{4\sqrt{\log n}}$. Then, for any $A_0 \subseteq V$ satisfying $c_2 \leq \pi(A_0) \leq c_3$, $\pi(A_t) \leq c_2$ within constant steps w.h.p.*

**(IV)** *Suppose that $\lambda \leq \frac{\epsilon_c(f)}{2K(f)}$ and $\|\pi\|_2 \leq \frac{\epsilon_c(f)^2}{32K(f)\sqrt{\log n}}$. Then, for any $A_0 \subseteq V$ satisfying $\pi(A_0) \leq \frac{\epsilon_c(f)}{8K(f)}$, $\pi(A_t) = 0$ within $t = O(\log n)$ steps w.h.p.*

**(V)** *Suppose that $H'_f(0) = 0$, $\lambda \leq \frac{1}{10K(f)}$ and $\|\pi\|_2 \leq \frac{1}{64K(f)\sqrt{\log n}}$. Then, for any $A_0 \subseteq V$ satisfying $\pi(A_0) \leq \frac{1}{7K(f)}$, it holds w.h.p. that $\pi(A_t) = 0$ within*

$$t = O\left(\log\log n + \frac{\log n}{\log \lambda^{-1}} + \frac{\log n}{\log(\|\pi\|_2\sqrt{\log n})^{-1}}\right) \textit{ steps.}$$

**Proof of Theorem 1.3(ii).** Since $\|\pi\|_2 \geq 1/\sqrt{n}$, we have $|\delta(A_0)| = \Omega(\sqrt{\log n/n})$. This implies that Phase (II) takes at most $O(\log n)$. Thus, we obtain the claim since we can merge Phases (II) to (IV) by taking appropriate constants $c_2, c_3$ in Phase (III). ◀

**Proof of Theorem 1.3(i).** Under the assumption of Theorem 1.3(i), for any positive constant $C$, a positive constant $C'$ exists such that $C(\lambda^2 + \|\pi\|_2\sqrt{\log n}) \leq C'\frac{\log n}{\sqrt{n}}$. Thus, we can combine Phase (I) and Theorem 1.3(ii), and we obtain the claim. ◀

**Proof of Theorem 1.5.** Combining Phases (II), (III) and (V), we obtain the claim. ◀

In the rest of this section, we show Phases (I) to (V) of Lemma 4.1. For notational convenience, let

$$\alpha := \pi(A),\ \alpha' := \pi(A'),\ \alpha_t := \pi(A_t), \delta := \delta(A) = 2\alpha - 1,\ \delta' := \delta(A'),\ \delta_t := \delta(A_t).$$

## 4.1 Phase (I): $0 \leq |\delta| \leq c_1 \log n/\sqrt{n}$

We use the following lemma to show Lemma 4.1(I).

▶ **Lemma 4.2** (Lemma 4.5 of [11]). *Consider a Markov chain $(X_t)_{t=1}^{\infty}$ with finite state space $\Omega$ and a function $\Psi : \Omega \to \{0, \ldots, n\}$. Let $C_3$ be arbitrary constant and $m = C_3\sqrt{n}\log n$. Suppose that $\Omega, \Psi$ and $m$ satisfies the following conditions:*

**(i)** *For any positive constant $h$, there exists a positive constant $C_1 < 1$ such that*

$$\mathbf{Pr}\left[\Psi(X_{t+1}) < h\sqrt{n} \,\middle|\, \Psi(X_t) \leq m\right] < C_1.$$

**(ii)** *Three positive constants $\gamma, C_2$ and $h$ exist such that, for any $x \in \Omega$ satisfying $h\sqrt{n} \leq \Psi(x) < m$,*

$$\mathbf{Pr}\left[\Psi(X_{t+1}) < (1+\gamma)\Psi(X_t) \mid X_t = x\right] < \exp\left(-C_2 \frac{\Psi(x)^2}{n}\right).$$

*Then, $\Psi(X_t) \geq m$ holds w.h.p. for some $t = O(\log n)$.*

Let us first prove the following lemma concerning the growth rate of $|\delta|$, which we will use in the proofs of (I) and (II) of Lemma 4.1.

▶ **Lemma 4.3.** *Consider a quasi-majority functional voting with respect to $f$ on an $n$-vertex $\lambda$-expander graph with degree distribution $\pi$. Let $\epsilon_h(f) := H_f'(1/2) - 1$ and $K(f) := \max\{K_2(f), K_2(H_f)\}$ be positive constants depending only on $f$. Suppose that $\lambda \leq \frac{\epsilon_h(f)}{2K(f)}$. Then, for any $A \subseteq V$ satisfying $\frac{2K(f)}{\epsilon_h(f)}\lambda^2 \leq |\delta| \leq \frac{\epsilon_h(f)}{K(f)}$,*

$$\mathbf{Pr}\left[|\delta'| \leq \left(1 + \frac{\epsilon_h(f)}{8}\right)|\delta|\right] \leq 2\exp\left(-\frac{\epsilon_h(f)^2\delta^2}{128\|\pi\|_2^2}\right).$$

**Proof.** Combining Lemma 2.5 and Taylor's theorem, we have

$$\begin{aligned}
\left|\mathbf{E}[\delta'] - H_f'\left(\frac{1}{2}\right)\delta\right| &= 2\left|\mathbf{E}[\alpha'] - \frac{1}{2} - H_f'\left(\frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)\right| \\
&= 2\left|\mathbf{E}\left[\alpha'\right] - H_f\left(\alpha\right) + H_f\left(\alpha\right) - H_f\left(\frac{1}{2}\right) - H_f'\left(\frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)\right| \\
&\leq 2K_2(f)\lambda\left(|\delta| + \lambda\right)\alpha(1-\alpha) + K_2(H_f)\left(\alpha - \frac{1}{2}\right)^2 \\
&\leq \left(\frac{K(f)}{2}\lambda + \frac{K(f)}{4}|\delta|\right)|\delta| + \frac{K(f)}{2}\lambda^2 \qquad (9)
\end{aligned}$$

Note that $H_f(1/2) = 1/2$ from the definition. From assumptions of $\lambda \leq \frac{\epsilon_h(f)}{2K(f)}$, $|\delta| \leq \frac{\epsilon_h(f)}{K(f)}$ and $\lambda^2 \leq \frac{\epsilon_h(f)}{2K(f)}|\delta|$, we have $\left|H_f'\left(\frac{1}{2}\right)\delta\right| - |\mathbf{E}[\delta']| \leq \left|H_f'\left(\frac{1}{2}\right)\delta - \mathbf{E}[\delta']\right| \leq \frac{3}{4}\epsilon_h(f)|\delta|$. Hence, it holds that

$$\left|\mathbf{E}[\delta']\right| \geq \left|H_f'\left(\frac{1}{2}\right)\delta\right| - \frac{3}{4}\epsilon_h(f)|\delta| = (1+\epsilon_h(f))|\delta| - \frac{3}{4}\epsilon_h(f)|\delta| = \left(1 + \frac{\epsilon_h(f)}{4}\right)|\delta|.$$

We observe that, for any $\kappa > 0$,

$$\mathbf{Pr}\left[|\delta'| \leq \left|\mathbf{E}[\delta']\right| - \kappa\right] \leq 2\exp\left(-\frac{\kappa^2}{2\|\pi\|_2^2}\right) \qquad (10)$$

from Corollary A.4 of the full version [38]. Note that $\delta' = \sum_{v\in V}\pi(v)(2X_v - 1)$ for independent indicator random variables $(X_v)_{v\in V}$ (see (5) for the definition of $X_v$). Thus,

$$\begin{aligned}
\mathbf{Pr}\left[|\delta'| \leq \left(1 + \frac{\epsilon_h(f)}{8}\right)|\delta|\right] &= \mathbf{Pr}\left[|\delta'| \leq \left(1 + \frac{\epsilon_h(f)}{4}\right)|\delta| - \frac{\epsilon_h(f)}{8}|\delta|\right] \\
&\leq \mathbf{Pr}\left[|\delta'| \leq \left|\mathbf{E}[\delta']\right| - \frac{\epsilon_h(f)}{8}|\delta|\right] \leq 2\exp\left(-\frac{\epsilon_h(f)^2\delta^2}{128\|\pi\|_2^2}\right)
\end{aligned}$$

and we obtain the claim. ◀

**Proof of Lemma 4.1(I).** We check the conditions (i) and (ii) of Lemma 4.2 with letting $\Psi(A) = \lfloor n|\delta(A)|\rfloor$ and $m = c_1\sqrt{n}\log n$.

**Condition (i).** First, we show the following claim that evaluates $\mathbf{Var}[\delta']$.

▷ Claim 4.4. Under the same assumption as Lemma 4.1(I),

$$\frac{\epsilon_{\mathrm{var}}(f)}{n} \le \mathbf{Var}[\delta'] \le \frac{5C_1^2}{n}$$

holds, where $\epsilon_{\mathrm{var}}(f) := f(1/2)(1 - f(1/2))$ is a positive constant depending only on $f$.

Proof of the claim. From Lemma 2.5 and assumptions, we have

$$\left|\frac{\mathbf{Var}[\delta']}{4} - \|\pi\|_2^2 g\left(\frac{1}{2}\right)\right| = \left|\mathbf{Var}[\alpha'] - \|\pi\|_2^2 g\left(\frac{1}{2}\right)\right| \le K_1(g)\left(\|\pi\|_2^2\frac{|\delta|}{2} + \|\pi\|_3^{3/2}\lambda\right)$$
$$\le \frac{K_1(g)}{n}\left(C_1^2 c_1\frac{\log n}{\sqrt{n}} + C_1\epsilon^{3/2}\right) = \frac{1}{n}\cdot o(1).$$

Note that $\mathbf{Var}[\delta'] = \mathbf{Var}[2\alpha' - 1] = 4\,\mathbf{Var}[\alpha']$. Since $\|\pi\|_2^2 \ge 1/n$, we have

$$\frac{\epsilon_{\mathrm{var}}(f)}{n} \le \frac{4\epsilon_{\mathrm{var}}(f) - o(1)}{n} \le \mathbf{Var}[\delta'] \le \frac{4C_1^2 + o(1)}{n} \le \frac{5C_1^2}{n}. \qquad \lhd$$

From Corollary A.6 of the full version [38] with letting $Y_v = \pi(v)(2X_v - 1)$, we have

$$\mathbf{Pr}\left[|\delta'| \le x\sqrt{\frac{\epsilon_{\mathrm{var}}(f)}{n}}\right] \le \mathbf{Pr}\left[|\delta'| \le x\sqrt{\mathbf{Var}[\delta']}\right] \le \Phi(x) + \frac{5.6\|\pi\|_3^3}{\mathbf{Var}[\delta']^{3/2}}$$
$$\le \Phi(x) + 5.6\frac{\epsilon^3}{n^{3/2}}\cdot\frac{n^{3/2}}{\epsilon_{\mathrm{var}}(f)^{3/2}} = \Phi(x) + o(1) \tag{11}$$

for any $x \in \mathbb{R}$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y$. Thus, for any constant $h > 0$, there exists some constant $C > 0$ such that $\mathbf{Pr}[\Psi(A') < h\sqrt{n} \mid \Psi(A) \le m] < C$, which verifies the condition (i).

**Condition (ii).** Set $h = \frac{2K(f)}{\epsilon_h(f)}C_1^2$ and assume $h\sqrt{n} \le \Psi(A) < m$. Then

$$\frac{2K(f)}{\epsilon_h(f)}\lambda^2 n \le \frac{2K(f)}{\epsilon_h(f)}C_1^2\sqrt{n} = h\sqrt{n} \le \Psi(A) \le |\delta|n = o(n).$$

Thus, we can apply Lemma 4.3 and positive constants $\gamma, C$ exist such that, for any $h\sqrt{n} \le \Psi(A) \le c_1\sqrt{n}\log n$, $\mathbf{Pr}[\Psi(A') < (1+\gamma)\Psi(A)] < \exp\left(-C\frac{\Psi(A)^2}{n}\right)$. Note that $\|\pi\|_2^2 = \Theta(1/n)$ from the assumption. Thus the condition (ii) holds and we can apply Lemma 4.2. ◀

## 4.2 Phase (II): $\frac{2\max\{K(f),8\}}{\epsilon_h(f)}\max\{\lambda^2, \|\pi\|_2\sqrt{\log n}\} \le |\delta| \le \frac{\epsilon_h(f)}{K(f)}$

**Proof of Lemma 4.1(II).** Since $|\delta| \ge \frac{16}{\epsilon_h(f)}\|\pi\|_2\sqrt{\log n}$ from assumptions, applying Lemma 4.3 yields $\mathbf{Pr}\left[|\delta'| \le \left(1 + \frac{\epsilon_h(f)}{8}\right)|\delta|\right] \le \frac{2}{n^2}$. Thus, it holds with probability larger than $(1 - 2/n^2)^t$ that $|\delta_t| \ge \left(1 + \frac{\epsilon_h(f)}{8}\right)^t |\delta_0|$ and we obtain the claim by substituting $t = O(\log|\delta_0|^{-1})$. ◀

## 4.3 Phase (III): $0 < c_2 \le \alpha \le c_3 < 1/2$

**Proof of Lemma 4.1(III).** We first observe that, for any $\kappa > 0$,

$$\mathbf{Pr}\left[|\alpha' - \mathbf{E}[\alpha']| \ge \kappa\|\pi\|_2\sqrt{\log n}\right] \le 2n^{-2\kappa} \tag{12}$$

from the Hoeffding theorem. Note that $\alpha' = \sum_{v\in V}\pi(v)X_v$ for independent indicator random variables $(X_v)_{v\in V}$. Hence, applying Lemma 2.5 yields

$$|\alpha' - H_f(\alpha)| \le |\alpha' - \mathbf{E}[\alpha']| + |\mathbf{E}[\alpha'] - H_f(\alpha)| \le \|\pi\|_2\sqrt{\log n} + \frac{K_2(f)}{4}(|\delta| + \lambda)\lambda \tag{13}$$

with probability larger than $1 - 2/n^2$. Then, for any $\alpha \in [c_2, c_3]$, it holds with probability larger than $1 - 2/n^2$ that

$$\alpha' \le H_f(\alpha) + \frac{K(f)}{2}\lambda + \|\pi\|_2\sqrt{\log n} \le \alpha - \epsilon(f) + \frac{\epsilon(f)}{4} + \frac{\epsilon(f)}{4} \le \alpha - \frac{\epsilon(f)}{2}.$$

Thus, for $\alpha_0 \in [c_2, c_3]$, $\alpha_t \le c_2$ within $t = 2(c_3 - c_2)/\epsilon(f) = O(1)$ steps w.h.p. ◀

## 4.4 Phase (IV): $0 \le \alpha \le \frac{\epsilon_c(f)}{8K(f)}$

We show the following lemma which is useful for proving (IV) and (V) of Lemma 4.1.

▶ **Lemma 4.5.** *Let $\epsilon \in (0, 1]$ be an arbitrary constant. Consider functional voting on an $n$-vertex connected graph with degree distribution $\pi$. Suppose that, for some $\alpha_* \in [0, 1]$ and $K \in [0, 1-\epsilon]$, $\mathbf{E}[\alpha'] \le K\alpha$ holds for any $A \subseteq V$ satisfying $\alpha \le \alpha_*$ and $\|\pi\|_2 \le \frac{\epsilon\alpha_*}{2\sqrt{\log n}}$. Then, for any $A_0 \subseteq V$ satisfying $\alpha_0 \le \alpha_*$, $\alpha_t = 0$ w.h.p. within $O\left(\frac{\log n}{\log K^{-1}}\right)$ steps.*

**Proof.** For any $\alpha \le \alpha_*$, from (12) and assumptions of $\mathbf{E}[\alpha'] \le \alpha$ and $\|\pi\|_2 \le \frac{\epsilon\alpha_*}{2\sqrt{\log n}}$, it holds with probability larger than $1 - 2/n^4$ that

$$\alpha' \le \mathbf{E}[\alpha'] + 2\|\pi\|_2\sqrt{\log n} \le K\alpha + \epsilon\alpha_* \le (1-\epsilon)\alpha_* + \epsilon\alpha_* = \alpha_*.$$

Thus, for any $\alpha_0 \le \alpha_*$, we have

$$\begin{aligned}
\mathbf{E}[\alpha_t] &= \sum_{x \le a_*} \mathbf{E}\left[\alpha_t | \alpha_{t-1} = x\right] \mathbf{Pr}\left[\alpha_{t-1} = x\right] + \sum_{x > a_*} \mathbf{E}\left[\alpha_t | \alpha_{t-1} = x\right] \mathbf{Pr}\left[\alpha_{t-1} = x\right] \\
&\le \sum_{x \le a_*} Kx\,\mathbf{Pr}\left[\alpha_{t-1} = x\right] + \mathbf{Pr}\left[\alpha_{t-1} > a_*\right] \le K\,\mathbf{E}[\alpha_{t-1}] + \frac{2t}{n^4} \\
&\le \cdots \le K^t\alpha_0 + \frac{2t^2}{n^4} \le K^t + \frac{2t^2}{n^4}.
\end{aligned}$$

This implies that, $\mathbf{E}[\alpha_t] = O(n^{-3})$ within $t = O\left(\frac{\log n}{\log K^{-1}}\right)$ steps. Let $\pi_{\min} := \min_{v\in V}\pi(v) \ge 1/(2|E|) \ge 1/n^2$. We obtain the claim from the Markov inequality, which yields $\mathbf{Pr}[\alpha_t = 0] = 1 - \mathbf{Pr}[\alpha_t \ge \pi_{\min}] \ge 1 - \frac{\mathbf{E}[\alpha_t]}{\pi_{\min}} = 1 - O(1/n)$. ◀

**Proof of Lemma 4.1 of (IV).** Combining Lemma 2.5 and Taylor's theorem,

$$\begin{aligned}
\left|\mathbf{E}[\alpha'] - H_f'(0)\alpha\right| &= \left|\mathbf{E}[\alpha'] - H_f(\alpha) + H_f(\alpha) - H_f(0) - H_f'(0)(\alpha - 0)\right| \\
&\le K_2(f)\lambda\left(|\delta| + \lambda\right)\alpha(1-\alpha) + \frac{K_2(H_f)}{2}\alpha^2 \\
&\le 2K(f)\lambda\alpha + \frac{K(f)}{2}\alpha^2.
\end{aligned} \tag{14}$$

Hence, for any $\alpha \le \frac{\epsilon_c(f)}{8K(f)}$, we have $\mathbf{E}[\alpha'] \le \left(H'_f(0) + 2K(f)\lambda + \frac{K(f)}{2}\alpha\right)\alpha \le \left(1 - \frac{\epsilon_c(f)}{2}\right)\alpha$. Letting $\epsilon = \epsilon_c(f)/2$, $K = 1 - \epsilon_c(f)/2$ and $\alpha_* = \frac{\epsilon_c(f)}{8K(f)}$, from the assumption, $\|\pi\|_2 \le \frac{\epsilon_c(f)^2}{32K(f)\sqrt{\log n}} = \frac{\epsilon\alpha_*}{2\sqrt{\log n}}$. Thus, we can apply Lemma 4.5 and we obtain the claim. ◀

## 4.5 Phase (V): $H'_f(0) = 0$ and $0 \le \alpha \le \frac{1}{7K(f)}$

**Proof of Lemma 4.1(V).** In this case, from (14),

$$\mathbf{E}[\alpha'] \le 2K(f)\lambda\alpha + \frac{K(f)}{2}\alpha^2. \tag{15}$$

We consider the following two cases.

**Case 1.** $\max\left\{\lambda, \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\right\} \le \alpha \le \frac{1}{7K(f)}$: In this case, combining (12) and (15), it holds with probability larger than $1 - 2/n^2$ that

$$\alpha' \le \left(\frac{2K(f)\lambda}{\alpha} + \frac{K(f)}{2} + \frac{\|\pi\|_2\sqrt{\log n}}{\alpha^2}\right)\alpha^2 \le \frac{7K(f)}{2}\alpha^2.$$

Applying this inequality iteratively, for any $\alpha_0 \le 7K(f)^{-1}$,

$$\alpha_t \le \frac{7K(f)}{2}\alpha_{t-1}^2 \le \cdots \le \frac{2}{7K(f)}\left(\frac{7K(f)}{2}\alpha_0\right)^{2^t} \le \frac{2}{7K(f)2^{2^t}}.$$

holds with probability larger than $(1 - 2/n^2)^t$. This implies that, within $t = O(\log\log n)$ steps, $\alpha_t \le \max\left\{\lambda, \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\right\}$ w.h.p. Note that $\max\left\{\lambda, \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\right\} \ge \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}} \ge \sqrt{\frac{\sqrt{\log n/n}}{K(f)}}$ since $\|\pi\|_2^2 \ge 1/n$.

**Case 2.** $\alpha \le \max\left\{\lambda, \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\right\}$: Set $\alpha_* = \max\left\{\lambda, \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\right\} \ge \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}$, $K = \frac{5K(f)}{2}\lambda + \frac{1}{2}\sqrt{K(f)\|\pi\|_2\sqrt{\log n}}$ and $\epsilon = 1/4$. Then, from $\lambda \le \frac{1}{10K(f)}$ and $\|\pi\|_2 \le \frac{1}{64K(f)\sqrt{\log n}}$, we have $K \le 1 - \epsilon$,

$$\|\pi\|_2 = (\sqrt{\|\pi\|_2})^2 \le \frac{\sqrt{\|\pi\|_2}}{8\sqrt{K(f)\sqrt{\log n}}} = \sqrt{\frac{\|\pi\|_2\sqrt{\log n}}{K(f)}}\frac{\epsilon}{2\sqrt{\log n}} \le \frac{\epsilon\alpha_*}{2\sqrt{\log n}},$$

$$\mathbf{E}[\alpha'] \le \left(2K(f)\lambda + \frac{K(f)}{2}\alpha\right)\alpha \le \left(2K(f)\lambda + \frac{K(f)}{2}\lambda + \frac{1}{2}\sqrt{K(f)\|\pi\|_2\sqrt{\log n}}\right)\alpha = K\alpha.$$

Thus, applying Lemma 4.5, we obtain the claim. ◀

# 5 Conclusion

In this paper we propose functional voting as a generalization of several known voting processes. We show that the consensus time is $O(\log n)$ for any quasi-majority functional voting on $O(n^{-1/2})$-expander graphs with balanced degree distributions. This result extends previous works concerning voting processes on expander graphs. Possible future direction of this work includes

1. Does $O(\log n)$ worst-case consensus time holds for quasi-majority functional voting on graphs with less expansion (i.e., $\lambda = \omega(n^{-1/2})$)?
2. Is there some relationship between best-of-$k$ and Majority?

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